THE FREE-BOUNDARY PROBLEMS IN INVISCID MAGNETOHYDRODYNAMICS WITH OR WITHOUT SURFACE TENSION

by

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Abstract

The free-boundary problems in magnetohydrodynamics (MHD) describe the motion of conducting fluids in electromagnetic fields. Such problems usually arise from the plasma confinement problems and some astrophysical phenomena, e.g., the propagation of solar wind. The thesis records the results for the local well-posedness (LWP) of the free-boundary problems in incompressible MHD with and without surface tension [52, 53, 28, 29] (joint with Xumin Gu and Chenyun Luo), compressible resistive MHD [82, 83], and compressible ideal MHD [50] (joint with Hans Lindblad).

For incompressible ideal MHD, we record a comprehensive study for the case with surface tension [53, 28, 29] which are the first breakthrough in the mathematical study of this direction. The proof relies on the tangential smoothing, penalization method and a new-developed cancellation structure enjoyed by the Alinhac good unknowns. When the surface tension is neglected, we present a minimal regularity result (for LWP) in a small fluid domain [52].

Compressible ideal MHD is a hyperbolic system with characteristic boundary conditions. When the magnetic field is parallel to the surface, the loss of normal derivatives cannot be compensated due to the failure of div-curl analysis. On the one hand, we observe that such derivative loss is exactly compensated by the magnetic diffusion. Based on this, we prove the LWP and the incompressible limit for compressible resistive MHD [82, 83]. On the other hand, we adopt the anisotropic Sobolev spaces together with the “modified" Alinac good unknowns to study compressible ideal MHD system. We establish the first result [50] on the nonlinear *a priori* estimates without loss of regularity for the free-boundary compressible ideal MHD system, which greatly improves the existing results proved by Nash-Moser iteration.

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Chapter 1

Mathematical Formulation and Backgrounds

We are concerned with 3D free-boundary magnetohydrodynamic (MHD) system

$$\begin{cases} \rho D_t u = -\nabla p + j \times B, \quad j := \nabla \times B & \text{in } \mathcal{D}, \\ D_t \rho + \rho(\nabla \cdot u) = 0 & \text{in } \mathcal{D}, \\ \partial_t B = -\nabla \times E, \quad E := -u \times B + \lambda j & \text{in } \mathcal{D}, \\ \nabla \cdot B = 0 & \text{in } \mathcal{D}, \end{cases} \tag{1.0.1}$$

which describes the motion of an inviscid conducting fluid (plasma) in an electro-magnetic field. Here $\mathcal{D} := \bigcup_{0 \le t \le T} \{t\} \times \mathcal{D}_t$ and $\mathcal{D}_t \subset \mathbb{R}^3$ is the domain occupied by the conducting fluid whose boundary $\partial\mathcal{D}_t$ moves with the velocity of the fluid. The operator $\nabla := (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ is the standard spatial derivative and the operator $D_t := \partial_t + u \cdot \nabla$ is the material derivative. The quantities u, p, ρ denote the fluid velocity, the fluid pressure and the fluid density. The quantities B, E, j denote the magnetic field, the induced electric field and the current density.

We always assume the density ρ satisfies $\rho \ge \bar{\rho}_0 > 0$ where $\bar{\rho}_0$ is a positive constant, i.e., we assume the fluid is a liquid. In the case of $\rho \equiv \bar{\rho}_0$, we say the fluid is incompressible and assume $\bar{\rho}_0 = 1$; otherwise we say the fluid is compressible. In the compressible case, the fluid pressure satisfies $p = p(\rho, S)$ with $\frac{\partial p}{\partial \rho} > 0$, where S, the entropy of the fluid, satisfies $\rho D_t S = 0$. In the thesis, we assume S to be a constant, i.e., we only consider the isentropic case; otherwise we say the compressible fluid is non-isentropic.

In physics, $j = \nabla \times B$ is the Amperè's law and $E = -u \times B + \lambda \nabla \times B$ is the Ohm's law where $\lambda \geq 0$ is the magnetic diffusivity constant. When $\lambda = 0$, we say (1.0.1) is ideal MHD system, otherwise we call (1.0.1) resistive MHD. Under the setting above, system (1.0.1) becomes

$$\begin{cases} \rho D_t u - (B \cdot \nabla) B = -\nabla P, \quad P := p + \frac{1}{2}|B|^2 & \text{in } \mathcal{D}, \\ D_t \rho + \rho(\nabla \cdot u) = 0 & \text{in } \mathcal{D}, \\ D_t B + \lambda \nabla \times (\nabla \times B) = (B \cdot \nabla) u - B(\nabla \cdot u) & \text{in } \mathcal{D}, \\ \nabla \cdot B = 0 & \text{in } \mathcal{D}, \end{cases} \tag{1.0.2}$$

where $P := p + \frac{1}{2}|B|^2$ is called the total pressure. In the incompressible case, the quantities u, p, B and the region $\mathcal{D}$ are the unknowns to be determined. In the compressible case, the quantities u, p, ρ, B and the region $\mathcal{D}$ are the unknowns to be determined with the equation of state $p = p(\rho)$.

We would like to study the Cauchy problem of (1.0.2) and thus the boundary conditions and the initial data need to be specified. The boundary conditions for *ideal* MHD ($\lambda = 0$) are

$$\text{Velocity}(\partial \mathcal{D}_t) = u \cdot \hat{n} \quad \text{on } \partial \mathcal{D} \tag{1.0.3a}$$

$$P = \sigma \mathcal{H} \quad \text{on } \partial \mathcal{D}, \tag{1.0.3b}$$

$$B \cdot \hat{n} = 0 \quad \text{on } \partial \mathcal{D}, \tag{1.0.3c}$$

where $\hat{n}$ denotes the unit exterior normal vector to $\partial \mathcal{D}_t$, $\mathcal{H}$ denotes the mean curvature of $\partial \mathcal{D}_t$ and the positive constant $\sigma \geq 0$ denotes the surface tension coefficient. Condition (1.0.3a) means the boundary moves with the motion of the fluid, and it can be equivalently expressed as "$D_t \in \mathcal{T}(\partial \mathcal{D})$" or "$(1, u)$ is tangent to $\partial \mathcal{D}$" where $\mathcal{T}(\partial \mathcal{D})$ denotes the tangential bundle of $\partial \mathcal{D}$. Condition (1.0.3b) is the pressure balance law and shows that outside the fluid region is a vacuum. Condition (1.0.3c) shows that the plasma is a perfect conductor.

Remark 1.0.1. The equation $\nabla \cdot B = 0$ is not an independent equation. When $\lambda = 0$, the boundary condition (1.0.3c) is not an imposed boundary condition. If else, the system would be over-determined. Instead, they are both *constraints for the initial data*, i.e., they automatically propagate to any $t > 0$ if initially holds. In fact, one can take the divergence in the third equation of (1.0.2) and use the second equation to derive $D_t(\rho^{-1}(\nabla \cdot B)) = 0$, and take D_t in (1.0.3c) to prove its propagation.

Remark 1.0.2. When the magnetic diffusivity constant $\lambda > 0$, the boundary condition (1.0.3c) should be replaced by the Dirichlet boundary condition

$$B = \mathbf{0} \quad \text{on } \partial\mathcal{D}. \tag{1.0.4}$$

See Section 1.1.1 for the illustration. Concerning the identity $\nabla \times (\nabla \times B) = -\triangle B + \nabla(\nabla \cdot B)$ and using $\nabla \cdot B = 0$, the third equation in (1.0.2) is parabolic when $\lambda > 0$ and thus (1.0.4) has to be an imposed condition.

Remark 1.0.3. When $\sigma = 0$ in (1.0.3b), i.e., the surface tension is neglected, we also need the Rayleigh-Taylor sign condition

$$-\nabla P \cdot \hat{n} \geq c_0 > 0 \tag{1.0.5}$$

where $c_0 > 0$ is a constant and $P := p + \frac{1}{2}|B|^2$ is the total pressure. When $B = \mathbf{0}$, Ebin [22] proved the ill-posedness of the free-boundary incompressible Euler equations when the Rayleigh-Taylor sign condition is violated. Hao-Luo [34] proved that the free-boundary incompressible MHD system is ill-posed when (1.0.5) fails. We also note that (1.0.5) is only required for initial data and it propagates in a short time interval because one can prove it is $C_{t,x}^{0,\frac{1}{4}}$ Hölder continuous by using Morrey's embedding. See [52, Lemma 5.5] for the proof.

Energy conservation/dissipation. System (1.0.2) equipped with the boundary conditions (1.0.3a)-(1.0.3c) and (1.0.4) gives the following energy conservation for $\lambda = 0$ and dissipation for $\lambda > 0$:

Define $Q(\rho) = \int_{\bar{\rho}_0}^{\rho} p(r)/r^2 \, \mathrm{d}r$, then we have

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\int_{\mathcal{D}_t} \rho|u|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathcal{D}_t} |B|^2 \,\mathrm{d}x + \int_{\mathcal{D}_t} \rho Q(\rho)\,\mathrm{d}x + \sigma \int_{\partial\mathcal{D}_t} \mathrm{d}S(\partial\mathcal{D}_t)\right) \\ =& -\lambda \int_{\mathcal{D}_t} |\nabla B|^2 \,\mathrm{d}x \le 0.\end{aligned} \tag{1.0.6}$$

See [53, 83] for the proof.

Equation of states. When the fluid is compressible, we need to specify the equation of state. In the thesis, we only consider the case of a liquid, and impose the following natural conditions for some fixed constant $A_0 > 1$ and $m \le 8$

$$A_0^{-1} \le |\rho^{(m)}(p)| \le A_0. \tag{1.0.7}$$

When proving the incompressible limit in Section 5.2, we need to require the following conditions for $1 \le m \le 6$

$$|\rho^{(m)}(p)| \le A_0, \;\; A_0^{-1}|\rho'(p)|^m \le |\rho^{(m)}(p)| \le A_0|\rho'(p)|^m. \tag{1.0.8}$$

Compatibility conditions on the initial data. To make the Cauchy problem solvable, we need to choose suitable initial data $(u_0, B_0, \rho_0, p_0, \mathcal{D}_0)$. In particular, the magnetic field should satisfy $\nabla \cdot B_0 = 0$ and $B_0 \cdot \hat{n}|_{\{0\}\times\mathcal{D}_0} = 0$ for $\lambda = 0$ ($B_0|_{\{0\}\times\mathcal{D}_0} = \mathbf{0}$ for $\lambda > 0$). The total pressure $P_0 := p_0 + \frac{1}{2}|B_0|^2$ should satisfy $P_0 = \sigma\mathcal{H}|_{\{0\}\times\partial\mathcal{D}_0}$. In the compressible case, we require the following k-th order compatibility conditions

$$D_t^j P|_{\{0\}\times\partial\mathcal{D}_0} = D_t^j(\sigma\mathcal{H})|_{\{0\}\times\partial\mathcal{D}_0} \text{ at } t = 0, \; \forall 0 \le j \le k, \tag{1.0.9}$$

and also the following one for $\lambda > 0$

$$D_t^j B|_{\{0\}\times\partial\mathcal{D}_0} = \mathbf{0} \text{ at } t = 0, \; \forall 0 \le j \le k. \tag{1.0.10}$$

Given a simply-connect domain $\mathcal{D}_0 \subset \mathbb{R}^3$ and the initial data (u_0, B_0, ρ_0, p_0) satisfying the constraints $\nabla \cdot B_0 = 0$ in $\mathcal{D}_0$ and $(B_0 \cdot \hat{n})|_{\{0\}\times\partial\mathcal{D}_0} = 0$ for $\lambda = 0$ ($B_0|_{\{0\}\times\partial\mathcal{D}_0} = \mathbf{0}$ for $\lambda > 0$), we want to find a set $\mathcal{D}$, the velocity u, the magnetic field B, and the density ρ solving (1.0.2) satisfying the boundary conditions in (1.0.3a)-(1.0.5). Specifically, we will record the following results in the thesis:

1. The minimal regularity $H^{2.5+\varepsilon}$ estimates of incompressible ideal MHD. See Chapter 4.1.
2. Local well-posedness, the zero surface tension limit, the $H^{3.5}$ (low regularity) estimates of incompressible ideal MHD with surface tension. See Chapter 4.2 and 4.3.
3. Local well-posedness and the incompressible limit of compressible resistive MHD. See Chapter 5.2.
4. Anisotropic *a priori* estimates of compressible ideal MHD. See Chapter 5.3.

1.1 Background in Physics

The free-boundary problems in MHD arise from the MHD current-vortex sheets and the plasma-vacuum interface model. The former one can be used to describe the heliopause (the theoretical boundary of the solar system) observed in the propagation of solar winds, the nightside magnetopause of the earth. The current-vortex sheets are mathematically formulated as the plasma-plasma interface problem: The motion of plasmas are governed by MHD system, and the jump conditions on the interface are

$$[P] := P^+ - P^- = \sigma\mathcal{H}, B^\pm \cdot \hat{n} = 0, [v]^\pm \cdot \hat{n} = 0. \tag{1.1.1}$$

In other words, there is no jump allowed in the normal direction.

The plasma-vacuum model describes the plasma confinement: The plasma is confined in a vacuum[1] in which there is another magnetic field B^-, and there is a free interface $\Gamma(t)$, moving with the motion

[1] Usually people use a low-density plasma, especially the vacuum, to confine a high-density plasma.

of plasma, between the plasma region $\Omega_+(t)$ and the vacuum region $\Omega_-(t)$. This model requires that (2.4.1) holds in the plasma region $\Omega_+(t)$ and the pre-Maxwell system holds in vacuum $\Omega_-(t)$:

$$\nabla \times B^- = \mathbf{0}, \quad \nabla \cdot B^- = 0. \tag{1.1.2}$$

On the interface $\Gamma(t)$, it is required that there is no jump for the pressure or the normal components of the magnetic fields:

$$B^\pm \cdot \hat{n} = 0, \quad [P] := p^+ + \frac{1}{2}|B^+|^2 - \frac{1}{2}|B^-|^2 = \sigma\mathcal{H} \tag{1.1.3}$$

Finally, there is a rigid wall W wrapping the vacuum region, on which the following boundary condition holds

$$B^- \times \hat{N} = \mathbf{J} \quad \text{on } W,$$

where $\mathbf{J}$ is the given outer surface current density (as an external input of energy) and $\hat{N}$ is the exterior normal to the rigid wall W. Note that for ideal MHD, the conditions div $B = 0$ and $B \cdot n = 0$ should also be constraints for initial data instead of imposed conditions. For details we refer to [24, Chapter 4, 6]. When the surface tension is not neglected, the model is used to characterize the motion of liquid metal which is useful in the fusion process. See Molokov [57] for detailed discussion.

Hence, the free-boundary problem (2.4.1) can be considered as the case that the vacuum magnetic field B^- vanishes. It characterizes the motion of an isolated perfect conducting fluid in an electromagnetic field.

Remark 1.1.1. For current-vortex sheets and plasma-vacuum models without surface tension, the Rayleigh-Taylor condition $-\nabla[P] \cdot \hat{n} \geq c_0 > 0$ may be not sufficient for the local well-posedness. Instead, the Syrovatskij type condition $|B^+ \times B^-| \geq c_0' > 0$ is required on the free interface, which in fact enhances extra 1/2-order regularity of the free interface. See [68, 73].

1.1.1 Illustration on the jump conditions on the free interface

The jump conditions (1.1.1) and (1.1.3) actually comes from the Rankine-Hugoniot conditions for hyperbolic conservation laws. One may rewrite the (non-isentropic) compressible MHD system (without surface tension) in the conservative form

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0 \\ \partial_t (\rho u) + \nabla \cdot \left(\rho u \otimes u + (p + \frac{1}{2}|B|^2) I_d - B \otimes B \right) = 0 \\ \partial_t B + \nabla \cdot (u \otimes B - B \otimes u) = 0 \\ \nabla \cdot B = 0 \\ \partial_t (\rho S) + \nabla \cdot (\rho S u) = 0. \end{cases} \tag{1.1.4}$$

If we exclude the possibility of MHD shocks (i.e., we do not allow the mass flow transferring across the interface), then $u' := u - V\hat{n}$ satisfies $u'_n = 0$ on the interface where V denotes the velocity of the moving interface. We then conclude the Rankine-Hugoniot conditions as follows

$$[p + \frac{1}{2} B_\tau^2] = 0,\ B_n [B_\tau] = 0,\ B_n [u'_\tau \cdot B_\tau] = 0,\ [B_n] = 0,\ B_n [u'_\tau] = 0. \tag{1.1.5}$$

Then we have two possibilities (for ideal MHD without surface tension)

1. MHD contact discontinuity. If the magnetic field intersects the interface ($B_n \neq 0$), then we have

   - jump: $[\rho] \neq 0$,
   - continuous: $[u] = \mathbf{0},\ [p] = 0,\ [B] = \mathbf{0}$.

   Examples are mostly observed in astrophysical phenomena, e.g., the solar wind, fast coronal mass ejections, where the magnetic fields typically originate in a star and intersect the surface.

2. Tangential discontinuities: If the magnetic field is parallel to the interface ($B_n = 0$), then

- jump: $[\rho] \neq 0,\ [u'_\tau] \neq 0,\ [p] \neq 0,\ [B_\tau] \neq 0,$

- continuous: $u'_n = 0,\ B_n = 0, [p + \frac{1}{2}|B_\tau|^2] = 0.$

Since $[B] \neq 0$ in this case, the surface current $j^* := [B] \times \hat{n} \neq \mathbf{0}$, and thus we call the interface as a "current"-vortex sheet. Examples mostly arise from laboratory plasmas aimed at thermonuclear energy production: confine a high-density plasma by a lower-density one to isolate it thermally from an outer wall. There are also astrophysical examples, e.g., the heliopause of solar system that separates the interstellar plasma compressed at the bow shock from the solar wind plasma compressed at the termination shock.

Remark 1.1.2. As for resistive MHD, the jump condition for $B^\pm$ must be the Dirichlet-type condition $[B] = \mathbf{0}$. Indeed, the divergence-free condition for B implies $[B] \cdot \hat{n} = 0$. When the electric resistivity is nonzero, the surface current is not allowed on the interface when doing the perturbation and thus $[B] \times \hat{n} = \mathbf{0}$. See [36] for details.

1.2 Overview of Previous Results

In the past a few decades, there have been numerous studies of free-boundary inviscid fluids. In the absence of magnetic field, the MHD system becomes Euler equations.

Free-boundary Euler equations The free-boundary Euler equations have been studied intensively by a lot of authors. The first breakthrough in solving the LWP for the incompressible irrotational problem for general initial data came in the work of Wu [79, 80]. In the case of nonzero vorticity, Christodoulou-Lindblad [13] first established the a priori estimates and then Lindblad [46, 47] proved the LWP by using Nash-Moser iteration. Coutand-Shkoller [16, 17] proved the LWP for incompressible Euler equations with or without surface tension and avoid the loss of regularity. We also refer to [85, 1, 65] and references therein.

The study of compressible perfect fluid is not quite developed as opposed to the incompressible case. Lindblad [48] established the first LWP result by Nash-Moser iteration. See also [72, 49, 51, 49, 23, 54] for the further study. In the case of nonzero surface tension, we refer to [15, 20].

Free-boundary MHD equations: Incompressible case The study of free-boundary MHD is more complicated than Euler equations due to the strong coupling between fluid and magnetic field and the failure of irrotational assumption. For incompressible ideal MHD, Hao-Luo [33, 35] established the a priori estimates and linearized LWP. Gu-Wang [30] proved the LWP. Luo-Zhang [52] proved the low regularity a priori estimates when the fluid domain is small. We also mention that Lee [43, 44] obtained a local solution via the vanishing viscosity-resistivity limit.

For the full plasma-vacuum model, Gu [25, 26] proved the LWP for the axi-symmetric case with nontrivial vacuum magnetic field in a non-simply connected vacuum domain under Rayleigh-Taylor sign condition. Hao [32] proved the LWP in the case of $\mathbf{J} = \mathbf{0}$. For the general case, all of the results require the Syrovatskij condition $|B \times \hat{B}| \geq c_0 > 0$ on the free interface. Under this condition, the results are due to Morando-Trakhinin-Trebeschi [58] and Sun-Wang-Zhang [67]. We also note that the study of the full plasma-vacuum model in ideal MHD under Rayleigh-Taylor sign condition is still an open problem when $\hat{B}$ is non-trivial with $\mathbf{J} \neq \mathbf{0}$. For incompressible current-vortex sheets, we refer to Coulombel-Morando-Secchi-Trebeschi [14] and Sun-Wang-Zhang [66].

For incompressible ideal MHD with surface tension, Luo-Zhang [53] proved the a priori estimates and Gu-Luo-Zhang [28, 29] proved the LWP and the zero surface tension limit. For incompressible dissipative MHD with surface tension, we refer to Chen-Ding [8] for the inviscid limit for viscous non-resistive MHD, Wang-Xin [78] for the GWP of the plasma-vacuum model for inviscid resistive MHD around a uniform transversal magnetic field, and Padula-Solonnikov [61] and Guo-Zeng-Ni [31] for viscous-resistive MHD.

Free-boundary MHD equations: Compressible case Compared with compressible Euler equations and incompressible MHD, compressible MHD has an extra coupling between the sound wave and the magnetic field which makes the analysis completely different. Here we emphasize that there is a normal derivative loss in the div-curl analysis of compressible MHD. On the one hand, the second author [82, 83] recently observed that the magnetic resistivity exactly compensates the derivative loss mentioned above. However, it is still hopeless to derive the vanishing resistivity limit. On the other hand, one can still expect to establish the tame estimates for the linearized equation. Based on this and Nash-Moser iteration, Trakhinin-Wang [74, 75] recently proved the LWP for free-boundary compressible ideal MHD with or without surface tension. We also mention that Chen-Wang [9] and Trakhinin [71] proved the LWP for the current-vortex sheets, and Secchi-Trakhinin [64] proved the LWP for the full plasma-vacuum problem for compressible ideal MHD under the non-collinearity condition. However, there is a big loss of regularity caused by the Nash-Moser iteration ($H^{\frac{m}{2}+6}$ loss with $H^{m+\frac{3}{2}}$ data for $m \geq 20$). Finding suitable estimates without loss of regularity is still a widely open problem. Our paper [50] was the first breakthrough in this direction. It also leaves open the possibility for the further study of current-vortex sheets and plasma-vacuum models which are the original models in the interface plasma physics. Also it may provide a new, comprehensive approach to study the nonlinear hyperbolic system with characteristic (free) boundary conditions arising in the study of inviscid fluids, e.g., the nonlinear stability and incompressible limit of compressible vortex sheets which are related to the suppression of the Kelvin-Helmholtz instability.

Chapter 2

Reformulation in Lagrangian Coordinates and Main Results

We use Lagrangian coordinates to reduce the free-boundary problem to a fixed-domain problem. We assume $\Omega := \mathbb{T}^2 \times (-1, 1)$ to be the reference domain and $\Gamma := \mathbb{T}^2 \times (\{-1\} \cup \{1\})$ to be the boundary. The coordinates on Ω is $y := (y', y_3) = (y_1, y_2, y_3)$. We define $\eta : [0, T] \times \Omega \to \mathcal{D}$ as the flow map of velocity field u, i.e.,

$$\partial_t \eta(t, y) = u(t, \eta(t, y)), \quad \eta(0, y) = \eta_0(y), \tag{2.0.1}$$

where η_0 is a diffeomorphism between Ω and $\mathcal{D}_0$. For technical simplicity we assume $\eta_0 = \text{Id}$, i.e., the initial domain is assumed to be $\mathcal{D}_0 = \mathbb{T}^2 \times (-1, 1)$. By chain rule, it is easy to see that the material derivative D_t becomes ∂_t in the (t, y) coordinates and the free-boundary $\partial\mathcal{D}_t$ becomes fixed ($\Gamma = \mathbb{T}^2 \times (\{-1\} \cup \{1\})$). We introduce the Lagrangian variables as follow: $v(t, y) := u(t, \eta(t, y))$, $b(t, y) := B(t, \eta(t, y))$, $q(t, y) := p(t, \eta(t, y))$, $Q(t, y) := P(t, \eta(t, y))$ and $R(t, y) := \rho(t, \eta(t, y))$.

In the thesis, we adapt Einstein summation convetion, i.e., the repeated indices imply taking summation on this index. All the Greek indices range over 1, 2, 3 and the Latin indices range over 1, 2. Let $\partial = \partial_y$ be the spatial derivative in Lagrangian coordinates and we define $\text{div}\, Y = \partial_\alpha Y^\alpha$ to be the (Lagrangian) divergence of the vector field Y. We introduce the matrix $A = [\partial\eta]^{-1}$, specifically

$A^{\mu\alpha} := \frac{\partial y^\mu}{\partial x^\alpha}$ where $x^\alpha = \eta^\alpha(t, y)$ is the α-th variable in Eulerian coordinates. From now on, we define $\nabla_A^\alpha = \frac{\partial}{\partial x^\alpha} = A^{\mu\alpha}\partial_\mu$ to be the covariant derivative in Lagrangian coordinates (or say Eulerian derivative), $\text{div}_A X := \nabla_A \cdot X = A^{\mu\alpha}\partial_\mu X_\alpha$ and $(\text{curl}_A X)_\alpha := \epsilon_{\alpha\beta\gamma}A^{\mu\beta}\partial_\mu X_\gamma$ to be the Eulerian divergence and curl of the vector field X. In addition, since $\eta(0,\cdot) = \text{Id}$, we have $A(0,\cdot) = I$, where I is the identity matrix, and (u_0, B_0, p_0) and (v_0, b_0, q_0) agree respectively.

In terms of η, v, b, q, R, the free-boundary MHD system reads

$$
\begin{cases}
\partial_t \eta = v & \text{in } [0,T]\times\Omega \\
R\partial_t v = (b\cdot\nabla_A) b - \nabla_A Q, \quad Q = q + \frac{1}{2}|b|^2 & \text{in } [0,T]\times\Omega \\
\partial_t R + R\text{div}_A v = 0 & \text{in } [0,T]\times\Omega \\
\partial_t b + \lambda \text{curl}_A \text{curl}_A b = (b\cdot\nabla_A) v - b\text{div}_A v & \text{in } [0,T]\times\Omega \\
\text{div}_A b = 0 & \text{in } [0,T]\times\Omega \\
b_\alpha A^{\mu\alpha} N_\alpha = 0 & \text{in } [0,T]\times\Omega \quad (\lambda = 0) \\
b = \mathbf{0} & \text{in } [0,T]\times\Omega \quad (\lambda > 0) \\
Q = 0, -\frac{\partial Q}{\partial N}|_\Gamma \geq c_0 > 0 & \text{on } [0,T]\times\Gamma \quad (\sigma = 0) \\
A^{\mu\alpha} N_\mu Q = -\sigma\sqrt{g}\triangle_g \eta^\alpha & \text{on } [0,T]\times\Gamma \quad (\sigma > 0) \\
(\eta, v, b, q, R)|_{t=0} = (\text{Id}, v_0, b_0, q_0, \rho_0), &
\end{cases}
\tag{2.0.2}
$$

where $N = (0,0,\pm 1)$ denotes the unit outer normal vector to $\Gamma = \mathbb{T}^2 \times \{\pm 1\}$ and $\triangle_g$ is the Laplace-Beltrami operator of the metric g_{ij} on $\mathcal{D}_t = \eta(t, \Gamma)$ induced by the embedding η. Specifically we have

$$
g_{ij} = \overline{\partial}_i \eta^\mu \overline{\partial}_j \eta_\mu, \quad \triangle_g(\cdot) = g^{-1}\overline{\partial}_i(\sqrt{g}g^{ij}\overline{\partial}_j(\cdot)), \quad g = \det[g_{ij}]. \tag{2.0.3}
$$

Throughout the thesis, we use $\overline{\partial}$ to emphasis that the derivative is tangential to Γ.

Let $J := \det[\partial\eta]$ and $\hat{A} := JA$. Then we have the Piola's identity $\partial_\mu \hat{A}^{\mu\alpha} = 0$ and J satisfies $\partial_t J = J\text{div}_A v$ which together with $\partial_t R + R\text{div}_A v = 0$ gives that $\rho_0 = RJ$.

Suppose D is the derivative ∂ or ∂_t, then we have the following identity

$$
DA^{\mu\alpha} = -A^{\mu\nu}\,\partial_\beta D\eta_\nu\, A^{\beta\alpha}. \tag{2.0.4}
$$

When $\lambda = 0$, the magnetic field b can be expressed in terms of b_0 and η so that the magnetic field is just a parametre instead of an independent unknown. This is called the "frozen effect of the magnetic field" which means each fluid particle never separates from the magnetic field line passing through it.

Lemma 2.0.1 ([50, Lemma 1.1]). We have $b = J^{-1}(b_0 \cdot \partial)\eta$ for non-resistive MHD.

2.1 Low-Regularity Estimates of Incompressible Ideal MHD

Under the setting above, the free-boundary incompressible ideal MHD system reads

$$\begin{cases} \partial_t \eta = v & \text{in } [0,T]\times\Omega \\ \partial_t v - (b_0\cdot\partial)^2\eta = -\nabla_A Q, \quad Q = q + \frac{1}{2}|b|^2 & \text{in } [0,T]\times\Omega \\ \text{div}_A v = 0 & \text{in } [0,T]\times\Omega \\ \text{div}\, b_0 = 0 & \text{in } [0,T]\times\Omega \\ b_0^3 = 0 & \text{in } [0,T]\times\Omega \\ Q = 0, \ -\frac{\partial Q_0}{\partial N}|_\Gamma \geq c_0 > 0 & \text{on } [0,T]\times\Gamma. \\ (\eta, v)|_{t=0} = (\text{Id}, v_0). \end{cases} \tag{2.1.1}$$

The local well-posedness of (2.1.1) was proved by Gu-Wang [30] in H^4 regularity. However, the low-regularity solution to (2.1.1) has not been studied before. In the absence of magnetic field, (2.1.1) reduces to the incompressible Euler equations whose local existence in $\mathbb{R}^3$ holds iff the regularity of initial data is strictly higher than $H^{2.5}(\mathbb{R}^3)$. See Bourgain-Li [6] for the ill-posedness result with $H^{2.5}(\mathbb{R}^3)$-data. On the other hand, Kukavica-Tuffaha-Vicol [40] proved the $H^{2.5+\varepsilon}$ regularity estimates for the free-boundary incompressible Euler equations in a bounded simply-connected domain. We are then interested to study if the similar low-regularity estimates can be established for incompressible ideal MHD. For simplicity of notations, we define $\|\cdot\|_s$ and $|\cdot|_s$ to be the standard Sobolev norms in Ω and on Γ respectively.

Theorem 2.1.1 ([52, Theorem 1.1]). Let Ω be the thin domain $\mathbb{T}^2 \times (-\bar{\varepsilon}, \bar{\varepsilon})$ for some $\bar{\varepsilon} \ll 1$ and $\delta \ll \frac{1}{2}$ be a given small constant. Let (η, v, q) be the solution to (2.1.1) with initial data

$(v_0, b_0) \in H^{2.5+\delta}(\Omega) \times H^{2.5+\delta}(\Omega)$ satisfying div $v_0 =$ div $b_0 = 0$ and $b_0^3|_{\partial\Omega} = 0$ and the Rayleigh-Taylor sign condition. Let

$$N(t) := \|\eta(t)\|_3^2 + \|v(t)\|_{2.5+\delta}^2 + \|(b_0 \cdot \partial)\eta(t)\|_{2.5+\delta}^2 + |\bar{\partial}^{2.5+\delta}\eta \cdot \hat{n}|_0^2. \tag{2.1.2}$$

Then there exists $\bar{T} = \bar{T}(N(0), \bar{\varepsilon}, c_0) > 0$ such that

$$\sup_{0 \le t \le \bar{T}} N(t) \le P(N(0)), \tag{2.1.3}$$

where $P(\cdots)$ always denotes a polynomial with positive coefficients of its arguments.

Remark 2.1.2. The smallness of the fluid domain is unavoidable. Based on the Cauchy invariance for Euler equations, one can gain $1/2$-order extra regularity for the flow map than the velocity [1, 40], which is then not possible for ideal MHD. In Gu-Wang [30], they adopted the Alinhac good unknowns to avoid the extra $1/2$-order regularity, but the least required regularity for such method has to be H^4, equivalently, the second fundamental form of the free surface must be continuous.

2.2 Well-posedness, Zero Surface Tension Limit, and Low-regularity Estimates of Incompressible Ideal MHD with Surface Tension

When the surface tension is not neglected, we have

$$\begin{cases} \partial_t \eta = v & \text{in } [0,T] \times \Omega \\ \partial_t v - (b_0 \cdot \partial)^2 \eta = -\nabla_A Q, \quad Q = q + \frac{1}{2}|b|^2 & \text{in } [0,T] \times \Omega \\ \text{div}_A v = 0 & \text{in } [0,T] \times \Omega \\ \text{div } b_0 = 0 & \text{in } [0,T] \times \Omega \\ b_0^3 = 0 & \text{in } [0,T] \times \Omega \\ A^{\mu\alpha} N_\mu Q = -\sigma \sqrt{g} \triangle_g \eta^\alpha & \text{on } [0,T] \times \Gamma \\ (\eta, v)|_{t=0} = (\text{Id}, v_0). \end{cases} \tag{2.2.1}$$

To the best of our knowledge, the following theorems are the *first breakthrough in the study of ideal MHD with surface tension.*

Theorem 2.2.1 (Local well-posedness [28, Theorem 1.2])**.** Let $v_0 \in H^{4.5}(\Omega) \cap H^5(\Gamma)$ and $b_0 \in$

$H^{4.5}(\Omega)$ be divergence-free vector fields with $(b_0 \cdot N)|_\Gamma = 0$. Then there exists $T > 0$, only depending on σ, v_0, b_0, such that (2.2.1) with initial data (v_0, b_0, q_0) has a unique strong solution (η, v, q) with energy estimate

$$\sup_{0\le t\le T} E(t) \le C(\sigma^{-1}, \|v_0\|_{4.5}, \|b_0\|_{4.5}, |v_0|_5), \tag{2.2.2}$$

where the energy functional $E(t)$ is

$$\begin{aligned} E(t) := \|\eta(t)\|_{4.5}^2 + \sum_{k=0}^{3}\left(\left\|\partial_t^k v(t), \partial_t^k (b_0\cdot\partial)\eta(t)\right\|_{4.5}^2\right) + \left\|\partial_t^4 v(t), \partial_t^4 (b_0\cdot\partial)\eta(t)\right\|_0^2 \\ + \sum_{k=0}^{3}\left|\overline{\partial}\left(\Pi\partial_t^{3-k}\overline{\partial}^k v(t)\right)\right|_0^2 + \left|\overline{\partial}\big(\Pi\overline{\partial}^3(b_0\cdot\partial)\eta(t)\big)\right|_0^2. \end{aligned} \tag{2.2.3}$$

Moreover, the $H^5(\Gamma)$-regularity of v on the free-surface can also be recovered, in the sense that there exists some $0 < T_1 \le T$, depending only on σ^{-1}, v_0, b_0, such that

$$\sup_{0\le t\le T_1} |\eta(t)|_5^2 + |v(t)|_5^2 \le C(\sigma^{-1}, \|v_0\|_{4.5}, \|b_0\|_{4.5}, |v_0|_5). \tag{2.2.4}$$

The proof of Theorem 2.2.1 relies on the adjusted tangential smoothing as an approximation scheme by combining the ideas of [30] and [16]. We shall first establish the *a priori* estimates, uniform in the smoothing parametre, for the approximate system. This will be proved by div-curl decomposition, tangential estimates which together with the surface tension equation also give the boundary energies. Then we solve the nonlinear approximate system by freezing the coefficients (linearization) and Picard iteration. The frozen-coefficient (linearized) problem is solved by the penalization method and Galerkin approximation. The enhanced boundary regularity relies on the BMO-coefficient elliptic estimates in [21]. Note that the energy estimate established in Theorem 2.2.1 relies on σ^{-1}. When the surface tension is almost negligible, one may ask if it is possible to establish the uniform-in-σ estimate. The answer is yes.

Theorem 2.2.2 (Zero surface tension limit [29]). Suppose the initial data (v_0, b_0) satisfies

1. div $b_0 = $ div $v_0 = 0$, $b_0^3|_\Gamma = 0$

2. $v_0, b_0 \in H^5(\Omega)$, $\sqrt{\sigma}v_0, \sqrt{\sigma}b_0 \in H^{5.5}(\Omega)$, $\sigma v_0^3, \sigma b_0^3 \in H^{5.5}(\Gamma)$ and $\sigma^{\frac{3}{2}} v_0^3, \sigma^{\frac{3}{2}} b_0^3 \in H^6(\Gamma)$

3. The Rayleigh-Taylor sign condition $-\partial Q_0/\partial N \geq c_0 > 0$ on Γ for all $\sigma > 0$.

4. The compatibility conditions up to 4-th order, where the j-th order condition reads $\partial_t^j q|_{t=0} = \sigma \partial_t^j \mathcal{H}|_{t=0}$ on Γ.

There exists $T' > 0$ independent of σ such that the solution $(v^\sigma, (b_0 \cdot \partial)\eta^\sigma, Q^\sigma)$ to (2.2.1) satisfies

$$\sup_{0\leq t\leq T'} E^\sigma(t) \leq C(c_0, \|(v_0, b_0)\|_5, \|(\sqrt{\sigma}v_0, \sqrt{\sigma}b_0)\|_{5.5}, |(\sigma v_0, \sigma b_0)|_{5.5}, |(\sigma^{\frac{3}{2}} v_0, \sigma^{\frac{3}{2}} b_0)|_6) \quad (2.2.5)$$

where $E^\sigma(t) := E_1^\sigma(t) + E_2^\sigma(t)$ and

$$E_1^\sigma(t) := \|\eta^\sigma(t)\|_5^2 + \sum_{k=0}^{5} \left(\left\| \partial_t^k v^\sigma(t) \right\|_{5-k}^2 + \left\| \partial_t^k (b_0 \cdot \partial)\eta^\sigma(t) \right\|_{5-k}^2 \right) + \left| \overline{\partial}^5 \eta^\sigma(t) \cdot \hat{n}(t) \right|_0^2 \quad (2.2.6)$$

and

$$\begin{aligned} E_2^\sigma(t) := \|\eta^\sigma(t)\|_{5.5}^2 &+ \sum_{k=0}^{4} \left(\left\| \partial_t^k v^\sigma(t) \right\|_{5.5-k}^2 + \left\| \partial_t^k (b_0 \cdot \partial)\eta^\sigma(t) \right\|_{5.5-k}^2 \right) \\ &+ \sum_{k=0}^{5} \left| \overline{\partial}^{6-k} \partial_t^k \eta^\sigma(t) \cdot \hat{n}(t) \right|_0^2 + \left| \overline{\partial}^5 (b_0 \cdot \partial)\eta^\sigma(t) \cdot \hat{n}(t) \right|_0^2 . \end{aligned} \quad (2.2.7)$$

Hence by the Arzelà-Ascoli lemma and Morrey's embedding, we have

$$(v^\sigma, (b_0 \cdot \partial)\eta^\sigma, Q^\sigma) \xrightarrow{C^1_{t,y}([0,T]\times\Omega)} (w, (b_0 \cdot \partial)\zeta, r), \text{ as } \sigma \to 0, \quad (2.2.8)$$

where $(w, (b_0 \cdot \partial)\zeta, r)$ solves (2.1.1) with initial data (v_0, b_0, Q_0). Moreover, the higher boundary regularity of v in (2.2.5) can also be recovered

$$\forall t \in (0, T'],\ |\sigma v^3(t)|_{5.5} + |\sigma b^3(t)|_{5.5} + |\sigma^{\frac{3}{2}} v^3(t)|_6 + |\sigma^{\frac{3}{2}} b^3(t)|_6 \leq P(E^\sigma(t)). \quad (2.2.9)$$

Remark 2.2.3. The initial data Q_0^σ is solved by the elliptic equation $-\triangle Q_0^\sigma = (\partial v_0)(\partial v_0) - (\partial b_0)(\partial b_0)$ with $Q_0^\sigma = \sigma \mathcal{H}_0$. When η_0 = Id, we have $\mathcal{H}_0 = 0$ and thus $Q_0^\sigma = Q_0$. For general diffeomorphism

$\eta_0 \neq \mathrm{Id}$, the initial data of Q^σ is no longer Q_0 for the $\sigma = 0$ problem. Yet we can still prove that $Q_0^\sigma \xrightarrow{C^1} Q_0$. For detailed discussion on the compatibility conditions. we refer to [29, Appendix A] .

When proving the local existence in Theorem 2.2.1, the normal trace of v is controlled by the BMO-coefficient elliptic estimates for the time-differentiated surface tension equation and thus σ^{-1} appears. In order for the uniform-in-σ estimates, we use the normal trace lemma (cf. Lemma 3.2.3) to convert the boundary normal trace estimate to the interior tangential estimate. Then we apply the Alinhac good unknowns to avoid the higher regularity of η. The Alinhac good unknowns also reveal a cancellation structure that simultaneously gives the non-weighted boundary regularity contributed by the Rayleigh-Taylor sign condition, the weighted boundary regularity contributed by the surface tension, and an anti-symmetric structure that eliminates the uncontrollable terms on the boundary.

Next, concerning the low-regularity solutions, we are able to generalize Disconzi-Kukavica [18] to incompressible MHD with surface tension. Due to the presence of surface tension, we need neither the smallness of the fluid domain nor the extra regularity of the flow map as in Theorem 2.1.1. Compared with Theorem 2.2.1 and 2.2.2, the second fundamental form of the free surface may not be L^∞ and thus the Alinhac good unknown method is no longer valid. Instead, we can use the Kato-Ponce inequalities (cf. Lemma 3.2.1) together with boundary elliptic estimates to overcome such difficulty.

Theorem 2.2.4 (Low-regularity estimates [53, Theorem 1.1])**.** Assume that $v_0 \in H^{3.5}(\Omega) \cap H^4(\Gamma)$ and $b_0 \in H^{3.5}(\Omega)$ to be divergence free vector fields with $b_0 \cdot N = 0$ on Γ. Assume that $(v, (b_0 \cdot \partial)\eta, Q)$ solves (2.2.1) with initial data v_0 and b_0. Define

$$\begin{aligned}\mathfrak{N}(t) =& \|\eta(t)\|_{3.5}^2 + \sum_{k=0}^{2} \left(\|\partial_t^k v(t)\|_{3.5-k}^2 + \|\partial_t^k (b_0 \cdot \partial)\eta(t)\|_{3.5-k}^2 \right) \\ &+ \|\partial_t^3 v(t)\|_0^2 + \|\partial_t^3 (b_0 \cdot \partial)\eta(t)\|_0^2 + \left|\overline{\partial}(\Pi \partial_t^2 v)\right|_0^2 + \left|\overline{\partial}(\Pi \overline{\partial}\partial_t v)\right|_0^2\end{aligned} \tag{2.2.10}$$

Then there exists a $T'' > 0$, chosen sufficiently small, such that $\mathfrak{N}(t) \le C_0$ for all $t \in [0, T]$, where C_0 only depends on σ^{-1}, $\|v_0\|_{3.5}$, $\|b_0\|_{3.5}$, $|v_0|_4$.

2.3 Well-posedness and Incompressible Limit of Compressible Resistive MHD

Next we take into account of the compressibility of the plasma that results in the coupling between sound wave and magnetic field. The system of compressible ideal MHD in the case of a liquid is a strictly hyperbolic system with characteristic boundary conditions. The failure of the uniform Kreiss-Lopatinskiĭ condition leads to a potential of normal derivative loss. Even worse, the div-curl analysis does not work in the control of normal derivatives. Concerning the degenerate boundary condition $B \cdot \hat{n} = 0$, the loss of normal derivatives may not be compensated. Indeed, Ohno-Shirota [59] proved the ill-posedness in standard Sobolev spaces H^l $(l \geq 2)$.

We found two ways to avoid such derivative loss. On the one hand, we found that the magnetic diffusion, together with the Christodoulou-Lindblad type ellipeic estimate (cf. Lemma 3.3.3), gives common control of both magnetic fields and sound waves, as well as the Lorentz force that appears to be a higher order term. On the other hand, inspired by Chen Shu-Xing [10], we can study the compressible ideal MHD system in the anisotropic Sobolev spaces instead of standard Sobolev spaces. The anisotropy is expected to compensate the loss of normal derivatives.

Let us first introduce the results about compressible resistive MHD

$$\begin{cases} \partial_t \eta = v & \text{in } [0,T]\times\Omega \\ \rho_0 \partial_t v = J\,(b\cdot\nabla_A)\,b - \nabla_{\mathbf{A}} Q,\ \ Q = q + \frac{1}{2}|b|^2 & \text{in } [0,T]\times\Omega \\ \frac{JR'(q)}{\rho_0}\partial_t q + \text{div}_A v = 0 & \text{in } [0,T]\times\Omega \\ q = q(R) \text{ strictly increasing} & \text{in } [0,T]\times\overline{\Omega} \\ \partial_t b + \lambda \text{curl}_A \text{curl}_A b = (b\cdot\nabla_A)\,v - b\,\text{div}_A v & \text{in } [0,T]\times\Omega \\ \text{div}_A b = 0 & \text{in } [0,T]\times\Omega \\ b = \mathbf{0},\ q = 0,\ -\frac{\partial Q_0}{\partial N}|_\Gamma \geq c_0 > 0 & \text{on } [0,T]\times\Gamma \\ (\eta, v, b, q)|_{t=0} = (\text{Id}, v_0, b_0, q_0). \end{cases} \tag{2.3.1}$$

Theorem 2.3.1 (Local well-posedness [83, Theorem 1.1])**.** Let the initial data $(v_0, b_0, q_0) \in H^4(\Omega) \times H^5(\Omega) \times H^4(\Omega)$ satisfy the compatibility conditions (1.0.9) up to 5-th order, div $b_0 = 0$ and $b_0|_\Gamma = \mathbf{0}$. Then there exists some $T_1 > 0$, such that the system (2.3.1) has a unique solution (η, v, b, q) in $[0, T_1]$

satisfying the following estimates

$$\sup_{0\le T\le T_1} \mathcal{E}(T) \le P\left(\|v_0\|_4, \|b_0\|_5, \|q_0\|_4\right), \tag{2.3.2}$$

where

$$\mathcal{E}(T) := \mathfrak{e}(T) + H(T) + W(T) + \sum_{k=0}^{4} \left\|\partial_t^{4-k}\left((b\cdot\nabla_A)\, b\right)\right\|_k^2, \tag{2.3.3}$$

where

$$\mathfrak{e}(T) := \|\eta\|_4^2 + \left|\overline{\partial}^4\eta\cdot\hat{n}\right|_0^2 + \sum_{k=0}^{4}\left(\left\|\partial_t^{4-k}v\right\|_k^2 + \left\|\partial_t^{4-k}b\right\|_k^2 + \left\|\partial_t^{4-k}q\right\|_k^2\right), \tag{2.3.4}$$

$$H(T) := \int_0^T \int_\Omega \left|\partial_t^5 b\right|^2 \,\mathrm{d}y\,\mathrm{d}t + \left\|\partial_t^4 b\right\|_1^2, \tag{2.3.5}$$

$$W(T) := \left\|\partial_t^5 q\right\|_0^2 + \left\|\partial_t^4 q\right\|_1^2. \tag{2.3.6}$$

Remark 2.3.2. One may not recover the full H^5 regularity for the magnetic field b due to the appearance of **free** boundary, otherwise the H^5-control of η is needed.

The proof of the local existence is based on the tangential smoothing as an approximation scheme introduced by Coutand-Shkoller [16]. To solve the approximate system, one can freeze the coefficient, then solve the linearized system by standard fixed-point argument, and finally use Picard iteration to solve the approximation system. The most difficult step is then the uniform (in the smoothing parametre) estimates for the approximate system. The velocity is still controlled via div-curl-tangential decomposition. In the control of divergence, one may find that the wave equation of q contains the term $\triangle_A(\frac{1}{2}|B|^2)$ in the source term and thus leads to the loss of one normal derivative. We then observe that the magnetic diffusion, which together with the divergence-free condition contributes to the Laplacian term $\triangle_A b$, exactly compensate such derivative loss. On the other hand, we also observe that the vanishing boundary condition, together with the Christodoulou-Lindblad elliptic estimates, helps us control the magnetic field b and the Lorentz force $(b\cdot\nabla_A)b$. Finally, one may combine the control of heat equation of b and the wave equation of q to close the estimates.

The sound speed $c^2 := q'(R)$ reflects the compressibility of a compressible fluid. In our setting, we may parametrize it by $\varepsilon := R'(q)|_{R=\bar{\rho}_0}$. Under this setting, we denote the unknowns to be $(v^\varepsilon, b^\varepsilon, q^\varepsilon, R^\varepsilon)$ and the process $\lim_{\varepsilon\to 0_+} R^\varepsilon(p^\varepsilon) = \bar{\rho}_0$ can be considered to be the incompressible limit. This is derived by establishing the energy estimate that is uniform in the sound speed. We may assume $\bar{\rho}_0 = 1$ for simplicity.

Let $(\mathbf{v}_0, \mathbf{b}_0)$ be the divergence-free and $\mathbf{b}_0|_\Gamma = \mathbf{0}$. Let $\mathbf{q}_0$ be the solution to

$$\triangle(\mathbf{q}_0 + \frac{1}{2}|\mathbf{b}_0|^2) = -\partial_\mu \mathbf{v}_0^\alpha \partial_\alpha \mathbf{v}_0^\mu + \partial_\mu \mathbf{b}_0^\alpha \partial_\alpha \mathbf{b}_0^\mu, \quad \mathbf{q}_0|_\Gamma = 0$$

and satisfy the Rayleigh-Taylor sign condition $-\partial_N(\mathbf{q}_0 + \frac{1}{2}|\mathbf{b}_0|^2) \geq c_0 > 0$. Let $(\mathbf{v}, \mathbf{b}, \mathbf{q})$ be the solution to the incompressible resistive MHD with initial data $(\mathbf{v}_0, \mathbf{b}_0)$

$$\begin{cases} \partial_t \zeta = \mathbf{v} & \text{in } [0,T]\times\Omega \\ \partial_t \mathbf{v} = \left(\mathbf{b}\cdot\nabla_{A(\zeta)}\right)\mathbf{b} - \nabla_{A(\zeta)}(\mathbf{q} + \frac{1}{2}|\mathbf{b}|^2) & \text{in } [0,T]\times\Omega \\ \text{div}_{A(\zeta)}\mathbf{v} = 0 & \text{in } [0,T]\times\Omega \\ \partial_t \mathbf{b} + \lambda \text{curl}_{A(\zeta)}\text{curl}_{A(\zeta)}\mathbf{b} = (b\cdot\nabla_A)\,\mathbf{v} & \text{in } [0,T]\times\Omega \\ \text{div}_{A(\zeta)}\mathbf{b} = 0 & \text{in } [0,T]\times\Omega \\ \mathbf{b} = \mathbf{0},\ \mathbf{q} = 0,\ -\frac{\partial \mathbf{q}_0}{\partial N}|_\Gamma \geq c_0 > 0 & \text{on } [0,T]\times\Gamma \\ (\zeta, \mathbf{v}, \mathbf{b}, \mathbf{q})|_{t=0} = (\text{Id}, \mathbf{v}_0, \mathbf{b}_0, \mathbf{q}_0). \end{cases} \tag{2.3.7}$$

Theorem 2.3.3 (Incompressible limit [82, Theorem 1.3])**.**

1. There exists $(v_0^\varepsilon, \mathbf{b}_0, \rho_0^\varepsilon, q_0^\varepsilon)$, the initial data of (2.3.1) with sound speed equal to ε^{-1}, satisfying the conditions mentioned in Theorem 2.3.1 and $(v_0^\varepsilon, \rho_0^\varepsilon) \xrightarrow{C^1} (\mathbf{v}_0, 1)$ as $\varepsilon \to 0$.

2. Let $(v^\varepsilon, b^\varepsilon, R^\varepsilon, q^\varepsilon)$ be the solution to (2.3.1) with initial data $(v_0^\varepsilon, \mathbf{b}_0, \rho_0^\varepsilon, q_0^\varepsilon)$. Then we have $(v^\varepsilon, b^\varepsilon, R^\varepsilon) \xrightarrow{C^1} (\mathbf{v}, \mathbf{b}, 1)$ as $\varepsilon \to 0$.

Remark 2.3.4. When passing to the incompressible limit, the pressure q^ε in the compressible system **never** converges to the incompressible counterpart. The reason is that the pressure in the incompressible system is a Lagrangian multiplier (cf. [16, Section 6-7]) instead of the solution to a wave equation.

Instead, it should be the enthalpy $h(R) := \int_1^R \frac{q'(r)}{r} dr$ that converges to q as $\varepsilon \to 0$.

To achieve the incompressible limit, it suffices to derive uniform-in-ε^{-1} estimates for (2.3.1) with the initial data $(v_0^\varepsilon, \mathbf{b}_0, \rho_0^\varepsilon, q_0^\varepsilon)$. One needs to be careful when doing the control of wave equation of q, because the time derivative is $\sqrt{R'(q)}$-weighted but the source term needs the non-weighted energy. The result is listed as follows.

Lemma 2.3.5 ([82, Theorem 1.1])**.** There exists some $T_1' > 0$ independent of ε, such that the $(\eta^\varepsilon, v^\varepsilon, b^\varepsilon, q^\varepsilon)$ in $[0, T_1']$ satisfying the following estimates

$$\sup_{0 \le T \le T_1'} \mathcal{E}^\varepsilon(T) \le P\left(\|v_0\|_4, \|b_0\|_5, \|q_0\|_4\right), \tag{2.3.8}$$

where

$$\mathcal{E}^\varepsilon(T) := \mathfrak{e}^\varepsilon(T) + H^\varepsilon(T) + W^\varepsilon(T) + \sum_{k=0}^4 \left\|\partial_t^{4-k}\left((b^\varepsilon \cdot \nabla_{A^\varepsilon})\, b^\varepsilon\right)\right\|_k^2, \tag{2.3.9}$$

where

$$\begin{aligned} \mathfrak{e}^\varepsilon(T) := \|\eta^\varepsilon\|_4^2 + \left|\overline{\partial}^4 \eta^\varepsilon \cdot \hat{n}\right|_0^2 + \sum_{k=1}^4 \left(\left\|\partial_t^{4-k} v^\varepsilon\right\|_k^2 + \left\|\sqrt{R'(q^\varepsilon)}\partial_t^{4-k} b^\varepsilon\right\|_k^2 + \left\|\partial_t^{4-k} q^\varepsilon\right\|_k^2 \right) \\ + \left\|\sqrt{R'(q^\varepsilon)}\partial_t^4 v^\varepsilon\right\|_k^2 + \left\|\partial_t^4 b^\varepsilon\right\|_k^2 + \left\|R'(q^\varepsilon)\partial_t^4 q^\varepsilon\right\|_k^2 \end{aligned} \tag{2.3.10}$$

$$H^\varepsilon(T) := \int_0^T \int_\Omega \left|\partial_t^5 b^\varepsilon\right|^2 \, \mathrm{d}y \, \mathrm{d}t + \left\|\partial_t^4 b^\varepsilon\right\|_1^2, \tag{2.3.11}$$

$$W^\varepsilon(T) := \left\|R'(q^\varepsilon)\partial_t^5 q^\varepsilon\right\|_0^2 + \left\|\sqrt{R'(q^\varepsilon)}\partial_t^4 q^\varepsilon\right\|_1^2. \tag{2.3.12}$$

2.4 Anisotropic *A priori* Estimates of Compressible Ideal MHD

$$\begin{cases} \partial_t \eta = v & \text{in } [0,T]\times\Omega \\ \rho_0 \partial_t v - (b_0\cdot\partial)(J^{-1}(b_0\cdot\partial)\eta) = -\nabla_{\mathbf{A}} Q, \quad Q = q + \frac{1}{2}|b|^2 & \text{in } [0,T]\times\Omega \\ \frac{JR'(q)}{\rho_0}\partial_t q + \text{div}_A v = 0 & \text{in } [0,T]\times\Omega \\ q = q(R) \text{ strictly increasing} & \text{in } [0,T]\times\overline{\Omega} \\ \text{div } b_0 = 0 & \text{in } [0,T]\times\Omega \\ b_0^3 = 0, \ Q = 0, \ -\frac{\partial Q_0}{\partial N}|_\Gamma \geq c_0 > 0 & \text{on } [0,T]\times\Gamma \\ (\eta, v, b, q)|_{t=0} = (\text{Id}, v_0, b_0, q_0). \end{cases} \tag{2.4.1}$$

So far, it is still difficult to pass the vanishing resistivity limit for (2.3.1) to derive the solution to (2.4.1). As stated before, it is not suitable to study (2.4.1) in the standard Sobolev space. On the other hand, Chen Shu-Xing [10] introduced the anisotropic Sobolev space to study hyperbolic systems with characteristic boundary conditions. Yanagisawa-Matsumura [81] and Secchi [62, 63] adopted this to proved the local existence of compressible ideal MHD system with perfect conducting wall conditions. As for the free-boundary problem, Trakhinin-Wang [74] proved the LWP by using Nash-Moser which leads to a big loss of regularity from the initial data to the solution. Therefore, we would like to generalize [62, 63] to the free-boundary problem to avoid the regularity loss.

Before stating our result, we shall define the anisotropic Sobolev space $H_*^m(\Omega)$ for $m \in \mathbb{N}^*$. Let $\sigma = \sigma(y_3)$ be a cutoff function on $[-1, 1]$ defined by $\sigma(y_3) = (1 - y_3)(1 + y_3)$. Then we define $H_*^m(\Omega)$ for $m \in \mathbb{N}^*$ as follows

$$H_*^m(\Omega) := \left\{ f \in L^2(\Omega) \middle| (\sigma\partial_3)^{i_4}\partial_1^{i_1}\partial_2^{i_2}\partial_3^{i_3} f \in L^2(\Omega), \ \ \forall i_1 + i_2 + 2i_3 + i_4 \leq m \right\},$$

equipped with the norm

$$\|f\|_{H_*^m(\Omega)}^2 := \sum_{i_1+i_2+2i_3+i_4\leq m} \|(\sigma\partial_3)^{i_4}\partial_1^{i_1}\partial_2^{i_2}\partial_3^{i_3} f\|_{L^2(\Omega)}^2.$$

For any multi-index $I := (i_0, i_1, i_2, i_3, i_4) \in \mathbb{N}^5$, we define

$$\partial_*^I := \partial_t^{i_0}(\sigma\partial_3)^{i_4}\partial_1^{i_1}\partial_2^{i_2}\partial_3^{i_3}, \ \ \langle I \rangle := i_0 + i_1 + i_2 + 2i_3 + i_4,$$

and define the **space-time anisotropic Sobolev norm** $\|\cdot\|_{m,*}$ by

$$\|f\|_{m,*}^2 := \sum_{\langle I\rangle \le m} \|\partial_*^I f\|_{L^2(\Omega)}^2 = \sum_{i_0\le m} \|\partial_t^{i_0} f\|_{H_*^{m-i_0}(\Omega)}^2.$$

We define $f_{(j)} = \partial_t^j f|_{t=0}$ for $j \in \mathbb{N}$. The main result is the following theorem.

Theorem 2.4.1 (Anisotropic regularity of compressible ideal MHD [50, Theorem 1.2])**.** Let the initial data be $(v_0, b_0, Q_0) \in H_*^8(\Omega)$ such that $(v_{(j)}, b_{(j)}, Q_{(j)}) \in H_*^{8-j}(\Omega)$ for $1 \le j \le 8$ and compatibility condition holds up to 7-th order, i.e., $Q_{(j)}|_\Gamma = 0$ for $0 \le j \le 7$. Then there exists some $T_2 > 0$, such that the solution (η, v, Q) to the system (2.4.1) satisfies the following estimates in $[0, T_2]$

$$\sup_{0\le t\le T_2} \mathfrak{E}(T) \le \mathcal{C}(\mathfrak{E}(0)). \tag{2.4.2}$$

Here the energy functional $\mathfrak{E}(t)$ is defined to be

$$\mathfrak{E}(t) := \|\eta(t,\cdot)\|_{8,*}^2 + \|v(t,\cdot)\|_{8,*}^2 + \|J^{-1}(b_0\cdot\partial)\eta(t,\cdot)\|_{8,*}^2 + \|Q(t,\cdot)\|_{8,*}^2 + \sum_{\langle I\rangle=8} \left|\partial_*^I \eta\cdot\hat{n}\right|_0^2, \tag{2.4.3}$$

and $\mathcal{C}(\mathfrak{E}(0)) > 0$ denotes a positive constant depending on $\mathfrak{E}(0)$.

Remark 2.4.2. There exists initial data $(v_0, b_0, Q_0) \in H^8(\Omega) \hookrightarrow H_*^8(\Omega)$ satisfying the compatibility conditions up to 7-th order, such that

$$\sum_{j=1}^{8} \|(v_{(j)}, b_{(j)}, Q_{(j)})\|_{H^{8-j}(\Omega)} \lesssim P(\|v_0\|_{H^8(\Omega)}, \|b_0\|_{H^8(\Omega)}, \|Q_0\|_{H^8(\Omega)}). \tag{2.4.4}$$

By the Sobolev embedding $H^{8-j}(\Omega) \hookrightarrow H_*^{8-j}(\Omega)$ for $0 \le j \le 8$, we have

$$\mathcal{E}(0) \lesssim P(\|v_0\|_{H^8(\Omega)}, \|b_0\|_{H^8(\Omega)}, \|Q_0\|_{H^8(\Omega)}). \tag{2.4.5}$$

Due to the anisotropy of the function space, it is not possible to establish

$$\sum_{j=1}^{8} \|(v_{(j)}, b_{(j)}, Q_{(j)})\|_{H_*^{8-j}(\Omega)} \lesssim P(\|v_0\|_{H_*^8(\Omega)}, \|b_0\|_{H_*^8(\Omega)}, \|Q_0\|_{H_*^8(\Omega)}). \tag{2.4.6}$$

The only way to prove Theorem 2.4.1 seems to be directly computing ∂_*^I-estimates. The interior terms are expected to contribute to the energy of v, b and q, while on the boundary, the top order derivatives of $v \cdot N$ and Q simultaneously appear. If ∂_*^I contains at least one normal derivatives, we can invoke the MHD system (2.4.1) to *replace the normal derivatives of the non-characteristic variables*(Q and $v \cdot N$) *by the tangential derivatives of the characteristic variables*, so that one normal derivative, as a "second-order" derivative, is replaced by $\mathfrak{D} = \overline{\partial}$ or $(b_0 \cdot \partial)$ or ∂_t. Then we use the divergence theorem to rewrite this boundary integral into the interior and integrate $\mathfrak{D}$ by part. Finally, using the anisotropy yields the desired estimates.

However, for the free-boundary problem, **the regularity of the free surface is limited, and in fact enters to the highest order.** To overcome this difficulty, we introduce the "modified" Alinhac good unknowns that take into account the covariance under the change of coordinates to avoid the derivative loss when commuting ∂_*^I with the covariant derivative ∇_A. In specific, the key observation is that the *essential* highest order term in $\partial_*^I(\nabla_A f)$ is not simply $\nabla_A(\partial_*^I f)$, but still has the form of ∇_A(good unknown of f). This was first observed by Alinhac [2]. In the study of free-surface fluid, this was first implicitly applied by Christodoulou-Lindblad [13], and then was used explicity [55, 77, 30, 54, 83, 84]. However, due to the anisotropy, the good unknowns applied in [55, 77, 30, 54, 83, 84] are no longer applicable. Our idea is to fully analyze the "covariant" structure of $\partial_*^I(\nabla_A^\alpha f)$ for $f = v_\alpha$ and $f = Q$ respectively, and then *modify the expression of the Alinhac good unknowns*. This idea never appeared in the previous related works. To achieve this, we need to repeatedly replace a normal derivative by a tangential one and need to produce a weight function σ by using the vanishing boundary value of Q, b_0^3 and the fundamental theorem of calculus in order to convert the normal derivative ∂_3 into the weighted (tangential) $\sigma\partial_3$ derivative.

Chapter 3

Preliminary Lemmas

In this chapter, we record all the lemmata that will be used in the proofs presented in the remaining chapters.

3.1 Geometric Identities

First we record the geometric identities related to the flow map η and the cofactor matrix A. They are repeatedly used in the proof, especially in the case of nonzero surface tension.

The explicit form of the matrix A is

$$A = J^{-1}\begin{pmatrix} \overline{\partial}_2\eta^2\partial_3\eta^3 - \partial_3\eta^2\overline{\partial}_2\eta^3 & \partial_3\eta^1\overline{\partial}_2\eta^3 - \overline{\partial}_2\eta^1\partial_3\eta^3 & \overline{\partial}_2\eta^1\partial_3\eta^2 - \partial_3\eta^1\overline{\partial}_2\eta^2 \\ \partial_3\eta^2\overline{\partial}_1\eta^3 - \overline{\partial}_1\eta^2\partial_3\eta^3 & \overline{\partial}_1\eta^1\partial_3\eta^3 - \partial_3\eta^1\overline{\partial}_1\eta^3 & \overline{\partial}_1\eta^1\overline{\partial}_1\eta^2 - \overline{\partial}_1\eta^1\partial_3\eta^2 \\ \overline{\partial}_1\eta^2\overline{\partial}_2\eta^3 - \overline{\partial}_2\eta^2\overline{\partial}_1\eta^3 & \overline{\partial}_2\eta^1\overline{\partial}_1\eta^3 - \overline{\partial}_1\eta^1\overline{\partial}_2\eta^3 & \overline{\partial}_1\eta^1\overline{\partial}_2\eta^2 - \overline{\partial}_2\eta^1\overline{\partial}_1\eta^2 \end{pmatrix} \tag{3.1.1}$$

Moreover, since $\mathbf{A} = JA$, and in view of (3.1.1), we can write

$$\mathbf{A}^{1i} = \epsilon^{ijk}\overline{\partial}_2\eta_j\partial_3\eta_k, \ \ \mathbf{A}^{2i} = -\epsilon^{ijk}\overline{\partial}_1\eta_j\partial_3\eta_k, \ \ \mathbf{A}^{3i} = \epsilon^{ijk}\overline{\partial}_1\eta_j\overline{\partial}_2\eta_k. \tag{3.1.2}$$

Here, ϵ^{ijk} is the sign of the 3-permutation $(ijk) \in S_3$. We will repeatedly use that fact that $\mathbf{A}^{1\cdot}, \mathbf{A}^{2\cdot}$ consist of $\overline{\partial}\eta \times \partial_3\eta$ and $\mathbf{A}^{3\cdot}$ consists of $\overline{\partial}_1\eta \times \overline{\partial}_2\eta$.

We also record the following identity: Suppose D is the derivative ∂ or ∂_t, then

$$DA^{li} = -A^{lr}\ \partial_k D\eta_r\ A^{ki}. \tag{3.1.3}$$

Lemma 3.1.1 ([18, Lemma 2.5])**.** Let $\hat{n}$ be the unit outer normal to $\eta(\Gamma)$ and $\mathcal{T},\mathcal{N}$ be the tangential and normal bundle of $\eta(\Gamma)$ respectively. Denote $\Pi : \mathcal{T}|_{\eta(\Gamma)} \to \mathcal{N}$ to be the canonical normal projection. Denote $\overline{\partial}$ to be ∂_t or $\overline{\partial}_1, \overline{\partial}_2$. Then

$$\hat{n} := n \circ \eta = \frac{A^\top N}{|A^\top N|}, \tag{3.1.4}$$

$$|A^\top N| = |(A^{31}, A^{32}, A^{33})| = \sqrt{g}, \tag{3.1.5}$$

$$\Pi^\alpha_\lambda = \hat{n}^\alpha \hat{n}_\lambda = \delta^\alpha_\lambda - g^{kl}\overline{\partial}_k \eta_\alpha \overline{\partial}_l \eta_\lambda, \tag{3.1.6}$$

$$\Pi^\alpha_\lambda = \Pi^\alpha_\mu \Pi^\mu_\lambda, \tag{3.1.7}$$

$$-\triangle_g(\eta^\alpha|_\Gamma) = \mathcal{H} \circ \eta \hat{n}^\alpha, \tag{3.1.8}$$

$$\sqrt{g}\triangle_g \eta^\alpha = \sqrt{g} g^{ij} \Pi^\alpha_\lambda \overline{\partial}_i \overline{\partial}_j \eta^\lambda = \sqrt{g} g^{ij} \overline{\partial}_i \overline{\partial}_j \eta^\alpha - \sqrt{g} g^{ij} g^{kl} \overline{\partial}_k \eta^\alpha \overline{\partial}_l \eta^\mu \overline{\partial}_i \overline{\partial}_j \eta_\mu, \tag{3.1.9}$$

$$\overline{\partial}(\sqrt{g}\triangle_g \eta^\alpha) = \overline{\partial}_i \left(\sqrt{g} g^{ij} \Pi^\alpha_\lambda \overline{\partial}\overline{\partial}_j \eta^\lambda + \sqrt{g}(g^{ij} g^{kl} - g^{ik} g^{lj}) \overline{\partial}_j \eta^\alpha \overline{\partial}_k \eta_\lambda \overline{\partial}\overline{\partial}_l \eta^\lambda \right), \tag{3.1.10}$$

$$\overline{\partial}\hat{n}_\mu = - g^{kl} \overline{\partial}_k \overline{\partial} \eta^\tau \hat{n}_\tau \overline{\partial}_l \eta_\mu, \tag{3.1.11}$$

$$\partial_t(\sqrt{g} g^{ij}) = \sqrt{g}(g^{ij} g^{kl} - 2 g^{lj} g^{ik}) \overline{\partial}_k v^\lambda \overline{\partial}_l \eta_\lambda. \tag{3.1.12}$$

Remark 3.1.2. Recall that $g_{ij} = \overline{\partial}_i \eta_\mu \overline{\partial}_j \eta^\mu$ and $g = \det[g_{ij}]$ and $[g^{ij}] = [g_{ij}]^{-1}$. This means that g_{ij}, g and g^{ij} are rational functions of $\overline{\partial}\eta$ and so is Π.

Notation 3.1.3. We shall use the notation $Q(\partial\eta)$ and $Q(\overline{\partial}\eta)$ to denote the rational functions of $\partial\eta$ and $\overline{\partial}\eta$, respectively. This Q notation allows us to record error terms in a concise way and so it will be used frequently throughout the rest of this paper. For example, for any tangential derivative $\overline{\partial}$, we have $\overline{\partial} Q(\overline{\partial}\eta) = \tilde{Q}^i_\alpha(\overline{\partial}\eta) \overline{\partial}\overline{\partial}_i \eta^\alpha$ where the term $\tilde{Q}^i_\alpha(\overline{\partial}\eta)$ is also a rational function of $\overline{\partial}\eta$. For more details of such notation, we refer readers to [16, Section 11] and [18, Remark 2.4].

3.2 Sobolev Inequalities

Lemma 3.2.1 (Kato-Ponce type Inequalities)**.** Let $J = (1 - \triangle)^{1/2}$, $s \geq 0$. Then the following estimates hold:

(1) $\forall s \geq 0$, we have

$$\|J^s(fg)\|_{L^2} \lesssim \|f\|_{W^{s,p_1}}\|g\|_{L^{p_2}} + \|f\|_{L^{q_1}}\|g\|_{W^{s,q_2}}, \tag{3.2.1}$$

with $1/2 = 1/p_1 + 1/p_2 = 1/q_1 + 1/q_2$ and $2 \leq p_1, q_2 < \infty$;

(2) $\forall s \in (0, 1)$, we have

$$\|J^s(fg) - f(J^s g) - (J^s f)g\|_{L^p} \lesssim \|f\|_{W^{s_1,p_1}}\|g\|_{W^{s-s_1,p_2}}, \tag{3.2.2}$$

where $0 < s_1 < s$ and $1/p_1 + 1/p_2 = 1/p$ with $1 < p < p_1, p_2 < \infty$;

(2') $\forall s \geq 1$, we have

$$\|J^s(fg) - (J^s f)g - f(J^s g)\|_{L^p} \lesssim \|f\|_{W^{1,p_1}}\|g\|_{W^{s-1,q_2}} + \|f\|_{W^{s-1,q_1}}\|g\|_{W^{1,q_2}} \tag{3.2.3}$$

for all the $1 < p < p_1, p_2, q_1, q_2 < \infty$ with $1/p_1 + 1/p_2 = 1/q_1 + 1/q_2 = 1/p$.

(3) $\forall s \geq 1$, we have

$$\|J^s(fg) - f(J^s g)\|_{L^2} \lesssim \|f\|_{W^{s,p_1}}\|g\|_{L^{p_2}} + \|f\|_{W^{1,q_1}}\|g\|_{W^{s-1,q_2}}, \tag{3.2.4}$$

where $1/2 = 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2$ with $1 < p < p_1, p_2 < \infty$;

(3') $\forall s \geq 0$ and $1 < p < \infty$, we have

$$\|J^s(fg) - f(J^s g)\|_{L^p} \lesssim \|\partial f\|_{L^\infty}\|J^{s-1}g\|_{L^p} + \|J^s f\|_{L^p}\|g\|_{L^\infty}; \tag{3.2.5}$$

(3'') For $1 < p < \infty$ and $1 < p_1, q_1, p_2, q_2 \leq \infty$ satisfying $1/p = 1/p_1 + 1/p_2 = 1/q_1 + 1/q_2$, the following hold:

- If $0 < s \le 1$, then

$$\|J^s(fg) - f(J^s g)\|_{L^p} \lesssim \|J^{s-1}\partial f\|_{L^{p_1}} \|g\|_{L^{p_2}}; \tag{3.2.6}$$

- If $s > 1$, then

$$\|J^s(fg) - f(J^s g)\|_{L^p} \lesssim \|J^{s-1}\partial f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|\partial f\|_{L^{q_1}} \|J^{s-2}\partial g\|_{L^{q_2}}. \tag{3.2.7}$$

Proof. See Li [45] for (3') and (3") and Kato-Ponce [38] for the others. □

Lemma 3.2.2 (Fractional Sobolev Interpolation [7])**.** Suppose Ω is a domain in $\mathbb{R}^d$. Suppose also $0 \le s_1 \le s \le s_2$ and $1 \le p, p_1, p_2 \le \infty$. If the condition

$$1 \le s_2 \in \mathbb{Z} \text{ and } p_2 = 1 \text{ and } s_2 - s_1 \le 1 - \frac{1}{p_1}$$

fails, then the following interpolation result holds for all $\theta \in (0, 1)$:

$$\|f\|_{W^{s,p}(\Omega)} \lesssim_{d,s_1,s_2,p_1,p_2,\Omega,\theta} \|f\|^{\theta}_{W^{s_1,p_1}(\Omega)} \|f\|^{1-\theta}_{W^{s_2,p_2}(\Omega)},$$

provided $s = \theta s_1 + (1-\theta)s_2$ and $1/p = \theta/p_1 + (1-\theta)/p_2$ hold.

Lemma 3.2.3 (Normal trace lemma [30, Lemma 3.4])**.** Let X be a smooth vector field. Then

$$\left|\overline{\partial} X \cdot N\right|_{-0.5} \lesssim \|\overline{\partial} X\|_0 + \|\text{div } X\|_0 \tag{3.2.8}$$

Lemma 3.2.4 (Harmonic trace lemma [69, Prop. 5.1.7])**.** Suppose that $s \ge 0.5$ and u solves the boundary-valued problem

$$\begin{cases} \triangle u = 0 & \text{in } \Omega, \\ u = g & \text{on } \Gamma \end{cases}$$

where $g \in H^s(\Gamma)$. Then it holds that

$$|g|_s \lesssim \|u\|_{s+0.5} \lesssim |g|_s$$

Lemma 3.2.5 (Anisotropic Sobolev trace lemma [60, Theorem 1])**.** Let $m \geq 1,\ m \in \mathbb{N}^*$, then we have the following trace lemma for the anisotropic Sobolev space.

1. If $f \in H_*^{m+1}(\Omega)$, then its trace $f|_\Gamma$ belongs to $H^m(\Gamma)$ and satisfies

$$|f|_m \lesssim \|f\|_{H_*^{m+1}(\Omega)}.$$

2. There exists a linear continuous operator $\mathfrak{R}_T : H^m(\Gamma) \to H_*^{m+1}(\Omega)$ such that $(\mathfrak{R}_T g)|_\Gamma = g$ and

$$\|\mathfrak{R}_T g\|_{H_*^{m+1}(\Omega)} \lesssim |g|_m.$$

Remark 3.2.6. The condition $m \geq 1$ is necessary and analogous result may not hold when $m = 0$. Indeed, we need to integrate one tangential derivative by part and thus $m \geq 1$ is necessary.

Lemma 3.2.7 (Anisotropic Sobolev embedding [74, Lemma 3.3])**.** We have the following inequalities

$$H^m(\Omega) \hookrightarrow H_*^m(\Omega) \hookrightarrow H^{\lfloor m/2 \rfloor}(\Omega), \quad \forall m \in \mathbb{N}^*$$

$$\|u\|_{L^\infty} \lesssim \|u\|_{H_*^3(\Omega)}, \quad \|u\|_{W^{1,\infty}} \lesssim \|u\|_{H_*^5(\Omega)}.$$

3.3 Elliptic Estimates

Lemma 3.3.1 (Hodge-type decomposition and the inverse theorem)**.**

(1) Let X be a smooth vector field and $s \geq 1$, then it holds that

$$\|X\|_s \lesssim \|X\|_0 + \|\text{curl } X\|_{s-1} + \|\text{div } X\|_{s-1} + |\overline{\partial} X \cdot N|_{s-1.5}. \tag{3.3.1}$$

(2) Let $\Omega \subseteq \mathbb{R}^3$ be a bounded H^{k+1}-domain with $k > 1.5$. Given $\mathbf{F}, G \in H^{l-1}(\Omega)$ with $\text{div } \mathbf{F} = 0$. Consider the equations

$$\text{curl } X = \mathbf{F}, \ \ \text{div } X = G \quad \text{in } \Omega. \tag{3.3.2}$$

If $\mathbf{F}$ satisfies $\int_\gamma \mathbf{F}\cdot N \,\mathrm{d}S = 0$ for each connected component γ of $\partial\Omega$ and $h \in H^{l-0.5}(\partial\Omega)$ satisfies $\int_{\partial\Omega} h \,\mathrm{d}S = \int_\Omega G \,\mathrm{d}y$, then $\forall 1 \le l \le k$, there exists a solution $X \in H^l(\Omega)$ to (3.3.2) with boundary condition $X \cdot N|_{\partial\Omega} = h$ such that

$$\|X\|_{H^l(\Omega)} \le C(|\partial\Omega|_{H^{k+0.5}})\left(\|\mathbf{F}\|_{H^{l-1}(\Omega)} + \|G\|_{H^{l-1}(\Omega)} + |h|_{H^{l-0.5}(\partial\Omega)}\right). \tag{3.3.3}$$

Such solution is unique if Ω is the disjoint union of simply connected open sets.

Proof. (1) This follows from the well-known identity $-\triangle X = \text{curl curl } X - \nabla \text{div } X$ and integrating by parts. (2) This is the main result of Cheng-Shkoller [12]. □

Lemma 3.3.2 (H^1 elliptic estimates [37, Lemma 3.2])**.** Assume $\mathfrak{B}^{\mu\nu}$ satisfies $\|\mathfrak{B}\|_{L^\infty} \le K$ and the ellipticity $\mathfrak{B}^{\mu\nu}(x)\xi_\mu\xi_\nu \ge \frac{1}{K}|\xi|^2$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^3$. Assume W to be an H^1 solution to

$$\begin{cases} \partial_\nu(\mathfrak{B}^{\mu\nu}\partial_\mu W) = \text{div } \pi & \text{in } \Omega \\ \mathfrak{B}^{\mu\nu}\partial_\nu W N_\mu = h & \text{on } \partial\Omega, \end{cases} \tag{3.3.4}$$

where $\pi, \text{div } \pi \in L^2(\Omega)$ and $h \in H^{-0.5}(\partial\Omega)$ with the compatibility condition

$$\int_{\partial\Omega} (\pi \cdot N - h) dS = 0.$$

If $\|\mathfrak{B} - I\|_{L^\infty} \le \varepsilon_0$ which is a sufficently small constant depending on K, then

$$\|W - \overline{W}\|_1 \lesssim \|\pi\|_0 + |h - \pi \cdot N|_{-0.5}, \text{ where } \overline{W} := \frac{1}{|\Omega|}\int_\Omega W dy. \tag{3.3.5}$$

Lemma 3.3.3 (Christodoulou-Lindblad elliptic estimate [54, Lemma 2.7])**.** If $f|_{\partial\Omega} = 0$, then the following elliptic estimate holds for $r \ge 2$.

$$\|\nabla_A f\|_r \le P(\|\eta\|_r)\Big(\|\triangle_A f\|_{r-1} + \|\overline{\partial}\eta\|_r \|f\|_r\Big). \tag{3.3.6}$$

When $r = 1$, $\|\eta\|_r$ should be replaced by $\|\partial\eta\|_1$.

3.4 Properties of Tangential Mollifiers

Let $\zeta = \zeta(y_1, y_2) \in C_c^\infty(\mathbb{R}^2)$ be a cut-off function such that Spt $\zeta = \overline{B(0,1)} \subseteq \mathbb{R}^2$, $0 \le \zeta \le 1$ and $\int_{\mathbb{R}^2} \zeta = 1$. The dilation is $\zeta_\kappa(y_1, y_2) = \frac{1}{\kappa^2}\zeta\left(\frac{y_1}{\kappa}, \frac{y_2}{\kappa}\right)$, $\kappa > 0$. Now we define

$$\Lambda_\kappa f(y_1, y_2, y_3) := \int_{\mathbb{R}^2} \zeta_\kappa(y_1 - z_1, y_2 - z_2) f(z_1, z_2)\, \mathrm{d}z_1\, \mathrm{d}z_2. \tag{3.4.1}$$

The following lemma records the basic properties of tangential smoothing.

Lemma 3.4.1 ([28, Lemma 2.7])**.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. For $\kappa > 0$, we have:

$$\|\Lambda_\kappa f\|_s \lesssim \|f\|_s, \quad \forall s \ge 0; \tag{3.4.2}$$

$$|\Lambda_\kappa f|_s \lesssim |f|_s, \quad \forall s \ge -0.5; \tag{3.4.3}$$

$$|\overline{\partial}\Lambda_\kappa f|_0 \lesssim \kappa^{-s}|f|_{1-s}, \quad \forall s \in [0,1]; \tag{3.4.4}$$

$$|f - \Lambda_\kappa f|_{L^\infty} \lesssim \sqrt{\kappa}|\overline{\partial} f|_{0.5} \tag{3.4.5}$$

$$|f - \Lambda_\kappa f|_{L^p} \lesssim \kappa|\overline{\partial} f|_{L^p}, \tag{3.4.6}$$

$$|f - \Lambda_\kappa f|_{L^2} \lesssim \sqrt{\kappa}|\overline{\partial}^{\frac{1}{2}} f|_0. \tag{3.4.7}$$

Define the commutator $[\Lambda_\kappa, f]g := \Lambda_\kappa(fg) - f\Lambda_\kappa(g)$. Then it satisfies

$$|[\Lambda_\kappa, f]g|_0 \lesssim |f|_{L^\infty}|g|_0, \tag{3.4.8}$$

$$|[\Lambda_\kappa, f]\overline{\partial} g|_0 \lesssim |f|_{W^{1,\infty}}|g|_0, \tag{3.4.9}$$

$$|[\Lambda_\kappa, f]\overline{\partial} g|_{0.5} \lesssim |f|_{W^{1,\infty}}|g|_{0.5}. \tag{3.4.10}$$

Chapter 4

Free-Boundary Incompressible MHD with or without Surface Tension

4.1 A Glimpse at Incompressible MHD without Surface Tension

We start with the simplest case, i.e., the incompressible ideal MHD without surface tension (2.1.1). The local well-posedness was proved by Gu-Wang [30] in H^4 regularity. However, the low-regularity solution to (2.1.1) has not been studied until our paper [52] appears. Below we present the proof of the $H^{2.5+\delta}$-estimates as stated in Theorem 2.1.1. We introduce the following a priori assumptions

Lemma 4.1.1 ([52, Lem 2.1 and Lem 5.5])**.** For every $0 < \varepsilon \leq 1$, there exists some $T_0 > 0$ sufficiently small, such that the following inequality holds in $[0, T_0]$:

$$\|A^\mu_\nu - \delta^\mu_\nu\|_{H^{1.5+\delta}(\Omega)} \leq \varepsilon, \quad \|A^\mu_\alpha A^\nu_\alpha - \delta^{\mu\nu}\|_{H^{1.5+\delta}(\Omega)} \leq \varepsilon. \tag{4.1.1}$$

$$-\partial_N Q \geq c_0/2 > 0. \tag{4.1.2}$$

4.1.1 Elliptic estimates of the pressure

In this section we derive the estimates for $\|Q\|_{3+\delta}$ and $\|Q_t\|_{2.5+\delta}$. These quantities are both required in Section 4.1.2. We denote $\mathcal{P} = P(\|v\|_{2.5+\delta}, \|b\|_{2.5+\delta})$ and so $\mathcal{P}_0 = P(\|v_0\|_{2.5+\delta}, \|b_0\|_{2.5+\delta})$.

Lemma 4.1.2. Assume Lemma 4.1.1 holds. Then the total pressure Q satisfies:

$$\|Q\|_{3+\delta} \lesssim \mathcal{P}_0 + \mathcal{P} + P(\|\eta\|_{3+\delta})\left(\|Q_0\|_{2+\delta} + \int_0^t \|Q_t\|_{2+\delta}\right), \tag{4.1.3}$$

and its time derivative Q_t satisfies:

$$\|Q_t\|_{2.5+\delta} \lesssim \mathcal{P}_0 + \mathcal{P} + P(\|v\|_{2.5+\delta})\left(\|Q_0\|_{2+\delta} + \int_0^t \|Q_t\|_{2+\delta}\right). \tag{4.1.4}$$

Proof. Applying $A^{\nu\alpha}\partial_\nu$ to the first equation of (2.1.1), we have:

$$A^{\nu\alpha}\partial_\nu(A^\mu_\alpha\partial_\mu Q) = -A^{\nu\alpha}\partial_\nu\partial_t v_\alpha + A^{\nu\alpha}\partial_\nu(b_0^\mu\partial_\mu b_\alpha). \tag{4.1.5}$$

Invoking Piola's identity, we get $-A^{\nu\alpha}\partial_\nu\partial_t v_\alpha = \partial_t A^{\nu\alpha}\partial_\nu v_\alpha$ and

$$A^{\nu\alpha}\partial_\nu(b_0^\mu\partial_\mu b_\alpha) = A^{\nu\alpha}\partial_\nu b_0^\mu\partial_\mu b_\alpha + \partial_\beta b_\gamma A^{\nu\gamma}A^{\beta\alpha}\partial_\nu b_\alpha - \partial_\beta b_0^\mu A^{\beta\alpha}\partial_\mu b_\alpha.$$

Thus, the total pressure Q satisfies

$$\begin{aligned}\partial^\mu\partial_\mu Q =& \partial_t A^{\nu\alpha}\partial_\nu v_\alpha + \partial_\nu((\delta^{\mu\nu} - A^\mu_\alpha A^{\nu\alpha})\partial_\mu Q) \\ &+ A^{\nu\alpha}\partial_\nu b_0^\mu\partial_\mu b_\alpha + \partial_\beta b_\gamma A^{\nu\gamma}A^{\beta\alpha}\partial_\nu b_\alpha - \partial_\beta b_0^\mu A^{\beta\alpha}\partial_\mu b_\alpha,\end{aligned} \tag{4.1.6}$$

with the boundary conditions

$$Q = 0 \text{ on } \Gamma \tag{4.1.7}$$

The standard elliptic estimate yields that

$$\begin{aligned}\|Q\|_{3+\delta} \lesssim& \underbrace{\|\partial_t A^{\nu\alpha}\partial_\nu v_\alpha\|_{1+\delta}}_{\mathcal{Q}_1} + \underbrace{\|(\delta^{\mu\nu} - A^\mu_\alpha A^{\nu\alpha})\partial_\mu Q\|_{2+\delta}}_{\mathcal{Q}_2} \\ &+ \underbrace{\|A^{\nu\alpha}\partial_\nu b_0^\mu\partial_\mu b_\alpha\|_{1+\delta} + \|\partial_\beta b_\gamma A^{\nu\gamma}A^{\beta\alpha}\partial_\nu b_\alpha\|_{1+\delta} + \|\partial_\beta b_0^\mu A^{\beta\alpha}\partial_\mu b_\alpha\|_{1+\delta}}_{\mathcal{Q}_3}\end{aligned} \tag{4.1.8}$$

Bounds for $\mathcal{Q}_1$: We have:

$$\|\partial_t A^{\nu\alpha}\partial_\nu v_\alpha\|_{1+\delta} \lesssim P(\|\eta\|_{2.5+\delta})\|v\|^2_{2.5+\delta}\|v\|^2_{2+\delta}. \tag{4.1.9}$$

Bounds for $\mathcal{Q}_2$: Invoking (3.2.1), we have:

$$\begin{aligned}&\|(\delta^{\mu\nu} - A^\mu_\alpha A^{\nu\alpha})\partial_\mu Q\|_{2+\delta} \\ &\lesssim \varepsilon\|Q\|_{3+\delta} + P(\|\eta\|_{3+\delta})\left(\|Q_0\|_{2+\delta} + \int_0^t \|Q_t\|_{2+\delta}\,\mathrm{d}s\right),\end{aligned} \tag{4.1.10}$$

Bounds for $\mathcal{Q}_3$: All the terms in $\mathcal{Q}_3$ can be controlled by $P(\|\eta\|_{2.5+\delta})\|b\|_{2.5+\delta}\|b_0\|_{2.5+\delta} + C\|b\|^2_{2.5+\delta}$ via the multiplicative Sobolev inequality. We only write the first term and the others are treated similarly.

$$\|A^{\nu\alpha}\partial_\nu b_0^\mu \partial_\mu b_\alpha\|_{1+\delta} \lesssim \|A^{\nu\alpha}\|_{1.5+\delta}\|\partial_\nu b_0^\mu \partial_\mu b_\alpha\|_{1+\delta} \lesssim P(\|\eta\|_{2.5+\delta})\|b\|_{2.5+\delta}\|b_0\|_{2.5+\delta}. \tag{4.1.11}$$

Summing up the bounds for $\mathcal{Q}_1$-$\mathcal{Q}_3$, then absorbing the ε-term to LHS, we conclude the estimates of Q as:

$$\|Q\|_{3+\delta} \lesssim \mathcal{P}_0 + \mathcal{P} + P(\|\eta\|_{3+\delta})\left(\|Q_0\|_{2+\delta} + \int_0^t \|Q_t\|_{2+\delta}\,\mathrm{d}s\right). \tag{4.1.12}$$

Now we prove the estimates of Q_t. Taking time derivative of (4.1.6), we obtain:

$$\begin{aligned}\partial^\mu\partial_\mu Q_t =\ & \partial_{tt}A^{\nu\alpha}\partial_\nu v_\alpha + \partial_t A^{\nu\alpha}\partial_\nu\partial_t v_\alpha \\ & - \partial_\nu(\partial_t A^\mu_\alpha A^{\nu\alpha}\partial_\mu Q) - \partial_\nu(A^\mu_\alpha \partial_t A^{\nu\alpha}\partial_\mu Q) + \partial_\nu((\delta^{\mu\nu} - A^\mu_\alpha A^\nu_\alpha)\partial_\mu Q_t) \\ & + A_t^{\nu\alpha}\partial_\nu b_0^\mu\partial_\mu b_\alpha + A^{\nu\alpha}\partial_\nu b_0^\mu \partial_t\partial_\mu b_\alpha + \partial_t(\partial_\beta b_\gamma \partial_\nu b_\alpha)A^{\nu\gamma}A^{\beta\alpha} \\ & + \partial_\beta b_\gamma \partial_t(A^{\nu\gamma}A^{\beta\alpha})\partial_\nu b_\alpha - \partial_\beta b_0^\mu A^{\beta\alpha}\partial_t\partial_\mu b_\alpha - \partial_\beta b_0^\mu A_t^{\beta\alpha}\partial_\mu b_\alpha.\end{aligned} \tag{4.1.13}$$

with the boundary condition $Q_t = 0$ on Γ. By the elliptic estimate and the multiplicative Sobolev inequality, we similarly have:

$$\|Q_t\|_{2.5+\delta} \lesssim \varepsilon\|Q_t\|_{2.5+\delta} + \mathcal{P}_0 + \mathcal{P} + P(\|v\|_{2.5+\delta})\left(\|Q_0\|_{2+\delta} + \int_0^t \|Q_t\|_{2+\delta}\right), \tag{4.1.14}$$

which yields (4.1.4) by letting ε sufficiently small. □

4.1.2 Tangential Estimates

In this section, we establish the tangential energy estimates.

Theorem 4.1.3. Let $S = \overline{\partial}^{2.5+\delta}$. Let $\bar{N}(t) = \|Sv\|_{L^2}^2 + \|Sb\|_{L^2}^2 + \frac{c_0}{4}\|A^3_\alpha S\eta^\alpha\|^2_{L^2(\Gamma)}$. Then there exists a $T > 0$ such that for each $t \in [0, T]$, such that

$$\bar{N}(t) \lesssim \mathcal{P}_0 + \int_0^t \mathcal{P} + \int_0^t P(\|Q\|_{3+\delta}, \|Q_t\|_{2.5+\delta}, \|\eta\|_{3+\delta})\,\mathrm{d}s \tag{4.1.15}$$

First, we derive the tangential estimates of v.

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega (Sv^\alpha)(Sv_\alpha)\,\mathrm{d}y = \int_\Omega (Sv^\alpha)(\partial_t Sv_\alpha)\,\mathrm{d}y \\ =&-\int_\Omega (Sv^\alpha)(S(A^\mu_\alpha \partial_\mu Q))\,\mathrm{d}y + \int_\Omega (Sv^\alpha)(S(b_\beta A^{\mu\beta}\partial_\mu b_\alpha))\,\mathrm{d}y =: I + J. \end{aligned} \tag{4.1.16}$$

To control I, we have:

$$\begin{aligned} I =& -\int_\Omega (Sv^\alpha)(S(A^\mu_\alpha \partial_\mu Q))\,\mathrm{d}y \\ =& \underbrace{-\int_\Omega (Sv^\alpha)(A^\mu_\alpha)(S\partial_\mu Q)\,\mathrm{d}y}_{I_1} \underbrace{-\int_\Omega (Sv^\alpha)(SA^\mu_\alpha)(\partial_\mu Q)\,\mathrm{d}y}_{I_2} \\ & \underbrace{-\int_\Omega (Sv^\alpha)[S(A^\mu_\alpha \partial_\mu Q) - A^\mu_\alpha(S\partial_\mu Q) - (SA^\mu_\alpha)\partial_\mu Q]\,\mathrm{d}y}_{I_3}. \end{aligned} \tag{4.1.17}$$

Control of I_3: This is a direct consequence of inequality (3.2.3),

$$\begin{aligned} I_3 \le& \|Sv\|_{L^2}(\|A^\mu_\alpha\|_{W^{1,6}}\|\partial_\mu Q\|_{W^{1.5+\delta,3}} + \|A^\mu_\alpha\|_{W^{1.5+\delta,3}}\|\partial_\mu Q\|_{W^{1,6}}) \\ \lesssim& \|v\|_{2.5+\delta}\|\eta\|^2_{3+\delta}\|Q\|_{3+\delta}. \end{aligned} \tag{4.1.18}$$

Control of I_1: We integrate ∂_μ by parts to get:

$$\begin{aligned} I_1 &= -\int_\Omega S v^\alpha A^\mu_\alpha (\partial_\mu SQ)\,\mathrm{d}y \\ &= \int_\Omega A^\mu_\alpha S\partial_\mu v^\alpha (SQ)\,\mathrm{d}y - \int_\Gamma (\underbrace{SQ}_{=0})(A^\mu_\alpha S v^\alpha N_\mu)\,\mathrm{d}S(\Gamma) \\ &= \int_\Omega \underbrace{S(A^\mu_\alpha \partial_\mu v^\alpha)}_{=0}(SQ)\,\mathrm{d}y - \int_\Omega (SA^\mu_\alpha)\partial_\mu v^\alpha (SQ)\,\mathrm{d}y \\ &\quad - \int_\Omega [S(A^\mu_\alpha \partial_\mu v^\alpha) - (SA^\mu_\alpha)\partial_\mu v^\alpha - A^\mu_\alpha S\partial_\mu v^\alpha](SQ)\,\mathrm{d}y, \end{aligned} \tag{4.1.19}$$

The last term in the third line is controlled by using (3.2.3):

$$\begin{aligned} &-\int_\Omega [S(A^\mu_\alpha \partial_\mu v^\alpha) - (SA^\mu_\alpha)\partial_\mu v^\alpha - A^\mu_\alpha S\partial_\mu v^\alpha](SQ)\,\mathrm{d}y \\ &\lesssim (\|A^\mu_\alpha\|_{W^{1.5+\delta,3}}\|\partial_\mu v^\alpha\|_{W^{1,3}} + \|A^\mu_\alpha\|_{W^{1,6}}\|\partial_\mu v^\alpha\|_{1.5+\delta})\|SQ\|_{L^3} \\ &\lesssim \|Q\|_{3+\delta}\|\eta\|^2_{3+\delta}\|v\|_{2.5+\delta}. \end{aligned} \tag{4.1.20}$$

For the second term in the last line of (4.1.19), we need to integrate $1/2$-tangential derivatives by parts and then apply (3.2.1):

$$\begin{aligned} &-\int_\Omega SA^\mu_\alpha \partial_\mu v^\alpha SQ\,\mathrm{d}y = \int_\Omega \overline{\partial}^{2+\delta} A^\mu_\alpha \overline{\partial}^{0.5}(SQ\partial_\mu v^\alpha) \\ &\lesssim \|A\|_{2+\delta}(\|SQ\|_{H^{0.5}}\|\partial_\mu v^\alpha\|_{L^\infty} + \|SQ\|_{L^3}\|\partial_\mu v^\alpha\|_{W^{0.5,6}}) \\ &\lesssim \|\eta\|^2_{3+\delta}\|Q\|_{3+\delta}\|v\|_{2.5+\delta}. \end{aligned} \tag{4.1.21}$$

Summing these up, we have:

$$I_1 \lesssim \|\eta\|^2_{3+\delta}\|Q\|_{3+\delta}\|v\|_{2.5+\delta}. \tag{4.1.22}$$

Control of I_2: Let $S_m := -(I-\overline{\triangle})^{0.25+0.5\delta}\partial_m$. Then one may decompose S as:

$$S = ((I-\overline{\triangle})^{1.25+0.5\delta} - (I-\overline{\triangle})^{0.25+0.5\delta}) + \underbrace{(I-\overline{\triangle})^{0.25+0.5\delta}}_{=:S_0}$$
$$= (I-\overline{\triangle})^{0.25+0.5\delta}(-\overline{\triangle}) + S_0 =: \sum_{m=1}^{2} S_m\partial_m + S_0. \tag{4.1.23}$$

For I_2 we have:

$$I_2 = -\sum_{m=1}^{2}\int_\Omega (Sv^\alpha)(S_m\partial_m A^\mu_\alpha)(\partial_\mu Q)\,\mathrm{d}y \underbrace{-\int_\Omega (Sv^\alpha)S_0 A^\mu_\alpha \partial_\mu Q\,\mathrm{d}y}_{R_1}$$
$$= \underbrace{\sum_{m=1}^{2}\int_\Omega (Sv^\alpha)(S_m\partial_\beta\partial_m\eta^\nu)(A^\mu_\nu A^\beta_\alpha)\partial_\mu Q\,\mathrm{d}y}_{I_{21}} + \tag{4.1.24}$$
$$\int_\Omega (Sv^\alpha)[S_m(A^\mu_\nu\partial_\beta\partial_m\eta^\nu A^\beta_\alpha) - (S_m\partial_\beta\partial_m\eta^\nu)(A^\mu_\nu A^\beta_\alpha)]\partial_\mu Q\,\mathrm{d}y + R_1$$

Here, R_1 is bounded by $P(\|\eta\|_{2.5+\delta})\|Q\|_{H^{1.5}}\|v\|_{2.5+\delta}$ via the multiplicative Sobolev inequality, while the last term in the third line of (4.1.24) can be controlled by using Kato-Ponce inequality (3.2.4)

$$\int_\Omega (Sv^\alpha)[S_m(A^\mu_\nu\partial_\beta\partial_m\eta^\nu A^\beta_\alpha) - (S_m\partial_\beta\partial_m\eta^\nu)(A^\mu_\nu A^\beta_\alpha)]\partial_\mu Q\,\mathrm{d}y$$
$$\lesssim (\|A^\mu_\nu A^\beta_\alpha\|_{W^{1,6}}\|\partial_\beta\partial_m\eta^\nu\|_{W^{0.5+\delta,3}} + \|\partial_\beta\partial_m\eta^\nu\|_{L^6}\|A^\mu_\nu A^\beta_\alpha\|_{W^{1.5+\delta,3}})\|\partial_\mu Q\|_{L^\infty}\|Sv^\alpha\|_{L^2}$$
$$\lesssim P(\|\eta\|_{3+\delta})\|Q\|_{2.5+\delta}\|v\|_{2.5+\delta}. \tag{4.1.25}$$

It remains to control I_{21}. Writing $\sum_{m=1}^2 S_m\partial_m = S - S_0$, we have:

$$I_{21} = \int_\Omega (Sv^\alpha)(S\partial_\beta\eta^\nu)(A^\mu_\nu A^\beta_\alpha)(\partial_\mu Q)\,\mathrm{d}y - \int_\Omega (Sv^\alpha)(S_0\partial_\beta\eta^\nu)(A^\mu_\nu A^\beta_\alpha)(\partial_\mu Q)\,\mathrm{d}y. \tag{4.1.26}$$

It is easy to see the second term in (4.1.26) can be bounded by $\|v\|_{2.5+\delta}\|Q\|_{H^{1.5}}P(\|\eta\|_{2.5+\delta})$. For the first term, we integrate ∂_β by parts to obtain:

$$
\begin{aligned}
I_{21} =& \underbrace{-\int_\Omega (\partial_\beta S v^\alpha)(S\eta^\nu)(A^\mu_\nu A^\beta_\alpha)(\partial_\mu Q)\,\mathrm{d}y}_{I_{211}} - \int_\Omega (Sv^\alpha)(S\eta^\nu)(\partial_\beta A^\mu_\nu)A^\beta_\alpha(\partial_\mu Q)\,\mathrm{d}y \\
& - \int_\Omega (Sv^\alpha)(S\eta^\nu)(A^\mu_\nu A^\beta_\alpha)(\partial_\beta\partial_\mu Q)\,\mathrm{d}y + \underbrace{\int_\Gamma (Sv^\alpha)(S\eta^\nu)A^\mu_\nu A^\beta_\alpha(\partial_\mu Q)N_\beta\,\mathrm{d}S(\Gamma)}_{I_{212}} + R_2 \\
& \lesssim I_{211} + I_{212} + \mathcal{P} + P(\|Q\|_{H^3}),
\end{aligned}
\tag{4.1.27}
$$

Now, we bound I_{211} by the Kato-Ponce commutator estimate (3.2.3), because we want to move the derivatives on v to a in order to control v.

$$
\begin{aligned}
I_{211} =& -\int_\Omega (S\partial_\beta v^\alpha A^\beta_\alpha)(A^\mu_\nu S\eta^\nu)(\partial_\mu Q)\,\mathrm{d}y \\
=& \int_\Omega (\partial_\beta v^\alpha) SA^\beta_\alpha (A^\mu_\nu S\eta^\nu)(\partial_\mu Q)\,\mathrm{d}y \\
& + \int_\Omega (A^\mu_\nu S\eta^\nu \partial_\mu Q)[S(A^\beta_\alpha \partial_\beta v^\alpha) - (SA^\beta_\alpha)\partial_\beta v^\alpha - A^\beta_\alpha S(\partial_\beta v^\alpha)]\,\mathrm{d}y.
\end{aligned}
\tag{4.1.28}
$$

The term on the second line of (4.1.28) is controlled by (3.2.1) after integrating 0.5 derivatives by parts, i.e.,

$$
\begin{aligned}
\int_\Omega (\partial_\beta v^\alpha) SA^\beta_\alpha (A^\mu_\nu S\eta^\nu)(\partial_\mu Q)\,\mathrm{d}y &= \int_\Omega \overline{\partial}^{1/2}(S\eta^\nu A^\mu_\nu \partial_\mu Q \partial_\beta v^\alpha)\overline{\partial}^{2+\delta} A^\beta_\alpha\,\mathrm{d}y \\
&\lesssim P(\|\eta\|_{3+\delta})\|v\|_{2.5+\delta}\|Q\|_{2.5+\delta}
\end{aligned}
\tag{4.1.29}
$$

In addition, we apply (3.2.3) to the term on the third line of (4.1.28) and get:

$$
\begin{aligned}
&\int_\Omega (A^\mu_\nu S\eta^\nu \partial_\mu Q)[S(A^\beta_\alpha \partial_\beta v^\alpha) - (SA^\beta_\alpha)\partial_\beta v^\alpha - A^\beta_\alpha S(\partial_\beta v^\alpha)]\,\mathrm{d}y \\
&\lesssim P(\|\eta\|_{3+\delta})\|v\|_{2.5+\delta}\|Q\|_{2.5+\delta}
\end{aligned}
\tag{4.1.30}
$$

Therefore,

$$
I_{211} \lesssim P(\|\eta\|_{3+\delta})\|v\|_{2.5+\delta}\|Q\|_{2.5+\delta}. \tag{4.1.31}
$$

Now we come to control I_{212}. We shall compute its time integral, which then allows us to integrate ∂_t by parts to eliminate 0.5 more derivatives falling on v.

$$\int_0^t I_{212}\,\mathrm{d}s = \int_0^t \int_\Gamma (\partial_t S\eta^\alpha)(S\eta^\nu) A^3_\nu A^\beta_\alpha N_\beta (\partial_3 Q)\,\mathrm{d}S(\Gamma)\,\mathrm{d}s$$

$$\leq -\frac{c_0}{4}\|A^3_\alpha S\eta^\alpha\|^2_{L^2(\Gamma)} \tag{4.1.32}$$

$$+ P(\|v_0\|_{2.5+\delta}, \|b_0\|_{2.5+\delta}) + \int_0^t P(\|\eta\|_{3+\delta}, \|Q\|_{2.5+\delta}, \|Q_t\|_{2.5+\delta})\,\mathrm{d}s.$$

Summing up (4.1.17), (4.1.22), (4.1.24), (4.1.27), (4.1.31), (4.1.32), we obtain:

$$\int_0^t I(s)\,\mathrm{d}s + \frac{c_0}{4}\|A^3_\alpha S\eta^\alpha\|^2_{L^2(\Gamma)} \lesssim \mathcal{P}_0 + \int_0^t \mathcal{P} + \int_0^t P(\|\eta\|_{3+\delta}, \|Q\|_{3+\delta}, \|Q_t\|_{2.5+\delta})\,\mathrm{d}s. \tag{4.1.33}$$

Control of J: We will use (3.2.5) in the following proof.

$$J = \int_\Omega (Sv^\alpha)(S(b_\beta A^{\mu\beta}\partial_\mu b_\alpha))\,\mathrm{d}y = \int_\Omega (Sv^\alpha)(S(b_0^\mu \partial_\mu b_\alpha))\,\mathrm{d}y$$

$$= \underbrace{\int_\Omega (Sv^\alpha) b_0^\mu S\partial_\mu b_\alpha\,\mathrm{d}y}_{J_1} + \int_\Omega Sv^\alpha [S(b_0^\mu \partial_\mu b_\alpha) - b_0^\mu S\partial_\mu b_\alpha S\partial_\mu b_\alpha]\,\mathrm{d}y \tag{4.1.34}$$

$$\lesssim J_1 + \|v\|_{2.5+\delta}\|b_0\|_{2.5+\delta}\|b\|_{2.5+\delta}.$$

The term J_1 cannot be controlled directly, but it actually cancels with the highest order term in the energy of b. We will see that in the next step.

We derive the tangential estimates of b in this subsection and then conclude the tangential energy estimates.Using (3.2.4), we have:

$$\frac{1}{2}\frac{d}{dt}\|Sb\|^2_{L^2} = \int_\Omega (Sb_\alpha) S(b_\beta A^{\mu\beta}\partial_\mu v^\alpha)\,\mathrm{d}y = \int_\Omega (Sb_\alpha) S(b_0^\mu \partial_\mu v^\alpha)\,\mathrm{d}y$$

$$= \underbrace{\int_\Omega (Sb_\alpha) b_0^\mu (S\partial_\mu v^\alpha)\,\mathrm{d}y}_{K_1} + \int_\Omega Sb_\alpha [S(b_0^\mu \partial_\mu v^\alpha) - b_0^\mu (S\partial_\mu v^\alpha)]\,\mathrm{d}y \tag{4.1.35}$$

$$\lesssim K_1 + \|v\|_{2.5+\delta}\|b_0\|_{2.5+\delta}\|b\|_{2.5+\delta}.$$

We find that J_1 cancels K_1: Integrating ∂_μ in $J_1 + K_1$ by parts, we have

$$\begin{aligned} J_1 + K_1 &= \int_\Omega (Sv^\alpha) b_0^\mu S\partial_\mu b_\alpha \, \mathrm{d}y + \int_\Omega (Sb_\alpha) b_0^\mu S\partial_\mu v^\alpha \, \mathrm{d}y \\ &= -\int_\Omega Sv^\alpha Sb_\alpha \underbrace{\partial_\mu b_0^\mu}_{\mathrm{div}\, b_0 = 0} \, \mathrm{d}y + \int_{\partial\Omega} Sv^\alpha Sb_\alpha \underbrace{b_\beta A^{\mu\beta} N_\mu}_{B\cdot N = 0} \, \mathrm{d}S(y) = 0. \end{aligned} \tag{4.1.36}$$

Combining (4.1.16), (4.1.33), (4.1.34), (4.1.35), (4.1.36), we derive

$$\begin{aligned} &\|Sv\|_{L^2}^2 + \|Sb\|_{L^2}^2 + \frac{c_0}{4}\|A^3_\alpha S\eta^\alpha\|^2_{L^2(\Gamma)} \\ \lesssim\, &\mathcal{P}_0 + \int_0^t \mathcal{P} + \int_0^t P(\|Q\|_{3+\delta}, \|Q_t\|_{2.5+\delta}, \|\eta\|_{3+\delta}) \, \mathrm{d}s \end{aligned} \tag{4.1.37}$$

which implies in (4.1.15).

4.1.3 The div-curl type estimates

$H^{2.5+\delta}$-estimates of v and b: We do the div-curl type estimate of v and b to derive the control of full $H^{2.5+\delta}$ norms. Although for Euler equations one can use the Cauchy invariance to give linear estimates for curl v and div v, there is no such analogue for MHD equations. Instead, inspired by Gu-Wang [30], we can derive the evolution equations of curl v to control the curl v and curl b simultaneously thanks to the identity $b = (b_0 \cdot \partial)\eta$. Then we apply the div-curl estimate to derive the control of full $H^{2.5+\delta}$ norms of v and b.

Let $X = (X^1, X^2, X^3)$ be a vector field. We denote the "curl operator" and the "div operator" in the Eulerian coordinate by

$$(\mathrm{curl}_A X)_\lambda = \epsilon_{\lambda\tau\alpha} A^{\mu\tau}\partial_\mu X^\alpha, \quad \text{and } \mathrm{div}_A X = A^\mu_\alpha \partial_\mu X^\alpha,$$

respectively, where $\epsilon_{\lambda\tau\alpha}$ is the sign of the permutation $(\lambda\tau\alpha) \in S_3$.

Proposition 4.1.4. For sufficiently small $T > 0$, the following estimates hold:

$$\|\text{curl } v\|_{1.5+\delta} + \|\text{curl } b\|_{1.5+\delta} \lesssim \varepsilon(\|v\|_{2.5+\delta} + \|b\|_{2.5+\delta}) + \mathcal{P}_0 + \int_0^t \mathcal{P};$$
$$\|\text{div } v\|_{1.5+\delta} + \|\text{div } b\|_{1.5+\delta} \lesssim \varepsilon(\|v\|_{2.5+\delta} + \|b\|_{2.5+\delta}), \tag{4.1.38}$$

whenever $t \in [0, T]$.

Proof. The divergence estimates are easy because $\text{div}_A v = 0$ and $\text{div}_A b = 0$, so:

$$\|\text{div } v\|_{1.5+\delta} = \|\underbrace{\text{div}_A v}_{=0} + (A_I - \text{div}_A)v\|_{1.5+\delta} \lesssim \varepsilon\|v\|_{2.5+\delta};$$
$$\|\text{div } b\|_{1.5+\delta} = \|\underbrace{\text{div}_A b}_{=0} + (A_I - \text{div}_A)b\|_{1.5+\delta} \lesssim \varepsilon\|b\|_{2.5+\delta}.$$

The estimates for $\|\text{curl } v\|_{1.5+\delta}$ and $\|\text{curl } b\|_{1.5+\delta}$ are more dedicate. Since

$$\|\text{curl } v\|_{1.5+\delta} + \|\text{curl } b\|_{1.5+\delta}$$
$$\leq \|\text{curl}_{I-A} v\|_{1.5+\delta} + \|\text{curl}_{I-A} b\|_{1.5+\delta} + \|\text{curl}_A v\|_{1.5+\delta} + \|\text{curl}_A b\|_{1.5+\delta} \tag{4.1.39}$$
$$\lesssim \varepsilon(\|v\|_{2.5+\delta} + \|b\|_{2.5+\delta}) + \|\text{curl}_A v\|_{1.5+\delta} + \|\text{curl}_A b\|_{1.5+\delta},$$

and so it suffices to control $\|\text{curl}_A v\|_{1.5+\delta}$ and $\|\text{curl}_A b\|_{1.5+\delta}$. As mentioned in the beginning of this subsection, we derive the evolution equation for $\text{curl}_A v$

$$(\text{curl}_A \partial_t v)_\lambda = (\text{curl}_A((b_0 \cdot \partial)^2 \eta))_\lambda. \tag{4.1.40}$$

Commuting ∂_t and $b_0 \cdot \partial$ with curl_A on both sides of (4.1.40), we have:

$$\partial_t(\text{curl}_A v)_\lambda - (b_0 \cdot \partial)\text{curl}_A((b_0 \cdot \partial)\eta)_\lambda = \epsilon_{\lambda\tau\alpha} \partial_t A^{\mu\tau} \partial_\mu v^\alpha + [\text{curl}_A, b_0 \cdot \partial]((b_0 \cdot \partial)\eta)_\lambda. \tag{4.1.41}$$

Taking $\partial^{1.5+\delta}$ derivatives, and then commuting it with ∂_t and $b_0 \cdot \partial$, respectively, we get the evolution equation of $\text{curl}_A v$:

$$\partial_t(\partial^{1.5+\delta}\text{curl}_A v)_\lambda - (b_0 \cdot \partial)(\partial^{1.5+\delta}\text{curl}_A(b_0 \cdot \partial)\eta)_\lambda = F_\lambda, \tag{4.1.42}$$

where

$$F_\lambda = [\partial^{1.5+\delta}, b_0\cdot\partial](\mathrm{curl}_A(b_0\cdot\partial)\eta)_\lambda + \partial^{1.5+\delta}(\epsilon_{\lambda\tau\alpha}\partial_t A^{\mu\tau}\partial_\mu v^\alpha + [\mathrm{curl}_A, b_0\cdot\partial]((b_0\cdot\partial)\eta)_\lambda). \quad (4.1.43)$$

Taking the L^2 inner product with $\partial^{1.5+\delta}$ and integrating ∂_ν by parts ,we have:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\partial^{1.5+\delta}\mathrm{curl}_A v|^2 + |\partial^{1.5+\delta}\mathrm{curl}_A(b_0\cdot\partial)\eta|^2\,\mathrm{d}y &= \underbrace{\int_\Omega F\cdot\partial^{1.5+\delta}\mathrm{curl}_A v\,\mathrm{d}y}_{\mathcal{B}_1} \\ &\underbrace{+\int_\Omega \partial^{1.5+\delta}(\mathrm{curl}_A(b_0\cdot\partial)\eta)\cdot[\partial^{1.5+\delta}\mathrm{curl}_A, b_0\cdot\partial]v\,\mathrm{d}y}_{\mathcal{B}_2} \\ &\underbrace{+\int_\Omega \partial^{1.5+\delta}(\mathrm{curl}_A(b_0\cdot\partial)\eta)^\lambda\partial^{1.5+\delta}(\epsilon_{\lambda\tau\alpha}\partial_t A^{\mu\tau}\partial_\mu(b_0\cdot\partial\eta^\alpha))\,\mathrm{d}y}_{\mathcal{B}_3},\end{aligned} \quad (4.1.44)$$

where the boundary term vanishes since $b_0^3 = 0$. The control of $\mathcal{B}_3$ is straightforward,

$$\mathcal{B}_3 \lesssim \|b\|_{2.5+\delta}^2\|A\|_{1.5+\delta}\|A_t\|_{1.5+\delta} \lesssim \|b\|_{2.5+\delta}^2\|v\|_{2.5+\delta}\|\eta\|_{2.5+\delta}^6. \quad (4.1.45)$$

To control $\mathcal{B}_2$, it suffices to control $\|[\partial^{1.5+\delta}\mathrm{curl}_A, b_0\cdot\partial]v\|_{L^2}$. We simplify the commutator term as follows:

$$\begin{aligned}[\partial^{1.5+\delta}\mathrm{curl}_A, b_0\cdot\partial]v &= \epsilon_{\lambda\tau\alpha}\underbrace{\left(\partial^{1.5+\delta}(A^{\mu\tau}\partial_\mu(b_0^\nu\partial_\nu v^\alpha)) - \partial_\nu\partial^{1.5+\delta}(b_0^\nu A^{\mu\tau}\partial_\mu v^\alpha)\right)}_{\mathcal{B}_{21}} \\ &+ \epsilon_{\lambda\tau\alpha}\underbrace{\left(\partial_\nu\partial^{1.5+\delta}(b_0^\nu A^{\mu\tau}\partial_\mu v^\alpha) - b_0^\nu\partial_\nu\partial^{1.5+\delta}(A^{\mu\tau}\partial_\mu v^\alpha)\right)}_{\mathcal{B}_{22}}.\end{aligned} \quad (4.1.46)$$

Invoking the Kato-Ponce commutator estimate (3.2.5), we can control $\mathcal{B}_{22}$ as

$$\|\partial_\nu\partial^{1.5+\delta}(b_0^\nu A^{\mu\tau}\partial_\mu v_\alpha) - b_0^\nu\partial_\nu\partial^{1.5+\delta}(A^{\mu\tau}\partial_\mu v_\alpha)\|_{L^2} \lesssim \|b_0\|_{2.5+\delta}\|v\|_{2.5+\delta}\|\eta\|_{2.5+\delta}^2. \quad (4.1.47)$$

For $\mathcal{B}_{21}$, we have

$$\mathcal{B}_{21} = \epsilon_{\lambda\tau\alpha}\partial^{1.5+\delta}(A^{\mu\tau}\partial_\mu(b_0^\nu\partial_\nu v^\alpha)) - \partial_\nu(b_0^\nu A^{\mu\tau}\partial_\mu v^\alpha))$$

$$= \epsilon_{\lambda\tau\alpha}\partial^{1.5+\delta}(A^{\mu\tau}\partial_\mu b_0^\nu\partial_\nu v^\alpha + \partial_\beta((b_0\cdot\partial)\eta_\gamma)A^{\mu\gamma}A^{\beta\tau}\partial_\mu v^\alpha - \underbrace{\partial_\beta b_0^\nu\partial_\nu\eta_\gamma A^{\mu\gamma}A^{\beta\tau}\partial_\mu v^\alpha}_{=\partial_\beta b_0^\nu\delta_\nu^\mu A^{\beta\tau}\partial_\mu v^\alpha}). \tag{4.1.48}$$

Therefore, the L^2 norm of $\mathcal{B}_{21}$ can be controlled by:

$$\|\mathcal{B}_{21}\|_{L^2} \lesssim P(\|\eta\|_{2.5+\delta})(\|b_0\|_{2.5+\delta} + \|b\|_{2.5+\delta})\|v\|_{2.5+\delta}. \tag{4.1.49}$$

It remains to control $\mathcal{B}_1$, specifically, we need to bound $\|F\|_{L^2}$ given by (4.1.43). The first term is controlled by using Kato-Ponce commutator estimate (3.2.5).

$$\|[\partial^{1.5+\delta}, b_0\cdot\partial](\text{curl}_A(b_0\cdot\partial)\eta)\|_{L^2} \lesssim P(\|\eta\|_{2.5+\delta})\|b_0\|_{2.5+\delta}\|v\|_{2.5+\delta}. \tag{4.1.50}$$

For the commutator term in (4.1.43), we can proceed similarly as in (4.1.48)

$$\|[\text{curl}_A, b_0\cdot\partial]((b_0\cdot\partial)\eta)\|_{1.5+\delta} \lesssim P(\|\eta\|_{2.5+\delta})\|b_0\|_{2.5+\delta}\|v\|_{2.5+\delta}. \tag{4.1.51}$$

The remaining term in F can be easily bounded by $P(\|\eta\|_{2.5+\delta})\|b_0\|_{2.5+\delta}\|v\|_{2.5+\delta}$.

Combining (4.1.46), (4.1.47), (4.1.49), (4.1.50) and (4.1.51), we have

$$\|\text{curl}_A v\|_{1.5+\delta} + \|\text{curl}_A b\|_{1.5+\delta} \lesssim \mathcal{P}_0 + \|b_0\|_{2.5+\delta}\int_0^t \mathcal{P}\,. \tag{4.1.52}$$

Therefore, invoking Lemma 4.1.1, we ends the proof by:

$$\|\text{curl}\, v\|_{1.5+\delta} + \|\text{curl}\, b\|_{1.5+\delta} \lesssim \varepsilon(\|v\|_{2.5+\delta} + \|b\|_{2.5+\delta}) + \mathcal{P}_0 + \int_0^t \mathcal{P}. \tag{4.1.53}$$

□

Now we can derive the estimate of full $H^{2.5+\delta}$ derivative estimate of v and b. First applying

Hodge's decomposition inequality, we get

$$\|v\|_{2.5+\delta} \lesssim \|v\|_{L^2} + \|\text{curl } v\|_{1.5+\delta} + \|\text{div } v\|_{1.5+\delta} + |(\overline{\partial} v) \cdot N|_{1+\delta}, \tag{4.1.54}$$

For the tangential term, we apply Lemma 3.2.3 to get:

$$|\overline{\partial} v \cdot N|_{1+\delta} \lesssim \|\overline{\partial}^{1.5+\delta} \partial v_3\|_0 + \|\text{div } v\|_{1.5+\delta}, \tag{4.1.55}$$

Combining (4.1.4) and (4.1.55) and absorbing $\varepsilon \|v\|_{2.5+\delta}$ to LHS, we have :

$$\|v\|_{2.5+\delta} \lesssim \mathcal{P}_0 + \int_0^t \mathcal{P} \, \mathrm{d}s + \|Sv\|_{L^2}. \tag{4.1.56}$$

The estimate of $\|b\|_{2.5+\delta}$ can be derived exactly in the same way as $\|v\|_{2.5+\delta}$.

$$\|b\|_{2.5+\delta} \lesssim \mathcal{P}_0 + \int_0^t \mathcal{P} \, \mathrm{d}s + \|Sb\|_{L^2}. \tag{4.1.57}$$

In conclusion, we have proved

Theorem 4.1.5. The following estimates hold in a sufficiently small $[0, T]$:

$$\|v\|_{2.5+\delta} + \|b\|_{2.5+\delta} \lesssim \mathcal{P}_0 + \int_0^t \mathcal{P} \, \mathrm{d}s + \|Sv\|_{L^2} + \|Sb\|_{L^2}. \tag{4.1.58}$$

□

$H^{3+\delta}$-estimate of η: We derive the $H^{3+\delta}$ estimate for η via the div-curl estimate:

$$\|\eta\|_{3+\delta} \lesssim \|\eta\|_{L^2} + \|\text{curl } \eta\|_{2+\delta} + \|\text{div } \eta\|_{2+\delta} + \|(\overline{\partial} \eta) \cdot N\|_{H^{1.5+\delta}(\partial\Omega)}. \tag{4.1.59}$$

The divergence part is easy to treat owing to the div-free condition $\text{div}_A v = 0$, i.e., the Eulerian divergence of v is identically zero.

$$\begin{aligned}\|\text{div } \eta\|_{2+\delta} &\lesssim \|\text{div } \partial\eta\|_{1+\delta} + \|\text{div } \eta\|_{1+\delta} \lesssim \|\text{div}_A \partial\eta\|_{1+\delta} + \|(\text{div}_{I-A})\partial\eta\|_{1+\delta} + \|\eta\|_{2+\delta} \\ &\lesssim \|\text{div}_A \partial\eta\|_{1+\delta} + \varepsilon\|\eta\|_{3+\delta} + \|\eta(0)\|_{2+\delta} + \int_0^t \|v\|_{2+\delta}.\end{aligned} \tag{4.1.60}$$

Now it remains to control $\text{div}_A \partial\eta$. We have:

$$\text{div}_A \partial\eta(t) = \text{div } \partial\eta(0) + \int_0^t \text{div}_{A_t} \partial\eta + \underbrace{\partial(\text{div}_A v)}_{\text{div}_A v = 0} - \text{div}_{\partial A} v \;\, \mathrm{d}s.$$

Therefore, it can be controlled as

$$\begin{aligned}\|\text{div}_A \partial\eta(t)\|_{1+\delta} &\leq \|\text{div } \partial\eta(0)\|_{1+\delta} + \int_0^t \|\text{div}_{A_t} \partial\eta\|_{1+\delta} + \|\text{div}_{\partial A} v\|_{1+\delta} \, \mathrm{d}s \\ &\lesssim \|\eta(0)\|_{3+\delta} + \int_0^t \|\eta\|_{3+\delta} \|v\|_{2.5+\delta} \, \mathrm{d}s.\end{aligned} \tag{4.1.61}$$

Summing up (4.1.60) and (4.1.61), then absorbing the ε-term to LHS, we get

$$\|\text{div } \eta\|_{2+\delta} \lesssim \|\eta(0)\|_{3+\delta} + \int_0^t P(\|\eta\|_{3+\delta}, \|v\|_{2.5+\delta}) \, \mathrm{d}s. \tag{4.1.62}$$

For the boundary estimate, we have:

$$\|(\overline{\partial}\eta) \cdot N\|_{H^{1.5+\delta}(\Gamma)} \lesssim_{c_0} \frac{c_0}{4} \|A^3_\alpha S \eta^\alpha\|_{L^2(\Gamma)} + \varepsilon\|\eta\|_{H^3} + \|\eta(0)\|_{H^2} + \int_0^t \|v\|_{H^2}. \tag{4.1.63}$$

Here we remark that the term $\frac{c_0}{4} \|A^3_\alpha S \eta^\alpha\|_{L^2(\Gamma)}$ is exactly the boundary energy term derived from the physical sign condition in the tangential estimate. It remains to control $\|\text{curl } \eta\|_{2+\delta}$, we start with

$$\|\text{curl } \eta\|_{2+\delta} \leq \|\text{curl}_A \partial\eta\|_{1+\delta} + \|\text{curl}_{I-A} \partial\eta\|_{1+\delta} + \|\text{curl } \eta\|_{1+\delta}. \tag{4.1.64}$$

Recall that the i-th component of $\text{curl}_A \partial\eta$ (resp. $\text{curl}_{I-A} \partial\eta$) is of the form $\epsilon_{ijk} A^{\mu j} \partial_\mu \partial\eta^k$ (resp. $\epsilon_{ijk}(\delta^{\mu j} - A^{\mu j})\partial_\mu \partial\eta^k$). So we apply the multiplicative Sobolev inequality to get:

$$\|\text{curl}_{I-A} \partial\eta\|_{1+\delta} \leq \|I - A\|_{1.5+\delta} \|\eta\|_{3+\delta} \leq \varepsilon\|\eta\|_{3+\delta}. \tag{4.1.65}$$

In addition, using multiplicative Sobolev inequality, Young's inequality and Jensen's inequality, we have:

$$
\begin{aligned}
\|\mathrm{curl}_A \partial\eta\|_{1+\delta} &\lesssim \|A\|_{1.5+\delta}\|\eta\|_{3+\delta} \lesssim \varepsilon^{-1}\|\eta\|_{2.5+\delta}^4 + \varepsilon\|\eta\|_{3+\delta}^2 \\
&\lesssim \varepsilon^{-1}\|\eta(0)\|_{2.5+\delta}^4 + \varepsilon^{-1}\int_0^t \|v\|_{2.5+\delta}^4 + \varepsilon\|\eta\|_{3+\delta}^2
\end{aligned}
\tag{4.1.66}
$$

holds for sufficiently small t. Also,

$$
\|\mathrm{curl}\ \eta(t)\|_{1+\delta} \lesssim \|\eta(t)\|_{2+\delta} \le \|\eta(0)\|_{2+\delta} + \int_0^t \|v\|_{2+\delta}, \tag{4.1.67}
$$

and hence

$$
\|\mathrm{curl}\ \eta\|_{2+\delta} \lesssim \varepsilon^{-1}P(\|\eta(0)\|_{2.5+\delta}) + \varepsilon P(\|\eta\|_{3+\delta}) + \varepsilon^{-1}\int_0^t P(\|v\|_{2.5+\delta}). \tag{4.1.68}
$$

Now summing up (4.1.62), (4.1.63) and (4.1.68), we get the $H^{3+\delta}$ estimates of η.

Theorem 4.1.6. In a sufficiently short time interval $[0, T]$, it holds that

$$
\|\eta\|_{3+\delta} \lesssim_{c_0} \frac{c_0}{4}\|A^3_\alpha S\eta^\alpha\|_{L^2(\Gamma)} + \varepsilon P(\|\eta\|_{3+\delta}) + \varepsilon^{-1}\left(P(\|\eta(0)\|_{2.5+\delta}) + \int_0^t P(\|v\|_{2.5+\delta})\right). \tag{4.1.69}
$$

4.1.4 Closing the estimates

Now we recall that

$$
N(t) := \|\eta(t)\|_{3+\delta}^2 + \|v(t)\|_{2.5+\delta}^2 + \|b(t)\|_{2.5+\delta}^2. \tag{4.1.70}
$$

From (4.1.15), (4.1.58) and (4.1.69), we have :

$$
\begin{aligned}
N(t) \lesssim\ & \varepsilon P(\|\eta(t)\|_{3+\delta}) + P(N(0)) + P(N(t))\int_0^t P(N(s))\,\mathrm{d}s \\
&+ \varepsilon^{-1}P(\|\eta(0)\|_{2.5+\delta}) + \varepsilon^{-1}\int_0^t \|v(s)\|_{2.5+\delta}\ \mathrm{d}s.
\end{aligned}
\tag{4.1.71}
$$

For fixed $\varepsilon \ll 1$, recall that $\Omega = \mathbb{T}^2 \times (0, \overline{\varepsilon})$ and $\eta(0) = Id$, one may choose $\overline{\varepsilon}$ sufficiently small so that $\varepsilon^{-1} P(\|\eta(0)\|_{2.5+\delta}) \leq 1$. Then by a Gronwall-type argument, we conclude that:

$$N(t) \lesssim 1 + P(N(0)), \quad \text{when} \quad t \in [0, \bar{T}], \tag{4.1.72}$$

for some $\bar{T} = \bar{T}(N(0), \overline{\varepsilon})$. The justification of the a priori assumptions is straightforward and we refer it to [52, Lem. 2.1 and Lem. 5.5]. Therefore, Theorem 2.1.1 is proven.

4.2 Well-posedness of the Free-Boundary Problem in Incompressible MHD with Surface Tension

When the surface tension is not neglected, we establish the first result on the local well-posedness theory of the free-boundary problem in incompressible ideal MHD. First we present the proof of Theorem 2.2.1 about the LWP of (2.2.1).

4.2.1 The nonlinear approximate system

For $\kappa > 0$, we denote Λ_κ to be the standard mollifier on $\mathbb{R}^2$ defined as (3.4.1). Define $\tilde{\eta}$ to be the smoothed version of η solved by the following elliptic system

$$\begin{cases} -\triangle \tilde{\eta} = -\triangle \eta, & \text{in } \Omega, \\ \tilde{\eta} = \Lambda_\kappa^2 \eta & \text{on } \partial\Omega, \end{cases} \tag{4.2.1}$$

and $\tilde{A} := [\partial \tilde{\eta}]^{-1}$, $\tilde{J} := \det[\partial \tilde{\eta}]$, $\tilde{\mathbf{A}} := \tilde{J} \tilde{A}$ and $\tilde{n} = \hat{n} \circ \tilde{\eta}$. Now we introduce the nonlinear κ-approximation system of (2.2.1).

$$\begin{cases} \partial_t \eta = v & \text{in } [0,T] \times \Omega; \\ \partial_t v - (b_0 \cdot \partial)^2 \eta + \nabla_{\tilde{\mathbf{A}}} Q = 0 & \text{in } [0,T] \times \Omega; \\ \text{div}_{\tilde{A}} v = 0, & \text{in } [0,T] \times \Omega; \\ \text{div}\, b_0 = 0 & \text{in } \{t=0\} \times \Omega; \\ v^3 = b_0^3 = 0 & \text{on } \Gamma_0; \\ \tilde{\mathbf{A}}^{3\alpha} Q = -\sigma \sqrt{g} (\triangle_g \eta \cdot \tilde{n}) \tilde{n}^\alpha + \kappa \left((1 - \overline{\triangle})(v \cdot \tilde{n}) \right) \tilde{n}^\alpha & \text{on } \Gamma; \\ b_0^3 = 0 & \text{on } \Gamma, \\ (\eta, v) = (\text{Id}, v_0) & \text{in } \{t=0\} \times \overline{\Omega}. \end{cases} \tag{4.2.2}$$

Here $\overline{\triangle} := \overline{\partial}_1^2 + \overline{\partial}_2^2$ is the (flat) tangential Laplacian. For technical simplicity, we only assume the upper boundary $\mathbb{T}^2 \times \{1\}$ correponds to the free surface, and the bottom $\Gamma_0 = \mathbb{T}^2 \times \{0\}$ satisfies the perfect conducting wall condition. The re-formulated boundary condition on Γ is used here since we find that it is more convenient to apply when studying (4.2.2). We remark here that in absence of $\kappa\left((1-\overline{\triangle})(v\cdot\tilde{n})\right)\tilde{n}^\alpha$ the boundary condition is just a reformulation of

$$\tilde{\mathbf{A}}^{3\alpha}Q = -\sigma\sqrt{g}\triangle_g\eta^\alpha. \tag{4.2.3}$$

Invoking (3.1.4) and the identity $\tilde{J}|\tilde{A}^T N| = \sqrt{\tilde{g}}$, where $\tilde{g} = g(\tilde{\eta})$, we have

$$\tilde{\mathbf{A}}^{3\alpha}/\sqrt{\tilde{g}} = \tilde{J}\tilde{A}^{\mu\alpha}N_\mu/\tilde{J}|\tilde{A}^T N| = \tilde{n}^\alpha, \tag{4.2.4}$$

and (4.2.3) becomes $Q\tilde{n}^\alpha = -\sigma\frac{\sqrt{g}}{\sqrt{\tilde{g}}}\triangle_g\eta^\alpha$. Also, due to $\tilde{n}\cdot\tilde{n} = 1$, we obtain $Q\tilde{n}^\alpha = Q(\tilde{n}\cdot\tilde{n})\tilde{n}^\alpha = -\sigma\frac{\sqrt{g}}{\sqrt{\tilde{g}}}(\triangle_g\eta\cdot\tilde{n})\tilde{n}^\alpha$. In the view of (4.2.4), this is equivalent to $\tilde{\mathbf{A}}^{3\alpha}Q = -\sigma\sqrt{g}(\triangle_g\eta\cdot\tilde{n})\tilde{n}^\alpha$ By adding the artificial viscosity term $\kappa\left((1-\overline{\triangle})(v\cdot\tilde{n})\right)\tilde{n}^\alpha$ on the RHS, the boundary condition of (4.2.2) is then achieved:

$$\tilde{\mathbf{A}}^{3\alpha}Q = -\sigma\sqrt{g}(\triangle_g\eta\cdot\tilde{n})\tilde{n}^\alpha + \kappa\left((1-\overline{\triangle})(v\cdot\tilde{n})\right)\tilde{n}^\alpha. \tag{4.2.5}$$

In addition, since $\tilde{\mathbf{A}}^{3\alpha}\tilde{n}_\alpha = \sqrt{\tilde{g}}$, (4.2.5) can be written as

$$\sqrt{\tilde{g}}Q = -\sigma\sqrt{g}(\triangle_g\eta\cdot\tilde{n}) + \kappa(1-\overline{\triangle})(v\cdot\tilde{n}). \tag{4.2.6}$$

Despite being equivalent to each other, (4.2.5) and (4.2.6) will be adapted to different scenarios. In fact, (4.2.5) will be used in Section 4.2.3 for the tangential energy estimate, whereas we find (4.2.6) more convenient when dealing with the boundary estimate in Section 4.2.2.3.

Remark 4.2.1 (Necessity of tangential smoothing)**.** It is often highly nontrivial to prove the local well-posedness for a free-boundary problem of inviscid fluid, especially when equipped with the Young-Laplace boundary condition, by a simple iteration scheme and fixed-point argument for the linearized equations. The reason is that the linearization breaks the subtle cancellation structure on the free surface and thus causes the loss of tangential derivatives of the flow map η, which also occurs for incompressible Euler equations with surface tension. For ideal MHD, one cannot directly define $\tilde{\eta} = \Lambda_\kappa^2 \eta$ as in [16]. Indeed, such construction is not applicable to MHD because we also need to control $\|[\Lambda_\kappa^2, (b_0 \cdot \partial)]\eta\|_{4.5}$ in which there is a normal derivative $b_0^3 \partial_3$ *in the interior* that is not compatible with the tangential mollification.

Remark 4.2.2 (Necessity of the artifical viscosity)**.** An essential reason for introducing such artificial viscosity term is that the presence of surface tension forces us to control all of the time derivatives. In particular, the pressure Q satisfies an elliptic equation and it appears that one can only get control of it by considering the Neumann boundary condition instead of Dirichlet boundary condition due to the presence of surface tension. The Neumann boundary condition contains the time derivative of v, and thus we have to include the time derivatives in our energy.

However, the full time derivatives of v and $(b_0 \cdot \partial)\eta$ only has $L^2(\Omega)$ regularity and we cannot get estimates of the full time derivatives of Q due to the low spatial regularity. Therefore, we do not have any control for the terms containing full time derivatives on the boundary due to the failure of Sobolev trace lemma. For the original system, one can use the subtle cancellation structure developed in [18, 53] to resolve this difficulty. But such cancellation structure no longer holds for the nonlinear κ-approximate problem due to the presence of tangential smoothing. Therefore, introducing the artificial viscosity term could produce κ-weighted higher order terms on the boundary, which enables us to finish the energy control.

The Young-Laplace boundary condition only gives us the information in the Eulerian normal direction. Therefore, the artificial viscosity can only be imposed in the smoothed Eulerian normal direction $\kappa\left((1-\overline{\triangle})(v\cdot\tilde{n})\right)\tilde{n}^\alpha$ instead of all the components, otherwise the system would be over-determined.

Remark 4.2.3 (Difficulty in vanishing viscosity limit)**.** Very recently, Gu-Lei [27] proved the LWP of incompressible elastodynamics with surface tension by proving the inviscid limit of visco-elastodynamics system in standard Sobolev spaces. We also note that the inviscid limit of free-boundary MHD was recently proved by Chen-Ding [8] in co-normal Sobolev spaces. However, analogous inviscid limit in standard Sobolev space is not applicable to MHD due to the existence of MHD boundary layers.

Our goal is to derive the uniform-in-κ a priori estimates for the system (4.2.2).

Proposition 4.2.4. Given the divergence-free vector fields $v_0 \in H^{4.5}(\Omega) \cap H^5(\Gamma)$ and $b_0 \in H^{4.5}(\Omega)$ satisfying $b_0^3 = 0$ on $\Gamma \cup \Gamma_0$, there exists some $T > 0$ independent of $\kappa > 0$, such that the solution $(\eta(\kappa), v(\kappa), q(\kappa))$ to (4.2.2) satisfies the following uniform-in-κ estimates

$$\sup_{0 \le t \le T} E_\kappa(t) \le \mathcal{C}, \tag{4.2.7}$$

where $\mathcal{C}$ is a constant depends on $\|v_0\|_{4.5}, \|b_0\|_{4.5}, |v_0|_5$, provided the following a priori assumption hold for all $t \in [0, T_1]$

$$\|\tilde{J}(t) - 1\|_{3.5} + \|\operatorname{Id} - \tilde{\mathbf{A}}(t)\|_{3.5} + \|\operatorname{Id} - \tilde{\mathbf{A}}^T \tilde{\mathbf{A}}(t)\|_{3.5} \le \varepsilon. \tag{4.2.8}$$

Here the energy functional E_κ of (4.2.2) is defined to be

$$E_\kappa = E_\kappa^{(1)} + E_\kappa^{(2)} + E_\kappa^{(3)}, \tag{4.2.9}$$

where

$$\begin{aligned} E_\kappa^{(1)} :=& \|\eta(\kappa)\|_{4.5}^2 + \sum_{k=0}^{3} \left(\left\| \partial_t^k v(\kappa) \right\|_{4.5-k}^2 + \left\| \partial_t^k (b_0 \cdot \partial) \eta(\kappa) \right\|_{4.5-k}^2 \right) \\ &+ \left\| \partial_t^4 v(\kappa) \right\|_0^2 + \left\| \partial_t^4 (b_0 \cdot \partial) \eta(\kappa) \right\|_0^2 \\ &+ \sum_{k=0}^{3} \left| \overline{\partial} \left(\Pi \partial_t^k \overline{\partial}^{3-k} v(\kappa) \right) \right|_0^2 + \left| \overline{\partial} \left(\Pi \overline{\partial}^3 (b_0 \cdot \partial) \eta(\kappa) \right) \right|_0^2, \end{aligned} \tag{4.2.10}$$

$$E_\kappa^{(2)} := \sigma \sum_{k=1}^{4} \int_0^T \left(\left| \sqrt{\kappa} \partial_t^k v(\kappa) \cdot \tilde{n}(\kappa) \right|_{5-k}^2 + \left| \sqrt{\kappa} (b_0 \cdot \partial) v(\kappa) \cdot \tilde{n}(\kappa) \right|_4^2 \right) \mathrm{d}t, \tag{4.2.11}$$

$$E_\kappa^{(3)} := \sum_{k=1}^{4} \int_0^T \left(\left\| \sqrt{\kappa} \partial_t^k v(\kappa) \right\|_{5.5-k}^2 + \left\| \sqrt{\kappa} \partial_t^k (b_0 \cdot \partial) \eta \right\|_{5.5-k}^2 \right) \mathrm{d}t. \tag{4.2.12}$$

By the Gronwall-type argument, we only need to show

$$E_\kappa(T) \le \mathcal{P}_0 + C(\varepsilon) E_\kappa(T) + \mathcal{P} \int_0^T \mathcal{P}, \tag{4.2.13}$$

Before going to the proof, we need the following preliminary estimates.

Lemma 4.2.5 ([28, Lem. 3.2])**.** We have

$$\|\tilde{\eta}\|_{4.5} \lesssim \|\eta\|_{4.5} \tag{4.2.14}$$

$$\|(b_0 \cdot \partial)\tilde{\eta}\|_{4.5} \lesssim P(\|b_0\|_{4.5}, \|(b_0 \cdot \partial)\eta\|_{4.5}, \|\eta\|_{4.5}). \tag{4.2.15}$$

Lemma 4.2.6 ([28, Lem. 3.3])**.** Assume that $\|\eta\|_{4.5}, \|v\|_{4.5} \le N_0$, where $N_0 \ge 1$. If $T \le \varepsilon / P(N_0)$ for some fixed polynomial P and η, v is defined on $[0, T]$, then the following inequality holds for $t \in [0, T]$:

$$\|\tilde{\mathbf{A}}^{\mu\alpha} - \delta^{\mu\alpha}\|_{3.5} + \|A^{\mu\alpha} - \delta^{\mu\alpha}\|_{3.5} + \|\tilde{A}^{\mu\alpha} - \delta^{\mu\alpha}\|_{3.5} \lesssim \varepsilon, \tag{4.2.16}$$

$$\forall 0 \le s \le 1.5, \;\; |\overline{\partial}^s(\tilde{n} - N)|_{L^\infty(\Gamma)} \lesssim \varepsilon, \quad |\overline{\partial}^s(\hat{n} - N)|_{L^\infty(\Gamma)} \lesssim \varepsilon, \tag{4.2.17}$$

$$|\tilde{n} - N|_3 \lesssim \varepsilon, \quad |\hat{n} - N|_3 \lesssim \varepsilon, \tag{4.2.18}$$

$$|\delta^{ij} - \sqrt{g} g^{ij}|_3 \le \varepsilon, \tag{4.2.19}$$

$$|\overline{\partial}\eta \cdot n|_3 \le \varepsilon, \quad |\overline{\partial}^2 \eta|_2 \le \varepsilon. \tag{4.2.20}$$

Remark 4.2.7. The inequalities in Lemma 4.2.6 can in fact be viewed as an extended list of the a priori assumptions. Moreover, (4.2.8) is in fact a direct consequence of (4.2.16).

Lemma 4.2.8 ([28, Lem. 3.4])**.** Let $k = 0, \cdots, 4$. Then

$$\|\partial\partial_t^k(\tilde{\eta}-\eta)\|_0 \lesssim \|\sqrt{\kappa}\partial_t^k\eta\|_{1.5}. \tag{4.2.21}$$

Further, for $\ell = 0, 1, 2$, there holds

$$\|\partial\partial_t^\ell(\tilde{\eta}-\eta)\|_{L^\infty} \lesssim \|\sqrt{\kappa}\partial_t^\ell\eta\|_{3.5}. \tag{4.2.22}$$

Finally, we state the following two lemmas that concern the boundary elliptic estimates of $\sqrt{\kappa}\tilde{\eta}$ and $\kappa(b_0\cdot\partial)\tilde{\eta}$. These lemmas will be adapted to control the boundary error terms generated when derivatives land on the Eulerian normal $\tilde{n}$.

Lemma 4.2.9 ([28, Lem. 3.5])**.** Let $\mathcal{M}_0 = P(\|v_0\|_{4.5}, \sqrt{\kappa}\|v_0\|_{8.5}, \sqrt{\kappa}\|b_0\|_{8.5}, \sqrt{\kappa}|v_0|_{10})$. Then

$$|\sqrt{\kappa}\eta|_5^2 \leq \mathcal{M}_0 + C(\varepsilon)E_\kappa(T) + \mathcal{P}\int_0^T \mathcal{P}, \tag{4.2.23}$$

$$\int_0^T |\sqrt{\kappa}v|_5^2 \leq \mathcal{M}_0 + C(\varepsilon)E_\kappa(T) + \mathcal{P}\int_0^T \mathcal{P}, \tag{4.2.24}$$

$$\int_0^T |\sqrt{\kappa}(b_0\cdot\partial)\eta|_5^2 \leq \mathcal{M}_0 + C(\varepsilon)E_\kappa(T) + \mathcal{P}\int_0^T \mathcal{P}. \tag{4.2.25}$$

4.2.2 A priori estimates of the approximate system

4.2.2.1 Elliptic estimates of pressure

We prove the following proposition in this section.

Proposition 4.2.10. The pressure Q in (4.2.2) and its time derivatives satisfy the following estimates

$$\|Q\|_{4.5} + \|\partial_t Q\|_{3.5} + \|\partial_t^2 Q\|_{2.5} + \|\partial_t^3 Q\|_1 \lesssim \mathcal{P}. \tag{4.2.26}$$

First, we give control of the pressure Q. Taking $\mathrm{div}_{\tilde{\mathbf{A}}}$ in the second equation of (4.2.2) we get the following elliptic system for Q:

$$-\triangle_{\tilde{\mathbf{A}}} Q := -\operatorname{div}_{\tilde{\mathbf{A}}}(\nabla_{\tilde{\mathbf{A}}} Q) = \left[\operatorname{div}_{\tilde{\mathbf{A}}}, \partial_t\right] v + \left[\operatorname{div}_{\tilde{\mathbf{A}}}, (b_0 \cdot \partial)\right] (b_0 \cdot \partial)\eta + (b_0 \cdot \partial)\operatorname{div}_{\tilde{\mathbf{A}}} ((b_0 \cdot \partial)\eta)$$

$$= -\partial_t \tilde{\mathbf{A}}^{\mu\alpha} \partial_\mu v_\alpha - ((b_0 \cdot \partial)\tilde{\mathbf{A}}^{\mu\alpha})\partial_\mu (b_0 \cdot \partial)\eta_\alpha + \tilde{\mathbf{A}}^{\mu\alpha}(\partial_\mu b_0 \cdot \partial)(b_0 \cdot \partial)\eta_\alpha$$

$$+ (b_0 \cdot \partial) \underbrace{\operatorname{div}_a ((b_0 \cdot \partial)\eta)}_{=\operatorname{div} b_0 = 0} + (b_0 \cdot \partial) \left((\tilde{\mathbf{A}}^{\mu\alpha} - A^{\mu\alpha})\partial_\mu (b_0 \cdot \partial)\eta_\alpha \right),$$

and thus

$$-\triangle Q =: -\operatorname{div}(\partial Q) = -\partial_\nu \left((\delta^{\mu\nu} - \tilde{\mathbf{A}}^{\mu\alpha}\tilde{\mathbf{A}}^{\nu\alpha})\partial_\mu Q \right) - \partial_t \tilde{\mathbf{A}}^{\mu\alpha}\partial_\mu v_\alpha - ((b_0 \cdot \partial)\tilde{\mathbf{A}}^{\mu\alpha})\partial_\mu (b_0 \cdot \partial)\eta_\alpha$$

$$+ \tilde{\mathbf{A}}^{\mu\alpha}(\partial_\mu b_0 \cdot \partial)(b_0 \cdot \partial)\eta_\alpha + (b_0 \cdot \partial) \left((\tilde{\mathbf{A}}^{\mu\alpha} - A^{\mu\alpha})\partial_\mu (b_0 \cdot \partial)\eta_\alpha \right). \tag{4.2.27}$$

We impose Neumann boundary condition to (4.2.27) by contracting $\tilde{\mathbf{A}}^{\mu\alpha} N_\mu = \tilde{\mathbf{A}}^{3\alpha}$ with the second equation of (4.2.2)

$$\frac{\partial Q}{\partial N} = (\delta^{\mu 3} - \tilde{\mathbf{A}}^{\mu\alpha}\tilde{\mathbf{A}}^{3\alpha})\partial_\mu Q - \tilde{\mathbf{A}}^{3\alpha}\partial_t v_\alpha + \tilde{\mathbf{A}}^{3\alpha}(b_0 \cdot \partial)^2 \eta_\alpha, \quad \text{on } \Gamma. \tag{4.2.28}$$

Also, since $\tilde{\mathbf{A}}^{31} = \tilde{\mathbf{A}}^{32} = 0$, $\tilde{\mathbf{A}}^{33} = 1$, $v_3 = 0$, and $b_0^3 = 0$ implies $(b_0 \cdot \partial)\eta_3 = b_0^j \overline{\partial}_j \eta_3 = 0$ on Γ_0, (4.2.28) yields

$$\frac{\partial Q}{\partial N} = 0, \quad \text{on } \Gamma_0. \tag{4.2.29}$$

By the standard elliptic estimates, we have

$$\|Q\|_{4.5} \lesssim \|\text{RHS of (4.2.27)}\|_{2.5} + |\text{RHS of (4.2.28)}|_3 + |Q|_0.$$

Here, $|Q|_0$ can be directly bounded by invoking the boundary condition of Q, i.e.,

$$Q = -\sigma \frac{\sqrt{g}}{\sqrt{\tilde{g}}}(\triangle_g \eta \cdot \tilde{n}) + \kappa \frac{1}{\sqrt{\tilde{g}}}(1 - \overline{\triangle})(v \cdot \tilde{n}), \tag{4.2.30}$$

and thus

$$|Q|_0 \lesssim \mathcal{P}. \tag{4.2.31}$$

Invoking the a priori assumption (4.2.8), we have

$$\|\text{RHS of (4.2.27)}\|_{2.5} \lesssim \varepsilon\|Q\|_{4.5} + P(\|b_0\|_{4.5}, \|(b_0\cdot\partial)\eta\|_{4.5}, \|\eta\|_{3.5}, \|v\|_{3.5}) \tag{4.2.32}$$

and

$$|\text{RHS of (4.2.28)}|_3 \lesssim \varepsilon\|Q\|_{4.5} + P(\|\eta\|_{4.5})\left(\|\partial_t v\|_{3.5} + \|b_0\|_{3.5}\|(b_0\cdot\partial)\eta\|_{4.5}\right). \tag{4.2.33}$$

Summing up (4.2.31)-(4.2.33) and choosing $\varepsilon > 0$ sufficiently small, we get

$$\|Q\|_{4.5} \lesssim \mathcal{P}. \tag{4.2.34}$$

Next we take ∂_t in (4.2.27)-(4.2.28) to get the equations of $\partial_t Q$:

$$\begin{aligned}
-\triangle\partial_t Q =& -\partial_\nu\left((\delta^{\mu\nu} - \tilde{\mathbf{A}}^{\mu\alpha}\tilde{\mathbf{A}}^{\nu\alpha})\partial_\mu\partial_t Q\right) - \partial_\nu\left((\delta^{\mu\nu} - \partial_t(\tilde{\mathbf{A}}^{\mu\alpha}\tilde{\mathbf{A}}^{\nu\alpha}))\partial_\mu Q\right)\\
&- \partial_t^2\tilde{\mathbf{A}}^{\mu\alpha}\partial_\mu v_\alpha - \partial_t\tilde{\mathbf{A}}^{\mu\alpha}\partial_t\partial_\mu v_\alpha\\
&+ \partial_t\left(\tilde{\mathbf{A}}^{\mu\alpha}(\partial_\mu b_0\cdot\partial)(b_0\cdot\partial)\eta_\alpha - ((b_0\cdot\partial)\tilde{\mathbf{A}}^{\mu\alpha})\partial_\mu(b_0\cdot\partial)\eta_\alpha\right)\\
&+ (b_0\cdot\partial)\left((\partial_t\tilde{\mathbf{A}} - \partial_t A)\partial((b_0\cdot\partial)\eta) + (\tilde{\mathbf{A}} - A)\partial((b_0\cdot\partial)v)\right),
\end{aligned} \tag{4.2.35}$$

with Neumann boundary condition

$$\begin{aligned}
\frac{\partial\partial_t Q}{\partial N} =& (\delta^{\mu 3} - \tilde{\mathbf{A}}^{\mu\alpha}\tilde{\mathbf{A}}^{3\alpha})\partial_\mu\partial_t Q - \partial_t(\tilde{\mathbf{A}}^{\mu\alpha}\tilde{\mathbf{A}}^{3\alpha})\partial_\mu Q\\
&- \tilde{\mathbf{A}}^{3\alpha}(\partial_t^2 v_\alpha - (b_0\cdot\partial)^2 v_\alpha) - \partial_t\tilde{\mathbf{A}}^{3\alpha}(\partial_t v - (b_0\cdot\partial)^2\eta)_\alpha, \quad \text{on } \Gamma,
\end{aligned} \tag{4.2.36}$$

and

$$\frac{\partial\partial_t Q}{\partial N} = 0, \quad \text{on } \Gamma_0. \tag{4.2.37}$$

Invoking the standard elliptic equation again, we have

$$\|\partial_t Q\|_{3.5} \lesssim \|\text{RHS of (4.2.35)}\|_{1.5} + |\text{RHS of (4.2.36)}|_2 + |\partial_t Q|_0.$$

The control of the first two terms follows similarly as above

$$\|\text{RHS of (4.2.35)}\|_{1.5} + |\text{RHS of (4.2.36)}|_2 \lesssim \mathcal{P}. \tag{4.2.38}$$

As for the boundary term, we take ∂_t in the surface tension equation to get

$$\partial_t Q = -\sigma \frac{\sqrt{g}}{\sqrt{\tilde{g}}}(\triangle_g v \cdot \tilde{n}) + \kappa \frac{1}{\sqrt{\tilde{g}}}(1-\overline{\triangle})(\partial_t v \cdot \tilde{n}) + \text{lower-order temrs}$$

and thus

$$\|\partial_t Q\|_0 \lesssim \mathcal{P}. \tag{4.2.39}$$

Summing up (4.2.38)-(4.2.39) and choosing $\varepsilon > 0$ to be sufficiently small, we get

$$\|\partial_t Q\|_{3.5} \lesssim \mathcal{P}. \tag{4.2.40}$$

Taking ∂_t again, we can silimarly get the estimates of $\|\partial_t^2 Q\|_{2.5}$:

$$\|\partial_t^2 Q\|_{2.5} \lesssim \mathcal{P}. \tag{4.2.41}$$

However, we cannot use the similar method to control $\|\partial_t^3 Q\|_1$ because the standard elliptic estimates requires at least H^2-regularity. Instead, we invoke Lemma 3.3.2 which allows us to perform the low regularity H^1-estimate for ∂_t^3-differentiated elliptic system (4.2.27)-(4.2.28). We need to first rewrite the elliptic equations into the divergence form. Recall that the elliptic equation (4.2.27) is derived by taking smoothed Eulerian divergence $\text{div}_{\tilde{\mathbf{A}}}$. This, together with Piola's identity gives that

$$-\partial_\nu(\tilde{\mathbf{A}}^{\nu\alpha}\tilde{\mathbf{A}}^{\mu\alpha}\partial_\mu Q) = \partial_\nu\left(\tilde{\mathbf{A}}^{\nu\alpha}(\partial_t v - (b_0\cdot\partial)^2\eta)_\alpha\right),$$

with the boundary condition

$$\tilde{\mathbf{A}}^{3\alpha}\tilde{\mathbf{A}}^{\mu\alpha}\partial_\mu Q = \tilde{\mathbf{A}}^{3\alpha}(\partial_t v - (b_0\cdot\partial)^2\eta)_\alpha, \quad \text{on } \Gamma,$$

and $\frac{\partial Q}{\partial N} = 0$ on Γ_0. Taking ∂_t^3 derivatives, we get

$$\partial_\nu(\tilde{\mathbf{A}}^{\nu\alpha}\tilde{\mathbf{A}}^{\mu\alpha}\partial_t^3\partial_\mu Q) = \partial_\nu\left(\left[\tilde{\mathbf{A}}^{\nu\alpha}\tilde{\mathbf{A}}^{\mu\alpha},\partial_t^3\right]\partial_\mu Q\right) + \partial_\nu\partial_t^3\left(\tilde{\mathbf{A}}^{\nu\alpha}(\partial_t v - (b_0\cdot\partial)^2\eta)_\alpha\right), \tag{4.2.42}$$

with the boundary condition

$$\tilde{\mathbf{A}}^{3\alpha}\tilde{\mathbf{A}}^{\mu\alpha}\partial_\mu\partial_t^3 Q = \left[\tilde{\mathbf{A}}^{3\alpha}\tilde{\mathbf{A}}^{\mu\alpha},\partial_t^3\right]\partial_\mu Q + \partial_t^3\left(\tilde{\mathbf{A}}^{3\alpha}(\partial_t v - (b_0\cdot\partial)^2\eta)_\alpha\right), \quad \text{on } \Gamma. \tag{4.2.43}$$

Now if we set

$$\mathfrak{B}^{\nu\mu} := \tilde{\mathbf{A}}^{\nu\alpha}\tilde{\mathbf{A}}^{\mu\alpha}, \quad h := \text{RHS of (4.2.43)}$$

and

$$\pi^\nu := \left[\tilde{\mathbf{A}}^{\nu\alpha}\tilde{\mathbf{A}}^{\mu\alpha},\partial_t^3\right]\partial_\mu Q + \partial_t^3\left(\tilde{\mathbf{A}}^{\nu\alpha}(\partial_t v - (b_0\cdot\partial)^2\eta)_\alpha\right)$$

then the elliptic system (4.2.42)-(4.2.43) is exactly of the form (3.3.4). The a priori assumption (4.2.8) shows that $\|\mathfrak{B} - \text{Id}\|_{L^\infty}$ is sufficiently small. Now it is straightforward to see that $\pi, \text{div}\,\pi \in L^2$, i.e.,

$$\|\pi\|_0 + \|\text{div}\,\pi\|_0 \lesssim \mathcal{P}. \tag{4.2.44}$$

Also, since

$$h - \pi\cdot N = 0, \tag{4.2.45}$$

then by Lemma 3.3.2 and invoking (4.2.34), (4.2.40), (4.2.41), we have

$$\left\|\partial_t^3 Q - \overline{\partial_t^3 Q}\right\|_1 \lesssim \|\pi\|_0 \lesssim \mathcal{P}. \tag{4.2.46}$$

Lastly, we need to control the H^1-norm of $\overline{\partial_t^3 Q}$ by $\mathcal{P}$.

$$\overline{\partial_t^3 Q} = \frac{1}{\text{vol}\,(\Omega)} \int_\Omega \partial_t^3 Q \,\mathrm{d}y = \frac{1}{\text{vol}\,(\Omega)} \int_\Omega \partial_t^3 Q \overline{\partial}_1 y_1 \,\mathrm{d}y = -\frac{1}{\text{vol}\,(\Omega)} \int_\Omega y_1 \overline{\partial}_1 \partial_t^3 Q$$

$$\leq C(\text{vol}\,(\Omega)) \|\overline{\partial}\partial_t^3 Q\|_0 \|y_1\|_0 = C(\text{vol}\,(\Omega)) \left\| \overline{\partial}(\partial_t^3 Q - \overline{\partial_t^3 Q}) \right\|_0 \|y_1\|_0 \tag{4.2.47}$$

$$\leq C(\text{vol}(\Omega)) \left\| \partial_t^3 Q - \overline{\partial_t^3 Q} \right\|_1 .$$

This concludes the control of $\|\partial_t^3 Q\|_1$, and we have

$$\|\partial_t^3 Q\|_1 \lesssim \mathcal{P}. \tag{4.2.48}$$

4.2.2.2 The div-curl estimates

Invoking Lemma 3.3.1, we have the following inequalities for $0 \leq k \leq 3$

$$\|v\|_{4.5}^2 \lesssim \|v\|_0^2 + \|\text{div } v\|_{3.5}^2 + \|\text{curl } v\|_{3.5}^2 + |\overline{\partial} v^3|_3^2, \tag{4.2.49}$$

$$\|(b_0 \cdot \partial)\eta\|_{4.5}^2 \lesssim \|(b_0 \cdot \partial)\eta\|_0^2 + \|\text{div } (b_0 \cdot \partial)\eta\|_{3.5}^2 + \|\text{curl } (b_0 \cdot \partial)\eta\|_{3.5}^2 + |\overline{\partial}(b_0 \cdot \partial)\eta^3|_3^2, \tag{4.2.50}$$

$$\|\partial_t^k v\|_{4.5-k}^2 \lesssim \|\partial_t^k v\|_0^2 + \|\text{div } \partial_t^k v\|_{3.5-k}^2 + \|\text{curl } \partial_t^k v\|_{3.5-k}^2 + |\overline{\partial}\partial_t^k v^3|_{3-k}^2, \tag{4.2.51}$$

$$\|\partial_t^k (b_0 \cdot \partial)\eta\|_{4.5-k}^2 \lesssim \|\partial_t^k (b_0 \cdot \partial)\eta\|_0^2 + \|\text{div } \partial_t^k (b_0 \cdot \partial)\eta\|_{3.5-k}^2 \tag{4.2.52}$$

$$+ \|\text{curl } \partial_t^k (b_0 \cdot \partial)\eta\|_{3.5-k}^2 + |\overline{\partial}\partial_t^k (b_0 \cdot \partial)\eta^3|_{3-k}^2.$$

Here, notice that we do not pick up terms on Γ_0 since $v^3 = 0$ and $(b_0 \cdot \partial)\eta^3 = b_0^i \overline{\partial}_i \eta^3 = 0$ there. Also, the L^2-norms in (4.2.49) and (4.2.50) are controlled by energy conservation law. We will omit the control of L^2-norms.

Divergence estimates. For the velocity vector field, one has

$$\text{div } v = \underbrace{\text{div}_{\tilde{A}} v}_{=0} + (\delta^{\mu\alpha} - \tilde{A}^{\mu\alpha})\partial_\mu v_\alpha = \text{div }_{\text{Id}-\tilde{A}} v, \tag{4.2.53}$$

and thus

$$\|\text{div } v\|_{3.5} \le \|\text{div}_{\tilde{A}} v\|_{3.5} + \|(\delta^{\mu\alpha} - \tilde{A}^{\mu\alpha})\partial_\mu v_\alpha\|_{3.5} \le 0 + \varepsilon \|v\|_{4.5}. \tag{4.2.54}$$

Time differentiating (4.2.53), one has

$$\|\text{div } \partial_t v\|_{2.5} \lesssim \varepsilon \|\partial_t^2 v\|_{2.5} + P(\|v_0\|_{3.5}) + \|\eta\|_{3.5} \int_0^T P(\|v\|_{4.5}), \tag{4.2.55}$$

where in the last step we write $\|v\|_{3.5}$ in terms of initial data plus time integral and use Young's inequality. The divergence estimates of $\|\partial_t^k v\|_{3.5-k}, k = 2, 3$ are parallel and so we omit the details.

$$\|\text{div } \partial_t^2 v\|_{1.5} + \|\text{div } \partial_t^3 v\|_{0.5} \lesssim \varepsilon(\|\partial_t^2 v\|_{2.5} + \|\partial_t^3 v\|_{1.5}) + \mathcal{P}_0 + \mathcal{P} \int_0^T \mathcal{P}. \tag{4.2.56}$$

As for $(b_0 \cdot \partial)\eta$, *one no longer has* $\text{div}_{\tilde{A}}((b_0 \cdot \partial)\eta) = 0$ *due to the tangential mollification.* Instead, one can compute the evolution equation verified by $\text{div}_{\tilde{A}}((b_0 \cdot \partial)\eta)$. Invoking $\text{div}_{\tilde{A}} v = 0$ and $\partial_t \eta = v$, we have

$$\partial_t(\text{div}_{\tilde{A}}((b_0 \cdot \partial)\eta)) = [\text{div}_{\tilde{A}}, (b_0 \cdot \partial)]v + \text{div }_{\partial_t \tilde{A}}(b_0 \cdot \partial)\eta. \tag{4.2.57}$$

The commutator $[\text{div}_{\tilde{A}}, (b_0 \cdot \partial)]v$ only contains first order derivative of v and $(b_0 \cdot \partial)\eta$. Using the identity

$$\partial \tilde{A}^{\mu\alpha} = -\tilde{A}^{\mu\gamma}\partial_\beta \partial \tilde{\eta}_\gamma \tilde{A}^{\beta\alpha}, \tag{4.2.58}$$

one has

$$[\text{div}_{\tilde{A}}, (b_0 \cdot \partial)]v = \partial_\beta((b_0 \cdot \partial)\tilde{\eta}_\gamma)\tilde{A}^{\mu\gamma}\tilde{A}^{\beta\alpha}\partial_\mu v_\alpha. \tag{4.2.59}$$

Moreover,

$$\text{div}_{\partial_t \tilde{A}}(b_0 \cdot \partial)\eta = \partial_t \tilde{A}^{\mu\alpha}\partial_\mu(b_0 \cdot \partial)\eta_\alpha = -a^{\mu\gamma}\partial_\beta \tilde{v}_\gamma \tilde{A}^{\beta\alpha}\partial_\mu(b_0 \cdot \partial)\eta_\alpha. \tag{4.2.60}$$

Taking $\partial^{3.5}$ in (4.2.57) and testing it with $\partial^{3.5}\text{div}_{\tilde{A}}(b_0 \cdot \partial)\eta$, we get

$$\begin{aligned} \|\text{div}_{\tilde{A}}(b_0 \cdot \partial)\eta\|_{3.5}^2 &\le \|\text{div}\, b_0\|_{3.5}^2 \\ &+ \int_0^T \|\text{div}_{\tilde{A}}(b_0 \cdot \partial)\eta\|_{3.5} \left(\|[\text{div}_{\tilde{A}}, (b_0 \cdot \partial)]v\|_{3.5} + \|\text{div}_{\partial_t \tilde{A}}(b_0 \cdot \partial)\eta\|_{3.5}\right). \end{aligned} \tag{4.2.61}$$

This suggests that we need to control $\int_0^T \|[\text{div}_{\tilde{A}}, (b_0 \cdot \partial)]v\|_{3.5}$ and $\int_0^T \|\text{div}_{\partial_t \tilde{A}}(b_0 \cdot \partial)\eta\|_{3.5}$ on the right hand side. In light of (4.2.59) and (4.2.60), we have

$$\int_0^T \|[\text{div}_{\tilde{A}}, (b_0 \cdot \partial)]v\|_{3.5} + \|\text{div}_{\partial_t \tilde{A}}(b_0 \cdot \partial)\eta\|_{3.5} \le \int_0^T \mathcal{P}.$$

Therefore,

$$\|\text{div}_{\tilde{A}}(b_0 \cdot \partial)\eta\|_{3.5}^2 \le \|\text{div}\, b_0\|_{3.5}^2 + \int_0^T \mathcal{P} \le \mathcal{P}_0 + \int_0^T \mathcal{P}, \tag{4.2.62}$$

which implies, after invoking (4.2.16), that

$$\|\text{div}\,(b_0 \cdot \partial)\eta\|_{3.5}^2 \lesssim \varepsilon^2 \|(b_0 \cdot \partial)\eta\|_{4.5}^2 + \mathcal{P}_0 + \int_0^T \mathcal{P}. \tag{4.2.63}$$

Similarly, one can take $\partial^{3.5-k}\partial_t^k$ for $1 \le k \le 3$ in (4.2.57) to get

$$\|\text{div}_{\tilde{A}}\partial_t^k(b_0 \cdot \partial)\eta\|_{3.5-k}^2 \lesssim \varepsilon^2 \|\partial_t^k(b_0 \cdot \partial)\eta\|_{4.5-k}^2 + \mathcal{P}_0 + \int_0^T \mathcal{P}. \tag{4.2.64}$$

Curl estimates. Taking $\text{curl}_{\tilde{A}}$ in the second equation of (4.2.2) yields

$$\partial_t(\text{curl}_{\tilde{\mathbf{A}}} v) - (b_0 \cdot \partial)\text{curl}_{\tilde{\mathbf{A}}}((b_0 \cdot \partial)\eta) = \text{curl}_{\partial_t \tilde{\mathbf{A}}} v + [\text{curl}_{\tilde{\mathbf{A}}}, (b_0 \cdot \partial)](b_0 \cdot \partial)\eta. \tag{4.2.65}$$

Then we take $\partial^{3.5}$, test it with $(b_0\cdot\partial)^{3.5}(\text{curl}_{\tilde{\mathbf{A}}}v)$ and integrate $(b_0\cdot\partial)$ by parts (recall that $b_0\cdot N|_{\partial\Omega}=0$ and div $b_0=0$) to get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\partial^{3.5}\text{curl}_{\tilde{\mathbf{A}}}v|^2+|\partial^{3.5}\text{curl}_{\tilde{\mathbf{A}}}(b_0\cdot\partial)\eta|^2\,\mathrm{d}y\\
=&\int_\Omega\left(\left[\partial^{3.5}(b_0\cdot\partial)\right]\text{curl}_{\tilde{\mathbf{A}}}(b_0\cdot\partial)\eta+\partial^{3.5}\left(\text{curl}_{\,\partial_t\tilde{\mathbf{A}}}v+[\text{curl}_{\tilde{\mathbf{A}}},(b_0\cdot\partial)](b_0\cdot\partial)\eta\right)\right)(\partial^{3.5}\text{curl}_{\tilde{\mathbf{A}}}v)\,\mathrm{d}y\\
&+\int_\Omega\partial^{3.5}(\text{curl}_{\tilde{\mathbf{A}}}(b_0\cdot\partial)\eta)\cdot\left(\left[\partial^{3.5}\text{curl}_{\tilde{\mathbf{A}}},(b_0\cdot\partial)\right]v+\partial^{3.5}(\text{curl}_{\,\partial_t\tilde{\mathbf{A}}}(b_0\cdot\partial)\eta)\right)\,\mathrm{d}y\\
\lesssim& P(\|b_0\|_{4.5},\|(b_0\cdot\partial)\eta\|_{4.5},\|v\|_{4.5},\|\tilde{\mathbf{A}}\|_{3.5},\|(b_0\cdot\partial)\tilde{\eta}\|_{4.5})\lesssim\mathcal{P},
\end{aligned}
\tag{4.2.66}
$$

and thus by the a priori assumption (4.2.8), we have

$$\|\text{curl } v\|_{3.5}^2+\|\text{curl } (b_0\cdot\partial)\eta\|_{3.5}^2\lesssim\varepsilon^2(\|v\|_{4.5}^2+\|(b_0\cdot\partial)\eta\|_{4.5}^2)+\int_0^T\mathcal{P}\,\mathrm{d}t. \tag{4.2.67}$$

Replacing $\partial^{3.5}$ by $\partial^{3.5-k}\partial_t^k$ for $1\le k\le 3$, we similarly get

$$\|\text{curl}_{\tilde{\mathbf{A}}}\partial_t^k(b_0\cdot\partial)\eta\|_{3.5-k}^2\lesssim\varepsilon^2\|\partial_t^k(b_0\cdot\partial)\eta\|_{4.5-k}^2+\mathcal{P}_0+\int_0^T\mathcal{P}\,\mathrm{d}t. \tag{4.2.68}$$

4.2.2.3 Boundary estimates

We need to control the boundary term $|\overline{\partial}\partial_t^k v\cdot N|_{3-k}$ and $|\overline{\partial}\partial_t^k(b_0\cdot\partial)\eta\cdot N|_{3-k}$. In the case of zero surface tension, one can use the normal trace theorem to reduce $|\overline{\partial}X\cdot N|_{s-1.5}$ to the interior tangential estimates $\|\overline{\partial}^s X\|_0$. But the interior tangential estimates, especially in the full spatial derivative case, cannot be controlled due to the appearance of surface tension.

Control of $|\overline{\partial}\partial_t^k v\cdot N|_{3-k}$

Theorem 4.2.11. For $k=0,1,2,3$, one has

$$|\overline{\partial}\partial_t^k v^3|_{3-k}^2\lesssim|\overline{\partial}(\Pi\overline{\partial}^{3-k}\partial_t^k v)|_0^2+\mathcal{P}\int_0^T\mathcal{P}. \tag{4.2.69}$$

First we study the case when $k=3$. Let us consider the projection of $\partial_t^3 v$ to the Eulerian normal direction, i.e., $(\Pi\partial_t^3 v)^3$ instead of Lagrangian normal direction. The reason is twofold.

1. Recall that (3.1.9) in Lemma 3.1.1 gives that

$$\sqrt{g}g^{ij}\triangle_g\eta^\alpha = \sigma\sqrt{g}g^{ij}\Pi^\alpha_\lambda\overline{\partial}^2_{ij}\eta^\lambda.$$

   So if we test ∂_t^4-differentiated version of (3.1.9) with $\partial_t^4 v$ and integrate by parts, then the term $|\overline{\partial}(\Pi\partial_t^3 v)|_0^2$ is produced as part of energy term,i.e.,

$$\int_\Gamma \sigma\sqrt{g}g^{ij}\Pi^\alpha_\lambda\partial_t^4\overline{\partial}^2_{ij}\eta^\lambda\cdot\partial_t^4 v_\alpha = -\frac{1}{2}\frac{d}{dt}\int_\Gamma\left|\overline{\partial}(\Pi\partial_t^3 v)\right|^2\,\mathrm{d}S+\cdots \tag{4.2.70}$$

2. The difference between X^3 and $(\Pi X)^3$ is small within a short period of time.

We make the above assertions precise. For any vector field X one has

$$\begin{aligned} X^3=\delta^3_\lambda X^\lambda =&(\delta^3_\lambda - g^{kl}\overline{\partial}_k\eta^3\overline{\partial}_l\eta_\lambda)X^\lambda + g^{kl}\overline{\partial}_k\eta^3\overline{\partial}_l\eta_\lambda X^\lambda \\ =&\Pi^3_\lambda X^\lambda + g^{kl}\overline{\partial}_k\eta^3\overline{\partial}_l\eta_\lambda X^\lambda = (\Pi X)^3 + g^{kl}\overline{\partial}_k\eta^3\overline{\partial}_l\eta_\lambda X^\lambda. \end{aligned} \tag{4.2.71}$$

Using $\overline{\partial}\eta^3=\int_0^T\overline{\partial}v^3\,\mathrm{d}t$ (this is true since $\overline{\partial}\eta^3=0$ initially), we can control the difference between $(\Pi X)^3$ and X^3 as

$$\begin{aligned} \left|\overline{\partial}\left((\Pi X)^3 - X^3\right)\right|_0^2 \lesssim& \left|g^{kl}\overline{\partial}_k\eta^3\overline{\partial}_l\eta_\lambda\overline{\partial}X^\lambda\right|_0^2 + \left|\overline{\partial}(g^{kl}\overline{\partial}_k\eta^3\overline{\partial}_l\eta_\lambda)X^\lambda\right|_0^2 \\ \lesssim&\|X\|_{1.5}^2 P(|\overline{\partial}\eta|_{L^\infty})\int_0^T\mathcal{P}. \end{aligned} \tag{4.2.72}$$

Let $X=\partial_t^3 v$. Since $\|\partial_t^3 v\|_{1.5}^2$ is included in the energy $E_\kappa^{(1)}$, then (4.2.72) implies

$$\left|\overline{\partial}\left((\Pi\partial_t^3 v)^3 - \partial_t^3 v^3\right)\right|_0^2 \lesssim \mathcal{P}\int_0^T\mathcal{P}, \tag{4.2.73}$$

and thus

$$\left|\overline{\partial}\partial_t^3 v^3\right|_0^2 \lesssim \left|\overline{\partial}(\Pi\partial_t^3 v)\right|_0^2 + \mathcal{P}\int_0^T\mathcal{P}. \tag{4.2.74}$$

Finally, (4.2.69) follows from a parallel argument.

Control of $|\overline{\partial}\partial_t^k(b_0\cdot\partial)\eta\cdot N|_{3-k}$. First, when $k\geq 1$, the control of $|\overline{\partial}\partial_t^k(b_0\cdot\partial)\eta\cdot N|_{3-k}$ requires to that of $|\overline{\partial}\partial_t^l v\cdot N|_{3-l}$ (modulo lower order terms generated when derivatives land on b_0) for $l=0,1,2$, which has been done in the previous subsection.

Thus it suffices to study the control of $|(b_0\cdot\partial)\eta^3|_4$. In [53], the boundary condition forms an elliptic equation $-\sigma\sqrt{g}\triangle_g\eta^\alpha = A^{3\alpha}Q$ and thus one can take $(b_0\cdot\partial)$ and then use elliptic estimates. However, the boundary condition now takes the form (4.2.6) in the smoothed approximate equations and there is no appropriate boundary H^2-control for $\kappa(b_0\cdot\partial)\overline{\triangle}(v\cdot\tilde{n})$ due to the lack of time integrals.

Our strategy here is to use the inequality (4.2.72) with $X=\overline{\partial}^3(b_0\cdot\partial)\eta$.

$$\left|\overline{\partial}\left((\Pi\overline{\partial}^3(b_0\cdot\partial)\eta)^3-\overline{\partial}^3(b_0\cdot\partial)\eta^3\right)\right|_0^2 \lesssim \|\overline{\partial}^3(b_0\cdot\partial)\eta\|_{1.5}^2 P(|\overline{\partial}\eta|_{L^\infty})\int_0^T \mathcal{P}\lesssim \mathcal{P}\int_0^T\mathcal{P}, \quad (4.2.75)$$

where the last inequality holds since $\|(b_0\cdot\partial)\eta\|_{4.5}^2$ is included in $E_\kappa^{(1)}$. Therefore,

$$\left|\overline{\partial}^4(b_0\cdot\partial)\eta^3\right|_0^2 \lesssim \left|\overline{\partial}(\Pi\overline{\partial}^3(b_0\cdot\partial)\eta)\right|_0^2+\mathcal{P}\int_0^T\mathcal{P}. \quad (4.2.76)$$

Remark 4.2.12. The term $\left|\overline{\partial}(\Pi\overline{\partial}^3(b_0\cdot\partial)\eta)\right|_0^2$ is part of the energy $E_\kappa^{(1)}$ defined in (4.2.9), which is a positive term generated by the $\overline{\partial}^3(b_0\cdot\partial)$ tangential energy estimate (See Section 4.2.3). There is no problem to study the $\overline{\partial}^3(b_0\cdot\partial)$-differentiated equations (4.2.2) since it is analogous to the $\overline{\partial}^3\partial_t$-differentiated equations. Indeed, as mentioned before, $(b_0\cdot\partial)\eta$ and $\partial_t\eta$ (which is v) have the same space-time regularity.

4.2.3 Tangential energy estimates

The purpose of this section is to investigate the a priori estimates for the tangentially differentiated approximate κ-problem (4.2.2). In particular, we will study the energy estimate for $\partial_t^4, \overline{\partial}\partial_t^3, \overline{\partial}^2\partial_t^2, \overline{\partial}^3\partial_t, \overline{\partial}^3(b_0\cdot\partial)$ differentiated κ-problem, respectively.

4.2.3.1 Control of full time derivatives

We do the L^2-estimate of $\partial_t^4 v$ and $\partial_t^4 (b_0 \cdot \partial)\eta$. This turns out to be the most difficult case compare to the cases with at least one tangential spatial derivatives that will be treated in Section 4.2.3.2. This is due to the fact that $\partial_t^4 v$ can only be controlled in $L^2(\Omega)$ and so one has to control some higher order interior terms instead. These interior terms will be treated by adapting the geometric cancellation scheme introduced in [18] together with an error term which can be controlled by terms in $E_\kappa^{(3)}(t)$.

For the sake of simplicity and clean arguments, we shall focus on treating the leading order terms. We henceforth adopt:

Notation 4.2.13. We use $\stackrel{L}{=}$ to denote equality modulo error terms that are effectively of lower order. For instance, $X \stackrel{L}{=} Y$ means that $X = Y + \mathcal{R}$, where $\mathcal{R}$ consists of lower order terms with respect to Y.

Invoking (4.2.2) and integrating $(b_0 \cdot \partial)$ by parts, we get

$$
\begin{aligned}
&\frac{1}{2}\int_0^T \frac{d}{dt}\int_\Omega |\partial_t^4 v|^2 + \left|\partial_t^4 (b_0\cdot\partial)\eta\right|^2 \,\mathrm{d}y \\
=&\int_0^T\int_\Omega \partial_t^4 v_\alpha \partial_t^5 v^\alpha \,\mathrm{d}y\,\mathrm{d}t + \int_0^T\int_\Omega \partial_t^4 (b_0\cdot\partial)\eta_\alpha \partial_t^4 (b_0\cdot\partial) v^\alpha \,\mathrm{d}y\,\mathrm{d}t \\
=&\int_0^T\int_\Omega \partial_t^4 v_\alpha \partial_t^4 (b_0\cdot\partial)^2\eta_\alpha \,\mathrm{d}y\,\mathrm{d}t - \int_0^T\int_\Omega \partial_t^4 v_\alpha \partial_t^4 (\tilde{\mathbf{A}}^{\mu\alpha}\partial_\mu Q) \,\mathrm{d}y\,\mathrm{d}t \\
&+\int_0^T\int_\Omega \partial_t^4 (b_0\cdot\partial)\eta_\alpha \partial_t^4 (b_0\cdot\partial) v^\alpha \,\mathrm{d}y\,\mathrm{d}t \qquad (4.2.77)\\
=&-\int_0^T\int_\Omega \partial_t^4 (b_0\cdot\partial) v_\alpha \partial_t^4 (b_0\cdot\partial)\eta_\alpha \,\mathrm{d}y\,\mathrm{d}t - \int_0^T\int_\Omega \partial_t^4 v_\alpha \partial_t^4 (\tilde{\mathbf{A}}^{\mu\alpha}\partial_\mu Q) \,\mathrm{d}y\,\mathrm{d}t \\
&+\int_0^T\int_\Omega \partial_t^4 (b_0\cdot\partial)\eta_\alpha \partial_t^4 (b_0\cdot\partial) v^\alpha \,\mathrm{d}y\,\mathrm{d}t \\
=&-\int_0^T\int_\Omega \partial_t^4 v_\alpha \partial_t^4 (\tilde{\mathbf{A}}^{\mu\alpha}\partial_\mu Q) \,\mathrm{d}y\,\mathrm{d}t =: I.
\end{aligned}
$$

Then we integrate ∂_μ by parts, I becomes

$$\int_0^T \int_\Omega \partial_t^4 \partial_\mu v_\alpha \partial_t^4 (\tilde{\mathbf{A}}^{\mu\alpha} Q) - \underbrace{\int_0^T \int_\Gamma \partial_t^4 v_\alpha \partial_t^4 (\tilde{\mathbf{A}}^{3\alpha} Q)}_{I_0} + \underbrace{\int_0^T \int_{\Gamma_0} \partial_t^4 v_\alpha \partial_t^4 (\tilde{\mathbf{A}}^{3\alpha} Q)}_{I_0'}$$

$$= \int_0^T \int_\Omega \tilde{\mathbf{A}}^{\mu\alpha} \partial_t^4 \partial_\mu v_\alpha \partial_t^4 Q + \underbrace{\int_0^T \int_\Omega \partial_t^4 \partial_\mu v_\alpha [\partial_t^4, \tilde{\mathbf{A}}^{\mu\alpha}] Q}_{I_1} + I_0$$

$$= \int_0^T \int_\Omega \underbrace{\partial_t^4 \mathrm{div}_{\tilde{\mathbf{A}}} v}_{=0} \partial_t^4 Q - \underbrace{\int_0^T \int_\Omega [\partial_t^4, \tilde{\mathbf{A}}^{\mu\alpha}] \partial_\mu v_\alpha \partial_t^4 Q}_{L} + I_1 + I_0 + I_0'. \tag{4.2.78}$$

$I_0' = 0$ since on Γ_0, we have $\tilde{\mathbf{A}}^{31} = \tilde{\mathbf{A}}^{32} = 0$, $\tilde{\mathbf{A}}^{33} = 1$, and $v_3 = 0$.

I_I yields a top order interior term when all 4 time derivatives land on $\tilde{\mathbf{A}}^{\mu\alpha}$, i.e.,

$$I_{11} = \int_0^T \int_\Omega \partial_t^4 \partial_\mu v_\alpha (\partial_t^4 \tilde{\mathbf{A}}^{\mu\alpha}) Q. \tag{4.2.79}$$

If $\tilde{\mathbf{A}}^{\mu\alpha}$ were $A^{\mu\alpha}$ then this term could have been controlled by the cancellation scheme developed in [18]. This motivate us to consider

$$\int_0^T \int_\Omega \partial_t^4 \partial_\mu v_\alpha (\partial_t^4 A^{\mu\alpha}) Q + \int_0^T \int_\Omega \partial_t^4 \partial_\mu v_\alpha \Big(\partial_t^4 (\tilde{\mathbf{A}}^{\mu\alpha} - A^{\mu\alpha}) \Big) Q = I_{111} + I_{112}. \tag{4.2.80}$$

Invoking (3.1.1) we get

$$\partial_t^4 (\tilde{\mathbf{A}} - A) = \sum_{i+j=3} b_{ij} \partial_t^i \partial \tilde{\eta} \times \partial \partial_t^j (\tilde{v} - v) + \sum_{i+j=3} b_{ij}' \partial_t^i (\partial \tilde{\eta} - \partial \eta) \times \partial \partial_t^j v,$$

and so $\|\partial_t^4(\tilde{\mathbf{A}} - A)\|_0$ consists the sum of $\|i_\ell\|_0, \ell = 1, \cdots, 8$, where

$$i_1 = (\partial\partial_t^2\tilde{v})\partial(\tilde{v} - v), \quad i_2 = (\partial\partial_t\tilde{v})\partial\partial_t(\tilde{v} - v), \quad i_3 = (\partial\tilde{v})\partial\partial_t^2(\tilde{v} - v),$$

$$i_4 = (\partial\tilde{\eta})\partial\partial_t^3(\tilde{v} - v), \;\; i_5 = \partial\partial_t^2(\tilde{v} - v)\partial v, \quad i_6 = \partial\partial_t(\tilde{v} - v)\partial\partial_t v,$$

$$i_7 = \partial(\tilde{v} - v)\partial\partial_t^2 v, \quad i_8 = \partial(\tilde{\eta} - \eta)\partial\partial_t^3 v.$$

The L^2-norm of these quantities can be controlled by invoking Lemma 4.2.8.

$$\sum_{N=1}^{8} \|i_N\|_0 \leq \sqrt{\kappa} P(\|v\|_{3.5}, \|\partial_t v\|_{3.5}, \|\partial_t^2 v\|_{1.5}, \|\partial_t^3 v\|_{1.5}, \|\eta\|_{3.5})$$

Summing these up and moving $\sqrt{\kappa}$ to $\|\partial_t^4 \partial v\|_0$, we obtain

$$I_{112} \leq \int_0^T \|\partial_t^4 \partial v\|_0 \|\partial_t^4(\tilde{\mathbf{A}}^{\mu\alpha} - A^{\mu\alpha})\|_0 \|Q\|_{L^\infty} \leq \frac{\varepsilon}{2} \int_0^T \|\sqrt{\kappa}\partial_t^4 \partial v\|_0^2 + \frac{1}{2\varepsilon} \int_0^T \mathcal{P}, \tag{4.2.81}$$

where the first term on the RHS contributes to $\varepsilon\mathcal{P}$, and we bound $\|Q\|_{L^\infty}$ by $\|Q\|_2 \leq \mathcal{P}$ through (4.2.26).

We next control I_{111}. The argument relies on exploiting the geometric structure to create cancellation among the leading order terms. Invoking (3.1.1) we have

$$\begin{aligned}
I_{111} =& \int_0^T \int_\Omega Q\epsilon^{\alpha\lambda\tau}\overline{\partial}_2\partial_t^3 v_\lambda \partial_3\eta_\tau \overline{\partial}_1\partial_t^4 v_\alpha - \int_0^T \int_\Omega Q\epsilon^{\alpha\lambda\tau}\overline{\partial}_1\partial_t^3 v_\lambda \partial_3\eta_\tau \overline{\partial}_2\partial_t^4 v_\alpha \\
&+ \int_0^T \int_\Omega Q\epsilon^{\alpha\lambda\tau}\partial_3\partial_t^3 v_\tau \overline{\partial}_2\eta_\lambda \overline{\partial}_1\partial_t^4 v_\alpha - \int_0^T \int_\Omega Q\epsilon^{\alpha\lambda\tau}\overline{\partial}_1\partial_t^3 v_\tau \overline{\partial}_2\eta_\lambda \partial_3\partial_t^4 v_\alpha \\
&+ \int_0^T \int_\Omega Q\epsilon^{\alpha\lambda\tau}\overline{\partial}_2\partial_t^3 v_\tau \overline{\partial}_1\eta_\lambda \partial_3\partial_t^4 v_\alpha - \int_0^T \int_\Omega Q\epsilon^{\alpha\lambda\tau}\partial_3\partial_t^3 v_\tau \overline{\partial}_1\eta_\lambda \overline{\partial}_2\partial_t^4 v_\alpha + I_{low} \\
=:& I_{1111} + I_{1112} + \cdots + I_{1116} + I_{low},
\end{aligned} \tag{4.2.82}$$

where I_{low} consists terms of the form $\int_0^T \int_\Omega Q\partial\partial_t^2 v \partial v \partial\partial_t^3 v$. This term can be treated by integrating

∂_t by parts,

$$\int_0^T \int_\Omega Q\partial\partial_t^2 v\partial v\partial\partial_t^4 v = \int_\Omega Q\partial\partial_t^2 v\partial v\partial\partial_t^3 v\Big|_0^T - \int_0^T \int_\Omega \partial_t(q\partial\partial_t^2 v\partial v)\partial\partial_t^3 v,$$

where the second term is controlled by $\int_0^T \mathcal{P}$, whereas

$$\left|\int_\Omega Q\partial\partial_t^2 v\partial v\partial\partial_t^3 v\Big|_0^T\right| \lesssim \mathcal{P}_0 + \varepsilon||\partial_t^3 v||_1^2 + \int_0^T \mathcal{P}.$$

To control the leading terms in (4.2.82), we consider $I_{1111}+I_{1112}$, $I_{1113}+I_{1114}$, and $I_{1115}+I_{1116}$. For $I_{1111}+I_{1112}$, integrating ∂_t by parts in I_{1112}, we have

$$\begin{aligned} I_{1111}+I_{1112} \le & \underbrace{\int_0^T \int_\Omega Q\epsilon^{\alpha\lambda\tau}\overline{\partial}_2\partial_t^3 v_\lambda\partial_3\eta_\tau\overline{\partial}_1\partial_t^4 v_\alpha - \int_0^T \int_\Omega Q\epsilon^{\alpha\lambda\tau}\overline{\partial}_1\partial_t^4 v_\lambda\partial_3\eta_\tau\overline{\partial}_2\partial_t^3 v_\alpha}_{=0} \\ & - \int_\Omega Q\epsilon^{\alpha\lambda\tau}\overline{\partial}_1\partial_t^3 v_\lambda\partial_3\eta_\tau\overline{\partial}_2\partial_t^3 v_\alpha\Big|_0^T + I'_{low}, \end{aligned} \tag{4.2.83}$$

where I'_{low} consists terms of the form $\int_0^T \int_\Omega Q\epsilon^{\alpha\lambda\tau}\partial_t(q\partial\eta)(\partial\partial_t^3 v)^2$ which can be controlled by $\int_0^T \mathcal{P}$. Next we treat the first term on the RHS of (4.2.83). Expanding in τ, we find

$$\mathcal{T} := -\int_\Omega Q\epsilon^{\alpha\lambda i}\overline{\partial}_1\partial_t^3 v_\lambda\partial_3\eta_i\overline{\partial}_2\partial_t^3 v_\alpha - \int_\Omega Q\epsilon^{\alpha\lambda 3}\overline{\partial}_1\partial_t^3 v_\lambda\partial_3\eta_3\overline{\partial}_2\partial_t^3 v_\alpha. \tag{4.2.84}$$

Since $\partial_3\eta_i|_{t=0} = 0$, we can write $\partial_3\eta_i = \int_0^T \partial_3 v_i$, and so

$$-\int_\Omega Q\epsilon^{\alpha\lambda i}\overline{\partial}_1\partial_t^3 v_\lambda\partial_3\eta_i\overline{\partial}_2\partial_t^3 v_\alpha \le \mathcal{P}\int_0^T \mathcal{P}. \tag{4.2.85}$$

In addition to this, we have $\partial_3\eta_3 = 1 + \int_0^T \partial_3 v_3$, and so

$$-\int_\Omega Q\epsilon^{\alpha\lambda 3}\overline{\partial}_1\partial_t^3 v_\lambda\partial_3\eta_3\overline{\partial}_2\partial_t^3 v_\alpha \le -\int_\Omega Q\epsilon^{\alpha\lambda 3}\overline{\partial}_1\partial_t^3 v_\lambda\overline{\partial}_2\partial_t^3 v_\alpha + \mathcal{P}\int_0^T \mathcal{P}. \tag{4.2.86}$$

To treat the first term on the RHS, we expand $\epsilon^{\alpha\lambda 3}$ and get

$$-\int_\Omega Q\epsilon^{\alpha\lambda 3}\overline{\partial}_1\partial_t^3 v_\lambda\overline{\partial}_2\partial_t^3 v_\alpha = -\int_\Omega Q(\overline{\partial}_1\partial_t^3 v_2\overline{\partial}_2\partial_t^3 v_1 - \overline{\partial}_1\partial_t^3 v_1\overline{\partial}_2\partial_t^3 v_2). \tag{4.2.87}$$

Integrating by parts $\overline{\partial}_2$ in the first term and $\overline{\partial}_1$ in the second term, we have

$$-\int_\Omega Q(\overline{\partial}_1\partial_t^3 v_2\overline{\partial}_2\partial_t^3 v_1-\overline{\partial}_1\partial_t^3 v_1\overline{\partial}_2\partial_t^3 v_2)$$

$$=\underbrace{\int_\Omega Q\overline{\partial}_1\overline{\partial}_2\partial_t^3 v_2\partial_t^3 v_1-\int_\Omega Q\partial_t^3 v_1\overline{\partial}_1\overline{\partial}_2\partial_t^3 v_2}_{=0}+\int_\Omega \overline{\partial}_2 Q\overline{\partial}_1\partial_t^3 v_2\partial_t^3 v_1-\int_\Omega \overline{\partial}_1 Q\partial_t^3 v_1\overline{\partial}_1\partial_t^3 v_2.$$

Here,

$$\Big|\int_\Omega \overline{\partial}_2 Q\overline{\partial}_1\partial_t^3 v_2\partial_t^3 v_1-\int_\Omega \overline{\partial}_1 Q\partial_t^3 v_1\overline{\partial}_1\partial_t^3 v_2\Big|\lesssim \varepsilon||\partial_t^3 v||_1^2+\mathcal{P}_0+\int_0^T\mathcal{P}.$$

Therefore,

$$I_{1111}+I_{1112}\le\varepsilon E(T)+\mathcal{P}_0+\mathcal{P}\int_0^T\mathcal{P}. \tag{4.2.88}$$

On the other hand, $I_{1113}+I_{1114}$ and $I_{1115}+I_{1116}$ are treated similarly with only one exception. Previously, we integrated $\overline{\partial}_1$ and $\overline{\partial}_2$ by parts in (4.2.87) and so there is no boundary terms. However, when controlling $I_{1113}+I_{1114}$, we need to integrate $\overline{\partial}_1$ and ∂_3 by parts when treating (4.2.87), and thus the following boundary term appears:

$$\int_\Gamma Q\partial_t^3 v_1\overline{\partial}_1\partial_t^3 v_3. \tag{4.2.89}$$

To control this term, we invoke the identity

$$\overline{\partial}_1\partial_t^3 v^3=\Pi_\lambda^3\overline{\partial}_1\partial_t^3 v^\lambda+g^{kl}\overline{\partial}_k\eta^3\overline{\partial}_l\eta_\lambda\overline{\partial}_1\partial_t^3 v^\lambda=\Pi_\lambda^3\overline{\partial}_1\partial_t^3 v^\lambda+g^{kl}\left(\int_0^T\overline{\partial}_k v^3\right)\overline{\partial}_l\eta_\lambda\overline{\partial}_1\partial_t^3 v^\lambda, \tag{4.2.90}$$

and thus (4.2.89) becomes

$$\int_\Gamma Q\partial_t^3 v_1 \Pi_\lambda^3 \overline{\partial}_1 \partial_t^3 v^\lambda + \int_\Gamma Q\partial_t^3 v_1 g^{kl}\left(\int_0^T \overline{\partial}_k v^3\right)\overline{\partial}_l \eta_\lambda \overline{\partial}_1 \partial_t^3 v^\lambda$$

$$\lesssim \varepsilon |\Pi\overline{\partial}\partial_t^3 v|_0^2 + |q|_{L^\infty}^2 |\partial_t^3 v|_0^2 + \left| q\partial_t^3 v_1 g^{kl}\left(\int_0^T \overline{\partial}_k v^3\right)\overline{\partial}_l \eta_\lambda \right|_{0.5} |\overline{\partial}\partial_t^3 v^\lambda|_{-0.5}$$

$$\lesssim \varepsilon |\Pi\overline{\partial}\partial_t^3 v|_0^2 + \mathcal{P}_0 + \mathcal{P}\int_0^T \mathcal{P}.$$

The extra term generated when analyzing $I_{1115} + I_{1116}$ is of the same type integral and thus can be treated by the same method. Therefore,

$$I_{111} \leq \varepsilon E(T) + \mathcal{P}_0 + \mathcal{P}\int_0^T \mathcal{P}. \tag{4.2.91}$$

Next we study

$$\begin{aligned} I_1 - I_{11} =& 4\int_0^T \int_\Omega \partial_t^4 \partial_\mu v_\alpha \partial_t^3 \tilde{\mathbf{A}}^{\mu\alpha} \partial_t Q + 6\int_0^T \int_\Omega \partial_t^4 \partial_\mu v_\alpha \partial_t^2 \tilde{\mathbf{A}}^{\mu\alpha} \partial_t^2 Q \\ &+ 4\int_0^T \int_\Omega \partial_t^4 \partial_\mu v_\alpha \partial_t \tilde{\mathbf{A}}^{\mu\alpha} \partial_t^3 Q = I_{12} + I_{13} + I_{14}. \end{aligned} \tag{4.2.92}$$

For I_{12}, we integrating ∂_t by parts and obtain

$$4\int_\Omega \partial_t^3 \partial_\mu v_\alpha \partial_t^3 \tilde{\mathbf{A}}^{\mu\alpha} \partial_t Q - 4\int_0^T \int_\Omega \partial_t^3 \partial_\mu v_\alpha \partial_t (\partial_t^3 \tilde{\mathbf{A}}^{\mu\alpha} \partial_t Q).$$

Here, the second term is $\leq \int_0^T \mathcal{P}$, and since $\partial_t^3 \tilde{\mathbf{A}} = Q(\partial\tilde{\eta})\partial\partial_t^2 \tilde{v} +$ lower order terms, the first term is bounded by $\varepsilon ||\partial_t^3 v||_1^2 + \mathcal{P}_0 + \int_0^T \mathcal{P}$. Then I_{13} is treated by a similar method and so we omit the details. However, we cannot integrate ∂_t by parts in order to control I_{14} as we do not have a bound for $\partial_t^4 Q$. We integrate ∂_μ by parts instead.

$$I_{14} = 4\int_0^T \int_\Gamma \partial_t^4 v_\alpha \partial_t \tilde{\mathbf{A}}^{3\alpha} \partial_t^3 Q - 4\int_0^T \int_\Omega \partial_t^4 v_\alpha \partial_\mu (\partial_t \tilde{\mathbf{A}}^{3\alpha} \partial_t^3 Q).$$

There is no problem to control the second integral by $\int_0^T \mathcal{P}$. For the first integral, invoking the boundary condition (4.2.6), we obtain

$$-4\sigma\int_0^T\int_\Gamma \partial_t^4 v_\alpha \partial_t \tilde{\mathbf{A}}^{3\alpha}\partial_t^3\Big(\frac{\sqrt{g}}{\sqrt{\tilde{g}}}\triangle_g\eta\cdot\tilde{n}\Big) \tag{4.2.93}$$

$$+4\int_0^T\int_\Gamma \kappa\partial_t^4 v_\alpha \partial_t \tilde{\mathbf{A}}^{3\alpha}\partial_t^3\Big(\frac{1}{\sqrt{\tilde{g}}}(1-\overline{\triangle})(v\cdot\tilde{n})\Big) =: I_{141}+I_{142}.$$

Invoking (3.1.9), I_{141} becomes

$$I_{141} = -4\sigma\int_0^T\int_\Gamma \partial_t^4 v_\alpha \partial_t \tilde{\mathbf{A}}^{3\alpha}\partial_t^3\Big(\frac{\sqrt{g}}{\sqrt{\tilde{g}}}g^{ij}\overline{\partial}_i\overline{\partial}_j\eta\cdot\tilde{n}\Big)$$

$$-4\sigma\int_0^T\int_\Gamma \partial_t^4 v_\alpha \partial_t \tilde{\mathbf{A}}^{3\alpha}\partial_t^3\Big(\frac{\sqrt{g}}{\sqrt{\tilde{g}}}g^{ij}g^{kl}\overline{\partial}_l\eta^\mu\overline{\partial}_i\overline{\partial}_j\eta_\mu\overline{\partial}_k\eta\cdot\tilde{n}\Big).$$

It suffices for us to consider the first integral only since the second integral is of the same type. Integrating by parts $\overline{\partial}_j$ first and then ∂_t, the first integral becomes

$$-4\sigma\int_0^T\int_\Gamma \partial_t^3\overline{\partial}_i v_\alpha \partial_t \tilde{\mathbf{A}}^{3\alpha}\Big(\frac{\sqrt{g}}{\sqrt{\tilde{g}}}g^{ij}\overline{\partial}_j\partial_t^3 v\cdot\tilde{n}\Big) - 4\sigma\int_\Gamma \partial_t^3\overline{\partial}_i v_\alpha \partial_t \tilde{\mathbf{A}}^{3\alpha}\Big(\frac{\sqrt{g}}{\sqrt{\tilde{g}}}g^{ij}\overline{\partial}_j\partial_t^2 v\cdot\tilde{n}\Big) + \mathcal{R}.$$

Since $\|\partial_t^3 v\|_{3.5}$ is part of $E_\kappa^{(1)}(t)$, the trace lemma implies that the first integral is bounded straightforwardly by $\int_0^T\mathcal{P}$. Moreover, for the second integral, we have

$$4\sigma\int_\Gamma \partial_t^3\overline{\partial}_i v_\alpha \partial_t \tilde{\mathbf{A}}^{3\alpha}\Big(\frac{\sqrt{g}}{\sqrt{\tilde{g}}}g^{ij}\overline{\partial}_j\partial_t^2 v\cdot\tilde{n}\Big) \lesssim \varepsilon\|\partial_t^3 v\|_{1.5}^2+\mathcal{P}_0+\int_0^T\mathcal{P}, \tag{4.2.94}$$

In addition,

$$I_{142} \overset{L}{=} -4\int_0^T\int_\Gamma (\sqrt{\kappa}\partial_t^4 v_\alpha)\partial_t\tilde{\mathbf{A}}^{3\alpha}\Big(\frac{1}{\sqrt{\tilde{g}}}\overline{\triangle}(\sqrt{\kappa}\partial_t^3 v\cdot\tilde{n})\Big).$$

Integrating $\overline{\partial}$ by parts,then

$$I_{142} \overset{L}{=} 4\int_0^T\int_\Gamma (\sqrt{\kappa}\overline{\partial}\partial_t^4 v_\alpha)\partial_t\tilde{\mathbf{A}}^{3\alpha}\frac{1}{\sqrt{\tilde{g}}}(\sqrt{\kappa}\overline{\partial}\partial_t^3 v\cdot\tilde{n})$$

$$\lesssim \varepsilon\int_0^T\|\sqrt{\kappa}\partial_t^4 v\|_{1.5}^2+\mathcal{P}_0+\int_0^T\mathcal{P}.$$

Now we analyze the boundary integral I_0 in (4.2.78). This is essentially identical to the case of the incompressible Euler equations [16, Sect. 12]. Indeed, as what appears in [53] concerning the a priori estimate, we found that the magnetic field plays no role in the estimate of I_0.

By plugging the boundary condition $\tilde{\mathbf{A}}^{3\alpha} Q = -\sigma\sqrt{g}(\triangle_g \eta\cdot\tilde{n})\tilde{n}^\alpha + \kappa\left((1-\overline{\triangle})(v\cdot\tilde{n})\right)\tilde{n}^\alpha$ in I_0 we obtain

$$\frac{1}{\sigma}I_0 = \int_0^T\int_\Gamma \partial_t^4 v_\alpha \partial_t^4(\sqrt{g}\triangle_g\eta\cdot\tilde{n}\tilde{n}^\alpha)\,\mathrm{d}S\,\mathrm{d}t - \frac{\kappa}{\sigma}\int_0^T\int_\Gamma \partial_t^4 v_\alpha\partial_t^4[(1-\overline{\triangle})(v\cdot\tilde{n})\tilde{n}^\alpha]\,\mathrm{d}S\,\mathrm{d}t, \tag{4.2.95}$$

where, after integrating one tangential derivative by parts, the second term becomes

$$-\frac{\kappa}{\sigma}\sum_{\ell=0,1}\left(\int_0^T\int_\Gamma \overline{\partial}^\ell\partial_t^4 v_\alpha\partial_t^4[\overline{\partial}^\ell(v\cdot\tilde{n})\tilde{n}^\alpha]\,\mathrm{d}S\,\mathrm{d}t + \int_0^T\int_\Gamma \partial_t^4 v_\alpha\partial_t^4[\overline{\partial}^\ell(v\cdot\tilde{n})\overline{\partial}^\ell\tilde{n}^\alpha]\,\mathrm{d}S\,\mathrm{d}t\right). \tag{4.2.96}$$

The first term on the RHS contributes to the energy term $\frac{\kappa}{\sigma}\int_0^T\int_\Gamma\left|\partial_t^4 v\cdot\tilde{n}\right|_1^2\,\mathrm{d}S\,\mathrm{d}t$ together with errors terms. The most difficult error term is

$$\kappa\int_0^T\int_\Gamma(\overline{\partial}\partial_t^4 v\cdot\tilde{n})(v\cdot\partial_t^4\overline{\partial}\tilde{n})\,\mathrm{d}S\,\mathrm{d}t, \tag{4.2.97}$$

where the other errors are either with the same type of integrand or are effectively of lower order by one derivative with the case above. Since $\overline{\partial}\tilde{n} = Q(\overline{\partial}\tilde{\eta})\overline{\partial}^2\tilde{\eta}\cdot\tilde{n}$, we have

$$\begin{aligned}
&\frac{\kappa}{\sigma}\int_0^T\int_\Gamma(\overline{\partial}\partial_t^4 v\cdot\tilde{n})(v\cdot\partial_t^4\overline{\partial}\tilde{n})\,\mathrm{d}S\,\mathrm{d}t \overset{L}{=} \frac{\kappa}{\sigma}\int_0^T\int_\Gamma(\overline{\partial}\partial_t^4 v\cdot\tilde{n})(v\cdot\overline{\partial}^2\partial_t^3\tilde{v}\cdot\tilde{n})\,\mathrm{d}S\,\mathrm{d}t\\
\le&\int_0^T P(|\overline{\partial}\tilde{\eta}|_{L^\infty(\Gamma)},|v|_{L^\infty(\Gamma)})|\sqrt{\kappa}\overline{\partial}\partial_t^4 v|_0|\sqrt{\kappa}\overline{\partial}^2\partial_t^3 v\cdot\tilde{n}|_0\\
\lesssim&\int_0^T|\sqrt{\kappa}\overline{\partial}\partial_t^4 v|_0^2 + \sup_t P(|\overline{\partial}\tilde{\eta}|_{L^\infty(\Gamma)},|v|_{L^\infty(\Gamma)}) + \left(\int_0^T|\sqrt{\kappa}\overline{\partial}^2\partial_t^3 v\cdot\tilde{n}|_0^2\right)^2\\
\lesssim&\int_0^T\|\sqrt{\kappa}\partial_t^4 v\|_{1.5}^2 + \left(\int_0^T\|\sqrt{\kappa}\partial_t^3 v\|_{2.5}^2\right)^2 + \sup_t P(|\overline{\partial}\tilde{\eta}|_{L^\infty(\Gamma)},|v|_{L^\infty(\Gamma)})\\
\le& E_\kappa^{(3)} + (E_\kappa^{(3)})^2 + \sup_t P(|\overline{\partial}\tilde{\eta}|_{L^\infty(\Gamma)},|v|_{L^\infty(\Gamma)}).
\end{aligned}$$

Here, the last term can be controlled appropriately because

$$|\overline{\partial}\tilde{\eta}|_{L^\infty(\Gamma)} \lesssim \|\eta\|_3 \le \|\eta_0\|_3 + \int_0^T \|v\|_3, |v|_{L^\infty(\Gamma)} \lesssim \|v\|_2 \le \|v_0\|_2 + \int_0^T \|v_t\|_2,$$

and so $\sup_t P(|\overline{\partial}\tilde{\eta}|_{L^\infty(\Gamma)}, |v|_{L^\infty(\Gamma)}) \le \mathcal{P}_0 + \mathcal{P}\int_0^T \mathcal{P}$. In addition, the second term on the RHS of (4.2.96) can be treated by the same argument.

Next we analyze the first term on the RHS of (4.2.95). Since $\hat{n}\cdot\hat{n} = 1$, invoking (3.1.8) in Lemma 3.1.1 and we obtain

$$\triangle_g \eta \cdot \hat{n}\hat{n}^\alpha = -\mathcal{H}\circ\eta\hat{n}^\alpha = \triangle_g \eta^\alpha, \tag{4.2.98}$$

and so we are able to rewrite

$$\begin{aligned}\sqrt{g}\triangle_g \eta\cdot\tilde{n}\tilde{n}^\alpha =& \sqrt{g}\triangle_g\eta\cdot\hat{n}\hat{n}^\alpha + \sqrt{g}\triangle_g\eta\cdot\tilde{n}(\tilde{n}^\alpha - \hat{n}^\alpha) + \sqrt{g}\triangle_g\eta\cdot(\tilde{n}-\hat{n})\hat{n}^\alpha \\ =& \sqrt{g}\triangle_g\eta^\alpha + \sqrt{g}\triangle_g\eta\cdot\tilde{n}(\tilde{n}^\alpha - \hat{n}^\alpha) + \sqrt{g}\triangle_g\eta\cdot(\tilde{n}-\hat{n})\hat{n}^\alpha. \end{aligned}\tag{4.2.99}$$

In light of this, the first term on the RHS of (4.2.95) becomes

$$\begin{aligned}&\int_0^T\int_\Gamma \partial_t^4 v_\alpha \partial_t^4(\sqrt{g}\triangle_g\eta^\alpha)\,\mathrm{d}S\,\mathrm{d}t + \int_0^T\int_\Gamma \partial_t^4 v_\alpha\partial_t^4[\sqrt{g}\triangle_g\eta\cdot\tilde{n}(\tilde{n}^\alpha-\hat{n}^\alpha)]\,\mathrm{d}S\,\mathrm{d}t \\ &+ \int_0^T\int_\Gamma \partial_t^4 v_\alpha\partial_t^4[\sqrt{g}\triangle_g\eta\cdot(\tilde{n}-\hat{n})\hat{n}^\alpha]\,\mathrm{d}S\,\mathrm{d}t. \end{aligned}\tag{4.2.100}$$

We shall study the main term $I_{00} = \int_0^T\int_\Gamma \partial_t^4 v_\alpha \partial_t^4(\sqrt{g}\triangle_g\eta^\alpha)\,\mathrm{d}S\,\mathrm{d}t$. The error terms involving $\tilde{n}-\hat{n}$ are treated using (3.4.6) and they are identical to the Euler case. We refer [16, (12.16)-(12.19)] for the details. Invoking (3.1.9)-(3.1.10), we have

$$\begin{aligned} I_{00} =& \int_0^T\int_\Gamma \partial_t^4 v_\alpha \partial_t^3\overline{\partial}_i\left(\sqrt{g}g^{ij}\Pi^\alpha_\lambda\overline{\partial}_j v^\lambda\right)\,\mathrm{d}S\,\mathrm{d}t \\ &+ \int_0^T\int_\Gamma \partial_t^4 v_\alpha\partial_t^3\overline{\partial}_i\left(\sqrt{g}(g^{ij}g^{kl}-g^{lj}g^{ik})\overline{\partial}_j\eta^\alpha\overline{\partial}_k\eta_\lambda\overline{\partial}_l v^\lambda\right). \end{aligned}\tag{4.2.101}$$

Integrating $\overline{\partial}_i$ by parts and expanding the parenthesis, we get

$$\begin{aligned}(4.2.101) = &- \int_0^T \int_\Gamma \sqrt{g} g^{ij} \Pi^\alpha_\lambda \partial_t^3 \overline{\partial}_j v^\lambda \partial_t^4 \overline{\partial}_i v_\alpha \\ &- \int_0^T \int_\Gamma \sqrt{g}(g^{ij}g^{kl} - g^{lj}g^{ik}) \overline{\partial}_j \eta^\alpha \overline{\partial}_k \eta_\lambda \partial_t^3 \overline{\partial}_l v^\lambda \partial_t^4 \overline{\partial}_i v_\alpha \qquad (4.2.102) \\ &- 3\int_0^T \int_\Gamma \partial_t(\sqrt{g} g^{ij} \Pi^\alpha_\lambda) \partial_t^2 \overline{\partial}_j v^\lambda \partial_t^4 \overline{\partial}_i v_\alpha \, \mathrm{d}S \, \mathrm{d}t \\ &- 3\int_0^T \int_\Gamma \partial_t \left(\sqrt{g}(g^{ij}g^{kl} - g^{lj}g^{ik}) \overline{\partial}_j \eta^\alpha \overline{\partial}_k \eta_\lambda\right) \partial_t^2 \overline{\partial}_l v^\lambda \partial_t^4 \overline{\partial}_i v_\alpha \\ &- 3\int_0^T \int_\Gamma \partial_t^2(\sqrt{g} g^{ij} \Pi^\alpha_\lambda) \partial_t \overline{\partial}_j v^\lambda \partial_t^4 \overline{\partial}_i v_\alpha \, \mathrm{d}S \, \mathrm{d}t \\ &- 3\int_0^T \int_\Gamma \partial_t^2 \left(\sqrt{g}(g^{ij}g^{kl} - g^{lj}g^{ik}) \overline{\partial}_j \eta^\alpha \overline{\partial}_k \eta_\lambda\right) \partial_t \overline{\partial}_l v^\lambda \partial_t^4 \overline{\partial}_i v_\alpha \\ &- \int_0^T \int_\Gamma \partial_t^3(\sqrt{g} g^{ij} \Pi^\alpha_\lambda) \overline{\partial}_j v^\lambda \partial_t^4 \overline{\partial}_i v_\alpha \, \mathrm{d}S \, \mathrm{d}t \\ &- \int_0^T \int_\Gamma \partial_t^3 \left(\sqrt{g}(g^{ij}g^{kl} - g^{lj}g^{ik}) \overline{\partial}_j \eta^\alpha \overline{\partial}_k \eta_\lambda\right) \overline{\partial}_l v^\lambda \partial_t^4 \overline{\partial}_i v_\alpha \\ =:& I_{01} + \cdots + I_{08}.\end{aligned}$$

The main terms are I_{01} and I_{02} which produces $|\overline{\partial}(\Pi \partial_t^3 v)|_0^2$ as a part of energy, and the others can be controlled by estimating $I_{03} + I_{04}, I_{05} + I_{06}, I_{07} + I_{08}$ and integrating ∂_t by parts. In I_{01}, we integrate ∂_t by parts and use (3.1.7)

$$\begin{aligned}I_{01} =& -\frac{1}{2}\int_\Gamma \sqrt{g} g^{ij} \Pi^\alpha_\lambda \partial_t^3 \overline{\partial}_j v^\lambda \partial_t^3 \overline{\partial}_i v_\alpha \Big|_0^T + \frac{1}{2}\int_0^T \int_\Gamma \partial_t(\sqrt{g} g^{ij} \Pi^\alpha_\lambda) \partial_t^3 \overline{\partial}_j v^\lambda \partial_t^3 \overline{\partial}_i v_\alpha \, \mathrm{d}S \, \mathrm{d}t \\ =& \frac{1}{2}\int_\Gamma \sqrt{g} g^{ij} \overline{\partial}_i (\Pi^\alpha_\mu \partial_t^3 v_\alpha) \overline{\partial}_j (\Pi^\mu_\lambda \partial_t^3 v^\lambda) + \int_\Gamma \sqrt{g} g^{ij} \overline{\partial} \Pi^\alpha_\mu \partial_t^3 v_\alpha \overline{\partial}_j (\Pi^\mu_\lambda \partial_t^3 v^\lambda) \\ &- \frac{1}{2}\int_\Gamma \overline{\partial}_i \Pi^\alpha_\mu \overline{\partial}_j \Pi^\mu_\lambda \partial_t^3 v_\alpha \partial_t^3 v^\lambda + \frac{1}{2}\int_0^T \int_\Gamma \partial_t(\sqrt{g} g^{ij} \Pi^\alpha_\lambda) \partial_t^3 \overline{\partial}_j v^\lambda \partial_t^3 \overline{\partial}_i v_\alpha \, \mathrm{d}S \, \mathrm{d}t + I_{01}|_{t=0} \\ =:& I_{011} + I_{012} + I_{013} + I_{014} + I_{01}|_{t=0}.\end{aligned} \tag{4.2.103}$$

The term I_{011} produces the energy term

$$I_{011} = -\frac{1}{2}\int_\Gamma \left|\overline{\partial}(\Pi\partial_t^3 v)\right|^2 \,\mathrm{d}S - \frac{1}{2}\int_\Gamma (\sqrt{g}g^{ij} - \delta^{ij})\overline{\partial}_i(\Pi^\alpha_\mu \partial_t^3 v_\alpha)\overline{\partial}_j(\Pi^\mu_\lambda \partial_t^3 v^\lambda)\,\mathrm{d}S$$

$$\lesssim -\frac{1}{2}\left|\overline{\partial}(\Pi\partial_t^3 v)\right|_0^2 + \left|\overline{\partial}(\Pi\partial_t^3 v)\right|_0^2 \left|\sqrt{g}g^{ij} - \delta^{ij}\right|_{1.5} \tag{4.2.104}$$

$$\lesssim -\frac{1}{2}\left|\overline{\partial}(\Pi\partial_t^3 v)\right|_0^2 + \left|\overline{\partial}(\Pi\partial_t^3 v)\right|_0^2 \int_0^T P(\|\partial_t\overline{\partial}\eta\|_2, \|\overline{\partial}\eta\|_2)\,\mathrm{d}t.$$

The terms $I_{012}, I_{013}, I_{014}$ can all be directly controlled. By $\overline{\partial}^2\eta|_{t=0} = 0$,

$$I_{012} \lesssim \left|\sqrt{g}g^{-1}\right|_{L^\infty} |\overline{\partial}\Pi|_{L^\infty}|\partial_t^3 v|_0|\overline{\partial}(\Pi\partial_t^3 v)|_0$$

$$\lesssim \varepsilon\left(\left|\overline{\partial}(\Pi\partial_t^3 v)\right|_0^2 + \|\partial_t^3 v\|_{1.5}^2\right) + \mathcal{P}_0 + \int_0^T P(\|\eta\|_4, \|v\|_4, \|\partial_t^4 v\|_0)\,\mathrm{d}t, \tag{4.2.105}$$

$$I_{013} \lesssim \varepsilon\|\partial_t^3 v\|_{1.5}^2 + P(\|\overline{\partial}\eta\|_2)\|\partial_t^3 v\|_0 \int_0^T \|\overline{\partial}^2 v\|_2\,\mathrm{d}t \tag{4.2.106}$$

$$I_{014} \lesssim \int_0^T \left|\partial_t^3\overline{\partial}v\right|_0^2 \left|\partial_t(\sqrt{g}g^{ij}\Pi)\right|_{L^\infty} \lesssim \int_0^T P(\|\partial_t^3 v\|_{1.5}, \|v\|_3, \|\eta\|_3)\,\mathrm{d}t. \tag{4.2.107}$$

Combining (4.2.103) with (4.2.104)-(4.2.107), we get the estimates of I_{01} as follows

$$I_{01} \lesssim \varepsilon\left(\left|\overline{\partial}(\Pi\partial_t^3 v)\right|_0^2 + \|\partial_t^3 v\|_{1.5}^2\right) + \mathcal{P}_0 + \mathcal{P}\int_0^T \mathcal{P}\,\mathrm{d}t. \tag{4.2.108}$$

Next we control $I_{02} := -\int_0^T\int_\Gamma \sqrt{g}(g^{ij}g^{kl} - g^{lj}g^{ik})\overline{\partial}_j\eta^\alpha\overline{\partial}_k\eta_\lambda\partial_t^3\overline{\partial}_l v^\lambda\partial_t^4\overline{\partial}_i v_\alpha$. We expand the summation on l, i and find that:

- $l = i$, this integral is zero thanks to the symmetry.
- $l = 1, i = 2$, the integrand is $\sqrt{g}^{-1}(\overline{\partial}_1\eta_\lambda\overline{\partial}_2\eta_\alpha - \overline{\partial}_1\eta_\alpha\overline{\partial}_2\eta_\lambda)\partial_t^3\overline{\partial}_1 v^\lambda\partial_t^4\overline{\partial}_2 v^\alpha$.
- $l = 2, i = 1$, the integrand is $-\sqrt{g}^{-1}(\overline{\partial}_1\eta_\lambda\overline{\partial}_2\eta_\alpha - \overline{\partial}_1\eta_\alpha\overline{\partial}_2\eta_\lambda)\partial_t^3\overline{\partial}_2 v^\lambda\partial_t^4\overline{\partial}_1 v^\alpha$.

Here, we use g^{-1} to denote $\det[g^{-1}] = g^{11}g^{22} - g^{12}g^{21}$. Therefore, we have

$$
\begin{aligned}
I_{02} =& -\int_0^T \int_\Gamma \frac{1}{\sqrt{g}} \left(\overline{\partial}_1\eta_\lambda\overline{\partial}_2\eta_\alpha - \overline{\partial}_1\eta_\alpha\overline{\partial}_2\eta_\lambda\right)\left(\partial_t^3\overline{\partial}_1 v^\lambda \partial_t^4\overline{\partial}_2 v^\alpha + \partial_t^3\overline{\partial}_2 v^\lambda \partial_t^4\overline{\partial}_1 v^\alpha\right) \mathrm{d}S\,\mathrm{d}t \\
=& \int_0^T \int_\Gamma \frac{1}{\sqrt{g}} \frac{d}{dt} \left(\det \underbrace{\begin{bmatrix} \overline{\partial}_1\eta_\mu\partial_t^3\overline{\partial}_1 v^\mu & \overline{\partial}_1\eta_\mu\partial_t^3\overline{\partial}_2 v^\mu \\ \overline{\partial}_2\eta_\mu\partial_t^3\overline{\partial}_1 v^\mu & \overline{\partial}_2\eta_\mu\partial_t^3\overline{\partial}_2 v^\mu \end{bmatrix}}_{=:\mathfrak{A}} \right) + \text{ lower order terms} \\
\overset{\partial_t}{=}& \int_\Gamma \frac{1}{\sqrt{g}} \det\mathfrak{A}\Big|_0^T - \int_0^T \int_\Gamma \partial_t\left(\frac{1}{\sqrt{g}}\right)\det\mathfrak{A}
\end{aligned}
\tag{4.2.109}
$$

The first term in the last line of (4.2.109) can be expanded into two terms

$$
\int_\Gamma \frac{1}{\sqrt{g}} \det\mathfrak{A} = \int_\Gamma \frac{1}{\sqrt{g}} \left(\overline{\partial}_1\eta_\mu\overline{\partial}_2\eta_\lambda\overline{\partial}_1\partial_t^3 v^\mu\overline{\partial}_2\partial_t^3 v^\lambda - \overline{\partial}_1\eta_\mu\overline{\partial}_2\eta_\lambda\overline{\partial}_2\partial_t^3 v^\mu\overline{\partial}_1\partial_t^3 v^\lambda\right). \tag{4.2.110}
$$

It can be seen that the top order terms cancel with each other if one integrates $\overline{\partial}_1$ by parts in the first term and $\overline{\partial}_2$ by parts in the second. The remaining terms are all of the form $-\int_\Gamma Q_{\mu\lambda}(\overline{\partial}\eta, \overline{\partial}^2\eta)\partial_t^3 v^\mu\overline{\partial}\partial_t^3 v^\lambda$, which can be controlled as

$$
\begin{aligned}
& -\int_\Gamma Q_{\mu\lambda}(\overline{\partial}\eta, \overline{\partial}^2\eta)\partial_t^3 v^\mu\overline{\partial}\partial_t^3 v^\lambda \\
\lesssim & P(|\overline{\partial}^2\eta|_{L^\infty}, |\overline{\partial}\eta|_{L^\infty})|\partial_t^3 v|_0|\overline{\partial}\partial_t^3 v|_0 \\
\lesssim & \varepsilon\|\partial_t^3 v\|_{1.5}^2 + \frac{1}{4\varepsilon}\|\partial_t^3 v\|_{0.5}\int_0^T P(\|\overline{\partial}^2 v\|_2)\,\mathrm{d}t.
\end{aligned}
\tag{4.2.111}
$$

The second term of (4.2.109) can be directly controlled, i.e.,

$$
\int_0^T \int_\Gamma \partial_t\left(\frac{1}{\sqrt{g}}\right)\det\mathbf{A} \lesssim |\partial_t\overline{\partial}\eta|_{L^\infty}|\overline{\partial}\eta|_{L^\infty}^2|\overline{\partial}\partial_t^3 v|_0^2\,\mathrm{d}t \lesssim \int_0^T \mathcal{P}. \tag{4.2.112}
$$

Therefore, we get the estimates of I_{02}:

$$
I_{02} \lesssim \varepsilon\|\partial_t^3 v\|_{1.5}^2 + \mathcal{P}_0 + \mathcal{P}\int_0^T \mathcal{P}\,\mathrm{d}t, \tag{4.2.113}
$$

Next we control the remaining terms in I_0, i.e., $I_{03}, \cdots, I_{08}$. The strategy here is to study $I_{03}+I_{04}, I_{05}+I_{06}, I_{07}+I_{08}$, where

$$\begin{aligned}
I_{03}+I_{04}=&-3\int_0^T\int_\Gamma \partial_t(Q(\overline{\partial}\eta))\partial_t^2\overline{\partial}v\partial_t^4\overline{\partial}v\,\mathrm{d}S\,\mathrm{d}t\\
&\overset{\partial_t}{=}3\int_0^T\int_\Gamma \partial_t^2(Q(\overline{\partial}\eta))\partial_t^2\overline{\partial}v\partial_t^3\overline{\partial}v+3\int_0^T\int_\Gamma \partial_t(Q(\overline{\partial}\eta))\partial_t^3\overline{\partial}v\partial_t^3\overline{\partial}v\\
&\quad+3\int_\Gamma \partial_t(Q(\overline{\partial}\eta))\partial_t^2\overline{\partial}v\partial_t^3\overline{\partial}v\Big|_0^T\\
&=3\int_0^T\int_\Gamma\left(\underbrace{Q(\overline{\partial}\eta)\overline{\partial}v}_{\partial_t(Q(\overline{\partial}\eta))}\overline{\partial}v+Q(\overline{\partial}\eta)\overline{\partial}\partial_t v\right)\partial_t^2\overline{\partial}v\partial_t^3\overline{\partial}v\\
&\quad+3\int_0^T\int_\Gamma Q(\overline{\partial}\eta)\overline{\partial}v\partial_t^3\overline{\partial}v\partial_t^3\overline{\partial}v+\int_\Gamma Q(\overline{\partial}\eta)\overline{\partial}v\partial_t^2\overline{\partial}v\partial_t^3\overline{\partial}v\,\mathrm{d}S\Big|_0^T\\
&\lesssim\varepsilon\|\partial_t^3 v\|_{1.5}^2+\mathcal{P}_0+\int_0^T\mathcal{P}.
\end{aligned}\tag{4.2.114}$$

Similarly, by plugging $\partial_t^3(Q(\overline{\partial}\eta))=Q(\overline{\partial}\eta)(\overline{\partial}\partial_t v\overline{\partial}v\overline{\partial}v+\overline{\partial}\partial_t v\overline{\partial}v+\overline{\partial}\partial_t^2 v)$ into $I_{05}+I_{06}$, we get

$$\begin{aligned}
I_{05}+I_{06}\overset{\partial_t}{=}&-\int_0^T\int_\Gamma \partial_t^3(Q(\overline{\partial}\eta))\partial_t\overline{\partial}v\partial_t^3\overline{\partial}v\,\mathrm{d}S\,\mathrm{d}t\\
&-\int_0^T\int_\Gamma \partial_t^2(Q(\overline{\partial}\eta))\partial_t^2\overline{\partial}v\partial_t^3\overline{\partial}v+\int_\Gamma \partial_t^2(Q(\overline{\partial}\eta))\partial_t\overline{\partial}v\partial_t^3\overline{\partial}v\Big|_0^T\\
&\lesssim\mathcal{P}_0+\int_0^T\mathcal{P}+\varepsilon\|\partial_t^3 v\|_{1.5}^2.
\end{aligned}\tag{4.2.115}$$

Following the same way as above, we can control $I_{07}+I_{08}$ by $\mathcal{P}_0+\int_0^T\mathcal{P}+\varepsilon\|\partial_t^3 v\|_{1.5}^2$ so we omit the details. Combining this with (4.2.102), (4.2.108), (4.2.113)-(4.2.115), we get the estimates of I_0 by

$$I_0+\left|\overline{\partial}\left(\Pi\partial_t^3 v\right)\right|_0^2\lesssim\varepsilon\|\partial_t^3 v\|_{1.5}^2+\mathcal{P}_0+\mathcal{P}\int_0^T\mathcal{P}.\tag{4.2.116}$$

Now the only term left to control in (4.2.78) is L. Expanding $[\partial_t^4, \tilde{\mathbf{A}}^{\mu\alpha}]$, we have

$$L = \int_0^T \int_\Omega \partial_t^4 \tilde{\mathbf{A}}^{\mu\alpha} \partial_\mu v_\alpha \partial_t^4 Q \, \mathrm{d}y \, \mathrm{d}t + 4 \int_0^T \int_\Omega \partial_t^3 \tilde{\mathbf{A}}^{\mu\alpha} \partial_t \partial_\mu v_\alpha \partial_t^4 Q \, \mathrm{d}y \, \mathrm{d}t$$

$$+ 6 \int_0^T \int_\Omega \partial_t^2 \tilde{\mathbf{A}}^{\mu\alpha} \partial_t^2 \partial_\mu v_\alpha \partial_t^4 Q \, \mathrm{d}y \, \mathrm{d}t + 4 \int_0^T \int_\Omega \partial_t \tilde{\mathbf{A}}^{\mu\alpha} \partial_t^3 \partial_\mu v_\alpha \partial_t^4 Q \, \mathrm{d}y \, \mathrm{d}t \tag{4.2.117}$$

$$=: L_{21} + L_{22} + L_{23} + L_{24}.$$

Despite having the right amount of derivatives, there is no direct control of $\|\partial_t^4 Q\|_0$ and so we have to make some extra efforts to control $L_{21}, \cdots, L_{24}$.

The hardest term to treat here is L_{21}. Since

$$\partial_t^4 \tilde{\mathbf{A}}^{\mu\alpha} \partial_\mu v_\alpha = \partial_t^4 (\tilde{J} \tilde{A}^{\mu\alpha}) \partial_\mu v_\alpha \overset{L}{=} -\tilde{A}^{\mu\nu} \partial_\beta \partial_t^3 \tilde{v}_\nu \tilde{\mathbf{A}}^{\beta\alpha} \partial_\mu v_\alpha \tag{4.2.118}$$

we have

$$L_{21} \overset{L}{=} \int_0^T \int_\Omega \tilde{A}^{\mu\nu} \partial_\beta \partial_t^3 \tilde{v}_\nu \tilde{\mathbf{A}}^{\beta\alpha} \partial_\mu v_\alpha \partial_t^4 Q. \tag{4.2.119}$$

Since

$$\tilde{\mathbf{A}}^{\beta\alpha} \partial_t^4 Q = \partial_t^4 (\tilde{\mathbf{A}}^{\beta\alpha} Q) - (\partial_t^4 \tilde{\mathbf{A}}^{\beta\alpha}) Q - 4(\partial_t^3 \tilde{\mathbf{A}}^{\beta\alpha}) \partial_t Q - 6(\partial_t^2 \tilde{\mathbf{A}}^{\beta\alpha}) \partial_t^2 Q - 4(\partial_t \tilde{\mathbf{A}}^{\beta\alpha}) \partial_t^3 Q,$$

and thus one can write the RHS of (4.2.119) as

$$\int_0^T \int_\Omega \tilde{A}^{\mu\nu} \partial_\beta \partial_t^3 \tilde{v}_\nu \partial_\mu v_\alpha \partial_t^4 (\tilde{\mathbf{A}}^{\beta\alpha} Q) - 4 \int_0^T \int_\Omega \tilde{A}^{\mu\nu} \partial_\beta \partial_t^3 \tilde{v}_\nu \partial_\mu v_\alpha \partial_t^3 \tilde{\mathbf{A}}^{\beta\alpha} \partial_t Q$$

$$- 6 \int_0^T \int_\Omega \tilde{A}^{\mu\nu} \partial_\beta \partial_t^3 \tilde{v}_\nu \partial_\mu v_\alpha \partial_t^2 \tilde{\mathbf{A}}^{\beta\alpha} \partial_t^2 Q - 4 \int_0^T \int_\Omega \tilde{A}^{\mu\nu} \partial_\beta \partial_t^3 \tilde{v}_\nu \partial_\mu v_\alpha \partial_t \tilde{\mathbf{A}}^{\beta\alpha} \partial_t^3 Q$$

$$=: L_{211} + L_{212} + L_{213} + L_{214}.$$

It is not hard to see that $L_{212}, L_{213}, L_{214}$ can all be controlled directly by $\int_0^T \mathcal{P}$ thanks to (4.2.26). To treat L_{211}, we integrate ∂_β by parts and get

$$\int_0^T \int_\Gamma \tilde{A}^{\mu\nu}\partial_t^3\tilde{v}_\nu\partial_\mu v_\alpha\partial_t^4(\tilde{\mathbf{A}}^{3\alpha}Q) - \int_0^T \int_\Omega \partial_t^3\tilde{v}_\nu\partial_\beta\left(\tilde{A}^{\mu\nu}\partial_\mu v_\alpha\partial_t^4(\tilde{\mathbf{A}}^{\beta\alpha}Q)\right) = L_{2111} + L_{2112}.$$

Since $L_{2112} \stackrel{L}{=} -\int_0^T\int_\Omega \partial_t^3\tilde{v}_\nu\tilde{A}^{\mu\nu}\partial_\mu v_\alpha\partial_t^4\partial_\beta(\tilde{\mathbf{A}}^{\beta\alpha}Q)$, we integrate ∂_t by parts in the last term and get

$$-\int_\Omega \partial_t^3\tilde{v}_\nu\tilde{A}^{\mu\nu}\partial_\mu v_\alpha\partial_t^3\partial_\beta(\tilde{\mathbf{A}}^{\beta\alpha}Q)\Big|_0^T + \int_0^T\int_\Omega \partial_t\left(\partial_t^3\tilde{v}_\nu\tilde{A}^{\mu\nu}\partial_\mu v_\alpha\right)\partial_t^3\partial_\beta(\tilde{\mathbf{A}}^{\beta\alpha}Q) \tag{4.2.120}$$

$$=: L_{21121} + L_{21122}.$$

Now, since $\partial_\beta\tilde{\mathbf{A}}^{\beta\alpha} = 0$, we can write

$$\partial_t^3\partial_\beta(\tilde{\mathbf{A}}^{\beta\alpha}Q) = -\partial_t^4 v^\alpha + \partial_t^3(b_0\cdot\partial)^2\eta^\alpha. \tag{4.2.121}$$

In light of this, we have $L_{21122} \le \int_0^T \mathcal{P}$ and

$$\begin{aligned}
L_{21121} =& -\int_\Omega \partial_t^3\tilde{v}_\nu\tilde{A}^{\mu\nu}\partial_\mu v_\alpha(-\partial_t^4 v^\alpha + \partial_t^3(b_0\cdot\partial)^2\eta^\alpha)\Big|_0^T \\
\lesssim& \mathcal{P}_0 + \|\tilde{A}^{\mu\nu}\partial_\mu v_\alpha\|_{L^\infty}\|\partial_t^3 v\|_0(\|\partial_t^4 v\|_0 + \|b_0\|_{L^\infty}\|\partial_t^3(b_0\cdot\partial)\eta\|_1) \\
\lesssim& \mathcal{P}_0 + \varepsilon(\|\partial_t^4 v\|_0^2 + \|\partial_t^3(b_0\cdot\partial)\eta\|_1^2) + P(\|\partial_t^3 v\|_0, \|\tilde{A}^{\mu\nu}\partial_\mu v_\alpha\|_2) \\
\le& \mathcal{P}_0 + \varepsilon(\|\partial_t^4 v\|_1^2 + \|\partial_t^3(b_0\cdot\partial)\eta\|_1^2) + \mathcal{P}\int_0^T \mathcal{P}.
\end{aligned}$$

Moreover, by plugging the boundary condition (4.2.6) to L_{2111} we obtain

$$\begin{aligned}
&-\sigma\int_0^T\int_\Gamma \tilde{A}^{\mu\nu}\partial_t^3\tilde{v}_\nu\partial_\mu v_\alpha\partial_t^4(\sqrt{g}\triangle_g\eta\cdot\tilde{n}\tilde{n}^\alpha \\
&+\kappa\int_0^T\int_\Gamma \tilde{A}^{\mu\nu}\partial_t^3\tilde{v}_\nu\partial_\mu v_\alpha\partial_t^4\left(\left((1-\overline{\triangle})(v\cdot\tilde{n})\right)\tilde{n}^\alpha\right) =: L_{21111} + L_{21112}.
\end{aligned}$$

Invoking (3.1.9), we have

$$L_{21111} = -\sigma\int_0^T\int_\Gamma \tilde{A}^{\mu\nu}\partial_t^3\tilde{v}_\nu\partial_\mu v_\alpha\partial_t^4(\sqrt{g}g^{ij}\overline{\partial}_i\overline{\partial}_j\eta\cdot\tilde{n}\tilde{n}^\alpha)$$

$$+\sigma\int_0^T\int_\Gamma \tilde{A}^{\mu\nu}\partial_t^3\tilde{v}_\nu\partial_\mu v_\alpha\partial_t^4(\sqrt{g}g^{ij}g^{kl}\overline{\partial}_l\eta^\mu\overline{\partial}_i\overline{\partial}_j\eta_\mu\overline{\partial}_k\eta\cdot\tilde{n}\tilde{n}^\alpha).$$

It suffices to control the first term only since the second term has a highest order contribution with the same type of integrand. Also,

$$-\sigma\int_0^T\int_\Gamma \tilde{A}^{\mu\nu}\partial_t^3\tilde{v}_\nu\partial_\mu v_\alpha\partial_t^4(\sqrt{g}g^{ij}\overline{\partial}_i\overline{\partial}_j\eta\cdot\tilde{n}\tilde{n}^\alpha)$$

$$\stackrel{L}{=} -\sigma\int_0^T\int_\Gamma \tilde{A}^{\mu\nu}\partial_t^3\tilde{v}_\nu\partial_\mu v_\alpha\sqrt{g}g^{ij}\overline{\partial}_i\overline{\partial}_j\partial_t^3 v\cdot\tilde{n}\tilde{n}^\alpha$$

$$-\sigma\int_0^T\int_\Gamma \tilde{A}^{\mu\nu}\partial_t^3\tilde{v}_\nu\partial_\mu v_\alpha\sqrt{g}g^{ij}\overline{\partial}_i\overline{\partial}_j\eta\cdot\tilde{n}(\partial_t^4\tilde{n}^\alpha).$$

Now, since

$$\partial_t^4\tilde{n} = Q(\overline{\partial}\tilde{\eta})\overline{\partial}\partial_t^3 v\cdot\tilde{n} + \text{lower-order terms}, \tag{4.2.122}$$

and so we have, after using the Sobolev embedding and trace lemma, that

$$\sigma\int_0^T\int_\Gamma\left|\tilde{A}^{\mu\nu}\partial_t^3\tilde{v}_\nu\partial_\mu v_\alpha\sqrt{g}g^{ij}\overline{\partial}_i\overline{\partial}_j\eta\cdot\tilde{n}(\partial_t^4\tilde{n}^\alpha)\right| \le \int_0^T\mathcal{P}. \tag{4.2.123}$$

In addition, by integrating $\overline{\partial}_i$ by parts and then using the trace lemma, we have

$$\sigma\int_0^T\int_\Gamma\left|\tilde{A}^{\mu\nu}\partial_t^3\tilde{v}_\nu\partial_\mu v_\alpha\sqrt{g}g^{ij}\overline{\partial}_i\overline{\partial}_j\partial_t^3 v\cdot\tilde{n}\tilde{n}^\alpha\right| \le \int_0^T\mathcal{P}. \tag{4.2.124}$$

Moreover, we still need to control L_{21112}. In light of (4.2.122), we only need to study the case when all four time derivatives land on $\overline{\triangle}v$, i.e.,

$$-\kappa\int_0^T\int_\Gamma \tilde{A}^{\mu\nu}\partial_t^3\tilde{v}_\nu\partial_\mu v_\alpha\overline{\triangle}(\partial_t^4 v\cdot\tilde{n})\tilde{n}^\alpha.$$

Integrating $\overline{\partial}$ by parts, this term has the contributes to

$$\kappa \int_0^T \int_\Gamma \tilde{A}^{\mu\nu} \partial_t^3 \overline{\partial} \tilde{v}_\nu \partial_\mu v_\alpha \overline{\partial} \partial_t^4 v \cdot \tilde{n} \tilde{n}^\alpha,$$

up to terms with the same type integrand, whose analysis (and bound) is identical. To control the main term, one has

$$\kappa \int_0^T \int_\Gamma \tilde{A}^{\mu\nu} \partial_t^3 \overline{\partial} \tilde{v}_\nu \partial_\mu v_\alpha \overline{\partial} \partial_t^4 v \cdot \tilde{n} \tilde{n}^\alpha = \sqrt{\kappa} \int_0^T \int_\Gamma Q(\overline{\partial} \tilde{\eta}, \partial v) \overline{\partial} \partial_t^3 v \sqrt{\kappa} \overline{\partial} \partial_t^4 v$$

$$\leq \sqrt{\kappa} \int_0^T Q(\|\overline{\partial} \tilde{\eta}\|_{L^\infty}, \|\partial v\|_{L^\infty}) \|\partial_t^3 v\|_{1.5} \|\sqrt{\kappa} \partial_t^4 v\|_{1.5} \leq \sqrt{\kappa} E_\kappa^{(3)} + \int_0^T \mathcal{P}.$$

Finally, combining (4.2.77) with the computations above, we finally get the control of full time derivatives

$$\left\|\partial_t^4 v\right\|_0^2 + \left\|\partial_t^4 (b_0 \cdot \partial) \eta\right\|_0^2 + \left|\overline{\partial}\left(\Pi \partial_t^3 v\right)\right|_0^2 \lesssim E_\kappa^{(3)} + (E_\kappa^{(3)})^2 + \mathcal{P}_0 + C(\varepsilon) E_\kappa(T) + \mathcal{P} \int_0^T \mathcal{P}. \quad (4.2.125)$$

4.2.3.2 Control of mixed space-time tangential derivatives

To finish the control of $E_\kappa(T)$, it remains to study the tangential energies generated by the $\overline{\partial}\partial_t^3$, $\overline{\partial}^2 \partial_t^2$, $\overline{\partial}^3 \partial_t$ and $\overline{\partial}^3 (b_0 \cdot \partial)$-differentiated κ-problem. Such energy estimate becomes much simpler when the tangential spatial derivative(s) ∂_t is taken into account. This is due to that we can avoid the terms associated to I_{11} in (4.2.79). This can be done by thanks to the extra 0.5 interior regularity.

The $\overline{\partial}\partial_t^3$-tangential energy: Similar to (4.2.77), we have

$$\begin{aligned} &\frac{1}{2} \int_0^T \frac{d}{dt} \int_\Omega \left|\overline{\partial} \partial_t^3 v\right|_0^2 + \left|\overline{\partial} \partial_t^3 (b_0 \cdot \partial) \eta\right|^2 \, \mathrm{d}y \, \mathrm{d}t \\ = & \underbrace{- \int_0^T \int_\Omega \overline{\partial} \partial_t^3 (\tilde{\mathbf{A}}^{\mu\alpha} \partial_\mu Q) \overline{\partial} \partial_t^3 v_\alpha \, \mathrm{d}y \, \mathrm{d}t}_{I^*} \\ & + \int_0^T \int_\Omega \overline{\partial} \partial_t^3 (b_0 \cdot \partial)^2 \eta_\alpha \overline{\partial} \partial_t^3 v_\alpha \, \mathrm{d}y \, \mathrm{d}t + \int_0^T \int_\Omega \overline{\partial} \partial_t^3 (b_0 \cdot \partial) \eta \alpha \overline{\partial} \partial_t^3 (b_0 \cdot \partial) v_\alpha \, \mathrm{d}y \, \mathrm{d}t. \end{aligned} \quad (4.2.126)$$

By integrating $(b_0 \cdot \partial)$ by parts in the second term, we can get the cancellation with the third term at the top order

$$\begin{aligned}
&\int_0^T \int_\Omega \overline{\partial}\partial_t^3 (b_0\cdot\partial)^2\eta_\alpha \overline{\partial}\partial_t^3 v_\alpha \,\mathrm{d}y\,\mathrm{d}t + \int_0^T \int_\Omega \overline{\partial}\partial_t^3 (b_0\cdot\partial)\eta\alpha \overline{\partial}\partial_t^3 (b_0\cdot\partial) v_\alpha \,\mathrm{d}y\,\mathrm{d}t \\
=& -\int_0^T \int_\Omega \overline{\partial}\partial_t^3 (b_0\cdot\partial)\eta_\alpha \overline{\partial}\partial_t^3 (b_0\cdot\partial) v_\alpha \,\mathrm{d}y\,\mathrm{d}t + \int_0^T \int_\Omega \overline{\partial}\partial_t^3 (b_0\cdot\partial)\eta_\alpha \overline{\partial}\partial_t^3 (b_0\cdot\partial) v_\alpha \,\mathrm{d}y\,\mathrm{d}t \\
&+ \int_0^T \int_\Omega \left[\overline{\partial}, (b_0\cdot\partial)\right] \partial_t^3 (b_0\cdot\partial)\eta^\alpha \cdot \overline{\partial}\partial_t^3 v_\alpha - \overline{\partial}\partial_t^3 (b_0\cdot\partial)\eta^\alpha \cdot \left[(b_0\cdot\partial), \overline{\partial}\right] \partial_t^3 v_\alpha \,\mathrm{d}y\,\mathrm{d}t \\
\lesssim& \int_0^T P(\|b_0\|_3, \|\partial_t^3 v\|_1, \|\partial_t^2 v\|_2)\,\mathrm{d}t
\end{aligned} \tag{4.2.127}$$

The main term I^* is treated a bit differently compare to I in (4.2.78). Specifically, one commutes $\tilde{\mathbf{A}}^{\mu\alpha}$ with $\overline{\partial}\partial_t^3$ first and then integrate by parts. This allows us to avoid the appearance of the higher order interior terms.

$$\begin{aligned}
I^* =& -\int_0^T \int_\Omega \overline{\partial}\partial_t^3 v_\alpha \tilde{\mathbf{A}}^{\mu\alpha} \overline{\partial}\partial_t^3 \partial_\mu Q \underbrace{- \int_0^T \int_\Omega \overline{\partial}\partial_t^3 v_\alpha \left[\overline{\partial}\partial_t^3, \tilde{\mathbf{A}}^{\mu\alpha}\right] \partial_\mu Q}_{L_1^*} \\
\overset{\partial_\mu}{=}& \int_0^T \int_\Omega \tilde{\mathbf{A}}^{\mu\alpha} \overline{\partial}\partial_t^3 \partial_\mu v_\alpha \overline{\partial}\partial_t^3 Q \underbrace{- \int_0^T \int_\Gamma \overline{\partial}\partial_t^3 v_\alpha \tilde{\mathbf{A}}^{3\alpha} \overline{\partial}\partial_t^3 Q}_{I_B^*} \underbrace{+ \int_0^T \int_{\Gamma_0} \overline{\partial}\partial_t^3 v_\alpha \tilde{\mathbf{A}}^{3\alpha} \overline{\partial}\partial_t^3 Q}_{I_B^{**}} + L_1^* \\
=& \int_0^T \int_\Omega \underbrace{\overline{\partial}\partial_t^3 (\mathrm{div}_{\tilde{\mathbf{A}}} v)}_{=0} \overline{\partial}\partial_t^3 Q \underbrace{+ \int_0^T \int_\Omega \left[\tilde{\mathbf{A}}^{\mu\alpha}, \overline{\partial}\partial_t^3\right] \partial_\mu v_\alpha \, \overline{\partial}\partial_t^3 Q}_{L_2^*} + I_B^* + I_B^{**} + L_1^*.
\end{aligned} \tag{4.2.128}$$

Here, $I_B^{**} = 0$ because $\tilde{\mathbf{A}}^{13} = \tilde{\mathbf{A}}^{23} = 0$, $\tilde{\mathbf{A}}^{33} = 1$ and $v_3 = 0$ on Γ_0. Also, L_1^* and L_2^* can be directly controlled. For simplicity we only list the computation of the highest order terms

$$
\begin{aligned}
L_1^* &= -\int_0^T \int_\Omega \overline{\partial}\partial_t^3 v_\alpha \, \left[\overline{\partial}\partial_t^3, \tilde{\mathbf{A}}^{\mu\alpha}\right] \partial_\mu Q \, \mathrm{d}y \, \mathrm{d}t \\
&\overset{L}{=} -\int_0^T \int_\Omega \overline{\partial}\partial_t^3 v_\alpha \, \overline{\partial}\partial_t^3 \tilde{\mathbf{A}}^{\mu\alpha} \partial_\mu Q \, \mathrm{d}y \, \mathrm{d}t \lesssim \int_0^T \mathcal{P} \, \mathrm{d}t.
\end{aligned} \tag{4.2.129}
$$

and

$$
\begin{aligned}
L_2^* &= \int_0^T \int_\Omega \left[\tilde{\mathbf{A}}^{\mu\alpha}, \overline{\partial}\partial_t^3\right] \partial_\mu v_\alpha \, \overline{\partial}\partial_t^3 Q \, \mathrm{d}y \, \mathrm{d}t \\
&\overset{L}{=} \int_0^T \int_\Omega \overline{\partial}\partial_t^3 \tilde{\mathbf{A}}^{\mu\alpha} \partial_\mu v_\alpha \, \overline{\partial}\partial_t^3 Q \, \mathrm{d}y \, \mathrm{d}t \lesssim \int_0^T \mathcal{P} \, \mathrm{d}t.
\end{aligned} \tag{4.2.130}
$$

Next we analyze the boundary integral I_B^*.

$$
\begin{aligned}
I_B^* = &-\int_0^T \int_\Gamma \overline{\partial}\partial_t^3 v_\alpha \overline{\partial}\partial_t^3 (\tilde{\mathbf{A}}^{3\alpha} Q) \, \mathrm{d}S \, \mathrm{d}t \\
&+ \int_0^T \int_\Gamma \overline{\partial}\partial_t^3 v_\alpha \overline{\partial}\partial_t^3 \tilde{\mathbf{A}}^{3\alpha} Q \, \mathrm{d}S \, \mathrm{d}t + \int_0^T \int_\Gamma \overline{\partial}\partial_t^3 v_\alpha \partial_t^3 \tilde{\mathbf{A}}^{3\alpha} \overline{\partial} Q \, \mathrm{d}S \, \mathrm{d}t \\
&+ 3\int_0^T \int_\Gamma \overline{\partial}\partial_t^3 v_\alpha \overline{\partial}\partial_t^2 \tilde{\mathbf{A}}^{3\alpha} \partial_t Q \, \mathrm{d}S \, \mathrm{d}t + 3\int_0^T \int_\Gamma \overline{\partial}\partial_t^3 v_\alpha \partial_t^2 \tilde{\mathbf{A}}^{3\alpha} \overline{\partial}\partial_t Q \, \mathrm{d}S \, \mathrm{d}t \\
&+ 3\int_0^T \int_\Gamma \overline{\partial}\partial_t^3 v_\alpha \overline{\partial}\partial_t \tilde{\mathbf{A}}^{3\alpha} \partial_t^2 Q \, \mathrm{d}S \, \mathrm{d}t + 3\int_0^T \int_\Gamma \overline{\partial}\partial_t^3 v_\alpha \partial_t \tilde{\mathbf{A}}^{3\alpha} \overline{\partial}\partial_t^2 Q \, \mathrm{d}S \, \mathrm{d}t \\
&+ \int_0^T \int_\Gamma \overline{\partial}\partial_t^3 v_\alpha \overline{\partial} \tilde{\mathbf{A}}^{3\alpha} \partial_t^3 Q \, \mathrm{d}S \, \mathrm{d}t \\
=: &J_0 + J_1 + \cdots + J_7.
\end{aligned} \tag{4.2.131}
$$

Since we have $H^{1.5}(\Omega)$ regularity for $\partial_t^3 v$ and $H^1(\Omega)$ regularity for $\partial_t^3 Q$, the top order terms contributed by J_1 to J_7 can all be directly controlled by the trace lemma. In the end, we have

$$
J_1 + \cdots + J_7 \lesssim \int_0^T \mathcal{P}. \tag{4.2.132}
$$

By plugging the boundary condition

$$
\tilde{\mathbf{A}}^{3\alpha} Q = -\sigma \sqrt{g} (\triangle_g \eta \cdot \tilde{n}) \tilde{n}^\alpha + \kappa \left((1 - \overline{\triangle})(v \cdot \tilde{n}) \right) \tilde{n}^\alpha
$$

in J_0, we obtain

$$\frac{1}{\sigma}J_0 = \int_0^T\int_\Gamma \overline{\partial}\partial_t^3(\sqrt{g}\triangle_g\eta\cdot\tilde{n}\tilde{n}^\alpha)\overline{\partial}\partial_t^3 v_\alpha \,\mathrm{d}S\,\mathrm{d}t - \frac{\kappa}{\sigma}\int_0^T\int_\Gamma \overline{\partial}\partial_t^3[\Big((1-\overline{\triangle})(v\cdot\tilde{n})\Big)\tilde{n}^\alpha]\overline{\partial}\partial_t^3 v_\alpha\,\mathrm{d}S\,\mathrm{d}t \tag{4.2.133}$$

For the second term, integrating $\overline{\partial}$ by parts, it contributes to the energy term

$$\frac{\kappa}{\sigma}\int_0^T\int_\Gamma |\partial_t^3 v\cdot\tilde{n}|_2^2\,\mathrm{d}S\,\mathrm{d}t, \tag{4.2.134}$$

and some error terms. Here, the most difficult error term reads

$$\frac{\kappa}{\sigma}\int_0^T\int_\Gamma (\overline{\partial}^2\partial_t^3 v\cdot\tilde{n})(v\cdot\partial_t^3\overline{\partial}^2\tilde{n})\,\mathrm{d}S\,\mathrm{d}t \tag{4.2.135}$$

which can be treated as follows:

$$\begin{aligned}
&\frac{\kappa}{\sigma}\int_0^T\int_\Gamma (\overline{\partial}^2\partial_t^3 v\cdot\tilde{n})(v\cdot\partial_t^3\overline{\partial}^2\tilde{n})\,\mathrm{d}S\,\mathrm{d}t \overset{L}{=} \frac{\kappa}{\sigma}\int_0^T\int_\Gamma (\overline{\partial}^2\partial_t^3 v\cdot\tilde{n})(v\cdot\overline{\partial}^3\partial_t^2\tilde{v}\cdot\tilde{n})\,\mathrm{d}S\,\mathrm{d}t\\
\le&\int_0^T P(|\overline{\partial}\tilde{\eta}|_{L^\infty(\Gamma)},|v|_{L^\infty(\Gamma)})|\sqrt{\kappa}\overline{\partial}^2\partial_t^3 v|_0|\sqrt{\kappa}\overline{\partial}^3\partial_t^2 v\cdot\tilde{n}|_0\\
\lesssim&\int_0^T \|\sqrt{\kappa}\partial_t^3 v\|_{2.5}^2 + \Big(\int_0^T\|\sqrt{\kappa}\partial_t^2 v\|_{3.5}^2\Big)^2 + \sup_t P(|\overline{\partial}\tilde{\eta}|_{L^\infty(\Gamma)},|v|_{L^\infty(\Gamma)})\\
\le& E_\kappa^{(3)} + (E_\kappa^{(3)})^2 + \mathcal{P}_0 + \mathcal{P}\int_0^T\mathcal{P}.
\end{aligned}$$

The first term in (4.2.133) is treated analogous to the first term in (4.2.95). The main term we need to study in this case reads

$$\begin{aligned}
&\int_0^T\int_\Gamma (\overline{\partial}\partial_t^3(\sqrt{g}\triangle_g\eta^\alpha))(\overline{\partial}\partial_t^3 v)\,\mathrm{d}S\,\mathrm{d}t\\
=&\int_0^T\int_\Gamma \overline{\partial}\partial_t^2\overline{\partial}_i\left(\sqrt{g}g^{ij}\Pi_\lambda^\alpha\overline{\partial}_j v^\lambda + \sqrt{g}(g^{ij}g^{kl}-g^{lj}g^{ik})\overline{\partial}_j\eta^\alpha\overline{\partial}_k\eta^\lambda\overline{\partial}_l v_\lambda\right)\overline{\partial}\partial_t^3 v^\lambda\,\mathrm{d}S\,\mathrm{d}t
\end{aligned}$$

Integrating $\overline{\partial}_i$ by parts, we get

$$
\begin{aligned}
J_{00} \overset{\overline{\partial}_i}{=} & -\int_0^T \int_\Gamma \sqrt{g} g^{ij} \Pi_\lambda^\alpha \overline{\partial}\partial_t^2 \overline{\partial}_j v^\lambda \overline{\partial}\partial_t^3 \overline{\partial}_i v_\alpha \, \mathrm{d}S \, \mathrm{d}t \\
& -\int_0^T \int_\Gamma \sqrt{g}(g^{ij}g^{kl} - g^{lj}g^{ik})\overline{\partial}_j \eta^\alpha \overline{\partial}_k \eta^\lambda \overline{\partial}\overline{\partial}_l \partial_t^2 v_\lambda \partial_t^3 \overline{\partial}_i \overline{\partial} v_\alpha \, \mathrm{d}S \, \mathrm{d}t + \mathcal{R}_0 \\
& =: J_{01} + J_{02} + \mathcal{R}_0,
\end{aligned}
\tag{4.2.136}
$$

where $\mathcal{R}_0$ consists terms that can be treated in the same way as in $I_{03}, \cdots, I_{08}$ in (4.2.102).

In J_{01}, we can integrate ∂_t by parts and mimic the proof of (4.2.103) to get

$$
J_{01} + \left|\overline{\partial}^2(\Pi \partial_t^2 v)\right|_0^2 \lesssim \varepsilon \left(\left|\overline{\partial}(\Pi \overline{\partial}\partial_t^2 v)\right|_0^2 + \|\partial_t^2 v\|_{2.5}^2 \right) + \mathcal{P}_0 + \mathcal{P}\int_0^T \mathcal{P} \, \mathrm{d}t \tag{4.2.137}
$$

J_{02} can also be controlled similarly as I_{02}. We find that the integrand is zero if $l = i$. So it suffices to compute the case $(l, i) = (1, 2)$ and $(2, 1)$. Similarly we get

$$
J_{02} = \int_\Gamma \frac{1}{\sqrt{g}} \det \begin{bmatrix} \overline{\partial}_1 \eta_\mu \overline{\partial}_1 \partial_t^2 \overline{\partial} v^\mu & \overline{\partial}_1 \eta_\mu \overline{\partial}_2 \partial_t^2 \overline{\partial} v^\mu \\ \overline{\partial}_2 \eta_\mu \overline{\partial}_1 \partial_t^2 \overline{\partial} v^\mu & \overline{\partial}_2 \eta_\mu \overline{\partial}_2 \partial_t^2 \overline{\partial} v^\mu \end{bmatrix} \mathrm{d}S \Bigg|_0^T + \int_0^T \mathcal{P} + \mathcal{R}. \tag{4.2.138}
$$

The main term can be computed as follows

$$
\begin{aligned}
& \int_\Gamma \frac{1}{\sqrt{g}} \det \begin{bmatrix} \overline{\partial}_1 \eta_\mu \overline{\partial}_1 \partial_t^2 \overline{\partial} v^\mu & \overline{\partial}_1 \eta_\mu \overline{\partial}_2 \partial_t^2 \overline{\partial} v^\mu \\ \overline{\partial}_2 \eta_\mu \overline{\partial}_1 \partial_t^2 \overline{\partial} v^\mu & \overline{\partial}_2 \eta_\mu \overline{\partial}_2 \partial_t^2 \overline{\partial} v^\mu \end{bmatrix} \mathrm{d}S \\
= & \int_\Gamma \frac{1}{\sqrt{g}} \left(\overline{\partial}_1 \eta_\mu \overline{\partial}_1 \partial_t^2 \overline{\partial} v^\mu \overline{\partial}_2 \eta_\mu \overline{\partial}_2 \partial_t^2 \overline{\partial} v^\mu - \overline{\partial}_1 \eta_\mu \overline{\partial}_2 \partial_t^2 \overline{\partial} v^\mu \overline{\partial}_2 \eta_\mu \overline{\partial}_1 \partial_t^2 \overline{\partial} v^\mu \right) \\
\overset{\overline{\partial}_1, \overline{\partial}_2}{=} & \int_\Gamma Q_{\mu\lambda}^i (\overline{\partial}\eta, \overline{\partial}^2 \eta) \overline{\partial}\partial_t^2 v^\mu \overline{\partial}_i \overline{\partial}\partial_t^2 v^\lambda \, \mathrm{d}S \\
\lesssim & \varepsilon \|\partial_t^2 v\|_{2.5}^2 + \mathcal{P}_0 + \int_0^T \mathcal{P},
\end{aligned}
\tag{4.2.139}
$$

and thus we get the control of J_{02}

$$
J_{02} \lesssim \varepsilon \left(\|\partial_t^2 v\|_{2.5}^2 \right) + \mathcal{P}_0 + \mathcal{P}\int_0^T \mathcal{P}. \tag{4.2.140}
$$

Combining (4.2.126)-(4.2.137) and (4.2.140), we get the $\overline{\partial}\partial_t^3$-tangential estimates as follows

$$\left\|\overline{\partial}\partial_t^3 v\right\|_0^2 + \left\|\overline{\partial}\partial_t^3 (b_0\cdot\partial)\eta\right\|_0^2 + \left|\overline{\partial}\left(\Pi\overline{\partial}\partial_t^2 v\right)\right|_0^2 \lesssim E_\kappa^{(3)} + (E_\kappa^{(3)})^2 + \varepsilon\|\partial_t^2 v\|_{2.5}^2 + \mathcal{P}_0 + \mathcal{P}\int_0^T \mathcal{P}. \tag{4.2.141}$$

The $\overline{\partial}^2\partial_t^2$, $\overline{\partial}^3\partial_t$ and $\overline{\partial}^3(b_0\cdot\partial)$-tangential energies: The control of the other tangential energies that involving at least one $\overline{\partial}$ is follows from the arguments above by replacing $\overline{\partial}\partial_t^3$ to the corresponding derivatives. Hence, we shall omit the details and only illustrate the major differences. First, we mention that the derivatives $\overline{\partial}^3\partial_t$ and $\overline{\partial}^3(b_0\cdot\partial)$ behave the same since both v and $(b_0\cdot\partial)\eta$ are of the same interior regularity. Second, one needs to pay attention to the terms that analogous to the error term generated by (4.2.133) during the construction of the energy term. In particular, we need to study the top order error term analogous to (4.2.135). Setting $\mathfrak{D} = \partial_t, \overline{\partial}$ or $(b_0\cdot\partial)$ and we consider

$$\frac{\kappa}{\sigma}\int_0^T\int_\Gamma (\overline{\partial}^3\mathfrak{D}^2 v\cdot\tilde{n})(v\cdot\overline{\partial}^3\mathfrak{D}^2\tilde{n})\,\mathrm{d}S\,\mathrm{d}t. \tag{4.2.142}$$

When $\mathfrak{D}^2 = \partial_t^2$ then (4.2.142) is treated similar to (4.2.135). This is due to that

$$\overline{\partial}^3\partial_t^2\tilde{n} = Q(\overline{\partial}\tilde{\eta})\overline{\partial}^4\partial_t\tilde{v}\cdot\tilde{n} + \text{lower-order terms},$$

and $\int_0^T |\partial_t^4 v|_{1.5}^2$ is included in $E_\kappa^{(3)}$. In the end, we obtain

$$\frac{\kappa}{\sigma}\int_0^T\int_\Gamma (\overline{\partial}^3\partial_t^2 v\cdot\tilde{n})(v\cdot\overline{\partial}^3\partial_t^2\tilde{n})\,\mathrm{d}S\,\mathrm{d}t \le E_\kappa^{(3)} + (E_\kappa^{(3)})^2 + \mathcal{P}_0 + \mathcal{P}\int_0^T \mathcal{P}.$$

On the other hand, when $\mathfrak{D}^2 = \overline{\partial}\partial_t, \overline{\partial}(b_0\cdot\partial)$, then using the fact that $\mathfrak{D}\tilde{n} = Q(\overline{\partial}\tilde{\eta})\mathfrak{D}\overline{\partial}\tilde{\eta}\cdot\tilde{n}$, we have

$$\frac{\kappa}{\sigma}\int_0^T\int_\Gamma (\overline{\partial}^4\partial_t v\cdot\tilde{n})(v\cdot\overline{\partial}^4\partial_t\tilde{n})\,\mathrm{d}S\,\mathrm{d}t \overset{L}{=} \frac{\kappa}{\sigma}\int_0^T\int_\Gamma (\overline{\partial}^4\partial_t v\cdot\tilde{n})(v\cdot\overline{\partial}^5 v\cdot\tilde{n})\,\mathrm{d}S\,\mathrm{d}t, \tag{4.2.143}$$

$$\frac{\kappa}{\sigma}\int_0^T\int_\Gamma (\overline{\partial}^4(b_0\cdot\partial)v\cdot\tilde{n})(v\cdot\overline{\partial}^4(b_0\cdot\partial)\tilde{n})\,\mathrm{d}S\,\mathrm{d}t \tag{4.2.144}$$

$$\overset{L}{=}\frac{\kappa}{\sigma}\int_0^T\int_\Gamma (\overline{\partial}^4(b_0\cdot\partial)v\cdot\tilde{n})(v\cdot\overline{\partial}^5(b_0\cdot\partial)\tilde{\eta}\cdot\tilde{n})\,\mathrm{d}S\,\mathrm{d}t.$$

The terms on the RHS requires $\int_0^T|\sqrt{\kappa}v|_5^2$ and $\int_0^T|\sqrt{\kappa}(b_0\cdot\partial)\eta|_5^2$, respectively, to control. However, owing to (4.2.23) and (4.2.25), both of them can be controlled by $\mathcal{M}_0+C(\varepsilon)E_\kappa(T)+\mathcal{P}\int_0^T\mathcal{P}$. Hence,

$$\begin{aligned}&\left\|\overline{\partial}^2\partial_t^2 v\right\|_0^2+\left\|\overline{\partial}^2\partial_t^2(b_0\cdot\partial)\eta\right\|_0^2+\left|\overline{\partial}\left(\Pi\overline{\partial}^2\partial_t v\right)\right|_0^2\\&+\left\|\overline{\partial}^3\partial_t v\right\|_0^2+\left\|\overline{\partial}^3\partial_t(b_0\cdot\partial)\eta\right\|_0^2+\left|\overline{\partial}\left(\Pi\overline{\partial}^3 v\right)\right|_0^2\\&+\left\|\overline{\partial}^3(b_0\cdot\partial)v\right\|_0^2+\left\|\overline{\partial}^3(b_0\cdot\partial)^2\eta\right\|_0^2+\left|\overline{\partial}\left(\Pi\overline{\partial}^3(b_0\cdot\partial)\eta\right)\right|_0^2\\&\lesssim E_\kappa^{(3)}+(E_\kappa^{(3)})^2+\varepsilon E_\kappa(T)+\mathcal{M}_0+\mathcal{P}\int_0^T\mathcal{P}.\end{aligned} \tag{4.2.145}$$

Notice that the RHS relies on $\mathcal{M}_0$, which is given in Lemma 4.2.9. In Section 4.2.5, in fact, we are able to control $\mathcal{M}_0$ by $\mathcal{C}(\|v_0\|_{4.5},\|b_0\|_{4.5},|v_0|_5)$.

4.2.4 Estimates for the higher order weighted interior norms

It remains to control $E_\kappa^{(3)}(T)$ in order to complete the proof of Proposition 4.2.4.

4.2.4.1 Full time derivatives

We shall first study the first two terms, i.e.,

$$\int_0^T\left(\left\|\sqrt{\kappa}\partial_t^4 v\right\|_{1.5}^2+\left\|\sqrt{\kappa}\partial_t^4(b_0\cdot\partial)\eta\right\|_{1.5}^2\right)\mathrm{d}t=K_1+K_2.$$

These terms appear to be the most difficult ones to control. In particular, they yield error terms that contribute to the top order and can only be controlled in $L^2([0,T])$.

The goal is to show:

$$K_1 + K_2 \le \mathcal{P}_0 + C(\varepsilon)E_\kappa(T) + \mathcal{P}\int_0^T \mathcal{P}. \tag{4.2.146}$$

The control of K_1, K_2 relies on the div-curl estimate

$$K_1 \le \int_0^T \left(\left\| \sqrt{\kappa}\text{div}\, \partial_t^4 v \right\|_{0.5}^2 + \left\| \sqrt{\kappa}\text{curl}\, \partial_t^4 v \right\|_{0.5}^2 + \left| \sqrt{\kappa}\partial_t^4 v^3 \right|_1^2 \right) \text{d}t =: K_{11} + K_{12} + K_{13}, \tag{4.2.147}$$

$$K_2 \le \int_0^T \left(\left\| \sqrt{\kappa}\text{div}\, \partial_t^4 (b_0\cdot\partial)\eta \right\|_{0.5}^2 + \left\| \sqrt{\kappa}\text{curl}\, \partial_t^4 (b_0\cdot\partial)\eta \right\|_{0.5}^2 + \left| \sqrt{\kappa}\partial_t^4 (b_0\cdot\partial)\eta^3 \right|_1^2 \right) \text{d}t$$

$$=: K_{21} + K_{22} + K_{23}. \tag{4.2.148}$$

For K_{13}, there holds $K_{13} \le \underbrace{\int_0^T |\sqrt{\kappa}\partial_t^4 v\cdot\tilde{n}|_1^2}_{\le E_\kappa^{(2)}} + \int_0^T |\sqrt{\kappa}\partial_t^4 v\cdot(N-\tilde{n})|_1^2$, and for the error term, we have $\int_0^T |\sqrt{\kappa}\partial_t^4 v\cdot(N-\tilde{n})|_1^2 \lesssim \int_0^T \|\sqrt{\kappa}\partial_t^4 v\|_{1.5}^2 \cdot |N-\tilde{n}|_{1+}^2 \lesssim \varepsilon^2\mathcal{P}$.To control K_{23}, since $\partial_t\eta = v$ we have $\int_0^T \left|\sqrt{\kappa}\partial_t^3(b_0\cdot\partial)v^3\right|_1^2$ and so it suffices to control $\int_0^T \left|\sqrt{\kappa}\partial_t^3 v^3\right|_2^2$. This term can then be treated similar to K_{13}.

For K_{11}, we have

$$K_{11} \le \int_0^T (\|\sqrt{\kappa}\text{div}_{\tilde{A}}\partial_t^4 v\|_{0.5}^2 + \|\sqrt{\kappa}\text{div}_{A-\tilde{A}}\partial_t^4 v\|_{0.5}^2). \tag{4.2.149}$$

Since $\|A-\tilde{A}\|_{1.5+}^2 \le \kappa P(\|\eta\|_{3.5})$, the error term can be controlled by

$$\int_0^T \|\sqrt{\kappa}\text{div}_{A-\tilde{A}}\partial_t^4 v\|_{0.5}^2 \lesssim \int_0^T \|A-\tilde{A}\|_{1.5+}^2 \|\sqrt{\kappa}\partial\partial_t^4 v\|_{0.5}^2, \tag{4.2.150}$$

which can be controlled by the RHS of (4.2.146) when κ is small. For the first term, since $\text{div}_{\tilde{A}} v = 0$ we have

$$\int_0^T \|\sqrt{\kappa}\text{div}_{\tilde{A}}\partial_t^4 v\|_{0.5}^2 \overset{L}{=} \int_0^T \|\sqrt{\kappa}(\partial_t\tilde{A}^{\mu\alpha})\partial_\mu\partial_t^3 v_\alpha\|_{0.5}^2 + \|\sqrt{\kappa}(\partial_t^4\tilde{A}^{\mu\alpha})\partial_\mu v_\alpha\|_{0.5}^2. \tag{4.2.151}$$

It is not hard to see that that $\int_0^T \|\sqrt{\kappa}\partial_t\tilde{A}\partial\partial_t^3 v\|_{0.5}^2 \le \int_0^T \mathcal{P}$ as $\partial_t^3 v \in H^{1.5}(\Omega)$ a priori. In addition, since $\partial_t \tilde{A}^{\mu\alpha} = Q(\partial\tilde{\eta})$, we obtain

$$\int_0^T \|\sqrt{\kappa}\partial_t^4 \tilde{A}\partial v\|_{0.5}^2 \stackrel{L}{=} \int_0^T \|\sqrt{\kappa}Q(\partial\tilde{\eta})\partial\partial_t^3 v\partial v\|_{0.5}^2 \leq \int_0^T \mathcal{P}, \tag{4.2.152}$$

The control of K_{21} is a bit more involved. We cannot commute ∂_t^4 to (4.2.57) as this would yield $\text{div}_{\partial_t^5 \tilde{A}}(b_0 \cdot \partial)\eta$ on the RHS which cannot be controlled. However, by writing $\text{div}\,\partial_t^4(b_0 \cdot \partial)\eta = \text{div}\,\partial_t^3(b_0 \cdot \partial)v$ and then we have

$$\int_0^T \|\sqrt{\kappa}\text{div}\,\partial_t^3(b_0 \cdot \partial)v\|_{0.5}^2 \leq \int_0^T \|\sqrt{\kappa}\text{div}_{\tilde{A}}\partial_t^3(b_0 \cdot \partial)v\|_{0.5}^2 + \int_0^T \|\sqrt{\kappa}\text{div}_{\tilde{A}-a}\partial_t^3(b_0 \cdot \partial)v\|_{0.5}^2. \tag{4.2.153}$$

The second term on the RHS is similar with (4.2.150). The first term can be similarly controlled by commuting $\text{div}_{\tilde{A}}$ with $\partial_t^3(b_0 \cdot \partial)$.

Bound for K_{12} and K_{22}: We would like to state the following strategy that will come in handy when dealing with the leading order terms in K_{12} and K_{22}. Let X be the term such that $\int_0^T \|\sqrt{\kappa}X\|_{0.5}^2$ is part of $E_\kappa^{(3)}$ and Y be a lower order term such that $\|Y\|_{1.5+}^2$ is controlled by $E_\kappa^{(1)}$. Then

$$\begin{aligned}\int_0^T\int_0^t \|\sqrt{\kappa}XY\|_{0.5}^2\,\mathrm{d}t \leq& T\int_0^T \|\sqrt{\kappa}XY\|_{0.5}^2 \lesssim T\sup_t \|Y\|_{1.5+}^2 \int_0^T \|\sqrt{\kappa}X\|_{0.5}^2\\ \leq& \frac{\varepsilon}{2}\Big(\int_0^T \|\sqrt{\kappa}X\|_{0.5}^2\Big)^2 + \frac{T^2}{2\varepsilon}\sup_t \|Y\|_{1.5+}^4,\end{aligned} \tag{4.2.154}$$

which is bounded by the RHS of (4.2.13) if T is sufficiently small.

K_{12} and K_{22} will be considered together via studying the evolution equation verified by $\text{curl}\,\partial_t^4 v$ and $\text{curl}\,\partial_t^4(b_0 \cdot \partial)\eta$. But this cannot be derived by taking ∂_t^4 to (4.2.65) as this yields $\text{curl}_{\partial_t^5 \tilde{\mathbf{A}}} v$ in the source term which cannot be controlled. Instead, we commute $\partial_t^4 \text{curl}_{\tilde{A}}$ to the equation $\partial_t v + (b_0 \cdot \partial)^2\eta = \nabla_{\tilde{A}} Q$ and get

$$\partial_t^4 \text{curl}_{\tilde{A}}\partial_t v + \partial_t^4 \text{curl}_{\tilde{A}}((b_0 \cdot \partial)^2\eta) = 0.$$

This yields the following evolution equation:

$$\partial_t \mathrm{curl}_{\tilde{A}} \partial_t^4 v + \mathrm{curl}_{\tilde{A}}((b_0 \cdot \partial)^2 \partial_t^4 \eta) = -\partial_t([\partial_t^3, \mathrm{curl}_{\tilde{A}}]\partial_t v) - [\partial_t^4, \mathrm{curl}_{\tilde{A}}](b_0 \cdot \partial)^2 \eta := f, \quad (4.2.155)$$

and, after expansion, the source term f becomes:

$$f = \partial_t \left(\sum_{1 \le j \le 3} \epsilon_{\alpha\beta\gamma} (\partial_t^j \tilde{A}^{\mu\beta}) \partial_\mu \partial_t^{4-j} v^\gamma \right) + \sum_{1 \le j \le 4} \epsilon_{\alpha\beta\gamma} (\partial_t^j \tilde{A}^{\mu\beta}) \partial_\mu (b_0 \cdot \partial)^2 \partial_t^{4-j} \eta^\gamma. \quad (4.2.156)$$

Then we integrate $(b_0 \cdot \partial)$ by parts in $\int_\Omega \kappa \Big(\mathrm{curl}_{\tilde{A}}((b_0 \cdot \partial)^2 \partial_t^4 \eta) \Big) \overline{\partial} (\mathrm{curl}_{\tilde{A}} \partial_t^4 v)$, then integrate $\overline{\partial}^{\frac{1}{2}}$ by parts, and integrate in time one more time, we get

$$\begin{aligned}
&\frac{1}{2} \int_0^T \|\sqrt{\kappa} \mathrm{curl}\,_{\tilde{A}} \partial_t^4 v\|_{0.5}^2 + \frac{1}{2} \int_0^T \|\sqrt{\kappa} \mathrm{curl}\,_{\tilde{A}} \partial_t^4 (b_0 \cdot \partial) \eta\|_{0.5}^2 \\
\lesssim & \int_0^T \mathcal{P}_0 + \int_0^T \int_0^t \|\sqrt{\kappa} f\|_{0.5} \|\sqrt{\kappa} \mathrm{curl}\,_{\tilde{A}} \partial_t^4 v\|_{0.5} \, \mathrm{d}t \\
&+ \int_0^T \int_0^t \|\sqrt{\kappa} [\mathrm{curl}\,_{\tilde{A}}, (b_0 \cdot \partial)] (b_0 \cdot \partial) \partial_t^4 \eta\|_{0.5} \|\sqrt{\kappa} \mathrm{curl}\,_{\tilde{A}} \partial_t^4 v\|_{0.5} \, \mathrm{d}t \\
&+ \int_0^T \int_0^t \|\sqrt{\kappa} [\mathrm{curl}\,_{\tilde{A}}, (b_0 \cdot \partial)] \partial_t^4 v\|_{0.5} \|\sqrt{\kappa} \mathrm{curl}\,_{\tilde{A}} (b_0 \cdot \partial) \partial_t^4 \eta\|_{0.5} \, \mathrm{d}t \\
&+ \int_0^T \int_0^t \|\sqrt{\kappa} \mathrm{curl}\,_{\partial_t \tilde{A}} \partial_t^4 (b_0 \cdot \partial) \eta\|_{0.5} \|\sqrt{\kappa} \mathrm{curl}\,_{\tilde{A}} (b_0 \cdot \partial) \partial_t^4 \eta\|_{0.5} \, \mathrm{d}t. \qquad (4.2.157)
\end{aligned}$$

We have

$$\begin{aligned}
&\int_0^T \int_0^t \|\sqrt{\kappa} [\mathrm{curl}\,_{\tilde{A}}, (b_0 \cdot \partial)] (b_0 \cdot \partial) \partial_t^4 \eta\|_{0.5}^2 \, \mathrm{d}t \\
\lesssim & \int_0^T \int_0^t \|\sqrt{\kappa} \epsilon_{\alpha\beta\gamma} \tilde{A}^{\mu\beta} (\partial_\mu b_0^\nu) (\partial_\nu (b_0 \cdot \partial) \partial_t^4 \eta^\gamma\|_{0.5}^2 \, \mathrm{d}t \\
&+ \int_0^T \int_0^t \|\sqrt{\kappa} \epsilon_{\alpha\beta\gamma} (b_0^\nu \partial_\nu \tilde{A}^{\mu\beta}) (\partial_\mu (b_0 \cdot \partial) \partial_t^4 \eta^\gamma\|_{0.5}^2 \, \mathrm{d}t,
\end{aligned}$$

which can be controlled by the RHS of (4.2.146) by adapting (4.2.154). The third and forth term are treated analogously. For $\int_0^T \int_0^t \|\sqrt{\kappa} f\|_{0.5}^2 \, \mathrm{d}t$, invoking (4.2.156), we need to consider

$$i = \sum_{1\le j\le 3} \int_0^T \int_0^t \|\sqrt{\kappa}\partial_t \big(\epsilon_{\alpha\beta\gamma}(\partial_t^j \tilde{A}^{\mu\beta})\partial_\mu \partial_t^{4-j} v^\gamma\big)\|_{0.5}^2 \, \mathrm{d}t, \tag{4.2.158}$$

$$ii = \sum_{1\le j\le 4} \int_0^T \int_0^t \|\sqrt{\kappa}\epsilon_{\alpha\beta\gamma}(\partial_t^j \tilde{A}^{\mu\beta})\partial_\mu (b_0\cdot\partial)^2 \partial_t^{4-j} \eta^\gamma\|_{0.5}^2 \, \mathrm{d}t. \tag{4.2.159}$$

Here, $i \stackrel{L}{=} \int_0^T \int_0^t \|\sqrt{\kappa}(\partial_t \tilde{A})(\partial\partial_t^4 v)\|_{0.5}^2$ is controlled by (4.2.154). Moreover,

$$ii \stackrel{L}{=} \int_0^T \int_0^t \|\sqrt{\kappa}\epsilon_{\alpha\beta\gamma}(\partial_t \tilde{A}^{\mu\beta})\partial_\mu (b_0\cdot\partial)^2 \partial_t^3 \eta^\gamma\|_{0.5}^2 \tag{4.2.160}$$

$$\lesssim \int_0^T \int_0^t \|\sqrt{\kappa}(\partial_t \tilde{A})\partial\partial_t^3[(b_0\cdot\partial)^2\eta]\|_{0.5}^2 \, \mathrm{d}t \le \int_0^T \mathcal{P}.$$

This concludes the control of $K_1 + K_2$.

Remark 4.2.14. There is an alternative way to control the last integral in (4.2.161). We may use the equation to replace $(b_0\cdot\partial)^2\eta$ by $\partial_t v + \nabla_{\tilde{A}} q$, and this allow us to control this integral without using $\int_0^T \|\sqrt{\kappa}\partial_t^3(b_0\cdot\partial)\eta\|_{2.5}^2$. In fact, one can show

$$\int_0^T \int_0^t \|\sqrt{\kappa}\partial_t^3 Q\|_{2.5}^2 \, \mathrm{d}t \le \mathcal{P}$$

by employing the elliptic estimate we used in Section 4.2.2.1, and so $\int_0^T \int_0^t \|\sqrt{\kappa}(\partial_t \tilde{A})\partial\partial_t^3[(b_0\cdot\partial)^2\eta]\|_{0.5}^2 \, \mathrm{d}t \le \int_0^T \mathcal{P}$.

4.2.4.2 Mixed space-time derivatives

The treatment for the remaining terms of $E_\kappa^{(3)}$ is parallel. We shall consider

$$\int_0^T \left(\left\|\sqrt{\kappa}\partial_t^k v\right\|_{5.5-k}^2 + \left\|\sqrt{\kappa}\partial_t^k (b_0\cdot\partial)\eta\right\|_{5.5-k}^2\right) \mathrm{d}t, \quad k = 1, 2, 3.$$

First, the boundary normal trace contributed by the time derivative(s) of $(b_0\cdot\partial)\eta$ reads $\int_0^T |\sqrt{\kappa}\partial_t^k (b_0\cdot\partial)\eta|_{5-k}$, $k = 1, 2, 3$. Generally speaking, for each fixed k, the control of the above term requires that of $\int_0^T |\sqrt{\kappa}\partial_t^{k-1} v|_{6-k}$, and this process stops when $k = 1$. In particular, for each fixed $k = 2, 3$, we write $\int_0^T |\sqrt{\kappa}\partial_t^k(b_0\cdot\partial)\eta|_{5-k}^2$ as $\int_0^T |\partial_t^{k-1}(b_0\cdot\partial)v|_{5-k}^2$, which can then be controlled together with $\int_0^T |\partial_t^i v|_{5-i}^2$ with $i=1,2$. On the other hand, when $k=1$, the control of $\int_0^T |\sqrt{\kappa}\partial_t(b_0\cdot\partial)\eta|_4$ requires

$$\int_0^T |\sqrt{\kappa}(b_0\cdot\partial)v|_4^2 \lesssim P(\|b_0\|_{4.5})\int_0^T |\sqrt{\kappa}v|_5^2,$$

where, in view of (4.2.23), we have $\int_0^T |\sqrt{\kappa}v|_5^2 \le \mathcal{M}_0 + C(\varepsilon)E_\kappa(T) + \mathcal{P}\int_0^T \mathcal{P}$.

Second, the control of the analogous terms of ii (defined in (4.2.159)) for $k=1,2,3$ requires a similar analysis as above. For each fixed k, we need to investigate

$$ii' = \sum_{1\le j\le k}\int_0^T\int_0^t \|\sqrt{\kappa}\epsilon_{\alpha\beta\gamma}(\partial_t^j\tilde{A}^{\mu\beta})\partial_\mu(b_0\cdot\partial)^2\partial_t^{k-j}\eta^\gamma\|_{4.5-k}^2\,\mathrm{d}t. \tag{4.2.161}$$

Again, it suffices to consider the most difficult term contributed by setting $j=1$, i.e.,

$$ii' = \int_0^T\int_0^t \|\sqrt{\kappa}\epsilon_{\alpha\beta\gamma}(\partial_t\tilde{A}^{\mu\beta})\partial_\mu(b_0\cdot\partial)^2\partial_t^{k-1}\eta^\gamma\|_{4.5-k}^2\,\mathrm{d}t \tag{4.2.162}$$

$$\lesssim \int_0^T\int_0^t P(\|v\|_{4.5},\|b_0\|_{4.5},\|\eta\|_{4.5})\|\sqrt{\kappa}\partial^3\partial_t^{k-1}\eta\|_{4.5-k}^2\,\mathrm{d}t. \tag{4.2.163}$$

In (4.2.163), it can be seen that when $k=2,3$, $\int_0^T\int_0^t \|\sqrt{\kappa}\partial^3\partial_t^{k-1}\eta\|_{4.5-k}\,\mathrm{d}t$ is bounded by $\int_0^T\int_0^t \|\sqrt{\kappa}\partial^3 v\|_{2.5}^2\,\mathrm{d}t$ and $\int_0^T\int_0^t \|\sqrt{\kappa}\partial^3\partial_t v\|_{1.5}^2\,\mathrm{d}t$, respectively. Moreover, when $k=1$, we need to consider (4.2.162) instead. The strategy here is to replace $(b_0\cdot\partial)^2\eta$ by $\partial_t v + \nabla_{\tilde{A}}Q$, and so

$$ii' = \int_0^T\int_0^t \|\sqrt{\kappa}\epsilon_{\alpha\beta\gamma}(\partial_t\tilde{A}^{\mu\beta})\partial_\mu\partial_t v^\gamma\|_{3.5}^2\,\mathrm{d}t + \int_0^T\int_0^t \|\sqrt{\kappa}\epsilon_{\alpha\beta\gamma}(\partial_t\tilde{A}^{\mu\beta})\partial_\mu\nabla_{\tilde{A}}^\gamma Q\|_{3.5}^2\,\mathrm{d}t, \tag{4.2.164}$$

where the first term is bounded by the RHS of (4.2.146) owing to (4.2.154). For the second term, since $v\in H^{4.5}(\Omega)$, so it suffices to consider the case when all derivatives land on $\nabla_{\tilde{A}}Q$, whose control requires that of

$$\int_0^T\int_0^t \|\sqrt{\kappa}\nabla_{\tilde{A}}Q\|_{4.5}^2\,\mathrm{d}t \tag{4.2.165}$$

after adapting (4.2.154). Actually, we have a slightly stronger bound by removing one time integral, i.e., $\int_0^T \|\sqrt{\kappa}\nabla_{\tilde{A}} Q\|_{4.5}^2$. By the div-curl estimate, one has

$$\int_0^T \|\sqrt{\kappa}\nabla_{\tilde{A}} Q\|_{4.5}^2 \lesssim \int_0^T \left(\|\sqrt{\kappa}\text{div}\,\nabla_{\tilde{A}} Q\|_{3.5}^2 + \|\sqrt{\kappa}\text{curl}\,\nabla_{\tilde{A}} Q\|_{3.5}^2 + |\sqrt{\kappa}N\cdot\nabla_{\tilde{A}} q\big|_3^2 + |\sqrt{\kappa}q|_0^2\right).$$

Here,

$$\int_0^T \|\sqrt{\kappa}\text{div}\,\nabla_{\tilde{A}} Q\|_{3.5}^2 \lesssim \int_0^T \|\sqrt{\kappa}\triangle_{\tilde{A}} Q\|_{3.5}^2 + \int_0^T \|\sqrt{\kappa}\text{div}\,_{\tilde{A}-\delta}\nabla_{\tilde{A}} Q\|_{3.5}^2, \tag{4.2.166}$$

and by (4.2.16), (4.2.26), (4.2.154), we have

$$\int_0^T \|\sqrt{\kappa}\text{div}\,_{\tilde{A}-\delta}\nabla_{\tilde{A}} Q\|_{3.5}^2 \lesssim \varepsilon\int_0^T \|\sqrt{\kappa}\nabla_{\tilde{A}} Q\|_{4.5}^2 + \int_0^T \mathcal{P}.$$

Similarly, because curl $_{\tilde{A}}\nabla_{\tilde{A}} Q = 0$, we have

$$\int_0^T \|\sqrt{\kappa}\text{curl}\,\nabla_{\tilde{A}} Q\|_{3.5}^2 \lesssim \varepsilon\int_0^T \|\sqrt{\kappa}\nabla_{\tilde{A}} Q\|_{4.5}^2 + \int_0^T \mathcal{P}.$$

Moreover, invoking (4.2.18), (4.2.154) and the trace lemma, then

$$\begin{aligned}\int_0^T |\sqrt{\kappa}N\cdot\nabla_{\tilde{A}} Q|_3^2 &\lesssim \int_0^T |\sqrt{\kappa}\tilde{n}\cdot\nabla_{\tilde{A}} Q|_3^2 + \int_0^T |\sqrt{\kappa}(N-\tilde{n})\cdot\nabla_{\tilde{A}} Q|_3^2\\ &\lesssim \int_0^T |\sqrt{\kappa}\tilde{n}\cdot\nabla_{\tilde{A}} Q|_3^2 + \varepsilon\int_0^T \|\sqrt{\kappa}\nabla_{\tilde{A}} Q\|_{4.5}^2 + \int_0^T \mathcal{P}.\end{aligned}$$

As a consequence, (4.2.166) becomes

$$\int_0^T \|\sqrt{\kappa}\nabla_{\tilde{A}} Q\|_{4.5}^2 \lesssim \int_0^T \left(\|\sqrt{\kappa}\triangle_{\tilde{A}} Q\|_{3.5}^2 + |\sqrt{\kappa}\tilde{n}\cdot\nabla_{\tilde{A}} Q\big|_3^2 + |\sqrt{\kappa}q|_0^2\right). \tag{4.2.167}$$

To control the RHS, we recall that Q verifies

$$-\triangle_{\tilde{A}} Q = -\partial_t \tilde{A}^{\mu\alpha}\partial_\mu v_\alpha + \partial_\beta((b_0\cdot\partial)\tilde{\eta}_\nu)\partial_\nu\tilde{A}^{\mu\nu}\tilde{A}^{\beta\alpha}\partial_\mu(b_0\cdot\partial)\eta_\alpha \tag{4.2.168}$$

with the Dirichlet and Neumann boundary conditions

$$\sqrt{\tilde{g}}Q = -\sigma\sqrt{g}(\triangle_g\eta\cdot\tilde{n}) + \kappa(1-\overline{\triangle})(v\cdot\tilde{n}), \tag{4.2.169}$$

$$\tilde{n}\cdot\nabla_{\tilde{A}}Q = -\partial_t v\cdot\tilde{n} + (b_0\cdot\partial)^2\eta\cdot\tilde{n}. \tag{4.2.170}$$

Now,

$$\int_0^T \|\sqrt{\kappa}\triangle_{\tilde{A}}Q\|_{3.5}^2 \le \int_0^T \kappa\|\partial_t\tilde{A}\partial v\|_{3.5}^2 + \int_0^T \kappa\|\partial((b_0\cdot\partial)\eta)(\partial_t a)(a\partial(b_0\cdot\partial)\eta)\|_{3.5}^2, \tag{4.2.171}$$

where the RHS is bounded by $\int_0^T \mathcal{P}$. Also, it is not hard to see, via the Dirichlet boundary condition, that $\int_0^T |\sqrt{\kappa}Q|_0^2 \le \int_0^T \mathcal{P}$.

Next, we control $\int_0^T |\sqrt{\kappa}\tilde{n}\cdot\nabla_{\tilde{A}}q|_3^2$. In view of the Neumann boundary condition (4.2.170), it contributes to

$$\int_0^T \kappa|\partial_t v\cdot\tilde{n}|_3^2, \quad \int_0^T |\sqrt{\kappa}(b_0\cdot\partial)^2\eta\cdot\tilde{n}|_3^2.$$

For the first term, since $\partial_t v\in H^{3.5}(\Omega)$ and $\eta\in H^{4.5}(\Omega)$ a priori, as well as $\overline{\partial}\tilde{n} = Q(\overline{\partial}\eta)\overline{\partial}^2\eta$, we have $\int_0^T \kappa|\partial_t v\cdot\tilde{n}|_3^2 \le \int_0^T \mathcal{P}$. Also, for the second term,

$$\int_0^T |\sqrt{\kappa}(b_0\cdot\partial)^2\eta\cdot\tilde{n}|_3^2 \stackrel{L}{=} \int_0^T |\sqrt{\kappa}(b_0\cdot\partial)^2\partial^3\eta\cdot\tilde{n}|_0^2 \le \int_0^T P(\|b_0\|_{4.5}, ||\eta||_{4.5})|\sqrt{\kappa}\overline{\partial}^5\eta|_0^2,$$

which can be controlled by $\mathcal{M}_0 + C(\varepsilon)E_\kappa(T) + \mathcal{P}\int_0^T \mathcal{P}$ owing to (4.2.23).

In summary, we have

$$E_\kappa^{(3)} \le \mathcal{M}_0 + C(\varepsilon)E_\kappa(T) + \mathcal{P}\int_0^T \mathcal{P}. \tag{4.2.172}$$

4.2.5 Closing the nonlinear energy estimate

We now conclude the proof of Proposition 4.2.4.

4.2.5.1 Regularity of initial data

Our first task is to remove the extra regularity assumptions on the initial data. These additional regularities are introduced in $\mathcal{M}_0$ (cf. Lemma 4.2.9). In addition to this, one has to control $\|q(0)\|_{4.5}, \|q_t(0)\|_{3.5}, \|q_{tt}(0)\|_{2.5}$ in terms of v_0 and b_0 by the elliptic estimate, and extra regularity on v_0 and b_0 shall appear due to the viscosity.

Note that Q_0 verifies the elliptic equation

$$\begin{cases} -\triangle Q_0 = (\partial v_0)(\partial v_0) - (\partial b_0)(\partial b_0) & \text{in } \Omega \\ Q_0 = \kappa(1-\overline{\triangle})v^3 & \text{on } \Gamma \\ \frac{\partial Q_0}{\partial N} = 0 & \text{on } \Gamma_0 \end{cases} \tag{4.2.173}$$

by standard elliptic estimates, we get $\|Q_0\|_{4.5} \lesssim \|\partial v_0\|_{2.5}^2 + \|\partial b_0\|_{2.5}^2 + \kappa\|v_0^3\|_{4.5} + \kappa|v_0^3|_6$. Moreover, note that the energy functional contains time derivatives of v and $(b_0 \cdot \partial)\eta$, so we need to express their initial data in terms of v_0 and b_0 as well. We invoke $\partial_t v(0) - (b_0 \cdot \partial) b_0 = -\partial Q_0$ to get $\|\partial_t v(0)\|_{3.5} \lesssim \|b_0\|_{3.5}\|b_0\|_{4.5} + \|Q_0\|_{4.5}$ and $\|\partial_t (b_0 \cdot \partial)\eta(0)\|_{3.5} \lesssim \|b_0\|_{3.5}\|v_0\|_{4.5}$. Similarly, we consider the ∂_t-differentiated elliptic equation of Q to get $\|\partial_t Q(0)\|_{3.5} \lesssim P(\|v_0\|_{4.5}, \|b_0\|_{4.5})(|v_0^3|_5 + \kappa|\partial_t v(0)|_5)$ and further $\|\partial_t^2 Q(0)\|_{2.5} + \|\partial_t^3 Q(0)\|_1 \lesssim P(\|v_0\|_{4.5}, \|b_0\|_{4.5}, |v_0|_5)(1 + \kappa|\overline{\triangle}\partial_t^2 v(0)|_2)$.

By Sobolev trace lemma, we need to bound $\kappa\|\partial_t^2 v(0)\|_{4.5}$ which requires the control of $\kappa(\|v_0\|_{6.5} + \|b_0\|_{5.5} + \|\partial_t Q(0)\|_{5.5})$. We replace 3.5 by 5.5 in the estimates of $\partial_t Q(0)$, and thus we need to control $\kappa^2|\partial_t v(0)|_7 \lesssim \kappa^2(\|b_0\|_{7.5}\|b_0\|_{8.5} + \|Q_0\|_{8.5})$. Finally, replacing 4.5 by 8.5 in the estimates of Q_0, we need to control $\kappa^2(\|v_0\|_{7.5}^2 + \|b_0\|_{7.5}^2) + \kappa^3(|v_0|_8 + |v_0|_{10})$.

We need to control κ-weighted norms of $\|v_0\|_{8.5}, \|b_0\|_{8.5}$ and $|v_0|_{10}$. However, our given initial data is $v_0 \in H^{4.5}(\Omega) \cap H^5(\Gamma)$ and $b_0 \in H^{4.5}$ and so we have to remove the additional regularity assumptions on the initial data. We define Ω_κ to be the regularized version of Ω tangentially mollified by $\zeta_{\exp^{-\kappa}}$ and define E_{Ω_κ} to be the extension operator from Ω to Ω_κ. Next we set

$$\mathbf{v}_0 := \zeta_{\exp^{-\kappa}} * E_{\Omega_\kappa}(v_0),\ \mathbf{b}_0 := \zeta_{\exp^{-\kappa}} * E_{\Omega_\kappa}(b_0),\ \mathbf{q}_0 := \zeta_{\exp^{-\kappa}} * E_{\Omega_\kappa}(Q_0).$$

Integrating by parts repeatedly to transfer derivatives to the mollifier $\zeta_{\exp^{-\kappa}}$, we get

$$\|\kappa \mathbf{v}_0\|_{8.5} + \|\kappa \mathbf{b}_0\|_{8.5} + \|\kappa \mathbf{q}_0\|_{8.5} + |\kappa \mathbf{v}_0|_{10} \lesssim \|v_0\|_{4.5} + \|b_0\|_{4.5} + \|Q_0\|_{4.5} + |v_0|_5 \le \mathcal{C}, \quad (4.2.174)$$

where $\mathcal{C}$ is the constant that appears in (4.2.7).

4.2.5.2 Nonlinear a priori estimates

We summarize the a priori estimates of the nonlinear approximate system (4.2.2).

1. (4.2.26) gives the elliptic estimates of Q and its time derivatives.
2. (4.2.54)-(4.2.56) and (4.2.63), (4.2.64) give the divergence estimate and (4.2.67)-(4.2.68) give the curl estimate.
3. (4.2.69) and (4.2.76) control the boundary normal traces.
4. (4.2.125), (4.2.141), (4.2.145) provide control of the mixed tangential derivatives of v and $(b_0 \cdot \partial)\eta$ and the normal traces of v. Note that these estimate depends on $E_\kappa^{(3)}$ on the RHS.
5. Finally, (4.2.172) provides the estimate for $E_\kappa^{(3)}$.

Thus, by combining these estimates and then invoking (4.2.174), we obtain

$$E_\kappa(T) - E_\kappa(0) \lesssim C(\varepsilon)E_\kappa(T) + \mathcal{C}(\|v_0\|_{4.5}, \|b_0\|_{4.5}) + P(E_\kappa(T)) \int_0^T E_\kappa(t)\,\mathrm{d}t. \quad (4.2.175)$$

We pick $\varepsilon > 0$ suitably small such that the ε-terms can be absorbed to LHS. Therefore, by the nonlinear Gronwall inequality, we know there exists some time $T > 0$ independent of κ, such that

$$\sup_{0 \le t \le T} E_\kappa(t) \le \mathcal{C}. \quad (4.2.176)$$

This concludes the proof for Proposition 4.2.4.

4.2.6 Well-posedness for the linearized approximate system

Since we obtained an uniform-in-κ a priori energy estimate for (4.2.2), our next goal is to construct a solution for this system for each fixed $\kappa > 0$.

Let $T > 0$. We define

$$\mathbf{X} = \{u \in L^\infty(0,T; H^{4.5}(\Omega)) : \sup_{[0,T]} \|u\|_{4.5} \le 2\|v_0\|_{4.5} + 1\}, \tag{4.2.177}$$

which is a closed subset of the space $L^\infty(0,T; H^{4.5}(\Omega))$.

To solve the approximate κ-problem (4.2.2) for each fixed $\kappa > 0$, we study the following linearized problem whose fixed-point provides the desired solutions. Fix an arbitrary function $\mathring{\eta} = \mathring{\eta}(t,y)$ whose time derivative $\mathring{\eta}_t \in \mathbf{X}$, we denote by $\mathring{A}$, $\mathring{g}$, $\mathring{J}$ and $\mathring{\mathbf{A}}$ the associated quantities in Lagrangian coordinates and $\tilde{\mathring{\eta}} := \Lambda_\kappa \mathring{\eta}$, $\tilde{\mathring{A}} := [\partial \tilde{\mathring{\eta}}]^{-1}$, $\tilde{\mathring{J}} := \det[\partial \tilde{\mathring{\eta}}]$, $\tilde{\mathring{\mathbf{A}}} := \tilde{\mathring{J}} \tilde{\mathring{A}}$ and $\tilde{\mathring{n}}$ to be the associated smoothed quantities.

We aim to construct η and v that solve

$$\begin{cases} \partial_t \eta = v & \text{in } [0,T]\times\Omega; \\ \partial_t v - (b_0\cdot\partial)^2\eta + \nabla_{\tilde{\mathring{\mathbf{A}}}} Q = 0 & \text{in } [0,T]\times\Omega; \\ \operatorname{div}_{\tilde{\mathring{\mathbf{A}}}} v = 0, \quad \operatorname{div} b_0 = 0 & \text{in } [0,T]\times\Omega; \\ v^3 = b_0^3 = 0 & \text{on } \Gamma_0; \\ \tilde{\mathring{\mathbf{A}}}^{3\alpha} Q = -\sigma\sqrt{\mathring{g}}(\triangle_{\mathring{g}}\mathring{\eta}\cdot\tilde{\mathring{n}})\tilde{n}^\alpha + \kappa(1-\overline{\triangle})(v\cdot\tilde{\mathring{n}})\tilde{\mathring{n}}^\alpha & \text{on } \Gamma; \\ (\eta, v) = (\mathrm{Id}, v_0) & \text{on } \{t=0\}\times\overline{\Omega}. \end{cases} \tag{4.2.178}$$

We show the existence of η, v by first establishing the existence of the weak solution and then boosting up their regularity. The construction of the solution for the nonlinear κ-problem will be postponed until the next subsection. We will adapt the method developed in [16] to study the weak solution for (4.2.178). Also, due to technical reasons, it is convenient for us to first construct the weak solution of (4.2.178) in $L^2(0,T; H^{-1}(\Omega))$ and then prove that this solution has $L^2(0,T; H^1(\Omega))$ regularity.

4.2.6.1 The penalized problem

The goal of this subsection is to study the penalized version (of the divergence-free condition on the velocity) of the linearized κ-problem (4.2.178). In particular, for $0 < \lambda \ll 1$, let w_λ, ξ_λ be the solutions for (4.2.178) with

$$\operatorname{div}_{\mathring{\tilde{\mathbf{A}}}} w_\lambda = -\lambda Q_\lambda \tag{4.2.179}$$

where Q_λ is defined to be the penalized pressure. In this case, (4.2.178) becomes

$$\begin{cases} \partial_t \xi_\lambda = w_\lambda & \text{in } [0,T]\times\Omega; \\ \partial_t w_\lambda - (b_0\cdot\partial)^2\xi_\lambda + \nabla_{\mathring{\tilde{\mathbf{A}}}} Q_\lambda = 0 & \text{in } [0,T]\times\Omega; \\ \operatorname{div}_{\mathring{\tilde{\mathbf{A}}}} w_\lambda = -\lambda Q_\lambda, \quad \operatorname{div} b_0 = 0 & \text{in } [0,T]\times\Omega; \\ w_\lambda^3 = b_0^3 = 0 & \text{on } \Gamma_0; \\ \mathring{\tilde{\mathbf{A}}}^{3\alpha} q_\lambda = -\sigma\sqrt{\mathring{g}}(\triangle_{\mathring{g}}\mathring{\eta}\cdot\mathring{\tilde{n}})\mathring{\tilde{n}}^\alpha + \kappa(1-\overline{\triangle})(v\cdot\mathring{\tilde{n}})\mathring{\tilde{n}}^\alpha & \text{on } \Gamma; \\ (\xi_\lambda, w_\lambda) = (\mathrm{Id}, v_0) & \text{on } \{t=0\}\times\overline{\Omega}. \end{cases} \tag{4.2.180}$$

Since each penalized problem is indexed by λ (recall κ is fixed), we shall denote them by "λ-problem" throughout the rest of this section.

Weak solution for the λ-problem. First of all, for each fixed λ, we will solve the λ-problem by the Galerkin approximation and obtain a weak solution. By introducing a basis $(e_k)_{k=1}^\infty$ of $L^2(\Omega)\cap H^1(\Omega)$, and considering the approximation

$$\partial_t \xi_m(t,y) = w_m(t,y), \tag{4.2.181}$$

$$w_m(t,y) = \sum_{k=1}^m z_k(t)e_k(y), \quad m\geq 2, \quad t\in[0,T], \tag{4.2.182}$$

one can form a system of ODE by multiplying a test vector field ϕ, whose component $\phi_\alpha \in \operatorname{span}(e_1,\cdots,e_m)$ to the λ-problem. Specifically, we have

$$\int_\Omega (w_m^\alpha)_t\phi_\alpha - \int_\Omega [(b_0\cdot\partial)^2\xi_m^\alpha]\phi_\alpha + \int_\Omega [\mathring{\mathbf{A}}^{\mu\alpha}\partial_\mu Q_m]\phi_\alpha = 0. \tag{4.2.183}$$

We recall that $(b_0 \cdot \partial)|_\Gamma$ is tangential to Γ. Owing to this and the boundary condition of q_m, we obtain, after integration by parts, that

$$\int_\Omega (w_m^\alpha)_t \phi_\alpha + \int_\Omega [(b_0 \cdot \partial)\xi_m^\alpha][(b_0 \cdot \partial)\phi_\alpha] + \kappa \sum_{l=0,1} \int_\Gamma \overline{\partial}^l (w_m \cdot \mathring{\tilde{n}}) \overline{\partial}^l (\phi \cdot \mathring{\tilde{n}})$$

$$- \int_\Omega Q_m [\mathring{\tilde{\mathbf{A}}}^{\mu\alpha} \partial_\mu \phi_\alpha] = \sigma \int_\Gamma (\sqrt{\mathring{g}} \triangle_{\mathring{g}} \mathring{\eta} \cdot \mathring{\tilde{n}})(\phi \cdot \mathring{\tilde{n}}), \tag{4.2.184}$$

$$w_m(0) = (v_0)_m, \ \ \xi_m(0) = \mathrm{Id}, \tag{4.2.185}$$

where $(v_0)_m$ is the projection of v_0 onto $\mathrm{span}(e_1, \cdots, e_m)$.

Let $\phi_\alpha = e_k, k = 1, \cdots, m$. Then (4.2.184)-(4.2.185) and (4.2.179) yield an ODE system, and the standard ODE theory gives the the existence and uniqueness of ξ_m and w_m in $[0, T_\lambda]$ for some $T_\lambda > 0$. *We mention that it is important to introduce the penalized pressure* (4.2.179)*, or else* (4.2.184) *would not form an ODE system.*

Setting $\phi = w_m$, and since $\sigma |\sqrt{\mathring{g}} \triangle_{\mathring{g}} \mathring{\eta}^\alpha|_0 \le \mathcal{N}_0$, where $\mathcal{N}_0$ denotes a generic polynomial function such that $\mathcal{N}_0 = P(\|\eta_0\|_{4.5}, \|v_0\|_{4.5}, \|b_0\|_{4.5})$, then (4.2.184) gives us

$$\|w_m\|_0^2 + \|(b_0 \cdot \partial)\xi_m\|_0^2 + \lambda \int_0^t \|q_m\|_0^2 + \kappa \int_0^t |w_m \cdot \mathring{\tilde{n}}|_1^2 \le \mathcal{N}_0, \quad t \in [0, T_\lambda] \tag{4.2.186}$$

Since the RHS of (4.2.186) is independent of λ, we know the solution (ξ_m, w_m) is defined on $[0, T]$ (possibly after setting T smaller). In addition, there is a subsequence, which is still denoted with the index m, satisfying

$$(b_0 \cdot \partial)\xi_m \rightharpoonup (b_0 \cdot \partial)\xi_\lambda, \quad w_m \rightharpoonup w_\lambda, \quad Q_m \rightharpoonup Q_\lambda, \quad \text{in } L^2(0, T; L^2(\Omega)), \tag{4.2.187}$$

$$w_m \cdot \mathring{\tilde{n}} \rightharpoonup w_\lambda \cdot \mathring{\tilde{n}}, \quad \text{in } L^2(0, T; H^1(\Gamma)), \tag{4.2.188}$$

where w_λ, $(b_0\cdot\partial)\xi_\lambda$, and q_λ verify the estimate

$$\|w_\lambda\|_0^2+\|(b_0\cdot\partial)\xi_\lambda\|_0^2+\lambda\int_0^t\|q_\lambda\|_0^2+\kappa\int_0^t|w_\lambda\cdot\mathring{\tilde{n}}|_1^2\le\mathcal{P}_0,\quad t\in[0,T]. \tag{4.2.189}$$

Now, let Y be a Banach space. We denote its dual by Y', and, for $\Psi\in H^s(\Omega)'=H^{-s}(\Omega)$ and $\Phi\in H^s(\Omega)$, the pairing between Ψ and Φ is denoted by $\langle\Psi,\Phi\rangle_s$. It follows from the ODE (4.2.184) defining w_m, that $\partial_t w_\lambda\in L^2(0,T;H^{-\frac12+})$, where $H^{-\frac12+}:=H^{-\frac12+\delta}$ for some $0<\delta\ll 1$, and $(b_0\cdot\partial)^2\xi_\lambda\in L^2(0,T;H^{-\frac12+})$ as well. Now, for $\phi\in L^2(0,T;H^{\frac12-})$, we have

$$\begin{aligned}&\int_0^T\langle\partial_t w_\lambda^\alpha,\phi_\alpha\rangle_{\frac12-}+\int_0^T\langle(b_0\cdot\partial)^2\xi_\lambda,\phi_\alpha\rangle_{\frac12-}\\&+\kappa\sum_{l=0,1}\int_0^T\int_\Gamma\overline{\partial}^l(w_\lambda\cdot\mathring{\tilde{n}})\overline{\partial}^l(\phi\cdot\mathring{\tilde{n}})-\int_0^T\langle Q_\lambda,\mathring{\tilde{\mathbf{A}}}^{\mu\alpha}\partial_\mu\phi_\alpha\rangle_{\frac12+}\\=&\sigma\int_0^T\int_\Gamma(\sqrt{\mathring{g}}\triangle_{\mathring{g}}\mathring{\eta}\cdot\mathring{\tilde{n}})(\phi\cdot\mathring{\tilde{n}}).\end{aligned} \tag{4.2.190}$$

In light of (4.2.190), we can see that $\partial_t w_\lambda\in L^2(0,T;H^{-\frac12+})$, and $q_\lambda\in L^2(0,T;H^{\frac12+})$, and the regularity of q_λ implies $\nabla_{\mathring{\tilde{\mathbf{A}}}}Q_\lambda\in L^2(0,T;H^{-\frac12-})$. Therefore, we have that

$$\partial_t w_\lambda-(b_0\cdot\partial)^2\xi_\lambda+\nabla_{\mathring{\tilde{\mathbf{A}}}}Q_\lambda=0 \tag{4.2.191}$$

holds in $L^2(0,T;H^{-\frac12-}(\Omega))\subset L^2(0,T;H^{-1}(\Omega))$. In addition, by commuting $\text{curl}_{\mathring{\tilde{\mathbf{A}}}}$ through (4.2.191) we get the following evolution equation

$$\partial_t(\text{curl}_{\mathring{\tilde{\mathbf{A}}}}w_\lambda)-(b_0\cdot\partial)\text{curl}_{\mathring{\tilde{\mathbf{A}}}}((b_0\cdot\partial)\xi_\lambda)=[\text{curl}_{\mathring{\tilde{\mathbf{A}}}},(b_0\cdot\partial)]((b_0\cdot\partial)\xi_\lambda)+\text{curl}_{\partial_t\mathring{\tilde{\mathbf{A}}}}w_\lambda. \tag{4.2.192}$$

The limit as $\lambda\to 0$. By (4.2.179) and (4.2.189) we have

$$\int_0^t\left(\|w_\lambda\|_0^2+\|(b_0\cdot\partial)\xi_\lambda\|_0^2+\frac{1}{\lambda}\|\text{div}_{\mathring{\tilde{\mathbf{A}}}}w_\lambda\|_0^2+\kappa|w_\lambda\cdot\mathring{\tilde{n}}|_1^2\right)dt\le\mathcal{N}_0,\quad t\in[0,T] \tag{4.2.193}$$

Thus, $\{w_\lambda\}$ and $\{(b_0 \cdot \partial)\xi_\lambda\}$ admit converging subsequences such that

$$w_\lambda \rightharpoonup v, \quad (b_0 \cdot \partial)\xi_\lambda \rightharpoonup (b_0 \cdot \partial)\eta, \quad \operatorname{div}_{\mathring{\tilde{\mathbf{A}}}} w_\lambda \rightharpoonup \operatorname{div}_{\mathring{\tilde{\mathbf{A}}}} v, \quad \text{in } L^2(0,T;L^2(\Omega)), \tag{4.2.194}$$

$$w_\lambda \cdot \mathring{\tilde{n}} \rightharpoonup v \cdot \mathring{\tilde{n}} \quad \text{in } L^2(0,T;H^1(\Gamma)). \tag{4.2.195}$$

Moreover, in view of (4.2.189), we must have that

$$\operatorname{div}_{\mathring{\tilde{\mathbf{A}}}} v = 0, \quad \text{in } L^2(0,T;L^2(\Omega)). \tag{4.2.196}$$

Also, this implies the evolution equation verified by div $(b_0 \cdot \partial)\eta$, i.e.,

$$\partial_t \operatorname{div}_{\mathring{\tilde{\mathbf{A}}}}((b_0 \cdot \partial)\eta) = [\operatorname{div}_{\mathring{\tilde{\mathbf{A}}}}, (b_0 \cdot \partial)]v + (\partial_t \mathring{\tilde{\mathbf{A}}}^{\mu\alpha})\partial_\mu((b_0 \cdot \partial)\eta_\alpha). \tag{4.2.197}$$

Our next goal is to show that (η, v) is a weak solution for (4.2.178) and we also need to get a bound for $\int_0^t \|v_t\|^2_{H^{-\frac{1}{2}+}(\Omega)}$ for $t \in [0,T]$. This ties to the $L^2(0,T;H^{\frac{1}{2}+})$ regularity of the pressure function Q (to be defined later in this section). First, we consider a vector field $f \in L^2(0,T;H^{\frac{1}{2}-})$. Define φ be the solution of

$$\triangle_{\mathring{\tilde{\mathbf{A}}}} \varphi = \operatorname{div}_{\mathring{\tilde{\mathbf{A}}}} f, \quad \text{in } \Omega, \tag{4.2.198}$$

$$\varphi = 0, \quad \text{on } \partial\Omega, \tag{4.2.199}$$

and let g, h be the vector fields such that $g = \nabla_{\mathring{\tilde{\mathbf{A}}}} \varphi$ and $h = f - g$. Here, it is clear that $g, h \in L^2(0,T;H^{\frac{1}{2}-})$ and $\operatorname{div}_{\mathring{\tilde{\mathbf{A}}}} h = 0$. Now, (4.2.190) yields, after replacing ϕ by h, that h verifies the following variational equation

$$\begin{aligned} &\int_0^T \langle \partial_t w_\lambda, h\rangle_{\frac{1}{2}-} + \int_0^T \langle (b_0 \cdot \partial)^2 \xi_\lambda, h\rangle_{\frac{1}{2}-} + \kappa \sum_{l=0,1} \int_0^T \int_\Gamma \overline{\partial}^l (w_\lambda \cdot \mathring{\tilde{n}}) \overline{\partial}^l (h \cdot \mathring{\tilde{n}}) \\ =&\sigma \int_0^T \int_\Gamma (\sqrt{\mathring{g}} \triangle_{\mathring{g}} \mathring{\eta} \cdot \mathring{\tilde{n}})(h \cdot \mathring{\tilde{n}}). \end{aligned} \tag{4.2.200}$$

On the other hand, since $\mathrm{div}_{\mathring{\mathbf{A}}} v = 0$, we have $\mathring{\mathbf{A}}^{\mu\alpha}\partial_\mu\partial_t v_\alpha = -(\partial_t \mathring{\tilde{\mathbf{A}}}^{\mu\alpha})\partial_\mu v_\alpha$. This identity and (4.2.199) yield

$$\langle v_t, g\rangle_{\frac{1}{2}-} = \langle v_t, \nabla_{\mathring{\mathbf{A}}}\varphi\rangle_{\frac{1}{2}-} = \int_\Omega (\partial_t \mathring{\tilde{\mathbf{A}}}^{\mu\alpha})\partial_\mu v_\alpha \varphi.$$

In light of this and (4.2.200), we obtain

$$\begin{aligned}
&\lim_{\lambda\to 0}\int_0^T \langle \partial_t w_\lambda, f\rangle_{\frac{1}{2}-} + \int_0^T \langle (b_0\cdot\partial)^2 \xi_\lambda^\alpha, f\rangle_{\frac{1}{2}-} \\
&= \int_0^T\int_\Omega (\partial_t \mathring{\tilde{\mathbf{A}}}^{\mu\alpha})\partial_\mu v_\alpha \varphi - \kappa\sum_{l=0,1}\int_0^T\int_\Gamma \overline{\partial}^l (v\cdot \mathring{\tilde{n}})\overline{\partial}^l (h\cdot \mathring{\tilde{n}}) \\
&+ \sigma\int_0^T\int_\Gamma (\sqrt{\mathring{g}}\triangle_{\mathring{g}}\mathring{\eta}\cdot \mathring{\tilde{n}})(h\cdot\mathring{\tilde{n}}),
\end{aligned} \tag{4.2.201}$$

and so

$$\lim_{\lambda\to 0}\int_0^T \|w_{\lambda t}\|^2_{H^{-\frac{1}{2}+}} + \|(b_0\cdot\partial)^2\xi_\lambda\|^2_{H^{-\frac{1}{2}+}} \le \mathcal{N}_0. \tag{4.2.202}$$

As a consequence, we have $w_{\lambda t} \rightharpoonup v_t$ and $(b_0\cdot\partial)^2\xi_\lambda \rightharpoonup (b_0\cdot\partial)^2\eta$ in $L^2(0,T;H^{-\frac{1}{2}+})$. The former ensures that $v \in C^0(0,T;L^2)$ and so the initial data of $w_\lambda(0)$ and $v(0)$ agrees and equals to v_0. Moreover, by employing [16, Lem. 7.4][1], there exists $q\in L^2(0,T;H^{\frac{1}{2}+})$, in terms of the pressure function, such that

$$\begin{aligned}
&\int_0^T \langle \partial_t v, \phi\rangle_{\frac{1}{2}-} + \int_0^T \langle (b_0\cdot\partial)^2\eta, \phi\rangle_{\frac{1}{2}-} - \int_0^T \langle Q, \mathring{\tilde{\mathbf{A}}}^{\mu\alpha}\partial_\mu \phi_\alpha\rangle_{\frac{1}{2}+} \\
&+ \kappa\sum_{l=0,1}\int_0^T\int_\Gamma \overline{\partial}^l (v\cdot\mathring{\tilde{n}})\overline{\partial}^l(\phi\cdot\mathring{\tilde{n}}) \\
&= \sigma\int_0^T\int_\Gamma (\sqrt{\mathring{g}}\triangle_{\mathring{g}}\mathring{\eta}\cdot\mathring{\tilde{n}})(\phi\cdot\mathring{\tilde{n}}).
\end{aligned} \tag{4.2.203}$$

[1]We need a small modification. Since we need our $q\in H^{\frac{1}{2}+}$, we need to consider the linear functional $\langle \mathrm{div}_{\mathring{\mathbf{A}}}\phi, p\rangle_{\frac{1}{2}+}$ defined on $X(t)$, where $X(t) = \{\phi\in H^{\frac{1}{2}-}(\Omega) : \mathrm{div}_{\mathring{\mathbf{A}}}\phi \in H^{-\frac{1}{2}-}(\Omega))\}$.

holds for any test function $\phi \in L^2(0,T;H^{\frac{1}{2}-})$. This yields that (η, v, Q) verifies

$$\partial_t v - (b_0\cdot\partial)^2\eta + \nabla_{\mathring{\tilde{\mathbf{A}}}} Q = 0, \quad \text{and } \operatorname{div}_{\mathring{\tilde{\mathbf{A}}}} v = 0, \quad \text{in } L^2(0,T;H^{-1}),$$

and so we've shown that η, v is indeed a weak solution for (4.2.178). Furthermore, (4.2.202) together with [16, Lem. 7.4] implies that

$$\int_0^T \|Q\|^2_{H^{\frac{1}{2}+}} \le \mathcal{P}_0. \tag{4.2.204}$$

Remark 4.2.15. The $\frac{1}{2}+$ interior regularity of Q is required here as this controls the $H^{0+}(\Gamma)$-norm of Q on the boundary. We refer Section 4.2.6.2 for the details. Finally, we consider the difference between (4.2.203) with v and v',

$$\begin{aligned}&\int_0^T \langle \partial_t(v-v'),\phi\rangle_{\frac{1}{2}-} + \int_0^T \langle (b_0\cdot\partial)^2(\eta-\eta'),\phi\rangle_{\frac{1}{2}+}\\ &+\kappa\sum_{l=0,1}\int_0^T\int_\Gamma \overline{\partial}^l((v-v')\cdot\mathring{\tilde{n}})\overline{\partial}^l(\phi\cdot\mathring{\tilde{n}}) - \int_0^T \langle (q-q'), \mathring{\tilde{\mathbf{A}}}^{\mu\alpha}\partial_\mu\phi\rangle_{\frac{1}{2}+} = 0.\end{aligned} \tag{4.2.205}$$

where (v',η') is assumed to be another solution with the initial data. The uniqueness of the weak solution follows from setting $\phi = v - v'$.

4.2.6.2 H^1 Regularity estimates of v, $(b_0\cdot\partial)\eta$ and Q

We shall show that v, $(b_0\cdot\partial)\eta$ and Q are in fact $L^2(0,T;H^1(\Omega))$. Let

$$e(t) := \int_0^t \|\eta\|_1^2 + \|v\|_1^2 + \|(b_0\cdot\partial)\eta\|_1^2\,\mathrm{d}t, \quad t\in[0,T]. \tag{4.2.206}$$

Our goal is to show

$$e(T) \le P(\mathcal{N}_0). \tag{4.2.207}$$

It suffices to consider $\int_0^T \|v\|_1^2$ and $\int_0^T \|(b_0\cdot\partial)\eta\|_1^2$ only since

$$\int_0^T \|\eta\|_1^2 \le \int_0^T \left(\|\eta_0\|_1^2 + \int_0^t \|v\|_1^2\,\mathrm{d}t\right)\mathrm{d}t.$$

Thanks to Lemma 3.3.1, it suffices for us to control $\int_0^T \|\mathrm{div}\, v\|_0^2, \int_0^T \|\mathrm{curl}\, v\|_0^2, \int_0^T |v^3|_{0.5}^2$, as well as $\int_0^T \|\mathrm{div}\,(b_0\cdot\partial)\eta\|_0^2, \int_0^T \|\mathrm{curl}\,(b_0\cdot\partial)\eta\|_0^2, \int_0^T |(b_0\cdot\partial)\eta^3|_{0.5}^2$, in order to control $\int_0^T \|v\|_1^2$ and $\int_0^T \|(b_0\cdot\partial)\eta\|_1^2$.

Control of the divergence and curl The estimates we need here are essentially the same as in Section 4.2.2.2 but without considering the time differentiated quantities. Firstly, since (4.2.16) in Lemma 4.2.6 remains true with $\tilde{\mathbf{A}}$ replaced by $\mathring{\tilde{\mathbf{A}}}$, then

$$\int_0^T \|\mathrm{div}\, v\|_0^2 \le \int_0^T \|(\mathring{\tilde{\mathbf{A}}}^{\mu\alpha} - \delta^{\mu\alpha})\partial_\mu v_\alpha\|_0^2 \le \varepsilon \int_0^T \|\partial v\|_0^2 \le \varepsilon e(T). \tag{4.2.208}$$

Secondly, because $\mathrm{div}_{\mathring{\tilde{\mathbf{A}}}}(b_0\cdot\partial)\eta$ verifies the evolution equation

$$\partial_t \mathrm{div}_{\mathring{\tilde{\mathbf{A}}}}((b_0\cdot\partial)\eta) = [\mathrm{div}_{\mathring{\tilde{\mathbf{A}}}}, (b_0\cdot\partial)]v + (\partial_t \mathring{\tilde{\mathbf{A}}}^{\mu\alpha})\partial_\mu((b_0\cdot\partial)\eta_\alpha). \tag{4.2.209}$$

So, one needs to bound $\int_0^T\int_0^t \|\text{RHS of (4.2.209)}\|_0^2\,\mathrm{d}t$ in order to control $\int_0^T \|\mathrm{div}_{\mathring{\tilde{\mathbf{A}}}}((b_0\cdot\partial)\eta)\|_0^2$. We have

$$\begin{aligned}\int_0^T\int_0^t \|\partial_t \mathring{\tilde{\mathbf{A}}}^{\mu\alpha}\partial_\mu((b_0\cdot\partial)\eta)\|_0^2\,\mathrm{d}t &\le \int_0^T\int_0^t \|\partial_t \mathring{\tilde{\mathbf{A}}}\|_{L^\infty}^2 \|\partial((b_0\cdot\partial)\eta)\|_0^2\,\mathrm{d}t \\ &\le \int_0^T\int_0^t \mathcal{N}_0 \|\partial((b_0\cdot\partial)\eta)\|_0^2\,\mathrm{d}t \le T\mathcal{N}_0 e(T).\end{aligned} \tag{4.2.210}$$

Moreover, $[\mathrm{div}_{\mathring{\tilde{\mathbf{A}}}}, (b_0\cdot\partial)]v = \mathring{\tilde{\mathbf{A}}}^{\mu\alpha}((\partial_\mu b_0)\cdot\partial)v_\alpha - ((b_0\cdot\partial)\mathring{\tilde{\mathbf{A}}}^{\mu\alpha})\partial_\mu v_\alpha$ yields

$$\int_0^T\int_0^t \|[\mathrm{div}_{\mathring{\tilde{\mathbf{A}}}}(b_0\cdot\partial)]v\|_0^2 \le \int_0^T\int_0^t \mathcal{N}_0\|\partial v\|_0^2\,\mathrm{d}t \le T\mathcal{N}_0 e(T). \tag{4.2.211}$$

Thus,

$$\int_0^T \|\mathrm{div}_{\mathring{\tilde{\mathbf{A}}}}((b_0\cdot\partial)\eta)\|_0^2 \le T\mathcal{N}_0 e(T). \tag{4.2.212}$$

In addition, using $\|\text{div}\,(b_0\cdot\partial)\eta\|_0^2 \le \|\text{div}_{\mathring{\tilde{\mathbf{A}}}}(b_0\cdot\partial)\eta\|_0^2 + \|\mathring{\tilde{\mathbf{A}}}-\delta\|_{L^\infty}^2\|\partial(b_0\cdot\partial)\eta\|_0^2$ and (4.2.16), we conclude that

$$\int_0^T \|\text{div}\,(b_0\cdot\partial)\eta\|_0^2 \le \varepsilon e(T) + T\mathcal{N}_0 e(T). \tag{4.2.213}$$

Thirdly, the evolution equation satisfied by $\text{curl}_{\mathring{\tilde{\mathbf{A}}}} v$ and $\text{curl}_{\mathring{\tilde{\mathbf{A}}}}(b_0\cdot\partial)\eta$ reads

$$\partial_t(\text{curl}_{\mathring{\tilde{\mathbf{A}}}} v)_\alpha - (b_0\cdot\partial)\text{curl}_{\mathring{\tilde{\mathbf{A}}}}((b_0\cdot\partial)\eta)_\alpha = [\text{curl}_{\mathring{\tilde{\mathbf{A}}}}, (b_0\cdot\partial)]((b_0\cdot\partial)\eta)_\alpha + \text{curl}_{\,\partial_t\mathring{\tilde{\mathbf{A}}}} v_\alpha, \tag{4.2.214}$$

and this yields the following $L^2([0,T];L^2(\Omega))$-energy identity after testing with $\text{curl}_{\mathring{\tilde{\mathbf{A}}}} v$ and integrating in space and time:

$$\begin{aligned}\|\text{curl}_{\mathring{\tilde{\mathbf{A}}}} v\|_0^2 + \|\text{curl}_{\mathring{\tilde{\mathbf{A}}}}(b_0\cdot\partial)\eta\|_0^2 &\lesssim \int_0^t \|[(b_0\cdot\partial),\text{curl}_{\mathring{\tilde{\mathbf{A}}}}](b_0\cdot\partial)\eta\|_0^2 + \int_0^t \|\text{curl}_{\,\partial_t\mathring{\tilde{\mathbf{A}}}} v\|_0^2 \\ &+ \int_0^t \|[(b_0\cdot\partial),\text{curl}_{\mathring{\tilde{\mathbf{A}}}}](b_0\cdot\partial)v\|_0^2 + \int_0^t \|\text{curl}_{\,\partial_t\mathring{\tilde{\mathbf{A}}}}(b_0\cdot\partial)\eta\|_0^2.\end{aligned}$$

Integrating in time one more time, we achieve

$$\begin{aligned}\int_0^T \left(\|\text{curl}_{\mathring{\tilde{\mathbf{A}}}} v\|_0^2 + \|\text{curl}_{\mathring{\tilde{\mathbf{A}}}}(b_0\cdot\partial)\eta\|_0^2\right) &\lesssim \int_0^T\int_0^t \left(\|[(b_0\cdot\partial),\text{curl}_{\mathring{\tilde{\mathbf{A}}}}](b_0\cdot\partial)\eta\|_0^2 + \|\text{curl}_{\,\partial_t\mathring{\tilde{\mathbf{A}}}} v\|_0^2\right)\mathrm{d}t \\ &+ \int_0^T\int_0^t \left(\|[(b_0\cdot\partial),\text{curl}_{\mathring{\tilde{\mathbf{A}}}}](b_0\cdot\partial)v\|_0^2 + \|\text{curl}_{\,\partial_t\mathring{\tilde{\mathbf{A}}}}(b_0\cdot\partial)\eta\|_0^2\right)\mathrm{d}t.\end{aligned}$$

It suffices to control the first two terms on the RHS since the third and fourth term can then be controlled by an analogous method with the same bound.

For the first term on the RHS, since one can express

$$[(b_0\cdot\partial),\text{curl}_{\mathring{\tilde{\mathbf{A}}}}](b_0\cdot\partial)\eta_\alpha = \epsilon_{\alpha\beta\gamma}((b_0\cdot\partial)\mathring{\tilde{\mathbf{A}}}^{\nu\beta})\partial_\nu\eta^\gamma - \epsilon_{\alpha\beta\gamma}\mathring{\tilde{\mathbf{A}}}^{\nu\beta}(\partial_\nu b_0\cdot\partial)\eta^\gamma$$

and so $\int_0^T\int_0^t \|[(b_0\cdot\partial),\text{curl}_{\mathring{\tilde{\mathbf{A}}}}](b_0\cdot\partial)\eta_\alpha\|_0\,\mathrm{d}t \lesssim T\mathcal{N}_0 e(T)$. Similarly, $\|[(b_0\cdot\partial),\text{curl}_{\mathring{\tilde{\mathbf{A}}}}](b_0\cdot\partial)v_\alpha\|_0 \le T\mathcal{N}_0 e(T)$. In addition, for the second term, we obtain $\int_0^T\int_0^t \|\text{curl}_{\,\partial_t\mathring{\tilde{\mathbf{A}}}} v\|_0\,\mathrm{d}t \le T\mathcal{N}_0 e(T)$. Summing these up, we obtain

$$\int_0^T \int_0^t \left(\|\mathrm{curl}_{\mathring{\tilde{\mathbf{A}}}} v\|_0^2 + \|\mathrm{curl}_{\mathring{\tilde{\mathbf{A}}}}(b_0 \cdot \partial)\eta\|_0^2\right) \mathrm{d}t \le T\mathcal{N}_0 e(T). \tag{4.2.215}$$

Control of the boundary terms First we remark here that (4.2.17), (4.2.18) remain true by replacing $\tilde{n}$ by $\mathring{\tilde{n}}$.

Control of $\int_0^T |v^3|_{0.5}^2$: It suffices to control $\int_0^T |v \cdot \mathring{\tilde{n}}|_{0.5}^2$ since

$$\int_0^T |v^3|_{0.5}^2 \le \int_0^T |v \cdot \mathring{\tilde{n}}|_{0.5}^2 + \int_0^T |v \cdot (\mathring{\tilde{n}} - N)|_{0.5}^2, \tag{4.2.216}$$

whence

$$\int_0^T |v \cdot (\mathring{\tilde{n}} - N)|_{0.5}^2 \le \int_0^T |v|_{0.5}^2 |\mathring{\tilde{n}} - N|_{1+}^2 \lesssim \varepsilon e(T). \tag{4.2.217}$$

Moreover, the control of $\int_0^T |v \cdot \mathring{\tilde{n}}|_{0.5}^2$ is a consequence of (4.2.193) as $\lambda \to 0$,

$$\int_0^T |v \cdot \mathring{\tilde{n}}|_{0.5}^2 \lesssim \frac{1}{\kappa} \int_0^T |v \cdot \mathring{\tilde{n}}|_1^2 \le \frac{\mathcal{N}_0}{\kappa}. \tag{4.2.218}$$

Control of $\int_0^T |(b_0 \cdot \partial)\eta^3|_{0.5}^2$: Similar to the control $\int_0^T |v^3|_{0.5}^2$, it suffices to bound $\int_0^T |(b_0 \cdot \partial)(\eta \cdot \mathring{\tilde{n}})|_{0.5}^2$ only. Since $(b_0 \cdot \partial)|_\Gamma = b_0 \cdot \overline{\partial}$ and $\overline{\partial}(\eta \cdot \mathring{\tilde{n}})|_{t=0} = \overline{\partial}\eta^3|_{t=0} = 0$, we have

$$(b_0 \cdot \partial)(\eta \cdot \mathring{\tilde{n}}) = \int_0^T \partial_t (b_0 \cdot \partial)(\eta \cdot \mathring{\tilde{n}})\, \mathrm{d}t. \tag{4.2.219}$$

Hence,

$$\int_0^T |(b_0 \cdot \partial)(\eta \cdot \mathring{\tilde{n}})|_{0.5}^2 \le \int_0^T \left| \int_0^t \partial_t (b_0 \cdot \partial)(\eta \cdot \mathring{\tilde{n}}) \right|_{0.5}^2 \mathrm{d}t \lesssim \int_0^T \int_0^t |\partial_t (b_0 \cdot \partial)(\eta \cdot \mathring{\tilde{n}})|_{0.5}^2 \,\mathrm{d}t,$$

by Jensen's inequality. Here, the term on the second line is equal to

$$\int_0^T \int_0^t |(b_0 \cdot \partial)(v \cdot \mathring{\tilde{n}})|_{0.5}^2 \,\mathrm{d}t + \int_0^T \int_0^t |(b_0 \cdot \partial)(\eta \cdot \partial_t \mathring{\tilde{n}})|_{0.5}^2 = I + II.$$

Since $\partial_t \mathring{\tilde{n}} = Q(\overline{\partial}\mathring{\tilde{\eta}})\overline{\partial}\mathring{\tilde{v}} \cdot \mathring{\tilde{n}}$, we have $II \leq T\mathcal{N}_0 e(T)$.

Next, we have $I \lesssim ||b_0||_{0.5} \int_0^T \int_0^t |v \cdot \mathring{\tilde{n}}|_2^2 \, \mathrm{d}t$. By employing the boundary condition we obtain the following elliptic equation of $v \cdot \mathring{\tilde{n}}$ on Γ:

$$\overline{\triangle}(v \cdot \mathring{\tilde{n}}) = \frac{1}{\kappa}\Big((v \cdot \mathring{\tilde{n}}) + \sqrt{\mathring{\tilde{g}}}Q + \sigma\sqrt{\mathring{\tilde{g}}}\triangle_{\mathring{g}}\mathring{\tilde{\eta}} \cdot \mathring{\tilde{n}}\Big). \tag{4.2.220}$$

By the virtual of the elliptic estimate, we have

$$\int_0^T \int_0^t |v \cdot \mathring{\tilde{n}}|_{2+}^2 \, \mathrm{d}t \leq \kappa^{-1} \int_0^T \int_0^t \Big(|v \cdot \mathring{\tilde{n}}|_{0+}^2 + |\sqrt{\mathring{\tilde{g}}}Q|_{0+}^2 + \sigma|\sqrt{\mathring{\tilde{g}}}\triangle_{\mathring{g}}\mathring{\tilde{\eta}}^3|_{0+}^2\Big) \, \mathrm{d}t. \tag{4.2.221}$$

It is clear that the third term can be controlled by $T\mathcal{N}_0$, and first term is bounded by $T\mathcal{N}_0 e(T)$ via the trace lemma. Therefore,

$$\int_0^T \int_0^t |v \cdot \mathring{\tilde{n}}|_{2+}^2 \leq \frac{1}{\kappa}(T\mathcal{N}_0 e(T) + \int_0^T \int_0^t \mathcal{N}_0 |q|_{0+}^2 \, \mathrm{d}t). \tag{4.2.222}$$

Here, in light of (4.2.204), we have $\int_0^t |q|_{0+}^2 \leq \mathcal{N}_0$. In summary, we have

$$e(T) \leq \kappa^{-1}\mathcal{N}_0 + \varepsilon e(T) + T\mathcal{N}_0 e(T), \tag{4.2.223}$$

and this implies (4.2.207) if T is chosen sufficiently small, say $T = \frac{\varepsilon}{\mathcal{N}_0}$.

The strong solution for the linearized equations Since $v, (b_0 \cdot \partial)\eta \in L^2(0, T; H^1(\Omega))$ and so $v_t, (b_0 \cdot \partial)^2\eta \in L^2(0, T; L^2(\Omega))$, we can now proceed as what has been done in Section 7 of [16] to bound Q in $L^2(0, T; H^1(\Omega))$. Alternatively, one may also adapt Lemma 3.3.2 to achieve the same objective. Therefore, we have obtained a strong solution for the linearized κ-problem (4.2.178). This allows us to further boost the regularity of the linearized solution to $H^{4.5}(\Omega)$ via classical methods in the upcoming section. Then we achieve a solution for the nonlinear κ-problem by approximating it by a sequence of linearized solutions.

4.2.7 Existence for the nonlinear approximate κ-problem

We construct a solution to the nonlinear κ-problem for *each fixed* $\kappa > 0$. Let $(\eta_0, v_0, Q_0) = (\mathrm{Id}, 0, 0)$. For each $m \geq 0$, Let $(\eta_{(m+1)}, v_{(m+1)}, Q_{(m+1)})$ be the solution for (4.2.178) with initial data (Id, v_0, Q_0), where the (linearized) coefficients are determined by $(\eta_{(m)}, v_{(m)}, Q_{(m)})$. The goal is to prove the sequence $\{(\eta_{(m)}, v_{(m)})\}_{m\geq 0}$ strongly converges and the limit verifies the nonlinear approximate κ-problem. This can be done by Picard iteration. We will first establish the $H^{4.5}$-energy estimate for $(\eta_{(m)}, v_{(m)})$, and then this estimate can be carried over to the difference between two successive systems (4.2.178) which yields the convergence of $(\eta_{(m)}, v_{(m)})$ as $m \to \infty$.

4.2.7.1 A priori estimate of the linearized approximate problem

Let $m \geq 0$ be fixed and assume the solutions $(\eta_{(l)}, v_{(l)}, q_{(l)})$ are known for all $l \leq m$. For the sake of clean notations, we will denote $(\eta_{(m+1)}, v_{(m+1)}, Q_{(m+1)})$ by (η, v) and $(\eta_{(m)}, v_{(m)}, Q_{(m)})$ by $(\mathring{\eta}, \mathring{v}, \mathring{q})$ if no confusion is raised.

Proposition 4.2.16. For each fixed $\kappa > 0$, there exists some $T_\kappa > 0$ such that the solution (η, v) for (4.2.178) satisfies

$$\sup_{0\leq t\leq T_\kappa} \widehat{E}(t) \leq \mathcal{C}, \tag{4.2.224}$$

where $\mathcal{C}$ is a constant depends on $\|v_0\|_{4.5}$, $\|b_0\|_{4.5}$, $|v_0|_5$, provided that

$$\|\mathring{\tilde{J}}(t) - 1\|_{3.5} + \|\mathrm{Id} - \mathring{\tilde{\mathbf{A}}}(t)\|_{3.5} + \|\mathrm{Id} - \mathring{\tilde{\mathbf{A}}}^T \mathring{\tilde{\mathbf{A}}}(t)\|_{3.5} \leq \varepsilon. \tag{4.2.225}$$

holds for all $t \in [0, T_\kappa]$. Here the energy functional $\widehat{E}$ of (4.2.178) is defined to be

$$\widehat{E}(t) = \widehat{E}^{(1)}(t) + \widehat{E}^{(2)}(t), \tag{4.2.226}$$

where

$$\widehat{E}^{(1)}(t) := \|\eta\|_{4.5}^2 + \sum_{k=0}^{3} \left\|\partial_t^k v, \partial_t^k (b_0\cdot\partial)\eta\right\|_{4.5-k}^2 + \left\|\partial_t^4 v, \partial_t^4 (b_0\cdot\partial)\eta\right\|_0^2$$

$$\widehat{E}^{(2)}(t) := \frac{\kappa}{\sigma}\int_0^T \left|\partial_t^4 v\cdot \mathring{\tilde{n}}\right|_1^2\,\mathrm{d}t + \kappa\left(\int_0^T \|\partial_t^4 v\|_{1.5}^2 + \int_0^T \|\partial_t^4 (b_0\cdot\partial)\eta\|_{1.5}^2\right).$$

It can be seen that $\widehat{E}(t)$ is significantly simpler than $E_\kappa(t)$ given in (4.2.9). In particular, *no* boundary terms appear in $\widehat{E}^{(1)}(t)$ since $-\sigma\sqrt{\mathring{g}}(\triangle_{\mathring{g}}\mathring{\eta}\cdot\mathring{\tilde{n}})\mathring{\tilde{n}}^{\alpha}$ is a fixed term in the linearized equations. In addition to this, we only need to perform the tangential energy estimate consists four time derivatives. Since κ is fixed, the boundary terms that involve at least two spatial derivatives can be controlled by study the elliptic equation generated by the boundary condition (i.e., (4.2.236)). Also, the following observation shall be used frequently throughout the rest of this section.;

Removing extra (tangential) spatial derivatives: By (3.4.4), we can absorb additional tangential spatial derivatives when necessary. This allows us to greatly simplify most of the estimates on the boundary. Thus, (4.2.224) is a direct consequence of

$$\sup_{0\le t\le T_\kappa}\widehat{E}(t) \lesssim_{\kappa^{-1}} \mathcal{C}(\|v_0\|_{4.5}, \|b_0\|_{4.5}, |v_0|_5) + C(\varepsilon)\sup_{0\le t\le T_\kappa}\widehat{E}(t) + (\sup_{0\le t\le T_\kappa}\mathcal{P})\int_0^T \mathcal{P}, \quad (4.2.227)$$

where $\mathcal{P} = P(\widehat{E}(t), \|\mathring{v}\|_{4.5}, \|(b_0\cdot\partial)\mathring{\eta}\|_{4.5})$. Also, we will drop the subscript κ and denote $T_\kappa = T$ for the sake of clean notations. Similar to (4.2.13) we shall assume that $\sup_{0\le t\le T}\widehat{E}(t) = \widehat{E}(T)$, and this allows us to drop sup $\sup_{0\le t\le T_\kappa}$ in (4.2.227). In other words, we only need to show

$$\widehat{E}(T) \lesssim_{\kappa^{-1}} \mathcal{P}_0 + C(\varepsilon)\widehat{E}(T) + \mathcal{P}\int_0^T \mathcal{P}, \quad (4.2.228)$$

where $\mathcal{P}_0 = \mathcal{P}(\widehat{E}(0), \|q(0)\|_{4.5}, \|q_t(0)\|_{3.5}, \|q_{tt}(0)\|_{2.5})$. We remark here that (4.2.228) does not have to be uniform in κ, and so the RHS may depend on $\frac{1}{\kappa}$. This fact allows us to greatly simplify some of the boundary estimates (See Section 4.2.7.1).

Interior estimates We control

$$\|\partial_t^k v\|_{4.5-k}^2, \quad \|\partial_t^k (b_0\cdot\partial)\eta\|_{4.5-k}^2, \quad k=0,1,2,3, \tag{4.2.229}$$

by applying the div-curl estimate:

$$\|\partial_t^k v\|_{4.5-k}^2 \lesssim \|\partial_t^k \text{div}\, v\|_{3.5-k}^2 + \|\partial_t^k \text{curl}\, v\|_{3.5-k}^2 + |\partial_t^k v^3|_{4-k}, \tag{4.2.230}$$

$$\|\partial_t^k (b_0\cdot\partial)\eta\|_{4.5-k}^2 \lesssim \|\partial_t^k \text{div}\, (b_0\cdot\partial)\eta\|_{3.5-k}^2 + \|\partial_t^k \text{curl}\, (b_0\cdot\partial)\eta\|_{3.5-k}^2 + |\partial_t^k (b_0\cdot\partial)\eta^3|_{4-k}. \tag{4.2.231}$$

These are identical to those in Section 4.2.2.2so we shall not repeat the proofs. We also need the estimates for the interior Sobolev norms of the pressure Q, which is identical to Section 4.2.2.1. Furthermore, the top order interior term in $\widehat{E}^{(2)}$ that

$$\kappa\Big(\int_0^T \|\partial_t^4 v\|_{1.5}^2 + \int_0^T \|\partial_t^4 (b_0\cdot\partial)\eta\|_{1.5}^2\Big) \le \mathcal{P}_0 + C(\varepsilon)\widehat{E}(T) + \mathcal{P}\int_0^T \mathcal{P} \tag{4.2.232}$$

is identical to Section 4.2.4.

Boundary estimates This subsection is devoted to control the boundary terms $|\partial_t^k v^3|_{4-k}$ and $|\partial_t^k (b_0\cdot\partial)\eta^3|_{4-k}$ for $k=0,1,2,3$. Our goal is to show

Lemma 4.2.17. For $k=0,1,2,3$, we have

$$|\partial_t^k v^3|_{4-k}^2 \lesssim_{\kappa^{-1}} \mathcal{P}_0 + C(\varepsilon)\widehat{E}(T) + \mathcal{P}\int_0^T \mathcal{P}, \tag{4.2.233}$$

$$|\partial_t^k (b_0\cdot\partial)\eta^3|_{4-k}^2 \lesssim_{\kappa^{-1}} \mathcal{P}_0 + C(\varepsilon)\widehat{E}(T) + \mathcal{P}\int_0^T \mathcal{P}. \tag{4.2.234}$$

Note that we no longer require the energy bound to be κ-independent. Hence, *we can use* (3.4.4) *to absorb extra tangential spatial derivatives on the smoothed variables.* Recall that the boundary condition in the linearized equations reads

$$\sqrt{\mathring{\tilde{g}}}Q = -\sigma\sqrt{\mathring{g}}\triangle_{\mathring{g}}\mathring{\eta}\cdot\mathring{\tilde{n}} + \kappa(1-\overline{\triangle})(v\cdot\mathring{\tilde{n}}). \tag{4.2.235}$$

This can be converted to an elliptic equation satisfied by $v\cdot\mathring{\tilde{n}}$, i.e.,

$$\overline{\triangle}(v\cdot\mathring{\tilde{n}}) = v\cdot\mathring{\tilde{n}} - \kappa^{-1}\left(\sqrt{\mathring{\tilde{g}}}q + \sigma\sqrt{\mathring{g}}\triangle_{\mathring{g}}\mathring{\eta}\cdot\mathring{\tilde{n}}\right). \tag{4.2.236}$$

Now, invoking the standard elliptic estimate and (3.1.9), we get

$$|v\cdot\mathring{\tilde{n}}|_4^2 \lesssim |v\cdot\mathring{\tilde{n}}|_2^2 + \kappa^{-1}\left(\left|\sqrt{\mathring{\tilde{g}}}Q\right|_2^2 + \sigma P(|\overline{\partial}\mathring{\eta}|_{L^\infty}, |\overline{\partial}^2\mathring{\eta}|_{L^\infty})|\mathring{\eta}|_4^2\right) \lesssim_{\kappa^{-1}} \mathcal{P}_0 + \int_0^T \mathcal{P}, \tag{4.2.237}$$

where the used the trace lemma and (4.2.26) in the second inequality.

Since $(b_0\cdot\partial) = b_0^j\overline{\partial}_j$ on Γ and hence $(b_0\cdot\partial)(\eta\cdot\mathring{\tilde{n}})|_{t=0} = 0$, we have

$$(b_0\cdot\partial)(\eta\cdot\mathring{\tilde{n}}) = \int_0^T \partial_t\Big((b_0\cdot\partial)(\eta\cdot\mathring{\tilde{n}})\Big) = \int_0^T (b_0\cdot\partial)(v\cdot\mathring{\tilde{n}}) + \int_0^T (b_0\cdot\partial)(\eta\cdot\partial_t\mathring{\tilde{n}}). \tag{4.2.238}$$

Since $\partial_t\mathring{\tilde{n}} = -\mathring{\tilde{g}}^{kl}\overline{\partial}_k\mathring{\tilde{v}}\cdot\mathring{\tilde{n}}\overline{\partial}_l\mathring{\tilde{\eta}} = Q(\overline{\partial}\mathring{\tilde{\eta}})\overline{\partial}\mathring{\tilde{v}}\cdot\mathring{\tilde{n}}$, and invoking (3.4.4) we have

$$\left|\int_0^T (b_0\cdot\partial)(\eta\cdot\partial_t\mathring{\tilde{n}})\right|_4^2 \lesssim T\int_0^T |(b_0\cdot\partial)(\eta\cdot\partial_t\mathring{\tilde{n}})|_4^2 \lesssim_{\kappa^{-1}} \int_0^T \mathcal{P} \tag{4.2.239}$$

Here, we need (3.4.4) to control the leading order term generated when $\overline{\partial}^4(b_0\cdot\partial)$ fall on $\overline{\partial}\mathring{\tilde{v}}$ (which is part of $\partial_t\mathring{\tilde{n}}$), i.e.,

$$\int_0^T Q(|\mathring{\tilde{\eta}}|_{L^\infty}, |\overline{\partial}\mathring{\tilde{\eta}}|_{L^\infty})|(b_0\cdot\partial)\overline{\partial}\mathring{\tilde{v}}|_4^2 \lesssim_{\kappa^{-1}} \int_0^T |b_0|_4^2 Q(|\mathring{\tilde{\eta}}|_{L^\infty}, |\overline{\partial}\mathring{\tilde{\eta}}|_{L^\infty})|\mathring{v}|_4^2.$$

In addition,

$$\left|\int_0^T (b_0\cdot\partial)(v\cdot\mathring{\tilde{n}})\right|_4^2 \lesssim T\int_0^T |(b_0\cdot\partial)(v\cdot\mathring{\tilde{n}})|_4^2, \tag{4.2.240}$$

and the RHS can be controlled by studying the elliptic equation satisfied by $(b_0\cdot\partial)(v\cdot\mathring{\tilde{n}})$. Taking

$(b_0 \cdot \partial)$ on (4.2.236) and we get

$$\overline{\triangle}(b_0\cdot\partial)(v\cdot\mathring{\tilde{n}}) = [\overline{\triangle}, (b_0\cdot\partial)](v\cdot\mathring{\tilde{n}})+(b_0\cdot\partial)(v\cdot\mathring{\tilde{n}})-\kappa^{-1}\left((b_0\cdot\partial)(\sqrt{\mathring{\tilde{g}}}q)+\sigma(b_0\cdot\partial)(\sqrt{\mathring{g}}\triangle_{\mathring{g}}\mathring{\eta}\cdot\mathring{\tilde{n}})\right), \tag{4.2.241}$$

then the elliptic estimate implies

$$\int_0^T |(b_0\cdot\partial)(v\cdot\mathring{\tilde{n}})|_4^2 \lesssim_{\kappa^{-1}} \int_0^T \mathcal{P}. \tag{4.2.242}$$

Thus,

$$|(b_0\cdot\partial)(\eta\cdot\mathring{\tilde{n}})|_4^2 \lesssim_{\kappa} \int_0^T \mathcal{P}. \tag{4.2.243}$$

We can obtain the bounds for $|v^3|_4^2$ and $|(b_0\cdot\partial)\eta^3|_4^2$ from (4.2.237) and (4.2.243), respectively. Indeed, we have

$$|v^3|_4^2 \leq |v\cdot\mathring{\tilde{n}}|_4^2 + |v\cdot(N-\mathring{\tilde{n}})|_4^2, \tag{4.2.244}$$

$$|(b_0\cdot\partial)\eta^3|_4^2 \leq |(b_0\cdot\partial)(\eta\cdot\mathring{\tilde{n}})|_4^2 + |(b_0\cdot\partial)(\eta\cdot(N-\mathring{\tilde{n}}))|_4^2. \tag{4.2.245}$$

Since $N-\mathring{\tilde{n}} = -\int_0^T \partial_t\mathring{\tilde{n}} = \int_0^T Q(\overline{\partial}\mathring{\tilde{\eta}})\overline{\partial}\mathring{\tilde{v}}\cdot\mathring{\tilde{n}}$, invoking the proof for (4.2.18) and (3.4.4), we have

$$|N-\mathring{\tilde{n}}|_5 \lesssim_{\kappa^{-1}} \int_0^T \mathcal{P}. \tag{4.2.246}$$

Therefore,

$$|v\cdot(N-\mathring{\tilde{n}})|_4^2 + |(b_0\cdot\partial)(\eta\cdot(N-\mathring{\tilde{n}}))|_4^2 \lesssim_{\kappa^{-1}} \mathcal{P}\int_0^T \mathcal{P}. \tag{4.2.247}$$

Now, we can take time derivative ∂_t in (4.2.236) to get the elliptic equation of $\partial_t(v\cdot\mathring{\tilde{n}})$ on the boundary, i.e.,

$$\overline{\triangle}\partial_t(v\cdot\mathring{\tilde{n}}) = \partial_t(v\cdot\mathring{\tilde{n}})-\kappa^{-1}\left(\partial_t(\sqrt{\mathring{\tilde{g}}}q)+\sigma\partial_t(\sqrt{\mathring{g}}\triangle_{\mathring{g}}\mathring{\eta}\cdot\mathring{\tilde{n}})\right). \tag{4.2.248}$$

Then standard elliptic estimate gives

$$|\partial_t(v\cdot\mathring{\tilde{n}})|_3^2 \lesssim_{\kappa^{-1}} \mathcal{P}_0+\int_0^T \mathcal{P}. \tag{4.2.249}$$

This estimate implies the estimate for $|\partial_t v^3|_3^2$ by writing $|\partial_t v^3|_3^2 \le |\partial_t(v\cdot\mathring{\tilde{n}})|_3^2+|\partial_t(v\cdot(N-\mathring{\tilde{n}}))|_3^2$. Moreover, in light of the estimate for $|v^3|_4^2$, we have

$$|\partial_t(b_0\cdot\partial)\eta^3|_3 = |(b_0\cdot\partial)v^3|_3^2 \le P(|b_0|_3)|v^3|_4^2 \lesssim_{\kappa^{-1}} \mathcal{P}_0+\mathcal{P}\int_0^T \mathcal{P}. \tag{4.2.250}$$

Similarly, by taking two time derivatives to (4.2.236), we can control $|\partial_t^2(v\cdot\mathring{\tilde{n}})|_2$ by the standard elliptic estimate $|\partial_t^2(v\cdot\mathring{\tilde{n}})|_2^2 \lesssim_{\kappa^{-1}} \mathcal{P}_0+\int_0^T \mathcal{P}$, and this yields

$$|\partial_t^2 v^3|_2^2 \lesssim_{\kappa^{-1}} \mathcal{P}_0+\mathcal{P}\int_0^T \mathcal{P}. \tag{4.2.251}$$

In addition to this, $|\partial_t^2(b_0\cdot\partial)\eta^3|_2^2$ reduces to $|\partial_t v^3|_3^2$, whose bound is given above. Also $|\partial_t^3(b_0\cdot\partial)\eta^3|_1^2$ reduces to $|\partial_t^2 v^3|_2^2$, which is just (4.2.251).

Finally, $|\partial_t^3 v^3|_1^2$ can be controlled with the help $\widehat{E}^{(2)}$. We can make use of the κ-weighted higher order terms to directly control the time integrated terms on the boundary. Specifically, by writing $|\partial_t^3 v^3|_1 \le \mathcal{P}_0+\int_0^T |\partial_t^4 v^3|_1$, we have

$$|\partial_t^3 v^3|_1^2 \lesssim \mathcal{P}_0+\left(\int_0^T |\partial_t^4 v^3|_1\right)^2 \lesssim \mathcal{P}_0+T\int_0^T |\partial_t^4 v^3|_1^2, \tag{4.2.252}$$

where $T\int_0^T |\partial_t^4 v^3|_1^2 \le T\int_0^T |\partial_t^4 v\cdot\mathring{\tilde{n}}|_1^2+T\int_0^T |\partial_t^4 v\cdot(N-\mathring{\tilde{n}})|_1^2$. Here, the second term on the RHS is $\lesssim_{\kappa^{-1}} TC(\varepsilon)\int_0^T \|\partial_t^4 v\|_{1.5}^2$ whereas the first term is $\lesssim_{\kappa^{-1}} T\int_0^T |\partial_t^4 v\cdot\mathring{\tilde{n}}|_1^2$. Therefore, by choosing T sufficiently small, we have

$$|\partial_t^3 v^3|_1^2 \lesssim_{\kappa^{-1}} \mathcal{P}_0+C(\varepsilon)\widehat{E}(T)+\mathcal{P}\int_0^T \mathcal{P}. \tag{4.2.253}$$

Tangential estimate with four time derivatives We still need to control $\|\partial_t^4 v\|_0^2$, $\|\partial_t^4(b_0\cdot\partial)\eta\|_0^2$ to finish the control of $\widehat{E}$. In fact, we only need to control $\|\partial_t^4 v\|_0^2$ since $\|\partial_t^4(b_0\cdot\partial)\eta\|_0^2$ reduces to $\|\partial_t^3 v\|_1^2$. Now we compute the L^2-estimate of $\partial_t^4 v$ and $\partial_t^4(b_0\cdot\partial)\eta$. Invoking (4.2.178) and integrating

$(b_0 \cdot \partial)$ by parts, we get

$$\frac{1}{2}\int_0^T \frac{d}{dt}\int_\Omega |\partial_t^4 v|^2 + \left|\partial_t^4 (b_0\cdot\partial)\eta\right|^2 \,\mathrm{d}y\,\mathrm{d}t = -\int_0^T\int_\Omega \partial_t^4 v_\alpha \partial_t^4 (\mathring{\tilde{\mathbf{A}}}^{\mu\alpha}\partial_\mu Q)\,\mathrm{d}y\,\mathrm{d}t$$

$$= -\int_0^T\int_\Omega \partial_t^4 v_\alpha \mathring{\tilde{\mathbf{A}}}^{\mu\alpha}\partial_t^4\partial_\mu Q\,\mathrm{d}y\,\mathrm{d}t \underbrace{-\int_0^T\int_\Omega \partial_t^4 v_\alpha \left[\partial_t^4, \mathring{\tilde{\mathbf{A}}}^{\mu\alpha}\right]\partial_\mu Q\,\mathrm{d}y\,\mathrm{d}t}_{\mathring{I}_1}$$

$$= \int_0^T\int_\Omega \mathring{\tilde{\mathbf{A}}}^{\mu\alpha}\partial_t^4\partial_\mu v_\alpha \partial_t^4 q\,\mathrm{d}y\,\mathrm{d}t \underbrace{-\int_0^T\int_\Gamma \partial_t^4 v_\alpha \mathring{\tilde{\mathbf{A}}}^{3\alpha}\partial_t^4 Q\,\mathrm{d}S\,\mathrm{d}t}_{\mathring{I}_B} \underbrace{+\int_0^T\int_{\Gamma_0}\partial_t^4 v_\alpha \mathring{\tilde{\mathbf{A}}}^{3\alpha}\partial_t^4 Q\,\mathrm{d}S\,\mathrm{d}t}_{=0} + \mathring{I}_1$$

$$= \int_0^T\int_\Omega \underbrace{\partial_t^4(\mathrm{div}_{\mathring{\tilde{\mathbf{A}}}} v)}_{=0}\partial_t^4 Q\,\mathrm{d}y\,\mathrm{d}t \underbrace{+\int_0^T\int_\Omega\left[\mathring{\tilde{\mathbf{A}}}^{\mu\alpha}, \partial_t^4\right]\partial_\mu v_\alpha\ \partial_t^4 Q\,\mathrm{d}y\,\mathrm{d}t}_{\mathring{I}_2} + \mathring{I}_B + \mathring{I}_1. \tag{4.2.254}$$

Here, $\mathring{I}_1$ and $\mathring{I}_2$ can be controlled by $\int_0^T \mathcal{P}$. We analyze the boundary integral I_B.

$$\begin{aligned}\mathring{I}_B =& -\int_0^T\int_\Gamma \partial_t^4 v_\alpha \mathring{\tilde{\mathbf{A}}}^{3\alpha}\partial_t^4 Q = -\int_0^T\int_\Gamma \sqrt{\mathring{\tilde{g}}}(\partial_t^4 v\cdot\mathring{\tilde{n}})(\partial_t^4 Q)\\ =&\sigma\int_0^T\int_\Gamma \sqrt{\mathring{\tilde{g}}}(\partial_t^4 v\cdot\mathring{\tilde{n}})\partial_t^4(\sqrt{\mathring{g}}\mathring{\tilde{g}}^{-\frac{1}{2}}\triangle_{\mathring{g}}\mathring{\eta}\cdot\mathring{\tilde{n}})\\ &-\kappa\int_0^T\int_\Gamma \sqrt{\mathring{\tilde{g}}}(\partial_t^4 v\cdot\mathring{\tilde{n}})\partial_t^4\left(\mathring{\tilde{g}}^{-\frac{1}{2}}(1-\overline{\triangle})(v\cdot\mathring{\tilde{n}})\right) := \mathring{I}_{B1} + \mathring{I}_{B2}.\end{aligned} \tag{4.2.255}$$

Invoking the identity (3.1.9), we have

$$\begin{aligned}\mathring{I}_{B1} =&\sigma\int_0^T\int_\Gamma \sqrt{\mathring{\tilde{g}}}(\partial_t^4 v\cdot\mathring{\tilde{n}})\partial_t^4(\sqrt{\mathring{g}}\mathring{\tilde{g}}^{-\frac{1}{2}}\mathring{g}^{ij}\overline{\partial}_i\overline{\partial}_j\mathring{\eta}\cdot\mathring{\tilde{n}})\\ &-\sigma\int_0^T\int_\Gamma \sqrt{\mathring{\tilde{g}}}(\partial_t^4 v\cdot\mathring{\tilde{n}})\partial_t^4(\sqrt{\mathring{g}}\mathring{\tilde{g}}^{-\frac{1}{2}}\mathring{g}^{ij}\mathring{g}^{kl}\overline{\partial}_l\mathring{\eta}^\mu\overline{\partial}_i\overline{\partial}_j\mathring{\eta}_\mu\overline{\partial}_k\mathring{\eta}\cdot\mathring{\tilde{n}})\\ =&\mathring{I}_{B11} + \mathring{I}_{B12}.\end{aligned} \tag{4.2.256}$$

Since $\mathring{I}_{B11} \stackrel{L}{=} \sigma \int_0^T \int_\Gamma (\partial_t^4 v \cdot \mathring{\tilde{n}})(\sqrt{\mathring{g}}\mathring{g}^{ij}\overline{\partial}_i\overline{\partial}_j\partial_t^3\mathring{v} \cdot \mathring{\tilde{n}})$, we integrate $\overline{\partial}_i$ by parts,

$$\mathring{I}_{B11} \stackrel{L}{=} -\sigma \int_0^T \int_\Gamma (\overline{\partial}_i\partial_t^4 v \cdot \mathring{\tilde{n}})(\sqrt{\mathring{g}}\mathring{g}^{ij}\overline{\partial}_j\partial_t^3\mathring{v} \cdot \mathring{\tilde{n}})$$

$$\lesssim_{\kappa^{-1}} \varepsilon \int_0^T \|\sqrt{\kappa}\partial_t^4 v\|_{1.5}^2 + \int_0^T \mathcal{P} \leq \varepsilon \widehat{E}(T) + \int_0^T \mathcal{P},$$

and $\mathring{I}_{B12}$ can be treated in the same fashion.

Next we study $\mathring{I}_{B2}$. We have

$$\mathring{I}_{B2} \stackrel{L}{=} \kappa \int_0^T \int_\Gamma (\partial_t^4 v \cdot \mathring{\tilde{n}})\overline{\triangle}(\partial_t^4 v \cdot \mathring{\tilde{n}}) + \kappa \int_0^T \int_\Gamma (\partial_t^4 v \cdot \mathring{\tilde{n}})\overline{\triangle}(v \cdot \partial_t^4 \mathring{\tilde{n}}) := \mathring{I}_{B21} + \mathring{I}_{B22}, \quad (4.2.257)$$

where $\mathring{I}_{B21}$ contributes to the positive energy term $\int_0^T |\partial_t^4 v \cdot \mathring{\tilde{n}}|_1^2$ after integrating $\overline{\partial}$ by parts and moving the resulting term to the LHS. In addition, since $\partial_t^4\mathring{\tilde{n}} = Q(\overline{\partial}\mathring{\tilde{\eta}})\overline{\partial}\partial_t^3\mathring{\tilde{v}} \cdot \mathring{\tilde{n}} +$ lower-order terms,

$$\mathring{I}_{B22} \stackrel{L}{=} -\kappa \int_0^T \int_\Gamma (\overline{\partial}\partial_t^4 v \cdot \mathring{\tilde{n}})(v \cdot Q(\overline{\partial}\mathring{\tilde{\eta}})\overline{\partial}^2\partial_t^3\mathring{\tilde{v}} \cdot \mathring{\tilde{n}})$$

$$\lesssim \varepsilon \int_0^T \|\partial_t^4 v\|_{1.5}^2 + \int_0^T |v|_{L^\infty}^2 Q(|\overline{\partial}\mathring{\tilde{\eta}}|_{L^\infty})|\overline{\partial}^2\partial_t^3\mathring{\tilde{v}}|_0^2$$

$$\lesssim_{\kappa^{-1}} \varepsilon \widehat{E}(T) + \int_0^T |v|_{L^\infty}^2 Q(|\overline{\partial}\mathring{\tilde{\eta}}|_{L^\infty})|\overline{\partial}\partial_t^3\mathring{v}|_0^2 \leq \varepsilon\widehat{E}(T) + \int_0^T \mathcal{P}. \quad (4.2.258)$$

This concludes the proof of Proposition 4.2.16.

4.2.7.2 Picard iteration

Now we prove that the sequence $\{(\eta_{(m)}, v_{(m)}, Q_{(m)})\}_{m\in\mathbb{N}^*}$ has a strongly convergent subsequence. We define $[f]_{(m)} := f_{(m+1)} - f_{(m)}$ for any function f and then $([\eta]_{(m)}, [v]_{(m)}, [Q]_{(m)})$ satisfies the following system

$$\begin{cases} \partial_t[\eta]_{(m)} = [v]_{(m)} & \text{in } \Omega, \\ \partial_t[v]_{(m)} - (b_0\cdot\partial)^2[\eta]_{(m)} + \nabla_{\tilde{\mathbf{A}}_{(m)}}[Q]_{(m)} = -\nabla_{[\tilde{\mathbf{A}}]_{(m-1)}} Q_{(m)} & \text{in } \Omega, \\ \text{div}_{\,\tilde{\mathbf{A}}_{(m)}}[v]_{(m)} = -\text{div}_{\,[\tilde{\mathbf{A}}]_{(m-1)}} v_{(m)} & \text{in } \Omega, \\ [Q]_{(m)} = \kappa(1-\overline{\triangle})([v]_{(m)}\cdot\tilde{n}_{(m)}) + h_{(m)} & \text{on } \Gamma, \\ ([\eta]_{(m)}, [v]_{(m)})|_{t=0} = (\mathbf{0}, \mathbf{0}). \end{cases} \quad (4.2.259)$$

where

$$
\begin{aligned}
h_{(m)} =& \kappa(1-\overline{\triangle})(v_{(m)}\cdot[\tilde{n}]_{(m-1)}) \\
& -\sigma\left(\sqrt{\tilde{g}_{(m)}}g^{ij}_{(m)}\Pi^{\lambda}_{(m)\alpha}\overline{\partial}_i\overline{\partial}_j\eta_{(m)\lambda}\tilde{n}^{\alpha}_{(m)}-\sqrt{\tilde{g}_{(m-1)}}g^{ij}_{(m-1)}\Pi^{\lambda}_{(m-1)\alpha}\overline{\partial}_i\overline{\partial}_j\eta_{(m-1)\lambda}\tilde{n}^{\alpha}_{(m-1)}\right)
\end{aligned}
$$

We also define the energy functional of $([\eta]_{(m)},[v]_{(m)},[Q]_{(m)})$ to be

$$
[\widehat{E}]_{(m)}:=[\widehat{E}]^{(1)}_{(m)}+[\widehat{E}]^{(2)}_{(m)}, \tag{4.2.260}
$$

where

$$
\begin{aligned}
&[\widehat{E}]^{(1)}_{(m)}(T):=\left\|[\eta]_{(m)}\right\|^2_{3.5}+\sum_{k=0}^{2}\left\|\partial_t^k[v]_{(m)},\partial_t^k(b_0\cdot\partial)[\eta]_{(m)}\right\|^2_{3.5-k}+\left\|\partial_t^3[v]_{(m)},\partial_t^3(b_0\cdot\partial)[\eta]_{(m)}\right\|^2_0 \\
&[\widehat{E}]^{(2)}_{(m)}(T):=\frac{\kappa}{\sigma}\int_0^T\left|\partial_t^3[v]_{(m)}\cdot\tilde{n}_{(m)}\right|^2_1\,\mathrm{d}t+\kappa\left(\int_0^T\|\partial_t^3[v]_{(m)}\|^2_{1.5}+\int_0^T\|\partial_t^3(b_0\cdot\partial)[\eta]_{(m)}\|^2_{1.5}\right).
\end{aligned} \tag{4.2.261}
$$

The div-curl estimates For $k=0,1,2$

$$
\begin{aligned}
\|\partial_t^k[v]_{(m)}\|_{3.5-k}\lesssim&\|\partial_t^k[v]_{(m)}\|_0+\|\text{div }\partial_t^k[v]_{(m)}\|_{2.5-k} \\
&+\|\text{curl }\partial_t^k[v]_{(m)}\|_{2.5-k}+|\overline{\partial}\partial_t^k[v]_{(m)}\cdot N|_{2-k},
\end{aligned} \tag{4.2.262}
$$

$$
\begin{aligned}
\|\partial_t^k(b_0\cdot\partial)[\eta]_{(m)}\|_{3.5-k}\lesssim&\|\partial_t^k(b_0\cdot\partial)[\eta]_{(m)}\|_0+\|\text{div }\partial_t^k(b_0\cdot\partial)[\eta]_{(m)}\|_{2.5-k} \\
&+\|\text{curl }\partial_t^k(b_0\cdot\partial)[\eta]_{(m)}\|_{2.5-k}+|\overline{\partial}\partial_t^k(b_0\cdot\partial)[\eta]_{(m)}\cdot N|_{2-k}.
\end{aligned} \tag{4.2.263}
$$

Again, each part in the div-curl estimates should follow in the same way as in Section 4.2.2.2 so we omit the proof. Similarly, to control the interior terms in $[\widehat{E}]^{(2)}_{(m)}$, we follow Section 4.2.4 to get

$$\begin{aligned}
&\kappa\left(\int_0^T \|\partial_t^3[v]_{(m)}\|_{1.5}^2 + \int_0^T \|\partial_t^3(b_0\cdot\partial)[\eta]_{(m)}\|_{1.5}^2\right) \\
\lesssim &\mathcal{P}_0 + \varepsilon[\widehat{E}]_{(m)}(T) + P([\widehat{E}]_{(m)}(T), \widehat{E}_{(m),(m-1)}(T)) \int_0^T P([\widehat{E}]_{(m),(m-1)}(t), \widehat{E}_{(m),(m-1)}(t))\,\mathrm{d}t.
\end{aligned} \tag{4.2.264}$$

Elliptic estimates of pressure Similarly as in Section 4.2.2.1, one can derive the elliptic equation verified by $[Q]_{(m)}$ and its time derivatives with Neumann boundary conditions. The only difference is that we need to control the contribution of $(\nabla_{[\tilde{\mathbf{A}}]_{(m-1)}} Q_{(m)})$ and its time derivatives, but this is straightforward. For example, we need to control $\|\text{div}_{\tilde{\mathbf{A}}_{(m)}}(\nabla_{[\tilde{\mathbf{A}}]_{(m-1)}} Q_{(m)})\|_{1.5}$ in the estimate of $\|[Q]_{(m)}\|_{3.5}$.

$$\|\text{div}_{\tilde{\mathbf{A}}_{(m)}}(\nabla_{[\tilde{\mathbf{A}}]_{(m-1)}} Q_{(m)})\|_{1.5} \lesssim P(\|[\tilde{\mathbf{A}}]_{(m-1)}\|_{2.5}, \|Q_{(m)}\|_{3.5}, \|\tilde{\mathbf{A}}_{(m)}\|_{2.5}),$$

and the boundary contribution

$$|\tilde{\mathbf{A}}_{(m)} N \cdot \nabla_{[\tilde{\mathbf{A}}]_{(m-1)}} Q_{(m)}|_2 \lesssim P(\|[\tilde{\mathbf{A}}]_{(m-1)}\|_{2.5}, \|Q_{(m)}\|_{3.5}, \|\tilde{\mathbf{A}}_{(m)}\|_{2.5}).$$

Boundary estimates The boundary estimates also follow in the same way as Section 4.2.7.1 because the energy is not required to be indepedent of κ. We can derive an elliptic equation on Γ, analogous with (4.2.236)

$$\kappa\overline{\triangle}([v]_{(m)}\cdot\tilde{n}_{(m)}) = \kappa([v]_{(m)}\cdot\tilde{n}_{(m)}) + h_{(m)} - [Q]_{(m)}. \tag{4.2.265}$$

Then using the boundary elliptic estimates, we get

$$\begin{aligned}
|[v]_{(m)}\cdot\tilde{n}_{(m)}|_3 &\lesssim_{\kappa^{-1}} |[v]_{(m)}\cdot\tilde{n}_{(m)}|_1 + |h_{(m)}|_1 + \|[Q]_{(m)}\|_{1.5} \\
&\lesssim \mathcal{P}_0 + P(\widehat{E}_{(m),(m-1)}(T)) \int_0^T P([\widehat{E}]_{(m),(m-1)}(t), \widehat{E}_{(m),(m-1)}(t))\,\mathrm{d}t.
\end{aligned} \tag{4.2.266}$$

As for the magnetic field, we use the fact that $(b_0 \cdot \partial) = b_0^j \overline{\partial}_j$ on Γ to get

$$(b_0 \cdot \partial)[\eta]_{(m)} \cdot \tilde{n}_{(m)} = 0 + \int_0^T (b_0 \cdot \partial)[v]_{(m)} \cdot \tilde{n}_{(m)} + (b_0 \cdot \partial)[\eta]_{(m)} \cdot \partial_t \tilde{n}_{(m)}.$$

Similarly as in Section 4.2.7.1, one can directly control the $H^3(\Gamma)$-norm of the second term. Then the first term can be controlled by using elliptic estimates in $(b_0 \cdot \partial)$-differentiated elliptic equation (4.2.265). We omit the detailed proof because there is no essential difference from the argument in Section 4.2.7.1.

$$|(b_0 \cdot \partial)[\eta]_{(m)} \cdot \tilde{n}_{(m)}|_3 \lesssim_{\kappa^{-1}} \int_0^T P([\widehat{E}]_{(m)}(t), \widehat{E}_{(m)}(t))\,\mathrm{d}t. \tag{4.2.267}$$

Taking one time derivative, we can similarly control the boundary norm of $\partial_t[v]_{(m)}$ and $\partial_t(b_0 \cdot \partial)[\eta]_{(m)}$. We skip the details.

$$\begin{aligned} &\left|\partial_t[v]_{(m)} \cdot \tilde{n}_{(m)}, \partial_t(b_0 \cdot \partial)[\eta]_{(m)}\right|_2 \\ \lesssim_{\kappa^{-1}} &\mathcal{P}_0 + P(\widehat{E}_{(m),(m-1)}(T)) \int_0^T P([\widehat{E}]_{(m),(m-1)}(t), \widehat{E}_{(m),(m-1)}(t))\,\mathrm{d}t. \end{aligned} \tag{4.2.268}$$

For the $H^1(\Gamma)$-norm of $\partial_t^2[v]_{(m)}$ and $\partial_t^2(b_0 \cdot \partial)[\eta]_{(m)}$, one can use the κ-weighted interior terms in $[\widehat{E}]_{(m)}^{(2)}$ and Sobolev trace lemma to get the control

$$\begin{aligned} &\left|\partial_t^2[v]_{(m)}^3, \partial_t^2(b_0 \cdot \partial)[\eta]_{(m)}^3\right|_1 \lesssim \left\|\partial_t^2[v]_{(m)}, \partial_t^2(b_0 \cdot \partial)[\eta]_{(m)}\right\|_{1.5} \\ \lesssim &\mathcal{P}_0 + \sqrt{\frac{T}{\kappa}} \left\|\sqrt{\kappa}\partial_t^3[v]_{(m)}, \sqrt{\kappa}\partial_t^3(b_0 \cdot \partial)[\eta]_{(m)}\right\|_{L_t^2 H_y^{1.5}} \lesssim_{\kappa^{-0.5}} \mathcal{P}_0 + \sqrt{T} P([\widehat{E}]_{(m)}^{(2)}(T)). \end{aligned} \tag{4.2.269}$$

Finally, we need to control the difference between $X \cdot N$ and $X \cdot \tilde{n}_{(m)}$, which should be done in the same way as (4.2.244)-(4.2.246), so we do not repeat the calculations. For $k = 0, 1$, we have for $X = [v]_{(m)}, (b_0 \cdot \partial)[\eta]_{(m)}$

$$|\partial_t^k X^3 - \partial_t^k (X \cdot \tilde{n}_{(m)})|_{3-k} \lesssim_{\kappa^{-1}} \int_0^T P([\widehat{E}]_{(m)}, \widehat{E}_{(m),(m-1)}(t))\,\mathrm{d}t. \tag{4.2.270}$$

Combining (4.2.266)-(4.2.270), we get the boundary estimates as

$$\sum_{k=0}^{2}\left|\partial_t^k([v]_{(m)}^3,(b_0\cdot\partial)[\eta]_{(m)}^3)\right|_{3-k} \tag{4.2.271}$$

$$\lesssim_{\kappa^{-1}}\mathcal{P}_0+P(\widehat{E}_{(m),(m-1)}(T))\int_0^T P([\widehat{E}]_{(m),(m-1)}(t),\widehat{E}_{(m),(m-1)}(t))\,\mathrm{d}t.$$

Estimates of full time derivatives Now it remains to control the L^2-norm of full time derivatives. By replacing ∂_t^4 in Section 4.2.7.1 by ∂_t^3, we can do analogous computation to control $\|\partial_t^3[v]_{(m)}\|_0$ and $\|\partial_t^3(b_0\cdot\partial)[\eta]_{(m)}\|_0$. The κ-weighted boundary terms in $[\widehat{E}]_{(m)}^{(2)}$ are produced in the analogues of (4.2.255). The only difference is that we should control the extra contribution (under time integral) of $\nabla_{[\tilde{\mathbf{A}}]_{(m-1)}}Q_{(m)}$ in the interior and the σ-coefficient part in the term $h_{(m)}$ on the boundary. These quantities can all be directly controlled

$$\|\partial_t^3\nabla_{[\tilde{\mathbf{A}}]_{(m-1)}}Q_{(m)}\|_0\lesssim P\Bigg(\|[v]_{(m-1)},\partial_t[v]_{(m-1)},\partial_t^2[v]_{(m-1)}\|_2,$$
$$\|\partial_t^3Q_{(m)}\|_1,\|\partial_t^2Q_{(m)},\partial_tQ_{(m)},Q_{(m)}\|_2\Bigg).$$

$$|\partial_t^3h_{(m),\sigma}|_0\lesssim P\left(|\partial_t^2v_{(m),(m-1)}|_2,|\overline{\partial}\eta_{(m),(m-1)},\overline{\partial}v_{(m),(m-1)},\overline{\partial}\partial_tv_{(m),(m-1)}|_{L^\infty}\right).$$

Therefore, one can get

$$\|\partial_t^3[v]_{(m)}\|_0^2+\|\partial_t^3(b_0\cdot\partial)[\eta]_{(m)}\|_0^2+\frac{\kappa}{\sigma}\int_0^T\left|\partial_t^3[v]_{(m)}\cdot\tilde{n}_{(m)}\right|_1^2\,\mathrm{d}t \tag{4.2.272}$$
$$\lesssim\mathcal{P}_0+\int_0^T P([\widehat{E}]_{(m)}(t),\widehat{E}_{(m),(m-1)}(t))\,\mathrm{d}t$$

4.2.7.3 Well-posedness of the nonlinear approximate problem

We conclude this section with the following proposition.

Proposition 4.2.18. For each fixed $\kappa>0$, $\exists T_\kappa'>0$ such that the nonlinear κ-problem (4.2.2) has a unique strong solution $(\eta(\kappa), v(\kappa), q(\kappa))$ in $[0, T'_\kappa]$ that satisfies

$$\sup_{0\le t\le T'_\kappa} \widehat{E}'(t) \le \mathcal{C} \tag{4.2.273}$$

where $\widehat{E}'(t) = \widehat{E}^{(1)'}(t) + \widehat{E}^{(2)'}(t)$

$$\widehat{E}^{(1)}(t) := \|\eta\|_{4.5}^2 + \sum_{k=0}^{3} \left\|\partial_t^k v, \partial_t^k (b_0\cdot\partial)\eta\right\|_{4.5-k}^2 + \left\|\partial_t^4 v, \partial_t^4 (b_0\cdot\partial)\eta\right\|_0^2$$

$$\widehat{E}^{(2)}(t) := \frac{\kappa}{\sigma}\int_0^T \left|\partial_t^4 v\cdot \mathring{\tilde{n}}\right|_1^2 \,\mathrm{d}t + \kappa\left(\int_0^T \|\partial_t^4 v\|_{1.5}^2 + \int_0^T \|\partial_t^4 (b_0\cdot\partial)\eta\|_{1.5}^2\right).$$

Proof. Summarizing (4.2.262)-(4.2.264), (4.2.271)-(4.2.272), we get

$$[\widehat{E}]_{(m)}(T) \lesssim_\kappa^{-1} \mathcal{P}_0 + \varepsilon[\widehat{E}](T) + TP([\widehat{E}]_{(m)}(T))$$
$$+ P([\widehat{E}]_{(m)}(T), \widehat{E}_{(m),(m-1)}(T)) \int_0^T P([\widehat{E}]_{(m),(m-1)}(t), \widehat{E}_{(m),(m-1)}(t)).$$

By Proposition 4.2.16, there exists some $T'_\kappa > 0$, such that $\forall t \in [0, T'_\kappa]$, $[\widehat{E}]_{(m)}(t) \le \frac{1}{4}[\widehat{E}]_{(m-1)}(t)$, which implies $[\widehat{E}]_{(m)}(t) \le 4^{-m}\mathcal{P}_0$. Let $m \to \infty$, we know the sequence $\{(\eta_{(m)}, v_{(m)}, Q_{(m)})\}$ must strongly converge. The strong limit is denoted by $(\eta(\kappa), v(\kappa), q(\kappa))$ which exactly solves (4.2.2). By taking $m \to \infty$ in the energy of linearized equation (4.2.178), one can also get the energy estimates. □

4.2.8 Local well-posedness

4.2.8.1 Uniqueness and continuous dependence on initial data

Combining the conclusions of Proposition 4.2.4 and Propostion 4.2.18 and letting $\kappa \to 0_+$, we actually prove that there exists some time $T' > 0$ (only depends on the initial data), such that the original system (2.2.1) has solution (η, v, q) satisfying the energy estimates

$$\sup_{0\le t\le T} E(t) \le \mathcal{C},$$

where $\mathcal{C} = \mathcal{C}(\|v_0\|_{4.5}, \|b_0\|_{4.5}, |v_0|_5)$, and the energy functional E is defined to be

$$E(t) := \|\eta\|_{4.5}^2 + \sum_{k=0}^{3} \left\| \partial_t^k v, \partial_t^k (b_0 \cdot \partial)\eta \right\|_{4.5-k}^2 + \left\| \partial_t^4 v, \partial_t^4 (b_0 \cdot \partial)\eta \right\|_0^2 + \sum_{k=0}^{3} \left| \overline{\partial} \left(\Pi \partial_t^k \overline{\partial}^{3-k} v \right) \right|_0^2 + \left| \overline{\partial} \left(\Pi \overline{\partial}^3 (b_0 \cdot \partial)\eta \right) \right|_0^2, \tag{4.2.274}$$

It remains to prove the uniqueness. Let $\{(\eta_{(m)}, v_{(m)}, Q_{(m)})\}_{m=1,2}$ be two solutions of (2.2.1) satisfying (4.2.274). Then we define

$$[\eta] := \eta_{(1)} - \eta_{(2)},\ [v] := v_{(1)} - v_{(2)},\ [Q] := Q_{(1)} - Q_{(2)},\ [A] := A_{(1)} - A_{(2)}.$$

Then $([\eta], [v], [Q])$ satisfies the following system

$$\begin{cases} \partial_t [\eta] = [v] & \text{in } [0,T] \times \Omega; \\ \partial_t [v] - (b_0 \cdot \partial)^2 [\eta] + \nabla_{A_{(1)}} [Q] = -\nabla_{[A]} Q_{(2)} & \text{in } [0,T] \times \Omega; \\ \text{div}_{A_{(1)}} [v] = -\text{div}_{[A]} v_{(2)}, & \text{in } [0,T] \times \Omega; \\ \text{div}\, b_0 = 0 & \text{in } [0,T] \times \Omega; \\ [v^3] = b_0^3 = 0 & \text{on } \Gamma_0; \\ [Q]\hat{n}_{(1)} = -\sigma g_{(1)}^{ij} \Pi_{(1)} \overline{\partial}_{ij}^2 [\eta] - \sigma \sqrt{g_{(1)}} \triangle_{[g]} \eta_{(2)} & \text{on } \Gamma; \\ b_0^3 = 0 & \text{on } \Gamma, \\ ([\eta], [v]) = (\mathbf{0}, \mathbf{0}) & \text{on } \{t = 0\} \times \overline{\Omega}. \end{cases} \tag{4.2.275}$$

Define

$$[E](t) := \|[\eta]\|_{3.5}^2 + \sum_{k=0}^{2} \left\| \partial_t^k [v], \partial_t^k (b_0 \cdot \partial)[\eta] \right\|_{3.5-k}^2 + \left\| \partial_t^3 [v], \partial_t^3 (b_0 \cdot \partial)[\eta] \right\|_0^2 + \sum_{k=0}^{2} \left| \overline{\partial} \left(\Pi_{(1)} \partial_t^k \overline{\partial}^{2-k} [v] \right) \right|_0^2 + \left| \overline{\partial} \left(\Pi_{(1)} \overline{\partial}^2 (b_0 \cdot \partial)\eta \right) \right|_0^2. \tag{4.2.276}$$

Then we can mimic the proof in Section 4.2.1 to get the energy estimates of $[E]$

$$[E](T) \lesssim P([E](T), E(T)) \int_0^T P([E](t), E(t))\, \mathrm{d}t,$$

which together with Gronwall-type inequality yields

$$\exists T \in [0, T'], \quad [E](t) = 0 \quad \forall t \in [0, T]$$

which establishes the local well-posedness of (2.2.1) in $[0, T]$. The continuous dependence on the initial data also follows from an identical argument.

4.2.8.2 Regularity of initial data and free surface

Finally, we need to prove the norms of time derivatives can be controlled by $\|v_0\|_{4.5}$, $\|b_0\|_{4.5}$ and $|v_0|_5$. This part is exactly the same as in [53, Section 7.1]. The conclusion is

$$|v(t)|_5 \lesssim P(E(t)) \quad \text{in } [0, T].$$

This concludes the proof of Theorem 2.2.1.

4.3 The Zero Surface Tension Limit

In the proof of Theorem 2.2.1, the energy estimate for $E(t)$ in (2.2.3) depends on σ^{-1}. When the surface tension is sufficiently small, the energy bound itself will go to infinity. Therefore, it is natural to ask if one can establish uniform-in-σ estimates such that the solutions to (2.2.1) converges to the solution to (2.1.1) as $\sigma \to 0$. The answer is yes if the Rayleigh-Taylor sign condition is also satisfied for the initial data.

The key point is that, in Section 4.2.2.3, the boundary normal traces of v and $(b_0 \cdot \partial)\eta$ (and their time derivatives) are controlled by the comparison with Eulerian normal projections, whose σ-**weighted** energy is contributed by the surface tension in the estimates of one more time derivative. The reason for that is the failure of $\overline{\partial}^{4.5}$-estimates which requires $\|\eta\|_5$ or $|\sqrt{\sigma}\eta|_5$ regularity. Even if one uses the BMO-coefficient elliptic estimates posteriori, the energy still depends on σ^{-1}.

We can use the Alinhac good unknowns to avoid the interior higher-order terms. As for the boundary terms, we can alternatively define the energy functional to be the form of $E_1 + \sigma E_2$ where E_1 is the H^m energy functional for the "$\sigma = 0$ problem" (2.1.1) and E_2 is the $H^{m+0.5}$ energy functional for (2.2.1) (i.e., replace the $H^{4.5}$-setting of $E(t)$ by $H^{m+0.5}$). In the proof, we may use $A = \partial\eta \times \partial\eta$ to find an anti-symmetric structure in order to cancel the highest order term. This is a 3D generalization of the 2D analogous structure discovered by Gu-Lei [27] in the study of elastodynamics with surface tension. Below we start to prove Theorem 2.2.2. For technical simplicity, we assume the energy functional E^σ to be the $H^5 + \sigma H^{5.5}$ setting, i.e., we assume the mean curvature of the free surface is Lipschitz. The div-curl estimates, elliptic estimates and the tangential estimates containing time derivatives are identical to Section 4.2.1. So we only present the proof of those different aspects. We will drop the script σ in the weight energy functional (2.2.5).

Remark 4.3.1 (Necessity of weight energy functional)**.** In the tangential estimates, especially in the boundary integrals, there are a lot of terms which have 5 derivatives weighted by the surface tension coefficients. Therefore, it is reasonable to include the weighted $H^{5.5}$-energy $\sigma E_2(t)$ to control these boundary terms via the trace lemma. To control the weighted higher order energy $\sigma E_2(t)$, we again do the div-curl decomposition, while the $\sqrt{\sigma}$-weighted normal traces, i.e., the $\sqrt{\sigma}$-weighted Lagrangian normal projections, are no longer reduced by using Lemma 3.2.3. Instead, we notice that the boundary energies contributed by the surface tension in the *non-weighted tangential estimates* are exactly the $\sqrt{\sigma}$-weighted Eulerian normal projections with the same order as those $\sqrt{\sigma}$-weighted normal traces. Therefore, it remains to control the gap between the Eulerian normal $\hat{n}$ and the Lagrangian normal N, which is expected to be small due to the short time and $\hat{n} = N$ at $t = 0$. Hence, the energy estimates for $E^\sigma(t) = E_1(t) + \sigma E_2(t)$ are closed.

4.3.1 Interior estimates for the full spatial derivatives: Alinhac good unknowns

The boundary normal traces $|v^3|_{4.5}$ and $|(b_0 \cdot \partial)\eta^3|_{4.5}$ are reduced to $\|\overline{\partial}^5(v, (b_0 \cdot \partial)\eta)\|_0$. However, we cannot directly commute $\overline{\partial}^5$ with the covariant derivative ∇_A because the commutator contains

$\overline{\partial}^5 A = \overline{\partial}^5 \partial\eta \times \partial\eta$ whose L^2 norm cannot be controlled. The reason is that the essential highest order term in $\overline{\partial}^5(\nabla_A f)$, i.e., the standard derivatives of a covariant derivative, is actually the covariant derivative of Alinhac good unknown $\mathbf{f} := \overline{\partial}^5 f - \overline{\partial}^5 \eta \cdot \nabla_A f$ instead of the term produced by simply commuting $\overline{\partial}^5$ with ∇_A. Specifically,

$$
\begin{aligned}
\overline{\partial}^5(\nabla_A^\alpha f) &= \nabla_A^\alpha(\overline{\partial}^5 f) + (\overline{\partial}^5 A^{\mu\alpha})\partial_\mu f + [\overline{\partial}^5, A^{\mu\alpha}, \partial_\mu f] \\
&= \nabla_A^\alpha(\overline{\partial}^5 f) - \overline{\partial}^4(A^{\mu\gamma}\overline{\partial}\partial_\beta \eta_\gamma A^{\beta\alpha})\partial_\mu f + [\overline{\partial}^5, A^{\mu\alpha}, \partial_\mu f] \\
&= \nabla_A^\alpha(\overline{\partial}^5 f) - A^{\beta\alpha}\partial_\beta \overline{\partial}^5 \eta_\gamma A^{\mu\gamma}\partial_\mu f - ([\overline{\partial}^4, A^{\mu\gamma}A^{\beta\alpha}]\overline{\partial}\partial_\beta \eta_\gamma)\partial_\mu f + [\overline{\partial}^5, A^{\mu\alpha}, \partial_\mu f] \\
&= \underbrace{\nabla_A^\alpha(\overline{\partial}^5 f - \overline{\partial}^5 \eta_\gamma A^{\mu\gamma}\partial_\mu f)}_{=\nabla_A^\alpha \mathbf{f}} \\
&\quad + \underbrace{\overline{\partial}^5 \eta_\gamma \nabla_A^\alpha(\nabla_A^\gamma g) - ([\overline{\partial}^4, A^{\mu\gamma}A^{\beta\alpha}]\overline{\partial}\partial_\beta \eta_\gamma)\partial_\mu f + [\overline{\partial}^5, A^{\mu\alpha}, \partial_\mu f]}_{=:C^\alpha(f)},
\end{aligned}
$$

We introduce the Alinhac good unknowns of v and q with respect to $\overline{\partial}^5$ by

$$\mathbf{V} := \overline{\partial}^5 v - \overline{\partial}^5 \eta \cdot \nabla_A v, \ \ \mathbf{Q} := \overline{\partial}^5 q - \overline{\partial}^5 \eta \cdot \nabla_A q. \tag{4.3.1}$$

Then direct computation (e.g., see [30, Section 4.2.4]) shows that the good unknowns enjoy the following properties

$$\overline{\partial}^5 \underbrace{(\nabla_A \cdot v)}_{=0} = \nabla_A \cdot \mathbf{V} + C(v), \ \ \overline{\partial}^5(\nabla_A q) = \nabla_A \mathbf{Q} + C(q) \tag{4.3.2}$$

and

$$\|C(f)\|_0 \lesssim P(\|\eta\|_5)\|f\|_5. \tag{4.3.3}$$

Under this setting, we take $\overline{\partial}^5$ in the second equation of (2.1.1) and invoke (4.3.1) to get the evolution equation of the Alinhac good unknowns

$$\partial_t \mathbf{V} = -\nabla_A \mathbf{Q} + (b_0\cdot\partial)(\overline{\partial}^5(b_0\cdot\partial)\eta) + \underbrace{\partial_t(\overline{\partial}^5\eta\cdot\nabla_A v) - C(q) + [\overline{\partial}^5, (b_0\cdot\partial)]((b_0\cdot\partial)\eta)}_{=:\mathfrak{f}_0}. \quad (4.3.4)$$

Taking $L^2(\Omega)$ inner product with $\mathbf{V}$, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{V}|^2\,\mathrm{d}y = -\int_\Omega \nabla_A\mathbf{Q}\cdot\mathbf{V}\,\mathrm{d}y + \int_\Omega \Big((b_0\cdot\partial)(\overline{\partial}^5(b_0\cdot\partial)\eta)\Big)\cdot\mathbf{V}\,\mathrm{d}y + \int_\Omega \mathfrak{f}_0\cdot\mathbf{V}\,\mathrm{d}y, \quad (4.3.5)$$

where the last term can be directly controlled

$$\int_\Omega \mathfrak{f}_0\cdot\mathbf{V}\,\mathrm{d}y \lesssim P(\|\eta\|_5, \|v\|_5, \|\partial_t v\|_4, \|q\|_5, \|b_0\|_5, \|(b_0\cdot\partial)\eta\|_5) \lesssim P(E_1(t)). \quad (4.3.6)$$

Then we integrate $(b_0\cdot\partial)$ by parts in the second integral of (4.3.5) to produce the tangential energy of the magnetic field $(b_0\cdot\partial)\eta$. Note that $b_0\cdot N = 0$ on $\partial\Omega$ and $\operatorname{div} b_0 = 0$, no boundary term appears in this step.

$$\begin{aligned}
&\int_\Omega \Big((b_0\cdot\partial)(\overline{\partial}^5(b_0\cdot\partial)\eta)\Big)\cdot\mathbf{V}\,\mathrm{d}y = -\int_\Omega (\overline{\partial}^5(b_0\cdot\partial)\eta)\cdot(b_0\cdot\partial)\mathbf{V}\,\mathrm{d}y \\
=& -\int_\Omega (\overline{\partial}^5(b_0\cdot\partial)\eta)\cdot(b_0\cdot\partial)(\overline{\partial}^5\partial_t\eta)\,\mathrm{d}y + \int_\Omega (\overline{\partial}^5(b_0\cdot\partial)\eta)\cdot(b_0\cdot\partial)(\overline{\partial}^5\eta\cdot\nabla_A v)\,\mathrm{d}y \\
=& -\frac{1}{2}\frac{d}{dt}\int_\Omega \left|\overline{\partial}^5((b_0\cdot\partial)\eta)\right|^2 \qquad (4.3.7)\\
&+ \int_\Omega (\overline{\partial}^5(b_0\cdot\partial)\eta^\alpha)([\overline{\partial}^5, (b_0\cdot\partial)]v_\alpha + (b_0\cdot\partial)(\overline{\partial}^5\eta\cdot\nabla_A v_\alpha))\,\mathrm{d}y \\
\lesssim& -\frac{1}{2}\frac{d}{dt}\int_\Omega \left|\overline{\partial}^5((b_0\cdot\partial)\eta)\right|^2 + P(E_1(t)).
\end{aligned}$$

Next we analyze the first integral of (4.3.5). Integrate by parts, using Piola's identity $\partial_\mu A^{\mu\alpha} = 0$

and invoking (4.3.2), we get

$$
\begin{aligned}
-\int_\Omega \nabla_A \mathbf{Q}\cdot\mathbf{V}\,\mathrm{d}y &= \int_\Omega \mathbf{Q}(\nabla_A\cdot\mathbf{V})\,\mathrm{d}y \underbrace{-\int_\Gamma A^{3\alpha}\mathbf{Q}\mathbf{V}_\alpha\,\mathrm{d}S}_{=:J} - \int_{\Gamma_0} \underbrace{A^{3\alpha}\mathbf{Q}\mathbf{V}_\alpha}_{=0}\,\mathrm{d}S \\
&= -\int_\Omega \mathbf{Q}\,C(v)\,\mathrm{d}y + J \lesssim \|\mathbf{Q}\|_0\|C(v)\|_0 + J \\
&\lesssim P(\|\eta\|_5, \|q\|_5, |v\|_5) + J,
\end{aligned}
\tag{4.3.8}
$$

where the boundary integral on Γ_0 vanishes due to $\eta|_{\Gamma_0} =$ Id and thus $A^{3\alpha}\mathbf{V}_\alpha = \overline{\partial}^5 v_3 = 0$. Therefore, it remains to analyze the boundary integral J.

4.3.2 Boundary estimates and cancellation structure

The boundary integral now reads

$$
\begin{aligned}
J &= -\int_\Gamma A^{3\alpha}\mathbf{Q}\mathbf{V}_\alpha\,\mathrm{d}S \\
&= -\int_\Gamma A^{3\alpha}\overline{\partial}^5 q\mathbf{V}_\alpha\,\mathrm{d}S \underbrace{+\int_\Gamma A^{3\alpha}(\overline{\partial}^5\eta\cdot\nabla_A q)\mathbf{V}_\alpha\,\mathrm{d}S}_{=:\,\mathrm{RT}} \\
&= \underbrace{-\int_\Gamma \overline{\partial}^5(A^{3\alpha}q)\mathbf{V}_\alpha\,\mathrm{d}S}_{=:\,\mathrm{ST}} + \int_\Gamma q(\overline{\partial}^5 A^{3\alpha})\mathbf{V}_\alpha\,\mathrm{d}S \\
&\quad + \int_\Gamma \sum_{k=1}^{4}\binom{5}{k}\overline{\partial}^{5-k}A^{3\alpha}\overline{\partial}^k q\mathbf{V}_\alpha\,\mathrm{d}S + \mathrm{RT} \\
&=: \mathrm{ST} + J_1 + J_2 + \mathrm{RT}.
\end{aligned}
\tag{4.3.9}
$$

4.3.2.1 Non-weighted boundary energy: Rayleigh-Taylor sign condition

The term RT together with the Rayleigh-Taylor sign condition yields the non-weighted boundary energy. Recall that $\mathbf{V} = \overline{\partial}^5 v - \overline{\partial}^5 \eta \cdot \nabla_A v$, then we have

$$
\begin{aligned}
\mathrm{RT} = & \int_\Gamma A^{3\alpha}\overline{\partial}^5\eta_\beta A^{3\beta}\partial_3 q\overline{\partial}^5 v_\alpha \,\mathrm{d}S \\
& - \int_\Gamma A^{3\alpha}\overline{\partial}^5\eta_\beta A^{3\beta}\partial_3 q\overline{\partial}^5\eta_\gamma A^{\mu\gamma}\partial_\mu v_\alpha \,\mathrm{d}S \\
& + \sum_{i=1}^{2}\int_\Gamma A^{3\alpha}\overline{\partial}^5\eta_\beta A^{i\beta}\overline{\partial}_i q(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S \\
=: & \ \mathrm{RT}_1 + \mathrm{RT}_2 + \mathrm{RT}_3.
\end{aligned}
\tag{4.3.10}
$$

The term RT_1 gives the boundary energy term by writting $v_\alpha = \partial_t\eta_\alpha$.

$$
\begin{aligned}
\mathrm{RT}_1 = & \int_\Gamma A^{3\alpha}\overline{\partial}^5\eta_\beta A^{3\beta}\partial_3 q\partial_t\overline{\partial}^5\eta_\alpha \,\mathrm{d}S \\
= & -\frac{1}{2}\frac{d}{dt}\int_\Gamma(-\partial_3 q)\left|A^{3\alpha}\overline{\partial}^5\eta_\alpha\right|^2 \mathrm{d}S \\
& + \int_\Gamma(\partial_t A^{3\alpha})A^{3\beta}\overline{\partial}^5\eta_\beta\partial_3 q\overline{\partial}^5\eta_\alpha \,\mathrm{d}S + \frac{1}{2}\int_\Gamma \partial_t\partial_3 q\left|A^{3\alpha}\overline{\partial}^5\eta_\alpha\right|^2 \mathrm{d}S \\
=: & -\frac{1}{2}\frac{d}{dt}\int_\Gamma(-\partial_3 q)\left|A^{3\alpha}\overline{\partial}^5\eta_\alpha\right|^2 \mathrm{d}S + \mathrm{RT}_{11} + \mathrm{RT}_{12}.
\end{aligned}
\tag{4.3.11}
$$

The term RT_{12} can be directly controlled by

$$\mathrm{RT}_{12} \lesssim |\partial_t\partial_3 q|_{L^\infty}\left|A^{3\alpha}\overline{\partial}^5\eta_\alpha\right|_0^2 \lesssim P(E_1(t)). \tag{4.3.12}$$

The term RT_{11} is exactly canlled by RT_2 after plugging $\partial_t A^{3\alpha} = -A^{3\gamma}\partial_\mu v_\gamma A^{\mu\alpha}$

$$\mathrm{RT}_{11} = -\int_\Gamma A^{3\gamma}\partial_\mu v_\gamma A^{\mu\alpha}A^{3\beta}\overline{\partial}^5\eta_\beta\partial_3 q\overline{\partial}^5\eta_\alpha \,\mathrm{d}S = -\mathrm{RT}_2. \tag{4.3.13}$$

Finally, invoking $q = -\sigma\sqrt{g}\Delta_g\eta\cdot\hat{n} = \sigma Q(\overline{\partial}\eta)\overline{\partial}^2\eta\cdot\hat{n}$, we can control RT_3 by the weighted energy and trace lemma

$$\begin{aligned}
\mathrm{RT}_3 &= \sigma \int_\Gamma A^{3\alpha}\overline{\partial}^5\eta_\beta A^{i\beta}\overline{\partial}_i(Q(\overline{\partial}\eta)\overline{\partial}^2\eta\cdot\hat{n})(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S \\
&\lesssim P(|\partial\eta|_{L^\infty})|\overline{\partial}^3\eta|_{L^\infty}|\sqrt{\sigma}\overline{\partial}^5\eta|_0(|\sqrt{\sigma}\overline{\partial}^5 v|_0 + |\sqrt{\sigma}\overline{\partial}^5\eta|_0|\nabla_A v|_{L^\infty}) \\
&\lesssim P(\|\eta\|_3)\|\eta\|_5 \underbrace{\|\sqrt{\sigma}\eta\|_{5.5}(\|\sqrt{\sigma}v\|_{5.5} + \|\sqrt{\sigma}\eta\|_{5.5}\|v\|_3)}_{\leq\sqrt{\sigma E_2}(\sqrt{\sigma E_2}+\sqrt{E_1}\sqrt{\sigma E_2})} \\
&\lesssim P(E_1(t))(\sigma E_2(t)).
\end{aligned} \tag{4.3.14}$$

Summarizing (4.3.10)-(4.3.14), we conclude the estimate of RT by

$$\int_0^T \mathrm{RT}\,\mathrm{d}t \lesssim -\frac{c_0}{4}\left|A^{3\alpha}\overline{\partial}^5\eta_\alpha\right|_0^2 + \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t. \tag{4.3.15}$$

4.3.2.2 Control of the weighted boundary energy: surface tension

Now we analyze the term ST, where the surface tension gives the $\sqrt{\sigma}$-weighted top order boundary energy. Invoking $A^{3\alpha}q = -\sigma\sqrt{g}g^{ij}\hat{n}^\alpha\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta^\beta$, we get

$$\begin{aligned}
\mathrm{ST} &= \sigma\int_\Gamma \sqrt{g}g^{ij}\hat{n}^\alpha\hat{n}^\beta\overline{\partial}^5\overline{\partial}_i\overline{\partial}_j\eta_\beta(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S \\
&\quad + 5\sigma\int_\Gamma \overline{\partial}(\sqrt{g}g^{ij}\hat{n}^\alpha\hat{n}^\beta)\overline{\partial}^4\overline{\partial}_i\overline{\partial}_j\eta_\beta(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S \\
&\quad + \sigma\int_\Gamma \overline{\partial}^5(\sqrt{g}g^{ij}\hat{n}^\alpha\hat{n}^\beta)\overline{\partial}_i\overline{\partial}_j\eta_\beta(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S \\
&\quad + \sum_{k=2}^4 \sigma\int_\Gamma \binom{5}{k}\overline{\partial}^k(\sqrt{g}g^{ij}\hat{n}^\alpha\hat{n}^\beta)\overline{\partial}^{5-k}\overline{\partial}_i\overline{\partial}_j\eta_\beta(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S \\
&=: \mathrm{ST}_1 + \mathrm{ST}_2 + \mathrm{ST}_3 + \mathrm{ST}_4.
\end{aligned} \tag{4.3.16}$$

The term ST_4 can be directly controlled with the help of $\sqrt{\sigma}$-weighted energy

$$\mathrm{ST}_4 \lesssim P(E_1(t))(\sigma E_2(t)). \tag{4.3.17}$$

In ST_1, we first integrate $\overline{\partial}_i$ by parts.

$$\mathrm{ST}_1 \overset{\overline{\partial}_i}{=} -\sigma \int_\Gamma \sqrt{g} g^{ij} \hat{n}^\alpha \hat{n}^\beta \overline{\partial}^5 \overline{\partial}_j \eta_\beta \overline{\partial}_i (\overline{\partial}^5 v_\alpha - \overline{\partial}^5 \eta \cdot \nabla_A v_\alpha) \, \mathrm{d}S$$
$$- \sigma \int_\Gamma \overline{\partial}_i (\sqrt{g} g^{ij} \hat{n}^\alpha \hat{n}^\beta) \overline{\partial}^5 \overline{\partial}_j \eta_\beta (\overline{\partial}^5 v_\alpha - \overline{\partial}^5 \eta \cdot \nabla_A v_\alpha) \, \mathrm{d}S \tag{4.3.18}$$
$$=: \mathrm{ST}_{11} + \mathrm{ST}_{12}.$$

In ST_{11}, we write $v_\alpha = \partial_t \eta_\alpha$ to produce the energy term

$$\mathrm{ST}_{11} = -\sigma \int_\Gamma \sqrt{g} g^{ij} \hat{n}^\alpha \hat{n}^\beta \overline{\partial}^5 \overline{\partial}_j \eta_\beta \overline{\partial}_i (\overline{\partial}^5 \partial_t \eta_\alpha - \overline{\partial}^5 \eta \cdot \nabla_A v_\alpha) \, \mathrm{d}S$$
$$= -\frac{\sigma}{2} \frac{d}{dt} \int_\Gamma \left| \overline{\partial}^5 \overline{\partial} \eta \cdot \hat{n} \right|^2 \mathrm{d}S - \frac{\sigma}{2} \int_\Gamma (\sqrt{g} g^{ij} - \delta^{ij}) \left(\overline{\partial}^5 \overline{\partial}_i \eta \cdot \hat{n} \right) \left(\overline{\partial}^5 \overline{\partial}_j \eta \cdot \hat{n} \right) \mathrm{d}S$$
$$- \frac{\sigma}{2} \int_\Gamma \sqrt{g} \partial_t g^{ij} (\overline{\partial}^5 \overline{\partial}_i \eta \cdot \hat{n}) (\overline{\partial}^5 \overline{\partial}_j \eta \cdot \hat{n}) \, \mathrm{d}S$$
$$+ \sigma \int_\Gamma g^{ij} \partial_t (\underbrace{\sqrt{g} \hat{n}^\alpha}_{= A^{3\alpha}}) \overline{\partial}^5 \overline{\partial}_i \eta_\alpha (\overline{\partial}^5 \overline{\partial}_j \eta \cdot \hat{n}) \, \mathrm{d}S \tag{4.3.19}$$
$$+ \sigma \int_\Gamma \sqrt{g} g^{ij} \hat{n}^\alpha \hat{n}^\beta \overline{\partial}^5 \overline{\partial}_j \eta_\beta \overline{\partial}_i \overline{\partial}^5 \eta_\gamma A^{\mu\gamma} \partial_\mu v_\alpha \, \mathrm{d}S$$
$$+ \sigma \int_\Gamma \sqrt{g} g^{ij} \hat{n}^\alpha (\hat{n}^\beta \overline{\partial}^5 \overline{\partial}_j \eta_\beta) (\overline{\partial}^5 \eta \cdot \overline{\partial}_i (\nabla_A v_\alpha))$$
$$=: -\frac{\sigma}{2} \frac{d}{dt} \int_\Gamma \left| \overline{\partial}^6 \eta \cdot \hat{n} \right|^2 \mathrm{d}S + \mathrm{ST}_{111} + \mathrm{ST}_{112} + \mathrm{ST}_{113} + \mathrm{ST}_{114} + \mathrm{ST}_{115}.$$

Invoking (4.2.19), the term ST_{111} can be absorbed by the weighted energy term

$$\mathrm{ST}_{111} \leq \varepsilon \left| \sqrt{\sigma} \overline{\partial}^6 \eta \cdot \hat{n} \right|_0^2. \tag{4.3.20}$$

The terms ST_{112} and ST_{115} can be directly controlled

$$\mathrm{ST}_{112} \lesssim |\sqrt{g} \partial_t g^{ij}|_{L^\infty} \left| \sqrt{\sigma} \overline{\partial}^6 \eta \cdot \hat{n} \right|_0^2 \lesssim P(E_1(t))(\sigma E_2(t)). \tag{4.3.21}$$

$$\mathrm{ST}_{115} \lesssim |\sqrt{g} g^{ij} \overline{\partial} (\nabla_A v)|_{L^\infty} \left| \sqrt{\sigma} \overline{\partial}^6 \eta \cdot \hat{n} \right|_0 \| \sqrt{\sigma} \eta \|_{5.5} \lesssim P(E_1(t))(\sigma E_2(t)). \tag{4.3.22}$$

In ST_{113}, we use (3.1.4)-(3.1.5), i.e., $\sqrt{g}\hat{n}^\alpha = A^{3\alpha}$ and $\partial_t A^{3\alpha} = -A^{3\gamma}\partial_\mu v_\gamma A^{\mu\alpha}$ to produce cancellation with ST_{114}

$$\mathrm{ST}_{113} = -\sigma\int_\Gamma g^{ij} A^{3\gamma}\partial_\mu v_\gamma A^{\mu\alpha}\overline{\partial}^5\overline{\partial}_i\eta_\alpha(\overline{\partial}^5\overline{\partial}_j\eta\cdot\hat{n})\,\mathrm{d}S = -\,\mathrm{ST}_{114}. \tag{4.3.23}$$

Summarizing (4.3.19)-(4.3.23), we conclude the estimate of ST_{11} by choosing $\varepsilon > 0$ sufficiently small

$$\int_0^T \mathrm{ST}_{11}\,\mathrm{d}t \lesssim -\frac{\sigma}{2}\left|\overline{\partial}^5\overline{\partial}\eta\cdot\hat{n}\right|_0^2\Big|_0^T + \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t. \tag{4.3.24}$$

Next we control ST_{12}. First we have

$$\begin{aligned}\mathrm{ST}_{12} = &-\sigma\int_\Gamma \overline{\partial}_i(\sqrt{g}g^{ij}\hat{n}^\alpha)(\overline{\partial}^5\overline{\partial}_j\eta\cdot\hat{n})(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\ &-\sigma\int_\Gamma \sqrt{g}g^{ij}\hat{n}^\alpha(\overline{\partial}_i\hat{n}^\beta)\overline{\partial}^5\overline{\partial}_j\eta_\beta(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\ =:&\ \mathrm{ST}_{121} + \mathrm{ST}_{122}.\end{aligned} \tag{4.3.25}$$

The term ST_{121} can be directly controlled

$$\mathrm{ST}_{121} \lesssim P(E_1(t))(\sigma E_2(t)). \tag{4.3.26}$$

To control ST_{122}, we first integrate $\overline{\partial}_j$ by parts.

$$\begin{aligned}\mathrm{ST}_{122} = &-\sigma\int_\Gamma \sqrt{g}g^{ij}\hat{n}^\alpha(\overline{\partial}_i\hat{n}^\beta)\overline{\partial}^5\eta_\beta\overline{\partial}_j\overline{\partial}^5 v_\alpha\,\mathrm{d}S\\ &+\sigma\int_\Gamma \sqrt{g}g^{ij}\hat{n}^\alpha(\overline{\partial}_i\hat{n}^\beta)\overline{\partial}^5\eta_\beta\overline{\partial}_j\overline{\partial}^5\eta_\gamma(A^{\mu\gamma}\partial_\mu v_\alpha)\,\mathrm{d}S\\ &+\sigma\int_\Gamma \sqrt{g}g^{ij}\hat{n}^\alpha(\overline{\partial}_i\hat{n}^\beta)\overline{\partial}^5\eta_\beta\overline{\partial}^5\eta_\gamma\overline{\partial}_j(A^{\mu\gamma}\partial_\mu v_\alpha)\,\mathrm{d}S\\ &+\sigma\int_\Gamma \overline{\partial}_j(\sqrt{g}g^{ij}\hat{n}^\alpha\overline{\partial}_i\hat{n}^\beta)\overline{\partial}^5\eta_\beta(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\ =:&\ \mathrm{ST}_{1221} + \mathrm{ST}_{1222} + \mathrm{ST}_{1223} + \mathrm{ST}_{1224}.\end{aligned} \tag{4.3.27}$$

The term ST_{1223} can be directly controlled by the weighted energy

$$\mathrm{ST}_{1223} + \mathrm{ST}_{1224} \lesssim P(E_1(t))(\sigma E_2(t)). \tag{4.3.28}$$

To control ST_{1221}, we write $v_\alpha = \partial_t \eta_\alpha$ and then integrate ∂_t by parts. When ∂_t falls on $\sqrt{g}\hat{n}^\alpha = A^{3\alpha}$, the structure analogous to (4.3.23) is again produced.

$$\begin{aligned}\int_0^T \mathrm{ST}_{1221}\,\mathrm{d}t &\overset{\partial_t}{=} \sigma \int_0^T \int_\Gamma \partial_t (g^{ij}\overline{\partial}_i \hat{n}^\beta \overline{\partial}^5 \eta_\beta)(\overline{\partial}_j \overline{\partial}^5 \eta_\alpha \hat{n}^\alpha)\,\mathrm{d}S \\ &\quad + \sigma \int_0^T \int_\Gamma \underbrace{(-A^{3\gamma}\partial_\mu v_\gamma A^{\mu\alpha})}_{\partial_t(\sqrt{g}\hat{n}^\alpha)} g^{ij}\overline{\partial}_i \hat{n}^\beta \overline{\partial}^5 \eta_\beta \overline{\partial}_j \overline{\partial}^5 \eta_\alpha\,\mathrm{d}S \\ &\lesssim \int_0^T P(E_1(t))(\|\sqrt{\sigma} v\|_{5.5} + \|\sqrt{\sigma}\eta\|_{5.5})|\sqrt{\sigma}\overline{\partial}^6 \eta \cdot \hat{n}|_0\,\mathrm{d}t + (-\mathrm{ST}_{1222}).\end{aligned} \tag{4.3.29}$$

Therefore, ST_{122} is controlled by

$$\int_0^T \mathrm{ST}_{122}\,\mathrm{d}t \lesssim \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t, \tag{4.3.30}$$

which together with (4.3.24) and (4.3.26) gives the control of ST_1

$$\int_0^T \mathrm{ST}_1\,\mathrm{d}t \lesssim -\frac{\sigma}{2}\int_\Gamma \left|\overline{\partial}^6 \eta \cdot \hat{n}\right|^2 \mathrm{d}S\bigg|_0^T + \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t. \tag{4.3.31}$$

It remains to control ST_2 and ST_3 in (4.3.16). From (4.3.18), we find that ST_2 has the same form as ST_{12}, so we omit the analysis of ST_2 and only list the result

$$\int_0^T \mathrm{ST}_2\,\mathrm{d}t \lesssim \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t. \tag{4.3.32}$$

As for ST_3, we have

$$\begin{aligned}\mathrm{ST}_3 =& \,\sigma\int_\Gamma \sqrt{g}g^{ij}\hat{n}^\alpha(\overline{\partial}^5\hat{n}^\beta)\overline{\partial}_i\overline{\partial}_j\eta_\beta(\overline{\partial}^5 v_\alpha-\overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\&+\sigma\int_\Gamma \sqrt{g}g^{ij}(\overline{\partial}^5\hat{n}^\alpha)\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta(\overline{\partial}^5 v_\alpha-\overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\&+\sigma\int_\Gamma \overline{\partial}^5(\sqrt{g}g^{ij})\hat{n}^\alpha\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta(\overline{\partial}^5 v_\alpha-\overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\&+\sum_{k=1}^4\sigma\int_\Gamma \overline{\partial}^k(\sqrt{g}g^{ij})\overline{\partial}^{5-k}(\hat{n}^\alpha\hat{n}^\beta)\overline{\partial}_i\overline{\partial}_j\eta_\beta(\overline{\partial}^5 v_\alpha-\overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\=:&\,\mathrm{ST}_{31}+\mathrm{ST}_{32}+\mathrm{ST}_{33}+\mathrm{ST}_{34},\end{aligned}\tag{4.3.33}$$

where ST_{34} can be directly controlled

$$\mathrm{ST}_{34}\lesssim P(|\eta|_{W^{3,\infty}})|\sqrt{\sigma}\eta|_5(|\sqrt{\sigma}\overline{\partial}^5 v|_0+|\sqrt{\sigma}\overline{\partial}^5\eta|_0|\nabla_A v|_{L^\infty})\lesssim P(E_1(t))(\sigma E_2(t)).\tag{4.3.34}$$

To control ST_{31} and ST_{32}, we need to invoke (3.1.11) to get

$$\overline{\partial}^5\hat{n}^\alpha=-\overline{\partial}^4\left(g^{kl}(\overline{\partial}\overline{\partial}_k\eta\cdot\hat{n})\overline{\partial}_l\eta^\alpha\right)=-g^{kl}(\overline{\partial}^5\overline{\partial}_k\eta\cdot\hat{n})\overline{\partial}_l\eta^\alpha-[\overline{\partial}^4,g^{kl}\overline{\partial}_l\eta^\alpha](\overline{\partial}\overline{\partial}_k\eta\cdot\hat{n}),$$

and thus plug it into ST_{31} and ST_{32}:

$$\begin{aligned}\mathrm{ST}_{31}=&-\sigma\int_\Gamma\sqrt{g}g^{ij}g^{kl}(\overline{\partial}^5\overline{\partial}_k\eta\cdot\hat{n})\overline{\partial}_l\eta^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta\hat{n}_\alpha(\overline{\partial}^5 v_\alpha-\overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\&-\sigma\int_\Gamma\sqrt{g}g^{ij}\left([\overline{\partial}^4,g^{kl}\overline{\partial}_l\eta^\alpha](\overline{\partial}\overline{\partial}_k\eta\cdot\hat{n})\right)\overline{\partial}_i\overline{\partial}_j\eta_\beta\hat{n}_\alpha(\overline{\partial}^5 v_\alpha-\overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S\\\lesssim&\,P(E_1(t))|\sqrt{\sigma}\overline{\partial}^6\eta\cdot\hat{n}|_0(|\sqrt{\sigma}\overline{\partial}^5 v|_0+|\sqrt{\sigma}\overline{\partial}^5\eta|_0|\nabla_A v|_{L^\infty})\\&+P(E_1(t))|\sqrt{\sigma}\overline{\partial}^5\eta|_0(|\sqrt{\sigma}\overline{\partial}^5 v|_0+|\sqrt{\sigma}\overline{\partial}^5\eta|_0|\nabla_A v|_{L^\infty})\\\lesssim&\,P(E_1(t))(\sigma E_2(t)),\end{aligned}\tag{4.3.35}$$

and similarly

$$
\begin{aligned}
\mathrm{ST}_{32} &\lesssim P(E_1(t))|\sqrt{\sigma}\overline{\partial}^6\eta\cdot\hat{n}|_0(|\sqrt{\sigma}\overline{\partial}^5 v|_0 + |\sqrt{\sigma}\overline{\partial}^5\eta|_0|\nabla_A v|_{L^\infty}) \\
&\quad + P(E_1(t))|\sqrt{\sigma}\overline{\partial}^5\eta|_0(|\sqrt{\sigma}\overline{\partial}^5 v|_0 + |\sqrt{\sigma}\overline{\partial}^5\eta|_0|\nabla_A v|_{L^\infty}) \\
&\lesssim P(E_1(t))(\sigma E_2(t)),
\end{aligned}
\tag{4.3.36}
$$

For ST_{33}, we use the identity (3.1.12) to get

$$
\begin{aligned}
&\overline{\partial}^5(\sqrt{g}g^{ij}) = \sqrt{g}\left(\frac{1}{2}g^{ij}g^{kl} - g^{ik}g^{jl}\right)(\overline{\partial}^5\overline{\partial}_k\eta^\mu\overline{\partial}_l\eta_\mu + \overline{\partial}_k\eta^\mu\overline{\partial}^5\overline{\partial}_l\eta_\mu) \\
&+ \underbrace{\left[\overline{\partial}^4, \overline{\partial}_l\eta^\mu\left(\frac{1}{2}g^{ij}g^{kl} - g^{ik}g^{jl}\right)\right]\overline{\partial}\overline{\partial}_k\eta_\mu + \left[\overline{\partial}^4, \overline{\partial}_k\eta^\mu\left(\frac{1}{2}g^{ij}g^{kl} - g^{ik}g^{jl}\right)\right]\overline{\partial}\overline{\partial}_l\eta_\mu}_{R_{33}^{ij}},
\end{aligned}
$$

and thus

$$
\begin{aligned}
\mathrm{ST}_{33} &= \sigma\int_\Gamma \sqrt{g}\left(\frac{1}{2}g^{ij}g^{kl} - g^{ik}g^{jl}\right)(\overline{\partial}^5\overline{\partial}_k\eta^\mu\overline{\partial}_l\eta_\mu + \overline{\partial}_k\eta^\mu\overline{\partial}^5\overline{\partial}_l\eta_\mu) \\
&\qquad \hat{n}^\alpha\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S \\
&\quad + \sigma\int_\Gamma R_{33}^{ij}\hat{n}^\alpha\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta(\overline{\partial}^5 v_\alpha - \overline{\partial}^5\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S \\
&=: \mathrm{ST}_{331} + \mathrm{ST}_{332}.
\end{aligned}
\tag{4.3.37}
$$

The term ST_{332} can be directly controlled

$$
\mathrm{ST}_{332} \lesssim P(E_1(t))|\sqrt{\sigma}\eta|_5(|\sqrt{\sigma}\overline{\partial}^5 v|_0 + |\sqrt{\sigma}\overline{\partial}^5\eta|_0|\nabla_A v|_{L^\infty}). \tag{4.3.38}
$$

In ST_{331}, we integrate the derivative $\overline{\partial}_k$ in $\overline{\partial}^5\overline{\partial}_k\eta^\mu$ (resp. $\overline{\partial}_l$ in $\overline{\partial}^5\overline{\partial}_l\eta_\mu$) by parts

$$\begin{aligned}
\mathrm{ST}_{331} =& -\sigma\int_\Gamma \sqrt{g}\left(\frac{1}{2}g^{ij}g^{kl}-g^{ik}g^{jl}\right)\overline{\partial}^5\eta^\mu\overline{\partial}_l\eta_\mu\hat{n}^\alpha\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta\overline{\partial}_k\mathbf{V}_\alpha\,\mathrm{d}S\\
&-\sigma\int_\Gamma \sqrt{g}\left(\frac{1}{2}g^{ij}g^{kl}-g^{ik}g^{jl}\right)\overline{\partial}_k\eta^\mu\overline{\partial}^5\eta_\mu\hat{n}^\alpha\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta\overline{\partial}_l\mathbf{V}_\alpha\,\mathrm{d}S\\
&-\sigma\int_\Gamma \overline{\partial}_k\left(\sqrt{g}\left(\frac{1}{2}g^{ij}g^{kl}-g^{ik}g^{jl}\right)\overline{\partial}_l\eta_\mu\hat{n}^\alpha\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta\right)\overline{\partial}^5\eta^\mu\mathbf{V}_\alpha\,\mathrm{d}S \qquad (4.3.39)\\
&-\sigma\int_\Gamma \overline{\partial}_l\left(\sqrt{g}\left(\frac{1}{2}g^{ij}g^{kl}-g^{ik}g^{jl}\right)\overline{\partial}_k\eta_\mu\hat{n}^\alpha\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta\right)\overline{\partial}^5\eta^\mu\mathbf{V}_\alpha\,\mathrm{d}S\\
=:&\ \mathrm{ST}_{3311}+\mathrm{ST}_{3312}+\mathrm{ST}_{3313}+\mathrm{ST}_{3314},
\end{aligned}$$

where ST_{3313} and ST_{3314} can be directly controlled

$$\mathrm{ST}_{3313}+\mathrm{ST}_{3314}\lesssim P(E_1(t))(\sigma E_2(t)). \qquad (4.3.40)$$

For ST_{3311} and ST_{3312}, we need to write $v_\alpha=\partial_t\eta_\alpha$ and then integrate ∂_t by parts. For simplicity we only show the control of ST_{3311}.

$$\int_0^T \mathrm{ST}_{3311}\,\mathrm{d}t$$

$$= -\sigma\int_0^T\int_\Gamma \sqrt{g}\left(\frac{1}{2}g^{ij}g^{kl}-g^{ik}g^{jl}\right)\overline{\partial}^5\eta^\mu\overline{\partial}_l\eta_\mu\hat{n}^\alpha\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta\overline{\partial}_k\overline{\partial}^5\partial_t\eta_\alpha\,\mathrm{d}S \tag{4.3.41}$$

$$+\sigma\int_0^T\int_\Gamma \sqrt{g}\left(\frac{1}{2}g^{ij}g^{kl}-g^{ik}g^{jl}\right)\overline{\partial}^5\eta^\mu\overline{\partial}_l\eta_\mu\hat{n}^\alpha\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta\overline{\partial}_k\overline{\partial}^5\eta_\gamma A^{\mu\gamma}\partial_\mu v_\alpha\,\mathrm{d}S \tag{4.3.42}$$

$$+\sigma\int_0^T\int_\Gamma \sqrt{g}\left(\frac{1}{2}g^{ij}g^{kl}-g^{ik}g^{jl}\right)\overline{\partial}^5\eta^\mu\overline{\partial}_l\eta_\mu\hat{n}^\alpha\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta\overline{\partial}^5\eta\cdot\overline{\partial}_k(\nabla_A v_\alpha)\,\mathrm{d}S \tag{4.3.43}$$

$$\overset{\partial_t}{=}\sigma\int_0^T\int_\Gamma \sqrt{g}\left(\frac{1}{2}g^{ij}g^{kl}-g^{ik}g^{jl}\right)\overline{\partial}^5 v^\mu\overline{\partial}_l\eta_\mu\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta(\overline{\partial}_k\overline{\partial}^5\eta_\alpha\hat{n}^\alpha)\,\mathrm{d}S \tag{4.3.44}$$

$$+\sigma\int_0^T\int_\Gamma \left(\frac{1}{2}g^{ij}g^{kl}-g^{ik}g^{jl}\right)\overline{\partial}^5\eta^\mu\overline{\partial}_l\eta_\mu\partial_t(\sqrt{g}\hat{n}^\alpha)\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta\overline{\partial}_k\overline{\partial}^5\eta_\alpha\,\mathrm{d}S \tag{4.3.45}$$

$$+\sigma\int_0^T\int_\Gamma \sqrt{g}\partial_t\left(\left(\frac{1}{2}g^{ij}g^{kl}-g^{ik}g^{jl}\right)\overline{\partial}_l\eta_\mu\hat{n}^\beta\overline{\partial}_i\overline{\partial}_j\eta_\beta\right)\overline{\partial}^5\eta^\mu(\hat{n}^\alpha\overline{\partial}_k\overline{\partial}^5\eta_\alpha)\,\mathrm{d}S \tag{4.3.46}$$

$$+ (4.3.42) + (4.3.43).$$

Note that $\partial_t A(\sqrt{g}\hat{n}^\alpha) = \partial_t A^{3\alpha} = -A^{3\gamma}\partial_\mu v_\gamma A^{\mu\alpha}$, we know (4.3.45) + (4.3.42) = 0. The remaining quantities (4.3.43), (4.3.44) and (4.3.46) are all directly controlled

$$(4.3.43) \lesssim \int_0^T |\sqrt{\sigma}\eta|_5^2 P(E_1(t)) \le \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t, \tag{4.3.47}$$

$$(4.3.44) \lesssim \int_0^T |\sqrt{\sigma}\overline{\partial}^6\eta\cdot\hat{n}|_0|\sqrt{\sigma}v|_5 P(E_1(t)) \le \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t, \tag{4.3.48}$$

$$(4.3.46) \lesssim \int_0^T |\sqrt{\sigma}\overline{\partial}^6\eta\cdot\hat{n}|_0|\sqrt{\sigma}\eta|_5 P(E_1(t)) \le \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t. \tag{4.3.49}$$

Combining (4.3.37)-(4.3.49), we get the control of ST_{33}

$$\int_0^T \mathrm{ST}_{33}\,\mathrm{d}t \lesssim \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t, \tag{4.3.50}$$

whic together with (4.3.34), (4.3.35) and (4.3.36) gives the control of ST_3

$$\int_0^T \mathrm{ST}_3 \, \mathrm{d}t \lesssim \int_0^T P(E_1(t))(\sigma E_2(t)) \, \mathrm{d}t. \tag{4.3.51}$$

Finally, (4.3.16), (4.3.17), (4.3.31), (4.3.32) and (4.3.51) yields the $\sqrt{\sigma}$-weighted boundary energy

$$\int_0^T \mathrm{ST} \, \mathrm{d}t \lesssim -\frac{\sigma}{2} \int_\Gamma \left|\overline{\partial}^6 \eta \cdot \hat{n}\right|^2 \, \mathrm{d}S\bigg|_0^T + \int_0^T P(E_1(t))(\sigma E_2(t)) \, \mathrm{d}t. \tag{4.3.52}$$

4.3.2.3 Control of the error terms

It remains to control J_1 and J_2 in (4.3.9). Note that $q = -\sigma\sqrt{g}\Delta_g \eta \cdot \hat{n} = \sigma Q(\overline{\partial}\eta)\overline{\partial}^2\eta \cdot \hat{n}$ on the boundary and $A^{3\alpha} = \overline{\partial}_1 \eta \times \overline{\partial}_2 \eta$. The term J_2 can be directly controlled

$$\begin{aligned}
J_2 = {} & 5\sigma \int_\Gamma \overline{\partial}^4 A^{3\alpha} \overline{\partial}(Q(\overline{\partial}\eta)\overline{\partial}^2\eta)(\overline{\partial}^5 v_\alpha - \overline{\partial}^5 \eta \cdot \nabla_A v_\alpha) \, \mathrm{d}S \\
& + 10\sigma \int_\Gamma \overline{\partial}^3 A^{3\alpha} \overline{\partial}^2(Q(\overline{\partial}\eta)\overline{\partial}^2\eta)(\overline{\partial}^5 v_\alpha - \overline{\partial}^5 \eta \cdot \nabla_A v_\alpha) \, \mathrm{d}S \\
& + 10\sigma \int_\Gamma \overline{\partial}^2 A^{3\alpha} \overline{\partial}^3(Q(\overline{\partial}\eta)\overline{\partial}^2\eta)(\overline{\partial}^5 v_\alpha - \overline{\partial}^5 \eta \cdot \nabla_A v_\alpha) \, \mathrm{d}S \\
& + 5\sigma \int_\Gamma \overline{\partial} A^{3\alpha} \overline{\partial}^4(Q(\overline{\partial}\eta)(\overline{\partial}^2\eta \cdot \hat{n}))(\overline{\partial}^5 v_\alpha - \overline{\partial}^5 \eta \cdot \nabla_A v_\alpha) \, \mathrm{d}S \\
\lesssim {} & P(|\overline{\partial}\eta|_{L^\infty})|\overline{\partial}^3\eta|_{L^\infty} \left(|\sqrt{\sigma}\eta|_5 + |\sqrt{\sigma}\overline{\partial}^6\eta \cdot n|_0\right)\left(|\sqrt{\sigma}\overline{\partial}^5 v|_0 + |\sqrt{\sigma}\overline{\partial}^5\eta|_0 |\nabla_A v|_{L^\infty}\right) \\
\lesssim {} & P(|\overline{\partial}\eta|_{L^\infty})|\overline{\partial}^3\eta|_{L^\infty}|\partial v|_{L^\infty} \left(\sqrt{E_1} + \sqrt{\sigma E_2}\right)\sqrt{\sigma E_2} \le P(E_1(t))(\sigma E_2(t)).
\end{aligned} \tag{4.3.53}$$

J_1 needs more decliate computation. Recall that $A^{3\alpha} = (\overline{\partial}_1 \eta \times \overline{\partial}_2 \eta)^\alpha$, we have that

$$
\begin{aligned}
J_1 =& \int_\Gamma \sigma \mathcal{H}(\overline{\partial}^5 \overline{\partial}_1 \eta \times \overline{\partial}_2 \eta) \cdot \overline{\partial}^5 \partial_t \eta \, \mathrm{d}S + \int_\Gamma \sigma \mathcal{H}(\overline{\partial}_1 \eta \times \overline{\partial}^5 \overline{\partial}_2 \eta) \cdot \overline{\partial}^5 \partial_t \eta \, \mathrm{d}S \\
& - \int_\Gamma \sigma \mathcal{H}(\overline{\partial}^5 \overline{\partial}_1 \eta \times \overline{\partial}_2 \eta) \cdot (\overline{\partial}^5 \eta_\gamma A^{\mu\gamma} \partial_\mu v) \, \mathrm{d}S \\
& - \int_\Gamma \sigma \mathcal{H}(\overline{\partial}_1 \eta \times \overline{\partial}^5 \overline{\partial}_2 \eta) \cdot (\overline{\partial}^5 \eta_\gamma A^{\mu\gamma} \partial_\mu v) \, \mathrm{d}S \\
& + \sum_{k=1}^{4} \int_\Gamma \sigma \mathcal{H}(\overline{\partial}^k \overline{\partial}_1 \eta \times \overline{\partial}^{4-k} \overline{\partial}_2 \eta) \cdot (\overline{\partial}^5 v - \overline{\partial}^5 \eta \cdot \nabla_A v) \, \mathrm{d}S \\
=:& \, J_{11} + J_{12} + J_{13} + J_{14} + J_{15}.
\end{aligned} \tag{4.3.54}
$$

Again, the term J_{15} is directly controlled by the weighted energy

$$
J_{15} \le P(E_1(t))(\sigma E_2(t)). \tag{4.3.55}
$$

Below we only show the control of J_{11} and J_{13}, and the control of J_{12} and J_{14} follows in the same way. For J_{11}, we integrate ∂_t by parts to get

$$
\begin{aligned}
\int_0^T J_{11} \, \mathrm{d}t =& - \int_0^T \int_\Gamma \sigma \mathcal{H}(\overline{\partial}^5 \overline{\partial}_1 v \times \overline{\partial}_2 \eta) \cdot \overline{\partial}^5 \eta \, \mathrm{d}S - \int_0^T \int_\Gamma \sigma \mathcal{H}(\overline{\partial}^5 \overline{\partial}_1 \eta \times \overline{\partial}_2 v) \cdot \overline{\partial}^5 \eta \, \mathrm{d}S \\
& - \int_0^T \int_\Gamma \sigma \partial_t \mathcal{H}(\overline{\partial}^5 \overline{\partial}_1 \eta \times \overline{\partial}_2 \eta) \cdot \overline{\partial}^5 \eta \, \mathrm{d}S + \int_\Gamma \sigma \mathcal{H}(\overline{\partial}^5 \overline{\partial}_1 \eta \times \overline{\partial}_2 \eta) \cdot \overline{\partial}^5 \eta \, \mathrm{d}S \Big|_0^T \\
=:& \, J_{111} + J_{112} + J_{113} + J_{114}.
\end{aligned} \tag{4.3.56}
$$

Next we integrate $\overline{\partial}_1$ by parts in J_{111} and use the vector identity $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = -(\mathbf{u}\times\mathbf{w})\cdot\mathbf{v}$ to get

$$
\begin{aligned}
J_{111} \overset{\overline{\partial}_1}{=} & \int_0^T \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5 v\times\overline{\partial}_2\eta)\cdot\overline{\partial}_1\overline{\partial}^5\eta\,\mathrm{d}S\,\mathrm{d}t \\
& + \int_0^T \int_\Gamma \sigma\overline{\partial}_1\mathcal{H}(\overline{\partial}^5 v\times\overline{\partial}_2\eta)\cdot\overline{\partial}^5\eta\,\mathrm{d}S\,\mathrm{d}t + \int_0^T \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5 v\times\overline{\partial}_1\overline{\partial}_2\eta)\cdot\overline{\partial}^5\eta\,\mathrm{d}S\,\mathrm{d}t \\
= & \underbrace{-\int_0^T \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2\eta)\cdot\overline{\partial}^5 v\,\mathrm{d}S\,\mathrm{d}t}_{=-\int_0^T J_{11}\,\mathrm{d}t} \\
& + \int_0^T \int_\Gamma \sigma\overline{\partial}_1\mathcal{H}(\overline{\partial}^5 v\times\overline{\partial}_2\eta)\cdot\overline{\partial}^5\eta\,\mathrm{d}S\,\mathrm{d}t + \int_0^T \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5 v\times\overline{\partial}_1\overline{\partial}_2\eta)\cdot\overline{\partial}^5\eta\,\mathrm{d}S\,\mathrm{d}t \\
\lesssim & -\int_0^T J_{11}\,\mathrm{d}t + \int_0^T P(E_1(t))(\sigma E_2(t)).
\end{aligned}
\tag{4.3.57}
$$

Therefore, we have

$$
\int_0^T J_{11}\,\mathrm{d}t \lesssim \frac{1}{2}(J_{112}+J_{113}+J_{114}) + \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t. \tag{4.3.58}
$$

Next we need to control J_{114} by $\mathcal{P}_0 + P(E_1(T))(\sigma E_2(T))\int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t$. For that we need the following identity $\delta^{\alpha\beta} = \hat{n}^\alpha\hat{n}^\beta + g^{\mathrm{ij}}\overline{\partial}_i\eta^\alpha\overline{\partial}_j\eta^\beta$ which yields

$$
\begin{aligned}
J_{114} = & \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2\eta)_\alpha\hat{n}^\alpha\hat{n}^\beta\overline{\partial}^5\eta_\beta\,\mathrm{d}S \\
& + \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2\eta)_\alpha g^{\mathrm{ij}}\overline{\partial}_i\eta^\alpha\overline{\partial}_j\eta^\beta\overline{\partial}^5\eta_\beta\,\mathrm{d}S \\
=: & J_{1141} + J_{1142}.
\end{aligned}
\tag{4.3.59}
$$

In J_{1141} we integrate $\overline{\partial}_1$ by parts

$$
\begin{aligned}
J_{1141} \overset{\overline{\partial}_1}{=} & -\int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\eta\times\overline{\partial}_2\eta)_\alpha\hat{n}^\alpha\hat{n}^\beta\overline{\partial}_1\overline{\partial}^5\eta_\beta\,\mathrm{d}S - \int_\Gamma \sigma\overline{\partial}_1(\mathcal{H}\hat{n}^\alpha\hat{n}^\beta\overline{\partial}_2\eta)\overline{\partial}^5\eta\overline{\partial}^5\eta\,\mathrm{d}S \\
\lesssim & P(|\eta|_{W^{3,\infty}})\left(|\sqrt{\sigma}\overline{\partial}^5\eta|_0|\sqrt{\sigma}\overline{\partial}^6\eta\cdot\hat{n}|_0 + |\sqrt{\sigma}\overline{\partial}^5\eta|_0^2\right) \\
\lesssim & P(E_1(T))(\sigma E_2(T))\int_0^T \|\sqrt{\sigma}\overline{\partial}^5 v(t)\|_0\,\mathrm{d}t,
\end{aligned}
\tag{4.3.60}
$$

where we used $\overline{\partial}^5\eta|_{t=0} = \mathbf{0}$. In J_{1142}, we notice that the integral vanishes if $i = 2$ due to $(\overline{\partial}^5\overline{\partial}_1\eta \times \overline{\partial}_2\eta)\cdot\overline{\partial}_2\eta = 0$. If $i = 1$, then

$$
\begin{aligned}
J_{1142} &= \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta \times \overline{\partial}_2\eta)\cdot\overline{\partial}_1\eta\, g^{1j}\overline{\partial}_j\eta^\beta\overline{\partial}^5\eta_\beta\,\mathrm{d}S \\
&= -\int_\Gamma \sigma\mathcal{H}\underbrace{(\overline{\partial}_1\eta \times \overline{\partial}_2\eta)}_{=\sqrt{g}\hat{n}}\cdot\overline{\partial}^5\overline{\partial}_1\eta\, g^{1j}\overline{\partial}_j\eta^\beta\overline{\partial}^5\eta_\beta\,\mathrm{d}S \\
&\lesssim |\sqrt{\sigma}\overline{\partial}^5\overline{\partial}_1\eta\cdot\hat{n}|_0|\sqrt{\sigma}\overline{\partial}^5\eta|_0 P(|\overline{\partial}\eta|_{L^\infty})|\overline{\partial}^2\eta|_{L^\infty}\,\mathrm{d}S \lesssim P(E_1)(\sigma E_2)\int_0^T P(E_1)\,\mathrm{d}t
\end{aligned}
\tag{4.3.61}
$$

The term J_{113} can be controlled in the same way as J_{114} so we omit the proof. Thus we already get

$$
J_{113} + J_{114} \lesssim P(E(T))\int_0^T P(E(t))\,\mathrm{d}t. \tag{4.3.62}
$$

It remains to analyze $\frac{1}{2}J_{112}$ which should be controlled together with J_{13}. Again we have

$$
\begin{aligned}
\frac{1}{2}J_{112} =& \frac{1}{2}\int_0^T\int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta \times \overline{\partial}_2 v)_\alpha\hat{n}^\alpha\hat{n}^\beta\overline{\partial}^5\eta_\beta\,\mathrm{d}S \\
&+ \frac{1}{2}\int_0^T\int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta \times \overline{\partial}_2 v)_\alpha g^{ij}\overline{\partial}_i\eta^\alpha\overline{\partial}_j\eta^\beta\overline{\partial}^5\eta_\beta\,\mathrm{d}S \\
=:& J_{1121} + J_{1122},
\end{aligned}
\tag{4.3.63}
$$

and the control of J_{1121} follows in the same way as (4.3.60) by integrating $\overline{\partial}_1$ by parts

$$
J_{1121} \lesssim \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t, \tag{4.3.64}
$$

For J_{1122} we need to do further decomposition

$$\begin{aligned} J_{1122} =& -\frac{1}{2}\int_0^T \int_\Gamma \sigma\mathcal{H}(\overline{\partial}_i \eta \times \overline{\partial}_2 v)_\alpha g^{ij}\overline{\partial}^5\overline{\partial}_1\eta^\alpha \overline{\partial}_j\eta^\beta\overline{\partial}^5\eta_\beta \,\mathrm{d}S\,\mathrm{d}t \\ =& -\frac{1}{2}\int_0^T \int_\Gamma \sigma\mathcal{H}\left((\overline{\partial}_i \eta \times \overline{\partial}_2 v)_\gamma \hat{n}^\gamma\hat{n}^\alpha\overline{\partial}^5\overline{\partial}_1\eta_\alpha\right) g^{ij}\overline{\partial}_j\eta^\beta\overline{\partial}^5\eta_\beta \,\mathrm{d}S\,\mathrm{d}t \\ & -\frac{1}{2}\int_0^T \int_\Gamma \sigma\mathcal{H}\left((\overline{\partial}_i \eta \times \overline{\partial}_2 v)_\gamma g^{kl}\overline{\partial}_k\eta^\gamma\overline{\partial}_l\eta^\alpha\overline{\partial}^5\overline{\partial}_1\eta_\alpha\right) g^{ij}\overline{\partial}_j\eta^\beta\overline{\partial}^5\eta_\beta \,\mathrm{d}S\,\mathrm{d}t \\ =:& J_{11221} + J_{11222}, \end{aligned} \tag{4.3.65}$$

where J_{11221} is directly controlled by

$$J_{11221} \lesssim \int_0^T P(E_1(t))|\sqrt{\sigma}\overline{\partial}^5\overline{\partial}_1\eta\cdot\hat{n}|_0|\sqrt{\sigma}\overline{\partial}^5\eta|_0\,\mathrm{d}t \lesssim \int_0^T P(E_1(t)) + \sigma E_2(t)\,\mathrm{d}t. \tag{4.3.66}$$

In J_{11222}, the integral vanishes if $i = k$, so we only need to investigate the cases $(i,k) = (1,2)$ and $(i,k) = (2,1)$, which contribute to

$$\begin{aligned} J_{11222} =& -\frac{1}{2}\int_0^T \int_\Gamma \sigma\mathcal{H}\left((\overline{\partial}_1 \eta \times \overline{\partial}_2 v)_\gamma g^{2l}\overline{\partial}_2\eta^\gamma\overline{\partial}_l\eta^\alpha\overline{\partial}^5\overline{\partial}_1\eta_\alpha\right) g^{1j}\overline{\partial}_j\eta^\beta\overline{\partial}^5\eta_\beta \,\mathrm{d}S\,\mathrm{d}t \\ & -\frac{1}{2}\int_0^T \int_\Gamma \sigma\mathcal{H}\left((\overline{\partial}_2 \eta \times \overline{\partial}_2 v)_\gamma g^{1l}\overline{\partial}_1\eta^\gamma\overline{\partial}_l\eta^\alpha\overline{\partial}^5\overline{\partial}_1\eta_\alpha\right) g^{2j}\overline{\partial}_j\eta^\beta\overline{\partial}^5\eta_\beta \,\mathrm{d}S\,\mathrm{d}t \\ =& -\frac{1}{2}\int_0^T \int_\Gamma \sigma\mathcal{H}\left((\overline{\partial}_1 \eta \times \overline{\partial}_2 v)\cdot\overline{\partial}_2\eta\right)\left(g^{2l}\overline{\partial}_l\eta\cdot\overline{\partial}^5\overline{\partial}_1\eta\right)\left(g^{1j}\overline{\partial}_j\eta\cdot\overline{\partial}^5\eta\right)\,\mathrm{d}S\,\mathrm{d}t \\ & +\frac{1}{2}\int_0^T \int_\Gamma \sigma\mathcal{H}\left((\overline{\partial}_1 \eta \times \overline{\partial}_2 v)\cdot\overline{\partial}_2\eta\right)\left(g^{1l}\overline{\partial}_l\eta\cdot\overline{\partial}^5\overline{\partial}_1\eta\right)\left(g^{2j}\overline{\partial}_j\eta\cdot\overline{\partial}^5\eta\right)\,\mathrm{d}S\,\mathrm{d}t. \end{aligned} \tag{4.3.67}$$

Next we analyze J_{13}. First we do the following decomposition

$$\begin{aligned} \overline{\partial}^5\eta_\gamma A^{\mu\gamma}\partial_\mu v_\alpha &= \overline{\partial}^5\eta_\beta\hat{n}^\beta\hat{n}_\gamma A^{\mu\gamma}\partial_\mu v_\alpha + \overline{\partial}^5\eta_\beta g^{ij}\overline{\partial}_i\eta^\beta \underbrace{\overline{\partial}_j\eta_\gamma A^{\mu\gamma}}_{=\delta_j^\mu}\partial_\mu v_\alpha \\ &= \overline{\partial}^5\eta_\beta\hat{n}^\beta\hat{n}_\gamma A^{\mu\gamma}\partial_\mu v_\alpha + \overline{\partial}^5\eta_\beta g^{ij}\overline{\partial}_i\eta^\beta\overline{\partial}_j v_\alpha, \end{aligned}$$

and thus

$$J_{13} = \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta \times \overline{\partial}_2\eta)\cdot(\overline{\partial}^5\eta_\beta\hat{n}^\beta\hat{n}_\gamma A^{\mu\gamma}\partial_\mu v)\,\mathrm{d}S$$

$$+\int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta \times \overline{\partial}_2\eta)\cdot(\overline{\partial}^5\eta_\beta g^{ij}\overline{\partial}_i\eta^\beta\overline{\partial}_j v)\,\mathrm{d}S \tag{4.3.68}$$

$$=: J_{131} + J_{132}.$$

The term J_{131} can be controlled similarly as J_{1141} in (4.3.60)

$$J_{131} \lesssim P(E_1(t))(\sigma E_2(t)). \tag{4.3.69}$$

In J_{132}, we need do further decomposition

$$J_{132} = \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta \times \overline{\partial}_2\eta)_\gamma\hat{n}^\gamma\hat{n}^\alpha(\overline{\partial}^5\eta_\beta g^{ij}\overline{\partial}_i\eta^\beta\overline{\partial}_j v_\alpha)\,\mathrm{d}S$$

$$+\int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta \times \overline{\partial}_2\eta)_\gamma g^{kl}\overline{\partial}_k\eta^\gamma\overline{\partial}_l\eta^\alpha(\overline{\partial}^5\eta_\beta g^{ij}\overline{\partial}_i\eta^\beta\overline{\partial}_j v_\alpha)\,\mathrm{d}S \tag{4.3.70}$$

$$=: J_{1321} + J_{1322}.$$

The integral in J_{1322} vanishes if $k = 2$. When $k = 1$, we again use the vector identity $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = -(\mathbf{u}\times\mathbf{w})\cdot\mathbf{v}$ and invoke $(\overline{\partial}_1\eta \times \overline{\partial}^2\eta) = \sqrt{g}\hat{n}$ to get

$$J_{1322} = \int_\Gamma \sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta \times \overline{\partial}_2\eta)_\gamma g^{1l}\overline{\partial}_1\eta^\gamma\overline{\partial}_l\eta^\alpha(\overline{\partial}^5\eta_\beta g^{ij}\overline{\partial}_i\eta^\beta\overline{\partial}_j v_\alpha)\,\mathrm{d}S$$

$$= -\int_\Gamma \sigma\mathcal{H}\left((\overline{\partial}_1\eta \times \overline{\partial}_2\eta)\cdot\overline{\partial}^5\overline{\partial}_1\eta\right) g^{1l}\overline{\partial}_l\eta^\alpha(\overline{\partial}^5\eta_\beta g^{ij}\overline{\partial}_i\eta^\beta\overline{\partial}_j v_\alpha)\,\mathrm{d}S \tag{4.3.71}$$

$$\lesssim |\sqrt{\sigma}\overline{\partial}^5\overline{\partial}_1\eta\cdot\hat{n}|_0|\sqrt{\sigma}\overline{\partial}^5\eta|_0|\mathcal{H}\ g^2\overline{\partial}\eta\ \overline{\partial}\eta\ \overline{\partial}v|_{L^\infty} \lesssim P(E_1(t))(\sigma E_2(t))$$

We recall $\hat{n}_\gamma = \sqrt{g}^{-1}(\overline{\partial}_1\eta \times \overline{\partial}_2\eta)_\gamma$ and use the vector identities $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = -(\mathbf{u}\times\mathbf{w})\cdot\mathbf{v}$ and

$\mathbf{u}\times(\mathbf{v}\times\mathbf{w})=(\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$ to get

$$(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2\eta)\cdot(\overline{\partial}_1\eta\times\overline{\partial}_2\eta)=-(\overline{\partial}^5\overline{\partial}_1\eta\times(\overline{\partial}_1\eta\times\overline{\partial}_2\eta))\cdot\overline{\partial}_2\eta$$

$$=-(\overline{\partial}^5\overline{\partial}_1\eta\cdot\overline{\partial}_2\eta)\underbrace{(\overline{\partial}_1\eta\cdot\overline{\partial}_2\eta)}_{=g_{12}=-(\det g)g^{12}}+(\overline{\partial}^5\overline{\partial}_1\eta\cdot\overline{\partial}_1\eta)\underbrace{(\overline{\partial}_2\eta\cdot\overline{\partial}_2\eta)}_{=g_{22}=(\det g)g^{11}}.$$

Plugging this into J_{1321} yields

$$\begin{aligned}J_{1321}&=\int_\Gamma\sigma\mathcal{H}(\overline{\partial}^5\overline{\partial}_1\eta\times\overline{\partial}_2\eta)\cdot(\overline{\partial}_1\eta\cdot\overline{\partial}_2\eta)\sqrt{g}^{-1}\hat{n}^\alpha(\overline{\partial}^5\eta_\beta g^{ij}\overline{\partial}_i\eta^\beta\overline{\partial}_j v_\alpha)\,\mathrm{d}S\\&=\int_\Gamma\sigma\mathcal{H}(g^{1l}\overline{\partial}_l\eta\cdot\overline{\partial}^5\overline{\partial}_1\eta)(g^{ij}\overline{\partial}_i\eta\cdot\overline{\partial}^5\eta)(\overline{\partial}_j v_\alpha\hat{n}^\alpha\sqrt{g})\,\mathrm{d}S\\&=\int_\Gamma\sigma\mathcal{H}(g^{1l}\overline{\partial}_l\eta\cdot\overline{\partial}^5\overline{\partial}_1\eta)(g^{1i}\overline{\partial}_i\eta\cdot\overline{\partial}^5\eta)(\overline{\partial}_1 v\cdot(\overline{\partial}_1\eta\times\overline{\partial}_2\eta))\,\mathrm{d}S\\&\quad+\int_\Gamma\sigma\mathcal{H}(g^{1l}\overline{\partial}_l\eta\cdot\overline{\partial}^5\overline{\partial}_1\eta)(g^{2i}\overline{\partial}_i\eta\cdot\overline{\partial}^5\eta)(\overline{\partial}_2 v\cdot(\overline{\partial}_1\eta\times\overline{\partial}_2\eta))\,\mathrm{d}S\\&=:J_{13211}+J_{13212}.\end{aligned}\tag{4.3.72}$$

Integraing $\overline{\partial}_1$ by parts in J_{13211}, the highest order term is exactly the same as J_{13211} itself but with a minus sign. Therefore,

$$\begin{aligned}J_{13211}&=-\frac{1}{2}\int_\Gamma\sigma(g^{1l}\overline{\partial}_l\eta\cdot\overline{\partial}^5\eta)(g^{1i}\overline{\partial}_i\eta\cdot\overline{\partial}^5\eta)\overline{\partial}_1(\mathcal{H}\overline{\partial}_1 v\cdot(\overline{\partial}_1\eta\times\overline{\partial}_2\eta))\,\mathrm{d}S\\&\lesssim|\sqrt{\sigma}\overline{\partial}^5\eta|_0^2P(|\overline{\partial}\eta|_{L^\infty})(|\overline{\partial}^2\eta\,\overline{\partial}v|_{L^\infty})\lesssim P(E_1(t))(\sigma E_2(t)).\end{aligned}\tag{4.3.73}$$

Now J_{13212} reads

$$J_{13212}=-\int_\Gamma\sigma\mathcal{H}(g^{1l}\overline{\partial}_l\eta\cdot\overline{\partial}^5\overline{\partial}_1\eta)(g^{2i}\overline{\partial}_i\eta\cdot\overline{\partial}^5\eta)((\overline{\partial}_1\eta\times\overline{\partial}_2 v)\cdot\overline{\partial}_2\eta)\,\mathrm{d}S,\tag{4.3.74}$$

which together with (4.3.67) yields that

$$\begin{aligned}J_{11222}+J_{13212}=&-\frac{1}{2}\int_\Gamma\sigma\mathcal{H}(g^{1l}\overline{\partial}_l\eta\cdot\overline{\partial}^5\overline{\partial}_1\eta)(g^{2i}\overline{\partial}_i\eta\cdot\overline{\partial}^5\eta)((\overline{\partial}_1\eta\times\overline{\partial}_2 v)\cdot\overline{\partial}_2\eta)\,\mathrm{d}S\\&-\frac{1}{2}\int_\Gamma\sigma\mathcal{H}(g^{1l}\overline{\partial}_l\eta\cdot\overline{\partial}^5\eta)(g^{2i}\overline{\partial}_i\eta\cdot\overline{\partial}^5\overline{\partial}_1\eta)((\overline{\partial}_1\eta\times\overline{\partial}_2 v)\cdot\overline{\partial}_2\eta)\,\mathrm{d}S,\end{aligned}\tag{4.3.75}$$

and thus integrating $\overline{\partial}_1$ by parts in the first integral yields the cancellation with the second integral due to the symmetry

$$\begin{aligned}
J_{11222}+J_{13212}=&\frac{1}{2}\int_\Gamma \sigma\mathcal{H}(\overline{\partial}_1(g^{1l}\overline{\partial}_l\eta)\cdot\overline{\partial}^5\eta)(g^{2i}\overline{\partial}_i\eta\cdot\overline{\partial}^5\eta)((\overline{\partial}_1\eta\times\overline{\partial}_2 v)\cdot\overline{\partial}_2\eta)\,\mathrm{d}S\\
&+\frac{1}{2}\int_\Gamma \sigma\mathcal{H}(g^{1l}\overline{\partial}_l\eta\cdot\overline{\partial}^5\eta)(\overline{\partial}_1(g^{2i}\overline{\partial}_i\eta)\cdot\overline{\partial}^5\eta)((\overline{\partial}_1\eta\times\overline{\partial}_2 v)\cdot\overline{\partial}_2\eta)\,\mathrm{d}S\\
&+\frac{1}{2}\int_\Gamma \sigma\mathcal{H}(g^{1l}\overline{\partial}_l\eta\cdot\overline{\partial}^5\eta)(g^{2i}\overline{\partial}_i\eta\cdot\overline{\partial}^5\eta)\overline{\partial}_1((\overline{\partial}_1\eta\times\overline{\partial}_2 v)\cdot\overline{\partial}_2\eta)\,\mathrm{d}S\\
\lesssim&|\sqrt{\sigma}\overline{\partial}^5\eta|_0^2P(E_1(t))\leq P(E_1(t))(\sigma E_2(t)).
\end{aligned}\tag{4.3.76}$$

Summarizing (4.3.54)-(4.3.56), (4.3.62)-(4.3.66), (4.3.68)-(4.3.76), we conclude the estimate of J_1 by

$$\int_0^T J_1\,\mathrm{d}t\lesssim P(E_1(T))(\sigma E_2(T))\int_0^T P(E_1(t))\,\mathrm{d}t+\int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t. \tag{4.3.77}$$

Finally, combining (4.3.9), (4.3.15), (4.3.52), (4.3.53) and (4.3.77), we conclude the $\overline{\partial}^5$-boundary estimate by

$$\begin{aligned}
\int_0^T J\,\mathrm{d}t\lesssim&-\frac{c_0}{4}\left|\overline{\partial}^5\eta\cdot\hat{n}\right|_0^2-\frac{\sigma}{2}\left|\overline{\partial}^6\eta\cdot\hat{n}\right|_0^2\\
&+P(E_1(T))(\sigma E_2(T))\int_0^T P(E_1(t))\,\mathrm{d}t+\int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t.
\end{aligned}\tag{4.3.78}$$

4.3.2.4 Finalizing the tangential estimate of spatial derivatives

Summarizing (4.3.5)-(4.3.8) and (4.3.78), we conclude the estimate of the Alinhac good unknowns by

$$\begin{aligned}
&\|\mathbf{V}\|_0^2+\frac{c_0}{4}\left|\overline{\partial}^5\eta\cdot\hat{n}\right|_0^2+\frac{\sigma}{2}\left|\overline{\partial}^6\eta\cdot\hat{n}\right|_0^2\Big|_{t=T}\\
\lesssim& P(\|v_0\|_5)+P(E_1(T))(\sigma E_2(T))\int_0^T P(E_1(t))\,\mathrm{d}t+\int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t.
\end{aligned}\tag{4.3.79}$$

Finally, from the definition of the good unknowns (4.3.1) and $\overline{\partial}^5\eta|_{t=0} = \mathbf{0}$, we know

$$\|\overline{\partial}^5 v(T)\|_0^2 \lesssim \|\mathbf{V}(T)\|_0^2 + \|\overline{\partial}^5\eta(T)\|_0^2\|\nabla_A v(T)\|_{L^\infty}^2 \lesssim \|\mathbf{V}(T)\|_0^2 + P(E_1(T))\int_0^T \|\overline{\partial}^5 v(t)\|_0^2\,\mathrm{d}t,$$

and thus

$$\begin{aligned}&\|\overline{\partial}^5 v\|_0^2 + \|\overline{\partial}^5(b_0\cdot\partial)\eta\|_0^2 + \frac{c_0}{4}\left|\overline{\partial}^5\eta\cdot\hat{n}\right|_0^2 + \frac{\sigma}{2}\left|\overline{\partial}^6\eta\cdot\hat{n}\right|_0^2\Big|_{t=T}\\ \lesssim{}& P(\|v_0\|_5) + P(E_1(T))(\sigma E_2(T))\int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t.\end{aligned}\tag{4.3.80}$$

4.3.3 Tangential estimates of time derivatives

Following the same method as in Chapter 4.2.3, we may derive the following tangential estimates of time derivatives. For the details one can refer to [29, Section 6].

$$\begin{aligned}&\sum_{k=1}^5 \|\overline{\partial}^{5-k}\partial_t^k v\|_0^2 + \|\overline{\partial}^{5-k}\partial_t^k(b_0\cdot\partial)\eta\|_0^2 + \frac{\sigma}{2}\left|\overline{\partial}^{6-k}\partial_t^{k-1}v\cdot\hat{n}\right|_0^2\\ \lesssim{}& \varepsilon(\sigma E_2(T)) + \mathcal{P}_0 + P(E_1(T))(\sigma E_2(T))\int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t.\end{aligned}\tag{4.3.81}$$

4.3.4 Control of weighted Sobolev norms

4.3.4.1 Weighted div-curl estimates

The estimate for $\|\sqrt{\sigma}v\|_{4.5}^2$ and $\|\sqrt{\sigma}(b_0\cdot\partial)\eta\|_{4.5}^2$ is done similarly as before so we again omit the details. For the divergence, we directly get

$$\sigma\|\text{div }v\|_{4.5}^2 + \sigma\|\text{div }(b_0\cdot\partial)\eta\|_{4.5}^2 \lesssim \varepsilon\sigma\left(\|v\|_{5.5}^2 + \|(b_0\cdot\partial)\eta\|_{5.5}^2\right),\tag{4.3.82}$$

and similarly

$$\begin{aligned}&\sum_{k=0}^4 \sigma\|\text{div }\partial_t^k v\|_{4.5-k}^2 + \sigma\|\text{div }\partial_t^k(b_0\cdot\partial)\eta\|_{4.5-k}^2\\ \lesssim{}& \sum_{k=0}^4 \varepsilon\sigma\left(\|\partial_t^k v\|_{5.5-k}^2 + \|\partial_t^k(b_0\cdot\partial)\eta\|_{5.5-k}^2\right) + P(E(0)) + \int_0^T P(E(t))\,\mathrm{d}t.\end{aligned}\tag{4.3.83}$$

For the curl we have

$$\|\sqrt{\sigma}\text{curl } v\|_{4.5}^2 + \|\sqrt{\sigma}\text{curl } (b_0\cdot\partial)\eta\|_{4.5}^2 \lesssim \varepsilon\sigma\left(\|v\|_{5.5}^2 + \|(b_0\cdot\partial)\eta\|_{5.5}^2\right) + \int_0^T P(E(t)), \tag{4.3.84}$$

and similarly

$$\begin{aligned}&\sum_{k=0}^4 \sigma\|\text{curl } \partial_t^k v\|_{4.5-k}^2 + \sigma\|\text{curl } \partial_t^k(b_0\cdot\partial)\eta\|_{4.5-k}^2 \\ \lesssim &\sum_{k=0}^4 \varepsilon\sigma\left(\|\partial_t^k v\|_{5.5-k}^2 + \|\partial_t^k(b_0\cdot\partial)\eta\|_{5.5-k}^2\right) + P(E(0)) + \int_0^T P(E(t))\,\mathrm{d}t.\end{aligned} \tag{4.3.85}$$

4.3.4.2 Control of the weighted boundary norms

We still need to control $\sqrt{\sigma}|\overline{\partial}^{5-k}\partial_t^k v\cdot N|_0$ and $\sqrt{\sigma}|\overline{\partial}^{5-k}\partial_t^k(b_0\cdot\partial)\eta\cdot N|_0$ for $0\le k\le 4$. For the boundary estimates of v, one can directly compare them with the energy terms contributed by surface tension. (cf. (4.3.80)- (4.3.81))

$$\sqrt{\sigma}\left|(\overline{\partial}^{5-k}\partial_t^k v\cdot N) - \overline{\partial}^{5-k}\partial_t^k v\cdot\hat{n}\right|_0 \lesssim \left\|\sqrt{\sigma}\partial_t^k v\right\|_{5.5-k}\int_0^T|\partial_t(\hat{n}-N)|_{L^\infty}^2\,\mathrm{d}t \tag{4.3.86}$$

As for $(b_0\cdot\partial)\eta$, when $k\ge 1$, we can directly control them by the norms of v

$$\begin{aligned}\sqrt{\sigma}|\overline{\partial}^{5-k}\partial_t^k(b_0\cdot\partial)\eta\cdot N|_0 &= \sqrt{\sigma}|\overline{\partial}^{5-k}\partial_t^{k-1}(b_0\cdot\partial)v\cdot N|_0 \\ &\lesssim \|b_0\|_{L^\infty}\|\sqrt{\sigma}\partial_t^{k-1}\partial v\|_{5.5-k} + \text{ lower order terms.}\end{aligned} \tag{4.3.87}$$

When $k=0$, again we need to compare it with the Eulerian normal projections

$$\sqrt{\sigma}\left|(\overline{\partial}^5((b_0\cdot\partial)\eta)\cdot N) - \overline{\partial}^5(b_0\cdot\partial)\eta\cdot\hat{n}\right|_0 \lesssim \left\|\sqrt{\sigma}(b_0\cdot\partial)\eta\right\|_{5.5}\int_0^T|\partial_t(\hat{n}-N)|_{L^\infty}^2\,\mathrm{d}t, \tag{4.3.88}$$

and thus it remains to control $\sqrt{\sigma}|\overline{\partial}^5(b_0\cdot\partial)\eta\cdot\hat{n}|_0$. In fact, this term naturally appears as a boundary energy term contributed by the surface tension in the $\overline{\partial}^4(b_0\cdot\partial)$ tangential estimate, which can be proceeded in the same way as $\overline{\partial}^4\partial_t$-estimate by just replacing ∂_t by $(b_0\cdot\partial)$. The reason for that is $(b_0\cdot\partial)\eta$ and η have the same spatial regularity, which is similar with the fact that $\partial_t\eta = v$ has the same spatial regularity as η. In other words, the tangential derivative $(b_0 \cdot \partial)$ (note that $b_0 \cdot N = 0$ on $\partial\Omega$!) plays the same role as a time derivative if it falls on the flow map η. We just list the result of $\overline{\partial}^4(b_0 \cdot \partial)$-estimate

$$
\begin{aligned}
&\|\overline{\partial}^4(b_0 \cdot \partial)v\|_0^2 + \|\overline{\partial}^4(b_0 \cdot \partial)^2\eta\|_0^2 + \frac{\sigma}{2}\left|\overline{\partial}^5(b_0 \cdot \partial)\eta \cdot \hat{n}\right|_0^2\Big|_{t=T} \\
\lesssim\ &\mathcal{P}_0 + P(E_1(T))(\sigma E_2(T)) \int_0^T P(E_1(t))(\sigma E_2(t))\,\mathrm{d}t.
\end{aligned}
\tag{4.3.89}
$$

4.3.5 The zero surface tension limit

Now we conclude the energy estimates. First, straightforward computation gives the div-curl control of the non-weighted Sobolev norms. Then the boundary normal traces are reduced to the interior tangential estimates which are established in (4.3.80) for spatial derivatives and (4.3.81) for time derivatives. In the control of the non-weighted Sobolev norms, the weighted energy $\sigma E_2(t)$ is needed to close the energy estimates. The $\sqrt{\sigma}$-weighted div-curl estimates are established in (4.3.83) and (4.3.85), while the boundary normal traces are no longer reduced to the interior tangential estimates via Lemma 3.2.3. Instead, we notice that, in the non-weighted tangential estimates, the surface tension contributes to $\sqrt{\sigma}$-weighted boundary energies which are exactly the *Eulerian* normal traces of the weighted variables. Therefore, it suffices to estimate the difference between the $\sqrt{\sigma}$-weighted Lagrangian normal traces and the $\sqrt{\sigma}$-weighted Eulerian normal traces, which is established in (4.3.86)-(4.3.88). Finally we get

$$
E(T) = E_1(T) + \sigma E_2(T) \lesssim \varepsilon E(t) + P(E(0)) + P(E(T)) \int_0^T P(E(t))\,\mathrm{d}t, \tag{4.3.90}
$$

which together with Gronwall inequality implies that there exists some $T' > 0$, independent of σ, such that

$$
\sup_{0 \le t \le T'} E(t) \le P(E(0)) \le \mathcal{C}. \tag{4.3.91}
$$

Finally, we recover the higher boundary regularity. This is done by the elliptic estimate in [21].

Taking ∂_t in the surface tension equation and letting $\alpha = 3$ yields

$$\sigma\sqrt{g}g^{ij}(\overline{\partial}^2_{ij}v^3 - \Gamma^k_{ij}\overline{\partial}_k v^3) = \sigma\partial_t(\sqrt{g}g^{ij})\overline{\partial}^2_{ij}\eta^3 - \sigma\partial_t(\sqrt{g}g^{ij}\Gamma^k_{ij})\overline{\partial}_k\eta^3 - \partial_t(A^{33}Q), \quad \text{on } \Gamma. \tag{4.3.92}$$

We then have

$$|\sigma v^3(T)|_{5.5} \lesssim |\text{RHS of (4.3.92)}|_{3.5} \lesssim P(E_1(T)) + P(E_1(T))\int_0^T |\sigma\overline{\partial}^2 v(t)|_{3.5}\,\mathrm{d}t, \tag{4.3.93}$$

where we use $\overline{\partial}^2\eta|_{t=0} = \mathbf{0}$ and $\overline{\partial}\eta^3|_{t=0} = 0$. Note that, when estimating |RHS of (4.3.92)$|_{3.5}$, the top order term $|Q_t|_{3.5} \lesssim \|Q_t\|_4$ is controlled by considering the Neumann boundary condition of Q_t, which then avoids circular arguments. Therefore, the standard Gronwall-type argument gives the control of $|\sigma v^3|_{5.5}$. As for $|\sigma^{\frac{3}{2}}v^3|_6$, one can multiply $\sqrt{\sigma}$ in (4.3.92) and again invoke the elliptic estimate to get

$$|\sigma^{\frac{3}{2}}v^3(T)|_6 \lesssim P(E_1(T))(\sigma E_2(T)) + P(E_1(T))(\sigma E_2(T))\int_0^T |\sigma^{\frac{3}{2}}\overline{\partial}^2 v^3(t)|_4\,\mathrm{d}t \tag{4.3.94}$$

which yields $|\sigma^{\frac{3}{2}}v^3(T)|_6 \leq P(E(T))$. The bounds for $b = (b_0\cdot\partial)\eta$ follow in the same way but just differentiating $(b_0\cdot\partial)$ instead of ∂_t in the surface tension equation.

Now we prove the zero surface tension limit. Assume $(w, (b_0\cdot\partial)\zeta, r)$ to be the solution to (2.1.1) and $(v^\sigma, (b_0\cdot\partial)\eta^\sigma, q^\sigma)$ to be the solution to (2.2.1) with $\sigma > 0$. Then Sobolev embedding implies

$$\|v^\sigma\|^2_{C^1([0,T]\times\Omega)} + \|(b_0\cdot\partial)\eta^\sigma\|^2_{C^1([0,T]\times\Omega)} + \|q^\sigma\|^2_{C^1([0,T]\times\Omega)} \lesssim \mathcal{C}.$$

By Morrey's embedding, we can prove $v^\sigma, (b_0\cdot\partial)\eta^\sigma, q^\sigma \in C^1_t H^4_y([0,T]\times\Omega) \hookrightarrow C^1_t C^{2,\frac{1}{2}}_y([0,T]\times\Omega)$, which implies the equi-continuity of $(v^\sigma, (b_0\cdot\partial)\eta^\sigma, q^\sigma)$ in $C^1([0,T]\times\Omega)$. By Arzelà-Ascoli lemma, we prove the uniform convergence (up to subsequence) of $(v^\sigma, (b_0\cdot\partial)\eta^\sigma, q^\sigma)$ as $\sigma \to 0_+$, and the limit is the solution $(w, (b_0\cdot\partial)\zeta, r)$ to (2.1.1). Theorem 2.2.2 is proven.

4.4 Low-Regularity Estimates of Incompressible MHD with Surface Tension

For the low-regularity solution, we can lower down the regularity to $H^{3.5}$ with the help of refined Kato-Ponce inequalities (cf. Lemma 3.2.1) applied to div-curl analysis and the BMO-coefficient elliptic estimates applied to the boundary normal traces.

4.4.1 Div-Curl estimates for time derivatives via refined Kato-Ponce inequalities

Due to the low-regularity issue, the div-curl analysis requires the refined Kato-Ponce inequalities recorded in Lemma 3.2.1 (3). First one gets

$$\begin{aligned}\|\text{div } v_t\|_{1.5} &= \|\text{div}_{\partial_t A} v\|_{1.5} + \|(\text{div}_{I-A}) v_t\|_{1.5} \\ &\lesssim P(\|v_0\|_{2.5}) + \int_0^t P(\|v_t(s)\|_{2.5}) ds + \varepsilon \|v_t\|_{2.5},\end{aligned} \tag{4.4.1}$$

and similarly,

$$\|\text{div } b_t\|_{1.5} \lesssim P(\|b_0\|_{2.5}) + \int_0^t P(\|b_t(s)\|_{2.5}) ds + \varepsilon \|b_t\|_{2.5}. \tag{4.4.2}$$

Now we start to control curl v_t and curl b_t. First, we have

$$\begin{aligned}\|\text{curl } v_t\|_{1.5} &\le \|\text{curl}_A v_t\|_{1.5} + \|(\text{curl}_{I-A}) v_t\|_{1.5} \lesssim \|\text{curl}_A v_t\|_{1.5} + \varepsilon \|v_t\|_{2.5} \\ \|\text{curl } b_t\|_{1.5} &\le \|\text{curl}_A b_t\|_{1.5} + \|(\text{curl}_{I-A}) b_t\|_{1.5} \lesssim \|\text{curl}_A b_t\|_{1.5} + \varepsilon \|b_t\|_{2.5}.\end{aligned} \tag{4.4.3}$$

The control of $\text{curl}_A v_t$ and $\text{curl}_A b_t$ is slightly different from that of $\text{curl}_A v$ and $\text{curl}_A b$. We start with the first equation of (2.2.1). Taking the time derivative at first, and then apply curl_A on both sides, we get

$$\partial_t (\text{curl}_A v_t)_\lambda - (\text{curl}_A (b_0 \cdot \partial)^2 v)_\lambda = \underbrace{(\text{curl}_{\partial_t A} v)_\lambda - \epsilon_{\lambda\tau\alpha} A^{\mu\tau} \partial_\mu (A_t^{\nu\alpha} \partial_\nu Q)}_{G^*}.$$

Commuting $(b_0 \cdot \partial)$ with curl_A on LHS, we have

$$\partial_t (\text{curl}_A v_t)_\lambda - (b_0 \cdot \partial)(\text{curl}_A (b_0 \cdot \partial) v)_\lambda = G^* + [\text{curl}_A, b_0 \cdot \partial] b_t.$$

Taking $\partial^{1.5}$ on both sides and commuting $b_0 \cdot \partial$ with curl_A, we get the evolution equation of curl v_t:

$$\partial_t \partial^{1.5}(\text{curl}_A v_t) - (b_0 \cdot \partial)\partial^{1.5}(\text{curl}_A b_t) = \underbrace{\partial^{1.5}(G^* + [\text{curl}_A, b_0 \cdot \partial] b_t) + [\partial^{1.5}, b_0 \cdot \partial]\text{curl}_A b_t}_{F^*}. \quad (4.4.4)$$

Next we again mimic [52, Prop. 5.2] and get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |\partial^{1.5}\text{curl}_A v_t|^2 + |\partial^{1.5}\text{curl}_A b_t|^2 dy &= \underbrace{\int_\Omega F^* \cdot \partial^{1.5}\text{curl}_A v_t \, dy}_{\mathcal{B}_1^*} \\ &+ \underbrace{\int_\Omega \partial^{1.5}(\text{curl}_A b_t) \cdot [\partial^{1.5}\text{curl}_A, b_0 \cdot \partial] v_t \, dy}_{\mathcal{B}_2^*} \\ &+ \underbrace{\int_\Omega \partial^{1.5}(\text{curl}_A b_t)^\lambda \partial^{1.5}(\epsilon_{\lambda\tau\alpha} A_t^{\mu\tau} \partial_\mu b_t^\alpha) dy}_{\mathcal{B}_3^*}. \end{aligned} \quad (4.4.5)$$

$\mathcal{B}_3^*$ can be controlled directly by the multiplicative Sobolev inequality:

$$\begin{aligned} \mathcal{B}_3^* &\lesssim \|\partial^{1.5}\text{curl}_A b_t\|_0 \|\partial^{1.5}(\epsilon_{\lambda\tau\alpha} A_t^{\mu\tau} \partial_\mu b_t^\alpha)\|_0 \\ &\lesssim \|A\|_2 \|b_t\|_{2.5} \|\partial_t A\|_2 \|b_t\|_{2.5} \lesssim \|v\|_3 \|b_t\|_{2.5}^2. \end{aligned} \quad (4.4.6)$$

To control $\mathcal{B}_2^*$, it suffices to control $\|[\partial^{1.5}\text{curl}_A, b_0 \cdot \partial] v_t\|_{L^2}$. First we simplify the commutator:

$$\begin{aligned} [\partial^{1.5}\text{curl}_A, b_0 \cdot \partial] v_t &= \epsilon_{\lambda\tau\alpha}\left(\partial^{1.5}(A^{\mu\tau}\partial_\mu(b_0^\nu \partial_\nu v_t^\alpha)) - b_0^\nu \partial_\nu \partial^{1.5}(A^{\mu\tau}\partial_\mu v_t^\alpha)\right) \\ &= \epsilon_{\lambda\tau\alpha}\underbrace{\left(\partial^{1.5}(A^{\mu\tau}\partial_\mu(b_0^\nu \partial_\nu v_t^\alpha)) - \partial_\nu \partial^{1.5}(b_0^\nu A^{\mu\tau}\partial_\mu v_t^\alpha)\right)}_{\mathcal{B}_{21}^*} \\ &+ \epsilon_{\lambda\tau\alpha}\underbrace{\left(\partial_\nu \partial^{1.5}(b_0^\nu A^{\mu\tau}\partial_\mu v_t^\alpha) - b_0^\nu \partial_\nu \partial^{1.5}(A^{\mu\tau}\partial_\mu v_t^\alpha)\right)}_{\mathcal{B}_{22}^*}. \end{aligned} \quad (4.4.7)$$

For $\mathcal{B}_{22}^*$, we invoke the refined Kato-Ponce estimate (3.2.7)

$$\|\mathcal{B}_{22}^*\|_{L^2} \lesssim \|b_0\|_{W^{1.5,3}} \|A^{\mu\tau}\partial_\mu v_t^\alpha\|_{L^6} + \|\partial b_0\|_{L^\infty} \|A^{\mu\tau}\partial_\mu v_t^\alpha\|_{1.5} \lesssim \|b_0\|_3 \|v_t\|_{2.5}. \quad (4.4.8)$$

For $\mathcal{B}_{21}^*$, we have

$$
\begin{aligned}
\mathcal{B}_{21}^* &= \epsilon_{\lambda\tau\alpha}\partial^{1.5}(A^{\mu\tau}\partial_\mu(b_0^\nu\partial_\nu v_t^\alpha)) - \partial_\nu(b_0^\nu A^{\mu\tau}\partial_\mu v_t^\alpha))\\
&= \epsilon_{\lambda\tau\alpha}\partial^{1.5}\left(A^{\mu\tau}\partial_\mu b_0^\nu\partial_\nu v_t^\alpha + A^{\mu\tau}b_0^\nu\partial_\mu\partial_\nu v_t^\alpha - b_0^\nu\partial_\nu A^{\mu\tau}\partial_\mu v_t^\alpha - b_0^\nu A^{\mu\tau}\partial_\mu\partial_\nu v_t^\alpha\right)\\
&= \epsilon_{\lambda\tau\alpha}\partial^{1.5}\left(A^{\mu\tau}\partial_\mu b_0^\nu\partial_\nu v_t^\alpha + b_0^\nu\partial_\beta\partial_\nu\eta_\gamma A^{\mu\gamma}A^{\beta\tau}\partial_\mu v_t^\alpha\right)\\
&= \epsilon_{\lambda\tau\alpha}\partial^{1.5}(A^{\mu\tau}\partial_\mu b_0^\nu\partial_\nu v_t^\alpha + \partial_\beta((b_0\cdot\partial)\eta_\gamma)A^{\mu\gamma}A^{\beta\tau}\partial_\mu v_t^\alpha - \partial_\beta b_0^\mu A^{\beta\tau}\partial_\mu v_t^\alpha),
\end{aligned}
\tag{4.4.9}
$$

Therefore, one can get:

$$
\|\mathcal{B}_{21}^*\|_{L^2} \lesssim \|b_0\|_3\|v_t\|_{2.5}. \tag{4.4.10}
$$

It remains to control $\mathcal{B}_1^*$, specifically, $\|F^*\|_{L^2}$. The two commutator terms can be controlled in the same way as $\mathcal{B}_{21}^*$ and straightforward computation

$$
\|[\partial^{1.5}, b_0\cdot\partial]\mathrm{curl}_A b_t\|_0 + \|\partial^{1.5}([\mathrm{curl}_A, b_0\cdot\partial]b_t)\|_0 \lesssim \|b_0\|_3\|v_t\|_{2.5}, \tag{4.4.11}
$$

and

$$
\|\mathrm{curl}_{\partial_t A} v\|_{1.5} + \|A^{\mu\tau}\partial_\mu(A_t^{\nu\alpha}\partial_\nu Q)\|_{1.5} \lesssim \|v\|_3(\|v\|_{3.5} + \|Q\|_{3.5}) + \|v\|_{3.5}\|Q\|_3. \tag{4.4.12}
$$

Combining (4.4.3)-(4.4.12), and absorbing the ε-term to LHS we have:

$$
\|\mathrm{curl}\ v_t\|_{1.5} + \|\mathrm{curl}\ b_t\|_{1.5} \lesssim \mathcal{P}_0 + \int_0^t \mathcal{P}. \tag{4.4.13}
$$

The boundary term $|b_t^3|_2$ can be directly controlled

$$
\|b_t^3\|_{2,\Gamma} = \|b_0\cdot\overline{\partial}\partial_t\eta\|_{2,\Gamma} \lesssim \|b_0\|_{2,\Gamma}|v^3|_{3,\Gamma}. \tag{4.4.14}
$$

Summing up (4.4.1), (4.4.2), (4.4.13) and (4.4.14), then absorbing the ε-term to LHS, we have

$$\begin{aligned} \|v_t\|_{2.5} &\lesssim \mathcal{P}_0 + \int_0^t \mathcal{P} + |v_t^3|_2; \\ \|b_t\|_{2.5} &\lesssim \mathcal{P}_0 + \int_0^t \mathcal{P}. \end{aligned} \tag{4.4.15}$$

Again, from Hodge's decomposition inequality applied to v_{tt} and b_{tt}, we have:

$$\begin{aligned} \|v_{tt}\|_{1.5} &\lesssim \|v_{tt}\|_0 + \|\text{curl } v_{tt}\|_{0.5} + \|\text{div } v_{tt}\|_{0.5} + |v_{tt}^3|_1; \\ \|b_{tt}\|_{1.5} &\lesssim \|b_{tt}\|_0 + \|\text{curl } b_{tt}\|_{0.5} + \|\text{div } b_{tt}\|_{0.5} + |b_{tt}^3|_1, \end{aligned} \tag{4.4.16}$$

We have

$$\|\text{div } v_{tt}\|_{0.5} \lesssim P(\|v\|_{2.5+\delta})(\|v\|_{1.5} + \|v_t\|_{1.5}) + \varepsilon\|v_{tt}\|_{1.5} \tag{4.4.17}$$

and similarly,

$$\|\text{div } b_{tt}\|_{0.5} \lesssim P(\|v\|_{2.5+\delta}, \|b\|_{2.5+\delta})(\|v\|_{1.5} + \|v_t\|_{1.5} + \|b_t\|_{1.5}) + \varepsilon\|b_{tt}\|_{1.5}, \tag{4.4.18}$$

where $\delta > 0$ can be arbitratily small.

The boundary term $|b_{tt}^3|_1$ is controlled in the same way as (4.4.14)

$$|b_{tt}^3|_1 = |b_0 \cdot \overline{\partial} v_t^3|_{1,\Gamma} \lesssim |b_0|_2 |v_t^3|_2. \tag{4.4.19}$$

It remains to control curl v_{tt} and curl b_{tt}. We have:

$$\partial_t(\text{curl}_A v_{tt}) - \text{curl}_A (b_0 \cdot \partial)^2 v_t = G^{**}, \tag{4.4.20}$$

where

$$G^{**} := -\text{curl}_{\partial_t^2 A} v_t - \text{curl}_{\partial_t A} v_{tt} + \text{curl}_{\partial_t^2 A}(b_0 \cdot \partial) b + 2\text{curl}_{\partial_t A}(b_0 \cdot \partial) b_t. \tag{4.4.21}$$

Commuting $(b.\partial)$ with curl_A on LHS of (4.4.20), taking $\partial^{0.5}$ derivative and then commuting it with

$b_0 \cdot \partial$, we get the evolution equation:

$$\begin{aligned}&\partial_t(\partial^{0.5}\text{curl}_A v_{tt}) - (b_0\cdot\partial)(\partial^{0.5}\text{curl}_A b_{tt})\\ =& \underbrace{\partial^{0.5}(G^{**} + [\text{curl}_A, b_0\cdot\partial]b_{tt}) + [\partial^{0.5}, b_0\cdot\partial](\text{curl}_A b_{tt})}_{=:F^{**}}.\end{aligned} \tag{4.4.22}$$

Analogous to (4.4.5), we can derive the following energy identity:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\partial^{0.5}\text{curl}_A v_{tt}|^2 + |\partial^{0.5}\text{curl}_A b_{tt}|^2 dy =& \underbrace{\int_\Omega F^{**}\cdot\partial^{0.5}\text{curl}_A v_{tt}\, dy}_{\mathcal{B}_1^{**}}\\ + \underbrace{\int_\Omega \partial^{0.5}(\text{curl}_A b_{tt})\cdot[\partial^{0.5}\text{curl}_A, b_0\cdot\partial]v_t\, dy}_{\mathcal{B}_2^{**}} +& \underbrace{\int_\Omega \partial^{0.5}(\text{curl}_A b_{tt})^\lambda \partial^{0.5}(\epsilon_{\lambda\tau\alpha}A_t^{\mu\tau}\partial_\mu b_{tt}^\alpha)dy}_{\mathcal{B}_3^{**}}.\end{aligned} \tag{4.4.23}$$

Then we have

$$\mathcal{B}_3^{**} \lesssim \|b_{tt}\|_{1.5}\|b_0\|_{1.5}\|v_t\|_{2.5}\|v\|_2. \tag{4.4.24}$$

For $\mathcal{B}_2^{**}$, it suffices to control $\|[\partial^{0.5}\text{curl}_A, b_0\cdot\partial]v_{tt}\|_{L^2}$. Analogous to (4.4.7), we have

$$\begin{aligned}[\partial^{0.5}\text{curl}_A, b_0\cdot\partial]v_{tt} =& \epsilon_{\lambda\tau\alpha}\underbrace{\left(\partial^{0.5}(A^{\mu\tau}\partial_\mu(b_0^\nu\partial_\nu v_{tt}^\alpha)) - \partial_\nu\partial^{0.5}(b_0^\nu A^{\mu\tau}\partial_\mu v_{tt}^\alpha)\right)}_{\mathcal{B}_{21}^{**}}\\ &+ \epsilon_{\lambda\tau\alpha}\underbrace{\left(\partial_\nu\partial^{0.5}(b_0^\nu A^{\mu\tau}\partial_\mu v_{tt}^\alpha) - b_0^\nu\partial_\nu\partial^{0.5}(A^{\mu\tau}\partial_\mu v_{tt}^\alpha)\right)}_{\mathcal{B}_{22}^{**}}.\end{aligned} \tag{4.4.25}$$

For $\mathcal{B}_{22}^{**}$, we invoke the refind Kato-Ponce type commutator estimate as in (4.4.8)

$$\|\mathcal{B}_{22}^{*}\|_{L^2} \lesssim \|b_0\|_{W^{1.5,6}}\|A^{\mu\tau}\partial_\mu v_{tt}^\alpha\|_{L^3} + \|\partial b_0\|_{L^\infty}\|A^{\mu\tau}\partial_\mu v_{tt}^\alpha\|_{1.5} \lesssim \|b_0\|_3\|v_{tt}\|_{1.5}. \tag{4.4.26}$$

For $\mathcal{B}_{21}^{**}$, we have

$$\mathcal{B}_{21}^{**} = \epsilon_{\lambda\tau\alpha}\partial^{0.5}(A^{\mu\tau}\partial_\mu b_0^\nu\partial_\nu v_{tt}^\alpha + \partial_\beta((b_0\cdot\partial)\eta_\gamma)A^{\mu\gamma}A^{\beta\tau}\partial_\mu v_{tt}^\alpha - \partial_\beta b_0^\mu A^{\beta\tau}\partial_\mu v_{tt}^\alpha), \tag{4.4.27}$$

Therefore, we get

$$\|\mathcal{B}_{21}^{*}\|_{L^2} \lesssim \|b_0\|_3\|v_{tt}\|_{1.5}. \tag{4.4.28}$$

It remains to control $\mathcal{B}_1^{**}$, specifically, $\|F^{**}\|_{L^2}$. The two commutator terms can be controlled by $\|b_0\|_3\|b_{tt}\|_{1.5}$ in the same way as $\mathcal{B}_1^*$. Therefore it remains to control $\|G^{**}\|_{0.5}$, which is directly controlled by using multiplicative Sobolev inequality

$$\|G^{**}\|_{0.5} \lesssim \|\partial_t^2 A\|_1(\|v_t\|_2 + \|b_0\|_3\|b\|_3) + \|\partial_t A\|_2(\|v_{tt}\|_{1.5}\|b_0\|_3\|b_t\|_{2.5}) \lesssim \mathcal{P}. \tag{4.4.29}$$

Combining (4.4.3), (4.4.5), (4.4.24), (4.4.26), (4.4.28), and(4.4.29), and absorbing the ε-term to LHS we have:

$$\|\text{curl } v_{tt}\|_{0.5} + \|\text{curl } b_{tt}\|_{0.5} \lesssim \mathcal{P}_0 + \int_0^t \mathcal{P} + \varepsilon(\|v_{tt}\|_{1.5} + \|b_{tt}\|_{1.5}). \tag{4.4.30}$$

Summing up (4.4.17), (4.4.18), (4.4.19) and (4.4.30), then absorbing the ε-term to LHS, and finally using Young's inequality and Jensen's inequality, we have

$$\begin{aligned}
\|v_{tt}\|_{1.5} &\lesssim \mathcal{P}_0 + \int_0^t \mathcal{P}\, ds + \|v_{tt}^3\|_{1,\Gamma} + P(\|v\|_{2.5+\delta}) \underbrace{(\|v\|_{1.5} + \|v_t\|_{1.5})}_{\lesssim \mathcal{P}_0 + \int_0^t \mathcal{P}} \\
&\lesssim \mathcal{P}_0 + \int_0^t \mathcal{P}\, ds + \|v_{tt}^3\|_{1,\Gamma} + P(\|v\|_{2.5+\delta}); \\
\|b_{tt}\|_{1.5} &\lesssim \mathcal{P}_0 + \int_0^t \mathcal{P}\, ds + P(\|v\|_{2.5+\delta}, \|b\|_{2.5+\delta}) \underbrace{(\|v\|_{1.5} + \|v_t\|_{1.5} + \|b_t\|_{1.5})}_{\lesssim \mathcal{P}_0 + \int_0^t \mathcal{P}} \\
&\lesssim \mathcal{P}_0 + \int_0^t \mathcal{P}\, ds + P(\|v\|_{2.5+\delta}, \|b\|_{2.5+\delta}),
\end{aligned} \tag{4.4.31}$$

where $\delta > 0$ can be arbitratily small.

The control of the boundary terms containing v and its time derivatives as well as the lower order terms (i.e., $\|v\|_{2.5+\delta}$ and $\|b\|_{2.5+\delta}$ are still needed. This will be done in the next subsection.

4.4.2 Boundary traces controlled by elliptic estimates

Taking ∂_t in the surface tension equation $A^{\mu\alpha}N_\mu Q+\sigma\sqrt{g}\triangle_g\eta^\alpha=0$ and letting $\alpha=3$ yield the following elliptic equation on Γ.

$$\sqrt{g}g^{ij}(\overline{\partial}_{ij}^2v^3-\Gamma_{ij}^k\overline{\partial}_kv^3)=\partial_t(\sqrt{g}g^{ij})\overline{\partial}_i\overline{\partial}_j\eta^3-\partial_t(\sqrt{g}g^{ij}\Gamma_{ij}^k)\overline{\partial}_k\eta^3-\frac{1}{\sigma}\partial_t(A^{\mu 3}Q)N_\mu. \tag{4.4.32}$$

By the critical Sobolev embedding one can easily prove $g\in BMO(\Gamma)$ and thus

$$\begin{aligned}|v^3|_3&\lesssim\|Q_t\|_1+P(\|v\|_{2.5+\delta},|Q|_{1.5})+P(\|v\|_{3.5})\int_0^t P(\|v\|_{3.5})\\&\lesssim\mathcal{P}_0+\mathcal{P}\int_0^t\mathcal{P}+P(\|v\|_{2.5+\delta}),\end{aligned}\tag{4.4.33}$$

As for $b=(b_0\cdot\partial)\eta$, we replace ∂_t by $(b_0\cdot\partial)$ in the step above and similarly get

$$|b^3|_3=|b_0\cdot\overline{\partial}\eta|_3\lesssim P(\|b_0\|_{3.5},\|Q(0)\|_{2.5})+\int_0^t\|Q_t\|_{2.5}. \tag{4.4.34}$$

Finally, we need to reduce the lower order term $\|v\|_{2.5+\delta}$ by interpolation. Since $\|v\|_{2.5+\delta}\le\frac{1}{2}+\frac{1}{2}\|v\|_{2.5+\delta}^2$, we may assume $P(\|v\|_{2.5+\delta})$ is the combination of terms of the form $\|v\|_{2.5+\delta}^d$ with $d\ge 2$. By Lemma 3.2.2, we have $\|v\|_{2.5+\delta}^d\lesssim\|v\|_3^{2\delta d}\|v\|_0^{(1-2\delta)d}$. Then choose δ sufficiently close to 0, for different d's, such that $p_d:=\frac{1}{d\delta}>1$. One can use ε-Young's inequality with p_d and its dual index to derive

$$\|v\|_{2.5+\delta}^k\lesssim\varepsilon\|v\|_3^2+\|v\|_0^b\lesssim\varepsilon\|v\|_{3.5}^2+P(\|v_0\|_{2.5})+\int_0^t P(\|v_t(s)\|_{2.5})ds,$$

for some $b>0$ and thus Theorem 2.2.4 is proven by Gronwall-type inequality.

Chapter 5

Free-Boundary Compressible MHD

5.1 Loss of Normal Derivatives for Compressible Ideal MHD

The free-boundary compressible ideal MHD system in the case of a liquid is a strictly hyperbolic system with characteristic boundary conditions. In the case of $B \cdot \hat{n}|_{\partial\mathcal{D}_t} = 0$, the uniform Kreiss-Lopatinskiĭ is violated and thus there is a potential of loss of normal derivatives. For incompressible ideal MHD and compressible Euler, one can use the div-curl decomposition (cf. Lemma 3.3.1) and the normal trace lemma (cf. Lemma 3.2.3) to control the normal derivatives. However this fails for compressible ideal MHD due to the extra coupling between the magnetic field and the sound wave, or namely the magnetoacoustic wave. We refer to [82, Sect 1.5] for detailed discussion.

Previously the local existence was proved by Trakhinin-Wang [74, 75] and see also Chen-Wang [9], Trakhinin [70, 71] and Secchi-Trakhinin [64] for the study of the current-vortex sheets and the plasma-vacuum model in compressible ideal MHD. It should be noted that, all the aforementioned results rely on the Nash-Moser iteration to prove the local existence, and thus one may not find a higher-order energy estimate without loss of regularity even for the linearized equations.

On the one hand, I was the first one that observed the magnetic diffusion could exactly compensate the normal derivative loss: the diffusion together with the Christodoulou-Lindblad elliptic estimate could give the common control of the heat equation of b and the wave equation of q as well as the Lorentz force which is a higher order term. Based on this, the local well-posedness and the incompressible limit are established [82, 83]. On the other hand, we proved the first result on the nonlinear a priori estimate without loss of regularity of free-boundary compressible ideal MHD [50]. We use the anisotropic Sobolev space that was first introduced by Chen [10], and the modified Alinhac good unknown method and delicate analysis of the structure of MHD system.

5.2 Well-posedness and Incompressible Limit of the Free-Boundary Problem in Compressible Resistive MHD

In this section we study the free-boundary problem in compressible resistive MHD. We will prove the local well-posedness and justify the incompressible limit. To prove the local existence, we shall first define the approximate system.

5.2.1 A priori estimates of the nonlinear approximate system

For $\kappa > 0$, we introduce the nonlinear κ-approximation system.

$$\begin{cases} \partial_t \eta = v + \psi & \text{in } \Omega, \\ \rho_0 \tilde{J}^{-1} \partial_t v = (b \cdot \nabla_{\tilde{A}}) b - \nabla_{\tilde{A}} Q, \quad Q = q + \frac{1}{2}|b|^2 & \text{in } \Omega, \\ \frac{\tilde{J} R'(q)}{\rho_0} \partial_t q + \text{div}_{\tilde{A}} v = 0 & \text{in } \Omega, \\ \partial_t b + \text{curl}_{\tilde{A}} \text{curl}_{\tilde{A}} b = (b \cdot \nabla_{\tilde{A}}) v - b \,\text{div}_{\tilde{A}} v, & \text{in } \Omega, \\ \text{div}_{\tilde{A}} b = 0 & \text{in } \Omega, \\ q = 0, \ b = \mathbf{0} & \text{on } \Gamma, \\ (\eta, v, b, q)|_{\{t=0\}} = (\text{Id}, v_0, b_0, q_0). \end{cases} \tag{5.2.1}$$

The quantities with "tilde" are defined in the same way as in Chapter 4.2. The term $\psi = \psi(\eta, v)$ is a correction term which solves the Laplacian equation

$$\begin{cases} \triangle \psi = 0 & \text{in } \Omega, \\ \psi = \overline{\triangle}^{-1} \mathbb{P}_{\neq 0} \left(\overline{\triangle} \eta_\beta \tilde{A}^{i\beta} \overline{\partial}_i \Lambda_\kappa^2 v - \overline{\triangle} \Lambda_\kappa^2 \eta_\beta \tilde{A}^{i\beta} \overline{\partial}_i v \right) & \text{on } \Gamma, \end{cases} \tag{5.2.2}$$

where $\mathbb{P}_{\neq 0}$ denotes the standard Littlewood-Paley projection in $\mathbb{T}^2$ which removes the zero-frequency part. $\overline{\triangle} := \partial_1^2 + \partial_2^2$ denotes the tangential Laplacian operator.

Remark 5.2.1.

1. Taking $\text{div}_{\tilde{A}}$ in the fourth equation yields $\partial_t(\text{div}_{\tilde{A}} b) + (\text{div}_{\tilde{A}} v)(\text{div}_{\tilde{A}} b) = 0$, which implies $\text{div}_{\tilde{A}} b = 0$ if $\text{div}\, b_0 = 0$ by the Gronwall-type argument. The identity $\text{curl}_{\tilde{A}}\text{curl}_{\tilde{A}} b = -\triangle_{\tilde{A}} b + \nabla_{\tilde{A}}\text{div}_{\tilde{A}} b$ and $\text{div}_{\tilde{A}} b = 0$ shows that b satisfies the heat equation $\partial_t b - \triangle_{\tilde{A}} b = (b\cdot\nabla_{\tilde{A}})v - b(\text{div}_{\tilde{A}} v)$ with $b|_\Gamma = \mathbf{0}$.

2. The corrector $\psi \to 0$ as $\kappa \to 0$ which is used to eliminate higher order boundary terms which appears in the tangential estimates of v. These terms are zero when $\kappa = 0$ but cannot be controlled when $\kappa > 0$. This is necesssary, otherwise we need higher regularity of η than v which is impossible for MHD.

3. The Littlewood-Paley projection is necessary because we will repeatedly use

$$|\overline{\triangle}^{-1}\mathbb{P}_{\neq 0} f|_s \approx |\mathbb{P}_{\neq 0} f|_{H^{s-2}} \approx |f|_{\dot{H}^{s-2}},$$

which can be proved by using Bernstein inequality.

4. The initial data is the same of origin system because the compatibility conditions stay unchanged after mollification by $\tilde{A}(0) = a(0) = \text{Id}$. Such initial data has been constructed in [82, Section 9], so they are omitted here.

5. The precise form of the commutators can be found in [82, Section 4.4]. Details are omitted here.

Now, we define the energy functional of (5.2.1)

$$\mathcal{E}_\kappa(T) := \mathfrak{e}_\kappa(T) + H_\kappa(T) + W_\kappa(T) + \left\|\partial_t^{4-k}\left(\left(b\cdot\nabla_{\tilde{A}}\right)b\right)\right\|_k^2, \tag{5.2.3}$$

where

$$\mathfrak{e}_\kappa(T) := \|\eta\|_4^2 + \left|\tilde{A}^{3\alpha}\overline{\partial}^4 \Lambda_\kappa \eta_\alpha\right|_0^2 + \sum_{k=0}^{4} \left(\left\|\partial_t^{4-k} v\right\|_k^2 + \left\|\partial_t^{4-k} b\right\|_k^2 + \left\|\partial_t^{4-k} q\right\|_k^2 \right), \tag{5.2.4}$$

$$H_\kappa(T) := \int_0^T \int_\Omega \left|\partial_t^5 b\right|^2 \,\mathrm{d}y\,\mathrm{d}t + \left\|\partial_t^4 b\right\|_1^2, \tag{5.2.5}$$

$$W_\kappa(T) := \left\|\partial_t^5 q\right\|_0^2 + \left\|\partial_t^4 q\right\|_1^2 \tag{5.2.6}$$

denote the energy functional of fluid, higher order heat equation of b, and wave equation of q, respectively. The context of this section is the uniform-in-κ a priori estimates of (5.2.1).

Proposition 5.2.2. There exists some $T > 0$ independent of κ, such that the energy functional E_κ satisfies

$$\sup_{0\le t\le T} \mathcal{E}_\kappa(t) \le P(\|v_0\|_4, \|b_0\|_5, \|q_0\|_4, \|\rho_0\|_4), \tag{5.2.7}$$

provided the following assumptions hold for all $t \in [0, T]$

$$-(\partial_N Q)(t) \ge c_0/2 \quad \text{on } \Gamma, \tag{5.2.8}$$

$$\|\tilde{J}(t) - 1\|_3 + \|\mathrm{Id} - \tilde{A}(t)\|_3 \le \varepsilon \quad \text{in } \Omega. \tag{5.2.9}$$

Remark 5.2.3. The a priori assumptions can be easily justified once the energy bounds are established by using $\tilde{A}(T) - \mathrm{Id} = \int_0^T \partial_t \tilde{A} = \int_0^T \tilde{A} : \partial_t \partial\tilde{\eta} : \tilde{A}\,\mathrm{d}t$ and the smallness of T. See Lemma 5.2.7.

In Section 5.2.2, we will prove the local well-posedness of (5.2.1) in an κ-dependent time interval $[0, T_\kappa]$. Therefore, the uniform-in-κ a priori estimate guarantees that the solution $(\eta(\kappa), v(\kappa), b(\kappa), q(\kappa))$ to (5.2.1) converges to the solution to the original system as $\kappa \to 0$, i.e., the local existence of the solution to free-boundary compressible resistive MHD system is established. For simplicity, we omit the κ and only write (η, v, b, q) in this manuscript.

5.2.1.1 Estimates of the correction term

First we bound the flow map and the correction term together with their smoothed version by the quantities in $\mathcal{E}_\kappa$. The following estimates will be repeatedly used.

Lemma 5.2.4 ([83, Lemma 3.2])**.** The following estimates for (v, ψ, η) holds.

$$\|\tilde{\eta}\|_4 \lesssim \|\eta\|_4, \tag{5.2.10}$$

$$\|\psi\|_4 \lesssim P(\|\eta\|_4, \|v\|_3), \tag{5.2.11}$$

$$\|\partial_t\psi\|_4 \lesssim P(\|\eta\|_4, \|v\|_4, \|\partial_t v\|_3), \tag{5.2.12}$$

$$\|\partial_t^2\psi\|_3 \lesssim P(\|\eta\|_4, \|v\|_4, \|\partial_t v\|_3, \|\partial_t^2 v\|_2), \tag{5.2.13}$$

$$\|\partial_t^3\psi\|_2 \lesssim P(\|\eta\|_4, \|v\|_4, \|\partial_t v\|_3, \|\partial_t^2 v\|_2, \|\partial_t^3 v\|_1), \tag{5.2.14}$$

$$\|\partial_t^4\psi\|_1 \lesssim P(\|\eta\|_4, \|v\|_4, \|\partial_t v\|_3, \|\partial_t^2 v\|_2, \|\partial_t^3 v\|_1, \|\partial_t^4 v\|_0). \tag{5.2.15}$$

and

$$\|\partial_t\tilde{\eta}\|_4 \lesssim \|\partial_t\eta\|_4 \lesssim P(\|\eta\|_4, \|v\|_4), \tag{5.2.16}$$

$$\|\partial_t^2\tilde{\eta}\|_3 \lesssim \|\partial_t^2\eta\|_3 \lesssim P(\|\eta\|_4, \|v\|_4, \|\partial_t v\|_3), \tag{5.2.17}$$

$$\|\partial_t^3\tilde{\eta}\|_2 \lesssim \|\partial_t^3\eta\|_2 \lesssim P(\|\eta\|_4, \|v\|_4, \|\partial_t v\|_3, \|\partial_t^2 v\|_2), \tag{5.2.18}$$

$$\|\partial_t^4\tilde{\eta}\|_1 \lesssim \|\partial_t^4\eta\|_1 \lesssim P(\|\eta\|_4, \|v\|_4, \|\partial_t v\|_3, \|\partial_t^2 v\|_2, \|\partial_t^3 v\|_1) \tag{5.2.19}$$

$$\|\partial_t^5\tilde{\eta}\|_0 \lesssim \|\partial_t^5\eta\|_0 \lesssim P(\|\eta\|_4, \|v\|_4, \|\partial_t v\|_3, \|\partial_t^2 v\|_2, \|\partial_t^3 v\|_1, \|\partial_t^4 v\|_0). \tag{5.2.20}$$

5.2.1.2 Estimates of the magnetic field

For ideal MHD, the magnetic field can be written as $b = J^{-1}(b_0 \cdot \partial)\eta$, which should be controlled together with the same derivatives of $v = \partial_t \eta$, and higher order terms are expected to vanish due to subtle cancellation. But for resistive MHD, invoking the well-known identity $-\triangle_{\tilde{A}} b = \text{curl}_{\tilde{A}}\text{curl}_{\tilde{A}} b - \nabla_{\tilde{A}}\text{div}_{\tilde{A}} b = \text{curl}_{\tilde{A}}\text{curl}_{\tilde{A}} b$, the magnetic field actually satisfies a heat equation. Thus, the magnetic diffusion, together with the boundary condition $b = \mathbf{0}$, allows us to control the higher order term $\triangle_{\tilde{A}} b$ directly, and $(b \cdot \nabla_{\tilde{A}})b$ (Lorentz force) with the help of Christodoulou-Lindblad type elliptic estimates Lemma 3.3.3.

Control of $\partial_t^k b$ when $k \le 2$ First we estimate $\|\partial_t^{4-k} b\|_k$. When $k \ge 1$, we have

$$\partial^{k-1}\partial_\alpha \partial_t^{4-k} b = \partial^{k-1}(\tilde{A}^\mu_\alpha \partial_\mu \partial_t^{4-k} b) + \partial^{k-1}((\delta^\mu_\alpha - \tilde{A}^\mu_\alpha)\partial_\mu \partial_t^{4-k} b),$$

which gives

$$\|\partial_t^{4-k} b\|_k^2 \lesssim \|\nabla_{\tilde{A}} \partial_t^{4-k} b\|_{k-1}^2 + \|\text{Id} - \tilde{A}\|_{k-1}^2 \|\partial_t^{4-k} b\|_k^2 \le \|\nabla_{\tilde{A}} \partial_t^{4-k} b\|_{k-1}^2 + \varepsilon^2 \|\partial_t^{4-k} b\|_k^2.$$

Here the ε-term can be absorbed into the LHS. When $k = 1, 2$, $\|\text{Id} - \tilde{A}\|_{k-1}$ should be replaced by $\|\text{Id} - \tilde{A}\|_{L^\infty}$. Therefore, we have that for $1 \le k \le 4$, $\|\partial_t^{4-k} b\|_k \lesssim \|\nabla_{\tilde{A}}(\partial_t^{4-k} b)\|_{k-1}$, which motivates us to use Lemma 3.3.3.

Applying Lemma 3.3.3 to b, we have

$$\|b\|_4 \approx \|\nabla_{\tilde{A}} b\|_3 \lesssim P(\|\tilde{\eta}\|_3)(\|\triangle_{\tilde{A}} b\|_2 + \|\overline{\partial}\tilde{\eta}\|_3 \|b\|_3) \tag{5.2.21}$$

Invoking the heat equation $\triangle_{\tilde{A}} b = \partial_t b - (b \cdot \nabla_{\tilde{A}})v + b\text{div}_{\tilde{A}} v$, we have

$$\begin{aligned}\|b\|_4 \lesssim & P(\|\tilde{\eta}\|_3)\left(\|\partial_t b\|_2 + \|(b \cdot \nabla_{\tilde{A}})v\|_2 + \|b\text{div}_{\tilde{A}} v\|_2 + \|\overline{\partial}\tilde{\eta}\|_3 \|b\|_3\right) \\ \lesssim & P(\|\tilde{\eta}\|_3) P\left(\|\partial_t b\|_2, \|b\|_2, \|u\|_3\right) + P(\|\tilde{\eta}\|_3)\|\overline{\partial}\tilde{\eta}\|_3 \|b\|_3\end{aligned} \tag{5.2.22}$$

The first term $P(\|\tilde{\eta}\|_3)P\,(\|\partial_t b\|_2, \|b\|_2, \|u\|_3)$ can be directly controlled by $\mathcal{P}_0 + \int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t$. For the second term, we notice that $\|\overline{\partial}\tilde{\eta}\|_3 \lesssim \|\overline{\partial}\tilde{\eta}\|_0 + \|\partial^3\overline{\partial}\tilde{\eta}\|_0$, and $\overline{\partial}_i\eta_\alpha|_{t=0} = \delta_{i\alpha}$, $\partial^3\overline{\partial}\eta|_{t=0} = 0$, so

$$\|\overline{\partial}\tilde{\eta}\|_3 \lesssim 1 + \int_0^T \|\partial_t\overline{\partial}\tilde{\eta}\|_3\,\mathrm{d}t.$$

Plugging this into the second term $\left(\|\overline{\partial}\eta\|_0 + \|\overline{\partial}\partial\tilde{\eta}\|_2\right) P(\|\tilde{\eta}\|_3)\|b\|_3$, we know

$$\begin{aligned}
&\left(\|\overline{\partial}\eta\|_0 + \|\overline{\partial}\partial\tilde{\eta}\|_2\right) P(\|\tilde{\eta}\|_3)\|b\|_3 \\
\lesssim& P(\|\tilde{\eta}\|_3)\|b\|_3 \int_0^T \|\partial_t\overline{\partial}\tilde{\eta}\|_3\,\mathrm{d}t + P(\|\tilde{\eta}\|_3)\left(\|b_0\|_3 + \int_0^T \|\partial_t b\|_3\,\mathrm{d}t\right) \\
\lesssim& \mathcal{P}_0 + P(\mathfrak{e}_\kappa(T))\int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t.
\end{aligned}$$

Therefore, (5.2.22) becomes

$$\|b\|_4 \lesssim \mathcal{P}_0 + P(\mathfrak{e}_\kappa(T))\int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t \tag{5.2.23}$$

Since ∂_t is tangential on Γ, $\partial_t b$ also vanishes on the boundary. Applying elliptic estimates as in (5.2.21), we get

$$\|\partial_t b\|_3 \approx \|\nabla_{\tilde{A}}\partial_t b\|_2 \lesssim P(\|\tilde{\eta}\|_2)\left(\|\triangle_{\tilde{A}}\partial_t b\|_1 + \|\overline{\partial}\tilde{\eta}\|_2\|\partial_t b\|_2\right) \tag{5.2.24}$$

$$\|\partial_t^2 b\|_2 \approx \|\nabla_{\tilde{A}}\partial_t^2 b\|_1 \lesssim P(\|\tilde{\eta}\|_2)\left(\|\triangle_{\tilde{A}}\partial_t^2 b\|_0 + \|\overline{\partial}\tilde{\eta}\|_2\|\partial_t^2 b\|_1\right). \tag{5.2.25}$$

Taking time derivatives in the heat equation of b, we have

$$\triangle_{\tilde{A}}\partial_t^k b = \partial_t^{k+1} b - \partial_t^k\left((b\cdot\nabla_{\tilde{A}})v - b\,\mathrm{div}_{\tilde{A}} v\right),$$

of which the RHS is of one less derivative than LHS. Therefore, we are able to control $\|\partial_t b\|_3$, $\|\partial_t^2 b\|_2$

in the same way as (5.2.23):

$$\|\partial_t b\|_3 + \|\partial_t^2 b\|_2 \lesssim \mathcal{P}_0 + P(\mathfrak{e}_\kappa(T)) \int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t \tag{5.2.26}$$

Control of $\partial_t^3 b$: Heat equation Note that $\|\partial_t^3 b\|_1 \approx \|\nabla_{\tilde{A}} \partial_t^3 b\|_0$ is a part of the energy of 3-rd order time-differentiated heat equation

$$\partial_t^4 b - \triangle_{\tilde{A}} \partial_t^3 b = \partial_t^3 \left((b \cdot \nabla_{\tilde{A}})v) - b\mathrm{div}_{\tilde{A}} v\right) + [\partial_t^3, \triangle_{\tilde{A}}]b,$$

of which the RHS only contain terms with ≤ 4 derivatives.

Taking L^2 inner product with $\tilde{J}\partial_t^4 b$, integrating in $y \in \Omega$ and $t \in [0, T]$, and then integrating by parts, one has

$$\begin{aligned}
LHS =& \int_0^T \int_\Omega \tilde{J} \left|\partial_t^4 b\right|^2 \,\mathrm{d}y\,\mathrm{d}t - \int_0^T \int_\Omega \partial_t^4 b \cdot \tilde{J} \triangle_{\tilde{A}} \partial_t^3 b\,\mathrm{d}y\,\mathrm{d}t \\
=& \int_0^T \int_\Omega \tilde{J} \left|\partial_t^4 b\right|^2 \,\mathrm{d}y\,\mathrm{d}t + \int_0^T \int_\Omega \tilde{J} \nabla_{\tilde{A}} \partial_t^4 b \cdot \nabla_{\tilde{A}} \partial_t^3 b\,\mathrm{d}y\,\mathrm{d}t \\
=& \int_0^T \int_\Omega \tilde{J} \left|\partial_t^4 b\right|^2 \,\mathrm{d}y\,\mathrm{d}t + \frac{1}{2} \int_\Omega \tilde{J} \left|\nabla_{\tilde{A}} \partial_t^3 b\right|^2 \,\mathrm{d}y \Big|_0^T \\
&- \frac{1}{2} \int_0^T \int_\Omega \partial_t \tilde{J} \left|\nabla_{\tilde{A}} \partial_t^3 b\right|^2 \,\mathrm{d}y\,\mathrm{d}t + \int_0^T \int_\Omega \tilde{J} \left[\nabla_{\tilde{A}}, \partial_t\right] \partial_t^3 b \cdot \nabla_{\tilde{A}} \partial_t^3 b\,\mathrm{d}y\,\mathrm{d}t \\
RHS =& \int_0^T \int_\Omega \partial_t^4 b \cdot \left(\partial_t^3 \left((b \cdot \nabla_{\tilde{A}})v) - b\mathrm{div}_{\tilde{A}} v\right) + [\partial_t^3, \triangle_{\tilde{A}}]b\right) \,\mathrm{d}y\,\mathrm{d}t
\end{aligned}$$

Therefore, one has

$$
\begin{aligned}
&\int_0^T \int_\Omega \tilde{J} \left|\partial_t^4 b\right|^2 \, \mathrm{d}y \, \mathrm{d}t + \frac{1}{2} \int_\Omega \tilde{J} \left|\nabla_{\tilde{A}} \partial_t^3 b(T)\right|^2 \, \mathrm{d}y \\
=&\frac{1}{2} \int_\Omega \tilde{J} \left|\nabla_{\tilde{A}} \partial_t^3 b(0)\right|^2 \, \mathrm{d}y + \frac{1}{2} \int_0^T \int_\Omega \partial_t \tilde{J} \left|\nabla_{\tilde{A}} \partial_t^3 b\right|^2 \, \mathrm{d}y \, \mathrm{d}t \\
&- \int_0^T \int_\Omega \tilde{J} \left[\nabla_{\tilde{A}}, \partial_t\right] \partial_t^3 b \cdot \nabla_{\tilde{A}} \partial_t^3 b \, \mathrm{d}y \, \mathrm{d}t \\
&+ \int_0^T \int_\Omega \partial_t^4 b \cdot \left(\partial_t^3 \left((b \cdot \nabla_{\tilde{A}}) v\right) - b \mathrm{div}_{\tilde{A}} v\right) + [\partial_t^3, \triangle_{\tilde{A}}] b\big) \, \mathrm{d}y \, \mathrm{d}t \\
\lesssim&\mathcal{P}_0 + \int_0^T P(\mathfrak{e}_\kappa(t)) \, \mathrm{d}t,
\end{aligned}
\tag{5.2.27}
$$

which gives the H^1 control of $\partial_t^3 b$.

Control of $\partial_t^4 b$: Higher order estimates needed There are two ways to control $\|\partial_t^4 b\|_0$. One way is to use Poincaré's inequality

$$
\|\partial_t^4 b\|_0 \lesssim \|\partial_t^4 b\|_1 \approx \|\nabla_{\tilde{A}} \partial_t^4 b\|_0 \tag{5.2.28}
$$

due to $\partial_t^4 b|_\Gamma = 0$. Another way is direct computation

$$
\begin{aligned}
\frac{1}{2} \|\partial_t^4 b\|_0^2 =&\frac{1}{2} \|\partial_t^4 b(0)\|_0^2 + \int_0^T \partial_t^4 b \cdot \partial_t^5 b \, \mathrm{d}t \\
&\lesssim \mathcal{P}_0 + \left\|\partial_t^5 b\right\|_{L_t^2 L_x^2([0,T]\times\Omega)} \left\|\partial_t^4 b\right\|_{L_t^2 L_x^2([0,T]\times\Omega)} \\
&\lesssim \mathcal{P}_0 + \varepsilon \int_0^T \int_\Omega \left|\partial_t^5 b\right|^2 \, \mathrm{d}y \, \mathrm{d}t + \frac{1}{4\varepsilon} \int_0^T \int_\Omega \left|\partial_t^4 b\right|^2 \, \mathrm{d}y \, \mathrm{d}t \\
&\lesssim \varepsilon \int_0^T \int_\Omega \left|\partial_t^5 b\right|^2 \, \mathrm{d}y \, \mathrm{d}t + \mathcal{P}_0 + \int_0^T P(\mathfrak{e}_\kappa(t)) \, \mathrm{d}t.
\end{aligned}
\tag{5.2.29}
$$

From (5.2.28) and (5.2.29), we find that either $\|\nabla_{\tilde{A}} \partial_t^4 b\|_0^2$ or $\|\partial_t^5 b\|_{L_t^2 L_x^2}$ is required to control $\|\partial_t^4 b\|_0^2$. On the other hand, we notice that these two terms exactly come from the energy functional of 4-th time-differentiated heat equation of b:

$$\partial_t^5 b - \triangle_{\tilde{A}} \partial_t^4 b = \left[\partial_t^4, \triangle_{\tilde{A}}\right] b + \partial_t^4 \left((b \cdot \nabla_{\tilde{A}})v - b\mathrm{div}_{\tilde{A}} v\right).$$

The energy estimate cannot be controlled in the same way as in Section 5.2.1.2 because the RHS of this heat equation contains 5-th order derivatives. Instead, we will seek for a common control of b and p via the heat equation and wave equation. This part will be postponed to Section 5.2.1.4.

Estimates of Lorentz force Later on we will see both the estimates of u and common control of higher order heat and wave equations require the control of 5-th derivatives of magnetic field, all of which are actually 4-th space-time derivatives of Lorentz force $(b \cdot \nabla_{\tilde{A}})b$. Notice that $b = 0$ on the boundary implies $(b \cdot \nabla_{\tilde{A}})b$ also vanishes on Γ. Therefore, we can apply the elliptic estimate Lemma 3.3.3 to $(b \cdot \nabla_{\tilde{A}})b$.

We start with $\|(b \cdot \nabla_{\tilde{A}})b\|_4$. Similarly as in (5.2.21), we have

$$\|(b \cdot \nabla_{\tilde{A}})b\|_4 \approx \|\nabla_{\tilde{A}}((b \cdot \nabla_{\tilde{A}})b)\|_3 \lesssim P(\|\tilde{\eta}\|_3)\left(\|\triangle_{\tilde{A}}((b \cdot \nabla_{\tilde{A}})b)\|_2 + \|\overline{\partial}\tilde{\eta}\|_3 \|(b \cdot \nabla_{\tilde{A}})b\|_3\right) \quad (5.2.30)$$

The second term $P(\|\tilde{\eta}\|_3)\|\overline{\partial}\tilde{\eta}\|_3\|(b \cdot \nabla_{\tilde{A}})b\|_3$ can again be controlled by $\mathcal{P}_0 + P(\mathfrak{e}_\kappa(T)) \int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t$ by writting $\|\overline{\partial}\tilde{\eta}\|_3 \lesssim 1 + \int_0^T \|\partial_t \overline{\partial}\tilde{\eta}\|_3$ as in (5.2.23). For the first term, we invoke the heat equation of b to get

$$\begin{aligned} \triangle_{\tilde{A}}\left((b \cdot \nabla_{\tilde{A}})b\right) =& (b \cdot \nabla_{\tilde{A}})\left(\triangle_{\tilde{A}} b\right) + [\triangle_{\tilde{A}}, b \cdot \nabla_{\tilde{A}}]b \\ =& (b \cdot \nabla_{\tilde{A}})\left(\partial_t b - (b \cdot \nabla_{\tilde{A}})v + b\mathrm{div}_{\tilde{A}} v\right) + [\triangle_{\tilde{A}}, b \cdot \nabla_{\tilde{A}}]b, \end{aligned}$$

of which the RHS only contains terms with ≤ 2 derivatives. So we have

$$\|\triangle_{\tilde{A}}\left((b \cdot \nabla_{\tilde{A}})b\right)\|_2 \lesssim P(\mathfrak{e}_\kappa(T)),$$

and thus

$$\|(b\cdot\nabla_{\tilde{A}})b\|_4 \lesssim P(\mathfrak{e}_\kappa(T)) + \mathcal{P}_0 + P(\mathfrak{e}_\kappa(T))\int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t. \tag{5.2.31}$$

When $k=1,2$, $\|\partial_t^k((b\cdot\nabla_{\tilde{A}})b)\|_{4-k}$ can be controlled in the same way as (5.2.24), (5.2.25) and (5.2.26):

$$\begin{aligned}
&\|\partial_t((b\cdot\nabla_{\tilde{A}})b)\|_3 + \|\partial_t^2((b\cdot\nabla_{\tilde{A}})b)\|_2\\
\lesssim & P(\|\tilde{\eta}\|_2)\left(\|\triangle_{\tilde{A}}\partial_t((b\cdot\nabla_{\tilde{A}})b)\|_1 + \|\triangle_{\tilde{A}}\partial_t^2((b\cdot\nabla_{\tilde{A}})b)\|_0\right)\\
&+ P(\|\tilde{\eta}\|_2)\|\overline{\partial}\tilde{\eta}\|_2\left(\|\partial_t((b\cdot\nabla_{\tilde{A}})b)\|_2 + \|\partial_t^2((b\cdot\nabla_{\tilde{A}})b)\|_1\right)\\
\lesssim & P(\|\tilde{\eta}\|_2)\Bigg(\|\partial_t(b\cdot\nabla_{\tilde{A}})(\triangle_{\tilde{A}}b)\|_1 + \|[\triangle_{\tilde{A}},\partial_t(b\cdot\nabla_{\tilde{A}})]b\|_1\\
&\quad + \|\partial_t^2(b\cdot\nabla_{\tilde{A}})\triangle_{\tilde{A}}b\|_0 + \|[\triangle_{\tilde{A}},\partial_t^2(b\cdot\nabla_{\tilde{A}})]b\|_0\Bigg)\\
&+ \mathcal{P}_0 + P(\mathfrak{e}_\kappa(T))\int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t\\
\lesssim & P(\mathfrak{e}_\kappa(T)) + \mathcal{P}_0 + P(\mathfrak{e}_\kappa(T))\int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t.
\end{aligned} \tag{5.2.32}$$

When $k=3$, we have $\partial(\partial_t^3(b\cdot\nabla_{\tilde{A}})b) = (b\cdot\nabla_{\tilde{A}})\partial\partial_t^3 b + [\partial\partial_t^3, b\cdot\nabla_{\tilde{A}}]b$, where the commutator only contains the terms of $\leq$ 4-th order derivative, so

$$\|\partial_t^3(b\cdot\nabla_{\tilde{A}})b\|_1 \lesssim \|b\|_2\|\nabla_{\tilde{A}}\partial_t^3 b\|_1 + P(\mathfrak{e}_\kappa(T)) \tag{5.2.33}$$

Then by elliptic estimates Lemma 3.3.3 and the heat equation,

$$\begin{aligned}
\|\nabla_{\tilde{A}}\partial_t^3 b\|_1 \lesssim & P(\|\tilde{\eta}\|_2)(\|\triangle_{\tilde{A}}\partial_t^3 b\|_0 + \|\overline{\partial}\tilde{\eta}\|_2\|\partial_t^3 b\|_1)\\
\lesssim & P(\|\tilde{\eta}\|_2)\left(\|\partial_t^4 b\|_0 + \|\partial_t^3((b\cdot\nabla_{\tilde{A}})v - b\,\mathrm{div}_{\tilde{A}}v)\|_0 + \|[\triangle_{\tilde{A}},\partial_t^3]b\|_0\right) + P(\mathfrak{e}_\kappa(T))\\
\lesssim & P(\mathfrak{e}_\kappa(T))
\end{aligned}$$

Similarly, for $k=4$, we have $\partial_t^4((b\cdot\nabla_{\tilde{A}})b)=(b\cdot\nabla_{\tilde{A}})\partial_t^4 b+[\partial_t^4, b\cdot\nabla_{\tilde{A}}]b$, where the commutator only contains the terms of $\leq$ 4-th order derivative, so

$$\begin{aligned}\|\partial_t^4(b\cdot\nabla_{\tilde{A}})b\|_0 \lesssim&\|(b\cdot\nabla_{\tilde{A}})\partial_t^4 b\|_0+\|[\partial_t^4, b\cdot\nabla_{\tilde{A}}]b\|_0\\ \lesssim&\|b\|_2\|\nabla_{\tilde{A}}\partial_t^4 b\|_0+P(\mathfrak{e}_\kappa(T)),\end{aligned} \tag{5.2.34}$$

where the term $\|\nabla_{\tilde{A}}\partial_t^4 b\|_0^2$ is exactly part of the energy functional $H_\kappa(T)$ of 4-th time-differentiated heat equation. Summing up (5.2.31), (5.2.32), (5.2.33) and (5.2.34), we get the estimates of Lorentz force

$$\sum_{k=0}^4\left\|\partial_t^{4-k}((b\cdot\nabla_{\tilde{A}})b)\right\|_k^2\lesssim\|b\|_2^2\left\|\nabla_{\tilde{A}}\partial_t^4 b\right\|_0^2+P(\mathfrak{e}_\kappa(T))+\mathcal{P}_0+P(\mathfrak{e}_\kappa(T))\int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t \tag{5.2.35}$$

Therefore, we find that the estimates of Lorentz force are again reduced to the control of higher order heat equation.

5.2.1.3 Estimates of the velocity and the pressure

In this part we control the space-time Sobolev norm of v and q. We first apply the Hodge-type div-curl decomposition (Lemma 3.3.1) to v (and its time derivatives). The curl part can be directly controlled by the counterpart of Lorentz force. The boundary term can be reduced to interior tangential estimates by using Sobolev trace Lemma. The divergence part together with the estimates of q can be reduced to the control of full time derivatives, which is also part of tangential estimates. One should keep in mind that, we no longer seek for subtle cancellation to eliminate higher order terms as what was done for ideal MHD, no matter in curl or tangential estimates. Instead, those higher order terms (with 5 derivatives) can be controlled either by Lorentz force, or by the combination of heat equation and wave equation, i.e., $H_\kappa(T)$ and $W_\kappa(T)$.

Let $X = v, \partial_t v, \partial_t^2 v, \partial_t^3 v$ and $s = 4, 3, 2, 1$ in Lemma 3.3.1 respectively. We have

$$
\begin{aligned}
\|v\|_4 &\lesssim \|v\|_0 + \|\text{div } v\|_3 + \|\text{curl } v\|_3 + |v \cdot N|_{3.5} \\
\|\partial_t v\|_3 &\lesssim \|\partial_t v\|_0 + \|\text{div } \partial_t v\|_2 + \|\text{curl } \partial_t v\|_2 + |\partial_t v \cdot N|_{2.5} \\
\|\partial_t^2 v\|_2 &\lesssim \|\partial_t^2 v\|_0 + \|\text{div } \partial_t^2 v\|_1 + \|\text{curl } \partial_t^2 v\|_1 + |\partial_t^2 v \cdot N|_{1.5} \\
\|\partial_t^3 v\|_1 &\lesssim \|\partial_t^3 v\|_0 + \|\text{div } \partial_t^3 v\|_0 + \|\text{curl } \partial_t^3 v\|_0 + |\partial_t^3 v \cdot N|_{0.5}.
\end{aligned}
\tag{5.2.36}
$$

First, the L^2-norms are of lower order. The L^2-norm of v has been controlled in the energy dissapation. While for $\|\partial_t v\|_0, \|\partial_t^2 v\|_0$ and $\|\partial_t^3 v\|_0$, we commute ∂_t through $\rho_0 \tilde{J}^{-1} \partial_t v = (b \cdot \nabla_{\tilde{A}}) b - \nabla_{\tilde{A}} Q$ and obtain

$$
\|\partial_t v(T)\|_0 + \|\partial_t^2 v(T)\|_0 + \|\partial_t^3 v(T)\|_0 \lesssim \mathcal{P}_0 + \int_0^T P(\mathfrak{e}_\kappa(t))\, \mathrm{d}t \tag{5.2.37}
$$

Boundary estimates: Reduced to tangential estimates The boundary part of div-curl decomposition can be reduced to the interior tangential estimates by invoking the normal trace Lemma 3.2.3

$$
|\overline{\partial} v^3|_{2.5} \lesssim \|\overline{\partial}^4 v\|_0 + \|\text{div } v\|_3. \tag{5.2.38}
$$

Similarly we have for $1 \le k \le 3$

$$
|\partial_t^k v^3|_{3.5-k} \lesssim \|\overline{\partial}^{4-k} \partial_t^k v\|_0 + \|\text{div } \partial_t^k v\|_{3-k} \tag{5.2.39}
$$

Curl control: Reduced to Lorentz force By the a priori assumption (5.2.9), we can estimate the Lagrangian vorticity via Eulerian vorticity plus a small error, for $1 \le k \le 4$

$$
\|\text{curl } \partial_t^{4-k} v\|_{k-1}^2 \lesssim \|\text{curl}_{\tilde{A}} \partial_t^{4-k} v\|_{k-1}^2 + \varepsilon^2 \|\partial_t^{4-k} v\|_k^2 \tag{5.2.40}
$$

Taking $\text{curl}_{\tilde{A}}$ in $\rho_0 \tilde{J}^{-1} \partial_t v = (b \cdot \nabla_{\tilde{A}}) b - \nabla_{\tilde{A}} Q$, we have

$$
\rho_0 \tilde{J}^{-1} \partial_t \text{curl}_{\tilde{A}} v = \text{curl}_{\tilde{A}} \left((b \cdot \nabla_{\tilde{A}}) b\right) + \left[\rho_0 \tilde{J}^{-1} \partial_t, \text{curl}_{\tilde{A}}\right] v, \tag{5.2.41}
$$

where the commutator only contains first order derivative of $v, \rho, \partial_t \eta$.

Taking $\partial_t^{4-k}\partial^{k-1}$ in (5.2.41), we get the evolution equation of $\text{curl}_{\tilde{A}} v$:

$$
\begin{aligned}
\rho_0 \tilde{J}^{-1} \partial_t (\partial^{k-1} \text{curl}_{\tilde{A}} \partial_t^{4-k} v) = & \, \partial_t^{4-k} \partial^{k-1} \text{curl} \, ((b \cdot \nabla_{\tilde{A}}) b) \\
& + \partial_t^{4-k} \partial^{k-1} \left(\left[\rho_0 \tilde{J}^{-1} \partial_t, \text{curl}_{\tilde{A}} \right] v \right) \\
& + [\partial_t^{4-k} \partial^{k-1}, \rho_0 \tilde{J}^{-1} \partial_t] \text{curl}_{\tilde{A}} v + \rho_0 \tilde{J}^{-1} \partial_t \partial^{k-1} ([\text{curl}_{\tilde{A}}, \partial_t^{4-k}] v) \\
=: & \partial_t^{4-k} \partial^{k-1} \text{curl} \, ((b \cdot \nabla_{\tilde{A}}) b) + F_k
\end{aligned}
\tag{5.2.42}
$$

then taking L^2-inner product with $\partial^{k-1}\text{curl}_{\tilde{A}}\partial_t^{4-k} v$, we have

$$
\begin{aligned}
& \frac{1}{2} \int_\Omega \rho_0 \tilde{J}^{-1} \left| \partial^{k-1} \text{curl}_{\tilde{A}} \partial_t^{4-k} v(T) \right|^2 \, dy - \frac{1}{2} \int_\Omega \rho_0 \tilde{J}^{-1} \left| \partial^{k-1} \text{curl}_{\tilde{A}} \partial_t^{4-k} v(0) \right|^2 \, dy \\
= & \frac{1}{2} \int_0^T \int_\Omega \partial_t \left(\rho_0 \tilde{J}^{-1} \right) \left| \partial^{k-1} \text{curl}_{\tilde{A}} \partial_t^{4-k} v \right|^2 \, dy \, dt \\
& + \int_0^T \int_\Omega \rho_0 \tilde{J}^{-1} \partial^{k-1} \text{curl}_{\tilde{A}} \partial_t^{4-k} v \cdot \partial^{k-1} \text{curl}_{\tilde{A}} \partial_t^{4-k} ((b \cdot \nabla_{\tilde{A}}) b) \, dy \, dt \\
& + \int_0^T \int_\Omega \rho_0 \tilde{J}^{-1} \partial^{k-1} \text{curl}_{\tilde{A}} \partial_t^{4-k} v \cdot F_k \, dy \, dt \\
\lesssim & \int_0^T \left\| \left(\rho_0 \tilde{J}^{-1} \right) \right\|_{L^\infty}^2 \left\| \partial_t^{4-k} v \right\|_k^2 \, dt + \int_0^T \left\| \partial_t^{4-k} v \right\|_k \left\| \partial_t^{4-k} \left((b \cdot \nabla_{\tilde{A}}) b \right) \right\|_k \, dt \\
& + \int_0^T \left\| \partial_t^{4-k} v \right\|_k \| F_k \|_{L^2} \, dt \\
\lesssim & \varepsilon T \sup_{0 \le t \le T} \left\| \partial_t^{4-k} \left((b \cdot \nabla_{\tilde{A}}) b \right) \right\|_k^2 + \int_0^T P(\mathfrak{e}_\kappa(t)) \, dt.
\end{aligned}
\tag{5.2.43}
$$

Here we used the fact that all terms in F_k are of ≤ 4 derivatives, and thus can be controlled by $P(\mathfrak{e}_\kappa(t))$.

Divergence Control: Reduction to full time derivatives Before going into the proof, we briefly describe the procedure of such reduction. The second and third equations of (5.2.1) give the following if we omit the coefficients $\rho_0 \tilde{J}^{-1}$ and $\frac{\tilde{J} R'(q)}{\rho_0}$:

$$\partial \partial_t^k q \approx \nabla_{\tilde{A}} \partial_t^k q \approx -\partial_t^{k+1} v + \partial_t^k \left((b \cdot \nabla_{\tilde{A}}) b - \frac{1}{2} |b|^2 \right),$$

and

$$\partial \partial_t^k v \xrightarrow{\text{div}} \partial_t^k \text{div}_{\tilde{A}} v + \text{curl} + \text{boundary} \sim \partial_t^{k+1} q + \text{curl} + \text{boundary}.$$

Since the terms containing magnetic field b can be reduced to lower order with the help of magnetic diffusion, the procedure above allows us to control div v by $\partial_t q$, and control ∂q by $\partial_t v$. In other words, we are able to trade one spatial derivative by one time derivative, and finally reduce the control to the full time derivative estimates.

$$\begin{gathered} \partial^4 q \xrightarrow{(5.2.1)} \partial^3 \partial_t v \xrightarrow{\text{div}} \partial^2 \partial_t^2 q \xrightarrow{(5.2.1)} \partial \partial_t^3 v \xrightarrow{\text{div}} \partial_t^4 q \\ \partial^3 \partial_t q \xrightarrow{(5.2.1)} \partial^2 \partial_t^2 v \xrightarrow{\text{div}} \partial \partial_t^3 q \xrightarrow{(5.2.1)} \partial_t^4 v. \end{gathered} \tag{5.2.44}$$

Step 1: Reduce q to $\partial_t v$

First we investigate $\|\partial_t^3 q\|_1$. We take ∂_t^3 in the second equation in (5.2.1) to get

$$\partial \partial_t^3 q = \partial_t^3 (\nabla_{\tilde{A}} q) + \nabla_{I-\tilde{A}} \partial_t^3 q = -\partial_t^3 \left(\rho_0 \tilde{J}^{-1} \partial_t v \right) + \partial_t^3 \left((b \cdot \nabla_{\tilde{A}}) b - \frac{1}{2} \nabla_{\tilde{A}} |b|^2 \right) + \nabla_{I-\tilde{A}} \partial_t^3 q,$$

where we have Therefore, $\partial_t^3 q$ is estimated as

$$\begin{aligned} \left\| \partial_t^3 q \right\|_1 \lesssim & \varepsilon \left\| \partial_t^3 q \right\|_1 + \left\| \partial_t^3 (\rho_0 \tilde{J}^{-1} \partial_t v) \right\|_0 + \left\| \partial_t^3 \left((b \cdot \nabla_{\tilde{A}}) b - \frac{1}{2} \nabla_{\tilde{A}} |b|^2 \right) \right\|_0 \\ \lesssim & \varepsilon \left\| \partial_t^3 q \right\|_1 + \left\| \rho_0 \tilde{J}^{-1} \right\|_{L^\infty} \left\| \partial_t^4 v \right\|_0 + \mathcal{P}_0 + \int_0^T P(\mathfrak{e}_\kappa(t)) \, \mathrm{d}t + L.O.T., \end{aligned} \tag{5.2.45}$$

where $\varepsilon > 0$ can be chosen suitably small in order for being absorbed by LHS. The $\mathcal{P}_0 + \int_0^T P(\mathfrak{e}_\kappa(t)) \, \mathrm{d}t$ comes from the magnetic field according to Section 5.2.1.2.

Similarly as in the derivation of (5.2.45), we get the following estimates

$$\|\partial_t^2 q\|_2 \lesssim \left\|\rho_0 \tilde{J}^{-1}\right\|_{L^\infty} \|\partial_t^3 v\|_1 + \mathcal{P}_0 + \int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t + L.O.T. \tag{5.2.46}$$

$$\|\partial_t q\|_3 \lesssim \left\|\rho_0 \tilde{J}^{-1}\right\|_{L^\infty} \|\partial_t^2 v\|_2 + \mathcal{P}_0 + \int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t + L.O.T. \tag{5.2.47}$$

$$\|q\|_4 \lesssim \left\|\rho_0 \tilde{J}^{-1}\right\|_{L^\infty} \|\partial_t v\|_3 + \mathcal{P}_0 + \int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t + L.O.T. \tag{5.2.48}$$

Step 2: Divergence estimates of v

The Eulerian divergence is $\text{div}_{\tilde{A}} X = \text{div } X + (\tilde{A}^{\mu\alpha} - \delta^{\mu\alpha})\partial_\mu X_\alpha$, which together with (5.2.9) implies

$$\begin{aligned} &\forall s > 2.5: \ \|\text{div } X\|_{s-1} \lesssim \|\text{div}_{\tilde{A}} X\|_{s-1} + \|I - \tilde{A}\|_{s-1}\|X\|_s \lesssim \|\text{div}_{\tilde{A}} X\|_{s-1} + \varepsilon\|X\|_s \\ &\forall 1 \le s \le 2.5: \ \|\text{div } X\|_{s-1} \lesssim \|\text{div}_{\tilde{A}} X\|_{s-1} + \|I - \tilde{A}\|_{L^\infty}\|X\|_s \lesssim \|\text{div}_{\tilde{A}} X\|_{s-1} + \varepsilon\|X\|_s. \end{aligned} \tag{5.2.49}$$

The ε-terms can be absorbed by $\|X\|_s$ on LHS by choosing $\varepsilon > 0$ sufficiently small. So it suffices to estimate the Eulerian divergence which satisfies $\text{div}_{\tilde{A}} v = -\frac{R'(q)\tilde{J}}{\rho_0}\partial_t q$. Taking time derivatives in this equation, we get

$$\text{div}_{\tilde{A}} \partial_t^k v = -\partial_t^k \left(\frac{R'(q)\tilde{J}}{\rho_0}\partial_t q\right) - [\partial_t^k, \tilde{A}^{\mu\alpha}]\partial_\mu v_\alpha \approx \frac{R'(q)\tilde{J}}{\rho_0}\partial_t^{k+1} q - [\partial_t^k, \tilde{A}^{\mu\alpha}]\partial_\mu v_\alpha.$$

Therefore, we have

$$\begin{aligned} \|\text{div}_{\tilde{A}} v\|_3 &\lesssim \|R'(q)\tilde{J}\|_{L^\infty}\|\partial_t q\|_3 + L.O.T. \\ \|\text{div}_{\tilde{A}} \partial_t v\|_2 &\lesssim \|R'(q)\tilde{J}\|_{L^\infty}\|\partial_t^2 q\|_2 + L.O.T. \\ \|\text{div}_{\tilde{A}} \partial_t^2 v\|_1 &\lesssim \|R'(q)\tilde{J}\|_{L^\infty}\|\partial_t^3 q\|_1 + L.O.T. \\ \|\text{div}_{\tilde{A}} \partial_t^3 v\|_0 &\lesssim \|R'(q)\tilde{J}\|_{L^\infty}\|\partial_t^4 q\|_0 + L.O.T. \end{aligned} \tag{5.2.50}$$

Combining (5.2.49) and (5.2.50), by choosing $\varepsilon > 0$ in (5.2.49) to be suitably small, we know the divergence estimates are all reduced to one more time derivative of q:

$$\|\text{div } v\|_3 \lesssim \varepsilon\|v\|_4 + \|R'(q)\tilde{J}\|_{L^\infty}\|\partial_t q\|_3 + L.O.T. \tag{5.2.51}$$

$$\|\text{div } \partial_t v\|_2 \lesssim \varepsilon\|v_t\|_3 + \|R'(q)\tilde{J}\|_{L^\infty}\|\partial_t^2 q\|_2 + L.O.T. \tag{5.2.52}$$

$$\|\text{div } \partial_t^2 v\|_1 \lesssim \varepsilon\|\partial_t^2 v\|_2 + \|R'(q)\tilde{J}\|_{L^\infty}\|\partial_t^3 q\|_1 + L.O.T. \tag{5.2.53}$$

$$\|\text{div } \partial_t^3 v\|_0 \lesssim \varepsilon\|\partial_t^3 v\|_1 + \|R'(q)\tilde{J}\|_{L^\infty}\|\partial_t^4 q\|_0 + L.O.T. \tag{5.2.54}$$

Combining (5.2.45)-(5.2.48). (5.2.51)-(5.2.54) with the previous analysis of curl and boundary estimates, the control of $\|\partial_t^{4-k} q\|_k$ and $\|\partial_t^{4-k} v\|_0$ are reduced to $\|\partial_t^4 v\|_0$ and $\|\partial_t^4 q\|_0$ together with the tangential estimates of v.

Tangential space-time derivative estimates Denote $\mathfrak{D} = \overline{\partial}$ or ∂_t. First we consider the case $\mathfrak{D}^4 = \partial_t^4, \partial_t^3\overline{\partial}, \partial_t^2\overline{\partial}^2, \partial_t\overline{\partial}^3$, i.e., there are at least one time derivative in the four tangential derivatives.

Direct computation gives

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \rho_0 \tilde{J}^{-1}\left|\mathfrak{D}^4 v\right|^2 \,\mathrm{d}y\bigg|_0^T = \int_0^T\int_\Omega \mathfrak{D}^4(\rho_0\tilde{J}^{-1}\partial_t v)\cdot\mathfrak{D}^4 v \,\mathrm{d}y\,\mathrm{d}t \\
&\underbrace{+\frac{1}{2}\int_0^T \partial_t\left(\rho_0\tilde{J}^{-1}\right)\left|\mathfrak{D}^4 v\right|^2 \,\mathrm{d}y\,\mathrm{d}t + \int_0^T\int_\Omega \left[\mathfrak{D}^4, \rho_0\tilde{J}^{-1}\right]\partial_t v\cdot\mathfrak{D}^4 v\,\mathrm{d}y\,\mathrm{d}t}_{L_1} \\
&= -\int_0^T\int_\Omega \mathfrak{D}^4(\nabla_{\tilde{A}} Q)\cdot\mathfrak{D}^4 v\,\mathrm{d}y\,\mathrm{d}t + \underbrace{\int_0^T\int_\Omega \mathfrak{D}^4\left((b\cdot\nabla_{\tilde{A}})b\right)\mathfrak{D}^4 v\,\mathrm{d}y\,\mathrm{d}t}_{L_2} + L_1,
\end{aligned} \tag{5.2.55}$$

where L_1 can be directtly bounded by $\int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t$, and L_2 can be controlled by the Lorentz force

$$L_2 \lesssim \int_0^T \left\|\mathfrak{D}^4 v\right\|_0\left\|\mathfrak{D}^4\left((b\cdot\nabla_{\tilde{A}})b\right)\right\|_0 \,\mathrm{d}t \lesssim \varepsilon\int_0^T \left\|\mathfrak{D}^4\left((b\cdot\nabla_{\tilde{A}})b\right)\right\|_0^2 \,\mathrm{d}t + \int_0^T \left\|\mathfrak{D}^4 v\right\|_0^2 \,\mathrm{d}t.$$

For the first term, we first commute $\mathfrak{D}^4$ with $\nabla_{\tilde{A}}$, then integrate $\nabla_{\tilde{A}}$ by parts to get

$$
\begin{aligned}
&-\int_0^T\int_\Omega \mathfrak{D}^4(\nabla_{\tilde{A}}Q)\cdot\mathfrak{D}^4 v\,\mathrm{d}y\,\mathrm{d}t\\
=&-\int_0^T\int_\Omega \nabla_{\tilde{A}}\mathfrak{D}^4 Q\cdot\mathfrak{D}^4 v\,\mathrm{d}y\,\mathrm{d}t+\underbrace{\int_0^T\int_\Omega\left[\mathfrak{D}^4,\tilde{A}^{\mu\alpha}\right]\partial_\mu Q\cdot\mathfrak{D}^4 v_\alpha\,\mathrm{d}y\,\mathrm{d}t}_{L_3}\\
=&-\int_0^T\int_\Gamma \tilde{A}^{3\alpha}\underbrace{\mathfrak{D}^4 Q}_{=0}\mathfrak{D}^4 v_\alpha\,\mathrm{d}S\,\mathrm{d}t+\underbrace{\int_0^T\int_\Omega\mathfrak{D}^4 Q\mathfrak{D}^4(\mathrm{div}_{\tilde{A}}v)\,\mathrm{d}y\,\mathrm{d}t}_{K_1}\\
&+\underbrace{\int_0^T\int_\Omega\mathfrak{D}^4 Q\cdot\left[\mathfrak{D}^4,\tilde{A}^{\mu\alpha}\right]\partial_\mu v_\alpha\,\mathrm{d}y\,\mathrm{d}t}_{L_4}+\underbrace{\int_0^T\int_\Omega\partial_\mu\tilde{A}^{\mu\alpha}\mathfrak{D}^4 Q\mathfrak{D}^4 v_\alpha}_{L_5}.
\end{aligned}
\tag{5.2.56}
$$

Notice that, $\mathfrak{D}^4=\mathfrak{D}^3\partial_t$ now contains at least one time derivative, and $\tilde{A}\sim\partial\tilde{\eta}\cdot\partial\tilde{\eta}$, so by the estimates correction term ψ, we know the L^2-norm of $\mathfrak{D}^4\tilde{A}\approx\mathfrak{D}^3\partial\partial_t\tilde{\eta}\cdot\partial\tilde{\eta}+L.O.T.$ can be controlled by $P(\mathfrak{e}_\kappa(t))$, and thus L_3, L_4 can be controlled by $\int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t$. The term L_5 is also directly bounded by $\int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t$.

Next we plug $\mathrm{div}_{\tilde{A}}v=-\frac{\tilde{J}R'(q)}{\rho_0}\partial_t q$ and $Q=q+\frac{1}{2}|b|^2$ into K_1 to get

$$
\begin{aligned}
K_1=&-\int_0^T\int_\Omega\mathfrak{D}^4 q\mathfrak{D}^4\left(\frac{\tilde{J}R'(q)}{\rho_0}\partial_t q\right)-\int_0^T\int_\Omega\mathfrak{D}^4\left(\frac{1}{2}|b|^2\right)\mathfrak{D}^4\left(\frac{\tilde{J}R'(q)}{\rho_0}\partial_t q\right)\\
=&-\frac{1}{2}\int_\Omega\frac{\tilde{J}R'(q)}{\rho_0}\left|\mathfrak{D}^4 q(t)\right|^2\,\mathrm{d}y\Big|_0^T\underbrace{-\int_0^T\int_\Omega\mathfrak{D}^4 Q\cdot\left[\mathfrak{D}^4,\frac{\tilde{J}R'(q)}{\rho_0}\right]\partial_t q\,\mathrm{d}y\,\mathrm{d}t}_{L_6}\\
&\underbrace{-\int_0^T\int_\Omega\frac{\tilde{J}R'(q)}{\rho_0}\mathfrak{D}^4\left(\frac{1}{2}|b|^2\right)\mathfrak{D}^4\partial_t q\,\mathrm{d}y\,\mathrm{d}t}_{K_2}.
\end{aligned}
\tag{5.2.57}
$$

From the computation above, we find that the energy term $\|\partial_t^4 q\|_0^2$ automatically appears if $\mathfrak{D}^4=\partial_t^4$.

The commutator term L_6 can be directly bounded by $\int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t$. The term K_2 satisfies

$$K_2 \lesssim \int_0^T \left\|\mathfrak{D}^4\left(\frac{1}{2}|b|^2\right)\right\|_0 \cdot \left\|\mathfrak{D}^4\partial_t q\right\|_0 \,\mathrm{d}t \lesssim \varepsilon\int_0^T \left\|\mathfrak{D}^4\partial_t q\right\|_0^2 \,\mathrm{d}t + \int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t, \tag{5.2.58}$$

therefore we need the energy of 4-th time differentiated wave equation of q and elliptic estimates Lemma 3.3.3 to bound K_2. This will be postponed to Section 5.2.1.4.

Tangential spatial derivative estimates: Alinhac good unknowns When $\mathfrak{D}^4 = \overline{\partial}^4$, the above analysis no longer works due to $[\overline{\partial}^4, \tilde{A}^{\mu\alpha}]\partial_\mu f$ being uncontrollable. According to the discussion in Section 1.4, we introduce the Alinhac good unknowns. In specific, we replace $\overline{\partial}^4$ by $\overline{\partial}^2\overline{\triangle}$ due to the special structure of correction term ψ. Then for any function f and its corresponding Alinhac good unknown

$$\mathbf{f} := \overline{\partial}^2\overline{\triangle} f - \overline{\partial}^2\overline{\triangle}\tilde{\eta}\cdot\nabla_{\tilde{A}} f,$$

the following equality holds

$$\begin{aligned}
\overline{\partial}^2\overline{\triangle}(\nabla_{\tilde{A}}^\alpha f) &= \nabla_{\tilde{A}}^\alpha(\overline{\partial}^2\overline{\triangle} f) + (\overline{\partial}^2\overline{\triangle}\tilde{A}^{\mu\alpha})\partial_\mu f + [\overline{\partial}^2\overline{\triangle}, \tilde{A}^{\mu\alpha}, \partial_\mu f] \\
&= \nabla_{\tilde{A}}^\alpha(\overline{\partial}^2\overline{\triangle} f) - \overline{\partial}\overline{\triangle}(\tilde{A}^{\mu\gamma}\overline{\partial}\partial_\beta\tilde{\eta}_\gamma\tilde{A}^{\beta\alpha})\partial_\mu f + [\overline{\partial}^2\overline{\triangle}, \tilde{A}^{\mu\alpha}, \partial_\mu f] \\
&= \nabla_{\tilde{A}}^\alpha(\overline{\partial}^2\overline{\triangle} f) - \tilde{A}^{\beta\alpha}\partial_\beta\overline{\partial}^2\overline{\triangle}\tilde{\eta}_\gamma\tilde{A}^{\mu\gamma}\partial_\mu f - ([\overline{\partial}\overline{\triangle}, \tilde{A}^{\mu\gamma}\tilde{A}^{\beta\alpha}]\overline{\partial}\partial_\beta\tilde{\eta}_\gamma)\partial_\mu f \\
&+ [\overline{\partial}^2\overline{\triangle}, \tilde{A}^{\mu\alpha}, \partial_\mu f] \\
&= \underbrace{\nabla_{\tilde{A}}^\alpha(\overline{\partial}^2\overline{\triangle} f - \overline{\partial}^2\overline{\triangle}\eta_\gamma\tilde{A}^{\mu\gamma}\partial_\mu f)}_{=\nabla_{\tilde{A}}^\alpha\mathbf{f}} \\
&+ \underbrace{\overline{\partial}^2\overline{\triangle}\eta_\gamma\nabla_{\tilde{A}}^\alpha(\nabla_{\tilde{A}}^\gamma f) - ([\overline{\partial}\overline{\triangle}, \tilde{A}^{\mu\gamma}\tilde{A}^{\beta\alpha}]\overline{\partial}\partial_\beta\tilde{\eta}_\gamma)\partial_\mu f + [\overline{\partial}^2\overline{\triangle}, \tilde{A}^{\mu\alpha}, \partial_\mu f]}_{=:C^\alpha(f)},
\end{aligned}$$

where $[\overline{\partial}^2\overline{\triangle}, g, h] := \overline{\partial}^2\overline{\triangle}(gh) - \overline{\partial}^2\overline{\triangle}(g)h - g\overline{\partial}^2\overline{\triangle}(h)$. Direct computation yields

$$\|\overline{\partial}^2\overline{\triangle}\eta_\gamma \nabla_{\tilde{A}}^\alpha(\nabla_{\tilde{A}}^\gamma f)\|_0 \lesssim \|\tilde{\eta}\|_4 \|\nabla_{\tilde{A}}^\alpha(\nabla_{\tilde{A}}^\gamma f)\|_{L^\infty}$$

$$\|([\overline{\partial}\overline{\triangle}, \tilde{A}^{\mu\gamma}\tilde{A}^{\beta\alpha}]\overline{\partial}\partial_\beta\tilde{\eta}_\gamma)\partial_\mu f\|_0 \lesssim \|[\overline{\partial}\overline{\triangle}, \tilde{A}^{\mu\gamma}\tilde{A}^{\beta\alpha}]\overline{\partial}\partial_\beta\tilde{\eta}_\gamma\|_0 \|f\|_{W^{1,\infty}} \lesssim P(\|\tilde{\eta}\|_4)\|f\|_3$$

$$\|[\overline{\partial}^2\overline{\triangle}, \tilde{A}^{\mu\alpha}, \partial_\mu f]\|_0 \lesssim P(\|\tilde{\eta}\|_4)\|f\|_4.$$

Therefore, Alinhac good unknown enjoys the following important properties:

$$\overline{\partial}^2\overline{\triangle}(\nabla_{\tilde{A}}^\alpha f) = \nabla_{\tilde{A}}^\alpha \mathbf{f} + C^\alpha(f) \tag{5.2.59}$$

with

$$\|C^\alpha(f)\| \lesssim P(\|\tilde{\eta}\|_4)\|f\|_4. \tag{5.2.60}$$

For (5.2.1), we define $\mathbf{V} = \overline{\partial}^2\overline{\triangle}v - \overline{\partial}^2\overline{\triangle}\tilde{\eta} \cdot \nabla_{\tilde{A}} v$ and $\mathbf{Q} = \overline{\partial}^2\overline{\triangle}Q - \overline{\partial}^2\overline{\triangle}\tilde{\eta} \cdot \nabla_{\tilde{A}} Q$ to be the Alinhac good unknowns for v and $Q = q + \frac{1}{2}|B|^2$. Taking $\overline{\partial}^2\overline{\triangle}$ in the second equation of (5.2.1), we get

$$\rho_0 \tilde{J}^{-1}\partial_t \mathbf{V} + \nabla_{\tilde{A}}\mathbf{Q} = \mathbf{F} \tag{5.2.61}$$

where

$$\mathbf{F} := \overline{\partial}^2\overline{\triangle}\left((b \cdot \nabla_{\tilde{A}})b\right) + [\rho_0\tilde{J}^{-1}, \overline{\partial}^2\overline{\triangle}]\partial_t v_\alpha + \rho_0\tilde{J}^{-1}\partial_t(\overline{\partial}^2\overline{\triangle}\tilde{\eta} \cdot \nabla_{\tilde{A}} v) + C(Q)$$

subject to

$$\mathbf{Q} = -\overline{\partial}^2\overline{\triangle}\tilde{\eta}_\beta \tilde{A}^{3\beta}(\partial_N Q) \quad on\ \Gamma. \tag{5.2.62}$$

and

$$\nabla_{\tilde{A}} \cdot \mathbf{V} = \overline{\partial}^2\overline{\triangle}(\mathrm{div}_{\tilde{A}} v) - C^\alpha(v_\alpha) \quad in\ \Omega. \tag{5.2.63}$$

Taking L^2 inner product with $\mathbf{V}$ and time integral, we have

$$\frac{1}{2}\int_\Omega \rho_0 \tilde{J}^{-1}|\mathbf{V}(t)|^2 \,\mathrm{d}y\Big|_0^T = -\int_0^T\int_\Omega \nabla_{\tilde{A}}\mathbf{Q}\cdot\mathbf{V}\,\mathrm{d}y\,\mathrm{d}t + \int_0^T\int_\Omega \frac{1}{2}\rho_0\partial_t\tilde{J}^{-1}|\mathbf{V}|^2 + \mathbf{F}\cdot\mathbf{V}\,\mathrm{d}y\,\mathrm{d}t. \tag{5.2.64}$$

By (5.2.59)-(5.2.60) and direct computation, we know the last term on RHS can be directly controlled:

$$\int_0^T\int_\Omega \frac{1}{2}\rho_0\partial_t\tilde{J}^{-1}|\mathbf{V}|^2 + \mathbf{F}\cdot\mathbf{V}\,\mathrm{d}y\,\mathrm{d}t \lesssim \int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t + \varepsilon\int_0^T \left\|\overline{\partial}^2\overline{\triangle}\left((b\cdot\nabla_{\tilde{A}})b\right)\right\|_0^2\,\mathrm{d}t. \tag{5.2.65}$$

We integrate $\nabla_{\tilde{A}}$ by parts to get

$$\begin{aligned}
&-\int_0^T\int_\Omega \nabla_{\tilde{A}}\mathbf{Q}\cdot\mathbf{V}\,\mathrm{d}y\,\mathrm{d}t = -\int_0^T\int_\Omega \tilde{A}^{\mu\alpha}\mathbf{Q}\cdot\mathbf{V}_\alpha\,\mathrm{d}y\,\mathrm{d}t\\
=&-\int_0^T\int_\Gamma \mathbf{Q}(\tilde{A}^{3\alpha}\mathbf{V}_\alpha)\,\mathrm{d}S\,\mathrm{d}t + \int_0^T\int_\Omega \mathbf{Q}(\nabla_{\tilde{A}}\cdot\mathbf{V})\,\mathrm{d}y\,\mathrm{d}t + \underbrace{\int_0^T\int_\Omega (\partial_\mu\tilde{A}^{\mu\alpha})\mathbf{Q}\mathbf{V}_\alpha\,\mathrm{d}y\,\mathrm{d}t}_{J_1}\\
=&\int_0^T\int_\Gamma (\partial_N Q)\overline{\partial}^2\overline{\triangle}\tilde{\eta}_\beta\tilde{A}^{3\beta}\tilde{A}^{3\alpha}\mathbf{V}_\alpha\,\mathrm{d}S\,\mathrm{d}t + \int_0^T\int_\Omega \mathbf{Q}\overline{\partial}^2\overline{\triangle}(\mathrm{div}_{\tilde{A}}v)\,\mathrm{d}y\,\mathrm{d}t\\
&-\int_0^T\int_\Omega \mathbf{Q}C^\alpha(v_\alpha)\,\mathrm{d}y\,\mathrm{d}t + J_1\\
=:&I_0 + I_1 + J_2 + J_1.
\end{aligned} \tag{5.2.66}$$

The term J_1, J_2 can be directly controlled by $\int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t$. Next we investigate I_1. Invoking $\mathrm{div}_{\tilde{A}}v = -\frac{\tilde{J}R'(q)}{\rho_0}\partial_t q$,

$$\begin{aligned}
I_1 =& \int_0^T\int_\Omega \mathbf{Q}\overline{\partial}^2\overline{\triangle}(\mathrm{div}_{\tilde{A}}v)\,\mathrm{d}y\,\mathrm{d}t\\
=&\int_0^T\int_\Omega \left(\overline{\partial}^2\overline{\triangle}q + \overline{\partial}^2\overline{\triangle}\left(\frac{1}{2}|b|^2\right) - \overline{\partial}^2\overline{\triangle}\tilde{\eta}\cdot\nabla_{\tilde{A}}Q\right)\overline{\partial}^2\overline{\triangle}\left(-\frac{\tilde{J}R'(q)}{\rho_0}\partial_t q\right)
\end{aligned} \tag{5.2.67}$$

$$
\begin{aligned}
=& -\int_0^T \int_\Omega \overline{\partial}^2\overline{\triangle} q \cdot \overline{\partial}^2\overline{\triangle}\left(-\frac{\tilde{J} R'(q)}{\rho_0}\partial_t q\right) \mathrm{d}y\,\mathrm{d}t \\
&+ \int_0^T \int_\Omega \overline{\partial}^2\overline{\triangle}\tilde{\eta}\cdot\nabla_{\tilde{A}} Q \overline{\partial}^2\overline{\triangle}\left(-\frac{\tilde{J} R'(q)}{\rho_0}\partial_t q\right) \mathrm{d}y\,\mathrm{d}t \\
&- \int_0^T \int_\Omega \overline{\partial}^2\overline{\triangle}\left(\frac{1}{2}|b|^2\right)\cdot \overline{\partial}^2\overline{\triangle}\left(-\frac{\tilde{J} R'(q)}{\rho_0}\partial_t q\right) \mathrm{d}y\,\mathrm{d}t \\
=:& I_{11} + I_{12} + I_{13}.
\end{aligned} \tag{5.2.68}
$$

The term I_{11} and I_{13} can be similarly computed as in K_1 and K_2 (5.2.58):

$$
I_{11} \lesssim -\frac{1}{2}\int_\Omega \frac{\tilde{J} R'(q)}{\rho_0}\left|\overline{\partial}^2\overline{\triangle} q\right|^2 \mathrm{d}y\Big|_0^T + \int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t, \tag{5.2.69}
$$

$$
I_{13} \lesssim \varepsilon \int_0^T \left\|\overline{\partial}^2\overline{\triangle}\partial_t q\right\|_0^2 \mathrm{d}t + \int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t. \tag{5.2.70}
$$

One can see that I_{11} has been controlled, while I_{13} requires the control of 5-th order wave equation of q to absorb that ε-term. This will again be postponed in Section 5.2.1.4. For I_{12}, we just need to integrate ∂_t by parts

$$
\begin{aligned}
I_{12} =& -\int_0^T \int_\Omega \overline{\partial}^2\overline{\triangle}\tilde{\eta}\cdot\nabla_{\tilde{A}} Q \overline{\partial}^2\overline{\triangle}\left(\frac{\tilde{J} R'(q)}{\rho_0}\partial_t q\right) \mathrm{d}y\,\mathrm{d}t \\
\approx& \int_\Omega \frac{\tilde{J} R'(q)}{\rho_0}\overline{\partial}^2\overline{\triangle}\tilde{\eta}\cdot\nabla_{\tilde{A}} Q \overline{\partial}^2\overline{\triangle} q\,\mathrm{d}y\Big|_{t=0}^{t=T} + \int_0^T\int_\Omega \frac{\tilde{J} R'(q)}{\rho_0}\overline{\partial}^2\overline{\triangle} q\,\partial_t \mathbf{Q}\,\mathrm{d}y\,\mathrm{d}t \\
\lesssim& \varepsilon \int_\Omega \frac{\tilde{J} R'(q)}{\rho_0}\left|\overline{\partial}^2\overline{\triangle} q(T)\right|^2 \mathrm{d}y + \frac{1}{8\varepsilon}\left(\|\tilde{\eta}(T)\|_4^4 + \|\nabla_{\tilde{A}} Q(T)\|_{L^\infty}^4\right) \\
&+ \mathcal{P}_0 + \int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t.
\end{aligned} \tag{5.2.71}
$$

Here in the last step we use ε-Young's inequality to deal with the first term in the second line. The second term can be directly controlled by using the estimates of $\|\partial_t\tilde{\eta}\|_4$. It remains to control the boundary integral I_0. Plugging $\mathbf{V}_\alpha = \overline{\partial}^2\overline{\triangle} v_\alpha - \overline{\partial}^2\overline{\triangle}\eta \cdot \nabla_{\tilde{A}} v_\alpha$ into I_0, we get

$$I_0 = \int_0^T \int_\Gamma (\partial_N Q)\tilde{A}^{3\alpha}\tilde{A}^{3\beta}\overline{\partial}^2\overline{\triangle}\tilde{\eta}_\beta(\overline{\partial}^2\overline{\triangle}\partial_t\eta_\alpha - \overline{\partial}^2\overline{\triangle}\psi - \overline{\partial}^2\overline{\triangle}\tilde{\eta}\cdot\nabla_{\tilde{A}} v_\alpha)\,\mathrm{d}S\,\mathrm{d}t. \tag{5.2.72}$$

The first term in (5.2.72) produces the Taylor sign term contributing to the boundary term in $E_\kappa(t)$ after commuting a Λ_κ:

$$\begin{aligned}
&\int_0^T \int_\Gamma (\partial_N Q)\tilde{A}^{3\alpha}\tilde{A}^{3\beta}\overline{\partial}^2\overline{\triangle}\tilde{\eta}_\beta\overline{\partial}^2\overline{\triangle}\partial_t\eta_\alpha\,\mathrm{d}S\,\mathrm{d}t\\
=&\int_0^T \int_\Gamma (\partial_N Q)\tilde{A}^{3\alpha}\tilde{A}^{3\beta}\overline{\partial}^2\overline{\triangle}\Lambda_\kappa\eta_\beta\overline{\partial}^2\overline{\triangle}\partial_t\Lambda_\kappa\eta_\alpha\,\mathrm{d}S\,\mathrm{d}t\\
&+\int_0^T \int_\Gamma \left(\overline{\partial}^2\overline{\triangle}\Lambda_\kappa\eta_\beta\right)\left(\left[\Lambda_\kappa, (\partial_N Q)\tilde{A}^{3\alpha}\tilde{A}^{3\beta}\right]\overline{\partial}^2\overline{\triangle}\partial_t\eta_\alpha\right)\,\mathrm{d}S\,\mathrm{d}t\\
=&\frac{1}{2}\int_\Gamma (\partial_N Q)\left|\tilde{A}^{3\alpha}\overline{\partial}^2\overline{\triangle}\Lambda_\kappa\eta_\alpha\right|^2\,\mathrm{d}S\Big|_0^T - \frac{1}{2}\int_0^T\int_\Gamma \partial_t(\partial_N Q)\left|\tilde{A}^{3\alpha}\overline{\partial}^2\overline{\triangle}\Lambda_\kappa\eta_\alpha\right|^2\,\mathrm{d}S\\
&\underbrace{-\int_0^T \int_\Gamma (\partial_N Q)\tilde{A}^{3\beta}\overline{\partial}^2\overline{\triangle}\Lambda_\kappa\eta_\beta\partial_t\tilde{A}^{3\alpha}\overline{\partial}^2\overline{\triangle}\Lambda_\kappa\eta_\alpha\,\mathrm{d}S\,\mathrm{d}t}_{I_{01}}\\
&+\underbrace{\int_0^T \int_\Gamma \left(\overline{\partial}^2\overline{\triangle}\Lambda_\kappa\eta_\beta\right)\left(\left[\Lambda_\kappa, (\partial_N Q)\tilde{A}^{3\alpha}\tilde{A}^{3\beta}\right]\overline{\partial}^2\overline{\triangle}\partial_t\eta_\alpha\right)\,\mathrm{d}S\,\mathrm{d}t}_{L_7}.
\end{aligned} \tag{5.2.73}$$

L_7 can be directly controlled after integrating $\overline{\partial}^{1/2}$ by parts

$$\begin{aligned}
L_7 =& \int_0^T \int_\Gamma \left(\overline{\partial}^{3/2}\overline{\triangle}\Lambda_\kappa\eta_\beta\right)\overline{\partial}^{1/2}\left(\left[\Lambda_\kappa, (\partial_N Q)\tilde{A}^{3\alpha}\tilde{A}^{3\beta}\right]\overline{\partial}(\overline{\partial}\overline{\triangle}\partial_t\eta_\alpha)\right)\,\mathrm{d}S\\
\lesssim& \int_0^T \|\eta\|_4\left|(\partial_N Q)\tilde{A}^{3\alpha}\tilde{A}^{3\beta}\right|_{W^{1,\infty}}\left|\overline{\partial}\overline{\triangle}\partial_t\eta_\alpha\right|_{1/2}\\
\lesssim& \int_0^T \|\eta\|_4\|Q\|_4\|\partial\tilde{A}\|_{L^\infty}\|\partial_t\eta\|_4 \lesssim \int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t.
\end{aligned} \tag{5.2.74}$$

In I_{01}, we have $\partial_t\tilde{A}^{3\alpha} = -\tilde{A}^{3\gamma}\partial_\mu\partial_t\tilde{\eta}_\gamma\tilde{A}^{\mu\alpha}$. Note that $\partial_t\eta = v + \psi$. The ψ term can be directly bounded by using the mollifier property, which the contribution of v cannot be bounded directly. Luckily, later on we will see that term can be cancelled together with another higher order term in (5.2.72) with the help of ψ. We have

$$
\begin{aligned}
B_1 =& \underbrace{\int_0^T \int_\Gamma (\partial_N Q)\tilde{A}^{3\beta}\overline{\partial}^2\overline{\triangle}\Lambda_\kappa \eta_\beta \tilde{A}^{3\gamma}\partial_3\partial_t\tilde{\eta}_\gamma \tilde{A}^{3\alpha}\overline{\partial}^2\overline{\triangle}\Lambda_\kappa\eta_\alpha \,\mathrm{d}S}_{L_8} \\
&+ \underbrace{\int_0^T \int_\Gamma (\partial_N Q)\tilde{A}^{3\beta}\overline{\partial}^2\overline{\triangle}\Lambda_\kappa \eta_\beta \tilde{A}^{3\gamma}\overline{\partial}_i\partial_t\Lambda_\kappa^2\psi_\gamma \tilde{A}^{i\alpha}\overline{\partial}^2\overline{\triangle}\Lambda_\kappa\eta_\alpha \,\mathrm{d}S}_{L_9} \\
&+ \underbrace{\int_0^T \int_\Gamma (\partial_N Q)\tilde{A}^{3\beta}\overline{\partial}^2\overline{\triangle}\Lambda_\kappa \eta_\beta \tilde{A}^{3\gamma}\overline{\partial}_i\Lambda_\kappa^2 v_\gamma \tilde{A}^{i\alpha}\overline{\partial}^2\overline{\triangle}\Lambda_\kappa\eta_\alpha \,\mathrm{d}S\,\mathrm{d}t}_{I_{02}}\,.
\end{aligned}
\tag{5.2.75}
$$

L_8 can be directly bounded by the boundary energy

$$
L_8 \lesssim \int_0^T \left|\tilde{A}^{3\beta}\overline{\partial}^2\overline{\triangle}\Lambda_\kappa\eta_\beta\right|_0^2 \cdot \left|(\partial_N Q)\tilde{A}^{3\gamma}\partial_3\partial_t\tilde{\eta}_\gamma\right|_{L^\infty} \lesssim \int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t \tag{5.2.76}
$$

L_9 can be bounded by using $|\overline{\partial}^2\overline{\triangle}\Lambda_\kappa\eta| \lesssim \kappa^{-1/2}|\eta|_{7/2}$ and sacrificing $\kappa^{-1/2}$.

$$
L_9 \lesssim \int_0^T \frac{1}{\sqrt{\kappa}}|\eta|_{7/2}\left|(\partial_N Q)\tilde{A}^{3\gamma}\tilde{A}^{i\alpha}\right|_{L^\infty}\left|\tilde{A}^{3\beta}\overline{\partial}^2\overline{\triangle}\Lambda_\kappa\eta_\beta\right|_0\left|\overline{\partial}\tilde{\psi}\right|_{L^\infty}\,\mathrm{d}t
$$

This can be compensated by estimating $|\overline{\partial}\psi|_{L^\infty}$ and $W^{1,4}(\mathbb{T}^2)\hookrightarrow L^\infty(\mathbb{T}^2)$. Since ψ removes the zero-frequency part (so the lowest frequency is ± 1 because the frequency on $\mathbb{T}^2$ is discrete), we know

$|\overline{\triangle}\psi|_{L^4}$ is comparable to $|\overline{\partial}\psi|_{W^{1,4}}$. Therefore,

$$
\begin{aligned}
\left|\overline{\partial}\psi\right|_{L^\infty} &\lesssim \left|\overline{\triangle}\psi\right|_{L^4} = \left|\mathbb{P}_{\neq 0}\left(\overline{\triangle}\eta_\beta \tilde{A}^{i\beta}\overline{\partial}_i \Lambda_\kappa^2 v - \overline{\triangle}\Lambda_\kappa^2\eta_\beta \tilde{A}^{i\beta}\overline{\partial}_i v\right)\right|_{L^4} \\
&\lesssim \left|\overline{\triangle}\eta_\beta \tilde{A}^{i\beta}\overline{\partial}_i \Lambda_\kappa^2 v - \overline{\triangle}\Lambda_\kappa^2\eta_\beta \tilde{A}^{i\beta}\overline{\partial}_i v\right|_{L^4} \\
&= \left|\overline{\triangle}(\eta_\beta - \Lambda_\kappa^2\eta_\beta) \tilde{A}^{i\beta}\overline{\partial}_i \Lambda_\kappa^2 v - \overline{\triangle}\Lambda_\kappa^2\eta_\beta \tilde{A}^{i\beta}\overline{\partial}_i (v - \Lambda_\kappa^2 v)\right|_{L^4} \\
&\lesssim \left|\overline{\triangle}\eta_\beta - \overline{\triangle}\tilde{\eta}_\beta\right|_{L^\infty} \left|\tilde{A}\right|_{L^\infty} \left|\overline{\partial}\tilde{v}\right|_{0.5} + \left|\overline{\triangle}\tilde{\eta}\right|_{1/2} \left|\tilde{A}\right|_{L^\infty} \left|\overline{\partial}(v - \Lambda_\kappa v)\right|_{L^\infty} \\
&\lesssim \sqrt{\kappa} P(\mathfrak{e}_\kappa(t)).
\end{aligned}
$$

Therefore we know L_9 can be bounded uniformly in κ

$$
L_9 \lesssim \int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t \tag{5.2.77}
$$

The estimate of I_{02} will be postponed after computing the third term in (5.2.72), for which we repeat the steps above to get

$$
\begin{aligned}
& -\int_0^T \int_\Gamma (\partial_N Q)\tilde{A}^{3\alpha}\tilde{A}^{3\beta}\overline{\partial}^2\overline{\triangle}\tilde{\eta}_\beta \overline{\partial}^2\overline{\triangle}\tilde{\eta}\cdot\nabla_{\tilde{A}} v_\alpha \, dS \\
=& -\int_0^T \int_\Gamma (\partial_N Q)\tilde{A}^{3\alpha}\tilde{A}^{3\beta}\overline{\partial}^2\overline{\triangle}\tilde{\eta}_\beta \overline{\partial}^2\overline{\triangle}\tilde{\eta}_\gamma \tilde{A}^{3\gamma}\partial_3 v_\alpha \, dS\,\mathrm{d}t \\
& \underbrace{-\int_0^T \int_\Gamma (\partial_N Q)\tilde{A}^{3\alpha}\tilde{A}^{3\beta}\overline{\partial}^2\overline{\triangle}\tilde{\eta}_\beta \overline{\partial}^2\overline{\triangle}\tilde{\eta}_\gamma \tilde{A}^{i\gamma}\overline{\partial}_i v_\alpha \, dS\,\mathrm{d}t}_{I_{03}}
\end{aligned} \tag{5.2.78}
$$

$$
= \int_0^T \int_\Gamma \left(-(\partial_N Q)\tilde{A}^{3\alpha}\partial_3 v_\alpha\right)\left(\tilde{A}^{3\beta}\overline{\partial}^2\overline{\triangle}\tilde{\eta}_\beta\right)\left(\tilde{A}^{3\gamma}\overline{\partial}^2\overline{\triangle}\tilde{\eta}_\gamma\right)\,\mathrm{d}S\,\mathrm{d}t + I_{03}
$$

The first term can be bounded by Taylor sign after commuting one Λ_κ:

$$
\left|\tilde{A}^{3\beta}\overline{\partial}^2\overline{\triangle}\Lambda_\kappa\eta_\beta\right|_0 \lesssim \left|\Lambda_\kappa\left(\tilde{A}^{3\beta}\overline{\partial}^2\overline{\triangle}\Lambda_\kappa\eta_\beta\right)\right|_0 + \left|\left[\Lambda_\kappa, \tilde{A}^{3\beta}\right]\overline{\partial}^2\overline{\triangle}\Lambda_\kappa\eta_\beta\right| \lesssim P(\mathfrak{e}_\kappa(t)).
$$

Therefore, it remains to control

$$I_{04} := -\int_0^T \int_\Gamma (\partial_N Q) \tilde{A}^{3\alpha} \tilde{A}^{3\beta} \overline{\partial}^2 \overline{\triangle} \eta_\beta \overline{\partial}^2 \overline{\triangle} \psi. \tag{5.2.79}$$

$$I_{02} := \int_0^T \int_\Gamma (\partial_N Q) \tilde{A}^{3\beta} \overline{\partial}^2 \overline{\triangle} \Lambda_\kappa \eta_\beta \tilde{A}^{3\gamma} \overline{\partial}_i \Lambda_\kappa^2 v_\gamma \tilde{A}^{i\alpha} \overline{\partial}^2 \overline{\triangle} \Lambda_\kappa \eta_\alpha \, \mathrm{d}S \, \mathrm{d}t \tag{5.2.80}$$

$$I_{03} := -\int_0^T \int_\Gamma (\partial_N Q) \tilde{A}^{3\alpha} \tilde{A}^{3\beta} \overline{\partial}^2 \overline{\triangle} \tilde{\eta}_\beta \overline{\partial}^2 \overline{\triangle} \tilde{\eta}_\gamma \tilde{A}^{i\gamma} \overline{\partial}_i v_\alpha \; dS \, \mathrm{d}t. \tag{5.2.81}$$

Plugging the expression of $\overline{\triangle}\psi$ into (5.2.79), we get

$$I_{04} = -\int_0^T \int_\Gamma (\partial_N Q) \tilde{A}^{3\alpha} \tilde{A}^{3\beta} \overline{\partial}^2 \overline{\triangle} \tilde{\eta}_\beta \overline{\partial}^2 \left(\overline{\triangle} \eta_\gamma \tilde{A}^{ir} \overline{\partial}_i \Lambda_\kappa^2 v_\alpha \right) \mathrm{d}S \, \mathrm{d}t \tag{5.2.82}$$

$$+ \int_0^T \int_\Gamma (\partial_N Q) \tilde{A}^{3\alpha} \tilde{A}^{3\beta} \overline{\partial}^2 \overline{\triangle} \tilde{\eta}_\beta \overline{\partial}^2 \tilde{\eta}_\gamma \tilde{A}^{i\gamma} \overline{\partial}_i v_\alpha \, \mathrm{d}S \, \mathrm{d}t \tag{5.2.83}$$

$$+ \int_0^T \int_\Gamma (\partial_N Q) \tilde{A}^{3\alpha} \tilde{A}^{3\beta} \overline{\partial}^2 \overline{\triangle} \tilde{\eta}_\beta \left(\left[\overline{\partial}^2, \tilde{A}^{i\gamma} \overline{\partial}_i v_\alpha \right] \overline{\triangle} \tilde{\eta}_\gamma \right) \mathrm{d}S \, \mathrm{d}t \tag{5.2.84}$$

$$+ \int_0^T \int_\Gamma (\partial_N Q) \tilde{A}^{3\alpha} \tilde{A}^{3\beta} \overline{\partial}^2 \overline{\triangle} \tilde{\eta}_\beta \overline{\partial}^2 \mathbb{P}_{=0} \left(\overline{\triangle} \eta_\beta \tilde{A}^{i\beta} \overline{\partial}_i \Lambda_\kappa^2 v - \overline{\triangle} \Lambda_\kappa^2 \eta_\beta \tilde{A}^{i\beta} \overline{\partial}_i v \right) \mathrm{d}S \, \mathrm{d}t. \tag{5.2.85}$$

Clearly, (5.2.83) exactly cancels with (5.2.81), (5.2.84) can be bounded by $\int_0^T P(\mathfrak{e}_\kappa(t)) \, \mathrm{d}t$, and (5.2.85) can be controlled by Bernstein's inequality $|\mathbb{P}_{\neq 0} f|_2 \approx |f|_0$.

$$\begin{aligned} (3.96) &\lesssim \int_0^T \left| (\partial_N Q) \tilde{A}^{3\alpha} \right|_{L^\infty} \left| \tilde{A}^{3\beta} \overline{\partial}^2 \overline{\triangle} \Lambda_\kappa \eta_\beta \right|_0 \left| \overline{\triangle} \eta_\beta \tilde{A}^{i\beta} \overline{\partial}_i \tilde{v} - \overline{\triangle} \tilde{\eta}_\beta \tilde{A}^{i\beta} \overline{\partial}_i v \right|_0 \mathrm{d}t \\ &\lesssim \int_0^T P(\mathfrak{e}_\kappa(t)) \, \mathrm{d}t. \end{aligned} \tag{5.2.86}$$

In (5.2.82), we move one Λ_κ on η_β to η_α to cancel I_{02}:

$$(3.93) = -\int_0^T \int_\Gamma (\partial_N Q)\tilde{A}^{3\beta}\overline{\partial}^2\overline{\triangle}\Lambda_\kappa\eta_\beta \left(\tilde{A}^{3\alpha}\overline{\partial}_i\Lambda_\kappa^2 v_\alpha\right)\left(\tilde{A}^{i\gamma}\overline{\partial}^2\overline{\triangle}\Lambda_\kappa\eta_\gamma\right) \tag{5.2.87}$$

$$-\int_0^T \int_\Gamma (\partial_N Q)\tilde{A}^{3\beta}\overline{\partial}^2\overline{\triangle}\Lambda_\kappa\eta_\beta \left(\left[\Lambda_\kappa, \tilde{A}^{3\alpha}\tilde{A}^{3\beta}\tilde{A}^{ir}\overline{\partial}_i\Lambda_\kappa^2 v_\alpha\right]\overline{\partial}^2\overline{\triangle}\eta_\gamma\right) \tag{5.2.88}$$

$$-\int_0^T \int_\Gamma (\partial_N Q)\tilde{A}^{3\alpha}\tilde{A}^{3\beta}\overline{\partial}^2\overline{\triangle}\tilde{\eta}_\beta \left(\left[\overline{\partial}^2, \tilde{A}^{i\gamma}\overline{\partial}_i\Lambda_\kappa^2 v_\alpha\right]\overline{\triangle}\eta_\gamma\right) \tag{5.2.89}$$

$$= -I_{02} + (3.99) + (3.100). \tag{5.2.90}$$

Summarising (5.2.72)-(5.2.78), (5.2.82)-I045 and I0414, we are able to control the boundary integral I_0 by invoking Taylor sign condition (5.2.8): $(\partial_N Q) \leq -\frac{c_0}{2}$

$$I_0 \lesssim -\frac{c_0}{4}\left|\tilde{A}^{3\alpha}\overline{\partial}^2\overline{\triangle}\Lambda_\kappa\eta_\alpha\right|_0^2\Bigg|_0^T + \int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t \tag{5.2.91}$$

Combining (5.2.91) with previous estimates (5.2.64)-(5.2.71), we finish the estimates of full tangential derivatives by

$$\begin{aligned} &\frac{1}{2}\int_\Omega \rho\left|\overline{\partial}^4 v\right|_0^2 \mathrm{d}y + \frac{1}{2}\int_\Omega \frac{\tilde{J}R'(q)}{\rho_0}\left|\overline{\partial}^4 q\right|_0^2 \mathrm{d}y + \frac{c_0}{4}\left|\tilde{A}^{3\alpha}\overline{\partial}^2\overline{\triangle}\Lambda_\kappa\eta_\alpha\right|_0^2 \\ \lesssim &\mathcal{P}_0 + \int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t + \varepsilon\int_0^T \left\|\overline{\partial}^2\overline{\triangle}\left((b\cdot\nabla_{\tilde{A}})b\right)\right\|_0^2 + \left\|\overline{\partial}^2\overline{\triangle}\partial_t q\right\|_0^2 \mathrm{d}t \end{aligned} \tag{5.2.92}$$

5.2.1.4 Control of the higher order heat and wave equations

Summarizing the previous energy estimates Before going to the next step, let us summarize what energy estimates we have gotten. First, from div-curl restimates((5.2.36), (5.2.38), (5.2.39), (5.2.42), (5.2.45)-(5.2.48), (5.2.51)-(5.2.54)) and tangential estimates ((5.2.55)-(5.2.58) and (5.2.92)) in Section

5.2.1.3, we got

$$
\begin{aligned}
&\sum_{k=0}^{4}\left\|\partial_t^{4-k}v\right\|_k^2+\left\|\partial_t^{4-k}q\right\|_k^2+\left|\tilde{A}^{3\alpha}\overline{\partial}^2\overline{\triangle}\Lambda_\kappa\eta_\alpha\right|_0^2\\
\lesssim&\varepsilon\left(\sum_{k=0}^{3}\left\|\partial_t^{4-k}v\right\|_k^2+\left\|\partial_t^{4-k}q\right\|_k^2\right)+\mathcal{P}_0+P(\mathfrak{e}_\kappa(T))\int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t \\
&+\varepsilon\sum_{k=0}^{4}\int_0^T\left\|\overline{\partial}^k\partial_t^{4-k}\left((b\cdot\nabla_{\tilde{A}})b\right)\right\|_0^2+\left\|\overline{\partial}^k\partial_t^{4-k}\partial_t q\right\|_0^2\,\mathrm{d}t
\end{aligned}
\tag{5.2.93}
$$

The magnetic field b has the following estimates by combining (5.2.23), (5.2.26), (5.2.27) and (5.2.29):

$$
\sum_{k=0}^{4}\left\|\partial_t^{4-k}b(T)\right\|_k^2\lesssim\mathcal{P}_0+P(\mathfrak{e}_\kappa(T))\int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t+\varepsilon H_\kappa(T). \tag{5.2.94}
$$

Summing up (5.2.93) and (5.2.94), we get the estimates of $E_\kappa(T)$ as

$$
\begin{aligned}
E_\kappa(T)\lesssim&\mathcal{P}_0+P(\mathfrak{e}_\kappa(T))\int_0^T P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t\\
&+\varepsilon\left(H_\kappa(T)+\sum_{k=0}^{4}\int_0^T\left\|\overline{\partial}^k\partial_t^{4-k}\left((b\cdot\nabla_{\tilde{A}})b\right)\right\|_0^2+\left\|\overline{\partial}^k\partial_t^{4-k}\partial_t q\right\|_0^2\,\mathrm{d}t\right)
\end{aligned}
\tag{5.2.95}
$$

(5.2.95) shows that we need H_κ, W_κ together with Lorentz force to absorb the ε-term in (5.2.95). From (5.2.35), we know Lorentz force can be controlled by $E_\kappa(T)$ plus a term in $H_\kappa(T)$

$$
\sum_{k=0}^{4}\left\|\partial_t^{4-k}((b\cdot\nabla_{\tilde{A}})b)\right\|_k^2\lesssim\|b\|_2^2\left\|\nabla_{\tilde{A}}\partial_t^4 b\right\|_0^2+P(\mathfrak{e}_\kappa(T))+\mathcal{P}_0+P(\mathfrak{e}_\kappa(T))\int_0^T P(\mathfrak{e}_\kappa(T))\,\mathrm{d}t. \tag{5.2.96}
$$

Also notice that $\partial_t q=0$ on Γ, which allows us to reduce the space-time control of $\partial_t q$ to the full time derivative case by using Lemma 3.3.3 (See Section 5.2.1.4). Therefore, all the estimates of the total energy $\mathcal{E}_\kappa$ in (5.2.3) are reduced to seek for a common control of $W_\kappa(T)$ and $H_\kappa(T)$, the energy functionals of 4-th time-differentiated heat and wave equations, by $\varepsilon\mathcal{E}_\kappa(T)+\mathcal{P}_0+P(\mathcal{E}_\kappa(T))\int_0^T P(\mathcal{E}_\kappa(t))\,\mathrm{d}t$.

Elliptic estimates of $\partial_t q$ Let us recall the heat equation of b and wave equation of q

$$\partial_t b - \triangle_{\tilde{A}} b = (b \cdot \nabla_{\tilde{A}}) v - b \text{div}_{\tilde{A}} v, \tag{5.2.97}$$

$$\begin{aligned}
&\frac{\tilde{J} R'(q)}{\rho_0} \partial_t^2 q - \triangle_{\tilde{A}} q \\
=& b \cdot \triangle_{\tilde{A}} b + R \partial_t \tilde{A}^{\mu\alpha} \partial_\mu v_\alpha - \nabla_{\tilde{A}}^\alpha b \cdot \nabla_{\tilde{A}} b_\alpha + |\nabla_{\tilde{A}} b|^2 \\
&+ \frac{R'(q)}{R} \left((\nabla_{\tilde{A}} Q - (b \cdot \nabla_{\tilde{A}}) b) \cdot \nabla_{\tilde{A}} q \right) + \left(\tilde{J} \frac{R'(q)}{\rho_0} - \frac{R \tilde{J} R''(q)}{\rho_0} \right) (\partial_t q)^2 \\
=:& b \cdot \triangle_{\tilde{A}} b + w_0.
\end{aligned} \tag{5.2.98}$$

Here we note that all the terms in w_0 only contain first-order derivative!

In (5.2.95), there are 4-th order space-time tangential derivatives of $\partial_t q$. It seems that we can directly consider the energy functional of $\mathfrak{D}^4$-differentiated wave equation of q (5.2.98). However, that also requires the control of commutator $[\mathfrak{D}^4, \text{div}_{\tilde{A}}] \nabla_{\tilde{A}} q$, which is uncontrollable when $\mathfrak{D}^4 = \overline{\partial}^4$. Therefore, we have to use Lemma 3.3.3 to reduce spatial derivatives to time derivatives.

We start with full spatial derivatives. Since $\|\partial_t q\|_4 \approx \|\nabla_{\tilde{A}} \partial_t q\|_3$, we have

$$\|\partial_t q\|_4 \lesssim P(\|\tilde{\eta}\|_3) \|\partial_t \triangle_{\tilde{A}} q\|_2 + P(\mathfrak{e}_\kappa(T)) \tag{5.2.99}$$

Invoking the ∂_t-differentiated wave equation, we find that

$$\partial_t \triangle_{\tilde{A}} q = \frac{\tilde{J} R'(q)}{\rho_0} \partial_t^3 q - b \cdot \partial_t \triangle_{\tilde{A}} b - \partial_t w_0.$$

Then using the heat equation (5.2.97) to reduce $\triangle_{\tilde{A}} b$ to lower order terms, we get

$$\partial_t \triangle_{\tilde{A}} q = \frac{\tilde{J} R'(q)}{\rho_0} \partial_t^3 q - b \cdot \partial_t \left(\partial_t b - (b \cdot \nabla_{\tilde{A}}) v + b \text{div}_{\tilde{A}} v \right) - \partial_t w_0.$$

Plugging this back to (5.2.99), we trade two spatial derivatives by two time derivatives

$$\|\partial_t q\|_4 \lesssim P(\|\tilde{\eta}\|_3)\|\partial_t^3 q\|_2 + P(\mathfrak{e}_\kappa(T)). \tag{5.2.100}$$

Repeating the same thing for $\|\partial_t^2 q\|_3, \|\partial_t^3 q\|_2$, we get the following reduction

$$\|\partial_t^2 q\|_3 \lesssim P(\|\tilde{\eta}\|_2)\|\partial_t^4 q\|_1 + P(\mathfrak{e}_\kappa(T)) \approx P(\|\tilde{\eta}\|_2)\|\nabla_{\tilde{A}}\partial_t^4 q\|_0 + P(\mathfrak{e}_\kappa(T)), \tag{5.2.101}$$

$$\|\partial_t^3 q\|_2 \lesssim P(\|\tilde{\eta}\|_2)\|\partial_t^5 q\|_0 + P(\mathfrak{e}_\kappa(T)). \tag{5.2.102}$$

From (5.2.100)-(5.2.102), we are able to reduce the energy estimates of $\partial_t q$ to $\|\nabla_{\tilde{A}}\partial_t^4 q\|_0$ and $\|\partial_t^5 q\|_0$, which motivates us to consider the 4-th time-differentiated wave equation (5.2.98) together with 4-th time differentiated heat equation (5.2.97).

4-th time differentiated heat and wave equation Taking ∂_t^4 in (5.2.97) and (5.2.98), we get

$$\begin{aligned}\partial_t^5 b - \triangle_{\tilde{A}}\partial_t^4 b =& \partial_t^4\left((b\cdot\nabla_{\tilde{A}})v - b\,\mathrm{div}_{\tilde{A}} v\right) + [\partial_t^4, \triangle_{\tilde{A}}]b \\ =& (b\cdot\nabla_{\tilde{A}})\partial_t^4 v + b\frac{\tilde{J}R'(q)}{\rho_0}\partial_t^5 q + \left[\partial_t^4, \triangle_{\tilde{A}}\right]b + \left[\partial_t^4, b\cdot\nabla_{\tilde{A}}\right]v \\ &+ \left[\partial_t^4, b\frac{\tilde{J}R'(q)}{\rho_0}\right]\partial_t q \\ =:& \, h_5\end{aligned} \tag{5.2.103}$$

In h_5, there are 5 derivatives of v. We can invoke the second equation of (5.2.1) to reduce to q and B, e.g., $\|\partial_t^5 v\|_0 \lesssim \|\partial_t^4((b\cdot\nabla_{\tilde{A}})b)\|_0 + \|\partial_t^4\nabla_{\tilde{A}} Q\|_0 + \cdots$, in which the leading order terms are $\nabla_{\tilde{A}}\partial_t^4 b$ and $\nabla_{\tilde{A}}\partial_t^4 q$, the same as part of W_κ and H_κ.

Taking $L_t^2 L_x^2$-inner product with $\partial_t^5 b$ and integrating by parts, we get

$$RHS = \int_0^T \int_\Omega h_5 \cdot \partial_t^5 b \, \mathrm{d}y \, \mathrm{d}t$$

$$\begin{aligned}
LHS =& \int_0^T \int_\Omega \left|\partial_t^5 b\right|^2 \mathrm{d}t - \int_0^T \int_\Omega \partial_t^5 b \cdot \triangle_{\tilde{A}} \partial_t^4 b \, \mathrm{d}y \, \mathrm{d}t \\
=& \int_0^T \int_\Omega \left|\partial_t^5 b\right|^2 \mathrm{d}t + \int_0^T \int_\Omega \partial_t \left(\nabla_{\tilde{A}} \partial_t^4 b\right) \cdot \left(\nabla_{\tilde{A}} \partial_t^4 b\right) \mathrm{d}y \, \mathrm{d}t \\
&+ \int_0^T \int_\Omega \partial_\mu \tilde{A}^{\mu\alpha} \left(\partial_t^5 b\right) \cdot \left(\nabla_{\tilde{A}} \partial_t^4 b\right) \mathrm{d}y \, \mathrm{d}t + \int_0^T \int_\Omega \left(\left[\nabla_{\tilde{A}}, \partial_t\right] \partial_t^4 b\right) \cdot \left(\nabla_{\tilde{A}} \partial_t^4 b\right) \mathrm{d}y \, \mathrm{d}t \\
&- \int_0^T \int_\Gamma \tilde{A}^{3\alpha} \partial_t^5 b \cdot \left(\nabla_{\tilde{A}} \partial_t^4 b\right)_\alpha \mathrm{d}S \, \mathrm{d}t
\end{aligned} \tag{5.2.104}$$

Since $b = \mathbf{0}$ on the boundary, we know the boundary integral vanishes. The first and second integrals give the energy functional $H_\kappa(T) - H_\kappa(0)$. Therefore, we have

$$\begin{aligned}
H_\kappa(T) - H_\kappa(0) =& \int_0^T \int_\Omega \left|\partial_t^5 b\right|^2 \mathrm{d}y \, \mathrm{d}t + \int_\Omega \left|\nabla_{\tilde{A}} \partial_t^4 b\right|^2 \mathrm{d}y \Big|_0^T \\
=& \int_0^T \int_\Omega h_5 \cdot \partial_t^5 b \, \mathrm{d}y \, \mathrm{d}t - \int_0^T \int_\Omega \partial_\mu \tilde{A}^{\mu\alpha} \left(\partial_t^5 b\right) \cdot \left(\nabla_{\tilde{A}} \partial_t^4 b\right) \mathrm{d}y \, \mathrm{d}t \\
&- \int_0^T \int_\Omega \left(\left[\nabla_{\tilde{A}}, \partial_t\right] \partial_t^4 b\right) \cdot \left(\nabla_{\tilde{A}} \partial_t^4 b\right) \mathrm{d}y \, \mathrm{d}t \\
\lesssim& \varepsilon \int_0^T \int_\Omega \left|\partial_t^5 b\right|^2 \mathrm{d}y \, \mathrm{d}t + \int_0^T \|h_5\|_0^2 \, \mathrm{d}t + \int_0^T \left\|\nabla_{\tilde{A}} \partial_t^4 b\right\|_0 \|\tilde{\eta}\|_4 \, \mathrm{d}t \\
&+ \int_0^T \|\partial_t \tilde{A}\|_{L^\infty} \left\|\partial_t^4 b\right\|_0 \left\|\nabla_{\tilde{A}} \partial_t^4 b\right\|_0 \mathrm{d}t \\
\lesssim& \varepsilon \int_0^T \int_\Omega \left|\partial_t^5 b(t)\right|^2 \mathrm{d}y \, \mathrm{d}t + \int_0^T P(\mathfrak{e}_\kappa(t)) + (H_\kappa(t) + W_\kappa(t)) \, \mathrm{d}t \\
\lesssim& \varepsilon H_\kappa(T) + \int_0^T P(\mathfrak{e}_\kappa(t)) + (H_\kappa(t) + W_\kappa(t)) \, \mathrm{d}t.
\end{aligned} \tag{5.2.105}$$

Here W_κ appears in the last term because $\partial_t^5 v$ contains $\nabla_{\tilde{A}} \partial_t^4 q$ which is part of $W_\kappa(t)$.

Next we ∂_t^4 differentiate (5.2.98) to get

$$\frac{\tilde{J}R'(q)}{\rho_0}\partial_t^6 q - \triangle_{\tilde{A}}\partial_t^4 q$$

$$= b \cdot \triangle_{\tilde{A}}\partial_t^4 b + \partial_t^4 w_0 + \left[b \cdot \triangle_{\tilde{A}}, \partial_t^4\right] + \left[\partial_t^4, \triangle_{\tilde{A}}\right] q + \left[\frac{\tilde{J}R'(q)}{\rho_0}, \partial_t^4\right]\partial_t^2 q.$$

Then plug the heat equation (5.2.103) $\triangle_{\tilde{A}} b = \partial_t^5 b - h_5$ to get

$$\begin{aligned}&\frac{\tilde{J}R'(q)}{\rho_0}\partial_t^6 q - \triangle_{\tilde{A}}\partial_t^4 q\\ =&b \cdot \left(\partial_t^5 b - h_5\right) + \partial_t^4 w_0 + \left[b \cdot \triangle_{\tilde{A}}, \partial_t^4\right] + \left[\partial_t^4, \triangle_{\tilde{A}}\right] q + \left[\frac{\tilde{J}R'(q)}{\rho_0}, \partial_t^4\right]\partial_t^2 q =: w_5\end{aligned} \tag{5.2.106}$$

Taking $L_t^2 L_x^2$ inner product with $\partial_t^5 q$, we have

$$\begin{aligned}RHS =& \int_0^T \int_\Omega w_5 \cdot \partial_t^5 q \,\mathrm{d}y\,\mathrm{d}t\\ LHS =& \int_0^T \int_\Omega \frac{\tilde{J}R'(q)}{\rho_0}\partial_t^6 q \partial_t^5 q \,\mathrm{d}t - \int_0^T \int_\Omega \partial_t^5 q \cdot \triangle_{\tilde{A}}\partial_t^4 q \,\mathrm{d}y\,\mathrm{d}t\\ =&\frac{1}{2}\int_\Omega \frac{\tilde{J}R'(q)}{\rho_0}\left|\partial_t^5 q\right|^2 \mathrm{d}y\Big|_0^T + \int_0^T \int_\Omega \partial_t\left(\nabla_{\tilde{A}}\partial_t^4 q\right)\cdot\left(\nabla_{\tilde{A}}\partial_t^4 q\right)\mathrm{d}y\,\mathrm{d}t\\ &+ \int_0^T \int_\Omega \partial_\mu \tilde{A}^{\mu\alpha}\left(\partial_t^5 q\right)\cdot\left(\nabla_{\tilde{A}}\partial_t^4 q\right)\mathrm{d}y\,\mathrm{d}t + \int_0^T \int_\Omega \left(\left[\nabla_{\tilde{A}}, \partial_t\right]\partial_t^4 q\right)\cdot\left(\nabla_{\tilde{A}}\partial_t^4 q\right)\mathrm{d}y\,\mathrm{d}t\\ &- \int_0^T \int_\Gamma \tilde{A}^{3\alpha}\underbrace{\partial_t^5 q}_{0}\cdot\left(\nabla_{\tilde{A}}\partial_t^4 q\right)_\alpha \mathrm{d}S\,\mathrm{d}t - \int_0^T \int_\Omega \frac{1}{2}\partial_t\left(\frac{\tilde{J}R'(q)}{\rho_0}\right)\left|\partial_t^5 q\right|^2 \mathrm{d}y\,\mathrm{d}t,\end{aligned} \tag{5.2.107}$$

and thus we have

$$W_\kappa(T) - W_\kappa(0) = \frac{1}{2}\int_\Omega \frac{\tilde{J}R'(q)}{\rho_0}\left|\partial_t^5 q\right|^2 \,\mathrm{d}y\Big|_0^T + \frac{1}{2}\int_\Omega \left|\nabla_{\tilde{A}}\partial_t^4 q\right|^2 \,\mathrm{d}y\Big|_0^T$$

$$= \int_0^T\int_\Omega w_5\cdot\partial_t^5 q\,\mathrm{d}y\,\mathrm{d}t + \int_0^T\int_\Omega \frac{1}{2}\partial_t\left(\frac{\tilde{J}R'(q)}{\rho_0}\right)\left|\partial_t^5 q\right|^2 \,\mathrm{d}y\,\mathrm{d}t \tag{5.2.108}$$

$$-\int_0^T\int_\Omega \partial_\mu \tilde{A}^{\mu\alpha}\left(\partial_t^5 q\right)\cdot\left(\nabla_{\tilde{A}}\partial_t^4 q\right)\,\mathrm{d}y\,\mathrm{d}t - \int_0^T\int_\Omega \left(\left[\nabla_{\tilde{A}},\partial_t\right]\partial_t^4 q\right)\cdot\left(\nabla_{\tilde{A}}\partial_t^4 q\right)\,\mathrm{d}y\,\mathrm{d}t.$$

The term $\|w_5\|_0^2$ can be controlled by $H_\kappa(T) + W_\kappa(T) + P(\mathfrak{e}_\kappa(T))$, because all the terms in w_5 are of ≤ 5 derivatives, and can be controlled by either heat or wave energy. The detailed estimate is referred to [82, (7.12)-(7.19)]. Therefore, we have

$$W_\kappa(T) - W_\kappa(0) \lesssim \varepsilon\left(W_\kappa(T) + H_\kappa(T)\right) + \int_0^T H_\kappa(t) + W_\kappa(t) + P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t \tag{5.2.109}$$

Summing up (5.2.105) and (5.2.109), we get the common control of H_κ and W_κ

$$\left(H_\kappa(t) + W_\kappa(t)\right)\Big|_0^T \lesssim \varepsilon\left(W_\kappa(T) + H_\kappa(T)\right) + \int_0^T H_\kappa(t) + W_\kappa(t) + P(\mathfrak{e}_\kappa(t))\,\mathrm{d}t \tag{5.2.110}$$

Closing the energy estimates Combining (5.2.95), (5.2.96) and (5.2.110), we get the inequality

$$\begin{aligned}\mathcal{E}_\kappa(T) - \mathcal{E}_\kappa(0) &= \left(\mathfrak{e}_\kappa + H_\kappa + W_\kappa + \sum_{k=0}^{4}\left\|\partial_t^{4-k}((b\cdot\nabla_{\tilde{A}})b)\right\|_k^2\right)\Bigg|_0^T \\ &\lesssim \varepsilon\left(H_\kappa(T) + W_\kappa(T)\right) + P(\mathfrak{e}_\kappa(T))\int_0^T P(\mathcal{E}_\kappa(t))\,\mathrm{d}t.\end{aligned} \tag{5.2.111}$$

By choosing $\varepsilon > 0$ sufficiently small, we get

$$\mathcal{E}_\kappa(T) - \mathcal{E}_\kappa(0) \lesssim P(\mathfrak{e}_\kappa(T))\int_0^T P(\mathcal{E}_\kappa(t))\,\mathrm{d}t. \tag{5.2.112}$$

Finally, by the Gronwall-type inequality, we know there exists some $T > 0$ only depending on $\|v_0\|_4, \|b_0\|_5, \|q_0\|_4, \|\rho_0\|_4$, such that

$$\sup_{0\leq t\leq T} \mathcal{E}_\kappa(t) \leq P(\mathcal{E}_\kappa(0)). \tag{5.2.113}$$

This finalizes the proof of Proposition 5.2.2, i.e., the uniform-in-κ a priori estimate for the nonlinear approximation system (5.2.1).

5.2.2 Well-posedness pf the nonlinear approximate system

In this section we are going to prove the local existence of the nonlinear κ-approximation system (5.2.1). The method is standard Picard type iteration. We start with the trivial solution $(\eta^{(0)}, v^{(0)}, b^{(0)}, q^{(0)}) = (\eta^{(1)}, v^{(1)}, b^{(1)}, q^{(1)}) = (\mathrm{Id}, 0, 0, 0)$. Suppose we have already constructed $\{(\eta^{(k)}, v^{(k)}, b^{(k)}, q^{(k)})\}_{0\leq k\leq n}$ for some given $n \in \mathbb{N}^*$. Inductively we define $(\eta^{(n+1)}, v^{(n+1)}, b^{(n+1)}, q^{(n+1)})$ by linearzing (5.2.1) near $a^{(n)} := [\partial\eta^{(n)}]^{-1}$.

$$\begin{cases}\partial_t\eta^{(n+1)} = v^{(n+1)} + \psi^{(n)} & \text{in } \Omega,\\ \frac{\rho_0}{\tilde{J}^{(n)}}\partial_t v^{(n+1)} = (b^{(n)}\cdot\nabla_{\tilde{A}^{(n)}})b^{(n+1)} - \nabla_{\tilde{A}^{(n)}} Q^{(n+1)}, & \text{in } \Omega,\\ \frac{\tilde{J}^{(n)}R'(q^{(n)})}{\rho_0}\partial_t q^{(n+1)} + \mathrm{div}_{\tilde{A}^{(n)}} v^{(n+1)} = 0 & \text{in } \Omega,\\ (\partial_t + \mathrm{curl}_{\tilde{A}^{(n)}}\mathrm{curl}_{\tilde{A}^{(n)}})b^{(n+1)} = (b^{(n)}\cdot\nabla_{\tilde{A}^{(n)}})v^{(n+1)} - b^{(n)}\mathrm{div}_{\tilde{A}^{(n)}} v^{(n+1)}, & \text{in } \Omega,\\ \mathrm{div}_{\tilde{A}^{(n)}} b^{(n+1)} = 0 & \text{in } \Omega,\\ q^{(n+1)} = 0,\ b^{(n+1)} = \mathbf{0} & \text{on } \Gamma,\\ (\eta^{(n+1)}, v^{(n+1)}, b^{(n+1)}, q^{(n+1)})|_{\{t=0\}} = (\mathrm{Id}, v_0, b_0, q_0). \end{cases} \tag{5.2.114}$$

Here $\tilde{A}^{(n)} := (\partial\tilde{\eta}^{(n)})^{-1}$ and the correction term $\psi^{(n)}$ is determined by (5.2.2) with $\eta = \eta^{(n)}, v = v^{(n)}, \tilde{A} = \tilde{A}^{(n)}$ in that equation. What we need to verify are

1. System (5.2.114) has a (unique) solution $(\eta^{(n+1)}, v^{(n+1)}, b^{(n+1)}, q^{(n+1)})$ (in a suitable function space).
2. The solution of (5.2.114) satifies an energy estimate uniformly in n.

3. The approximate solutions $\{(\eta^{(n)}, v^{(n)}, b^{(n)}, q^{(n)})\}_{n=0}^{\infty}$ converge strongly.

We denote $(\eta^{(n)}, v^{(n)}, b^{(n)}, q^{(n)})$ by $(\mathring{\eta}, \mathring{v}, \mathring{b}, \mathring{q})$, and $(\eta^{(n+1)}, v^{(n+1)}, b^{(n+1)}, q^{(n+1)})$ by (η, v, b, q) for the simplicity of notations. Then (5.2.114) becomes

$$\begin{cases} \partial_t \eta = v + \mathring{\psi} & \text{in } \Omega, \\ \rho_0 \mathring{J}^{-1} \partial_t v = (\mathring{b} \cdot \nabla_{\tilde{\mathring{A}}}) b - \nabla_{\tilde{\mathring{A}}} Q, \quad Q = q + \frac{1}{2}|b|^2 & \text{in } \Omega, \\ \frac{\tilde{\mathring{J}} R'(\mathring{q})}{\rho_0} \partial_t q + \operatorname{div}_{\tilde{\mathring{A}}} v = 0 & \text{in } \Omega, \\ \partial_t b + \operatorname{curl}_{\tilde{\mathring{A}}} \operatorname{curl}_{\tilde{\mathring{A}}} b = (\mathring{b} \cdot \nabla_{\tilde{\mathring{A}}}) v - \mathring{b} \operatorname{div}_{\tilde{\mathring{A}}} v, & \text{in } \Omega, \\ \operatorname{div}_{\tilde{\mathring{A}}} b = 0 & \text{in } \Omega, \\ q = 0, \ b = \mathbf{0} & \text{on } \Gamma, \\ (\eta, v, b, q)|_{\{t=0\}} = (\mathrm{Id}, v_0, b_0, q_0). \end{cases} \tag{5.2.115}$$

The divergence-free condition for b is still a constraint for initial data and b still satisfies a heat equation.

5.2.2.1 A priori estimates of the linearized approximation system

We first prove the a priori estimate of the linearized system (5.2.114) (or equivalently (5.2.115)) because such a priori bound helps us to choose a suitable function space when proving the existence of the linearized system by fixed-point argument.

Define the energy functional for $(\eta^{(n+1)}, v^{(n+1)}, b^{(n+1)}, q^{(n+1)})$ by

$$\mathcal{E}^{(n+1)}(T) := \mathfrak{e}^{(n+1)}(T) + H^{(n+1)}(T) + W^{(n+1)}(T) + \sum_{k=0}^{4} \left\| \partial_t^{4-k} \left((b^{(n)} \cdot \nabla_{\tilde{A}^{(n)}}) b^{(n+1)} \right) \right\|_k^2, \tag{5.2.116}$$

where

$$\mathfrak{e}^{(n+1)}(T) := \left\|\eta^{(n+1)}\right\|_4^2 + \sum_{k=0}^{4}\left\|\partial_t^{4-k}v^{(n+1)}\right\|_k^2 + \sum_{k=0}^{4}\left\|\partial_t^{4-k}b^{(n+1)}\right\|_k^2 + \sum_{k=0}^{4}\left\|\partial_t^{4-k}q^{(n+1)}\right\|_k^2 \tag{5.2.117}$$

$$H^{(n+1)}(T) := \int_0^T\int_\Omega \left|\partial_t^5 b^{(n+1)}\right|^2 \,\mathrm{d}y\,\mathrm{d}t + \left\|\partial_t^4 b^{(n+1)}\right\|_1^2 \tag{5.2.118}$$

$$W^{(n+1)}(T) := \sum_{k=0}^{4}\left\|\nabla_{\tilde{A}^{(n)}}\partial_t^{4-k}q^{(n+1)}\right\|_k^2 + \left\|\partial_t^5 q^{(n+1)}\right\|_0^2. \tag{5.2.119}$$

The conclusion is

Proposition 5.2.5. Suppose $(\eta^{(n+1)}, v^{(n+1)}, b^{(n+1)}, q^{(n+1)})$ satisfies (5.2.114), then there exists $T_\kappa >$ 0 sufficiently small, independent of n. such that

$$\sup_{0\le t\le T_\kappa} \mathcal{E}^{(n+1)}(t) \le \mathcal{P}_0. \tag{5.2.120}$$

Remark 5.2.6. Compared with $\mathcal{E}_\kappa$ in (5.2.3), we find that there are extra terms in $W^{(n+1)}(T)$. We note that these extra terms are not needed in the uniform-in-n a priori estimates bacause the elliptic estimates of $\partial_t q$ helps us reduce $\|\partial_t^{4-k}q\|_{k+1}$ to the L^2-norm of $\partial_t^5 q$ and $\nabla_{\tilde{\mathring{A}}}\partial_t^4 q$, and $\|\nabla_{\tilde{\mathring{A}}}q\|_4$ is not needed. **However, these terms are needed when we verify the fixed-point argument in the construction of the solution to** (5.2.115)**:** The H^4-norm of v has to be controlled by

$$v(T) = v_0 + \int_0^T \|\partial_t v(t)\|_4 \,\mathrm{d}t,$$

and thus the H^4-norm of $\nabla_{\tilde{\mathring{A}}}Q$ is definitely needed.

5.2.2.2 Estimates of the frozen coefficients

We prove Proposition 5.2.5 by induction on n. When $n = -1, 0$, it auotmatically holds for the trivial solution. Suppose the energy bound (5.2.120) holds for all $\mathcal{E}^{(k)}$ with $1 \le k \le n$. Then we have the following estimates for $\mathring{A}, \mathring{\eta}, \mathring{J}$.

Lemma 5.2.7. Let $T \in (0, T_\kappa)$. Then there exists some $\varepsilon \in (0,1)$ sufficiently small and constant $C > 1$ such that

$$\mathring{\psi} \in L_t^\infty([0,T]; H^4(\Omega)),\ \ \partial_t^l \mathring{\psi} \in L_t^\infty([0,T]; H^{5-l}(\Omega)),\ \ \forall 1 \le l \le 4; \tag{5.2.121}$$

$$\|\mathring{J} - 1\|_3 + \|\mathring{\tilde{J}} - 1\|_3 + \|\mathrm{Id} - \mathring{\tilde{A}}\|_3 + \|\mathrm{Id} - \mathring{A}\|_3 \le \varepsilon; \tag{5.2.122}$$

$$\partial_t \mathring{\eta} \in L^\infty([0,T]; H^4(\Omega)),\ \ \partial_t^{l+1} \mathring{\eta} \in L^\infty([0,T]; H^{5-l}(\Omega)),\ \ \forall 1 \le l \le 4; \tag{5.2.123}$$

$$\mathring{J}, \partial_t \mathring{J} \in L_t^\infty([0,T]; H^3(\Omega)),\ \ \partial_t^{1+l} \mathring{J} \in L_t^\infty([0,T]; H^{4-l}(\Omega)),\ \ \forall 1 \le l \le 4; \tag{5.2.124}$$

$$1/C \le \frac{\mathring{\tilde{J}} R'(\mathring{q})}{\rho_0}, \rho_0 \mathring{\tilde{J}}^{-1} \le C,\ \partial_t^l \left(\frac{\mathring{\tilde{J}} R'(\mathring{q})}{\rho_0}, \rho_0 \mathring{\tilde{J}}^{-1} \right) \in L^\infty([0,T]; H^{5-l}(\Omega)),\ \ \forall 1 \le l \le 5. \tag{5.2.125}$$

Proof. (5.2.121) follows in the same way as Lemma 5.2.4. $\mathring{J} = \det[\partial \mathring{\eta}]$ and $\mathring{A} = [\partial \mathring{\eta}]^{-1}$ prove (5.2.123) and (5.2.124) because the elements are multilinear functions of $\partial \mathring{\eta}$. The smallness of $\tilde{J} - 1$ and $\mathrm{Id} - \mathring{A}$ follows from $\mathring{J} = \det[\partial \mathring{\eta}]$ and

$$\mathrm{Id} - \mathring{A} = -\int_0^T \partial_t \mathring{A} = \int_0^T \mathring{A} : (\partial(\mathring{v} + \psi^{(n-1)})) : \mathring{A}\, \mathrm{d}t$$

and choosing ε (depending on T_κ) sufficiently small. (5.2.125) is similarly proven. □

5.2.2.3 Control of $\mathcal{E}^{(n+1)}$

The control of $\mathcal{E}^{(n+1)}$ follows nearly in the same way as the nonlinear functional $\mathcal{E}_\kappa(T)$ except the extra term $\|\nabla_{\mathring{\tilde{A}}} q\|_4$ and boundary integral in the tangential estimates.

Step 1: Estimates of magnetic field and Lorentz force

Since $b = \mathbf{0}$ on the boundary and $\mathrm{div}_{\mathring{\tilde{A}}} b = 0$ in Ω, we are able to directly mimic the proof in

Section 5.2.1.2 to get the analogues of (5.2.23)-(5.2.29):

$$\sum_{k=0}^{4} \left\| \partial_t^{4-k} b(T) \right\|_k^2 \lesssim \mathcal{P}_0 + P(\mathfrak{e}^{(n+1)}(T)) \int_0^T P(\mathfrak{e}^{(n+1)}(t))\, \mathrm{d}t + \varepsilon H^{(n+1)}(T) \tag{5.2.126}$$

and an analogue of (5.2.31)

$$\begin{aligned} &\sum_{k=0}^{4} \left\| \partial_t^{4-k} ((\mathring{b} \cdot \nabla_{\mathring{\tilde{A}}}) b) \right\|_k^2 \\ \lesssim &\|b\|_2^2 \left\| \nabla_{\mathring{\tilde{A}}} \partial_t^4 b \right\|_0^2 + P(\mathfrak{e}^{(n+1)}(T)) + \mathcal{P}_0 + P(\mathfrak{e}^{(n+1)}(T)) \int_0^T P(\mathfrak{e}^{(n+1)}(t))\, \mathrm{d}t. \end{aligned} \tag{5.2.127}$$

Step 2: Div-Curl estimates of v

By (5.2.122), we know the div-curl estimates follow in the same way as Section 5.2.1.3-5.2.1.3. For $1 \le k \le 4$, we have

$$\frac{1}{2} \int_{\Omega} \rho_0 \mathring{\tilde{J}}^{-1} \left| \mathrm{curl}_{\mathring{\tilde{A}}} \partial_t^{4-k} v(t) \right|^2 \mathrm{d}y \Big|_0^T \lesssim \varepsilon T \sup_{0 \le t \le T} \left\| \partial_t^4 ((\mathring{b} \cdot \nabla_{\mathring{\tilde{A}}}) b) \right\|_k^2 + \int_0^T P(\mathfrak{e}^{(n+1)}(t))\, \mathrm{d}t. \tag{5.2.128}$$

$$\left\| \mathrm{div}_{\mathring{\tilde{A}}} \partial_t^{4-k} v \right\|_{k-1} \lesssim \varepsilon \left\| \partial_t^{4-k} v \right\|_k + \left\| \partial_t^{5-k} q \right\|_{k-1} + L.O.T. \tag{5.2.129}$$

$$\left| \partial_t^{4-k} v^3 \right|_{k-1/2} \lesssim \left\| \overline{\partial}^k \partial_t^{4-k} v \right\|_0 + \|\mathrm{div}\, \partial_t^{4-k} v\|_{k-1}. \tag{5.2.130}$$

$$\left\| \partial_t^{4-k} q \right\|_k \lesssim \sum_{k=1}^{4} \left\| \partial_t^{5-k} v \right\|_{k-1} + \mathcal{P}_0 + \int_0^T P(\mathfrak{e}^{(n+1)}(t))\, \mathrm{d}t + L.O.T. \tag{5.2.131}$$

Step 3: Space-Time tangential estimates

Let $\mathfrak{D} = \overline{\partial}$ or ∂_t. When $\mathfrak{D}^4$ contains at least one time derivative, we are able to directly ocmmute $\mathring{\tilde{A}}$ with $\mathfrak{D}^4$ because $\partial_t \mathring{\tilde{\eta}}$ has the same regularity as $\mathring{\tilde{\eta}}$, see Lemma 5.2.7. Since the boundary condition of (5.2.115) is the same as (5.2.1), we are able to mimic the proof of the nonlinear functional. The result is

$$\sum_{k=0}^{3} \left\|\overline{\partial}^k \partial_t^{4-k} v\right\|_k^2 + \left\|\overline{\partial}^k \partial_t^{4-k} q\right\|_k^2$$

$$\lesssim \varepsilon \left(\sum_{k=0}^{3} \left\|\partial_t^{4-k} v\right\|_k^2 + \left\|\partial_t^{4-k} q\right\|_k^2\right) + \mathcal{P}_0 + P(\mathfrak{e}^{(n+1)}(T)) \int_0^T P(\mathfrak{e}^{(n+1)}(t))\,\mathrm{d}t \tag{5.2.132}$$

$$+ \varepsilon \sum_{k=0}^{4} \int_0^T \left\|\overline{\partial}^k \partial_t^{4-k} \left((b \cdot \nabla_{\tilde{A}}) b\right)\right\|_0^2 + \left\|\overline{\partial}^k \partial_t^{4-k} \partial_t q\right\|_0^2 \,\mathrm{d}t$$

Step 4: Tangential spatial derivative estimates

This part contains a non-trivial boundary integral. In the nonlinear estimates, that boundary term together with Taylor sign condition gives the boundary part of nonlinear functional $E_\kappa(T)$. However, here we no longer need Taylor sign condition. Instead, we can sacrifise $1/\kappa$ to directly control the boundary integral by using the mollifier property, because the derivative loss is only tangential.

Similarly as in Section 5.2.1.3, we rewrite the equation in terms of Alinhac good unknonws. Define the Alinhac good unknowns of v, Q in (5.2.115) by

$$\mathring{\mathbf{V}} := \overline{\partial}^2 \overline{\triangle} v - \overline{\partial}^2 \overline{\triangle} \mathring{\tilde{\eta}} \cdot \nabla_{\mathring{\tilde{A}}} v, \quad \mathring{\mathbf{Q}} := \overline{\partial}^2 \overline{\triangle} Q - \overline{\partial}^2 \overline{\triangle} \mathring{\tilde{\eta}} \cdot \nabla_{\mathring{\tilde{A}}} Q.$$

Then we take $\overline{\partial}^2 \overline{\triangle}$ in the second equation of (5.2.115)

$$\rho_0 \mathring{\tilde{J}}^{-1} \partial_t \mathring{\mathbf{V}} + \nabla_{\mathring{\tilde{A}}} \mathring{\mathbf{Q}} = \mathring{\mathbf{F}} \tag{5.2.133}$$

where

$$\mathring{\mathbf{F}} := \overline{\partial}^2 \overline{\triangle}((\mathring{b} \cdot \nabla_{\mathring{\tilde{A}}}) b) + [\rho_0 \mathring{\tilde{J}}^{-1}, \overline{\partial}^2 \overline{\triangle}] \partial_t v + \rho_0 \mathring{\tilde{J}}^{-1} \partial_t (\overline{\partial}^2 \overline{\triangle} \mathring{\tilde{\eta}} \nabla_{\mathring{\tilde{A}}} v) + \mathring{C}(Q).$$

The equation is subjected to

$$\mathring{\mathbf{Q}} = -\overline{\partial}^2 \overline{\triangle} \mathring{\tilde{\eta}}_\beta \mathring{\tilde{A}}^{3\beta} (\partial_N Q) \ \ on \ \Gamma, \tag{5.2.134}$$

and

$$\nabla_{\tilde{\mathring{A}}} \cdot \mathring{\mathbf{V}} = \overline{\partial}^2 \overline{\triangle}(\operatorname{div}_{\tilde{\mathring{A}}} v) - \mathring{C}^{\alpha}(v_\alpha) \;\; in\ \Omega. \tag{5.2.135}$$

Multiplying $\tilde{\mathring{J}}\mathring{\mathbf{V}}$ and take space-time integral, we have

$$\begin{aligned} &\frac{1}{2}\int_\Omega \rho_0 \left|\partial_t \mathring{\mathbf{V}}(t)\right|^2 \,\mathrm{d}y\bigg|_0^T = -\int_0^T \int_\Omega \tilde{\mathring{J}} \nabla_{\tilde{\mathring{A}}} \mathring{\mathbf{Q}} \cdot \mathring{\mathbf{V}} \,\mathrm{d}y\,\mathrm{d}t + \int_0^T \int_\Omega \mathring{\mathbf{F}} \cdot \mathring{\mathbf{V}} \,\mathrm{d}y\,\mathrm{d}t \\ &= \int_0^T \tilde{\mathring{J}} (\partial_N Q) \overline{\partial}^2 \overline{\triangle} \tilde{\mathring{\eta}}_\beta \tilde{\mathring{A}}^{3\beta} \tilde{\mathring{A}}^{3\alpha} \mathring{\mathbf{V}}_\alpha \,\mathrm{d}S\,\mathrm{d}t + \int_0^T \tilde{\mathring{J}} \mathring{\mathbf{Q}} \overline{\partial}^2 \overline{\triangle} \left(\operatorname{div}_{\tilde{\mathring{A}}} v\right) \,\mathrm{d}y\,\mathrm{d}t \\ &- \int_0^T \int_\Omega Q \mathring{C}(v) \,\mathrm{d}y\,\mathrm{d}t \\ &=: LI_0 + LI_1 + LJ_1. \end{aligned} \tag{5.2.136}$$

Mimicing the estimates (5.2.68)-(5.2.71), we are able to control LI_1 as

$$\begin{aligned} LI_1 \lesssim & -\frac{1}{2}\int_\Omega \frac{\tilde{\mathring{J}} R'(\mathring{q})}{\rho_0} \left|\overline{\partial}^2 \overline{\triangle} q\right|^2 \,\mathrm{d}y\bigg|_0^T + \varepsilon \int_0^T \left\|\overline{\partial}^2 \overline{\triangle} \partial_t q\right\|_0^2 \,\mathrm{d}t \\ & + \mathcal{P}_0 + \int_0^T P(\mathfrak{e}^{(n+1)}(t)) \,\mathrm{d}t. \end{aligned} \tag{5.2.137}$$

For the boundary integral LI_0, we integral $\overline{\partial}^{1/2}$ by parts to get

$$\begin{aligned} LI_0 &= \int_0^T \tilde{\mathring{J}} (\partial_N Q) \overline{\partial}^2 \overline{\triangle} \tilde{\mathring{\eta}}_\beta \tilde{\mathring{A}}^{3\beta} \tilde{\mathring{A}}^{3\alpha} \mathring{\mathbf{V}}_\alpha \,\mathrm{d}S\,\mathrm{d}t \\ &= \int_0^T \overline{\partial}^{1/2} \left(\tilde{\mathring{J}} (\partial_N Q) \overline{\partial}^2 \overline{\triangle} \tilde{\mathring{\eta}}_\beta \tilde{\mathring{A}}^{3\beta} \tilde{\mathring{A}}^{3\alpha}\right) \overline{\partial}^{-1/2} \mathring{\mathbf{V}}_\alpha \,\mathrm{d}S\,\mathrm{d}t \\ &\lesssim \int_0^T \left(|(\partial_N Q)|_{L^\infty} \left|\tilde{\mathring{J}} \tilde{\mathring{A}}\right|_{L^\infty}^2 \left|\overline{\partial}^2 \overline{\triangle} \tilde{\mathring{\eta}}\right|_{1/2} + \left|(\partial_N Q) \tilde{\mathring{J}} \tilde{\mathring{A}}^{3\beta} \tilde{\mathring{A}}^{3\alpha}\right|_{W^{\frac{1}{2},4}} \left|\overline{\partial}^2 \overline{\triangle} \tilde{\mathring{\eta}}_\beta\right|_{L^4} \right) \left|\mathring{\mathbf{V}}\right|_{-1/2} \,\mathrm{d}t. \end{aligned}$$

By the mollifier property $|\overline{\partial}^2 \overline{\triangle} \tilde{\mathring{\eta}}|_{1/2} \lesssim \kappa^{-1} |\mathring{\eta}|_{7/2}$ and $H^{1/2}(\mathbb{T}^2) \hookrightarrow L^4(\mathbb{T}^2)$, we are able to control LI_0 by

$$LI_0 \lesssim \frac{1}{\kappa} P\left(\|Q\|_3, \|v\|_4, \|\mathring{\eta}\|_4\right). \tag{5.2.138}$$

This together with (5.2.138) gives the tangential spatial estimates

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega} \rho_0 \left|\overline{\partial}^4 v\right|_0^2 \,\mathrm{d}y + \frac{1}{2}\int_{\Omega} \frac{\mathring{\tilde{J}} R'(\mathring{q})}{\rho_0} \left|\overline{\partial}^4 q\right|_0^2 \,\mathrm{d}y \\
\lesssim & \mathcal{P}_0 + \int_0^T P(\mathfrak{e}^{(n+1)}(t))\,\mathrm{d}t + \varepsilon \int_0^T \left\|\overline{\partial}^2 \overline{\triangle}\left((\mathring{b}\cdot\nabla_{\mathring{\tilde{A}}})b\right)\right\|_0^2 + \left\|\overline{\partial}^2\overline{\triangle}\partial_t q\right\|_0^2 \,\mathrm{d}t
\end{aligned}
\tag{5.2.139}
$$

Step 5: Elliptic estimates of q

The control of $\|\partial_t^{5-k} q\|_k$ is the same as Section 5.2.1.4 so we omit the proof. However, we still need to control $\|\nabla_{\mathring{\tilde{A}}} q\|_4$. By Lemma 3.3.3, we have

$$
\|\nabla_{\mathring{\tilde{A}}} q\|_4 \lesssim P(\|\mathring{\tilde{\eta}}\|_4)(\|\triangle_{\mathring{\tilde{A}}} q\|_3 + \|\overline{\partial}\mathring{\tilde{\eta}}\|_4 \|q\|_4) \lesssim P(\|\mathring{\tilde{\eta}}\|_4)\|\triangle_{\mathring{\tilde{A}}} q\|_3 + \frac{1}{\kappa} P(\mathfrak{e}^{(n+1)}(T)). \tag{5.2.140}
$$

Taking $\mathrm{div}_{\mathring{\tilde{A}}}$ in the second equation of (5.2.115), we get the wave equation of q

$$
\begin{aligned}
&\frac{\mathring{\tilde{J}} R'(\mathring{q})}{\rho_0} \partial_t^2 q - \triangle_{\mathring{\tilde{A}}} q \\
=& b \cdot \triangle_{\mathring{\tilde{A}}} b + R \partial_t \mathring{\tilde{A}}^{\mu\alpha} \partial_\mu v_\alpha - \left[\mathrm{div}_{\mathring{\tilde{A}}}, (\mathring{b}\cdot\nabla_{\mathring{\tilde{A}}})\right] b + |\nabla_{\mathring{\tilde{A}}} b|^2 \\
&+ \frac{\mathring{\tilde{J}} R'(\mathring{q})}{\rho_0}\left((\nabla_{\mathring{\tilde{A}}} Q - (\mathring{b}\cdot\nabla_{\mathring{\tilde{A}}})b)\cdot\nabla_{\mathring{\tilde{A}}} q\right) + \left(\mathring{\tilde{J}} \frac{R'(\mathring{q})}{\rho_0} - \mathring{\tilde{J}} R''(q)\rho_0\right)(\partial_t q)^2 \\
=:& b \cdot \triangle_{\mathring{\tilde{A}}} b + w_{00}.
\end{aligned}
\tag{5.2.141}
$$

So $\|\triangle_{\mathring{\tilde{A}}} q\|_3$ can be reduced to $\|b\cdot\triangle_{\mathring{\tilde{A}}} b\|_3 + \|w_{00}\|_3$. Then $\|\triangle_{\mathring{\tilde{A}}} b\|_3$ can again be reduced to the terms with no more than 4 derivatives by the heat equation

$$
\partial_t b = \triangle_{\mathring{\tilde{A}}} b = (\mathring{b}\cdot\nabla_{\mathring{\tilde{A}}})v - \mathring{b}\,\mathrm{div}_{\mathring{\tilde{A}}} v. \tag{5.2.142}
$$

Therefore we are able to reduce $\|\nabla_{\mathring{\tilde{A}}} q\|_4$ to the finished estimates by sacrifising a $1/\kappa$ with the help of mollifier.

Combining (5.2.140) with (5.2.100)-(5.2.102) (but replacing $\tilde{A}$ by $\mathring{\tilde{A}}$), we get

$$\left\|\nabla_{\mathring{\tilde{A}}} q\right\|_4 + \sum_{k=2}^{4} \left\|\partial_t^{5-k} q\right\|_{k-1} \lesssim \left(1+\frac{1}{\kappa}\right)\left(P(\mathfrak{e}^{(n+1)}(T)) + \left\|\nabla_{\mathring{\tilde{A}}} \partial_t^4 q\right\|_0 + \left\|\partial_t^5 q\right\|_0\right) \tag{5.2.143}$$

Step 6: Common control of higher order heat and wave equation

We differentiate ∂_t^4 in (5.2.141) and (5.2.142) to get

$$\begin{aligned}
&\partial_t^5 b - \triangle_{\mathring{\tilde{A}}} \partial_t^4 b = \partial_t^4 \left((\mathring{b} \cdot \nabla_{\mathring{\tilde{A}}}) v - \mathring{b} \operatorname{div}_{\mathring{\tilde{A}}} v \right) + [\partial_t^4, \triangle_{\mathring{\tilde{A}}}] b \\
=&(\mathring{b} \cdot \nabla_{\mathring{\tilde{A}}}) \partial_t^4 v + b \frac{\mathring{\tilde{J}} R'(q)}{\rho_0} \partial_t^5 q + \left[\partial_t^4, \triangle_{\mathring{\tilde{A}}}\right] b + \left[\partial_t^4, \mathring{b} \cdot \nabla_{\mathring{\tilde{A}}}\right] v + \left[\partial_t^4, b \frac{\mathring{\tilde{J}} R'(q)}{\rho_0}\right] \partial_t q \\
=&: h_{55}
\end{aligned} \tag{5.2.144}$$

and

$$\begin{aligned}
&\frac{\mathring{\tilde{J}} R'(\mathring{q})}{\rho_0} \partial_t^6 q - \triangle_{\mathring{\tilde{A}}} \partial_t^4 q \\
&= b \cdot \triangle_{\mathring{\tilde{A}}} \partial_t^4 b + \partial_t^4 w_0 + \left[b \cdot \triangle_{\mathring{\tilde{A}}}, \partial_t^4 \right] + \left[\partial_t^4, \triangle_{\mathring{\tilde{A}}} \right] q + \left[\frac{\mathring{\tilde{J}} R'(\mathring{q})}{\rho_0}, \partial_t^4 \right] \partial_t^2 q.
\end{aligned}$$

Then plug the heat equation (5.2.144) $\triangle_{\mathring{\tilde{A}}} b = \partial_t^5 b - h_{55}$ to get

$$\begin{aligned}
&\frac{\mathring{\tilde{J}} R'(q)}{\rho_0} \partial_t^6 q - \triangle_{\mathring{\tilde{A}}} \partial_t^4 q \\
=& b \cdot \left(\partial_t^5 b - h_{55} \right) + \partial_t^4 w_0 + \left[b \cdot \triangle_{\mathring{\tilde{A}}}, \partial_t^4 \right] + \left[\partial_t^4, \triangle_{\mathring{\tilde{A}}} \right] q + \left[\frac{\mathring{\tilde{J}} R'(q)}{\rho_0}, \partial_t^4 \right] \partial_t^2 q \\
=:& w_{55}
\end{aligned} \tag{5.2.145}$$

Similarly as in Section 5.2.1.4, we are able to get a common control of the energy functional of these 2 equations. Define

$$\widetilde{W}^{(n+1)} := \left\|\partial_t^5 q\right\|_0^2 + \left\|\nabla_{\mathring{\tilde{A}}} \partial_t^4 q\right\|_0^2,$$

then we have the analogue of (5.2.110)

$$\begin{aligned}&\left(H^{(n+1)}(T)+\widetilde{W}^{(n+1)}(T)\right)-\left(H^{(n+1)}(0)+\widetilde{W}^{(n+1)}(0)\right)\\ \lesssim_{\varepsilon}&\left(H^{(n+1)}(T)+\widetilde{W}^{(n+1)}(T)\right)+\int_0^T H^{(n+1)}(t)+\widetilde{W}^{(n+1)}(t)+P(\mathfrak{e}^{(n+1)}(t))\,\mathrm{d}t\end{aligned} \tag{5.2.146}$$

Step 7: Finalizing the a priori estimates

Summing up (5.2.126), (5.2.127), (5.2.128), (5.2.129), (5.2.130), (5.2.131), (5.2.132), (5.2.139), (5.2.143) and (5.2.146), we get

$$\mathcal{E}^{(n+1)(T)}-\mathcal{E}^{(n+1)(0)}\lesssim_{1/\kappa}\varepsilon\mathcal{E}^{(n+1)(T)}+P(\mathfrak{e}^{(n+1)}(T))+\int_0^T P(\mathcal{E}^{(n+1)}(t))\,\mathrm{d}t.$$

By Gronwall inequality, we can find some $T_\kappa>0$ independent of n, such that

$$\sup_{0\le t\le T_\kappa}\mathcal{E}^{(n+1)}(t)\le P(\mathcal{E}^{(n+1)}(0))\lesssim\mathcal{P}_0.$$

This finalizes the proof of Proposition 5.2.5.

5.2.2.4 Well-posedness of the linearized approximation system

This part presents a fixed-point argument to solve the linearized system (5.2.115)

$$\begin{cases}\partial_t\eta=v+\mathring{\psi} & \text{in } \Omega,\\ \rho_0\mathring{\tilde{J}}^{-1}\partial_t v=(\mathring{b}\cdot\nabla_{\mathring{\tilde{A}}})b-\nabla_{\mathring{\tilde{A}}}Q,\ \ Q=q+\frac12|b|^2 & \text{in } \Omega,\\ \frac{\mathring{\tilde{J}}R'(\mathring{q})}{\rho_0}\partial_t q+\mathrm{div}_{\mathring{\tilde{A}}}v=0 & \text{in } \Omega,\\ \partial_t b+\mathrm{curl}_{\mathring{\tilde{A}}}\mathrm{curl}_{\mathring{\tilde{A}}}b=(\mathring{b}\cdot\nabla_{\mathring{\tilde{A}}})v-\mathring{b}\,\mathrm{div}_{\mathring{\tilde{A}}}v, & \text{in } \Omega,\\ \mathrm{div}_{\mathring{\tilde{A}}}b=0 & \text{in } \Omega,\\ q=0,\ b=\mathbf{0} & \text{on } \Gamma,\\ (\eta,v,b,q)|_{\{t=0\}}=(\mathrm{Id},v_0,b_0,q_0).\end{cases}$$

Define the norm $\|\cdot\|_{\mathbf{X}^r}$ by

$$\|f\|_{\mathbf{X}^r}^2:=\sum_{m=0}^r\sum_{k+l=m}\left\|\partial_t^k\partial^l f\right\|_0^2$$

and a Banach space on $[0,T]\times\Omega$

$$\mathbf{X}(M,T) := \left\{ (\xi, w, h, \pi) \,\middle|\, (\xi, w, h, \pi)\Big|_{t=0} = (\mathrm{Id}, v_0, b_0, q_0)\,,\, \|(\xi, w, h, \pi)\|_{\mathbf{X}} \le M \right\}$$

where

$$\|(\xi, w, h, \pi)\|_{\mathbf{X}}^2 := \left\| \left(\xi, \partial_t\xi, w, h, \nabla_{\tilde{\mathring{A}}} h, \pi, \partial_t\pi, \nabla_{\tilde{\mathring{A}}}\pi\right) \right\|_{L_t^\infty \mathbf{X}^4}^2 + \|\partial_t^5 h\|_{L_t^2 L_x^2}^2$$

Next we define the solution map

$$\Xi : \mathbf{X}(M,T) \to \mathbf{X}(M,T)$$

$$(\xi, w, h, \pi) \mapsto (\eta, v, b, q)$$

as follows:

1. Define η by $\partial_t\eta = w + \mathring{\psi}$ with $\eta(0) =$Id
2. Define v by $\rho_0 \tilde{\mathring{J}}^{-1} \partial_t v := (\mathring{b}\cdot\nabla_{\tilde{\mathring{A}}})h - \nabla_{\tilde{\mathring{A}}}(\pi + \frac{1}{2}|h|^2)$. with $v(0) = v_0$
3. Define b, q by the coupled system of heat equation and wave equation

$$\begin{cases} \partial_t b + \mathrm{curl}_{\tilde{\mathring{A}}}\mathrm{curl}_{\tilde{\mathring{A}}} b = (\mathring{b}\cdot\nabla_{\tilde{\mathring{A}}})v - \mathring{b}\,\mathrm{div}_{\tilde{\mathring{A}}} v \\ \mathrm{div}_{\tilde{\mathring{A}}} b = 0 \\ b|_\Gamma = \mathbf{0} \\ b(0) = b_0 \end{cases} \tag{5.2.147}$$

and

$$\begin{cases} R'(\mathring{q})\partial_t^2 q - \triangle_{\tilde{\mathring{A}}} q = \triangle_{\tilde{\mathring{A}}}\left(\frac{1}{2}|b|^2\right) - \left[\mathrm{div}_{\tilde{\mathring{A}}}, (\mathring{b}\cdot\nabla_{\tilde{\mathring{A}}})\right] b \\ \qquad\qquad + \rho_0 \tilde{\mathring{J}}^{-1} \partial_t \tilde{\mathring{A}}^{\mu\alpha} \partial_\mu v_\alpha + \tilde{\mathring{A}}^{\mu\alpha} \partial_\mu(\rho_0 \tilde{\mathring{J}}^{-1})\partial_t v_\alpha - \tilde{\mathring{J}}^{-1} \partial_t \left(\tilde{\mathring{J}} R'(\mathring{q})\right)\partial_t q \\ q|_\Gamma = 0, \\ (q(0), \partial_t q(0)) = (q_0, q_1). \end{cases} \tag{5.2.148}$$

We need to verify the following things to prove the existence and uniqueness of the system (5.2.115).

1. The image of $\mathbf{X}(M,T)$ under Ξ still lies in $\mathbf{X}(M,T)$.

2. Ξ is a contraction on $\mathbf{X}(M,T)$.

We first prove Ξ is a self-mapping of $\mathbf{X}(M,T)$. The velocity is directly controlled by

$$\rho_0 \mathring{\tilde{J}}^{-1} \partial_t v := (\mathring{b} \cdot \nabla_{\mathring{\tilde{A}}})h - \nabla_{\mathring{\tilde{A}}}(\pi + \frac{1}{2}|h|^2).$$

$$\begin{aligned} \|\partial_t^{4-k} v(T)\|_k^2 \lesssim & \|\partial_t^{4-k} v(0)\|_0^2 + \int_0^T \left\| \partial_t^{4-k} \left((\mathring{b} \cdot \nabla_{\mathring{\tilde{A}}})h - \nabla_{\mathring{\tilde{A}}}(\pi + \frac{1}{2}|h|^2) \right) \right\|_k^2 \\ \lesssim & \|\partial_t^{4-k} v(0)\|_0^2 + \int_0^T \left\| \nabla_{\mathring{\tilde{A}}} h \right\|_{\mathbf{X}^4}^2 + \left\| \nabla_{\mathring{\tilde{A}}} \pi \right\|_{\mathbf{X}^4}^2 \,\mathrm{d}t \end{aligned} \tag{5.2.149}$$

And thus the bound for $\|\partial_t \eta\|_{\mathbf{X}^4}$ and $\|\eta\|_{\mathbf{X}^4}$ directly follows.

Next we control $\|b\|_{\mathbf{X}^4}$ by elliptic estimates as in Section 5.2.1.2. For example

$$\|b\|_4 \approx \|\nabla_{\mathring{\tilde{A}}} b\|_3 \lesssim P(\|\mathring{\tilde{\eta}}\|_3) \left(\|\triangle_{\tilde{A}} b\|_2 + \|\overline{\partial} \mathring{\tilde{\eta}}\|_3 \|b\|_3 \right).$$

Then invoking $\triangle_{\mathring{\tilde{A}}} b = \partial_t b - (\mathring{b} \cdot \nabla_{\mathring{\tilde{A}}})v + \mathring{b}\,\mathrm{div}_{\mathring{\tilde{A}}} v$ to get

$$\|b\|_4 \lesssim P(\|\mathring{\tilde{\eta}}\|_3) \left((\|b\|_2 + \|\mathring{b}\|_2)\|v\|_3 + \|\overline{\partial} \mathring{\tilde{\eta}}\|_3 \|b\|_3 \right).$$

Combining the estimates of v above, we are able to write

$$\|b\|_4 \lesssim P(\|\mathring{\tilde{\eta}}\|_3)\|\overline{\partial} \mathring{\tilde{\eta}}\|_3 \|b\|_3 + \|v(0)\|_{\mathbf{X}^3} + \int_0^T \left\| \nabla_{\mathring{\tilde{A}}} h \right\|_{\mathbf{X}^3} + \left\| \nabla_{\mathring{\tilde{A}}} \pi \right\|_{\mathbf{X}^3} \,\mathrm{d}t$$

Then one can repeat the same steps for $\|b\|_3$ to get

$$\|b\|_4 \lesssim \mathcal{P}_0 + P(\|\mathring{\tilde{\eta}}\|_3) \int_0^T \left\| \nabla_{\mathring{\tilde{A}}} h \right\|_{\mathbf{X}^3} + \left\| \nabla_{\mathring{\tilde{A}}} \pi \right\|_{\mathbf{X}^3} \,\mathrm{d}t \tag{5.2.150}$$

Similar estimates hold for $\|\partial_t^{4-k} b\|_k$ for $1 \le k \le 4$, while $\|\partial_t^4 b\|_0^2$ is again reduced to $\int_0^T \|\partial_t^5 b\|_0^2 \,\mathrm{d}t$ as before.

One can mimic the proof above to estimate the space-time derivative of $\nabla_{\mathring{\tilde{A}}} b$ or $\partial_t b$. One exception is $\|\nabla_{\mathring{\tilde{A}}} b\|_4$, for which we have to use the mollifier property.

$$\|\nabla_{\mathring{\tilde{A}}} b\|_4 \lesssim P(\|\mathring{\eta}\|_4)\left(\|\triangle_{\mathring{\tilde{A}}} b\|_3 + \frac{1}{\kappa}\|\mathring{\eta}\|_4\|b\|_4\right).$$

Again, invoking the heat equation and the $\mathbf{X}^4$ estimates of v, we get

$$\|\nabla_{\mathring{\tilde{A}}} b\|_4 \lesssim \mathcal{P}_0 + P(\|\mathring{\eta}\|_4)\int_0^T \left\|\nabla_{\mathring{\tilde{A}}} h\right\|_{\mathbf{X}^4} + \left\|\nabla_{\mathring{\tilde{A}}} \pi\right\|_{\mathbf{X}^4} \,\mathrm{d}t$$

Similar estimates holds for the space-time derivatives except $\|\partial_t^5 b\|_{L_t^2 L_x^2}$ and $\|\nabla_{\mathring{\tilde{A}}} \partial_t^4 b\|_0$.

$$\sum_{k=1}^{4}\left\|\nabla_{\mathring{\tilde{A}}}\partial_t^{4-k} b\right\|_k^2 \lesssim \mathcal{P}_0 + P(\|\mathring{\eta}\|_4)\int_0^T P\left(\left\|\nabla_{\mathring{\tilde{A}}} h\right\|_{\mathbf{X}^4}, \left\|\nabla_{\mathring{\tilde{A}}} \pi\right\|_{\mathbf{X}^4}\right) \mathrm{d}t. \tag{5.2.151}$$

Analogously, we can apply the elliptic estimates and wave equation to q in order to reduce the estimates to the full time derivatives. For example

$$\|q\|_4 \approx \|\nabla_{\mathring{\tilde{A}}} q\|_3 \lesssim P(\|\mathring{\tilde{\eta}}\|_3)\left(\|\triangle_{\mathring{\tilde{A}}} q\|_2 + \|\overline{\partial}\mathring{\tilde{\eta}}\|_3\|q\|_3\right)$$

Invoking the wave equation and heat equation

$$\triangle_{\tilde{A}} q = \partial_t^2 q - \triangle_{\tilde{A}}(1/2|b|^2) + \cdots = \partial_t^2 q - \partial_t b - (\mathring{b}\cdot\nabla_{\mathring{\tilde{A}}})v + \mathring{b}\,\mathrm{div}_{\mathring{\tilde{A}}} v + \cdots,$$

we are able to reduce $\|\triangle_{\mathring{\tilde{A}}} q\|_2$ to $\|\partial_t^2 q\|_2$ plus the terms with ≤ 3 derivatives. Repeat the steps above, we are able to reduce $\|q\|_{\mathbf{X}^4}$ to $\|\partial_t^4 q\|_0$ and $\|\partial_t^3 q\|_1$. Similarly,

$$\|\nabla_{\mathring{\tilde{A}}} q\|_4 \lesssim P(\|\mathring{\tilde{\eta}}\|_4)\left(\|\triangle_{\mathring{\tilde{A}}} q\|_3 + \kappa^{-1}\|\mathring{\eta}\|_4\|q\|_4\right)$$

Therefore, the control of $\|\nabla_{\mathring{\tilde{A}}} q\|_{\mathbf{X}^4}$ and $\|\partial_t q\|_{\mathbf{X}^4}$ are reduced to $\|\partial_t^5 q\|_0$ and $\|\nabla_{\mathring{\tilde{A}}}\partial_t^4 q\|_0$.

The final step is to seek for a common control of 4-th order time-differetiated heat and wave equations. The proof is the same as in Section 5.2.1.4 and step 6 in Section 5.2.2.3. The only thing we would like to remark here is that there are terms like $\partial_t^5 v$ and $\partial\partial_t^4 v$ appearing in the time integral of the source term. In this case, we can invoke the equation of v to eliminate one time derivative and reduce to the $\mathbf{X}^4$ norm of $\nabla_{\tilde{A}}\pi$ and $(\mathring{b} \cdot \nabla_{\mathring{\tilde{A}}})h$.

$$\int_0^T \int_\Omega \left|\partial_t^5 b\right|^2 \,\mathrm{d}y\,\mathrm{d}t + \left(\left\|\nabla_{\mathring{\tilde{A}}}\partial_t^4 b\right\|_0^2 + \left\|\partial_t^5 q\right\|_0^2 + \left\|\nabla_{\mathring{\tilde{A}}}\partial_t^4 q\right\|_0^2\right)\Big|_0^T$$

$$\lesssim \varepsilon \int_0^T \int_\Omega \left|\partial_t^5 b\right|^2 \,\mathrm{d}y\,\mathrm{d}t + \int_0^T P\left(\left\|\partial_t^5 b\right\|_{L_t^2 L_x^2}, \left\|\left(v, \nabla_{\mathring{\tilde{A}}} b, b, \partial_t q, q, \nabla_{\mathring{\tilde{A}}} q\right)\right\|_{\mathbf{X}^4}\right)\mathrm{d}t \tag{5.2.152}$$

$$+ \mathcal{P}_0 + \int_0^T P\left(\left\|\nabla_{\mathring{\tilde{A}}} h\right\|_{\mathbf{X}^4}, \left\|\nabla_{\mathring{\tilde{A}}}\pi\right\|_{\mathbf{X}^4}, \left\|\partial_t \pi\right\|_{\mathbf{X}^4}\right)\mathrm{d}t$$

By choosing $\varepsilon > 0$ sufficiently small, we can absorb the ε-term to LHS.

Summarizing these steps above, we find that, there exists some $T_\kappa > 0$ sufficiently small and M chosen suitably large, such that

$$\left\|\left(\eta, \partial_t\eta, b, \nabla_{\mathring{\tilde{A}}} b, q, \partial_t q, \nabla_{\mathring{\tilde{A}}} q\right)\right\|_{\mathbf{X}^4} < \infty. \tag{5.2.153}$$

Next we prove Ξ is a contraction. Pick any $(\xi_i, w_i, h_i, \pi_i) \mapsto (\eta_i, v_i, b_i, q_i)$ and define $[f] := f_1 - f_2$. Then by the linearity of the equations above, we know $([\eta], [v], [b], [q])$ satisfies the same equation with (ξ, w, h, π) replaced by $([\xi], [w], [h], [\pi])$ and zero initial data. Thus $([\eta], [v], [b], [q])$ satisfies

$$\|([\eta], [v], [b], [q])\|_{\mathbf{X}} \lesssim_{\kappa^{-1}} \int_0^T P\left(\|([\xi], [w], [h], [\pi])\|_{\mathbf{X}}\right)\mathrm{d}t.$$

Choosing a suitably small $T_\kappa > 0$ such that

$$\|([\eta], [v], [b], [q])\|_{\mathbf{X}} \le \frac{1}{2}\|([\xi], [w], [h], [\pi])\|_{\mathbf{X}},$$

we know Ξ is indeed a contraction. By Contraction Mapping Theorem, Ξ has a unique fixed point (η, v, b, q), and thus the local existence and uniqueness of the solution to the linearized equation (5.2.115) is established.

5.2.2.5 Iteration to the nonlinear approximation system

For each n, we have already established the local existence and uniqueness of solution $(\eta^{(n+1)}, v^{(n+1)}, b^{(n+1)}, q^{(n+1)})$ to the n-th linearized approximation system

$$\begin{cases} \partial_t \eta^{(n+1)} = v^{(n+1)} + \psi^{(n)} & \text{in } \Omega, \\ \frac{\rho_0}{\tilde{J}^{(n)}} \partial_t v^{(n+1)} = (b^{(n)} \cdot \nabla_{a^{(n)}}) b^{(n+1)} - \nabla_{\tilde{A}^{(n)}} Q^{(n+1)}, & \text{in } \Omega, \\ \frac{\tilde{J}^{(n)} R'(q^{(n)})}{\rho_0} \partial_t q^{(n+1)} + \text{div}_{\tilde{A}^{(n)}} v^{(n+1)} = 0 & \text{in } \Omega, \\ (\partial_t + \text{curl}_{\tilde{A}^{(n)}} \text{curl}_{\tilde{A}^{(n)}}) b^{(n+1)} = (b^{(n)} \cdot \nabla_{a^{(n)}}) v^{(n+1)} - b^{(n)} \text{div}_{\tilde{A}^{(n)}} v^{(n+1)}, & \text{in } \Omega, \\ \text{div}_{\tilde{A}^{(n)}} b^{(n+1)} = 0 & \text{in } \Omega, \\ q^{(n+1)} = 0, \ b^{(n+1)} = \mathbf{0} & \text{on } \Gamma, \\ (\eta^{(n+1)}, v^{(n+1)}, b^{(n+1)}, q^{(n+1)})|_{\{t=0\}} = (\text{Id}, v_0, b_0, q_0). \end{cases}$$

This part shows the Picard-type iteration of the sequence $\{(\eta^{(n)}, v^{(n)}, b^{(n)}, q^{(n)})\}_{n\in\mathbb{N}}$ which gives a subsequential limit (η, v, b, q) converging in H^3-norm. Such limit (η, v, b, q) exactly solves the nonlinear κ-approximation problem (5.2.1).

Define $[\eta]^{(n)} := \eta^{(n+1)} - \eta^{(n)}$, $[v]^{(n)} := v^{(n+1)} - v^{(n)}$, $[b]^{(n)} := b^{(n+1)} - b^{(n)}$, $[q]^{(n)} := q^{(n+1)} - q^{(n)}$, and $[a]^{(n)} := a^{(n)} - a^{(n-1)}$, $[A]^{(n)} := A^{(n)} - A^{(n-1)}$, $[\psi]^{(n)} := \psi^{(n)} - \psi^{(n-1)}$. Then these quantities satisfy the following system consisting of:

The equation of momentum

$$\begin{aligned} \rho_0 \partial_t [v]^{(n)} = &\left(b^{(n)} \cdot \nabla_{\tilde{\mathbf{A}}^{(n)}}\right) [b]^{(n)} - \nabla_{\tilde{\mathbf{A}}^{(n)}} [Q]^{(n)} \\ &+ b^{(n)} \cdot \nabla_{[\tilde{\mathbf{A}}]^{(n)}} b^{(n)} + [b]^{(n-1)} \cdot \nabla_{\mathring{\tilde{\mathbf{A}}}^{(n-1)}} b^{(n)} - \nabla_{\tilde{\mathbf{A}}^{(n)}} Q^{(n)}. \end{aligned} \tag{5.2.154}$$

Continuity equation:

$$r^{(n)} \partial_t [q]^{(n)} + \text{div}_{\tilde{A}^{(n)}} [v]^{(n)} = -\text{div}_{[\tilde{A}]^{(n)}} v^{(n)} + [r]^{(n)} \partial_t q^{(n)}, \tag{5.2.155}$$

here $r^{(n)} := \mathring{\tilde{J}}^{(n)} R'(q^{(n)})/\rho_0$.

Equation of magnetic field:

$$
\begin{aligned}
\partial_t [b]^{(n)} - \triangle_{\tilde{A}^{(n)}} [b]^{(n)} =& (b^{(n)} \cdot \nabla_{\tilde{A}^{(n)}})[v]^{(n)} - b^{(n)} \mathrm{div}_{\tilde{A}^{(n)}} [v]^{(n)} \\
&+ b^{(n)} \cdot \nabla_{[\tilde{A}]^{(n)}} v^{(n)} - b^{(n)} \mathrm{div}_{[\tilde{A}]^{(n)}} v^{(n)} \\
&+ [b]^{(n-1)} \cdot \nabla_{\overset{\circ}{\tilde{A}}{}^{(n-1)}} v^{(n)} - [b]^{(n-1)} \mathrm{div}_{\overset{\circ}{\tilde{A}}{}^{(n-1)}} v^{(n)} \\
&+ \mathrm{div}_{\tilde{A}^{(n)}} \left(\nabla_{[\tilde{A}]^{(n)}} b^{(n)} \right) + \mathrm{div}_{[\tilde{A}]^{(n)}} \left(\nabla_{\overset{\circ}{\tilde{A}}{}^{(n-1)}} b^{(n)} \right).
\end{aligned}
\tag{5.2.156}
$$

Divergence-free condition for b:

$$
\mathrm{div}_{\tilde{A}^{(n)}} [b]^{(n)} = -\mathrm{div}_{[\tilde{A}]^{(n)}} b^{(n)}. \tag{5.2.157}
$$

The initial data of $([\eta], [v], [b], [q]) = (\mathbf{0}, \mathbf{0}, \mathbf{0}, 0)$. The boundary conditions are

$$
[b]^{(n)} = \mathbf{0}, [q]^{(n)} = 0. \tag{5.2.158}
$$

Define the energy functional

$$
[\mathcal{E}]^{(n)}(T) := [\mathfrak{e}]^{(n)}(T) + [H]^{(n)}(T) + [W]^{(n)}(T) + \sum_{k=0}^{3} \left\| \partial_t^{3-k} \nabla_{\tilde{A}^{(n)}} [b]^{(n)} \right\|_k^2, \tag{5.2.159}
$$

where

$$
[\mathfrak{e}]^{(n)}(T) := \sum_{k=0}^{3} \left(\left\| \partial_t^{3-k} [v]^{(n)} \right\|_k^2 + \left\| \partial_t^{3-k} [b]^{(n)} \right\|_k^2 + \left\| \partial_t^{3-k} [q]^{(n)} \right\|_k^2 \right), \tag{5.2.160}
$$

$$
[H]^{(n)}(T) := \int_0^T \left\| \partial_t^4 [b]^n \right\|_0^2 \, \mathrm{d}t + \left\| \partial_t^3 \nabla_{\tilde{A}^{(n)}} [b]^{(n)} \right\|_0^2, \tag{5.2.161}
$$

$$
[W]^{(n)}(T) := \left\| \partial_t^4 [q]^{(n)} \right\|_0^2 + \left\| \partial_t^3 \nabla_{\tilde{A}^{(n)}} [q]^{(n)} \right\|_0^2. \tag{5.2.162}
$$

The conclusion is

Proposition 5.2.8. For n sufficiently large and $T_\kappa > 0$ suitably small, we have that $\forall T \in [0, T_\kappa]$

$$[\mathfrak{e}]^{(n)}(T) \leq \frac{1}{4}\left([E]^{(n-1)}(T) + [E]^{(n-2)}(T)\right).$$

□

By Proposition 5.2.8, we know $[\mathfrak{e}]^{(n)} \leq \frac{1}{2^n} P_\kappa(\mathcal{P}_0)$, and thus yields the limit for each fixed $\kappa > 0$:

$$\left(\eta^{(n)}, v^{(n)}, b^{(n)}, q^{(n)}\right) \xrightarrow{\text{converge strongly}} (\eta(\kappa), v(\kappa), b(\kappa), q(\kappa)) \quad as\ n \to \infty.$$

Such limit exactly solves the nonlinear approximation system 5.2.1.

Corollary 5.2.9. The limit $(\eta(\kappa), v(\kappa), b(\kappa), q(\kappa))$ gotten in Proposition 5.2.8 is the unique strong solution to the nonlinear approximation system (5.2.1) and satisfies the energy estimates in $[0, T_\kappa]$

$$\sup_{0 \leq T \leq T_\kappa} \widetilde{\mathcal{E}}_\kappa(T) \leq 2\left(\|v_0\|_4^2 + \|b_0\|_5^2 + \|q_0\|_4^2\right),$$

where

$$\widetilde{\mathcal{E}}_\kappa(T) := \widetilde{\mathfrak{e}}_\kappa(T) + \widetilde{H}_\kappa(T) + \widetilde{W}_\kappa(T) + \sum_{k=0}^{4}\left\|\partial_t^{4-k}\left((b(\kappa)\cdot\nabla_{\tilde{A}})b(\kappa)\right)\right\|_k^2, \tag{5.2.163}$$

and

$$\widetilde{\mathfrak{e}}_\kappa(T) := \|\eta\|_4^2 + \sum_{k=0}^{4}\left\|\partial_t^{4-k} v(\kappa)\right\|_k^2 + \sum_{k=0}^{4}\left\|\partial_t^{4-k} b(\kappa)\right\|_k^2 + \sum_{k=0}^{4}\left\|\partial_t^{4-k} q(\kappa)\right\|_k^2 \tag{5.2.164}$$

$$\widetilde{H}_\kappa(T) := \int_0^T \int_\Omega \left|\partial_t^5 b(\kappa)\right|^2 \,\mathrm{d}y\,\mathrm{d}t + \left\|\partial_t^4 b(\kappa)\right\|_1^2 \tag{5.2.165}$$

$$\widetilde{W}_\kappa(T) := \sum_{k=0}^{4}\left\|\nabla_{\tilde{A}}\partial_t^{4-k} q(\kappa)\right\|_k^2 + \left\|\partial_t^5 q(\kappa)\right\|_0^2. \tag{5.2.166}$$

□

The proof process is nearly the same as in the a priori estimates part, so we do not write all details here but still state the main steps.

Step 1: Correction Terms

First we estimate the coefficients and correction terms.

$[\psi]^{(n)}$ satisfies $-\triangle[\psi]^{(n)} = 0$ with the boundary condition

$$[\psi]^{(n)} = \overline{\triangle}^{-1}\mathbb{P}_{\neq 0}\bigg(\overline{\triangle}[\eta]_\beta^{(n-1)}\tilde{A}^{(n)i\beta}\overline{\partial}_i\Lambda_\kappa^2 v^{(n)} + \overline{\partial}\eta_\beta^{(n-1)}[\tilde{A}]^{(n)i\beta}\overline{\partial}_i\Lambda_\kappa^2 v^{(n)}$$

$$+ \overline{\partial}\eta_\beta^{(n-1)}\tilde{A}^{(n-1)i\beta}\overline{\partial}_i\Lambda_\kappa^2[v]^{(n-1)} - \overline{\triangle}\Lambda_\kappa^2[\eta]_\beta^{(n-1)}\tilde{A}^{(n)i\beta}\overline{\partial}_i v^{(n)}$$

$$- \overline{\triangle}\Lambda_\kappa^2\eta_\beta^{(n-1)}[\tilde{A}]^{(n)i\beta}\overline{\partial}_i v^{(n)} - \overline{\triangle}\Lambda_\kappa^2\eta_\beta^{(n-1)}\tilde{A}^{(n-1)i\beta}\overline{\partial}_i[v]^{(n-1)}\bigg).$$

By the standard elliptic estimates, we have the control for $[\psi]^{(n)}$

$$\|[\psi]^{(n)}\|_3^2 \lesssim |[\psi]^{(n)}|_{2.5} \lesssim \mathcal{P}_0\left(\|[\eta]^{(n-1)}\|_3^2 + \|[v]^{(n-1)}\|_2^2 + \|[\tilde{A}]^{(n)}\|_1^2\right). \tag{5.2.167}$$

On the other hand, we have

$$[a]^{(n)\mu\nu}(T) = \int_0^T \partial_t(a^{(n)\mu\nu} - a^{(n-1)\mu\nu})\,\mathrm{d}t$$

$$= -\int_0^T [a]^{(n)\mu\gamma}\partial_\beta\partial_t\eta_\gamma^{(n)}a^{(n)\beta\nu} + a^{(n-1)\mu\gamma}\partial_\beta\partial_t[\eta]_\gamma^{(n-1)}a^{(n)\beta\nu} + a^{(n-1)\mu\gamma}\partial_\beta\partial_t\eta_\gamma^{(n-1)}[a]^{(n)\beta\nu},$$

which gives

$$\|[a]^{(n)}(T)\|_2 \lesssim \mathcal{P}_0\int_0^T \|[a]^{(n)}(t)\|_2^2(\|[v]^{(n-1)}\|_3 + \|[\psi]^{(n-1)}\|_3))\,\mathrm{d}t. \tag{5.2.168}$$

Therefore we get

$$\sup_{[0,T]}\|[a]^{(n)}\|_2^2 \lesssim \mathcal{P}_0 T^2\left(\|[a]^{(n)},[a]^{(n-1)}\|_{L_t^\infty H^2} + \|[v]^{(n-1)},[v]^{(n-2)},[\eta]^{(n-2)}\|_{L_t^\infty H^3}^2\right), \tag{5.2.169}$$

and the bound for $[\eta]$ via $\partial_t[\eta]^{(n)} = [v]^{(n)} + [\psi]^{(n)}$:

$$\sup_{[0,T]}\|[\eta]^{(n)}\|_3^2 \lesssim \mathcal{P}_0 T^2\left(\|[a]^{(n)}\|_{L_t^\infty H^2}^2 + \|[v]^{(n)},[v]^{(n-1)},[\eta]^{(n-1)}\|_{L_t^\infty H^3}^2\right) \tag{5.2.170}$$

Similar as in Lemma 5.2.4, we control the time derivatives of $[\eta]$ and $[\psi]$

$$\|[\partial_t \psi]^{(n)}\|_3^2 \lesssim \mathcal{P}_0 \left(\|[a]^{(n)}\|_2^2 + \|[\partial_t v]^{(n-1)}\|_2^2 + \|[v]^{(n-1)}, [\eta]^{(n-1)}\|_3^2\right) \tag{5.2.171}$$

$$\|[\partial_t^2 \psi]^{(n)}\|_2^2 \lesssim \mathcal{P}_0 \left(\|[a]^{(n)}\|_2^2 + \|[\partial_t^2 v]^{(n-1)}\|_1^2 + \|[\partial_t v]^{(n-1)}\|_2^2 + \|[v]^{(n-1)}, [\eta]^{(n-1)}\|_3^2\right) \tag{5.2.172}$$

$$\|[\partial_t^3 \psi]^{(n)}\|_1^2 \lesssim \mathcal{P}_0 \big(\|[a]^{(n)}\|_2^2 + \|[\partial_t^2 v]^{(n-1)}\|_1^2 + \|[\partial_t^3 v]^{(n-1)}\|_0^2$$

$$+ \|[\partial_t v]^{(n-1)}\|_2^2 + \|[v]^{(n-1)}, [\eta]^{(n-1)}\|_3^2\big) \tag{5.2.173}$$

$$\|[\partial_t \eta]^{(n)}\|_3^2 \lesssim \mathcal{P}_0 T^2 \left(\|[a]^{(n)}, [\partial_t v]^{(n)}, [\partial_t v]^{(n-1)}\|_{L_t^\infty H^2}^2 + \|[v]^{(n)}, [v]^{(n-1)}, [\eta]^{(n-1)}\|_{L_t^\infty H^3}^2\right) \tag{5.2.174}$$

$$\|[\partial_t^2 \eta]^{(n)}\|_2^2 \lesssim \mathcal{P}_0 T^2 \big(\|[\partial_t^2 v]^{(n),(n-1)}\|_{L_t^\infty H^1}^2 + \|[a]^{(n)}, [\partial_t v]^{(n),(n-1)}\|_{L_t^\infty H^2}$$

$$+ \|[v]^{(n),(n-1)}, [\eta]^{(n-1)}\|_{L_t^\infty H^3}^2\big) \tag{5.2.175}$$

$$\|[\partial_t^3 \eta]^{(n)}\|_1^2 \lesssim \mathcal{P}_0 \big(\|[\partial_t^2 v]^{(n),(n-1)}\|_{L_t^\infty H^1}^2 + \|[a]^{(n)}, [\partial_t v]^{(n),(n-1)}\|_{L_t^\infty H^2}^2$$

$$+ \|[v]^{(n),(n-1)}, [\eta]^{(n-1)}\|_{L_t^\infty H^3}^2\big). \tag{5.2.176}$$

$$\|[\partial_t^4 \eta]^{(n)}\|_0^2 \lesssim \mathcal{P}_0 \Bigg(\|[\partial_t^3 v]^{(n,n-1)}\|_{L_t^\infty L_x^2}^2 + \|[\partial_t^2 v]^{(n,n-1)}\|_{L_t^\infty H^1}^2 + \|[a]^{(n)}, [\partial_t v]^{(n,n-1)}\|_{L_t^\infty H^2}^2 \tag{5.2.177}$$

$$+ \|[v]^{(n,n-1)}, [\eta]^{(n-1)}\|_{L_t^\infty H^3}^2 \Bigg). \tag{5.2.178}$$

Step 2: Magnetic field and Lorentz force

The first step is still the elliptic estimates of $[b]^{(n)}$. We show an example of $\|[b]^{(n)}\|_3$:

$$\|[b]^{(n)}\|_3 \lesssim P(\|\mathring{\tilde{\eta}}^{(n)}\|_2) \left(\|\triangle_{\tilde{A}^{(n)}} [b]^{(n)}\|_1 + P(\|\overline{\partial}\mathring{\tilde{\eta}}\|_2)\|[b]^{(n)}\|_2\right).$$

One can still use the heat equation (5.2.156) to eliminate the Laplacian terms, but now we have two more higher order terms when "[·]" falls on $\text{div}_{\mathring{\tilde{A}}}$ or $\mathring{\tilde{A}}$. Such terms can be controlled directly by $\mathcal{E}^{(n-1,n,n+1)}$ and thus by $\mathcal{P}_0$. In specific, such terms are

$$\left\|\text{div}_{\tilde{A}^{(n)}}\left(\nabla_{[\tilde{A}]^{(n)}} b^{(n)}\right)\right\|_1 + \left\|\text{div}_{[\tilde{A}]^{(n)}}\left(\nabla_{\mathring{\tilde{A}}^{(n-1)}} b^{(n)}\right)\right\|_1.$$

The leading order part in these two terms can be written as $[\tilde{A}]^{(n)}$ times the top order derivatives (4-th order) of $b^{(n)}$ which has been controlled uniformly in n in Proposition 5.2.5. For example,

$$\left\|\text{div}_{\tilde{A}^{(n)}}\left(\nabla_{[\tilde{A}]^{(n)}} b^{(n)}\right)\right\|_1 \lesssim \left\|[\tilde{A}]^{(n)}\right\|_2 \left\|b^{(n)}\right\|_4 \times \cdots \lesssim \mathcal{P}_0 \left\|[\tilde{A}]^{(n)}\right\|_2$$

Therefore, the control of $[b]^{(n)}$ can be controlled in the same manner as before. Similar estimates hold for $\partial_t[b]^{(n)}$. The control of $\|\partial_t^2[b]\|_1$ and $\|\partial_t^3[b]\|_0$ is reduced to the estimates of heat equation (5.2.156). The proof is the same as Section 5.2.1.2 so we omit it here.

The Lorentz force is controlled in a silimar way. For example,

$$\left\|\nabla_{\tilde{A}^{(n)}}[b]^{(n)}\right\|_3 \lesssim P\left(\|\mathring{\tilde{\eta}}^{(n)}\|_3\right)\left(\left\|\triangle_{\tilde{A}^{(n)}}[b]^{(n)}\right\|_2 + \left\|\overline{\partial}\mathring{\tilde{\eta}}^{(n)}\right\|_3 \left\|[b]^{(n)}\right\|_3\right)$$

We again use the heat equation (5.2.156) to eliminate the Laplacian term, and the extra terms can be controlled in the same way as above. (Note that $\|\nabla_{\mathring{\tilde{A}}} b\|_4$ is controlled in Proposition 5.2.5). Therefore,

$$\left\|\nabla_{\tilde{A}^{(n)}}[b]^{(n)}\right\|_3 \lesssim \kappa^{-1}\mathcal{P}_0 \left\|[\tilde{A}]^{(n)}\right\|_2.$$

Similar estimates hold for the time derivatives of Lorentz force.

Step 3: Div-Curl estimates

The control of $[v]^{(n)}$ and $[q]^{(n)}$ also follows the same way as Section 5.2.1.3. The equation of curl $_{\tilde{A}^{(n)}}[v]^{(n)}$ is

$$\rho_0\partial_t \text{curl}_{\tilde{\mathbf{A}}^{(n)}}[v]^{(n)} = \text{curl}_{\tilde{\mathbf{A}}^{(n)}}\left((b^{(n)}\cdot\nabla_{\tilde{\mathbf{A}}^{(n)}})[b]^{(n)}\right) + \left[\rho_0\partial_t, \text{curl}_{\tilde{\mathbf{A}}^{(n)}}\right][v]^{(n)} + \text{curl}_{\tilde{\mathbf{A}}^{(n)}}\left([b]^{(n-1)}\cdot\nabla_{\mathring{\tilde{\mathbf{A}}}^{(n-1)}}b^{(n)} + b^{(n-1)}\cdot\nabla_{[\tilde{\mathbf{A}}]^{(n)}}b^{(n)} - \nabla_{[\tilde{\mathbf{A}}]^{(n)}}Q^{(n)}\right) \tag{5.2.179}$$

The first two terms in the second line is controlled in the same way as before(just consider curl $_{\tilde{\mathbf{A}}^{(n)}}$ as the covariant derivative $\nabla_{\tilde{\mathbf{A}}^{(n)}}$. Also

$$\left\|\text{curl}_{\tilde{\mathbf{A}}^{(n)}}\left(\nabla_{[\tilde{\mathbf{A}}]^{(n)}}Q^{(n)}\right)\right\|_2 \lesssim \|[a]^{(n)}\|_2\|Q^{(n)}\|_4\mathcal{P}_0 \lesssim \|[a]^{(n)}\|_2\mathcal{P}_0.$$

Therefore,

$$\|\text{curl }[v]^{(n)}\|_2^2 \lesssim \varepsilon\|[v]^{(n)}\|_3^2 + P_\kappa(\mathcal{P}_0)T^2\sup_{[0,T]}[\mathcal{E}]^{(n),(n-1)}(t).$$

And similarly

$$\|\text{curl }[\partial_t v]^{(n)}\|_1^2 \lesssim \varepsilon\|[\partial_t v]^{(n)}\|_2^2 + P_\kappa(\mathcal{P}_0)T^2\sup_{[0,T]}[\mathcal{E}]^{(n),(n-1)}(t),$$

$$\|\text{curl }[\partial_t^2 v]^{(n)}\|_0^2 \lesssim \varepsilon\|[\partial_t^2 v]^{(n)}\|_1^2 + P_\kappa(\mathcal{P}_0)T^2\sup_{[0,T]}[\mathcal{E}]^{(n),(n-1)}(t).$$

Invoking (5.2.155), we are still able to reduce that control to $\partial_t^3 v$ and $\partial_t^3 q$.

Step 4: Space-time tangential estimates

Let $\mathfrak{D}^3 = \overline{\partial}^2\partial_t, \overline{\partial}\partial_t^2, \partial_t^3$. Follwing Section 5.2.1.3, we derive the estimates

$$\sum_{k=1}^{3}\left\|\overline{\partial}^{3-k}\partial_t^k[v]^{(n)}\right\|_0^2 + \left\|\overline{\partial}^{3-k}\partial_t^k[q]^{(n)}\right\|_0^2 \lesssim \int_0^T P([\mathcal{E}]^{(n),(n-1),(n-2)}(t))\,\mathrm{d}t. \tag{5.2.180}$$

Step 5: Spatial tangential estimates

We adopt the same method as in Section 5.2.1.3. For each n, we define the Alinhac good unknowns by

$$\mathbf{V}^{(n+1)} = \overline{\partial}^3 v^{(n+1)} - \overline{\partial}^3 \tilde{\eta}^{(n)} \cdot \nabla_{\tilde{A}^{(n)}} v^{(n+1)}, \quad \mathbf{Q}^{(n+1)} = \overline{\partial}^3 Q^{(n+1)} - \overline{\partial}^3 \tilde{\eta}^{(n)} \cdot \nabla_{\tilde{A}^{(n)}} Q^{(n+1)}. \tag{5.2.181}$$

Their difference is denoted by

$$[\mathbf{V}]^{(n)} := \mathbf{V}^{(n+1)} - \mathbf{V}^{(n)}, [\mathbf{Q}]^{(n)} := \mathbf{Q}^{(n+1)} - \mathbf{Q}^{(n)}.$$

Similarly as in Section 5.2.1.3, we can derive the analogue of (5.2.61) as

$$\rho_0 \partial_t [\mathbf{V}]^{(n)} + \nabla_{\tilde{A}^{(n)}} [\mathbf{Q}]^{(n)} = -\nabla_{[\tilde{A}]^{(n)}} \mathbf{Q}^{(n)} + \mathbf{F}^{(n)}, \tag{5.2.182}$$

subject to the boundary condition

$$[\mathbf{Q}]^{(n)}|_{\Gamma} = -\left(\overline{\partial}^3 \tilde{\eta}_\beta^{(n)} \tilde{A}^{(n)3\beta} \partial_3 [Q]^{(n)} + \overline{\partial}^3 [\tilde{\eta}]_\beta^{(n-1)} \tilde{A}^{(n)3\beta} (\partial_N Q)^{(n)} + \overline{\partial}^3 \tilde{\eta}_\beta^{(n-1)} [\tilde{A}]^{(n)3\beta} (\partial_N Q)^{(n)}\right), \tag{5.2.183}$$

and

$$\nabla_{\tilde{A}^{(n)}} \cdot [\mathbf{V}]^{(n)} = -\nabla_{[\tilde{A}]^{(n)}} \cdot \mathbf{V}^{(n)} + \mathbf{G}^{(n)}, \tag{5.2.184}$$

where

$$
\begin{aligned}
\mathbf{F}^{(n)\alpha} &= [\rho_0, \overline{\partial}^3]\partial_t [v]^{(n)\alpha} + \overline{\partial}^3 \left((b^{(n)} \cdot \nabla_{\tilde{A}^{(n)}})[b]^{(n)} + (b^{(n)} \cdot \nabla_{[\tilde{A}]^{(n)}}) b^{(n)} + ([b]^{(n-1)} \cdot \nabla_{\tilde{A}^{(n-1)}}) b^{(n)} \right) \\
&+ \rho_0 \partial_t \left(\overline{\partial}^3 [\tilde{\eta}]_\beta^{(n-1)} \tilde{A}^{(n)\mu\beta} \partial_\mu v_\alpha^{(n+1)} + \overline{\partial}^3 \tilde{\eta}_\beta^{(n-1)} [\tilde{A}]^{(n)\mu\beta} \partial_\mu v_\alpha^{(n+1)} + \overline{\partial}^3 \tilde{\eta}_\beta^{(n-1)} \tilde{A}^{(n)\mu\beta} \partial_\mu [v]_\alpha^{(n)} \right) \\
&+ [\tilde{A}]^{(n)\mu\beta} \partial_\mu (\tilde{A}^{(n)\gamma\alpha} \partial_\gamma Q^{(n+1)}) \overline{\partial}^3 \tilde{\eta}_\beta^{(n)} + \tilde{A}^{(n-1)\mu\beta} \partial_\mu ([\tilde{A}]^{(n)\gamma\alpha} \partial_\gamma Q^{(n+1)}) \overline{\partial}^3 \tilde{\eta}_\beta^{(n)} \\
&+ \tilde{A}^{(n-1)\mu\beta} \partial_\mu (\tilde{A}^{(n-1)\gamma\alpha} \partial_\gamma [Q]^{(n)}) \overline{\partial}^3 \tilde{\eta}_\beta^{(n)} + \tilde{A}^{(n-1)\mu\beta} \partial_\mu ([\tilde{A}]^{(n)\gamma\alpha} \partial_\gamma h^{(n)}) \overline{\partial}^3 [\tilde{\eta}]_\beta^{(n-1)} \\
&- \left[\overline{\partial}^2, [\tilde{A}]^{(n)\mu\beta} \tilde{A}^{(n)\gamma\alpha} \overline{\partial} \right] \partial_\gamma \tilde{\eta}_\beta^{(n)} \partial_\mu Q^{(n+1)} - \left[\overline{\partial}^2, \tilde{A}^{(n-1)\mu\beta} [\tilde{A}]^{(n)\gamma\alpha} \overline{\partial} \right] \partial_\gamma \tilde{\eta}_\beta^{(n)} \partial_\mu Q^{(n+1)} \\
&- \left[\overline{\partial}^2, \tilde{A}^{(n-1)\mu\beta} \tilde{A}^{(n-1)\gamma\alpha} \overline{\partial} \right] \partial_\gamma [\tilde{\eta}]_\beta^{(n-1)} \partial_\mu Q^{(n+1)} - \left[\overline{\partial}^2, \tilde{A}^{(n-1)\mu\beta} \tilde{A}^{(n-1)\gamma\alpha} \overline{\partial} \right] \partial_\gamma \tilde{\eta}_\beta^{(n-1)} \partial_\mu [Q]^{(n)} \\
&- \left[\overline{\partial}^3, [\tilde{A}]^{(n)\mu\alpha}, \partial_\mu Q^{(n+1)} \right] - \left[\overline{\partial}^3, \tilde{A}^{(n-1)\mu\alpha}, \partial_\mu [Q]^{(n)} \right]
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{G}^{(n)} &= \overline{\partial}^3 (\operatorname{div}_{\tilde{A}^{(n)}} [v]^{(n)} - \operatorname{div}_{[\tilde{A}]^{(n)}} v^{(n)}) \\
&\quad - \left[\overline{\partial}^2, [\tilde{A}]^{(n)\mu\beta} \tilde{A}^{(n)\gamma\alpha} \overline{\partial} \right] \partial_\gamma \tilde{\eta}_\beta^{(n)} \partial_\mu v_\alpha^{(n+1)} - \left[\overline{\partial}^2, \tilde{A}^{(n-1)\mu\beta} [\tilde{A}]^{(n)\gamma\alpha} \overline{\partial} \right] \partial_\gamma \tilde{\eta}_\beta^{(n)} \partial_\mu v_\alpha^{(n+1)} \\
&\quad - \left[\overline{\partial}^2, \tilde{A}^{(n-1)\mu\beta} \tilde{A}^{(n-1)\gamma\alpha} \overline{\partial} \right] \partial_\gamma [\tilde{\eta}]_\beta^{(n-1)} \partial_\mu v_\alpha^{(n+1)} \\
&\quad - \left[\overline{\partial}^2, \tilde{A}^{(n-1)\mu\beta} \tilde{A}^{(n)\gamma\alpha} \overline{\partial} \right] \partial_\gamma \tilde{\eta}_\beta^{(n-1)} \partial_\mu [v]_\alpha^{(n)} \\
&\quad - \left[\overline{\partial}^3, [\tilde{A}]^{(n)\mu\alpha}, \partial_\mu v_\alpha^{(n+1)} \right] - \left[\overline{\partial}^3, \tilde{A}^{(n-1)\mu\alpha}, \partial_\mu [v]_\alpha^{(n)} \right] \\
&\quad + [\tilde{A}]^{(n)\mu\beta} \partial_\mu (\tilde{A}^{(n)\gamma\alpha} \partial_\gamma v_\alpha^{(n+1)}) \overline{\partial}^3 \tilde{\eta}_\beta^{(n)} + \tilde{A}^{(n-1)\mu\beta} \partial_\mu ([\tilde{A}]^{(n)\gamma\alpha} \partial_\gamma v_\alpha^{(n+1)}) \overline{\partial}^3 \tilde{\eta}_\beta^{(n)} \\
&\quad + \tilde{A}^{(n-1)\mu\beta} \partial_\mu (\tilde{A}^{(n-1)\gamma\alpha} \partial_\gamma [v]_\alpha^{(n)}) \overline{\partial}^3 \tilde{\eta}_\beta^{(n)} + \tilde{A}^{(n-1)\mu\beta} \partial_\mu ([\tilde{A}]^{(n)\gamma\alpha} \partial_\gamma v_\alpha^{(n)}) \overline{\partial}^3 [\tilde{\eta}]_\beta^{(n-1)}.
\end{aligned}
$$

Multiplying $[\mathbf{V}]^{(n)}$ in (5.2.182) and integrate by parts, we get

$$\frac{1}{2}\frac{d}{dt}\left\|\mathbf{V}^{(n)}\right\|_0^2 = \int_\Omega [\mathbf{Q}]^{(n)}\left(\nabla_{\tilde{A}^{(n)}}\cdot[\mathbf{V}]^{(n)} - \partial_\mu \tilde{A}^{\mu\alpha}[\mathbf{V}]^{(n)}_\alpha\right)\,\mathrm{d}y$$

$$+\int_\Omega (\mathbf{F}^{(n)} - \nabla_{[\tilde{A}]^{(n)}}\mathbf{Q}^{(n)})\cdot[\mathbf{V}]^{(n)}\,\mathrm{d}y$$

$$-\int_\Gamma [\mathbf{Q}]^{(n)}\tilde{A}^{(n)3\alpha}[\mathbf{V}]^{(n)}_\alpha\,\mathrm{d}S.$$

Similarly as in Section 5.2.1.3, we are able to control the first three terms by using $[Q] = [q]+\frac{1}{2}[|b|^2]$

$$-\frac{1}{2}\frac{d}{dt}\left\|\overline{\partial}^4[q]^{(n)}\right\|_0^2 + \mathcal{P}_0 P([\mathcal{E}]^{(n),(n-1)}(t)).$$

For the boundary term, we integrate $\overline{\partial}^{1/2}$ by parts as in (5.2.138) to get

$$-\int_\Gamma [\mathbf{Q}]^{(n)}\tilde{A}^{(n)3\alpha}[\mathbf{V}]^{(n)}_\alpha\,\mathrm{d}S$$

$$=\int_\Gamma \partial_3[Q]^{(n)}\tilde{A}^{(n)3\alpha}[\mathbf{V}]^{(n)}_\alpha\left(\overline{\partial}^3\tilde{\eta}^{(n)}_\beta\tilde{A}^{(n)3\beta} + \overline{\partial}^3[\tilde{\eta}]^{(n-1)}_\beta\tilde{A}^{(n)3\beta} + \overline{\partial}^3\tilde{\eta}^{(n-1)}_\beta[\tilde{A}]^{(n)3\beta}\right)$$

$$\lesssim |[\mathbf{V}]^{(n)}|_{\dot{H}^{-0.5}}\left(\frac{1}{\kappa}\mathcal{P}_0\left|[\eta]^{(n-1)}\right|_{2.5} + \left\|[\tilde{A}]\right\|_2\right).$$

This finalizes the tangential estimates.

Step 6: Elliptic estimates of $[\partial_t q]^{(n)}$

Since $[q]^{(n)}$ vanishes on the boundary, we can still use Lemma 3.3.3 to reduce the spatial derivative to time derivative by replacing the Laplacian term with ∂_t^2 plus source terms. We only list the wave equation of $[q]^{(n)}$ and omit the computation.

$$
\begin{aligned}
& -\overset{\circ}{\tilde{J}}^{(n)} R'(q^{(n)})\partial_t^2[q]^{(n)} - \triangle_{\tilde{A}^{(n)}}[q]^{(n)} \\
= & \frac{1}{2}\triangle_{\tilde{A}^{(n)}}[|b|^2]^{(n)} - \text{div}_{\tilde{A}^{(n)}}\left((b^{(n)}\cdot\nabla_{\tilde{A}^{(n)}})[b]^{(n)}\right) + \partial_t\left(\overset{\circ}{\tilde{J}}^{(n)} R'(q^{(n)})\right)\partial_t[q]^{(n)} \\
& + \text{div}_{\tilde{A}^{(n)}}\left(\nabla_{\tilde{\mathbf{A}}^{(n)}}Q^{(n)} - (b^{(n)}\cdot\nabla_{[\tilde{\mathbf{A}}]^{(n)}})b^{(n)} - ([b]^{(n-1)}\cdot\nabla_{\tilde{\mathbf{A}}^{(n-1)}})b^{(n)}\right) \\
& - \left(\text{div}_{[\tilde{A}]^{(n)}}v^{(n)} + [\overset{\circ}{\tilde{J}}R'(q)]^{(n)}\partial_t q^{(n)}\right).
\end{aligned}
\tag{5.2.185}
$$

Note that $\text{div}_{\tilde{A}^{(n)}}\left((b^{(n)}\cdot\nabla_{\tilde{A}^{(n)}})[b]^{(n)}\right)$ only contains first order derivative of $[b]^{(n)}$ because of the divergence-free condition on $b^{(n)}$.

Step 7: Common control of heat and wave equations

Differentiate ∂_t^3 in (5.2.156) and (5.2.185), we are able to get similar estimates of $[W]^{(n+1)}$ and $H^{(n+1)}$ as in Section 5.2.1.4. We omit the proof here.

Finally, we conclude that

$$
[\mathcal{E}]^{(n+1)} \lesssim_\kappa \mathcal{P}_0 T^2\left([\mathcal{E}]^{(n)} + [\mathcal{E}]^{(n-1)}\right),
$$

where we pick T_κ suitably small such that the coefficient $\leq 1/4$. This ends the proof of Proposition 5.2.8 and Corollary 5.2.9.

5.2.3 Local well-posedness of the original system

As stated in Corollary 5.2.9, the local well-posedness of the nonlinear approximation system (5.2.1) is established in an κ-dependent time interval $[0, T_\kappa]$. Combining the uniform-in-κ nonlinear a priori estimates Proposition 5.2.2, we know that there exists a κ-**independent** time $T_1 > 0$, such that the local existence of the solution (η, v, b, q) to the original equation (2.3.1) holds in $[0, T_1]$ by letting $\kappa \to 0$. It remains to prove the uniqueness of the solution. Let us recall the original equation first

$$\begin{cases} \partial_t \eta = v & \text{in } \Omega, \\ \rho_0 J^{-1} \partial_t v = (b \cdot \nabla_A) b - \nabla_A Q, \quad Q = q + \frac{1}{2}|b|^2 & \text{in } \Omega, \\ \frac{JR'(q)}{\rho_0} \partial_t q + \text{div}\,_a v = 0 & \text{in } \Omega, \\ \partial_t b + \text{curl}\,_A \text{curl}\,_A b = (b \cdot \nabla_A) v - b \text{div}\,_a v, & \text{in } \Omega, \\ \text{div}\,_a b = 0 & \text{in } \Omega, \\ q = 0,\ b = \mathbf{0},\ -(\partial_N Q)|_{t=0} \geq c_0 > 0 & \text{on } \Gamma, \\ (\eta, v, b, q)|_{\{t=0\}} = (\text{Id}, v_0, b_0, q_0). \end{cases}$$

Suppose $(\eta^i, v^i.b^i, q^i),\ i = 1, 2$ solves (2.3.1) with the same initial data $(\text{Id}, v_0, b_0, q_0)$. Then we consider the system of $([\eta], [v], [b], [q])$ by setting $[f] := f^1 - f^2$. Then we have

The flow map:

$$\partial_t [\eta] = [v].$$

The momentum equation:

$$\rho_0 (J^1)^{-1} \partial_t [v] = (b^1 \cdot \nabla_{a^1})[b] - \nabla_{a^1}[Q] - \rho_0 [J^{-1}] \partial_t v^2 + (b^1 \cdot \nabla_{[a]}) b^2 + [b] \cdot \nabla_{a^2} b^2 - \nabla_{[a]} Q^2.$$

The continuity equation:

$$\frac{J^1 R'(q^1)}{\rho_0} \partial_t [q] + \text{div}\,_{a^1}[v] = \left[\frac{JR'(q)}{\rho_0}\right] \partial_t q^2 - \text{div}\,_{[a]} v^2 \quad \text{in } \Omega.$$

The equation of magnetic field:

$$\begin{aligned} \partial_t [b] - \triangle_{a^1}[b] =& (b^1 \cdot \nabla_{a^1})[v] - b \text{div}\,_a [v] \\ & + \text{div}\,_{a^1} \left(\nabla_{[a]} b^2\right) + \text{div}\,_{[a]} \left(\nabla_{a^2} b^2\right) \\ & + (b^1 \cdot \nabla_{[a]}) v^2 + ([b] \cdot \nabla_{a^2}) v^2 \\ & - b^1 \text{div}\,_{[a]} v^2 - [b] \text{div}\,_{a^2} v^2, \end{aligned}$$

and

$$\text{div}_{a^1}[b] = -\text{div}_{[a]} b^2.$$

The boundary conditions:

$$[q] = 0,\ [b] = \mathbf{0},\ -(\partial_N Q)_1 \text{ and } -(\partial_N Q)_2|_{t=0} \geq c_0 > 0,$$

and zero initial data.

Define the energy functional

$$[\mathcal{E}](T) := [\mathfrak{e}](T) + [H](T) + [W](T) + \left\| \partial_t^{2-k} \left(\left(b^1 \cdot \nabla_{a^1} \right) [b] \right) \right\|_k^2,$$

where

$$[\mathfrak{e}](t) := \|[\eta]\|_2^2 + \left| \tilde{A}^{3\alpha} \overline{\partial}^2 [\eta]_\alpha \right|_0^2 + \sum_{k=0}^{2} \left(\left\| \partial_t^{2-k}[v] \right\|_k^2 + \left\| \partial_t^{2-k}[b] \right\|_k^2 + \left\| \partial_t^{2-k}[q] \right\|_k^2 \right),$$

$$[H](T) := \int_0^T \int_\Omega \left| \partial_t^3 [b] \right|^2 \, \mathrm{d}y \, \mathrm{d}t + \left\| \partial_t^2 [b] \right\|_1^2,$$

$$[W](T) := \left\| \partial_t^3 [q] \right\|_0^2 + \left\| \partial_t^2 [q] \right\|_1^2.$$

The energy estimate of $[\mathcal{E}]$ is almost the same as $\mathcal{E}_\kappa$ except that $[Q]$ no longer satisfies Taylor sign condition. So what we need to do is to investigate the boundary integral

$$\int_\Gamma [\mathbf{Q}] (a^1)^{3\alpha} [\mathbf{V}]_\alpha \, \mathrm{d}S,$$

where we define the Alinhac good unknowns

$$\mathbf{V}^i = \overline{\partial}^2 v^i - \overline{\partial}^2 \eta^i \cdot \nabla_{a^i} v^i,\ \ \mathbf{Q}^i = \overline{\partial}^2 Q^i - \overline{\partial}^2 \eta^i \cdot \nabla_{a^i} Q^i,$$

and

$$[\mathbf{V}] := \mathbf{V}^1 - \mathbf{V}^2,\ \ [\mathbf{Q}] := \mathbf{Q}^1 - \mathbf{Q}^2.$$

The boundary terms then becomes

$$\begin{aligned}
\int_\Gamma [\mathbf{Q}](a^1)^{3\alpha}[\mathbf{V}]_\alpha =& -\int_\Gamma \partial_3[Q]\overline{\partial}^2\eta^2_\beta(a^2)^{3\beta}(a^2)^{3\alpha}[\mathbf{V}]_\alpha\,\mathrm{d}S \\
& -\int_\Gamma (\partial_N Q)^1(\overline{\partial}^2[\eta]_\beta(a^1)^{3\beta}+\overline{\partial}^2\eta^2_\beta[a]^{3\beta})(a^1)^{3\alpha}[\mathbf{V}]_\alpha\,\mathrm{d}S \\
& \lesssim -\frac{1}{2}\frac{d}{dt}\int_\Gamma (\partial_N Q)^1|(a^1)^{3\alpha}\overline{\partial}^2[\eta]_\alpha|_0^2\,\mathrm{d}S \\
& \quad -\int_\Gamma (\partial_N Q)^1(a^1)^{3\gamma}\overline{\partial}^2[\eta]_\gamma(\overline{\partial}^2\eta^2_\beta[a]^{\mu\beta}\partial_\mu v^1_\alpha-\overline{\partial}^2\eta^2_\beta(a^2)^{\mu\beta}\partial_\mu[v]_\alpha)(a^1)^{3\alpha}\,\mathrm{d}S \\
& \quad -\int_\Gamma (\partial_N Q)^1(\overline{\partial}^2[\eta]_\beta(a^1)^{3\beta}+\overline{\partial}^2\eta^2_\beta[a]^{3\beta})(a^1)^{3\alpha}[\mathbf{V}]_\alpha\,\mathrm{d}S \\
& \lesssim -\frac{c_0}{2}\frac{d}{dt}\int_\Gamma |(a^1)^{3\alpha}\overline{\partial}^2[\eta]_\alpha|_0^2\,\mathrm{d}S + P(\text{initial data})P([\mathcal{E}](t)).
\end{aligned}$$

Here in the second step we use the precise formula of $[\mathbf{V}]$, and in the third step we use Taylor sign condition for Q^1. Thus similarly we get

$$\sup_{t\in[0,T_1]}[\mathcal{E}](t)\le \text{initial data}+\int_0^{T_0}P([\mathcal{E}](t))\,\mathrm{d}t.$$

Since the initial data of the system of $([\eta],[v],[b],[q])$ is 0, we know $[\mathcal{E}](t)=0$ for all $t\in[0,T_1]$. Conclusively, the local well-posedness of (2.3.1) is established in Lagrangian coordinates with Sobolev initial data.

5.2.4 The Incompressible limit

The incompressible limit requires the energy estimate for (η, v, b, q) that is uniform in the sound speed, or equivalently, does not rely on $1/R'(q)$. The problem arises in the control of the wave equation

$$R'(q)\partial_t q - \triangle_A q = \cdots,$$

and its time differentiatied versions (cf. Section 5.2.1.4). Note that LHS only gives the energy of $\sqrt{R'(q)}\partial_t q$ but the RHS requires the control of $\partial_t q$. To avoid the loss of weight function, we can add $R'(q)$ in the control of full time derivatives. Rigorously speaking, we first parametrize the reciprocal of the sound speed to be $\varepsilon := R'(q)|_{R=1}$. Under this setting, we denote the unknowns to be $(v^\varepsilon, b^\varepsilon, q^\varepsilon, R^\varepsilon)$ and $\lim_{\varepsilon\to 0_+} R^\varepsilon(p^\varepsilon) = 1$. Then, for (2.3.1) with the initial data $(v_0^\varepsilon, \mathbf{b}_0, \rho_0^\varepsilon, q_0^\varepsilon)$ whose sound speed is ε^{-1}, we define the weighted energy to be

$$\mathcal{E}^\varepsilon(T) := \mathfrak{e}^\varepsilon(T) + H^\varepsilon(T) + W^\varepsilon(T) + \sum_{k=0}^{4} \left\| \partial_t^{4-k}\left((b^\varepsilon\cdot\nabla_{A^\varepsilon})\, b^\varepsilon\right)\right\|_k^2, \tag{5.2.186}$$

where

$$\mathfrak{e}^\varepsilon(T) := \|\eta^\varepsilon\|_4^2 + \left|\overline{\partial}^4\eta^\varepsilon\cdot\hat{n}\right|_0^2 + \sum_{k=1}^{4}\left(\left\|\partial_t^{4-k}v^\varepsilon\right\|_k^2 + \left\|\sqrt{R'(q^\varepsilon)}\partial_t^{4-k}b^\varepsilon\right\|_k^2 + \left\|\partial_t^{4-k}q^\varepsilon\right\|_k^2\right)$$

$$+ \left\|\sqrt{R'(q^\varepsilon)}\partial_t^4 v^\varepsilon\right\|_k^2 + \left\|\partial_t^4 b^\varepsilon\right\|_k^2 + \left\|R'(q^\varepsilon)\partial_t^4 q^\varepsilon\right\|_k^2 \tag{5.2.187}$$

$$H^\varepsilon(T) := \int_0^T\int_\Omega \left|\partial_t^5 b^\varepsilon\right|^2\,\mathrm{d}y\,\mathrm{d}t + \left\|\partial_t^4 b^\varepsilon\right\|_1^2, \tag{5.2.188}$$

$$W^\varepsilon(T) := \left\|R'(q^\varepsilon)\partial_t^5 q^\varepsilon\right\|_0^2 + \left\|\sqrt{R'(q^\varepsilon)}\partial_t^4 q^\varepsilon\right\|_1^2. \tag{5.2.189}$$

Following the same method as in Section 5.2.1, we can prove that: there exists some $T_1' > 0$, such that the $(\eta^\varepsilon, v^\varepsilon, b^\varepsilon, q^\varepsilon)$ in $[0, T_1']$ satisfying the following estimates

$$\sup_{0\le T\le T_1'} \mathcal{E}^\varepsilon(T) \le P\left(\|v_0\|_4, \|b_0\|_5, \|q_0\|_4\right). \tag{5.2.190}$$

Let $(\mathbf{v}_0, \mathbf{b}_0)$ be the divergence-free vector fields with $\mathbf{b}_0|_\Gamma = \mathbf{0}$. Let $\mathbf{q_0}$ be the solution to

$$\triangle(\mathbf{q}_0 + \frac{1}{2}|\mathbf{b}_0|^2) = -\partial_\mu \mathbf{v}_0^\alpha \partial_\alpha \mathbf{v}_0^\mu + \partial_\mu \mathbf{b}_0^\alpha \partial_\alpha \mathbf{b}_0^\mu,\ \ \mathbf{q}_0|_\Gamma = 0$$

and satisfy the Rayleigh-Taylor sign condition $-\partial_N(\mathbf{q_0} + \frac{1}{2}|\mathbf{b}_0|^2) \ge c_0 > 0$. Let $(\mathbf{v}, \mathbf{b}, \mathbf{q})$ be the solution to the incompressible resistive MHD equations with initial data $(\mathbf{v_0}, \mathbf{b_0})$

$$\begin{cases} \partial_t \zeta = v & \text{in } [0,T]\times\Omega \\ \partial_t v = \left(\mathbf{b}\cdot\nabla_{A(\zeta)}\right)\mathbf{b} - \nabla_{A(\zeta)}(\mathbf{q}+\frac{1}{2}|\mathbf{b}|^2) & \text{in } [0,T]\times\Omega \\ \text{div}_{A(\zeta)} v = 0 & \text{in } [0,T]\times\Omega \\ \partial_t \mathbf{b} + \lambda \text{curl}_{A(\zeta)}\text{curl}_{A(\zeta)}\mathbf{b} = (b\cdot\nabla_A)\,\mathbf{v} & \text{in } [0,T]\times\Omega \\ \text{div}_{A(\zeta)}\mathbf{b} = 0 & \text{in } [0,T]\times\Omega \\ \mathbf{b}=\mathbf{0},\ \mathbf{q}=0,\ -\frac{\partial \mathbf{q_0}}{\partial N}|_\Gamma \geq c_0 > 0 & \text{on } [0,T]\times\Gamma \\ (\zeta,\mathbf{v},\mathbf{b},\mathbf{q})|_{t=0} = (\text{Id}, \mathbf{v_0}, \mathbf{b_0}, \mathbf{q_0}). \end{cases} \tag{5.2.191}$$

Therefore, by the compactness argument, we can pass the limit as $\varepsilon \to 0$ to the incompressible counterpart. This concludes the proof of Theorem 2.3.1.

1. There exists $(v_0^\varepsilon, \mathbf{b}_0, \rho_0^\varepsilon, q_0^\varepsilon)$, the initial data of (2.3.1) with sound speed equal to ε^{-1}, satisfying the conditions mentioned in Theorem 2.3.1 and $(v_0^\varepsilon, \rho_0^\varepsilon) \xrightarrow{C^1} (\mathbf{v}_0, 1)$ as $\varepsilon \to 0$.

2. Let $(v^\varepsilon, b^\varepsilon, R^\varepsilon, q^\varepsilon)$ be the solution to (2.3.1) with initial data $(v_0^\varepsilon, \mathbf{b}_0, \rho_0^\varepsilon, q_0^\varepsilon)$. Then we have $(v^\varepsilon, b^\varepsilon, R^\varepsilon) \xrightarrow{C^1} (\mathbf{v}, \mathbf{b}, 1)$ as $\varepsilon \to 0$.

5.3 Anisotropic Regularity of the Free-Boundary Problem in Compressible Ideal MHD

Now we turn to prove the a priori estimates of the free-boundary compressible ideal MHD system in the anisotropic Sobolev space, i.e., Theorem 2.4.1. We first impose the following a priori assumptions: There exists some $T_1 > 0$, such that the solution (η, v, Q) to the system (2.4.1) satisfies

$$\|J-1\|_{7,*} \leq \frac{1}{4} \tag{5.3.1}$$

$$-\frac{\partial Q}{\partial N} \geq \frac{3}{4} c_0. \tag{5.3.2}$$

5.3.1 Control of purely non-weighted normal derivatives

We first consider the case of purely normal derivatives. We aim tio prove

Proposition 5.3.1. The following energy inequality holds

$$\|\partial_3^4 v\|_0^2 + \left\|\partial_3^4\left(J^{-1}(b_0\cdot\partial)\eta\right)\right\|_0^2 + \|\partial_3^4 q\|_0^2 + \frac{c_0}{4}\left|A^{3\alpha}\partial_3^4\eta_\alpha\right|_0^2\Big|_{t=T}$$
$$\lesssim \mathcal{P}_0 + P(\mathfrak{E}(T))\int_0^T P(\mathfrak{E}(t))\,\mathrm{d}t. \tag{5.3.3}$$

5.3.1.1 Evolution equation of Alinhac good unknowns

We first compute the estimates of purely normal derivatives. When $\langle I\rangle = 8$, the purely non-weighted normal derivative should be $\partial_*^I = \partial_3^4$. First we introduce the following Alinhac good unknowns of v and Q with respect to ∂_3^4

$$\mathbf{V}_\alpha := \partial_3^4 v_\alpha - \partial_3^4\eta_\nu\, A^{\mu\nu}\,\partial_\mu v_\alpha,\;\; \mathbf{Q} := \partial_3^4 Q - \partial_3^4\eta_\nu\, A^{\mu\nu}\,\partial_\mu Q. \tag{5.3.4}$$

Then we have that for any function f

$$\begin{aligned}\partial_3^4(\nabla_A^i f) &= \nabla_A^\alpha(\partial_3^4 f) + (\partial_3^4 A^{\mu\alpha})\partial_\mu f + [\partial_3^4, A^{\mu\alpha}, \partial_\mu f]\\ &= \nabla_A^i(\partial_3^4 f) - \partial_3^3(A^{\mu\nu}\,\partial_3\partial_\beta\eta_\nu\, A^{\beta\alpha})\partial_\mu f + [\partial_3^4, A^{\mu\alpha}, \partial_\mu f]\\ &= \nabla_A^i(\underbrace{\partial_3^4 f - \partial_3^4\eta_\nu\, A^{\mu\nu}\,\partial_\mu f}_{\text{good unknowns}})\\ &\quad + \underbrace{\partial_3^4\eta_\nu\nabla_A^i(\nabla_A^p f) - ([\partial_3^3, A^{\mu\nu}A^{\beta\alpha}]\partial_3\partial_\beta\eta_\nu)\partial_\mu f + [\partial_3^4, A^{\mu\alpha}, \partial_\mu f]}_{=:C^i(f)},\end{aligned} \tag{5.3.5}$$

and thus

$$\nabla_A\cdot\mathbf{V} = \partial_3^4(\mathrm{div}\,_A v) - C^i(v_\alpha),\quad \nabla_A\mathbf{Q} = \partial_3^4(\nabla_A Q) - C(Q), \tag{5.3.6}$$

where the commutator satisfies the estimate

$$\|C(f)\|_4 \lesssim P(\|\eta\|_4)\|f\|_4. \tag{5.3.7}$$

Now taking ∂_3^4 yields the evolution equation of the Alinhac good unknowns

$$R\partial_t \mathbf{V} - J^{-1}(b_0 \cdot \partial)\partial_3^4 \left(J^{-1}(b_0 \cdot \partial)\eta\right) + \nabla_A \mathbf{Q}$$
$$= \underbrace{\left[R, \partial_3^4\right] \partial_t v + \left[\partial_3^4, J^{-1}(b_0 \cdot \partial)\right] b - C(Q) - R\partial_t(\partial_3^4 \eta \cdot \nabla_A v)}_{=:\mathbf{F}}. \tag{5.3.8}$$

Taking $L^2(\Omega)$-inner product of (5.3.8) and $J\mathbf{V}$ and using $\rho_0 = RJ$ yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho_0 \left|\mathbf{V}\right|^2 \, \mathrm{d}y = \int_\Omega (b_0 \cdot \partial)\partial_3^4 (J^{-1}(b_0 \cdot \partial)\eta) \cdot \mathbf{V} - \int_\Omega (\nabla_\mathbf{A} \mathbf{Q}) \cdot \mathbf{V} + \int_\Omega J\mathbf{F} \cdot \mathbf{V}. \tag{5.3.9}$$

5.3.1.2 Interior estimates

The third integral on the RHS of (5.3.9) can be directly controlled

$$\int_\Omega J\mathbf{F} \cdot \mathbf{V} \lesssim \|J\mathbf{F}\|_0 \|\mathbf{V}\|_0 \lesssim P(\|\rho_0\|_4, \|b_0\|_4, \|\eta\|_4, \|J^{-1}(b_0 \cdot \partial)\eta\|_4, \|Q\|_4, \|v\|_4, \|\partial_t v\|_3)\|\mathbf{V}\|_0. \tag{5.3.10}$$

The first integral on the RHS of (5.3.9) gives the energy of magnetic field $b = J^{-1}(b_0 \cdot \partial)\eta$ after integrating $(b_0 \cdot \partial)$ by parts. Note that $b_0^3|_\Gamma = 0$ and $\operatorname{div} b_0 = 0$, there will be no boundary integral. In specific, we have

$$\int_\Omega (b_0 \cdot \partial)\partial_3^4 (J^{-1}(b_0 \cdot \partial)\eta) \cdot \mathbf{V} \, \mathrm{d}y = -\int_\Omega \partial_3^4 (J^{-1}(b_0 \cdot \partial)\eta) \cdot (b_0 \cdot \partial)\mathbf{V} \, \mathrm{d}y$$
$$= -\int_\Omega \partial_3^4 (J^{-1}(b_0 \cdot \partial)\eta) \cdot (b_0 \cdot \partial)\partial_3^4 v \, \mathrm{d}y + \underbrace{\int_\Omega \partial_3^4 (J^{-1}(b_0 \cdot \partial)\eta) \cdot (b_0 \cdot \partial)(\partial_3^4 \eta \cdot \nabla_A v) \, \mathrm{d}y}_{=:L_1}$$
$$= -\int_\Omega J\partial_3^4 (J^{-1}(b_0 \cdot \partial)\eta) \cdot \partial_3^4 \partial_t (J^{-1}(b_0 \cdot \partial)\eta) \, \mathrm{d}y \tag{5.3.11}$$
$$\underbrace{-\int_\Omega J\partial_3^4 (J^{-1}(b_0 \cdot \partial)\eta) \cdot \left[J^{-1}(b_0 \cdot \partial), \partial_3^4 \partial_t\right] \eta \, \mathrm{d}y}_{K_1} + L_1$$
$$= -\frac{1}{2}\frac{d}{dt}\int_\Omega J \left|\partial_3^4 (b_0 \cdot \partial)\eta\right|^2 \, \mathrm{d}y + \frac{1}{2}\int_\Omega \partial_t J \left|\partial_3^4 (b_0 \cdot \partial)\eta\right|^2 \, \mathrm{d}y + K_1 + L_1.$$

The term L_1 can be directly controlled

$$L_1 \lesssim P\left(\|(b_0\cdot\partial)\eta\|_4, \|\eta\|_4, \|b_0\|_4, \|v\|_4\right). \tag{5.3.12}$$

The term K_1 produces a higher order term when $\partial_3^4\partial_t$ falls on J^{-1}. We invoke $\partial_t J = J\,\mathrm{div}_{\tilde{A}} v$ to get

$$\begin{aligned}
&-\left[J^{-1}(b_0\cdot\partial), \partial_3^4\partial_t\right]\eta \\
=&\partial_3^4\partial_t(J^{-1})\,(b_0\cdot\partial)\eta + \sum_{N=0}^{3}\partial_3^N\partial_t(J^{-1})\,\partial_3^{4-N}b_0\cdot\partial\eta \\
&+\sum_{M=0}^{3}\partial_t\left(\partial_3^M(J^{-1}b_0^l)\,\partial_\mu\partial_3^{4-M}\eta\right) \\
=&-J^{-1}\partial_3^4(\mathrm{div}_{\tilde{A}}v)\,(b_0\cdot\partial)\eta + \left([\partial_3^4, J^{-1}]\mathrm{div}_{\tilde{A}}v\right)(b_0\cdot\partial)\eta \\
&+\sum_{N=0}^{3}\partial_3^N\partial_t(J^{-1})(\partial_3^{4-N}b_0^l)(\partial_\mu\eta) + \sum_{M=0}^{3}\partial_t\left(\partial_3^M(J^{-1}b_0^l)\partial_\mu\partial_3^{4-M}\eta\right) \\
=:&-J^{-1}\partial_3^4(\mathrm{div}_{\tilde{A}}v)\,(b_0\cdot\partial)\eta + KL_1
\end{aligned} \tag{5.3.13}$$

and thus

$$\begin{aligned}
K_1 =&\underbrace{-\int_\Omega J\partial_3^4(J^{-1}(b_0\cdot\partial)\eta)\cdot\left(J^{-1}(b_0\cdot\partial)\eta\right)\partial_3^4(\mathrm{div}_{\tilde{A}}v)\,\mathrm{d}y}_{K_{11}} \\
&+\int_\Omega J\partial_3^4(J^{-1}(b_0\cdot\partial)\eta)\cdot(KL_1) \\
\lesssim&K_{11} + \|J\|_{L^\infty}\|J^{-1}(b_0\cdot\partial)\eta\|_4\|KL_1\|_0 \\
\lesssim&K_{11} + P\left(\|(b_0\cdot\partial)\eta\|_4, \|\eta\|_4, \|b_0\|_4\right).
\end{aligned} \tag{5.3.14}$$

Summarizing (5.3.11)-(5.3.14), we get the following estimates

$$\begin{aligned}
&\int_\Omega (b_0\cdot\partial)\partial_3^4(J^{-1}(b_0\cdot\partial)\eta)\cdot\mathbf{V}\,\mathrm{d}y \\
\lesssim&-\frac{1}{2}\frac{d}{dt}\int_\Omega J\left|\partial_3^4(b_0\cdot\partial)\eta\right|^2\,\mathrm{d}y + K_{11} + P\left(\|(b_0\cdot\partial)\eta\|_4, \|\eta\|_4, \|b_0\|_4, \|v\|_4\right).
\end{aligned} \tag{5.3.15}$$

We note that the term K_{11} cannot be directly controlled, but will be cancelled by another term produced by $-\int_\Omega (\nabla_{\mathbf{A}} \mathbf{Q}) \cdot \mathbf{V}$. Next we analyze the second integral on the RHS of (5.3.9). Integraing by parts and invoking Piola's identity $\partial_\mu \mathbf{A}^{li} = 0$, we get

$$-\int_\Omega (\nabla_{\mathbf{A}} \mathbf{Q}) \cdot \mathbf{V} \, \mathrm{d}y = \int_\Omega J\mathbf{Q}(\nabla_A \cdot \mathbf{V}) \, \mathrm{d}y - \int_\Gamma J\mathbf{Q} A^{\mu\alpha} N_\mu \mathbf{V}_\alpha \, \mathrm{d}S =: I + IB. \tag{5.3.16}$$

Plugging (5.3.4) and (5.3.6) as well as $Q = q + \frac{1}{2}|b|^2$ into I, we get

$$\begin{aligned} I =& \int_\Omega J \, \partial_3^4 q \, \partial_3^4 (\mathrm{div}_{\tilde{A}} v) + \int_\Omega J \, \partial_3^4 \left(\frac{1}{2} \left| J^{-1} (b_0 \cdot \partial) \eta \right|^2 \right) \partial_3^4 (\mathrm{div}_{\tilde{A}} v) \\ & - \int_\Omega \partial_3^4 \eta_\nu \, \mathbf{A}^{\mu\nu} \partial_\mu Q \, \partial_3^4 (\mathrm{div}_{\tilde{A}} v) - \int_\Omega \partial_3^4 Q \, C(v) \\ =: & I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{5.3.17}$$

The term I_4 can be directly controlled by using (5.3.7)

$$I_4 \lesssim \|Q\|_4 \|C(v)\|_0 \lesssim P(\|\eta\|_4) \|Q\|_4 \|v\|_4. \tag{5.3.18}$$

The term I_1 gives the energy of q by invoking $\mathrm{div}_{\tilde{A}} v = -\frac{\partial_t R}{R} = -\frac{JR'(q)}{\rho_0} \partial_t q$

$$\begin{aligned} I_1 =& - \int_\Omega J \, \partial_3^4 q \, \partial_3^4 \left(\frac{JR'(q)}{\rho_0} \partial_t q \right) \\ =& - \frac{1}{2} \frac{d}{dt} \int_\Omega \frac{J^2 R'(q)}{\rho_0} \left| \partial_3^4 q \right|^2 \, \mathrm{d}y + \frac{1}{2} \int_\Omega \partial_t \left(\frac{J^2 R'(q)}{\rho_0} \right) \left| \partial_3^4 q \right|^2 \\ & - \int_\Omega J \, \partial_3^4 q \left(\left[\partial_3^4, \frac{JR'(q)}{\rho_0} \right] \partial_t q \right) \\ \lesssim & - \frac{1}{2} \frac{d}{dt} \int_\Omega \frac{J^2 R'(q)}{\rho_0} \left| \partial_3^4 q \right|^2 \, \mathrm{d}y + P(\|q\|_{8,*}, \|\rho_0\|_4, \|\eta\|_4). \end{aligned} \tag{5.3.19}$$

The term I_2 will produce another higher order term to cancel with K_{11}

$$
\begin{aligned}
I_2 =& \underbrace{\int_\Omega J \partial_3^4 \left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\left(J^{-1}(b_0\cdot\partial)\eta\right)\partial_3^4(\operatorname{div}_{\tilde{A}} v)}_{\text{exactly cancel with } K_{11}} \\
&+\int_\Omega \sum_{N=1}^{3}\binom{4}{N} J \partial_3^N \left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\partial_3^{4-N}\left(J^{-1}(b_0\cdot\partial)\eta\right)\partial_3^4(\operatorname{div}_{\tilde{A}} v) \\
=& -K_{11} - \int_\Omega \sum_{N=1}^{3}\binom{4}{N}\left(\frac{J^2 R'(q)}{\rho_0}\right)\partial_3^N \left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\partial_3^{4-N}\left(J^{-1}(b_0\cdot\partial)\eta\right)\partial_3^4\partial_t q \\
&+\int_\Omega \sum_{N=1}^{3}\binom{4}{N} J \partial_3^N \left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\partial_3^{4-N}\left(J^{-1}(b_0\cdot\partial)\eta\right)\left(\left[\partial_3^4, \frac{JR'(q)}{\rho_0}\right]\partial_t q\right) \\
=:& -K_{11} + I_{21} + I_{22}.
\end{aligned}
\tag{5.3.20}
$$

We should control I_{21} by integrating ∂_t by parts under time integral

$$
\begin{aligned}
\int_0^T I_{21} \overset{\partial_t}{=} & \int_0^T \int_\Omega \sum_{N=1}^{3}\binom{4}{N}\partial_t\left(\frac{J^2 R'(q)}{\rho_0}\right)\partial_3^N \left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\partial_3^{4-N}\left(J^{-1}(b_0\cdot\partial)\eta\right)\partial_3^4 q \\
&+\int_0^T \int_\Omega \sum_{N=1}^{3}\binom{4}{N}\left(\frac{J^2 R'(q)}{\rho_0}\right)\partial_t\partial_3^N \left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\partial_3^{4-N}\left(J^{-1}(b_0\cdot\partial)\eta\right)\partial_3^4 q \\
&-\int_\Omega \sum_{N=1}^{3}\binom{4}{N}\left(\frac{J^2 R'(q)}{\rho_0}\right)\partial_3^N \left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\partial_3^{4-N}\left(J^{-1}(b_0\cdot\partial)\eta\right)\partial_3^4 q\Big|_0^T \\
\lesssim & \int_0^T P(\|J^{-1}(b_0\cdot\partial)\eta\|_4, \|\partial_t(J^{-1}(b_0\cdot\partial)\eta)\|_3, \|q\|_4) + \mathcal{P}_0 + \|J^{-1}(b_0\cdot\partial)\eta\|_3^2\|\partial_3^4 q\|_0 \\
\lesssim & \mathcal{P}_0 + \int_0^T P(\mathfrak{E}(t))\,\mathrm{d}t + \varepsilon\|\partial_3^4 q\|_0^2.
\end{aligned}
\tag{5.3.21}
$$

Then I_{22} can be directly controlled since at most three ∂_3's fall on $\partial_t q$.

$$
I_{22} \lesssim \|J^{-1}(b_0\cdot\partial)\eta\|_3^2\|q\|_{7,*}. \tag{5.3.22}
$$

The term I_3 should also be controlled under time integral. We have

$$\int_0^T I_3 = \int_0^T \int_\Omega \frac{JR'(q)}{\rho_0} \partial_3^4 \eta_\nu \, \mathbf{A}^{\mu\nu} \partial_\mu Q \, \partial_3^4 \partial_t q + \underbrace{\int_0^T \int_\Omega \partial_3^4 \eta_\nu \, \mathbf{A}^{\mu\nu} \partial_\mu Q \left[\partial_3^4, \frac{JR'(q)}{\rho_0} \right] \partial_t q}_{L_2}$$

$$\begin{aligned} &\overset{\partial_t}{=} - \int_0^T \int_\Omega \partial_t \left(\frac{JR'(q)}{\rho_0} \partial_3^4 \eta_\nu \, \mathbf{A}^{\mu\nu} \partial_\mu Q \right) \partial_3^4 q + \int_\Omega \frac{JR'(q)}{\rho_0} \partial_3^4 \eta_\nu \, \mathbf{A}^{\mu\nu} \partial_\mu Q \, \partial_3^4 q \Big|_0^T + L_2 \\ &\lesssim \mathcal{P}_0 + \int_0^T P(\mathfrak{E}(t)) \, \mathrm{d}t + \left\| \frac{JR'(q)}{\rho_0} A \, \partial Q \right\|_{L^\infty} \|\partial_3^4 q\|_0 \int_0^T \|\partial_3^4 v(t)\|_0 \, \mathrm{d}t \\ &\lesssim \mathcal{P}_0 + P(\mathfrak{E}(t)) \int_0^T P(\mathfrak{E}(t)) \, \mathrm{d}t, \end{aligned} \tag{5.3.23}$$

where we use $\partial^4 \eta|_{t=0} = 0$ in the last step. Summarizing (5.3.18)-(5.3.23) and choosing $\varepsilon > 0$ suitably small, we get the estimates of I under time integral

$$\int_0^T I \, \mathrm{d}t \lesssim -\frac{1}{2} \int_\Omega \frac{J^2 R'(q)}{\rho_0} \left| \partial_3^4 q \right|^2 \, \mathrm{d}y \Big|_0^T + \mathcal{P}_0 + P(\mathfrak{E}(t)) \int_0^T P(\mathfrak{E}(t)) \, \mathrm{d}t. \tag{5.3.24}$$

5.3.1.3 Boundary estimates

To finish the estimates of purely non-weighted normal derivative, it remains to control the boundary integral IB in (5.3.16) which reads

$$\begin{aligned} - \int_\Gamma J \mathbf{Q} A^{\mu\alpha} N_\mu \mathbf{V}_\alpha \, \mathrm{d}S = & - \int_\Gamma \mathbf{A}^{3\alpha} N_3 \, \partial_3^4 Q \, \mathbf{V}_\alpha \, \mathrm{d}S \\ & + \int_\Gamma \mathbf{A}^{3\alpha} N_3 \partial_3^4 \eta_\nu \, A^{3\nu} \partial_3 Q \, \partial_3^4 v_\alpha \, \mathrm{d}S \\ & - \int_\Gamma \mathbf{A}^{3\alpha} N_3 \partial_3^4 \eta_\nu \, A^{3\nu} \partial_3 Q \, (\partial_3^4 \eta_\gamma \, A^{\beta\gamma} \, \partial_\beta v_\alpha) \, \mathrm{d}S \\ =: & IB_0 + IB_1 + IB_2. \end{aligned} \tag{5.3.25}$$

First, IB_1 will produce the boundary energy with the help of Rayleigh-Taylor sign condition (5.3.2) and the error terms will be cancelled with IB_2. In specific, we have

$$\begin{aligned}
IB_1 =& -\int_\Gamma \left(-\frac{\partial Q}{\partial N}\right) JA^{3\alpha}\partial_3^4\eta_\nu\, A^{3\nu}\partial_3^4\partial_t\eta_\alpha\,\mathrm{d}S \\
=& -\frac{1}{2}\frac{d}{dt}\int_\Gamma \left(-J\frac{\partial Q}{\partial N}\right)\left|A^{3\alpha}\partial_3^4\eta_\alpha\right|^2\,\mathrm{d}S \\
& -\frac{1}{2}\int_\Gamma \partial_t\left(J\frac{\partial Q}{\partial N}\right)\left|A^{3\alpha}\partial_3^4\eta_\alpha\right|^2\,\mathrm{d}S + \int_\Gamma \left(-J\frac{\partial Q}{\partial N}\right)\partial_t A^{3\alpha}\,\partial_3^4\eta_\nu\, A^{3\nu}\partial_3^4\eta_\alpha\,\mathrm{d}S \\
=:& IB_{11} + IB_{12} + IB_{13}.
\end{aligned} \tag{5.3.26}$$

Invoking Rayleigh-Taylor sign condition, we get

$$\int_0^T IB_{11}\,\mathrm{d}t \lesssim -\frac{c_0}{4}\int_\Gamma \left|A^{3\alpha}\partial_3^4\eta_\alpha\right|^2\,\mathrm{d}S\Big|_0^T, \tag{5.3.27}$$

and thus the term IB_{12} can be directly controlled by the boundary energy

$$IB_{12} \lesssim |\partial_t(J\partial_3 Q)|_{L^\infty}\left|A^{3\alpha}\partial_3^4\eta_\alpha\right|_0^2 \lesssim P(\mathfrak{E}(t)). \tag{5.3.28}$$

Then we plug $\partial_t A^{3\alpha} = -A^{3\gamma}\,\partial_\beta v_\gamma\, A^{\beta\alpha}$ into IB_{13} to get

$$IB_{13} = \int_\Gamma \left(\frac{\partial Q}{\partial N}\right) A^{3\gamma}\partial_\beta v_\gamma\,\mathbf{A}^{\beta\alpha}\partial_3^4\eta_\nu\, A^{3\nu}\partial_3^4\eta_\alpha\,\mathrm{d}S, \tag{5.3.29}$$

and this term exactly cancel with IB_2 if we replace the indices (α,γ) by (γ,α).

It now remains to control IB_0. We have

$$IB_0 = -\int_\Gamma N_3 J\,\partial_3^4 Q\,(A^{3\alpha}\partial_3^4 v_\alpha)\,\mathrm{d}S + \int_\Gamma \mathbf{A}^{3\alpha}N_3\partial_3^4 Q\,\partial_3^4\eta_\nu\, A^{\mu\nu}\partial_\mu v_\alpha\,\mathrm{d}S =: IB_{01} + IB_{02}. \tag{5.3.30}$$

To control IB_0, we shall differentiate the following relations

$$A^{3\alpha}\partial_3 v_\alpha = \mathrm{div}_{\tilde{A}} v - A^{1\alpha}\overline{\partial}_1 v_\alpha - A^{2\alpha}\overline{\partial}_2 v_\alpha = -\frac{JR'(q)}{\rho_0}\partial_t q - A^{1\alpha}\overline{\partial}_1 v_\alpha - A^{2\alpha}\overline{\partial}_2 v_\alpha. \tag{5.3.31}$$

In IB_{01}, we use the relation (5.3.31) to get

$$\begin{aligned} A^{3\alpha}\partial_3^4 v_\alpha =& \partial_3^3(A^{3\alpha}\partial_3 v_\alpha) - \partial_3^3 A^{3\alpha}\,\partial_3 v_\alpha - 3\partial_3^2 A^{3\alpha}\,\partial_3^2 v_\alpha - 3\partial_3 A^{3\alpha}\,\partial_3^3 v_\alpha \\ =& -\partial_3^3\left(\frac{JR'(q)}{\rho_0}\partial_t q\right) - \sum_{L=1}^{2}\partial_3^3(A^{L\alpha}\overline{\partial}_L v_\alpha) \\ & -\partial_3^3 A^{3\alpha}\,\partial_3 v_\alpha - 3\partial_3^2 A^{3\alpha}\,\partial_3^2 v_\alpha - 3\partial_3 A^{3\alpha}\,\partial_3^3 v_\alpha, \end{aligned} \tag{5.3.32}$$

and thus IB_{01} becomes

$$\begin{aligned} IB_{01} =& \int_\Gamma N_3 J\ \partial_3^4 Q\ \partial_3^3\left(\frac{JR'(q)}{\rho_0}\partial_t q\right) + \sum_{L=1}^{2}\int_\Gamma N_3 J\ \partial_3^4 Q\ \partial_3^3(A^{L\alpha}\overline{\partial}_L v_\alpha) \\ & + \int_\Gamma N_3 J \partial_3^4 Q\ \left(\partial_3^3 A^{3\alpha}\,\partial_3 v_\alpha + 3\partial_3^2 A^{3\alpha}\,\partial_3^2 v_\alpha + 3\partial_3 A^{3\alpha}\,\partial_3^3 v_\alpha\right) \\ =:& IB_{011} + IB_{012} + IB_{013}. \end{aligned} \tag{5.3.33}$$

In IB_{012}, the highest order term contains $\partial_3^3 A^{L\alpha} = \partial_3^4\eta \times \overline{\partial}\eta + \cdots$ which cannot be directly controlled. However, this term can produce cancellation with IB_{02}.

$$\begin{aligned} \partial_3^3 A^{L\alpha} =& -\partial_3^2(A^{L\nu}\,\partial_3\partial_\beta\eta_\nu\,A^{\beta\alpha}) \\ =& -A^{L\nu}\partial_3^4\eta_\nu A^{3\alpha} - \sum_{M=1}^{2} A^{L\nu}\partial_3^3\overline{\partial}_M\eta_\nu A^{M\alpha} - [\partial_3^2, A^{L\nu}A^{\beta\alpha}]\partial_3\partial_\beta\eta_\nu, \end{aligned} \tag{5.3.34}$$

and thus IB_{012} can be written as

$$IB_{012} = -\sum_{L=1}^{2}\int_\Gamma \mathbf{A}^{3\alpha} N_3\partial_3^4 Q\ \partial_3^4\eta_\nu\ A^{L\nu}\overline{\partial}_L v_\alpha \tag{5.3.35}$$

$$-\sum_{L=1}^{2}\int_\Gamma N_3 J\partial_3^4 Q\left(\sum_{M=1}^{2} A^{L\nu}\partial_3^3\overline{\partial}_M\eta_\nu\ A^{M\alpha} + [\partial_3^2, A^{L\nu}A^{\beta\alpha}]\partial_3\partial_\beta\eta_\nu\right). \tag{5.3.36}$$

On the other hand, we write IB_{02} as

$$IB_{02} = \int_{\Gamma} \mathbf{A}^{3\alpha} N_3 \partial_3^4 Q \, \partial_3^4 \eta_\nu \, A^{3\nu} \partial_3 v_\alpha \, \mathrm{d}S \tag{5.3.37}$$

$$+ \sum_{L=1}^{2} \int_{\Gamma} \mathbf{A}^{3\alpha} N_3 \partial_3^4 Q \, \partial_3^4 \eta_\nu \, A^{L\nu} \overline{\partial}_L v_\alpha \, \mathrm{d}S. \tag{5.3.38}$$

Therefore, (5.3.38) exactly cancels with the main term (5.3.35) in IB_{012}.

Now it remains to control IB_{011}, IB_{013} and (5.3.36), (5.3.37). Invoking

$$\mathbf{A}^{3\alpha} \partial_3 Q = -\sum_{L=1}^{2} \mathbf{A}^{Li} \overline{\partial}_L Q - \rho_0 \partial_t v^i + (b_0 \cdot \partial)(J^{-1}(b_0 \cdot \partial)\eta^i), \tag{5.3.39}$$

we get

$$\begin{aligned}
\mathbf{A}^{3\alpha} \partial_3^4 Q =& \partial_3^3(\mathbf{A}^{3\alpha} \partial_3 Q) - \partial_3^3 \mathbf{A}^{3\alpha} \, \partial_3 Q - 3\partial_3^2 \mathbf{A}^{3\alpha} \, \partial_3^2 Q - 3\partial_3 \mathbf{A}^{3\alpha} \, \partial_3^3 Q \\
=& \partial_3^3 \left(-\rho_0 \partial_t v^i + (b_0 \cdot \partial)(J^{-1}(b_0 \cdot \partial)\eta)\right) - \sum_{L=1}^{2} \partial_3^3(\mathbf{A}^{Li} \overline{\partial}_L Q) \\
& - \partial_3^3 \mathbf{A}^{3\alpha} \, \partial_3 Q - 3\partial_3^2 \mathbf{A}^{3\alpha} \, \partial_3^2 Q - 3\partial_3 \mathbf{A}^{3\alpha} \, \partial_3^3 Q.
\end{aligned} \tag{5.3.40}$$

Note that

- The term $\mathbf{A}^{3\alpha}$ is of the form $\overline{\partial}\eta \times \overline{\partial}\eta$, so the leading order term in $\partial_3^3 A^{3\alpha}$ should be $(\partial_3^3 \overline{\partial}\eta)(\overline{\partial}\eta)$.
- The highest order term in $\partial_3^3(\mathbf{A}^{Li} \overline{\partial}_L Q)$ is $\partial_3^3 \mathbf{A}^{Li} \, \overline{\partial}_L Q = 0$ due to $\overline{\partial}_L Q|_{\Gamma=0}$.
- The highest order term in $\partial_3^3((b_0 \cdot \partial)(J^{-1}(b_0 \cdot \partial)\eta))$ is $(b_0 \cdot \overline{\partial})\partial_3^3(J^{-1}(b_0 \cdot \partial)\eta)$ because $b_0^3|_\Gamma = 0$ makes $(b_0 \cdot \partial)$ tangential on the boundary.

Therefore, we can rewrite $\partial_3^4 Q$ to be the terms of at most 3 normal derivatives and one tangential

derivative:
$$\partial_3^4 Q = \underbrace{J^{-1}\mathbf{A}^{3\alpha}\partial_3\eta_\alpha}_{=1}\partial_3^4 Q = J^{-1}\partial_3\eta_\alpha(\mathbf{A}^{3\alpha}\partial_3^4 Q)$$
$$\begin{aligned}=&J^{-1}\partial_3\eta_\alpha\Bigg(\partial_3^3\left(-\rho_0\partial_t v^i + (b_0\cdot\overline{\partial})(J^{-1}(b_0\cdot\partial)\eta)\right)\\ &-\sum_{L=1}^{2}\sum_{N=0}^{2}\binom{3}{N}(\partial_3^N\mathbf{A}^{Li})(\partial_3^{3-N}\overline{\partial}_L Q)\\ &-\partial_3^3\mathbf{A}^{3\alpha}\,\partial_3 Q - 3\partial_3^2\mathbf{A}^{3\alpha}\,\partial_3^2 Q - 3\partial_3\mathbf{A}^{3\alpha}\,\partial_3^3 Q\Bigg).\end{aligned} \tag{5.3.41}$$

In (5.3.37), we need to rewrite $A^{3\nu}\partial_3^4\eta_\nu$ by using $A^{3\nu}\partial_3\eta_\nu = 1$ in $\bar{\Omega}$ (and thus $\partial_3^3(A^{3\nu}\partial_3\eta_\nu) = 0$)
$$A^{3\nu}\partial_3^4\eta_\nu = -\partial_3^3 A^{3\nu}\,\partial_3\eta_\nu - 3\partial_3^2 A^{3\nu}\,\partial_3^2\eta_\nu - 3\partial_3 A^{3\nu}\,\partial_3^3\eta_\nu. \tag{5.3.42}$$

In the light of (5.3.40)-(5.3.42), we are able to write IB_{011}, IB_{013} and (5.3.36), (5.3.37) in the form of
$$\int_\Gamma N_3(\partial_3^3\mathfrak{D} f)(\partial_3^3\mathfrak{D} g)h\,\mathrm{d}S + \text{lower order terms}, \tag{5.3.43}$$
where $\mathfrak{D} = \overline{\partial}$ or ∂_t or $b_0\cdot\overline{\partial}$, and f, g can be $\eta, v, q, J^{-1}(b_0\cdot\partial)\eta$, and h contains at most first order derivative of η, v. Then (5.3.43) can be controlled as follows
$$\begin{aligned}&\int_\Gamma N_3(\partial_3^3\mathfrak{D} f)(\partial_3^3\mathfrak{D} g)h\,\mathrm{d}S\\ =&\left(\int_\Omega(\partial_3^4\mathfrak{D} f)(\partial_3^3\mathfrak{D} g)h - \int_\Omega(\partial_3^3\mathfrak{D} f)(\partial_3^4\mathfrak{D} g)h - \int_\Omega(\partial_3^3\mathfrak{D} f)(\partial_3^3\mathfrak{D} g)(\partial_3 h)\right)\\ \overset{\mathfrak{D}}{=}&-\int_\Omega(\partial_3^4 f)(\partial_3^3\mathfrak{D}^2 g)h - \int_\Omega(\partial_3^4 f)(\partial_3^3\mathfrak{D} g)(\mathfrak{D} h)\\ &+\int_\Omega(\partial_3^3\mathfrak{D}^2 f)(\partial_3^4 g)h + \int_\Omega(\partial_3^3\mathfrak{D} f)(\partial_3^4 g)(\mathfrak{D} h) - \int_\Omega(\partial_3^3\mathfrak{D} f)(\partial_3^3\mathfrak{D} g)(\partial_3 h)\\ \lesssim&(\|\partial_3^4 f\|_0 + \|\partial_3^3\mathfrak{D}^2 f\|_0)(\|\partial_3^4 g\|_0 + \|\partial_3^3\mathfrak{D}^2 g\|_0)\|\partial h\|_{L^\infty} \lesssim \|f\|_{8,*}\|g\|_{8,*}\|h\|_3,\end{aligned} \tag{5.3.44}$$
which gives the control of IB_{011}, IB_{013} and (5.3.36), (5.3.37).

Remark 5.3.2. If we integrate $\mathfrak{D} = \partial_t$ by parts in (5.3.44) (such term appears in $\partial_3^3\partial_t v$ from $\partial_3^4 Q$), then we should proceed the estimate under time integral:

$$
\begin{aligned}
&\int_{\Omega}(\partial_3^4 v)(\partial_3^3\mathfrak{D} g)h \lesssim \varepsilon\|\partial_3^4 v\|_0^2+\frac{1}{8\varepsilon}\|\partial_3^3\mathfrak{D} g\|_0^4+\frac{1}{8\varepsilon}\|h\|_{L^\infty}^4\\
\lesssim&\varepsilon\|\partial_3^4 v\|_0^2+\frac{1}{8\varepsilon}\left(\|g(0)\|_{7,*}^4+\|h(0)\|_2^4+\int_0^T\|\partial_3^3\mathfrak{D}\partial_t g(t)\|_0^4+\|\partial_t h(t)\|_2^4\right) \qquad (5.3.45)\\
\lesssim&\varepsilon\|\partial_3^4 v\|_0^2+\mathcal{P}_0+\int_0^T P(\|g\|_{8,*},\|h\|_{5,*})\,\mathrm{d}t.
\end{aligned}
$$

According to (5.3.44)-(5.3.45), we can finalize the estimates of the boundary integral IB as follows

$$
IB \lesssim \varepsilon\|\partial_3^4 v\|_0^2-\frac{c_0}{4}\frac{d}{dt}\int_{\Gamma}\left|A^{3\alpha}\partial_3^4\eta_\alpha\right|^2\,\mathrm{d}S+P(\mathfrak{E}(t)). \qquad (5.3.46)
$$

5.3.1.4 Energy estimates of purely normal derivatives

Now, (5.3.46) together with (5.3.9), (5.3.10), (5.3.15), (5.3.24) gives the estimates of Alinhac good unknowns of v, Q in the case of purely non-weighted normal derivatives

$$
\begin{aligned}
&\|\mathbf{V}\|_0^2+\left\|\partial_3^4\left(J^{-1}(b_0\cdot\partial)\eta\right)\right\|_0^2+\|\partial_3^4 q\|_0^2+\frac{c_0}{4}\left|A^{3\alpha}\partial_3^4\eta_\alpha\right|_0^2\bigg|_{t=T}\\
\lesssim&\ \varepsilon\|\partial_3^4 v\|_0^2+\mathcal{P}_0+P(\mathfrak{E}(T))\int_0^T P(\mathfrak{E}(t))\,\mathrm{d}t.
\end{aligned} \qquad (5.3.47)
$$

Finally, by the definition of Alinhac good unknown (5.3.4) and $\partial_3^4\eta|_{t=0}=\mathbf{0}$, $\partial_3^4 v$ is controlled by

$$
\|\partial_3^4 v\|_0^2 \lesssim \|\mathbf{V}\|_0^2+\|a\partial v\|_{L^\infty}^2\int_0^T\|\partial_3^4 v\|_0^2\,\mathrm{d}t \lesssim \|\mathbf{V}\|_0+P(\mathfrak{E}(T))\int_0^T P(\mathfrak{E}(t))\,\mathrm{d}t, \qquad (5.3.48)
$$

and thus by choosing $\varepsilon>0$ sufficiently small, we get

$$
\begin{aligned}
&\|\partial_3^4 v\|_0^2+\left\|\partial_3^4\left(J^{-1}(b_0\cdot\partial)\eta\right)\right\|_0^2+\|\partial_3^4 q\|_0^2+\frac{c_0}{4}\left|A^{3\alpha}\partial_3^4\eta_\alpha\right|_0^2\bigg|_{t=T}\\
\lesssim&\ \mathcal{P}_0+P(\mathfrak{E}(T))\int_0^T P(\mathfrak{E}(t))\,\mathrm{d}t.
\end{aligned} \qquad (5.3.49)
$$

5.3.2 The case of full tangential spatial derivatives

Now we consider the purely tangential derivatives. In this case, the top order derivative becomes $\partial_*^I = \partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}$ with $i_0 + i_1 + i_2 = 8$. We will prove the following estimates by a modified Alinhac good unknown method.

Proposition 5.3.3. The following holds for any sufficiently small $\varepsilon > 0$

$$\sum_{i_3=i_4=0} \|\partial_*^I v\|_0^2 + \left\|\partial_*^I \left(J^{-1}(b_0\cdot\partial)\eta\right)\right\|_0^2 + \|\partial_*^I q\|_0^2 + \frac{c_0}{4}\left|A^{3\alpha}\partial_*^I\eta_\alpha\right|_0^2\bigg|_{t=T}$$
$$\lesssim \varepsilon\|\partial_3\partial_t^6 v\|_0^2 + \mathcal{P}_0 + P(\mathfrak{E}(T))\int_0^T P(\mathfrak{E}(t))\,\mathrm{d}t. \tag{5.3.50}$$

For simplicity, we mainly study the case $i_0 = 0$, i.e., $\partial_*^I = \overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}$ with $i_1 + i_2 = 8$. For sake of clean notations, we denote $\overline{\partial}^8 = \overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}$. In fact, most of the steps of the proof in this section are completely applicable to the case of $i_0 > 0$.

5.3.2.1 Derivation of “modified Alinhac good unknowns" in anisotropic Sobolev space

We still use Alinhac good unknowns to control the tangential derivatives. However, we cannot directly replace ∂_3^4 by $\overline{\partial}^8$ in (5.3.4) because the commutator contains the terms like $\overline{\partial}^7\partial\eta$, $\overline{\partial}^7\partial v$ and $\overline{\partial}^7\partial Q$ whose L^2-norm cannot be controlled in H_*^8. In specific, we have

$$\begin{aligned}\overline{\partial}^8(\nabla_A^i f) =& \nabla_A^\alpha(\overline{\partial}^8 f) + (\overline{\partial}^8 A^{\mu\alpha})\partial_\mu f + [\overline{\partial}^8, A^{\mu\alpha}, \partial_\mu f] \\ =& \nabla_A^i(\overline{\partial}^8 f) - \overline{\partial}^7(A^{\mu\nu}\,\overline{\partial}\partial_\beta\eta_\nu\,A^{\beta\alpha})\partial_\mu f + [\overline{\partial}^8, A^{\mu\alpha}, \partial_\mu f] \\ =& \nabla_A^i(\overline{\partial}^8 f - \overline{\partial}^8\eta_\nu\,A^{\mu\nu}\,\partial_\mu f) + \overline{\partial}^8\eta_\nu\,\nabla_A^i(\nabla_A^\nu f) \\ & - ([\overline{\partial}^7, A^{\mu\nu}A^{\beta\alpha}]\overline{\partial}\partial_\beta\eta_\nu)\partial_\mu f + [\overline{\partial}^8, A^{\mu\alpha}, \partial_\mu f].\end{aligned} \tag{5.3.51}$$

We notice that the $L^2(\Omega)$-norm of the following quantities coming from the last two terms of (5.3.51) cannot be controlled because $\overline{\partial}^7$ may fall on $a = \partial\eta \times \partial\eta$ and ∂f.

$$e_1 := -\overline{\partial}^7(A^{\mu\nu}A^{\beta\alpha})\,\overline{\partial}\partial_\beta\eta_\nu\,\partial_\mu f, \quad e_2 := -7\overline{\partial}(A^{\mu\nu}A^{\beta\alpha})\,\overline{\partial}^7\partial_\beta\eta_\nu\,\partial_\mu f$$
$$e_3 := 8(\overline{\partial}^7 A^{\mu\alpha})(\overline{\partial}\partial_\mu f), \quad e_4 := 8(\overline{\partial}A^{\mu\alpha})(\overline{\partial}^7\partial_\mu f). \tag{5.3.52}$$

Here $8\overline{\partial}^7$ means there are 8 terms of the form $\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}$ with $i_1 + i_2 = 7$. We will repeatedly use similar notations throughout the manuscript.

Our idea to overcome this difficulty is mainly based on the following three techniques:

1. Modify the definition of "Alinhac good unknowns": Rewrite these quantities in terms of $\nabla_A^\alpha(\cdots) +$ L^2-bounded terms, and then merge the terms inside the covariant derivative ∇_A^α into the "Alinhac good unknowns".

2. Produce a weighted normal derivative to replace a non-weighted one: There are terms like $(\overline{\partial}^7\partial_3\eta)(\overline{\partial}Q)$. Since $Q|_\Gamma = 0$, we know $\overline{\partial}Q|_\Gamma = 0$. Therefore, we can estimate the L^∞-norm of $\overline{\partial}Q$ by fundamental theorem of calculus: (Suppose $y_3 > 0$ without loss of generality)

$$|\overline{\partial}Q(t,y_3)|_{L^\infty(\mathbb{T}^2)} = \left|0 + \int_1^{y_3}\overline{\partial}\partial_3 Q(t,\zeta_3)d\zeta_3\right|_{L^\infty(\mathbb{T}^2)}$$
$$\leq (1-y_3)\|\overline{\partial}\partial_3 Q\|_{L^\infty} \leq \sigma(y_3)\|\overline{\partial}\partial_3 Q\|_{L^\infty},$$

   then we move the $\sigma(y_3)$ to $\overline{\partial}^7\partial_3\eta$ to get a weighted normal derivative $(\sigma\partial_3)^1\overline{\partial}^7\eta$ whose L^2-norm can be directly bounded in H_*^8.

3. Replace $\nabla_{\mathbf{A}}Q$ (contains a normal derivative) by $-\rho_0\partial_t v + (b_0\cdot\partial)(J^{-1}(b_0\cdot\partial)\eta)$ (only contains tangential derivative) in order to make the order of the derivatives lower thanks to the anisotropy of H_*^m.

Now we analyze these extra terms from the commutator. We start with $8(\overline{\partial}^7 A^{\mu\alpha})(\overline{\partial}\partial_\mu f)$ and

$8(\overline{\partial}A^{\mu\alpha})(\overline{\partial}^7\partial_\mu f)$ coming from $[\overline{\partial}^8, A^{\mu\alpha}, \partial_\mu f]$ in (5.3.51). Since $\overline{\partial}A^{\mu\alpha} = -A^{\mu\nu}\,\overline{\partial}\partial_\beta\eta_\nu\,A^{\beta\alpha}$, we have

$$\overline{\partial}^7 A^{\mu\alpha} = -A^{\mu\nu}\,\overline{\partial}^7\partial_\beta\eta_\nu\,A^{\beta\alpha} - [\overline{\partial}^6, A^{\mu\nu}A^{\beta\alpha}]\partial_\beta\eta_\nu,$$

where the highest order term in $[\overline{\partial}^6, A^{\mu\nu}A^{\beta\alpha}]\partial_\beta\eta_\nu$ is $\overline{\partial}^6\partial_\beta\eta_\nu$ whose L^2-norm can be directly bounded by $\|\eta\|_{8,*}$. Therefore, we have

$$\begin{aligned}
8(\overline{\partial}^7 A^{\mu\alpha})(\overline{\partial}\partial_\mu f) &= -8(A^{\beta\alpha}\,\partial_\beta\overline{\partial}^7\eta_\nu\,A^{\mu\nu})\overline{\partial}\partial_\mu f - 8([\overline{\partial}^6, A^{\mu\nu}A^{\beta\alpha}]\partial_\beta\eta_\nu)\overline{\partial}\partial_\mu f \\
&= -8\nabla_A^\alpha(\overline{\partial}^7\eta_\nu\,A^{\mu\nu}\overline{\partial}\partial_\mu f)\underbrace{+8\nabla_A^\alpha(\nabla_A^\nu\overline{\partial}f)\overline{\partial}^7\eta_\nu - 8([\overline{\partial}^6, A^{\mu\nu}A^{\beta\alpha}]\partial_\beta\eta_\nu)\overline{\partial}\partial_\mu f}_{=:C_1(f)} \qquad (5.3.53)\\
&=: -8\nabla_A^\alpha(\overline{\partial}^7\eta\cdot\nabla_A\overline{\partial}f) + C_1(f),
\end{aligned}$$

where $C_1(f)$ can be controlled by using $H^{1/2}\hookrightarrow L^3$ and $H^1\hookrightarrow L^6$ in 3D domain

$$\begin{aligned}
C_1(f) &\lesssim \|A\|_{L^\infty}^2\|\partial^2\overline{\partial}f\|_{L^6}\|\overline{\partial}^7\eta\|_{L^3} + \|A\,\overline{\partial}f\,\partial A\|_{L^\infty}\|\overline{\partial}^7\eta\|_{L^2} + P(\|\eta\|_{8,*})\|\overline{\partial}\partial f\|_{L^\infty} \\
&\lesssim \|A\|_{L^\infty}^2\|\partial^2\overline{\partial}f\|_1\|\langle\overline{\partial}\rangle^{1/2}\overline{\partial}^7\eta\|_0 + P(\|\eta\|_{8,*})\|\overline{\partial}\partial f\|_{L^\infty} \\
&\lesssim \|A\|_{L^\infty}^2\|\partial^2\overline{\partial}f\|_1\|\overline{\partial}^7\eta\|_0^{1/2}\|\overline{\partial}^8\eta\|_0^{1/2} + P(\|\eta\|_{8,*})\|\overline{\partial}\partial f\|_{L^\infty} \lesssim P(\|\eta\|_{8,*})\|f\|_{7,*}.
\end{aligned}$$

The term $8(\overline{\partial}A^{\mu\alpha})(\overline{\partial}^7\partial_\mu f)$ should be treated differently in the case of $f = v_\alpha$ and $f = Q$ respectively.

- When $f = v_\alpha$, then this term becomes

$$\begin{aligned}
8(\overline{\partial}A^{\mu\alpha})(\overline{\partial}^7\partial_\mu v_\alpha) &= -8A^{\mu\nu}\,\overline{\partial}\partial_\beta\eta_\nu\,A^{\beta\alpha}\overline{\partial}^7\partial_\mu v_\alpha = -8A^{\mu\alpha}\,\overline{\partial}\partial_\beta\eta_\alpha\,A^{\beta\nu}\,\overline{\partial}^7\partial_\mu v_\nu \\
&= -8\nabla_A^\alpha(\overline{\partial}^7 v_\nu\,A^{\beta\nu}\,\overline{\partial}\partial_\beta\eta_\alpha) + \underbrace{8\nabla_A^\alpha(\overline{\partial}\partial_\beta\eta_\alpha\,A^{\beta\nu})\overline{\partial}^7 v_\nu}_{=:C_2(v)} \qquad (5.3.54)\\
&=: -8\nabla_A^\alpha(\overline{\partial}^7 v\cdot\nabla_A\overline{\partial}\eta_\alpha) + C_2(v),
\end{aligned}$$

  and similarly we have $\|C_2(v)\|_0 \lesssim P(\|\eta\|_{7,*})\|v\|_{8,*}$.

- When $f = Q$, we cannot mimic the simplification as above. Instead, we need to invoke the MHD equation to replace $\nabla_A Q$ by tangential derivatives. We consider

$$
\begin{aligned}
&8(J\overline{\partial}A^{\mu\alpha})(\overline{\partial}^7\partial_\mu Q) = -8(\mathbf{A}^{\mu\nu}\,\overline{\partial}\partial_\beta\eta_\nu\, A^{\beta\alpha})\overline{\partial}^7\partial_\mu Q \\
=&-8\overline{\partial}^7(\mathbf{A}^{\mu\nu}\partial_\mu Q)\, A^{\beta\alpha}\overline{\partial}\partial_\beta\eta_\nu + 8(\overline{\partial}^7\mathbf{A}^{\mu\nu})(\partial_\mu Q)(\overline{\partial}\partial_\beta\eta_\nu\, A^{\beta\alpha}) \\
&+8\sum_{N=1}^{6}\binom{7}{N}(\overline{\partial}^N\mathbf{A}^{\mu\nu})(\overline{\partial}^{7-N}\partial_\mu Q)(\overline{\partial}\partial_\beta\eta_\nu\, A^{\beta\alpha}) \\
=&8\overline{\partial}^7\left(\rho_0\partial_t v^\nu - (b_0\cdot\partial)(J^{-1}(b_0\cdot\partial)\eta^\nu)\right)A^{\beta\alpha}\overline{\partial}\partial_\beta\eta_\nu + 8(\overline{\partial}^7\mathbf{A}^{\mu\nu})(\partial_\mu Q)(\overline{\partial}\partial_\beta\eta_\nu\, A^{\beta\alpha}) \\
&+8\sum_{N=1}^{6}\binom{7}{N}(\overline{\partial}^N\mathbf{A}^{\mu\nu})(\overline{\partial}^{7-N}\partial_\mu Q)(\overline{\partial}\partial_\beta\eta_\nu\, A^{\beta\alpha}) \\
=:&C_{21}+C_{22}+C_{23}.
\end{aligned}
\tag{5.3.55}
$$

The L^2-norm of C_{23} can be directly controlled

$$
\|C_{23}\|_0 \lesssim \|\eta\|_{8,*}\|Q\|_{8,*}P(\|\eta\|_{7,*}). \tag{5.3.56}
$$

The L^2-norm of C_{22} can be directly controlled when $l = 3$ because $\mathbf{A}^{3\nu}$ consists of $\overline{\partial}\eta\times\overline{\partial}\eta$. When $l = 1, 2$, we need to invoke the second technique above, i.e., using $\overline{\partial}Q|_\Gamma = 0$ to produce a weight function $\sigma(y_3)$.

$$
\begin{aligned}
\|C_{22}\|_0 \lesssim&\|\overline{\partial}^7\mathbf{A}^{3\nu}\|_0\|\partial_3 Q\,\overline{\partial}\partial\eta\, a\|_{L^\infty} + \sum_{L=1}^{2}\|(\overline{\partial}^7\mathbf{A}^{L\nu})(\overline{\partial}_L Q)(\overline{\partial}\partial_\beta\eta_\nu\, A^{\beta\alpha})\|_0 \\
\lesssim&P(\|\eta\|_{7,*})\|\overline{\partial}^8\eta\|_0\|Q\|_3 + \sum_{L=1}^{2}\|(\overline{\partial}^7\mathbf{A}^{L\nu})(\sigma(y_3)\partial_3\overline{\partial}_L Q)(\overline{\partial}\partial_\beta\eta_\nu\, A^{\beta\alpha})\|_0 \\
\lesssim&P(\|\eta\|_{7,*})\|\overline{\partial}^8\eta\|_0\|Q\|_3 + \sum_{L=1}^{2}\|\sigma\overline{\partial}^7\mathbf{A}^{L\nu}\|_0\|(\partial_3\overline{\partial}_L Q)(\overline{\partial}\partial_\beta\eta_\nu\, A^{\beta\alpha})\|_{L^\infty} \\
\lesssim&P(\|\eta\|_{7,*})\|Q\|_{7,*}\left(\|\overline{\partial}^8\eta\|_0 + \|(\sigma\partial_3)\overline{\partial}^7\eta\|_0\right),
\end{aligned}
\tag{5.3.57}
$$

where we use the fact that $\mathbf{A}^{L\nu}$ consists of $(\partial_3\eta)(\overline{\partial}\eta)$ in the last step.

Finally, C_{21} can also be directly bounded because the top order derivatives are $\overline{\partial}^7\partial_t$ and $\overline{\partial}^7(b_0\cdot\partial)$. Note that $b_0^3|_\Gamma = 0$ yields the following estimates by using the second technique mentioned above.

$$\|b_0^3\partial_3\overline{\partial}^7(J^{-1}(b_0\cdot\partial)\eta)\|_0 \lesssim \|\partial b_0\|_2\|(\sigma\partial_3)\overline{\partial}^7(J^{-1}(b_0\cdot\partial)\eta)\|_0,$$

and thus

$$C_{21} \lesssim P(\|\eta\|_{7,*})(\|\rho_0\|_{7,*}\|v\|_{8,*} + \|b_0\|_{7,*}\|(b_0\cdot\partial)\eta\|_{8,*}). \tag{5.3.58}$$

Therefore, we have the estimates for $C_2(Q) := 8\overline{\partial}A^{\mu\alpha}\overline{\partial}^7\partial_\mu Q$

$$\|C_2(Q)\|_0 \lesssim P(\|\eta\|_{8,*}, \|v\|_{8,*}, \|b_0\|_{7,*}, \|J^{-1}(b_0\cdot\partial)\eta\|_{8,*}, \|\rho_0\|_{7,*}, \|Q\|_{8,*}). \tag{5.3.59}$$

Next we analyze $-(\overline{\partial}^7(A^{\mu\nu}A^{\beta\alpha})\,\overline{\partial}\partial_\beta\eta_\nu)\partial_\mu f$ coming from $-([\overline{\partial}^7, A^{\mu\nu}A^{\beta\alpha}]\overline{\partial}\partial_\beta\eta_\nu)\partial_\mu f$. There are two terms of top order derivatives:

$$-\overline{\partial}^7(A^{\mu\nu}A^{\beta\alpha})\,\overline{\partial}\partial_\beta\eta_\nu\,\partial_\mu f = -(\overline{\partial}^7A^{\mu\nu})A^{\beta\alpha}\,\overline{\partial}\partial_\beta\eta_\nu\,\partial_\mu f - A^{\mu\nu}(\overline{\partial}^7A^{\beta\alpha})\,\overline{\partial}\partial_\beta\eta_\nu\,\partial_\mu f$$
$$- \sum_{N=1}^{6}\binom{7}{N}(\overline{\partial}^NA^{\mu\nu})(\overline{\partial}^{6-N}A^{\beta\alpha})\overline{\partial}\partial_\beta\eta_\nu\,\partial_\mu f,$$

where the L^2-norm of the last term can be directly controlled

$$\left\|\sum_{N=1}^{6}\binom{7}{N}(\overline{\partial}^NA^{\mu\nu})(\overline{\partial}^{6-N}A^{\beta\alpha})\overline{\partial}\partial_\beta\eta_\nu\,\partial_\mu f\right\|_0 \lesssim P(\|\eta\|_{8,*})\|f\|_3.$$

Similarly as (5.3.53), the term $-A^{\mu\nu}(\overline{\partial}^7A^{\beta\alpha})\overline{\partial}\partial_\beta\eta_\nu\,\partial_\mu f$ can be written as the covariant derivatives plus L^2-bounded terms

$$
\begin{aligned}
&- A^{\mu\nu}(\overline{\partial}^7 A^{\beta\alpha})\overline{\partial}\partial_\beta\eta_\nu\,\partial_\mu f \\
=&A^{\mu\nu}A^{\beta\nu}(\partial_k\overline{\partial}^7\eta_\nu)A^{ki}\,\overline{\partial}\partial_\beta\eta_\nu\,\partial_\mu f + ([\overline{\partial}^6, A^{\beta\nu}A^{ki}]\overline{\partial}\partial_k\eta_\nu)A^{\mu\nu}\,\overline{\partial}\partial_\beta\eta_\nu\,\partial_\mu f \\
=&\nabla_A^\alpha(\overline{\partial}^7\eta_\nu\,A^{\beta\nu}\,\overline{\partial}\partial_\beta\eta_\nu\,A^{\mu\nu}\,\partial_\mu f) \\
&\underbrace{-\overline{\partial}^7\eta_\nu\,\nabla_A^\alpha(A^{\beta\nu}\,\overline{\partial}\partial_\beta\eta_\nu\,A^{\mu\nu}\,\partial_\mu f) + ([\overline{\partial}^6, A^{\beta\nu}A^{ki}]\overline{\partial}\partial_k\eta_\nu)A^{\mu\nu}\,\overline{\partial}\partial_\beta\eta_\nu\,\partial_\mu f}_{=:C_3(f)} \\
=:&\nabla_A^\alpha(\overline{\partial}^7\eta\cdot\nabla_A\overline{\partial}\eta\cdot\nabla_A f) + C_3(f),
\end{aligned}
\tag{5.3.60}
$$

where $C_3(f)$ can be directly controlled similarly as $C_1(f)$

$$
\|C_3(f)\|_0 \lesssim P(\|\eta\|_{8,*})\|\partial f\|_2.
$$

We then compute $-(\overline{\partial}^7 A^{\mu\nu})A^{\beta\alpha}\,\overline{\partial}\partial_\beta\eta_\nu\,\partial_\mu f$.

- When $f = v_\alpha$: Similarly as in (5.3.60), we have

$$
\begin{aligned}
&- (\overline{\partial}^7 A^{\mu\nu})A^{\beta\alpha}\,\overline{\partial}\partial_\beta\eta_\nu\,\partial_\mu v_\alpha \\
=&A^{\mu\nu}(\overline{\partial}^7\partial_k\eta_\nu)A^{kr}A^{\beta\alpha}\,\overline{\partial}\partial_\beta\eta_\nu\,\partial_\mu v_\alpha - ([\overline{\partial}^6, A^{\mu\nu}A^{kr}]\overline{\partial}\partial_k\eta_\nu)A^{\beta\alpha}\,\overline{\partial}\partial_\beta\eta_\nu\,\partial_\mu v_\alpha \\
=&A^{\mu\nu}(\overline{\partial}^7\partial_k\eta_\nu)A^{ki}A^{mr}\,\overline{\partial}\partial_\beta\eta_\alpha\,\partial_\mu v_r - ([\overline{\partial}^6, A^{\mu\nu}A^{kr}]\overline{\partial}\partial_k\eta_\nu)A^{\beta\alpha}\,\overline{\partial}\partial_\beta\eta_\nu\,\partial_\mu v_\alpha \\
=&\nabla_A^\alpha(\overline{\partial}^7\eta_\nu\,A^{\mu\nu}\,\partial_\mu v_r\,A^{mr}\,\overline{\partial}\partial_\beta\eta_\alpha) \\
&\underbrace{-\nabla_A^\alpha(A^{\mu\nu}A^{mr}\overline{\partial}\partial_\beta\eta_\alpha\,\partial_\mu v_r)\overline{\partial}^7\eta_\nu - ([\overline{\partial}^6, A^{\mu\nu}A^{kr}]\overline{\partial}\partial_k\eta_\nu)A^{\beta\alpha}\,\overline{\partial}\partial_\beta\eta_\nu\,\partial_\mu v_\alpha}_{=:C_4(v)} \\
=:&\nabla_A^\alpha(\overline{\partial}^7\eta\cdot\nabla_A v\cdot\nabla_A\overline{\partial}\eta_\alpha) + C_4(v),
\end{aligned}
\tag{5.3.61}
$$

where $C_4(v)$ can be directly controlled similarly as $C_1(f)$

$$\|C_4(v)\|_0 \lesssim P(\|\eta\|_{8,*})\|\partial v\|_2.$$

- When $f = Q$: If $l = 3$, then this term can be directly controlled since $A^{3\nu} = J^{-1}\overline{\partial}\eta \times \overline{\partial}\eta$ only contains first-order tangential derivatives. If $l = 1, 2$, then we can mimic the treatment of C_{22}, i.e., using $\overline{\partial}_L Q|_\Gamma = 0$ and fundamental theorem of calculus to produce a weight function $\sigma(y_3)$ and move that to $\overline{\partial}^7 A^{\mu\nu}$. Define $C_4(Q) := -(\overline{\partial}^7 A^{\mu\nu}) A^{\beta\alpha}\,\overline{\partial}\partial_\beta \eta_\nu\,\partial_\mu Q$, then

$$\|C_4(Q)\|_0 \lesssim \|(\overline{\partial}^7 A^{3\nu}) A^{\beta\alpha}\,\overline{\partial}\partial_\beta\eta_\nu\,\partial_3 Q\|_0 + \sum_{L=1}^{2} \|(\overline{\partial}^7 A^{L\nu}) A^{\beta\alpha}\,\overline{\partial}\partial_\beta\eta_\nu\,\overline{\partial}_L Q\|_0$$

$$\lesssim \|\overline{\partial}^8\eta\|_0 \|Q\|_3 P(\|\partial\eta\|_2, \|\overline{\partial}\partial\eta\|_2) + \sum_{L=1}^{2} \|\sigma\overline{\partial}^7 A^{L\nu}\|_0 \|A^{\beta\alpha}\,\overline{\partial}\partial_\beta\eta_\nu\,\overline{\partial}_L\partial_3 Q\|_{L^\infty} \tag{5.3.62}$$

$$\lesssim \left(\|\overline{\partial}^8\eta\|_0 + \|(\sigma\partial_3)\overline{\partial}^7\eta\|_0\right) P(\|Q\|_3, \|\overline{\partial}Q\|_3, \|\eta\|_{7,*}).$$

Next we analyze $-7\overline{\partial}(A^{\mu\nu}A^{\beta\alpha})\overline{\partial}^7\partial_\beta\eta_\nu\,\partial_\mu f$ coming from $-[\overline{\partial}^7, A^{\mu\nu}A^{\beta\alpha}]\overline{\partial}\partial_\beta\eta_\nu\,\partial_\mu f$. This term cannot be directly controlled when $m = 3$. We should analyze it term by term. First we have

$$-7\overline{\partial}(A^{\mu\nu}A^{\beta\alpha})\overline{\partial}^7\partial_\beta\eta_\nu\,\partial_\mu f = -7\overline{\partial}A^{\mu\nu}\,A^{\beta\alpha}\,\overline{\partial}^7\partial_\beta\eta_\nu\,\partial_\mu f - 7A^{\mu\nu}\,\overline{\partial}A^{\beta\alpha}\,\overline{\partial}^7\partial_\beta\eta_\nu\,\partial_\mu f$$

$$=7A^{\mu\xi}\,\partial_\kappa\overline{\partial}\eta_\xi\,A^{\kappa\nu}A^{\beta\alpha}\,\overline{\partial}^7\partial_\beta\eta_\nu\,\partial_\mu f + 7A^{\mu\nu}A^{\beta\xi}\,\partial_\kappa\overline{\partial}\eta_\xi\,A^{\kappa\alpha}\,\overline{\partial}^7\partial_\beta\eta_\nu\,\partial_\mu f.$$

The first term can be directly rewritten as follows

$$7A^{\mu\xi}\,\partial_\kappa\overline{\partial}\eta_\xi\,A^{\kappa\nu}A^{\beta\alpha}\,\overline{\partial}^7\partial_\beta\eta_\nu\,\partial_\mu f$$

$$=7\nabla_A^\alpha(\overline{\partial}^7\eta_\nu\,A^{\kappa\nu}\,\partial_\kappa\overline{\partial}\eta_\xi\,A^{\mu\xi}\,\partial_\mu f)\underbrace{-7\nabla_A^\alpha(A^{\kappa\nu}\,\partial_\kappa\overline{\partial}\eta_\xi\,A^{\mu\xi}\,\partial_\mu f)\overline{\partial}^7\eta_\nu}_{C_5(f)} \tag{5.3.63}$$

$$=:7\nabla_A^\alpha(\overline{\partial}^7\eta\cdot\nabla_A\overline{\partial}\eta\cdot\nabla_A f) + C_5(f),$$

where $C_5(f)$ can be similarly controlled as $C_1(f)$

$$\|C_5(f)\|_0 \lesssim P(\|\eta\|_{8,*})\|\partial f\|_3.$$

Then we analyze $7A^{\mu\nu}A^{\beta\xi}(\partial_\kappa\overline{\partial}\eta_\xi)A^{\kappa\alpha}(\overline{\partial}^7\partial_\beta\eta_\nu)\partial_\mu f$, which needs different treatment for $f = v_\alpha$ and $f = Q$ respectively.

- When $f = v_\alpha$, we have the following simplification

$$7A^{\mu\nu}A^{\beta\xi}\partial_k\overline{\partial}\eta_\xi\, A^{\kappa\alpha}\,\overline{\partial}^7\partial_\beta\eta_\nu\,\partial_\mu v_\alpha = 7A^{\mu\nu}A^{\beta\alpha}\,\partial_\kappa\overline{\partial}\eta_\alpha\, A^{\kappa\xi}\,\overline{\partial}^7\partial_\beta\eta_\nu\,\partial_\mu v_\xi$$

$$=7\nabla_A^\alpha(\overline{\partial}^7\eta_\nu\, A^{\mu\nu}\,\partial_\mu v_\xi\, A^{\kappa\xi}\,\overline{\partial}\partial_\kappa\eta_\alpha) + \underbrace{7\nabla_A^\alpha(A^{\mu\nu}\,\partial_\mu v_\xi\, A^{\kappa\xi}\,\overline{\partial}\partial_\kappa\eta_\alpha)\overline{\partial}^7\eta_\nu}_{C_6(v)} \tag{5.3.64}$$

$$=:7\nabla_A^\alpha(\overline{\partial}^7\eta\cdot\nabla_A v\cdot\nabla_A\overline{\partial}\eta_\alpha) + C_6(v),$$

and $\|C_6(v)\|_0 \lesssim P(\|\eta\|_{8,*})\|v\|_3$ follows from direct computation.

- When $f = Q$, this term becomes

$$C_6(Q) := -7A^{\mu\nu}(\overline{\partial}A^{\beta\alpha})(\overline{\partial}^7\partial_\beta\eta_\nu)\partial_\mu Q$$

$$=-7\left(\underbrace{\overline{\partial}^7(A^{\mu\nu}\partial_\beta\eta_\nu)}_{=\overline{\partial}^7\delta^l_m=0}-(\overline{\partial}^7A^{\mu\nu})(\partial_\beta\eta_\nu)-\sum_{N=1}^{6}\binom{7}{N}(\overline{\partial}^N A^{\mu\nu})(\overline{\partial}^{7-N}\partial_\beta\eta_\nu)\right)\overline{\partial}A^{\beta\alpha}\,\partial_\mu Q$$

$$=7(\overline{\partial}^7A^{3\nu})\partial_\beta\eta_\nu\,\overline{\partial}A^{\beta\alpha}\,\partial_3 Q + \sum_{L=1}^{2}(\overline{\partial}^7A^{L\nu})\partial_\beta\eta_\nu\,\overline{\partial}A^{\beta\alpha}\,\overline{\partial}_L Q$$

$$+7\sum_{N=1}^{6}\binom{7}{N}(\overline{\partial}^N A^{\mu\nu})(\overline{\partial}^{7-N}\partial_\beta\eta_\nu)\overline{\partial}A^{\beta\alpha}\,\partial_\mu Q$$

$$=:C_{61}+C_{62}+C_{63}. \tag{5.3.65}$$

Since $A^{3\nu} = J^{-1}\overline{\partial}\eta\times\overline{\partial}\eta$, we know the top order term is of the form $\overline{\partial}^8\eta\cdot\overline{\partial}\eta$ and thus C_{61} can be directly controlled

$$\|C_{61}\|_0 \lesssim P(\|\eta\|_{8,*})\|\partial_3 Q\|_2.$$

The term C_{62} can be treated in the same way as $C_4(Q)$ in (5.3.62) by using $\overline{\partial}_L Q|_\Gamma = 0$ to produce a weight function σ

$$\|C_{62}\|_0 \lesssim (\|(\sigma\partial_3)\overline{\partial}^7\eta\| + \|\overline{\partial}^8\eta\|_0) P(\|\eta\|_{7,*})\|\overline{\partial}\partial_3 Q\|_2 \lesssim P(\|\eta\|_{8,*})\|Q\|_{7,*}.$$

Finally, C_{63} can be directly controlled

$$\|C_{63}\|_0 \lesssim P(\|\eta\|_{8,*})\|\partial Q\|_2,$$

and thus

$$\|C_6(Q)\|_0 \lesssim P(\|\eta\|_{8,*})\|Q\|_{7,*}. \tag{5.3.66}$$

Now we plug (5.3.53)-(5.3.54), (5.3.59)-(5.3.66) into (5.3.51) and define the "modified Alinhac good unknowns" of v and Q with respect to $\overline{\partial}^8$ as

$$\begin{aligned}
\mathbf{V}_\alpha^* := {} & \overline{\partial}^8 v_\alpha - \overline{\partial}^8\eta\cdot\nabla_A v_\alpha \\
& - 8\overline{\partial}^7\eta\cdot\nabla_A\overline{\partial}v_\alpha - 8\overline{\partial}^7 v\cdot\nabla_A\overline{\partial}\eta_\alpha \\
& + \overline{\partial}^7\eta\cdot\nabla_A\overline{\partial}\eta\cdot\nabla_A v_\alpha + \overline{\partial}^7\eta\cdot\nabla_A v\cdot\nabla_A\overline{\partial}\eta_\alpha \\
& + 7\overline{\partial}^7\eta\cdot\nabla_A\overline{\partial}\eta\cdot\nabla_A v_\alpha + 7\overline{\partial}^7\eta\cdot\nabla_A v\cdot\nabla_A\overline{\partial}\eta_\alpha \\
= {} & \overline{\partial}^8 v_\alpha - \overline{\partial}^8\eta\cdot\nabla_A v_\alpha \\
& - 8\overline{\partial}^7\eta\cdot\nabla_A\overline{\partial}v_\alpha - 8\overline{\partial}^7 v\cdot\nabla_A\overline{\partial}\eta_\alpha + 8\overline{\partial}^7\eta\cdot\nabla_A\overline{\partial}\eta\cdot\nabla_A v_\alpha + 8\overline{\partial}^7\eta\cdot\nabla_A v\cdot\nabla_A\overline{\partial}\eta_\alpha,
\end{aligned} \tag{5.3.67}$$

and

$$\mathbf{Q}^* := \overline{\partial}^8 Q - \overline{\partial}^8\eta\cdot\nabla_A Q - 8\overline{\partial}^7\eta\cdot\nabla_A\overline{\partial}Q + 8\overline{\partial}^7\eta\cdot\nabla_A\overline{\partial}\eta\cdot\nabla_A Q. \tag{5.3.68}$$

Then the modified good unknowns satisfy the following relations

$$\overline{\partial}^8(\text{div}_{\tilde{A}} v) = \nabla_A \cdot \mathbf{V}^* + \sum_{M=0}^{6} C_M(v), \quad \overline{\partial}^8(\nabla_A Q) = \nabla_A \mathbf{Q}^* + \sum_{M=0}^{6} C_M(Q), \tag{5.3.69}$$

where $C_0(f)$ comes from the directly controllable terms in the RHS of (5.3.51)

$$\begin{aligned} C_0(f) :=& \overline{\partial}^8 \eta_r \, \nabla_A^\alpha(\nabla_A^r f) - \sum_{N=2}^{6} \binom{7}{N} \overline{\partial}^N (A^{\mu\nu} A^{\beta\alpha}) \overline{\partial}^{7-N} (\overline{\partial} \partial_\beta \eta_r) \partial_\mu f \\ &+ \sum_{N=2}^{6} \binom{8}{N} (\overline{\partial}^N A^{\mu\alpha})(\overline{\partial}^{8-N} \partial_\mu f), \end{aligned} \tag{5.3.70}$$

satisfies

$$\|C_0(f)\|_0 \lesssim P(\|\eta\|_{8,*})\|f\|_{8,*},$$

and $C_1 \sim C_6$ are constructed in (5.3.53)-(5.3.54), (5.3.59)-(5.3.66).

Now we denote $C^*(f) := C_0(f) + C_1(f) + \cdots + C_6(f)$ and the "extra modification terms" in the modified Alinhac good unknowns by

$$(\triangle_v^*)_i := -8\overline{\partial}^7\eta \cdot \nabla_A \overline{\partial} v_\alpha - 8\overline{\partial}^7 v \cdot \nabla_A \overline{\partial}\eta_\alpha + 8\overline{\partial}^7\eta \cdot \nabla_A \overline{\partial}\eta \cdot \nabla_A v_\alpha + 8\overline{\partial}^7\eta \cdot \nabla_A v \cdot \nabla_A \overline{\partial}\eta_\alpha,$$

$$\triangle_Q^* := -8\overline{\partial}^7\eta \cdot \nabla_A \overline{\partial} Q + 8\overline{\partial}^7\eta \cdot \nabla_A \overline{\partial}\eta \cdot \nabla_A Q.$$

Then the modified Alinhac good unknowns become

$$\mathbf{V}^* = \overline{\partial}^8 v - \overline{\partial}^8 \eta \cdot \nabla_A v + \triangle_v^*, \ \ \mathbf{Q}^* = \overline{\partial}^8 Q - \overline{\partial}^8 \eta \cdot \nabla_A Q + \triangle_Q^*.$$

Remark 5.3.4. There are more modification terms in $\mathbf{V}^*$ than in $\mathbf{Q}^*$. The reason is that we can replace $\nabla_A Q$ which contains a normal derivative with tangential derivative ($\partial_t v$ and $(b_0 \cdot \partial)(J^{-1}(b_0 \cdot \partial)\eta)$) by invoking the MHD equation. However, similar relation only holds for $\text{div}_{\tilde{A}} v$ instead of $\nabla_A v$. Therefore, for those terms in the commutators containing v, we have to rewrite them to be the covariant derivatives of the modifition terms plus $L^2(\Omega)$-bounded terms.

It is straightforward to see that the $L^2(\Omega)$ norms of $\triangle_v^*, \triangle_Q^*, \partial_t(\triangle_v^*)$ and $\partial_t(\triangle_Q)$ can be controlled by $P(\mathfrak{E}(t))$

$$\|\partial_t(\triangle_v^*)\|_0 \lesssim \|\overline{\partial}^7 v\|_0(\|\nabla_A\overline{\partial}v\|_2 + \|\nabla_A\overline{\partial}\eta\|_2\|\nabla_A v\|_2) + \|\overline{\partial}^7\partial_t v\|_0\|\nabla_A\overline{\partial}\eta\|_2$$

$$+ \|\overline{\partial}^7\eta\|_0(\|\nabla_A\overline{\partial}\partial_t v\|_2 + \|\nabla_A\overline{\partial}\eta\|_2\|\nabla_A\partial_t v\|_2 + \|\nabla_A\overline{\partial}v\|_2\|\nabla_A v\|_2) \tag{5.3.71}$$

$$\lesssim P(\|\eta\|_{8,*}, \|v\|_{8,*}),$$

$$\|\partial_t(\triangle_Q^*)\|_0 \lesssim \|\overline{\partial}^7 v\|_0(\|\nabla_A\overline{\partial}Q\|_2 + \|\nabla_A\overline{\partial}\eta\|_2\|\nabla_A Q\|_2)$$

$$+ \|\overline{\partial}^7\eta\|_0(\|\nabla_A\overline{\partial}\partial_t Q\|_2 + \|\nabla_A\overline{\partial}\eta\|_2\|\nabla_A\partial_t Q\|_2 + \|\nabla_A\overline{\partial}v\|_2\|\nabla_A Q\|_2) \tag{5.3.72}$$

$$\lesssim P(\|\eta\|_{8,*}, \|v\|_{7,*}, \|Q\|_{8,*}),$$

$$\|\triangle_Q^*\|_0 + \|\triangle_v^*\|_0 \lesssim P(\|\eta\|_{7,*}, \|v\|_{7,*}, \|Q\|_{7,*}). \tag{5.3.73}$$

Now we take $\overline{\partial}^8$ and invoking (5.3.69) to get the evolution equation of $\mathbf{V}^*$ and $\mathbf{Q}^*$

$$R\partial_t\mathbf{V}^* - J^{-1}(b_0\cdot\partial)\overline{\partial}^8\left(J^{-1}(b_0\cdot\partial)\eta\right) + \nabla_A\mathbf{Q}^*$$
$$= \left[R, \overline{\partial}^8\right]\partial_t v + \left[\overline{\partial}^8, J^{-1}(b_0\cdot\partial)\right]\left(J^{-1}(b_0\cdot\partial)\eta\right) - C^*(Q) + R\partial_t(-\overline{\partial}^8\eta\cdot\nabla_A v + \triangle_v^*) \tag{5.3.74}$$

We denote the RHS of (5.3.74) by $\mathbf{F}^*$. Similarly as in Section 5.3.1, we compute the L^2-inner product of (5.3.74) and $J\mathbf{V}^*$ to get the energy identity

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho_0\,|\mathbf{V}^*|^2\,\mathrm{d}y = \int_\Omega (b_0\cdot\partial)\overline{\partial}^8(J^{-1}(b_0\cdot\partial)\eta)\cdot\mathbf{V}^* - \int_\Omega(\nabla_{\mathbf{A}}\mathbf{Q}^*)\cdot\mathbf{V}^* + \int_\Omega J\mathbf{F}^*\cdot\mathbf{V}^*. \tag{5.3.75}$$

5.3.2.2 Interior estimates

Using (5.3.71), the third integral on RHS of (5.3.75) is controlled directly

$$\int_\Omega J\mathbf{F}^*\cdot\mathbf{V}^* \lesssim \|J\mathbf{F}^*\|_0\|\mathbf{V}^*\|_0 \lesssim P\left(\|(\rho_0, \eta, v, Q, b_0, (b_0\cdot\partial)\eta)\|_{8,*}\right)\|\mathbf{V}^*\|_0. \tag{5.3.76}$$

The first integral on RHS of (5.3.75) can be similarly treated as (5.3.11)-(5.3.15) by replacing ∂_3^4 by $\overline{\partial}^8$ and $\|\cdot\|_4$-norm by $\|\cdot\|_{8,*}$-norm. We omit the details

$$\int_\Omega (b_0\cdot\partial)\overline{\partial}^8(J^{-1}(b_0\cdot\partial)\eta)\cdot\mathbf{V}^*\,\mathrm{d}y \lesssim -\frac{1}{2}\frac{d}{dt}\int_\Omega J\left|\overline{\partial}^8((b_0\cdot\partial)\eta)\right|^2\,\mathrm{d}y + K_{11}^* + P(\mathfrak{E}(t)), \quad (5.3.77)$$

where K_{11}^* is defined to be

$$K_{11}^* := -\int_\Omega J\overline{\partial}^8(J^{-1}(b_0\cdot\partial)\eta)\cdot\left(J^{-1}(b_0\cdot\partial)\eta\right)\overline{\partial}^8(\mathrm{div}_{\tilde{A}}v)\,\mathrm{d}y. \quad (5.3.78)$$

Next we analyze the term $-\int_\Omega J\nabla_A\mathbf{Q}\cdot\mathbf{V}$. Integrating by parts and using Piola's identity, we get

$$-\int_\Omega(\nabla_{\mathbf{A}}\mathbf{Q}^*)\cdot\mathbf{V}^* = \int_\Omega J\mathbf{Q}(\nabla_A\cdot\mathbf{V}^*) - \int_\Gamma J\mathbf{Q}A^{\mu\alpha}N_\mu\mathbf{V}_\alpha^*\,\mathrm{d}S =: I^* + IB^*. \quad (5.3.79)$$

Invoking (5.3.67), (5.3.69) and $Q = q + \frac{1}{2}|J^{-1}(b_0\cdot\partial)\eta|^2$, we get

$$\begin{aligned} I^* =& \int_\Omega J\overline{\partial}^8 q\,\overline{\partial}^8(\mathrm{div}_{\tilde{A}}v) + \int_\Omega J\overline{\partial}^8\left(\frac{1}{2}\left|J^{-1}(b_0\cdot\partial)\eta\right|^2\right)\overline{\partial}^8(\mathrm{div}_{\tilde{A}}v) \\ &+ \int_\Omega(-\overline{\partial}^8\eta_\nu\,\mathbf{A}^{\mu\nu}\,\partial_\mu Q + \triangle_Q^*)\overline{\partial}^8(\mathrm{div}_{\tilde{A}}v) - \int_\Omega\overline{\partial}^8 Q\,C^*(v) \quad (5.3.80) \\ =:& I_1^* + I_2^* + I_3^* + I_4^*, \end{aligned}$$

where I_4^* can be directly controlled by using the estimates of $C^*(v)$

$$I_4^* \lesssim \|\overline{\partial}^8 Q\|_0\|C^*(v)\|_0 \lesssim P(\|\eta\|_{8,*})\|\overline{\partial}^8 Q\|_0\|v\|_{8,*}. \quad (5.3.81)$$

Similarly, I_2^* produces another higher order term to cancel with K_{11}^*

$$
\begin{aligned}
I_2 =& \underbrace{\int_\Omega J\overline{\partial}^8\left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\left(J^{-1}(b_0\cdot\partial)\eta\right)\overline{\partial}^8(\mathrm{div}_{\tilde{A}}v)}_{\text{exactly cancel with } K_{11}^*} \\
&+\sum_{N=1}^{7}\binom{8}{N}\int_\Omega J\overline{\partial}^N\left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\overline{\partial}^{8-N}\left(J^{-1}(b_0\cdot\partial)\eta\right)\overline{\partial}^8(\mathrm{div}_{\tilde{A}}v) \\
=&-K_{11}^*-\sum_{N=1}^{7}\binom{8}{N}\int_\Omega \frac{J^2R'(q)}{\rho_0}\overline{\partial}^N\left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\overline{\partial}^{8-N}\left(J^{-1}(b_0\cdot\partial)\eta\right)\overline{\partial}^8\partial_t q \\
&-\sum_{N=1}^{7}\binom{8}{N}\int_\Omega J\overline{\partial}^N\left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\overline{\partial}^{8-N}\left(J^{-1}(b_0\cdot\partial)\eta\right)\left(\left[\overline{\partial}^8,\frac{JR'(q)}{\rho_0}\right]\partial_t q\right) \\
=:&-K_{11}^*+I_{21}^*+I_{22}^*
\end{aligned}
\tag{5.3.82}
$$

Similarly as in (5.3.21)-(5.3.22), the term I_{21}^* should be controlled by integrating ∂_t by parts under time integral and I_{22}^* can be directly controlled. We omit the details

$$\int_0^T I_{21}^* \lesssim \varepsilon\|\overline{\partial}^8 q\|_0^2+\mathcal{P}_0+\int_0^T P(\mathfrak{E}(t))\,\mathrm{d}t \tag{5.3.83}$$

$$I_{22}^* \lesssim \|J^{-1}(b_0\cdot\partial)\eta\|_{7,*}^2\|q\|_{8,*}. \tag{5.3.84}$$

The term I_1^* produces the energy term $\|\overline{\partial}^8 q\|_0^2$ as in (5.3.19).

$$I_1^* \lesssim -\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{J^2R'(q)}{\rho_0}|\overline{\partial}^8 q|^2+P(\|q\|_{8,*},\|\rho_0\|_{8,*},\|\eta\|_{8,*}). \tag{5.3.85}$$

I_3^* can be controlled by integrating ∂_t by parts under time integral after invoking $\mathrm{div}_{\tilde{A}}v\ =$

$-\frac{JR'(q)}{\rho_0}\partial_t q$ and (5.3.72)-(5.3.73).

$$\int_0^T I_3^* = \int_0^T \int_\Omega \frac{JR'(q)}{\rho_0}(\overline{\partial}^8 \eta_\nu \, \mathbf{A}^{\mu\nu} \, \partial_\mu Q - \triangle_Q^*)\overline{\partial}^8 \partial_t q$$

$$+ \underbrace{\int_0^T \int_\Omega (\overline{\partial}^8 \eta_\nu \, \mathbf{A}^{\mu\nu} \, \partial_\mu Q - \triangle_Q^*) \left(\left[\overline{\partial}^8, \frac{JR'(q)}{\rho_0} \right] \partial_t q \right)}_{L_2^*}$$

$$\overset{\partial_t}{=} - \int_0^T \int_\Omega \partial_t \left(\frac{JR'(q)}{\rho_0} \overline{\partial}^8 \eta_\nu \, \mathbf{A}^{\mu\nu} \, \partial_\mu Q - \triangle_Q^* \right) \overline{\partial}^8 q \tag{5.3.86}$$

$$+ \int_\Omega \frac{JR'(q)}{\rho_0}(\overline{\partial}^8 \eta_\nu \, \mathbf{A}^{\mu\nu} \, \partial_\mu Q - \triangle_Q^*)\overline{\partial}^8 q \Big|_0^T + L_2^*$$

$$\lesssim \varepsilon \|\overline{\partial}^8 q\|_0^2 + \mathcal{P}_0 + \int_0^T P(\mathfrak{E}(t)) \, \mathrm{d}t,$$

Summarizing (5.3.80)-(5.3.86) and choosing $\varepsilon > 0$ to be sufficiently small, we get the estimates of I^* under time integral

$$\int_0^T I^* \, \mathrm{d}t \lesssim -\frac{1}{2} \int_\Omega \frac{J^2 R'(q)}{\rho_0} \left| \overline{\partial}^8 q \right|^2 \mathrm{d}y \Big|_0^T + \mathcal{P}_0 + \int_0^T P(\mathfrak{E}(t)) \, \mathrm{d}t. \tag{5.3.87}$$

5.3.2.3 Boundary estimates

Now it remains to deal with the boundary integral IB^*. Since $Q|_\Gamma = 0$, we know

$$\mathbf{Q}^*|_\Gamma = -\overline{\partial}^8 \eta_\nu \, A^{3\nu} \, \partial_3 Q + \triangle_Q^*,$$

and

$$\triangle_Q^*|_\Gamma = -8\overline{\partial}^7 \eta_\nu \, A^{3\nu} \, \overline{\partial}\partial_3 Q + 8\overline{\partial}^7 \eta \cdot \nabla_A \overline{\partial}\eta_r \, A^{3\nu} \, \partial_3 Q.$$

Then the boundary integral IB^* reads

$$\begin{aligned}IB^* = &\int_\Gamma \mathbf{A}^{3\alpha} N_3 \overline{\partial}^8 \eta_\nu \, A^{3\nu} \, \partial_3 Q \, \overline{\partial}^8 v_\alpha \, \mathrm{d}S - \int_\Gamma \mathbf{A}^{3\alpha} N_3 (\overline{\partial}^8 \eta_\nu A^{3\nu} \partial_3 Q)(\overline{\partial}^8 \eta \cdot \nabla_A v_\alpha) \, \mathrm{d}S \\ &- \int_\Gamma \mathbf{A}^{3\alpha} N_3 \triangle_Q^* \, \overline{\partial}^8 v_\alpha \, \mathrm{d}S + \int_\Gamma \mathbf{A}^{3\alpha} N_3 \triangle_Q^* \, \overline{\partial}^8 \eta \cdot \nabla_A v_\alpha \, \mathrm{d}S \\ &- \int_\Gamma \mathbf{A}^{3\alpha} N_3 \triangle_Q^* (\triangle_v^*)_i \, \mathrm{d}S + \int_\Gamma \mathbf{A}^{3\alpha} N_3 (\overline{\partial}^8 \eta_\nu \, A^{3\nu} \, \partial_3 Q)(\triangle_v^*)_i \, \mathrm{d}S \\ =: &IB_1^* + IB_2^* + IB_3^* + IB_4^* + IB_5^* + IB_6^*.\end{aligned} \tag{5.3.88}$$

Before going to the proof, we would like to state our basic strategy to deal with the boundary control

- IB_1^* together with the Raylor-Taylor sign condition gives the boundary energy $|A^{3\alpha}\overline{\partial}^8\eta_\alpha|_0^2$ and the extra terms can be cancelled by IB_2^*. This step also appears in the study of Euler equations [13, 17, 49, 51, 54] and incompressible MHD [33, 30, 25, 26] and compressible resistive MHD [83]. It actually gives the control of the second fundamental form of the free surface [13].
- IB_3^*: We can write $\overline{\partial}^8 v_\alpha = \overline{\partial}^8 \partial_t \eta_\alpha$ and integrate ∂_t by parts. When ∂_t falls on $\triangle_Q^*$, the boundary integral can be directly controlled by using trace lemma. When ∂_t falls on $\mathbf{A}^{3\alpha}$, such terms exactly cancel with the top order term in IB_4^*.
- IB_5^* and IB_6^*: Direct computation together with the trace lemma gives the control.

We first compute IB_1^*. Similarly as (5.3.26), we have

$$\begin{aligned}IB_1^* = &- \int_\Gamma \left(-\frac{\partial Q}{\partial N}\right) J A^{3\alpha} \overline{\partial}^8 \eta_\nu \, A^{3\nu} \, \overline{\partial}^8 \partial_t \eta_\alpha \, \mathrm{d}S \\ = &- \frac{1}{2} \frac{d}{dt} \int_\Gamma \left(-J \frac{\partial Q}{\partial N}\right) \left| A^{3\alpha} \overline{\partial}^8 \eta_\alpha \right|^2 \, \mathrm{d}S \\ &- \frac{1}{2} \int_\Gamma \partial_t \left(J \frac{\partial Q}{\partial N}\right) \left| A^{3\alpha} \overline{\partial}^8 \eta_\alpha \right|^2 \, \mathrm{d}S + \int_\Gamma \left(-J \frac{\partial Q}{\partial N}\right) \partial_t A^{3\alpha} \, \overline{\partial}^8 \eta_\nu \, A^{3\nu} \, \overline{\partial}^8 \eta_\alpha \, \mathrm{d}S \\ =: &IB_{11}^* + IB_{12}^* + IB_{13}^*,\end{aligned} \tag{5.3.89}$$

The term IB^*_{11} together with the Rayleigh-Taylor sign condition gives the boundary energy

$$\int_0^T IB^*_{11} \le -\frac{c_0}{4}\left|A^{3\alpha}\overline{\partial}^8\eta_\alpha\right|_0^2\Big|_0^T, \tag{5.3.90}$$

and IB^*_{12} can be directly controlled by the boundary energy

$$IB^*_{12} \lesssim \left|A^{3\alpha}\overline{\partial}^8\eta_\alpha\right|_0^2 \left|\partial_t\left(J\frac{\partial Q}{\partial N}\right)\right|_{L^\infty} \lesssim P(\mathfrak{E}(t)). \tag{5.3.91}$$

Then we plug $\partial_t A^{3\alpha} = -A^{3\nu}\partial_\beta v_\nu A^{\beta\alpha}$ into IB^*_{13} to get the cancellation structure

$$IB^*_{13} = \int_\Gamma J\frac{\partial Q}{\partial N}A^{3\nu}\,\partial_\beta v_\nu A^{\beta\alpha}\,\overline{\partial}^8\eta_\nu\, A^{3\nu}\overline{\partial}^8\eta_\alpha = -IB^*_2 \tag{5.3.92}$$

Next we analyze IB^*_3. We write $v_\alpha = \partial_t\eta_\alpha$ and integrate this ∂_t by parts

$$\begin{aligned}
\int_0^T IB^*_3 =& -\int_0^T\int_\Gamma JA^{3\alpha}N_3\triangle^*_Q\,\overline{\partial}^8\partial_t\eta_\alpha\,\mathrm{d}S\,\mathrm{d}t\\
&\overset{\partial_t}{=}\int_0^T\int_\Gamma J\;\partial_t A^{3\alpha}\;N_3\triangle^*_Q\,\overline{\partial}^8\eta_\alpha\,\mathrm{d}S\,\mathrm{d}t\\
&\quad-\int_0^T\int_\Gamma A^{3\alpha}N_3\,\partial_t(J\triangle^*_Q)\,\overline{\partial}^8\eta_\alpha\,\mathrm{d}S\,\mathrm{d}t - \int_\Gamma JA^{3\alpha}N_3\,\triangle^*_Q\,\overline{\partial}^8\eta_\alpha\,\mathrm{d}S\Big|_0^T\\
&=: IB^*_{31}+IB^*_{32}+IB^*_{33}.
\end{aligned} \tag{5.3.93}$$

Again, plug $\partial_t A^{3\alpha} = -A^{3\nu}\partial_\beta v_\nu A^{\beta\alpha}$ into IB^*_{31} to get the cancellation with IB^*_4

$$\begin{aligned}
IB^*_{31} =& -\int_0^T\int_\Gamma JA^{3\nu}\;\partial_\beta v_\nu A^{\beta\alpha}\;N_3\;\triangle^*_Q\,\overline{\partial}^8\eta_\alpha\,\mathrm{d}S\,\mathrm{d}t\\
=& -\int_0^T\int_\Gamma JA^{3\alpha}\;\partial_\beta v_\nu A^{\beta\alpha}\;N_3\;\triangle^*_Q\,\overline{\partial}^8\eta_r\,\mathrm{d}S\,\mathrm{d}t = -IB^*_4.
\end{aligned} \tag{5.3.94}$$

For IB^*_{33}, we use the fact that $\overline{\partial}^7\eta|_{t=0} = 0$ (and thus $\triangle^*_Q|_\Gamma = 0$ when $t = 0$) together with

Lemma 3.2.5 to get

$$\begin{aligned}
&\int_\Gamma JA^{3\alpha} N_3 \triangle_Q^* \,\overline{\partial}^8 \eta_\alpha \,\mathrm{d}S\Big|_{t=T} \\
&= -\int_\Gamma JA^{3\alpha} N_3 (8\overline{\partial}^7 \eta_\nu \, A^{3\nu}\, \overline{\partial}\partial_3 Q - 8\overline{\partial}^7 \eta \cdot \nabla_A \overline{\partial}\eta_\nu \, A^{3\nu}\, \partial_3 Q)\overline{\partial}^8 \eta_\alpha \,\mathrm{d}S \\
&\lesssim \Big|A^{3\alpha}\overline{\partial}^8\eta_\alpha\Big|_0 |J|_{L^\infty}(|A^{3\nu}\overline{\partial}\partial_3 Q|_{L^\infty} + |(\nabla_A\overline{\partial}\eta_\nu)A^{3\nu}\partial_3 Q|_{L^\infty})\int_0^T |\overline{\partial}^7 v(t)|_0 \,\mathrm{d}t \\
&\lesssim \Big|A^{3\alpha}\overline{\partial}^8\eta_\alpha\Big|_0 P(\|\eta\|_{8,*}, \|Q\|_{8,*})\int_0^T \|v(t)\|_{8,*}\,\mathrm{d}t.
\end{aligned} \tag{5.3.95}$$

In IB_{32}^*, we invoke the relation (5.3.39) to get

$$\begin{aligned}
\partial_t(J\triangle_Q^*)|_\Gamma =& -8\overline{\partial}^7 v_\nu \,\mathbf{A}^{3\nu}\, \overline{\partial}\partial_3 Q + 8\overline{\partial}^7 v\cdot\nabla_A\overline{\partial}\eta_r \,\mathbf{A}^{3r}\partial_3 Q \\
& - 8\overline{\partial}^7\eta_\nu \,\partial_t(\mathbf{A}^{3\nu}\overline{\partial}\partial_3 Q) + 8\overline{\partial}^7\eta\cdot\partial_t(\nabla_A\overline{\partial}\eta_r\,\mathbf{A}^{3r}\partial_3 Q) \\
=& -8\overline{\partial}^7 v_\nu \,\mathbf{A}^{3\nu}\, \overline{\partial}\partial_3 Q + 8\overline{\partial}^7 v\cdot\nabla_A\overline{\partial}\eta_r \,\mathbf{A}^{3r}\partial_3 Q \\
& - 8\overline{\partial}^7\eta_\nu\partial_t\overline{\partial}(\mathbf{A}^{3\nu}\partial_3 Q) + 8\overline{\partial}^7\eta_\nu\partial_t(\overline{\partial}\mathbf{A}^{3\nu}\,\partial_3 Q) + 8\overline{\partial}^7\eta\cdot\partial_t(\nabla_A\overline{\partial}\eta_r\,\mathbf{A}^{3r}\,\partial_3 Q) \\
\overset{(5.3.39)}{=}& -8\overline{\partial}^7 v_\nu \,\mathbf{A}^{3\nu}\, \overline{\partial}\partial_3 Q + 8\overline{\partial}^7 v\cdot\nabla_A\overline{\partial}\eta_r \,\mathbf{A}^{3r}\partial_3 Q \\
& + 8\overline{\partial}^7\eta_\nu\partial_t\overline{\partial}\left(\rho_0\partial_t v^\nu - (b_0\cdot\overline{\partial})(J^{-1}(b_0\cdot\partial)\eta)^\nu\right) + 8\overline{\partial}^7\eta_\nu\,\partial_t(\overline{\partial}\mathbf{A}^{3\nu}\,\partial_3 Q) \\
& + 8\overline{\partial}^7\eta\cdot\partial_t(\nabla_A\overline{\partial}\eta_\nu\,\mathbf{A}^{3\nu}\partial_3 Q).
\end{aligned}$$

Then we use $H^{\frac{3}{2}}(\mathbb{T}^2)\hookrightarrow L^\infty(\mathbb{T}^2)$, Lemma 3.2.5 and standard Sobolev trace lemma to get

$$\left|\partial_t(J\triangle_Q^*)|_\Gamma\right|_0 \lesssim P(\|\eta\|_{8,*}, \|v\|_{8,*}, \|Q\|_{8,*}, \|b\|_{8,*}, \|\rho_0\|_3),$$

and thus

$$IB^*_{32} \lesssim \int_0^T \left|A^{3\alpha}\overline{\partial}^8\eta_\alpha\right|_0 P(\|\eta\|_{8,*}, \|v\|_{8,*}, \|Q\|_{8,*}, \|b\|_{8,*}, \|\rho_0\|_3)\,\mathrm{d}t. \tag{5.3.96}$$

From (5.3.88), we know it suffices to control the product of "error part" IB^*_5

$$IB^*_5 \lesssim |\mathbf{A}^{3\alpha}|_{L^\infty}|\triangle^*_Q|_\Gamma|_0|(\triangle^*_v)_\alpha|_0,$$

and the RHS can be directly controlled by Lemma 3.2.5 and standard trace lemma

$$\left|\triangle^*_Q|_\Gamma\right|_0 \lesssim \left|\overline{\partial}^7\eta_\nu\right|_0 \left(|A^{3\nu}\overline{\partial}\partial_3 Q|_{L^\infty} + |\nabla_A\overline{\partial}\eta_\nu\ A^{3\nu}\partial_3 Q|_{L^\infty}\right) \lesssim P(\|\eta\|_{8,*}, \|Q\|_{7,*}),$$

$$\begin{aligned}|\triangle^*_v|_0 \lesssim & |\overline{\partial}^7\eta|_0 \left(|\nabla_A\overline{\partial}v|_{L^\infty} + |\nabla_A\overline{\partial}\eta\cdot\nabla_A v|_{L^\infty} + |\nabla_A v\cdot\nabla_A\overline{\partial}\eta|_{L^\infty}\right) + |\overline{\partial}^7 v|_0|\nabla_A\overline{\partial}\eta|_{L^\infty} \\ \lesssim & P(\|\eta\|_{8,*}, \|v\|_{8,*}).\end{aligned}$$

Therefore,

$$IB^*_5 \lesssim P(\|\eta\|_{8,*}, \|v\|_{8,*}, \|Q\|_{7,*}), \tag{5.3.97}$$

and similarly

$$IB^*_6 \lesssim |\mathbf{A}^{3\alpha}\partial_3 Q|_{L^\infty}|A^{3\nu}\overline{\partial}^8\eta_\nu|_0|(\triangle^*_v)_i|_0. \tag{5.3.98}$$

Summarizing (5.3.88)-(5.3.98) gives the control of the boundary integral

$$\int_0^T IB^* \lesssim -\frac{c_0}{4}\left|A^{3\alpha}\overline{\partial}^8\eta_\alpha\right|_0^2 + \mathcal{P}_0 + P(\mathfrak{E}(T))\int_0^T P(\mathfrak{E}(t))\,\mathrm{d}t. \tag{5.3.99}$$

Combining (5.3.75), (5.3.76), (5.3.77), (5.3.87) and (5.3.99) and choosing $\varepsilon > 0$ in (5.3.83) to be suitably small, we get the following energy inequality

$$\|\mathbf{V}^*\|_0^2 + \left\|\overline{\partial}^8 b\right\|_0^2 + \|\overline{\partial}^8 q\|_0^2 + \frac{c_0}{4}\left|A^{3\alpha}\overline{\partial}^8\eta_\alpha\right|_0^2\Big|_{t=T} \lesssim \mathcal{P}_0 + P(\mathfrak{E}(T))\int_0^T P(\mathfrak{E}(t))\,\mathrm{d}t. \tag{5.3.100}$$

Finally, invoking (5.3.67), we get the $\overline{\partial}^8$-estimates of v by using $\partial^m \eta|_{t=0} = 0$ for any $m \geq 2, m \in \mathbb{N}^*$

$$\|\overline{\partial}^8 v\|_0 \lesssim \|\mathbf{V}^*\|_0 + P(\|v\|_{7,*}, \|\eta\|_{7,*}) \int_0^T P(\|v\|_{8,*}), \tag{5.3.101}$$

and thus

$$\begin{aligned} &\|\overline{\partial}^8 v\|_0^2 + \left\|\overline{\partial}^8 \left(J^{-1}(b_0 \cdot \partial)\eta\right)\right\|_0^2 + \|\overline{\partial}^8 q\|_0^2 + \frac{c_0}{4} \left|A^{3\alpha}\overline{\partial}^8 \eta_\alpha\right|_0^2 \Big|_{t=T} \\ &\lesssim \mathcal{P}_0 + P(\mathfrak{E}(T)) \int_0^T P(\mathfrak{E}(t))\, \mathrm{d}t. \end{aligned} \tag{5.3.102}$$

5.3.3 The case of one time derivative $\overline{\partial}^7 \partial_t$

If we replace $\partial_*^I = \overline{\partial}^8$ by $\overline{\partial}^7 \partial_t$, then most of steps in the proof above are still applicable because we do not integrate the derivative(s) in $\mathfrak{D}^8$ by parts. However, we still need to do the following modifications due to the presence of time derivative.

5.3.3.1 Extra difficulty: non-vanishing initial data of $\partial_*^I \eta$

If $\partial_*^I = \overline{\partial}^7 \partial_t$, then we can no longer derive $\overline{\partial}^7 \partial_t \eta|_{t=0} = \mathbf{0}$ from $\eta|_{t=0} = \mathrm{Id}$ due to the presence of time derivative and $\partial_t \eta = v$. This property is used in the analysis of IB_{33}^* and the control of the difference between $\mathbf{V}^*$ and $\partial_*^I v$. Before we analyze the analogues of IB_{33}^* and (5.3.101) in the case of $\partial_*^I = \overline{\partial}^7 \partial_t$, we have to find out the precise form of the modified Alinhac good unknowns when $\partial_*^I = \overline{\partial}^7 \partial_t$.

5.3.3.2 The modified Alinhac good unknowns

Recall the "extra modification terms" $\triangle_Q^*, \triangle_v^*$ in (5.3.74) come from the bad terms (5.3.52). Now we replace $\overline{\partial}^8$ by $\overline{\partial}^7 \partial_t$. In e_1, e_2, e_3 in (5.3.52), if we replace $\overline{\partial}^7$ by $\overline{\partial}^6 \partial_t$ (i.e., the time derivative falls on the higher order term), then their L^2 norms can be directly controlled since $\partial_t a$ has the same spatial regularity as a. Therefore, the remaining quantities whose L^2-norms cannot be directly controlled in the case of $\partial_*^I = \overline{\partial}^7\partial_t$ are

$$e_1 := -\overline{\partial}^7(A^{\mu\nu}A^{\beta\alpha})\partial_t\partial_\beta\eta_\nu\,\partial_\mu f,\ \ e_2 := -7\partial_t(A^{\mu\nu}A^{\beta\alpha})\,\overline{\partial}^7\partial_\beta\eta_\nu\,\partial_\mu f$$
$$e_3 := 8(\overline{\partial}^7 A^{\mu\alpha})\partial_t\partial_\mu f,\ \ e_4 := (\partial_t A^{\mu\alpha})(\overline{\partial}^7\partial_\mu f) + 7(\overline{\partial}A^{\mu\alpha})(\overline{\partial}^6\partial_t\partial_\mu f). \tag{5.3.103}$$

Then the corresponding Alinhac good unknowns now becomes (with the abuse of terminology)

$$\mathbf{V}^* = \overline{\partial}^7\partial_t v - \overline{\partial}^7\partial_t\eta\cdot\nabla_A v + \triangle_v^*,\ \ \mathbf{Q}^* = \overline{\partial}^7\partial_t Q - \overline{\partial}^7\partial_t\eta\cdot\nabla_A Q + \triangle_Q^*., \tag{5.3.104}$$

where

$$(\triangle_v^*)_i := -\,8\overline{\partial}^7\eta\cdot\nabla_A\partial_t v_\alpha - 8\overline{\partial}^7 v\cdot\nabla_A v_\alpha + 16\overline{\partial}^7\eta\cdot\nabla_A v\cdot\nabla_A v_\alpha,$$
$$\triangle_Q^* := -\,8\overline{\partial}^7\eta\cdot\nabla_A\partial_t Q + 8\overline{\partial}^7\eta\cdot\nabla_A v\cdot\nabla_A Q, \tag{5.3.105}$$

and

$$\overline{\partial}^7\partial_t(\mathrm{div}_{\tilde{A}} v) = \nabla_A\cdot\mathbf{V}^* + C^*(v),\quad \overline{\partial}^7\partial_t(\nabla_A Q) = \nabla_A\mathbf{Q}^* + C^*(Q), \tag{5.3.106}$$

with

$$\|C^*(f)\|_0 \lesssim P(\mathfrak{E}(t))\|f\|_{8,*}.$$

Now, the analogue of IB_{33}^* becomes the following quantity (recall such term comes from the product of $\triangle_Q$ and $\overline{\partial}^7\partial_t v$

$$\int_\Gamma JA^{3\alpha}N_3(8\overline{\partial}^7\eta_\nu\ A^{3\nu}\ \partial_t\partial_3 Q - 8\overline{\partial}^7\eta\cdot\nabla_A\partial_t\eta_r\ A^{3\nu}\ \partial_3 Q)\overline{\partial}^7\partial_t\eta_\alpha\,\mathrm{d}S, \tag{5.3.107}$$

and we can still use $\overline{\partial}^7\eta|_{t=0} = \mathbf{0}$.

The analogue of (5.3.101) is

$$\|\overline{\partial}^7\partial_t v\|_0 \lesssim \|\mathbf{V}^*\|_0 + \|\overline{\partial}^7 v\|_0\|\nabla_A v\|_{L^\infty} + \|\overline{\partial}^7\eta\|_0\left(8\|\nabla_A\partial_t v\|_{L^\infty} + 16\|\nabla_A v\|_{L^\infty}^2\right)$$
$$\lesssim \|\mathbf{V}^*\|_0^2 + \mathcal{P}_0 + P(\mathfrak{E}(T))\int_0^T P(\mathfrak{E}(t))\,\mathrm{d}t. \tag{5.3.108}$$

The remaining analysis should follow in the same way as before, so we omit those details. The result is

$$
\begin{aligned}
&\|\overline{\partial}^7\partial_t v\|_0^2 + \left\|\overline{\partial}^7\partial_t\left(J^{-1}(b_0\cdot\partial)\eta\right)\right\|_0^2 + \|\overline{\partial}^7\partial_t q\|_0^2 + \frac{c_0}{4}\left|A^{3\alpha}\overline{\partial}^7\partial_t\eta_\alpha\right|_0^2\Bigg|_{t=T} \\
\lesssim\,& \mathcal{P}_0 + P(\mathfrak{E}(T))\int_0^T P(\mathfrak{E}(t))\,\mathrm{d}t.
\end{aligned}
\tag{5.3.109}
$$

5.3.4 The case of 2∼7 time derivatives

If the number of time derivatives in ∂_*^I is between 2 and 7, i.e,. ∂_*^I contains at least one spatial and two time derivatives, we can still mimic most steps in Section 5.3.3. In this case we write $\partial_*^I = \mathfrak{D}^6\partial_t^2$ where $\mathfrak{D} = \overline{\partial}$ or ∂_t and $\mathfrak{D}^6$ contains at least one $\overline{\partial}$.

The extra time derivatives allow us to eliminate most of the “extra modification terms" in the modified Alinhac good unknowns as in (5.3.74), (5.3.104)-(5.3.105) and thus much simplify the analysis of Alinhac good unknowns and tne boundary control. The reason is that the L^2-norm of the analogues of $e_1 \sim e_3$ in (5.3.52) can be directly controlled in the case of $\mathfrak{D}^8 = \mathfrak{D}^6\partial_t^2$. In specific, we have

$$
\begin{aligned}
\mathfrak{D}^6\partial_t^2(\nabla_A^i f) =& \nabla_A^\alpha(\mathfrak{D}^6\partial_t^2 f) + (\mathfrak{D}^6\partial_t^2 A^{\mu\alpha})\partial_\mu f + [\mathfrak{D}^6\partial_t^2, A^{\mu\alpha}, \partial_\mu f] \\
=& \nabla_A^i(\mathfrak{D}^6\partial_t^2 f) - \mathfrak{D}^6\partial_t(A^{\mu\nu}\ \partial_t\partial_\beta\eta_r\ A^{\beta\alpha})\partial_\mu f + [\mathfrak{D}^6\partial_t^2, A^{\mu\alpha}, \partial_\mu f] \\
=& \nabla_A^i(\mathfrak{D}^6\partial_t^2 f - \mathfrak{D}^6\partial_t^2\eta_\nu\ A^{\mu\nu}\partial_\mu f) \\
&+ \underbrace{\mathfrak{D}^6\partial_t^2\eta_\nu\ \nabla_A^\alpha(\nabla_A^\nu f) - ([\mathfrak{D}^6\partial_t, A^{\mu\nu}A^{\beta\alpha}]\partial_t\partial_\beta\eta_\nu)\partial_\mu f}_{C_0(f)} + [\mathfrak{D}^6\partial_t^2, A^{\mu\alpha}, \partial_\mu f]
\end{aligned}
\tag{5.3.110}
$$

and

$$\|C_0(f)\|_0 \lesssim P(\|\eta\|_{8,*}, \|v\|_{8,*})\|f\|_{8,*}.$$

Therefore, the analogous analysis of $C_1, C_3 \sim C_6$ in Section 5.3.2 are no longer needed here. The only problematic term is $-2(\partial_t A^{\mu\alpha})(\mathfrak{D}^6\partial_t\partial_\mu f) - 6(\mathfrak{D}A^{\mu\alpha})(\mathfrak{D}^5\partial_t^2\partial_\mu f)$ which comes from

$[\mathfrak{D}^6\partial_t^2, A^{\mu\alpha}, \partial_\mu f]$. By mimicing the treatment of $C_2(Q)$ and $C_2(v)$ in (5.3.54)-(5.3.55), we can define the modified Alinhac good unknowns in the case of $\partial_*^I = \partial_t^N\overline{\partial}^{8-N}$ $(2 \le N \le 7)$ as the following

$$\mathbf{V}^* = \mathfrak{D}^6\partial_t^2 v - \mathfrak{D}^6\partial_t^2\eta \cdot \nabla_A v + \triangle_v^*, \ \ \mathbf{Q}^* = \mathfrak{D}^6\partial_t^2 Q - \mathfrak{D}^6\partial_t^2\eta \cdot \nabla_A Q, \tag{5.3.111}$$

where

$$(\triangle_v^*)_i := -6\mathfrak{D}^5\partial_t^2 v \cdot \nabla_A \mathfrak{D}\eta_\alpha - 2\mathfrak{D}^6\partial_t v \cdot \nabla_A v_\alpha \tag{5.3.112}$$

and

$$\mathfrak{D}^6\partial_t^2(\mathrm{div}_{\tilde{A}} v) = \nabla_A \cdot \mathbf{V}^* + C^*(v), \quad \mathfrak{D}^6\partial_t^2(\nabla_A Q) = \nabla_A \mathbf{Q}^* + C^*(Q), \tag{5.3.113}$$

with

$$\|C^*(f)\|_0 \lesssim P(\mathfrak{E}(t))\|f\|_{8,*}.$$

In this case, $\triangle_Q^* = 0$, and thus the boundary integrals IB_3^*, IB_4^*, IB_5^* all vanish. The analogues of IB_1^*, IB_2^*, IB_6^* in this case can still be controlled in the same way as before. In the control of the difference between $\mathbf{V}^*$ and $\mathfrak{D}^6\partial_t^2$, we have by (5.3.111)-(5.3.112) that

$$\begin{aligned}
\|\mathfrak{D}^6\partial_t^2 v\|_0 \lesssim & \|\mathbf{V}^*\|_0 + \|\mathfrak{D}^6\partial_t v\|_0\|\nabla_A v\|_{L^\infty} + \|\mathfrak{D}^5\partial_t^2 v\|_0\|\nabla_A\mathfrak{D}\eta\|_{L^\infty} \\
\lesssim & \|\mathbf{V}^*\|_0 + \|\mathfrak{D}^6\partial_t v\|_0^2 + \|\nabla_A v\|_2^2 + \|\mathfrak{D}^5\partial_t^2 v\|_0^2 + \|\nabla_A\mathfrak{D}\eta\|_2^2 \\
\lesssim & \lesssim \|\mathbf{V}^*\|_0 + \mathcal{P}_0 + \int_0^T P(\mathfrak{E}(t))\,\mathrm{d}t
\end{aligned} \tag{5.3.114}$$

The remaining analysis should follow in the same way as in Section 5.3.2 and 5.3.3 so we omit the details. The result is

$$\begin{aligned}
& \|\mathfrak{D}^6\partial_t^2 v\|_0^2 + \left\|\mathfrak{D}^6\partial_t^2\left(J^{-1}(b_0\cdot\partial)\eta\right)\right\|_0^2 + \|\mathfrak{D}^6\partial_t^2 q\|_0^2 + \frac{c_0}{4}\left|A^{3\alpha}\mathfrak{D}^6\partial_t^2\eta_\alpha\right|_0^2\Big|_{t=T} \\
\lesssim & \mathcal{P}_0 + P(\mathfrak{E}(T))\int_0^T P(\mathfrak{E}(t))\,\mathrm{d}t,
\end{aligned} \tag{5.3.115}$$

where $\mathfrak{D}^6$ contains at least one spatial derivative $\overline{\partial}$.

5.3.5 The case of full time derivatives

In the case of full time derivatives, the modified Alinhac good unknown is still defined similarly as in (5.3.111)-(5.3.113):

$$\mathbf{V}^* = \partial_t^8 v - \partial_t^8 \eta \cdot \nabla_A v + \triangle_v^*, \;\; \mathbf{Q}^* = \partial_t^8 Q - \partial_t^8 \eta \cdot \nabla_A Q, \tag{5.3.116}$$

where

$$(\triangle_v^*)_i := -8\partial_t^7 v \cdot \nabla_A v_\alpha \tag{5.3.117}$$

and

$$\partial_t^8(\mathrm{div}_{\tilde{A}} v) = \nabla_A \cdot \mathbf{V}^* + C^*(v), \quad \partial_t^8(\nabla_A Q) = \nabla_A \mathbf{Q}^* + C^*(Q), \tag{5.3.118}$$

with

$$\|C^*(f)\|_0 \lesssim P(\mathfrak{E}(t))\|f\|_{8,*}.$$

Extra difficulty: trace lemma is no longer applicable When $\partial_*^I = \partial_t^8$, there are terms of the form $\partial_t^7 v$ in the boundary integrals. In the case of full time derivative, one cannot apply Lemma 3.2.5 to control $|\partial_t^7 v|_0$. This difficulty appears in the estimates of the analogue of IB_6^*. Instead, we need to write IB_6^* in terms of interior integrals by using a similar technique in (5.3.44).

$$
\begin{aligned}
IB_6^* =& -8\int_\Gamma \mathbf{A}^{3\alpha} N_3\ \partial_t^8 \eta_\nu\ A^{3\nu}\partial_3 Q\ \partial_t^7 v_\gamma\ A^{\mu\gamma}\partial_\mu v_\alpha\,\mathrm{d}S \\
=& -8\int_\Gamma \mathbf{A}^{3\alpha} N_3\ \partial_t^7 v_\nu\ A^{3\nu}\partial_3 Q\ \partial_t^7 v_\gamma\ A^{\mu\gamma}\partial_\mu v_\alpha\,\mathrm{d}S \\
=& -8\int_\Omega \mathbf{A}^{3\alpha}\partial_3\partial_t^7 v_\nu\ A^{3\nu}\partial_3 Q\ \partial_t^7 v_\gamma\ A^{\mu\gamma}\partial_\mu v_\alpha\,\mathrm{d}y \\
& -8\int_\Omega \mathbf{A}^{3\alpha}\partial_t^7 v_\nu\ A^{3\nu}\partial_3 Q\ \partial_3\partial_t^7 v_\gamma\ A^{\mu\gamma}\partial_\mu v_\alpha\,\mathrm{d}y \\
& -8\int_\Omega \partial_t^7 v_\nu\ \partial_t^7 v_\gamma\ \partial_3(\mathbf{A}^{3\alpha} A^{3\nu}\partial_3 Q\ A^{\mu\gamma}\partial_\mu v_\alpha)\,\mathrm{d}y \\
=:& IB_{61}^* + IB_{62}^* + IB_{63}^*.
\end{aligned}
\tag{5.3.119}
$$

The term IB_{63}^* can be directly controlled

$$
IB_{63}^* \lesssim P(\|\partial_t^7 v\|_0, \|\partial v\|_3, \|\partial Q\|_3, \|A\|_3) \lesssim P(\|v\|_{8,*}, \|Q\|_{8,*}, \|\eta\|_4). \tag{5.3.120}
$$

The term IB_{61}^*, IB_{62}^* should be controlled by integrating ∂_t by parts under time integral.

$$
\begin{aligned}
\int_0^T IB_{61}^* =& -8\int_0^T\int_\Omega \mathbf{A}^{3\alpha}\partial_3\partial_t^7 v_\nu\ A^{3\nu}\partial_3 Q\ \partial_t^7 v_\gamma\ A^{\mu\gamma}\partial_\mu v_\alpha\,\mathrm{d}y\,\mathrm{d}t \\
\overset{\partial_t}{=}& -8\int_\Omega \mathbf{A}^{3\alpha}\partial_3\partial_t^6 v_\nu\ A^{3\nu}\partial_3 Q\ \partial_t^7 v_\gamma\ A^{\mu\gamma}\partial_\mu v_\alpha\,\mathrm{d}y \\
& +8\int_0^T\int_\Omega \mathbf{A}^{3\alpha}\partial_3\partial_t^6 v_\nu\ A^{3\nu}\partial_3 Q\ \partial_t^8 v_\gamma\ A^{\mu\gamma}\partial_\mu v_\alpha\,\mathrm{d}y\,\mathrm{d}t \\
& +8\int_0^T\int_\Omega \mathbf{A}^{3\alpha}\partial_3\partial_t^6 v_\nu\ \partial_t^7 v_\gamma\ \partial_t(A^{3\nu}\partial_3 Q\ A^{\mu\gamma}\partial_\mu v_\alpha)\,\mathrm{d}y\,\mathrm{d}t \\
\lesssim& \varepsilon\|\partial_3\partial_t^6 v\|_0^2 + \mathcal{P}_0 + \int_0^T P(\mathfrak{E}(t))\,\mathrm{d}t,
\end{aligned}
\tag{5.3.121}
$$

IB_{62}^* can be controlled in the same way, so we omit the details. Summarizing the estimates above, we get the energy inequality of the full time derivatives

$$\begin{aligned}
&\|\partial_t^8 v\|_0^2 + \left\|\partial_t^8 \left(J^{-1}(b_0\cdot\partial)\eta\right)\right\|_0^2 + \|\partial_t^8 q\|_0^2 + \frac{c_0}{4}\left|A^{3\alpha}\partial_t^8\eta_\alpha\right|_0^2\Bigg|_{t=T} \\
\lesssim\, &\varepsilon\|\partial_3\partial_t^6 v\|_0^2 + \mathcal{P}_0 + P(\mathfrak{E}(T))\int_0^T P(\mathfrak{E}(t))\,\mathrm{d}t,
\end{aligned} \tag{5.3.122}$$

which together with (5.3.102), (5.3.109), (5.3.115) concludes the proof of Proposition 5.3.3.

5.3.6 Control of purely spatial derivatives

The case of mixed non-weighted derivatives correspond to $\partial_*^I = \partial_t^{i_0}(\sigma\partial_3)^{i_4}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}$ with $1\leq i_3\leq 3,\ i_4=0$. In this case, the modified Alinhac good unknowns introduced in Section 5.3.2 are still needed when commuting ∂_*^I with ∇_A. On the other hand, the highest order term $\partial_*^I Q$ no longer vanishes on the boundary due to the presence of normal derivatives, so we need to use the method in Section 5.3.1 to deal with the boundary integral. Therefore, we should combine the methods in Section 5.3.1 and Section 5.3.2 to get the control of mixed non-weighted derivatives. The result of this section is

Proposition 5.3.5. The following energy inequality holds for sufficiently small $\varepsilon>0$

$$\begin{aligned}
&\sum_{1\leq i_3\leq 3,\ i_4=0}\|\partial_*^I v\|_0^2 + \left\|\partial_*^I \left(J^{-1}(b_0\cdot\partial)\eta\right)\right\|_0^2 + \|\partial_*^I q\|_0^2 + \frac{c_0}{4}\left|A^{3\alpha}\partial_*^I\eta_\alpha\right|_0^2\Bigg|_{t=T} \\
\lesssim\, &\varepsilon\|\partial_3^4 v\|_0^2 + \mathcal{P}_0 + P(\mathfrak{E}(T))\int_0^T P(\mathfrak{E}(t))\,\mathrm{d}t.
\end{aligned} \tag{5.3.123}$$

We still start with the control of purely spatial derivatives. Let $N=1,2,3$ and we consider $\partial_*^I = \partial_3^N\overline{\partial}^{8-2N}$.

5.3.6.1 The modified Alinhac good unknowns

Similarly as in Section 5.3.2.1, we have

$$
\begin{aligned}
&\partial_3^N\overline{\partial}^{8-2N}(\nabla_A^i f) = \nabla_A^\alpha(\partial_3^N\overline{\partial}^{8-2N} f) + (\partial_3^N\overline{\partial}^{8-2N} A^{\mu\alpha})\partial_\mu f + [\partial_3^N\overline{\partial}^{8-2N}, A^{\mu\alpha}, \partial_\mu f] \\
=&\nabla_A^i(\partial_3^N\overline{\partial}^{8-2N} f) - \partial_3^N\overline{\partial}^{7-2N}(A^{\mu\nu}\,\overline{\partial}\partial_\beta\eta_r\, A^{\beta\alpha})\partial_\mu f + [\partial_3^N\overline{\partial}^{8-2N}, A^{\mu\alpha}, \partial_\mu f] \\
=&\nabla_A^i(\partial_3^N\overline{\partial}^{8-2N} f - \partial_3^N\overline{\partial}^{8-2N}\eta_r\, A^{\mu\nu}\,\partial_\mu f) + (\partial_3^N\overline{\partial}^{8-2N}\eta_r)\nabla_A^i(\nabla_A^r f) \\
&- ([\partial_3^N\overline{\partial}^{7-2N}, A^{\mu\nu}A^{\beta\alpha}]\overline{\partial}\partial_\beta\eta_r)\partial_\mu f + [\partial_3^N\overline{\partial}^{8-2N}, A^{\mu\alpha}, \partial_\mu f],
\end{aligned}
\tag{5.3.124}
$$

where the last line still contains the terms whose $L^2(\Omega)$-norms cannot be directly bounded under the setting of anisotropic Sobolev space $H_*^8(\Omega)$. The reason is that $\partial_3^N\overline{\partial}^{7-2N}$ may fall on $A = \partial\eta\times\partial\eta$ and $\partial_\mu f$. The following quantities are exactly these terms.

$$
\begin{aligned}
&e_1^\sharp := -\partial_3^N\overline{\partial}^{7-2N}(A^{\mu\nu}A^{\beta\alpha})(\overline{\partial}\partial_\beta\eta_\nu)\partial_\mu f,\ \ e_2^\sharp := -(7-2N)\overline{\partial}(A^{\mu\nu}A^{\beta\alpha})(\partial_3^N\overline{\partial}^{7-2N}\partial_\beta\eta_\nu)\partial_\mu f, \\
&e_3^\sharp := (8-2N)(\partial_3^N\overline{\partial}^{7-2N}A^{\mu\alpha})(\overline{\partial}\partial_\mu f),\ \ e_4^\sharp := (8-2N)(\overline{\partial}A^{\mu\alpha})(\partial_3^N\overline{\partial}^{7-2N}\partial_\mu f).
\end{aligned}
\tag{5.3.125}
$$

One can mimic the derivation of (5.3.67) and (5.3.68) to define the "modified Alinhac good unknowns" of v and Q with respect to $\partial_3^N\overline{\partial}^{8-2N}$ to be

$$
\begin{aligned}
\mathbf{V}_\alpha^\sharp :=&\partial_3^N\overline{\partial}^{8-2N}v_\alpha - \partial_3^N\overline{\partial}^{8-2N}\eta\cdot\nabla_A v_\alpha \\
&- (8-2N)\partial_3^N\overline{\partial}^{7-2N}\eta\cdot\nabla_A\overline{\partial}v_\alpha - (8-2N)\partial_3^N\overline{\partial}^{7-2N}v\cdot\nabla_A\overline{\partial}\eta_\alpha \\
&+ (8-2N)\partial_3^N\overline{\partial}^{7-2N}\eta\cdot\nabla_A\overline{\partial}\eta\cdot\nabla_A v_\alpha + (8-2N)\partial_3^N\overline{\partial}^{7-2N}\eta\cdot\nabla_A v\cdot\nabla_A\overline{\partial}\eta_\alpha,
\end{aligned}
\tag{5.3.126}
$$

and

$$
\begin{aligned}
\mathbf{Q}^\sharp :=&\partial_3^N\overline{\partial}^{8-2N}Q - \partial_3^N\overline{\partial}^{8-2N}\eta\cdot\nabla_A Q \\
&- (8-2N)\partial_3^N\overline{\partial}^{7-2N}\eta\cdot\nabla_A\overline{\partial}Q + (8-2N)\partial_3^N\overline{\partial}^{7-2N}\eta\cdot\nabla_A\overline{\partial}\eta\cdot\nabla_A Q.
\end{aligned}
\tag{5.3.127}
$$

Then $\mathbf{V}^\sharp$ and $\mathbf{Q}^\sharp$ satisfy the following relations

$$\partial_3^N\overline{\partial}^{8-2N}(\text{div}_{\tilde{A}}v) = \nabla_A\cdot\mathbf{V}^\sharp + C^\sharp(v), \quad \partial_3^N\overline{\partial}^{8-2N}(\nabla_A Q) = \nabla_A\mathbf{Q}^* + C^\sharp(Q), \tag{5.3.128}$$

where the commutator $C^\sharp$ satisfies

$$\|C^\sharp(f)\|_0 \lesssim P(\mathfrak{E}(t))\|f\|_{8,*}. \tag{5.3.129}$$

Denote $\triangle_v^\sharp$ and $\triangle_Q^\sharp$ to be

$$(\triangle_v^\sharp)_i := -(8-2N)\partial_3^N\overline{\partial}^{7-2N}\eta\cdot\nabla_A\overline{\partial}v_\alpha - (8-2N)\partial_3^N\overline{\partial}^{7-2N}v\cdot\nabla_A\overline{\partial}\eta_\alpha$$

$$+ (8-2N)\partial_3^N\overline{\partial}^{7-2N}\eta\cdot\nabla_A\overline{\partial}\eta\cdot\nabla_A v_\alpha + (8-2N)\partial_3^N\overline{\partial}^{7-2N}\eta\cdot\nabla_A v\cdot\nabla_A\overline{\partial}\eta_\alpha,$$

$$\triangle_Q^\sharp := -(8-2N)\partial_3^N\overline{\partial}^{7-2N}\eta\cdot\nabla_A\overline{\partial}Q + (8-2N)\partial_3^N\overline{\partial}^{7-2N}\eta\cdot\nabla_A\overline{\partial}\eta\cdot\nabla_A Q.$$

Then we can derive the evolution equation of $\mathbf{V}^\sharp$ and $\mathbf{Q}^\sharp$

$$\begin{aligned} &R\partial_t\mathbf{V}^\sharp - J^{-1}(b_0\cdot\partial)\partial_3^N\overline{\partial}^{8-2N}\left(J^{-1}(b_0\cdot\partial)\eta\right) + \nabla_A\mathbf{Q}^\sharp \\ =&[R, \partial_3^N\overline{\partial}^{8-2N}]\partial_t v + \left[J^{-1}(b_0\cdot\partial), \partial_3^N\overline{\partial}^{8-2N}\right]\left(J^{-1}(b_0\cdot\partial)\eta\right) \\ &+ C^\sharp(Q) + R\partial_t(-\partial_3^N\overline{\partial}^{8-2N}\eta\cdot\nabla_A v + \triangle_v^\sharp). \end{aligned} \tag{5.3.130}$$

Denote the RHS of (5.3.130) to be $\mathbf{F}^\sharp$, then direct computation yields that

$$\|\mathbf{F}^\sharp\|_0 \lesssim P\left(\|\eta\|_{8,*}, \|v\|_{8,*}, \|Q\|_{8,*}\right).$$

Now we take $L^2(\Omega)$ inner product of (5.3.130) and $J\mathbf{V}^\sharp$ to get the following energy identity

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega \rho_0|\mathbf{V}^\sharp|^2\,\mathrm{d}y =& \int_\Omega (b_0\cdot\partial)\partial_3^N\overline{\partial}^{8-2N}\left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\mathbf{V}^\sharp \\ &- \int_\Omega (\nabla_{\mathbf{A}}Q)\cdot\mathbf{V}^\sharp + \int_\Omega J\mathbf{F}^\sharp\cdot\mathbf{V}^\sharp. \end{aligned} \tag{5.3.131}$$

5.3.6.2 Interior estimates

The last integral on RHS of (5.3.131) is directly controlled

$$\int_\Omega J\mathbf{F}^\sharp \cdot \mathbf{V}^\sharp \lesssim \int_\Omega \|\mathbf{F}^\sharp\|_0 \|\mathbf{V}^\sharp\|_0. \tag{5.3.132}$$

Then for the first term on RHS of (5.3.131) we integrate $(b_0 \cdot \partial)$ by parts to produce the energy of magnetic field. Again, there is one term which cannot be directly controlled but will cancel with another term produced by $-\int_\Omega (\nabla_{\mathbf{A}} Q) \cdot \mathbf{V}^\sharp$. The proof follows in the same way as (5.3.15) so we omit the details.

$$\begin{aligned} &\int_\Omega (b_0 \cdot \partial)\partial_3^N \overline{\partial}^{8-2N} \left(J^{-1}(b_0 \cdot \partial)\eta\right) \cdot \mathbf{V}^\sharp \\ \lesssim & -\frac{1}{2}\frac{d}{dt}\int_\Omega J \left|\partial_3^N \overline{\partial}^{8-2N} (J^{-1}(b_0 \cdot \partial)\eta)\right|^2 + K_{11}^\sharp + P\left(\|(\eta, v, b_0, (b_0 \cdot \partial))\|_{8,*}\right), \end{aligned} \tag{5.3.133}$$

where

$$K_{11}^\sharp := -\int_\Omega J\partial_3^N \overline{\partial}^{8-2N} (J^{-1}(b_0 \cdot \partial)\eta) \cdot \left(J^{-1}(b_0 \cdot \partial)\eta\right) \partial_3^N \overline{\partial}^{8-2N} (\text{div}_{\tilde{A}} v) \, \mathrm{d}y. \tag{5.3.134}$$

Next we analyze the term $-\int_\Omega (\nabla_{\mathbf{A}} Q) \cdot \mathbf{V}^\sharp$. Integrating by parts and using Piola's identity $\partial_\mu \mathbf{A}^{li} = 0$, we get

$$-\int_\Omega (\nabla_{\mathbf{A}} Q) \cdot \mathbf{V}^\sharp = \int_\Omega J\mathbf{Q}^\sharp (\nabla_A \cdot \mathbf{V}^\sharp) - \int_\Gamma J\mathbf{Q}^\sharp A^{\mu\alpha} N_\mu \mathbf{V}_\alpha^\sharp \, \mathrm{d}S =: I^\sharp + IB^\sharp. \tag{5.3.135}$$

Plugging (5.3.128) into $I^\sharp$, we get

$$\begin{aligned} I^\sharp = & \int_\Omega J\partial_3^N \overline{\partial}^{8-2N} q \partial_3^N \overline{\partial}^{8-2N} (\text{div}_{\tilde{A}} v) \\ & + \int_\Omega J\partial_3^N \overline{\partial}^{8-2N} \left(\frac{1}{2}\left|J^{-1}(b_0 \cdot \partial)\eta\right|^2\right) \partial_3^N \overline{\partial}^{8-2N} (\text{div}_{\tilde{A}} v) \\ & + \int_\Omega \left(-(\partial_3^N \overline{\partial}^{8-2N} \eta_\nu)\mathbf{A}^{\mu\nu} \partial_\mu Q + \triangle_Q^\sharp\right) \partial_3^N \overline{\partial}^{8-2N} (\text{div}_{\tilde{A}} v) - \int_\Omega (\partial_3^N \overline{\partial}^{8-2N} Q) C^\sharp(v) \\ =: & \, I_1^\sharp + I_2^\sharp + I_3^\sharp + I_4^\sharp, \end{aligned} \tag{5.3.136}$$

where $I_4^\sharp$ can be directly controlled by using the estimates of $C^\sharp(v)$

$$I_4^\sharp \lesssim \|\partial_3^N \overline{\partial}^{8-2N} Q\|_0 \|C^\sharp(v)\|_0 \lesssim P(\|\eta\|_{8,*}) \|\partial_3^N \overline{\partial}^{8-2N} Q\|_0 \|v\|_{8,*}. \tag{5.3.137}$$

The term $I_1^\sharp$ produces the energy of fluid pressure

$$I_1^\sharp \lesssim -\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{J^2 R'(q)}{\rho_0} \left|\partial_3^N \overline{\partial}^{8-2N} q\right|^2 + P(\|q\|_{8,*}, \|\rho_0\|_{8,*}, \|\eta\|_{8,*}). \tag{5.3.138}$$

Similarly as in (5.3.82), the term $I_2^\sharp$ produces the cancellation with $K_{11}^\sharp$.

$$\begin{aligned}
I_2^\sharp = & \underbrace{\int_\Omega J \partial_3^N \overline{\partial}^{8-2N} \left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot\left(J^{-1}(b_0\cdot\partial)\eta\right) \partial_3^N \overline{\partial}^{8-2N} (\text{div}_{\tilde{A}} v)}_{\text{exactly cancel with } K_{11}^\sharp} \\
& + C_{N_1,N_2}\int_\Omega J \partial_3^{N_1}\overline{\partial}^{N_2}\left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot \partial_3^{N-N_1}\overline{\partial}^{8-2N-N_2}\left(J^{-1}(b_0\cdot\partial)\eta\right)\partial_3^N\overline{\partial}^{8-2N}(\text{div}_{\tilde{A}} v) \\
= & -K_{11}^\sharp \\
& - C_{N_1,N_2}\int_\Omega \frac{J^2 R'(q)}{\rho_0}\partial_3^{N_1}\overline{\partial}^{N_2}\left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot \partial_3^{N-N_1}\overline{\partial}^{8-2N-N_2}\left(J^{-1}(b_0\cdot\partial)\eta\right)(\partial_3^N\overline{\partial}^{8-2N}\partial_t q) \\
& - C_{N_1,N_2}\int_\Omega J \partial_3^{N_1}\overline{\partial}^{N_2}\left(J^{-1}(b_0\cdot\partial)\eta\right)\cdot \partial_3^{N-N_1}\overline{\partial}^{8-2N-N_2}\left(J^{-1}(b_0\cdot\partial)\eta\right)\left(\left[\overline{\partial}^8, \frac{JR'(q)}{\rho_0}\right]\partial_t q\right) \\
=: & -K_{11}^\sharp + I_{21}^\sharp + I_{22}^\sharp
\end{aligned} \tag{5.3.139}$$

and by direct computation we have

$$\int_0^T I_{21}^\sharp \lesssim \varepsilon \|\partial_3^N \overline{\partial}^{8-2N} q\|_0^2 + \mathcal{P}_0 + \int_0^T P(\mathfrak{E}(t))\,\mathrm{d}t \tag{5.3.140}$$

$$I_{22}^\sharp \lesssim \|J^{-1}(b_0\cdot\partial)\eta\|_{7,*}^2 \|q\|_{8,*}. \tag{5.3.141}$$

Then $I_3^\sharp$ can be controlled by integrating ∂_t by parts under time integral after invoking $\text{div}_A v = -\frac{JR'(q)}{\rho_0}\partial_t q$. The proof is similar to (5.3.86) so we do not repeat the proof.

$$\int_0^T I_3^\sharp \lesssim \varepsilon \|\overline{\partial}^{8-2N}\partial_3^N q\|_0^2 + \mathcal{P}_0 + \int_0^T P(\mathfrak{E}(t))\,\mathrm{d}t. \tag{5.3.142}$$

Summarizing (5.3.136)-(5.3.142) and choosing $\varepsilon > 0$ sufficiently small, we get the interior estimates

$$\int_0^T I^\sharp \, \mathrm{d}t \lesssim -\frac{1}{2}\int_\Omega \frac{J^2 R'(q)}{\rho_0} \left|\partial_3^N \overline{\partial}^{8-2N} q\right|^2 \, \mathrm{d}y \Big|_0^T + \mathcal{P}_0 + \int_0^T P(\mathfrak{E}(t))\, \mathrm{d}t. \tag{5.3.143}$$

Therefore, it suffices to analyze the boundary integral $IB^\sharp$.

5.3.6.3 Boundary estimates

Invoking (5.3.126)-(5.3.127), the boundary integral now reads

$$\begin{aligned}
IB^\sharp =& -\int_\Gamma \mathbf{Q}^\sharp J A^{3\alpha} N_3 \mathbf{V}_\alpha^\sharp \, \mathrm{d}S \\
=& -\int_\Gamma J A^{3\alpha} N_3 (\partial_3^N \overline{\partial}^{8-2N} Q) \mathbf{V}_\alpha^\sharp \, \mathrm{d}S \\
& + \int_\Gamma \mathbf{A}^{3\alpha} N_3 (\partial_3^N \overline{\partial}^{8-2N} \eta_\nu) A^{3\nu} \partial_3 Q \ \partial_3^N \overline{\partial}^{8-2N} v_\alpha \, \mathrm{d}S \\
& - \int_\Gamma \mathbf{A}^{3\alpha} N_3 (\partial_3^N \overline{\partial}^{8-2N} \eta_\nu \ A^{3\nu} \partial_3 Q)(\partial_3^N \overline{\partial}^{8-2N} \eta \cdot \nabla_A v_\alpha) \, \mathrm{d}S \\
& - \int_\Gamma \mathbf{A}^{3\alpha} N_3 (\triangle_Q^\sharp)(\partial_3^N \overline{\partial}^{8-2N} v_\alpha) \, \mathrm{d}S + \int_\Gamma \mathbf{A}^{3\alpha} N_3 (\triangle_Q^\sharp) \partial_3^N \overline{\partial}^{8-2N} \eta \cdot \nabla_A v_\alpha \, \mathrm{d}S \\
& - \int_\Gamma \mathbf{A}^{3\alpha} N_3 \triangle_Q^\sharp (\triangle_v^\sharp)_i \, \mathrm{d}S + \int_\Gamma \mathbf{A}^{3\alpha} N_3 (\partial_3^N \overline{\partial}^{8-2N} \eta_\nu \ A^{3\nu} \ \partial_3 Q)(\triangle_v^\sharp)_i \, \mathrm{d}S \\
=:& IB_0^\sharp + IB_1^\sharp + IB_2^\sharp + IB_3^\sharp + IB_4^\sharp + IB_5^\sharp + IB_6^\sharp.
\end{aligned} \tag{5.3.144}$$

To control $IB^\sharp$, we only need to combine the techniques used in Section 5.3.1.3 and Section 5.3.2.3:

- $IB_1^\sharp, IB_2^\sharp$ together with the Rayleigh-Taylor sign condition produces the boundary energy $|A^{3\alpha}\partial_3^N \overline{\partial}^{8-2N} \eta_\alpha|_0^2$, similarly as $IB_1 + IB_2$ in Section 5.3.1.3 and $IB_1^* + IB_2^*$ Section 5.3.2.3.
- The term $IB_0^\sharp$ is the analogue of IB_0 in Section 5.3.1.3 and can be controlled with similar method as in Section 5.3.1.3.

- $IB_3^\sharp \sim IB_6^\sharp$ are the analogues of $IB_3^* \sim IB_6^*$ in Section 5.3.2.3. These terms can be controlled exactly in the same way as $IB_3^* \sim IB_6^*$.

First, $IB_1^\sharp$ and $IB_2^\sharp$ give the boundary energy with the help of Rayleigh-Taylor sign condition. The proof is exactly the same as in Section 5.3.1.3 and Section 5.3.2.3 by merely replacing ∂_3^4 or $\overline{\partial}^8$ with $\partial_3^N\overline{\partial}^{8-2N}$, so we do not repeat the computations here.

$$\int_0^T IB_1^\sharp + IB_2^\sharp = -\frac{c_0}{4}\left|A^{3\alpha}\partial_3^N\overline{\partial}^{8-2N}\eta_\alpha\right|_0^2\Big|_0^T + \int_0^T P(\mathfrak{E}(t))\,\mathrm{d}t. \tag{5.3.145}$$

We then analyze $IB_0^\sharp$. Invoking (5.3.126), we have

$$\begin{aligned} IB_0^\sharp = & -\int_\Gamma N_3(J\partial_3^N\overline{\partial}^{8-2N}Q)(A^{3\alpha}\,\partial_3^N\overline{\partial}^{8-2N}v_\alpha)\,\mathrm{d}S \\ & + \int_\Gamma JA^{3\alpha}N_3(\partial_3^N\overline{\partial}^{8-2N}Q)(\partial_3^N\overline{\partial}^{8-2N}\eta\cdot\nabla_A v_\alpha)\,\mathrm{d}S \\ & - \int_\Gamma JA^{3\alpha}N_3(\partial_3^N\overline{\partial}^{8-2N}Q)(\triangle_v^\sharp)_\alpha\,\mathrm{d}S \\ =: & IB_{01}^\sharp + IB_{02}^\sharp + IB_{03}^\sharp. \end{aligned} \tag{5.3.146}$$

We note that $IB_{01}^\sharp$ and $IB_{02}^\sharp$ are the analogues of IB_{01} and IB_{02} in Section 5.3.1.3, so we do not repeat all the details here. The extra term $IB_{03}^\sharp$ can be directly controlled (cf. (5.3.153) below).

We differentiate the continuity equation (5.3.31) by $\partial_3^N\overline{\partial}^{8-2N}$ to simplify the top order term containing v in $IB_{01}^\sharp$:

$$\begin{aligned} A^{3\alpha}\partial_3^N\overline{\partial}^{8-2N}v_\alpha = & -\partial_3^{N-1}\overline{\partial}^{8-2N}\left(\frac{JR'(q)}{\rho_0}\partial_t q\right) - \sum_{L=1}^2 \partial_3^{N-1}\overline{\partial}^{8-2N}(A^{L\alpha}\overline{\partial}_L v_\alpha) \\ & - \sum_{N_1+N_2\geq 1, N_1\leq N-1}\binom{N-1}{N_1}\binom{8-2N}{N_2}\left(\partial_3^{N_1}\overline{\partial}^{N_2}A^{3\alpha}\right)\left(\partial_3^{N-N_1}\overline{\partial}^{8-2N-N_2}v_\alpha\right), \end{aligned} \tag{5.3.147}$$

where the term contains $\partial_3^{N-1}\overline{\partial}^{8-2N}A^{L\alpha} = \partial_3^N\overline{\partial}^{8-2N}\eta\times\overline{\partial}\eta + L.O.T.$ which cannot be controlled on the boundary. Invoking (3.1.3) with $D = \overline{\partial}$, we expand this problematic term to be

$$
\begin{aligned}
(\partial_3^{N-1}\overline{\partial}^{8-2N} A^{L\alpha})\overline{\partial}_L v_\alpha =& - \left(\partial_3^{N-1}\overline{\partial}^{7-2N}(A^{L\nu}\ \overline{\partial}\partial_\beta \eta_\nu\ A^{\beta\alpha})\right)\overline{\partial}_L v_\alpha \\
=& - A^{L\nu}\ \partial_3^N\overline{\partial}^{8-2N}\eta_\nu\ A^{3\alpha}\overline{\partial}_L v_\alpha \\
&- \sum_{M=1}^{2} A^{L\nu}(\partial_3^{N-1}\overline{\partial}^{8-2N}\overline{\partial}_M\eta_\nu)A^{M\alpha}\overline{\partial}_L v_\alpha \\
&- ([\partial_3^{N-1}\overline{\partial}^{7-2N}, A^{L\nu}A^{\beta\alpha}]\overline{\partial}\partial_\beta\eta_\nu)\overline{\partial}_L v_\alpha .
\end{aligned}
\tag{5.3.148}
$$

On the other hand, in $IB_{02}^\sharp$ we have

$$
A^{3\alpha}\partial_3^N\overline{\partial}^{8-2N}\eta\cdot\nabla_A v_\alpha = A^{3\alpha}\sum_{L=1}^{2}\partial_3^N\overline{\partial}^{8-2N}\eta_\nu A^{L\nu}\overline{\partial}_L v_\alpha + A^{3\alpha}\partial_3^N\overline{\partial}^{8-2N}\eta_\nu A^{3\nu}\partial_3 v_\alpha, \tag{5.3.149}
$$

where the first term exactly cancels with the first term in the RHS of (5.3.148). In fact, this is the analogue of (5.3.35)-(5.3.38) by merely replacing ∂_3^4 with $\partial_3^N\overline{\partial}^{8-2N}$. Thus we get the cancellation of the top order terms in $IB_{01}^\sharp$ and $IB_{02}^\sharp$.

The second term in (5.3.149) could be treated similarly as in (5.3.42) by invoking $A^{3\nu}\partial_3\eta_\nu = 1$

$$
\partial_3^N\overline{\partial}^{8-2N}\eta_\nu A^{3\nu} \stackrel{L}{=} \underbrace{\partial_3^{N-1}\overline{\partial}^{8-2N}(\partial_3\eta_\nu A^{3\nu})}_{=0} - (\partial_3^{N-1}\overline{\partial}^{8-2N}A^{3\nu})\partial_3\eta_\nu - (\overline{\partial}A^{3\nu})(\partial_3^N\overline{\partial}^{7-2N}\eta_\nu). \tag{5.3.150}
$$

To control $IB_0^\sharp$, we still need to analyze $\partial_3^N\overline{\partial}^{8-2N}Q$. Following the aruments in (5.3.39)-(5.3.41) and replacing ∂_3^4 with $\partial_3^N\overline{\partial}^{8-2N}$, we can reduce one normal derivative of Q to one tangential derivative of v and $(b_0\cdot\partial)\eta$ via

$$
\begin{aligned}
\partial_3^N\overline{\partial}^{8-2N}Q =& J^{-1}\partial_3\eta_\alpha\left(\rho_0\partial_3^{N-1}\overline{\partial}^{8-2N}\partial_t v^\alpha + (b_0\cdot\overline{\partial})\partial_3^{N-1}\overline{\partial}^{8-2N}(J^{-1}(b_0\cdot\partial)\eta^\alpha)\right) \\
&- \sum_{L=1}^{2}\mathbf{A}^{Li}(\partial_3^{N-1}\overline{\partial}^{8-2N}\overline{\partial}_L Q) \\
&- (N-1)(\partial_3^{N-1}\overline{\partial}^{8-2N}\mathbf{A}^{3\alpha})(\partial_3 Q) + \text{ lower order terms.}
\end{aligned}
\tag{5.3.151}
$$

Plugging the expression of $\triangle_v^\sharp$ and (5.3.147)-(5.3.151) into (5.3.146), we find that every highest order term in $IB_0^\sharp$ must be one of the following forms

$$K_1^\sharp := \int_\Gamma N_3(\partial_3^{N-1}\overline{\partial}^{8-2N}\mathfrak{D} f)(\partial_3^{N-1}\overline{\partial}^{9-2N} g)(\partial h) r \, \mathrm{d}S,$$

$$K_2^\sharp := \int_\Gamma N_3(\partial_3^{N-1}\overline{\partial}^{8-2N}\mathfrak{D} f)(\partial_3^{N}\overline{\partial}^{7-2N} g)(\partial\overline{\partial} h) r \, \mathrm{d}S,$$

$$K_3^\sharp := \int_\Gamma N_3(\partial_3^{N-1}\overline{\partial}^{8-2N}\mathfrak{D} f)(\partial_3^{N}\overline{\partial}^{7-2N} g)(\partial h) r \, \mathrm{d}S,$$

where $\mathfrak{D} = \overline{\partial}$ or ∂_t or $(b_0 \cdot \overline{\partial})$, the functions f, g, h can be $\eta, v, Q, J^{-1}(b_0 \cdot \partial)\eta$, and r contains at most one derivative of η, v or Q. We note that the term $K_2^\sharp$ comes from $IB_{03}^\sharp$ where $\triangle_v^\sharp$ contributes to $\partial_3^N \overline{\partial}^{7-2N} g \cdot \partial\overline{\partial} h \cdot r$.

Since $1 \leq N \leq 3$, we know $7 - 2N \geq 1$ and thus we can directly apply lemma 3.2.5 to control $K_1^\sharp \sim K_3^\sharp$.

$$\begin{aligned} K_1^\sharp \lesssim & |\partial_3^{N-1}\mathfrak{D} f|_{8-2N} |\partial_3^{N-1} g|_{9-2N} |\partial h \, r|_{L^\infty} \lesssim \|\partial_3^{N-1}\mathfrak{D} f\|_{H_*^{9-2N}} \|\partial_3^{N-1} g\|_{H_*^{10-2N}} \|\partial h \, r\|_{H^2} \\ \lesssim & \|f\|_{2(N-1)+1+(9-2N),*} \|g\|_{2(N-1)+(10-2N),*} \|h\|_3 \|r\|_2 = \|f\|_{8,*} \|g\|_{8,*} \|h\|_3 \|r\|_2. \end{aligned} \tag{5.3.152}$$

and

$$\begin{aligned} K_2^\sharp \lesssim & |\partial_3^{N-1}\mathfrak{D} f|_{8-2N} |\partial_3^{N} g|_{7-2N} |(\partial\overline{\partial} h) r|_{L^\infty} \lesssim |\partial_3^{N-1}\mathfrak{D} f|_{8-2N} |\partial_3^{N} g|_{7-2N} |(\partial\overline{\partial} h) r|_{1.5} \\ \lesssim & \|\partial_3^{N-1}\mathfrak{D} f\|_{H_*^{9-2N}} \|\partial_3^{N} g\|_{H_*^{8-2N}} \|(\partial\overline{\partial} h) r\|_2 \lesssim \|f\|_{8,*} \|g\|_{8,*} \|h\|_{7,*} \|r\|_2, \end{aligned} \tag{5.3.153}$$

and

$$\begin{aligned} K_3^\sharp \lesssim & |\partial_3^{N-1}\mathfrak{D} f|_{8-2N} |\partial_3^{N} g|_{7-2N} |(\partial h) r|_{L^\infty} \lesssim |\partial_3^{N-1}\mathfrak{D} f|_{8-2N} |\partial_3^{N} g|_{7-2N} |(\partial h) r|_{1.5} \\ \lesssim & \|\partial_3^{N-1}\mathfrak{D} f\|_{H_*^{9-2N}} \|\partial_3^{N} g\|_{H_*^{8-2N}} \|(\partial h) r\|_2 \lesssim \|f\|_{8,*} \|g\|_{8,*} \|h\|_3 \|r\|_2. \end{aligned} \tag{5.3.154}$$

One can use either trace lemma or similar techniques as in (5.3.43)-(5.3.45) to analyze the remaining terms which are all of lower order than $K_1^\sharp \sim K_3^\sharp$. This completes the control of $IB_0^\sharp$.

The analysis of $IB_3^\sharp \sim IB_6^\sharp$ can be proceeded exactly in the same way as $IB_3^* \sim IB_6^*$. Since these quantities involving the modification terms $\triangle_Q^\sharp, \triangle_v^\sharp$ are of lower order, we do not repeat the details again. We can finally prove that

$$\int_0^T IB_3^\sharp + IB_4^\sharp \, \mathrm{d}t \lesssim \int_0^T \left| A^{3\alpha}\partial_3^N \overline{\partial}^{8-2N}\eta_\alpha \right|_0 P\left(\|(\eta, v, b)\|_{8,*}, \|Q\|_{8,*}, \|\rho_0\|_3\right) \mathrm{d}t, \tag{5.3.155}$$

$$+ \left| A^{3\alpha}\partial_3^N \overline{\partial}^{8-2N}\eta_\alpha \right|_0 P(\|\eta\|_{8,*}, \|Q\|_{8,*}) \int_0^T \|v(t)\|_{8,*} \, \mathrm{d}t$$

$$IB_5^\sharp \lesssim |\mathbf{A}^{3\alpha}|_{L^\infty} |\triangle_Q^\sharp|_0 |(\triangle_v^\sharp)_i|_0 \lesssim P(\|\eta\|_{8,*}, \|v\|_{8,*}, \|Q\|_{7,*}), \tag{5.3.156}$$

$$IB_6^\sharp \lesssim |\mathbf{A}^{3\alpha}\partial_3 Q|_{L^\infty} |A^{3\nu}\partial_3^N \overline{\partial}^{8-2N}\eta_\nu|_0 |(\triangle_v^\sharp)_i|_0 \lesssim P(\|\eta\|_{8,*}, \|v\|_{8,*}, \|Q\|_{7,*}). \tag{5.3.157}$$

Summarizing the estimates above, we get the control of the boundary integral

$$\int_0^T IB^\sharp \lesssim -\frac{c_0}{4} \left| A^{3\alpha}\partial_3^N \overline{\partial}^{8-2N}\eta_\alpha \right|_0^2 + \mathcal{P}_0 + P(\mathfrak{E}(T)) \int_0^T P(\mathfrak{E}(t)) \, \mathrm{d}t. \tag{5.3.158}$$

Combining (5.3.131)-(5.3.133), (5.3.143), (5.3.158) and choosing $\varepsilon > 0$ in (5.3.140) to be suitably small, we get the following inequality

$$\begin{aligned} &\|\mathbf{V}^\sharp\|_0^2 + \left\| \partial_3^N \overline{\partial}^{8-2N} \left(J^{-1}(b_0 \cdot \partial)\eta\right) \right\|_0^2 + \|\partial_3^N \overline{\partial}^{8-2N} q\|_0^2 + \frac{c_0}{4} \left| A^{3\alpha}\partial_3^N \overline{\partial}^{8-2N}\eta_\alpha \right|_0^2 \Big|_{t=T} \\ &\lesssim \mathcal{P}_0 + P(\mathfrak{E}(T)) \int_0^T P(\mathfrak{E}(t)) \, \mathrm{d}t. \end{aligned} \tag{5.3.159}$$

Finally, invoking (5.3.126), we get the $\partial_3^N \overline{\partial}^{8-2N}$ $(N = 1, 2, 3)$-estimates of v by using $\partial^m \eta|_{t=0} = 0$ for any $m \geq 2, m \in \mathbb{N}^*$,

$$\|\partial_3^N \overline{\partial}^{8-2N} v\|_0 \lesssim \|\mathbf{V}^\sharp\|_0 + P(\|v\|_{7,*}, \|\eta\|_{7,*}) \int_0^T P(\|v\|_{8,*}), \tag{5.3.160}$$

and thus

$$\begin{aligned} &\left\| \partial_3^N \overline{\partial}^{8-2N} \left(v, J^{-1}(b_0 \cdot \partial)\eta, q\right) \right\|_0^2 + \frac{c_0}{4} \left| A^{3\alpha}\partial_3^N \overline{\partial}^{8-2N}\eta_\alpha \right|_0^2 \Big|_{t=T} \\ &\lesssim \mathcal{P}_0 + P(\mathfrak{E}(T)) \int_0^T P(\mathfrak{E}(t)) \, \mathrm{d}t. \end{aligned} \tag{5.3.161}$$

5.3.7 Control of time derivatives

In the case of $\partial_*^I = \partial_3^N \overline{\partial}^{8-2N-k} \partial_t^k$ for $1 \le k \le 8-2N$, most of steps in the proof are still applicable. However, the presence of time derivative(s) could simplify the "modified Alinhac good unknowns". We note that most of the modifications are essentially similar to Section (5.3.3) $\sim$ Section 5.3.5, so we omit the proof.

5.3.7.1 One time derivative

When $k = 1$, the modified Alinhac good unknowns can be defined by replacing $8\overline{\partial}^7$ by $(8 - 2N)\partial_3^N \overline{\partial}^{7-2N}$ in Section 5.3.3.2, i.e.,

$$\mathbf{V}^\sharp = \partial_3^N \overline{\partial}^{7-2N} \partial_t v - \partial_3^N \overline{\partial}^{7-2N} \partial_t \eta \cdot \nabla_A v + \triangle_v^\sharp, \;\; \mathbf{Q}^\sharp = \partial_3^N \overline{\partial}^{7-2N} \partial_t Q - \partial_3^N \overline{\partial}^{7-2N} \partial_t \eta \cdot \nabla_A Q + \triangle_Q^\sharp., \tag{5.3.162}$$

where

$$\begin{aligned}(\triangle_v^\sharp)_i :=& - (8-2N)\partial_3^N \overline{\partial}^{7-2N} \eta \cdot \nabla_A \partial_t v_\alpha - (8-2N)\partial_3^N \overline{\partial}^{7-2N} v \cdot \nabla_A v_\alpha \\ &+ (16-4N)\partial_3^N \overline{\partial}^{7-2N} \cdot \nabla_A v \cdot \nabla_A v_\alpha,\end{aligned} \tag{5.3.163}$$

$$\triangle_Q^\sharp := - (8-2N)\partial_3^N \overline{\partial}^{7-2N} \eta \cdot \nabla_A \partial_t Q + (8-2N)\partial_3^N \overline{\partial}^{7-2N} \eta \cdot \nabla_A v \cdot \nabla_A Q,$$

and

$$\partial_3^N \overline{\partial}^{7-2N} \partial_t (\mathrm{div}_{\tilde{A}} v) = \nabla_A \cdot \mathbf{V}^\sharp + C^\sharp(v), \quad \partial_3^N \overline{\partial}^{7-2N} \partial_t (\nabla_A Q) = \nabla_A \mathbf{Q}^\sharp + C^\sharp(Q), \tag{5.3.164}$$

with

$$\|C^\sharp(f)\|_0 \lesssim P(\mathfrak{E}(t))\|f\|_{8,*}.$$

The difference between $\partial_3^N \overline{\partial}^{7-2N} v$ and $\mathbf{V}^\sharp$ should be controlled in the same way as (5.3.108) by replacing $\overline{\partial}^7$ with $\partial_3^N\overline{\partial}^{7-2N}$

$$\|\partial_3^N\overline{\partial}^{7-2N}\partial_t v\|_0 \lesssim \|\mathbf{V}^*\|_0^2 + \mathcal{P}_0 + P(\mathfrak{E}(T))\int_0^T P(\mathfrak{E}(t))\,\mathrm{d}t, \tag{5.3.165}$$

and thus

$$\begin{aligned}&\left\|\partial_3^N\overline{\partial}^{7-2N}\partial_t\left(v, J^{-1}(b_0\cdot\partial)\eta, q\right)\right\|_0^2 + \frac{c_0}{4}\left|A^{3\alpha}\partial_3^N\overline{\partial}^{7-2N}\partial_t\eta_\alpha\right|_0^2\Bigg|_{t=T}\\ \lesssim&\mathcal{P}_0 + P(\mathfrak{E}(T))\int_0^T P(\mathfrak{E}(t))\,\mathrm{d}t.\end{aligned} \tag{5.3.166}$$

5.3.7.2 2∼(7-2N) time derivatives

When $2\le k\le 7-2N$, we can mimic the analysis in Section 5.3.4: We just need to replace $\mathfrak{D}^6\partial_t^2$ by $\partial_3^N\mathfrak{D}^{6-2N}\partial_t^2$ where $\mathfrak{D}$ denotes $\overline{\partial}$ or ∂_t and $\mathfrak{D}^{6-2N}$ contains at least one $\overline{\partial}$. The analogous problematic term becomes $-2(\partial_t A^{\mu\alpha})(\partial_3^N\mathfrak{D}^{6-2N}\partial_t\partial_\mu f) - (6-2N)(\mathfrak{D}A^{\mu\alpha})(\partial_3^N\mathfrak{D}^{5-2N}\partial_t^2\partial_\mu f)$ which comes from $[\partial_3^N\mathfrak{D}^{6-2N}\partial_t^2, A^{\mu\alpha}, \partial_\mu f]$. Following (5.3.111)-(5.3.113), we can similarly define

$$\mathbf{V}^\sharp = \partial_3^N\mathfrak{D}^{6-2N}\partial_t^2 v - \partial_3^N\mathfrak{D}^{6-2N}\partial_t^2\eta\cdot\nabla_A v + \triangle_v^\sharp,\ \ \mathbf{Q}^\sharp = \partial_3^N\mathfrak{D}^{6-2N}\partial_t^2 Q - \partial_3^N\mathfrak{D}^{6-2N}\partial_t^2\eta\cdot\nabla_A Q, \tag{5.3.167}$$

where

$$(\triangle_v^\sharp)_i := -(6-2N)\partial_3^N\mathfrak{D}^{5-2N}\partial_t^2 v\cdot\nabla_A\mathfrak{D}\eta_\alpha - 2\partial_3^N\mathfrak{D}^{6-2N}\partial_t v\cdot\nabla_A v_\alpha \tag{5.3.168}$$

and

$$\partial_3^N\mathfrak{D}^{6-2N}\partial_t^2(\mathrm{div}_{\tilde{A}} v) = \nabla_A\cdot\mathbf{V}^\sharp + C^\sharp(v),\quad \partial_3^N\mathfrak{D}^{6-2N}\partial_t^2(\nabla_A Q) = \nabla_A\mathbf{Q}^* + C^\sharp(Q), \tag{5.3.169}$$

with

$$\|C^\sharp(f)\|_0 \lesssim P(\mathfrak{E}(t))\|f\|_{8,*}.$$

Again we have $\triangle_Q^\sharp$ in this case, and thus the analogues of $IB_3^\sharp\sim IB_5^\sharp$ all vanish. The boundary integrals $IB_0^\sharp, IB_1^\sharp, IB_2^\sharp, IB_6^\sharp$ and the interior terms can be controlled in the same way as Section 5.3.6.

Finally, one has

$$\left\|\partial_3^N \mathfrak{D}^{6-2N} \partial_t^2 \left(v, J^{-1}(b_0\cdot\partial)\eta, q\right)\right\|_0^2 + \frac{c_0}{4}\left|A^{3\alpha}\partial_3^N\mathfrak{D}^{6-2N}\partial_t^2\eta_\alpha\right|_0^2\Bigg|_{t=T} \lesssim \mathcal{P}_0 + P(\mathfrak{E}(T))\int_0^T P(\mathfrak{E}(t))\,\mathrm{d}t, \tag{5.3.170}$$

where $\mathfrak{D}^{6-2N}$ contains at least one spatial derivative $\overline{\partial}$.

5.3.7.3 Full time derivatives

When $\partial_*^I = \partial_3^N\partial_t^{8-2N}$ for $N = 1, 2, 3$, there is not tangential spatial derivative on the boundary and thus Lemma 3.2.5 is no longer applicable. In this case, the modified Alinhac good unknowns become

$$\mathbf{V}^\sharp = \partial_3^N\partial_t^{8-2N} v - \partial_3^N\partial_t^{8-2N}\eta\cdot\nabla_A v + \triangle_v^\sharp,\ \ \mathbf{Q}^\sharp = \partial_3^N\partial_t^{8-2N} Q - \partial_3^N\partial_t^{8-2N}\eta\cdot\nabla_A Q, \tag{5.3.171}$$

where

$$(\triangle_v^\sharp)_i := -(8-2N)\partial_3^N\partial_t^{8-2N} v\cdot\nabla_A v_\alpha \tag{5.3.172}$$

and

$$\partial_3^N\partial_t^{8-2N}(\mathrm{div}_{\tilde{A}} v) = \nabla_A\cdot\mathbf{V}^\sharp + C^\sharp(v),\quad \partial_3^N\partial_t^{8-2N}(\nabla_A Q) = \nabla_A\mathbf{Q}^\sharp + C^\sharp(Q), \tag{5.3.173}$$

with

$$\|C^\sharp(f)\|_0 \lesssim P(\mathfrak{E}(t))\|f\|_{8,*}.$$

The proof follows in the same way as Section 5.3.5 after replacing ∂_t^7 by ∂_t^{7-2N} and the coefficient 8 by $(8-2N)$. So we no longer repeat the details. Finally, we get

$$\|\partial_3^N\partial_t^{8-2N} v\|_0^2 + \left\|\partial_3^N\partial_t^{8-2N}\left(J^{-1}(b_0\cdot\partial)\eta\right)\right\|_0^2 + \|\partial_3^N\partial_t^{8-2N} q\|_0^2 + \frac{c_0}{4}\left|A^{3\alpha}\partial_3^N\partial_t^{8-2N}\eta_\alpha\right|_0^2\Bigg|_{t=T} \lesssim \varepsilon\|\partial_3^{N+1}\partial_t^{6-2N} v\|_0^2 + \mathcal{P}_0 + P(\mathfrak{E}(T))\int_0^T P(\mathfrak{E}(t))\,\mathrm{d}t, \tag{5.3.174}$$

which together with (5.3.161), (5.3.166), (5.3.170) concludes the proof of Proposition 5.3.5.

5.3.8 Control of weighted normal derivatives

Now we consider the most general case $\partial_*^I = \partial_t^{i_0}(\sigma\partial_3)^{i_4}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}$ with $i_1 + i_2 + 2i_3 + i_4 = 8$ and $i_4 > 0$. The presence of the weighted normal derivatives $(\sigma\partial_3)^{i_4}$ makes the following difference from the non-weighted case.

1. Extra terms are produced when we commute ∂_*^I with ∂_3 because σ is a function of y_3. Once ∂_3 falls on the weight function, we will lose a weight and $(\sigma\partial_3)$ becomes ∂_3, which causes a loss of derivative. This appears when we commute ∂_*^I with ∇_A^α that falls on Q or v_α and commute ∂_*^I with $(b_0 \cdot \partial)$.

2. There is no boundary integral because the weight function σ vanishes on Γ.

To overcome the difficulty mentioned above, we can again use the techniques, similar with those in the previous sections.

- Invoke the MHD equation and the continuity equation to replace $\nabla_A Q$ and $\mathrm{div}_{\tilde{A}} v$ by tangential derivatives.

- Produce a weight funtion by using $b_0^3|_\Gamma = 0$ and $\overline{\partial} Q|_\Gamma = 0$.

- In particular, if ∂_*^I does not contain time derivative, we need to add an extra modification term in the good unknown of v.

First we analyze $[(b_0 \cdot \partial), \partial_t^{i_0}(\sigma\partial_3)^{i_4}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}]f$. Compared with the non-weighted case, we need to control the extra term

$$b_0^3\partial_3(\sigma^{i_4})\,(\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3+i_4}f) = i_4 b_0^3(\partial_3\sigma)\left(\partial_t^{i_0}(\sigma\partial_3)^{i_4-1}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3+1}\right)f.$$

Using $b_0^3|_\Gamma = 0$, one can produce a weight function σ as in (5.3.58). Therefore

$$\left\| b_0^3(\partial_3\sigma)\left(\partial_t^{i_0}(\sigma\partial_3)^{i_4-1}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3+1}f\right)\right\|_0$$

$$\lesssim \|\partial_3 b_0\|_{L^\infty}\|(\sigma\partial_3)\partial_t^{i_0}(\sigma\partial_3)^{i_4-1}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}f\|_0 \le \|b_0\|_3\|f\|_{8,*}.$$

Next we analyze the commutator between $\partial_*^I = \partial_t^{i_0}(\sigma\partial_3)^{i_4}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}$ and $\nabla_A f$. Compared with the non-weighted case, we shall analyze an extra term C_σ below. In specific, one has

$$\begin{aligned}
&\partial_t^{i_0}(\sigma\partial_3)^{i_4}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}(A^{\mu\alpha}\partial_\mu f) = \sigma^{i_4}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3+i_4}(A^{\mu\alpha}\partial_\mu f)\\
=&\sigma^{i_4}\left(A^{\mu\alpha}\partial_\mu(\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3+i_4})f\right) + \underbrace{\sigma^{i_4}[\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3+i_4}, A^{\mu\alpha}]\partial_\mu f}_{\mathring{C}}\\
=&A^{\mu\alpha}\partial_\mu\left(\sigma^{i_4}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3+i_4}f\right) - (i_4\partial_3\sigma)\underbrace{A^{3\alpha}\left((\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3+1}f\right)}_{C_\sigma} + \mathring{C}.
\end{aligned} \tag{5.3.175}$$

The term $\mathring{C}$ consists of the commutators produced in the same way as the non-weighted case. It can be analyzed in the same way as in previous sections by just considering $(\sigma\partial_3)$ as a tangential derivative on the boundary. As for the extra term, we do the following computation

$$\begin{aligned}
&A^{3\alpha}\left((\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3+1}f\right)\\
=&(\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}(A^{3\alpha}\partial_3 f) - \left[(\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}, A^{3\alpha}\right]\partial_3 f\\
=:&C_1^\sigma(f) + C_2^\sigma(f).
\end{aligned} \tag{5.3.176}$$

Note that $i_0+i_1+i_2+i_4 = 8-2i_3$. We know the leading order terms in C_2^σ are $\left((\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}A^{3\alpha}\right)f$ and $(\mathfrak{D}A^{3\alpha})(\mathfrak{D}^{6-2i_3}\partial_3^{i_3+1}f)$, where $\mathfrak{D}$ represents any one of $(\sigma\partial_3), \partial_t, \overline{\partial}_1, \overline{\partial}_2$. Recall that $A^{3\alpha}$ cosists of $\overline{\partial}\eta\cdot\overline{\partial}\eta$. This shows that the highest order term in $\left((\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}A^{3\alpha}\right)\partial_3 f$ is $(\mathfrak{D}^{8-2i_3}\partial_3^{i_3}\eta)(\overline{\partial}\eta)f$ whose $L^2(\Omega)$ norm can be directly controlled by $\|\eta\|_{8,*}\|\overline{\partial}\eta\|_{L^\infty}\|\partial_3 f\|_{L^\infty}$. As for the second term, we have

$$\|(\mathfrak{D}A^{3\alpha})(\mathfrak{D}^{6-2i_3}\partial_3^{i_3+1}f)\|_0 \lesssim \|(\mathfrak{D}\overline{\partial}\eta)(\overline{\partial}\eta)\|_{L^\infty}\|f\|_{8,*}.$$

Therefore, C_2^σ can be directly controlled.

The control of C_1^σ is more complicated. We should use the structure of MHD system (2.4.1) to replace one normal derivative by one tangential derivative.

$$A^{3\alpha}\partial_3 Q = -\sum_{L=1}^{2} A^{L\alpha}\overline{\partial}_L Q - R\partial_t v^\alpha + J^{-1}(b_0\cdot\partial)(J^{-1}(b_0\cdot\partial)\eta)^\alpha \tag{5.3.177}$$

$$A^{3\alpha}\partial_3 v_\alpha = \text{div}_{\tilde{A}} v - A^{1\alpha}\overline{\partial}_1 v_\alpha - A^{2\alpha}\overline{\partial}_2 v_\alpha = -\frac{JR'(q)}{\rho_0}\partial_t q - \sum_{L=1}^{2} A^{L\alpha}\overline{\partial}_L v_\alpha \tag{5.3.178}$$

When $f = Q$, we plug (5.3.177) into $C_1^\sigma(Q)$ to get

$$\begin{aligned} C_1^\sigma(Q) =& (\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}(A^{3\alpha}\partial_3 Q) \\ =& -\sum_{L=1}^{2}(\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}(A^{L\alpha}\overline{\partial}_L Q) \\ &- (\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}(R\partial_t v^\alpha) \\ &+ (\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}\left(J^{-1}(b_0\cdot\partial)(J^{-1}(b_0\cdot\partial)\eta^\alpha)\right) \\ =:& C_{11}^\sigma(Q) + C_{12}^\sigma(Q) + C_{13}^\sigma(Q). \end{aligned} \tag{5.3.179}$$

When $f = v_\alpha$, we plug (5.3.178) into $C_1^\sigma(v)$ to get

$$\begin{aligned} C_1^\sigma(v) =& (\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}(A^{3\alpha}\partial_3 v_\alpha) \\ =& -\sum_{L=1}^{2}(\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}(A^{L\alpha}\overline{\partial}_L v_\alpha) - (\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}\left(\frac{JR'(q)}{\rho_0}\partial_t q\right) \\ =:& C_{11}^\sigma(v) + C_{12}^\sigma(v). \end{aligned} \tag{5.3.180}$$

The terms $C^{\sigma}_{12}(Q)$ and $C^{\sigma}_{12}(v)$ can be directly controlled. Note that $i_0+i_1+i_2+(i_4-1) = 7-2i_3$, so

$$\|C^{\sigma}_{12}(Q)\|_0 \lesssim \|R\|_{7,*}\|v\|_{8,*} \lesssim \|q\|_{7,*}\|v\|_{8,*}, \tag{5.3.181}$$

$$\|C^{\sigma}_{12}(v)\|_0 \lesssim \|\rho_0\|_{7,*}\|q\|_{8,*}. \tag{5.3.182}$$

Using $b_0^3|_{\Gamma} = 0$, one can produce a weight function σ as in (5.3.58) when all the derivatives fall on $J^{-1}(b_0\cdot\partial)\eta$.

$$\begin{aligned}\|C^{\sigma}_{13}(Q)\|_0 \lesssim& \|J^{-1}(b_0\cdot\partial)(\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}(J^{-1}(b_0\cdot\partial)\eta)\|_0\\ &+\left\|\left[(\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}, J^{-1}(b_0\cdot\partial)\right](J^{-1}(b_0\cdot\partial)\eta)\right\|_0\\ \lesssim&\|\partial_3(J^{-1}b_0)\|_{L^\infty}\|(\sigma\partial_3)^{i_4}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}(J^{-1}(b_0\cdot\partial)\eta)\|_0\\ \lesssim&\|b_0\|_{7,*}\|J^{-1}(b_0\cdot\partial)\eta\|_{8,*}.\end{aligned} \tag{5.3.183}$$

As for C^{σ}_{11}, the highest order term can be merged into the modified Alinhac good unknowns. One has

$$\begin{aligned}C^{\sigma}_{11}(f) :=& -\sum_{L=1}^{2}(\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}(A^{L\alpha}\overline{\partial}_L f)\\ =&-\sum_{L=1}^{2}\left((\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}A^{L\alpha}\right)\overline{\partial}_L f\\ &\underbrace{-\sum_{L=1}^{2}\left[(\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}, A^{L\alpha}\right]\overline{\partial}_L f}_{C^{\sigma}_{111}(f)}\end{aligned} \tag{5.3.184}$$

which again is equal to

$$
\begin{aligned}
&-\sum_{L=1}^{2} A^{L\nu}\left((\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}\partial_\beta\eta_\nu\right)A^{\beta\alpha}\overline{\partial}_L f \\
&\underbrace{-\sum_{L=1}^{2}\left(\left[(\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}, A^{L\nu}A^{\beta\alpha}\right]\partial_\beta\eta_\nu\right)\overline{\partial}_L f}_{C_{112}^{\sigma}(f)} + C_{111}^{\sigma}(f).
\end{aligned}
\tag{5.3.185}
$$

Since $i_0+i_1+i_2+i_4=8-2i_3$, one can directly control the $L^2(\Omega)$-norms of $C_{111}^{\sigma}(f),\ C_{112}^{\sigma}(f)$ by $P(\|\eta\|_{8,*})\|f\|_{8,*}$. For the first term in the RHS of (5.3.184), one can proceed in the following ways

- $f=Q$: Since $\overline{\partial}_L Q|_\Gamma=0$, one can produce a weight function as in (5.3.62) and thus

$$
\begin{aligned}
&\left\|A^{L\nu}\left((\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}\partial_\beta\eta_\nu\right)A^{\beta\alpha}\overline{\partial}_L Q\right\|_0 \\
\lesssim&\sum_{M=1}^{2}\left\|A^{L\nu}\left((\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}\overline{\partial}_M\eta_\nu\right)A^{M\alpha}\overline{\partial}_L Q\right\|_0 \\
&+\|A^{L\nu}A^{3\alpha}\overline{\partial}\partial_3 Q\|_{L^\infty}\|(\sigma\partial_3)^{i_4}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}\eta_\nu\|_0 \\
\lesssim& P(\|\eta\|_3)\|Q\|_{7,*}\|\eta\|_{8,*}.
\end{aligned}
\tag{5.3.186}
$$

- $f=v_\alpha$: When ∂_*^I contains time derivative ($i_0>0$), then it can be directly controlled due to $\partial_t\eta=v$

$$
\left\|A^{L\nu}\left((\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}\partial_\beta\eta_\nu\right)A^{\beta\alpha}\overline{\partial}_L v_\alpha\right\|_0 \lesssim P(\|\eta\|_3)\|v\|_{5,*}\|v\|_{8,*}. \tag{5.3.187}
$$

If $i_0 = 0$, then it can be written in the form of covariant derivative plus a controllable term.

$$
\begin{aligned}
& - A^{L\nu} \left((\sigma\partial_3)^{i_4-1} \overline{\partial}_1^{i_1} \overline{\partial}_2^{i_2} \partial_3^{i_3} \partial_\beta \eta_\nu \right) A^{\beta\alpha} \overline{\partial}_L v_\alpha \\
= & - A^{\beta\alpha} \partial_\beta \left((\sigma\partial_3)^{i_4-1} \overline{\partial}_1^{i_1} \overline{\partial}_2^{i_2} \partial_3^{i_3} \eta_\nu A^{L\nu} \overline{\partial}_L v_\alpha \right) \\
& + A^{3\alpha} (\partial_3 \sigma) \left((i_4 - 1)(\sigma\partial_3)^{i_4-2} \overline{\partial}_1^{i_1} \overline{\partial}_2^{i_2} \partial_3^{i_3+1} \eta_\nu \right) A^{L\nu} \overline{\partial}_L v_\alpha \\
& + \nabla_A^\alpha (A^{L\nu} \overline{\partial}_L v_\alpha) \left((\sigma\partial_3)^{i_4-1} \overline{\partial}_1^{i_1} \overline{\partial}_2^{i_2} \partial_3^{i_3} \eta_\nu \right) \\
=: & - \nabla_A^\alpha \left((\sigma\partial_3)^{i_4-1} \overline{\partial}_1^{i_1} \overline{\partial}_2^{i_2} \partial_3^{i_3} \eta_\nu A^{L\nu} \overline{\partial}_L v_\alpha \right) + C_{113}^\sigma (v_\alpha).
\end{aligned}
\tag{5.3.188}
$$

We note that the first term in $C_{113}^\sigma(f)$ appears when ∂_k ($k = 3$) falls on the weight function and $i_4 \geq 2$ and can also be directly controlled by $P(\|\eta\|_{8,*}) \|f\|_{8,*}$.

Next we merge the covariant derivative terms in C_σ into the modified Alinhac good unknowns, i.e., for $\partial_*^I = \partial_t^{i_0} (\sigma\partial_3)^{i_4} \overline{\partial}_1^{i_1} \overline{\partial}_2^{i_2} \partial_3^{i_3}$ we define $\mathbf{V}_\alpha^\sigma$ to be

$$
\begin{cases}
\partial_*^I v_\alpha - \partial_*^I \eta \cdot \nabla_A v_\alpha + (\triangle_v^\sigma)_\alpha, & i_0 \geq 1 \\
\partial_*^I v_\alpha - \partial_*^I \eta \cdot \nabla_A v_\alpha + (\triangle_v^\sigma)_\alpha + \sum_{L=1}^{2} \left((i_4 \partial_3 \sigma)(\sigma\partial_3)^{i_4-1} \overline{\partial}_1^{i_1} \overline{\partial}_2^{i_2} \partial_3^{i_3} \eta_\nu \right) A^{L\nu} \overline{\partial}_L v_\alpha, & i_0 = 0,
\end{cases}
\tag{5.3.189}
$$

and

$$
\mathbf{Q}^\sigma := \partial_*^I Q - \partial_*^I \eta \cdot \nabla_A Q + \triangle_Q^\sigma. \tag{5.3.190}
$$

Then one has

$$
\partial_*^I (\nabla_A \cdot v) = \nabla_A \cdot \mathbf{V}^\sigma + C^\sigma(v), \tag{5.3.191}
$$

$$
\partial_*^I (\nabla_A Q) = \nabla_A \mathbf{Q}^\sigma + C^\sigma(Q), \tag{5.3.192}
$$

with $\|C^\sigma(f)\|_0 \lesssim P(\mathfrak{E}(t)) \|f\|_{8,*}$. Here the "extra modification terms" $\triangle_v^\sigma$ and $\triangle_Q^\sigma$ comes from the analysis of $\mathring{C}$ in (5.3.175) whose precise expressions can be derived in the same way as before. The term $\left((i_4\partial_3\sigma)(\sigma\partial_3)^{i_4-1}\partial_t^{i_0}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}\eta_\nu\right)A^{L\nu}\overline{\partial}_L f$ comes from (5.3.175) and (5.3.188). Finally, the commutator $C^\sigma(f)$ consists of the commutator part in $\mathring{C}$, $C^\sigma_{111}(f)\sim C^\sigma_{113}(f), C^\sigma_{12}(f)$ and $C^\sigma_{13}(Q)$.

Recall that $\sigma|_\Gamma = 0$ and $\overline{\partial}Q|_\Gamma = 0$ imply $\mathbf{Q}^\sigma|_\Gamma = 0$. Therefore the boundary integral $\int_\Gamma N_3\mathbf{A}^{3\alpha}\mathbf{Q}^\sigma\mathbf{V}^\sigma_\alpha\,\mathrm{d}S$ vanishes. Hence, we can get the following estimates for $\partial_*^I := \partial_t^{i_0}(\sigma\partial_3)^{i_4}\overline{\partial}_1^{i_1}\overline{\partial}_2^{i_2}\partial_3^{i_3}$

$$\|\partial_*^I v\|_0^2 + \left\|\partial_*^I\left(J^{-1}(b_0\cdot\partial)\eta\right)\right\|_0^2 + \|\partial_*^I q\|_0^2\Big|_{t=T} \lesssim \mathcal{P}_0 + P(\mathfrak{E}(T))\int_0^T P(\mathfrak{E}(t))\,\mathrm{d}t. \tag{5.3.193}$$

5.3.9 A priori estimates of the compressible MHD system

5.3.9.1 Finalizing the energy estimates

Combining the L^2-energy conservation with (5.3.3), (5.3.50), (5.3.123) and (5.3.193), and then choosing $\varepsilon > 0$ to be suitably small, we finally get the following energy inequality

$$\mathfrak{E}(T) - \mathfrak{E}(0) \lesssim \mathcal{P}_0 + P(\mathfrak{E}(T))\int_0^T P(\mathfrak{E}(t))\,\mathrm{d}t \tag{5.3.194}$$

under the a priori assumptions (5.3.1)-(5.3.2). By the Gronwall-type inequality, one can find some $T_2 > 0$ depending only on the initial data, such that

$$\sup_{0\le t\le T_2}\mathfrak{E}(t) \le \mathcal{C}(\mathfrak{E}(0)), \tag{5.3.195}$$

where $\mathcal{C}(\mathfrak{E}(0))$ denotes a positive constant depending on $\mathfrak{E}(0)$. This completes the a priori estimates of (2.4.1).

5.3.9.2 Justification of the a priori assumptions

It suffices to justify the a priori assumptions (5.3.1)-(5.3.2). First, invoking $\partial_t J = J\mathrm{div}_{\tilde{A}}v$ and $J|_{t=0} = 1$, we get

$$\|J-1\|_{7,*} \le \int_0^T \|J\mathrm{div}_{\tilde{A}}v\|_{7,*}\,\mathrm{d}t \lesssim \int_0^T P(\|\partial\eta\|_{L^\infty})\|\partial_t q\|_{7,*}\,\mathrm{d}t$$

Therefore choosing $T > 0$ to be sufficiently small yields (5.3.1). The Rayleigh-Taylor sign condition in $[0, T_1]$ can be justified by proving $\partial Q/\partial N$ is a Hölder-continuous function in t and y variables. In specific, from the energy estimates we know that

$$\frac{\partial Q}{\partial N} \in L^\infty([0,T]; H^{\frac{5}{2}}(\Gamma)),\ \partial_t\left(\frac{\partial Q}{\partial N}\right) \in L^\infty([0,T]; H^{\frac{3}{2}}(\Gamma)).$$

By using the 2D Sobolev embedding $H^{\frac{1}{2}}(\Gamma) \hookrightarrow L^4(\Gamma)$ and Morrey's embedding $W^{1,4} \hookrightarrow C^{0,\frac{1}{4}}$ in 3D domain, we get its Hölder continuity

$$\frac{\partial Q}{\partial N} \in W^{1,\infty}([0,T]; H^{\frac{3}{2}}(\Gamma)) \hookrightarrow W^{1,4}([0,T]\times\Gamma) \hookrightarrow C^{0,\frac{1}{4}}_{t,x}([0,T]\times\Gamma).$$

Therefore, (5.3.2) holds in a positive time interval provided that $-\dfrac{\partial Q_0}{\partial N} \geq c_0 > 0$ holds initially. Theorem 2.4.1 is proved.

5.3.10 Initial data satisfying the compatibility conditions

Define $f_{(j)} := \partial_t^j f|_{t=0}$ to be the initial data of $\partial_t^j f$ for $j \in \mathbb{N}$. Finally, we need to prove the existence of initial data satisfying the following properties:

- The compatibility conditions (1.0.9) up to 7-th order.
- The constraints $\nabla \cdot B_0 = 0,\ B_0 \cdot n|_{\{0\}\times\partial\mathcal{D}_0} = 0$ and the Rayleigh-Taylor sign condition (1.0.5) on $\{0\}\times\partial\mathcal{D}_0$.
- The norms of the initial datum of the time derivatives of (v, b, Q) can be controlled by the norms of initial data (v_0, b_0, Q_0).

We note that the compatibility conditions up to order m can be expressed in Lagrangian coordinates by using the formal power series solution to (2.4.1) in t:

$$\hat{v}(t,y) = \sum v_{(j)}(y)\frac{t^j}{j!},\ \hat{b}(t,y) = \sum b_{(j)}(y)\frac{t^j}{j!},\ \hat{Q}(t,y) = \sum Q_{(j)}(y)\frac{t^j}{j!},$$

satisfying $Q_{(j)}|_\Gamma = 0$ for $j = 0, 1, \cdots, m$. Since the solutions are in H_*^8, the compatibility conditions have to be expressed in a weak form

$$Q_{(j)}(y) \in H_0^1(\Omega), \;\; 0 \le j \le m. \tag{5.3.196}$$

From $(v_0, b_0, Q_0) \in H_*^8(\Omega)$ and the system (2.4.1), one can only get $(v_{(j)}, b_{(j)}, Q_{(j)}) \in H_*^{8-2j}(\Omega)$ for $0 \le j \le 4$. To guarantee $(v_{(j)}, b_{(j)}, Q_{(j)}) \in H_*^{8-j}(\Omega)$ and $Q_{(j)} \in H_0^1(\Omega)$, the initial data should be constructed in $H^8(\Omega)$ with

$$\sum_{j=1}^{8} \|(v_{(j)}, b_{(j)}, Q_{(j)})\|_{8-j} \lesssim P(\|v_0\|_8, \|b_0\|_8, \|Q_0\|_8),$$

instead of in $H_*^8(\Omega)$. See [74, Lemma 4.1] for the proof.

On the one hand, by Lemma 3.2.7 we know $(v_{(j)}, b_{(j)}, Q_{(j)}) \in H^{8-j}(\Omega) \hookrightarrow H_*^{8-j}(\Omega)$ which satisfies our requirement and implies $\mathfrak{E}(0) \lesssim P(\|v_0\|_8, \|b_0\|_8, \|Q_0\|_8)$. On the other hand, if we directly construct the initial data $(v_0, b_0, q_0) \in H_*^8(\Omega)$ such that $(v_{(j)}, b_{(j)}, Q_{(j)}) \in H_*^{8-j}(\Omega)$, then it is not clear in which sense the boundary conditions and the compatibility conditions are satisfied. For example, $Q_{(7)} \in H_*^1(\Omega)$, but the trace of such function in that space has no meaning in general. This also explains why we require $Q_{(7)} \in H_0^1(\Omega)$ in (5.3.196). Therefore, the initial data (v_0, b_0, Q_0) has to be constructed in the standard Sobolev space $H^8(\Omega)$.

Chapter 6

Open Problems

The last chapter records some open problems in the study of free-interface problems in MHD. Recall in Chapter 1.1 that such free-boundary problems arise from the current-vortex sheets and the plasma-vacuum models. In the case of compressible ideal MHD, all the previous results about the local existence [9, 74, 75, 70, 71, 64] rely on the Nash-Moser iteration method which yields a big loss of regularity from the initial data to the solution. It is natural to ask if one can prove the local existence such that there is no loss of regularity for the higher order energy. Our paper [50] is the first breakthrough in this direction. Specifically, the following problems are unsolved

Problem 6.0.1.

1. Use the energy functional in [50] to prove the local existence.
2. Extend to current-vortex sheets and plasma-vacuum model with Syrovatskij condition and prove the incompressible limit. (Magnetic shear suppresses the Kelvin-Helmholtz instability.)
3. Extend to current-vortex sheets and plasma-vacuum model with surface tension but drop the Syrovatskij condition and prove the incompressible limit. (Surface tension eliminates the Kelvin-Helmholtz instability.)

Problem 6.0.1 is never a simple generalization of [50]. For problem 1, one may need to find a new way to define the approximate system *other than* the tangential smoothing method. From the proof of Theorem 2.3.1 in Chapter 5.2, we find the “corrector" ψ in the approximate system is necessary in order for the uniform-in-κ estimates. In fact, this cannot be avoided because it is not likely to have higher regularity of η than v and thus the analogous proofs for Euler equation (cf. [16, 17, 15]) are no longer valid. Due to the structure of ψ in (5.2.2), it is not likely to close the uniform-in-κ estimate in the control of less than 2 tangential derivatives (the contribution of ψ on the boundary is $\overline{\triangle}\psi$).

For problem 2∼3, an extra difficulty arises from the nonvanishing boundary condition for the total pressure Q. Recall in (5.3.62) that we use $Q|_{\Gamma} = 0$ to produce a weight function in the derivation of “modified" Alinhac good unknown, which seems unavoidable if one use Lagrangian coordinates. To overcome this difficulty, one might alternatively assume the free interface as a graph of function ψ. Under this setting, the extension of ψ to the interior, namely φ, still has full Sobolev regularity even if all the unknowns only have anisotropic Sobolev regularity. However, it is still unknown if one can derive the a prori estimates due to the extremely complicated computation on the free interface. Even so, it is still highly non-trivial to prove the local existence with this energy. Indeed, in the iteration scheme, the boundary energy of ψ (or its smoothed version) cannot be derived for the frozen-coefficient linearized system. One may alternatively apply the idea of Wu [79] to control the evolution of the free interface. So far, all the aforementioned results only proved the tame estimates for the linearized system, which is far from what we desire. Further, the study of the incompressible limit of compressible vortex sheets is completely open, while the incompressible counterpart is related to the “suppression effects” (contributed by magnetic fields, elasticity, surface tension, etc) on the Kelvin-Helmholtz instability.

Next, concerning the plasma-vacuum model, we raise the following question

Problem 6.0.2. What should be the necessary and sufficient “stablization condition" on the interface for the local existence?

The Syrovatskij condition $|B^+ \times B^-| \geq c_0 > 0$ which comes from the study of current-vortex sheets is shown to be sufficient (cf. [64, 67]), but it is still unknown if it is necessary. On the other hand, Gu [25, 26] proved the local existence for the axi-symmetric case under the Rayleigh-Taylor sign condition. Mathematically, if we assume the free interface as a graph of function ψ and the initial data is in H^r, then the Rayleigh-Taylor sign condition gives $H^r(\Gamma)$ regularity for ψ but the Syrovatskij condition gives $H^{r+\frac{1}{2}}(\Gamma)$ regularity because the latter one allows us the rewrite $\nabla\psi$ in terms of the linear combination of $B^{\pm}$. But so far, there is no result that shows the latter one implies the former one. See also Trakhinin [73] for detailed discussion.

Yet there are more interesting and deep problems. Recall in Chapter 1.1.1 we exclude the case of MHD shocks, for which one of the jump condition becomes $[u_n'] \neq 0$. Blokhin-Trakhinin [5] studies the stability of MHD shocks. However, it is still widely unknown about the formation mechanisms. The only related result is due to An-Chen-Yin [3] for the instantaneous fast MHD shock driven by low-regularity data, whereas the general case is completely unknown and extremely difficult due to the multiple speeds (fast and slow magnetosonic waves, entropy wave and sound wave). The study of MHD transonic shocks is also completely open. However, this phenomenon happens when the solar winds pass across the termination shock [24, Chap. 20].

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