Compressible Gravity-Capillary Water Waves with Vorticity: Local Well-Posedness, Incompressible and Zero-Surface-Tension Limits

CHENYUN LUO[*] JUNYAN ZHANG [†]

Abstract

We consider 3D compressible isentropic Euler equations describing the motion of a liquid in an unbounded initial domain with a moving boundary and a fixed flat bottom at finite depth. The liquid is under the influence of gravity and surface tension and it is not assumed to be irrotational. We prove the local well-posedness by introducing carefully-designed approximate equations which are asymptotically consistent with the a priori energy estimates. The energy estimates yield no regularity loss and are uniform in Mach number. Also, they are uniform in surface tension coefficient if the Rayleigh-Taylor sign condition holds initially. We can thus simultaneously obtain incompressible and vanishing-surface-tension limits. The method developed in this paper is a unified and robust hyperbolic approach to free-boundary problems in compressible Euler equations. It can be applied to some important complex fluid models as it relies on neither parabolic regularization nor irrotational assumption. This paper joined with our previous works [45, 46] rigorously proves the local well-posedness and the incompressible limit for a compressible gravity water wave with or without surface tension.

Keywords: Compressible water waves, Inviscid fluids, Free-boundary problem, Well-posedness, Incompressible limit.

Contents

[*]Department of Mathematics, The Chinese University of Hong Kong, Shatin, NT, Hong Kong. CL is supported in part by the Hong Kong RGC grant CUHK-24304621. Email: `cluo@math.cuhk.edu.hk`

[†]Department of Mathematics, National University of Singapore, Singapore. Email: `zhangjy@nus.edu.sg`

1 Introduction

In this paper we study the motion of water waves in $\mathbb{R}^3$ described by the compressible Euler equations:

$$\begin{cases} \rho(\partial_t + u\cdot\nabla)u = -\nabla p - \rho g e_3, & \text{in } \mathcal{D} \\ \partial_t \rho + \nabla\cdot(\rho u) = 0 & \text{in } \mathcal{D} \\ p = p(\rho) & \text{in } \mathcal{D} \end{cases} \tag{1.1}$$

where $\mathcal{D} = \bigcup_{0\le t\le T} \{t\}\times\mathcal{D}_t$ with $\mathcal{D}_t := \{(x_1,x_2,x_3)\in\mathbb{R}^3 : -b < x_3 < \psi(t,x_1,x_2)\}$ with $b > 10$ a given constant representing the unbounded domain with finite depth occupied by the fluid at each fixed time t, whose boundary $\partial\mathcal{D}_t$ is determined by a moving surface represented via the graph $\Sigma_t := \{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_3 = \psi(t,x_1,x_2)\}$ and a flat bottom $\Sigma_b := \{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_3 = -b\}$. In the first two equations of (1.1), u, ρ, p represent respectively the fluid's velocity, density, and pressure. Also, we assume that the fluid is under influence of the gravity $\rho g e_3$, with $g > 0$ and $e_3 = (0,0,1)^\top$. The third equation of (1.1) is known to be the equation of states which satisfies

$$p'(\rho) > 0, \quad \text{for } \rho \ge \bar\rho_0, \tag{1.2}$$

where $\bar\rho_0 := \rho|_{\partial\mathcal{D}}$ is a positive constant (we set $\bar\rho_0 = 1$ for simplicity), which is in the case of an isentropic *liquid*[1]. The equation of states is required to close the system of compressible Euler equations. Another type of boundary condition is $\rho|_{\partial\mathcal{D}} = 0$ which is in the case of a *gas* and is not discussed in this paper.

The initial and boundary conditions of the system (1.1) are

$$\mathcal{D}_0 = \{x : (0,x)\in\mathcal{D}\}, \quad \text{and } u = u_0, \rho = \rho_0 \quad \text{on } \{t=0\}\times\mathcal{D}_0, \tag{1.3}$$

$$D_t|_{\partial\mathcal{D}} \in T(\partial\mathcal{D}), \quad v_3|_{\Sigma_b} = 0, \quad p|_{\Sigma_t} = \sigma\mathcal{H}, \tag{1.4}$$

where $T(\partial\mathcal{D})$ stands for the tangent bundle of $\partial\mathcal{D}$. The first condition in (1.4) is the kinematic boundary condition, which indicates that the free surface boundary moves with the normal component of the velocity (see (1.16) for an explicit illustration). The second condition is the slip condition imposed on the flat bottom Σ_b. The last condition in (1.4) shows that the pressure is balanced by surface tension on the moving surface Σ_t. Here, $\sigma > 0$ is called the surface tension constant, and $\mathcal{H}$ denotes the mean curvature of the free boundary of the fluid domain. Note that $\mathcal{H}$, $T(\partial\mathcal{D})$ and p are functions of the unknowns u, ρ and $\mathcal{D}$. So these quantities are not known a priori, and hence have to be determined alongside a solution to the problem. Let $D_t := \partial_t + u\cdot\nabla$ be the material derivative. The equations modeling the motion of compressible gravity-capillary water waves read

$$\begin{cases} \rho D_t u = -\nabla p - \rho g e_3, & \text{in } \mathcal{D}, \\ \partial_t\rho + \nabla\cdot(\rho u) = 0, & \text{in } \mathcal{D}, \\ p = p(\rho), & \text{in } \mathcal{D}, \\ (u,\rho,\mathcal{D})|_{t=0} = (u_0,\rho_0,\mathcal{D}_0), \end{cases} \tag{1.5}$$

equipped with the boundary conditions

$$\begin{cases} p = \sigma\mathcal{H} & \text{on } \bigcup_{0\le t\le T}\{t\}\times\Sigma_t, \\ v_3 = 0 & \text{on } [0,T]\times\Sigma_b, \\ D_t|_{\partial\mathcal{D}} \in T(\partial\mathcal{D}). \end{cases} \tag{1.6}$$

System (1.5) together with (1.6) admits a conserved quantity

$$\mathsf{E}_0(t) := \frac{1}{2}\int_{\mathcal{D}_t} \rho|u|^2\,\mathrm{d}x + \int_{\mathcal{D}_t}\rho\mathsf{Q}(\rho)\,\mathrm{d}x + \int_{\mathcal{D}_t}(\rho-1)gx_3\,\mathrm{d}x + \int_{\Sigma_t} g|\psi|^2 + \sigma\left(\sqrt{1+|\overline{\nabla}\psi|^2} - 1\right)\mathrm{d}x',$$

where $\mathsf{Q}(\rho) := \int_1^\rho p(r)r^{-2}\,\mathrm{d}r$ and $\mathrm{d}x' := \mathrm{d}x_1\,\mathrm{d}x_2$. A direct calculation (cf. [65, Section 6.1]) shows $\mathsf{E}_0'(t) = 0$. Note that we need a *localized* initial data such that $\mathsf{E}_0(0) < +\infty$ which can be achieved similarly as in [45, Section 7] .

[1] In general, the equation of state is $p = p(\rho, S)$ where S denotes the entropy of the fluid and satisfies $(\partial_t + u\cdot\nabla)S = 0$. It is required to have $\partial p/\partial\rho > 0$. When S is a constant, we say the fluid is isentropic. Also, the assumptions $p'(\rho) > 0$ and $\rho \ge \bar\rho_0$ ensure the hyperbolicity of (1.1).

1.1 Fixing the fluid domain

We shall convert (1.5)-(1.6) into a system of equations defined on the fixed domain

$$\Omega = \{(x_1, x_2, x_3) : -b < x_3 < 0\}.$$

One way to achieve this would be to consider the Lagrangian coordinates. Nevertheless, here, we consider a family of diffeomorphism $\Phi(t, \cdot) : \Omega \to \mathcal{D}_t$ characterized by the moving surface boundary. In particular, let

$$\Phi(t, x_1, x_2, x_3) = (t, x_1, x_2, \varphi(t, x_1, x_2, x_3)), \tag{1.7}$$

where

$$\varphi(t, x_1, x_2, x_3) = x_3 + \chi(x_3)\psi(t, x_1, x_2), \tag{1.8}$$

and $\chi \in C_c^\infty(-b, 0]$ is a smooth cut-off function satisfying the following bound for some small constant $\delta_0 > 0$.

$$\sum_{j=1}^{5} \|\chi^{(k)}\|_{L^\infty(-b,0]} \leq \frac{1}{\|\psi_0\|_\infty + 1}, \quad \chi = 1 \quad \text{on } (-\delta_0, 0], \tag{1.9}$$

We will write $x' = (x_1, x_2)$ throughout the rest of this paper. It can be seen that

$$\partial_3\varphi(t, x', x_3) = 1 + \chi'(x_3)\psi(t, x') > 0, \quad t \in [0, T],$$

for some small $T > 0$, which ensures that $\Phi(t)$ is a diffeomorphism.

Let $x = (x', x_3) \in \Omega$. We denote respectively by

$$v(t, x) = u(t, \Phi(t, x)), \quad \rho(t, x) = \mathrm{\rho}(t, \Phi(t, x)), \quad q(t, x) = p(t, \Phi(t, x)) \tag{1.10}$$

the velocity, density, and pressure defined on the fixed domain Ω. Also, we introduce the differential operators

$$\partial_t^\varphi = \partial_t - \frac{\partial_t\varphi}{\partial_3\varphi}\partial_3, \tag{1.11}$$

$$\nabla_a^\varphi = \partial_a^\varphi = \partial_a - \frac{\partial_a\varphi}{\partial_3\varphi}\partial_3, \quad a = 1, 2, \tag{1.12}$$

$$\nabla_3^\varphi = \partial_3^\varphi = \frac{1}{\partial_3\varphi}\partial_3, \tag{1.13}$$

and thus there hold

$$\partial_\alpha u \circ \Phi = \partial_\alpha^\varphi v, \quad \partial_\alpha \mathrm{\rho} \circ \Phi = \partial_\alpha^\varphi \rho, \quad \partial_\alpha p \circ \Phi = \partial_\alpha^\varphi q, \quad \alpha = t, 1, 2, 3. \tag{1.14}$$

Moreover, setting

$$\overline{\nabla} = \overline{\partial} := (\partial_1, \partial_2),$$

the boundary condition (1.6) is turned into

$$q = -\sigma\overline{\nabla} \cdot \left(\frac{\overline{\nabla}\psi}{\sqrt{1 + |\overline{\nabla}\psi|^2}} \right), \quad \text{on } [0, T] \times \Sigma, \tag{1.15}$$

$$\partial_t\psi = v \cdot N, \quad N = (-\partial_1\psi, -\partial_2\psi, 1)^\top, \quad \text{on } [0, T] \times \Sigma, \tag{1.16}$$

$$v_3 = 0, \quad \text{on } [0, T] \times \Sigma_b, \tag{1.17}$$

respectively, where $\Sigma = \{x_3 = 0\}$ and $\Sigma_b = \{x_3 = -b\}$. Let $D_t^\varphi := \partial_t^\varphi + v \cdot \nabla^\varphi$. Then the system (1.5) and (1.6) are converted into

$$\begin{cases} \rho D_t^\varphi v + \nabla^\varphi q = -\rho g e_3 & \text{in } [0, T] \times \Omega, \\ \partial_t^\varphi \rho + \nabla^\varphi \cdot (\rho v) = 0 & \text{in } [0, T] \times \Omega, \\ q = q(\rho) & \text{in } [0, T] \times \Omega, \\ q = -\sigma\overline{\nabla} \cdot \left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}} \right) & \text{on } [0, T] \times \Sigma, \\ \partial_t\psi = v \cdot N & \text{on } [0, T] \times \Sigma, \\ v_3 = 0 & \text{on } [0, T] \times \Sigma_b, \\ (v, \rho, \psi)|_{t=0} = (v_0, \rho_0, \psi_0) = (u_0, \mathrm{\rho}_0, \psi_0). \end{cases} \tag{1.18}$$

The second equation of (1.18), i.e., the continuity equation, can be re-expressed as

$$D_t^\varphi \rho + \rho \nabla^\varphi \cdot v = 0. \tag{1.19}$$

Let $\mathcal{F} := \log \rho$. Since $q'(\rho) > 0$ indicates $\mathcal{F}'(q) > 0$, then (1.19) is equivalent to

$$\mathcal{F}'(q) D_t^\varphi q + \nabla^\varphi \cdot v = 0. \tag{1.20}$$

Also, by invoking (1.11)-(1.13), we can alternatively write the material derivative D_t^φ as

$$D_t^\varphi = \partial_t + \overline{v} \cdot \overline{\nabla} + \frac{1}{\partial_3 \varphi}(v \cdot \mathbf{N} - \partial_t \varphi)\partial_3, \tag{1.21}$$

where $\overline{v} \cdot \overline{\nabla} = v_1 \partial_1 + v_2 \partial_2$, and $\mathbf{N} := (-\partial_1 \varphi, -\partial_2 \varphi, 1)$. This formulation provides a good motivation to define the smoothed material derivative in Section 3 and the linearized material derivative in Section 5.

1.2 The new formulation with modified pressure

Since the gravity term $\rho g e_3 \notin L^2(\Omega)$, we then use $\partial_i^\varphi \varphi = \delta_{i3}$ to rewrite the momentum equation as

$$\rho D_t^\varphi v + \nabla^\varphi \check{q} = -(\rho - 1) g e_3,$$

where

$$\check{q} := q + g\varphi, \tag{1.22}$$

is the "modified" pressure balanced by gravity. Under this setting, the fluid pressure gradient $\nabla^\varphi \check{q}$ becomes an $L^2(\Omega)$ function and the source term becomes $(\rho - 1) g e_3$ which is also in $L^2(\Omega)$ if we assume the initial data $\rho_0 - 1 \in L^2(\Omega)$. We then directly calculate that $D_t^\varphi \varphi = v_3$, so the continuity equation (1.20) now becomes

$$\mathcal{F}'(q) D_t^\varphi \check{q} + \nabla^\varphi \cdot v = \mathcal{F}'(q) g D_t^\varphi \varphi = \mathcal{F}'(q) g v_3, \tag{1.23}$$

and thus the compressible gravity-capillary water wave system is now reformulated as follows

$$\begin{cases} \rho D_t^\varphi v + \nabla^\varphi \check{q} = -(\rho - 1) g e_3 & \text{in } [0,T] \times \Omega, \\ \mathcal{F}'(q) D_t^\varphi \check{q} + \nabla^\varphi \cdot v = \mathcal{F}'(q) g v_3 & \text{in } [0,T] \times \Omega, \\ q = q(\rho), \check{q} = q + g\varphi & \text{in } [0,T] \times \Omega, \\ \check{q} = g\psi - \sigma \overline{\nabla} \cdot \left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}} \right) & \text{on } [0,T] \times \Sigma, \\ \partial_t \psi = v \cdot N & \text{on } [0,T] \times \Sigma, \\ v_3 = 0 & \text{on } [0,T] \times \Sigma_b, \\ (v, \rho, \psi)|_{t=0} = (v_0, \rho_0, \psi_0). \end{cases} \tag{1.24}$$

1.3 The equation of states and sound speed

Part of this paper is devoted to studying the behavior of the solution of (1.24) as either the sound speed goes to infinity or the surface tension σ coefficient goes to 0. The former is known to be the incompressible limit, and the latter is known to be the zero surface tension limit. Mathematically, it is convenient to view the sound speed $c_s := \sqrt{q'(\rho)}$ as a family of parameters. As in [16, 17, 18, 43, 45], we consider a family $\{q_{\lambda'}(\rho)\}$ parametrized by $\lambda' \in (0, \infty)$, where

$$(\lambda')^2 := q'_{\lambda'}(\rho)|_{\rho=1}. \tag{1.25}$$

Here and in the sequel, we slightly abuse the terminology and call λ' the sound speed. A typical choice of the equation of states $q_{\lambda'}(\rho)$ would be the Tait type equation

$$q_{\lambda'}(\rho) = \gamma^{-1}(\lambda')^2(\rho^\gamma - 1), \quad \gamma \geq 1. \tag{1.26}$$

When viewing the density as a function of the pressure, this indicates

$$\rho_{\lambda'}(q) = \left(\frac{\gamma}{(\lambda')^2}q + 1\right)^{\frac{1}{\gamma}}, \quad \text{and } \log(\rho_{\lambda'}(q)) = \gamma^{-1}\log\left(\frac{\gamma}{(\lambda')^2}q + 1\right). \tag{1.27}$$

Hence, we can view $\mathcal{F}(q)$ as a parametrized family $\{\mathcal{F}_\lambda(q)\}$ as well, where $\lambda = \frac{1}{\lambda'}$. Indeed, we have

$$\mathcal{F}_\lambda(q) = \gamma^{-1}\log(\lambda^2\gamma q + 1). \tag{1.28}$$

We again slightly abuse the terminology and call λ the Mach number[2]. Furthermore, there exists $C > 0$ such that

$$C^{-1}\lambda^2 \le \mathcal{F}_\lambda'(q) \le C\lambda^2. \tag{1.29}$$

Also, we assume

$$|\mathcal{F}_\lambda^{(s)}(q)| \le C, \quad |\mathcal{F}_\lambda^{(s)}(q)| \le C|\mathcal{F}_\lambda'(q)|^s \le C\mathcal{F}_\lambda'(q) \tag{1.30}$$

holds for $0 \le s \le 4$.

Remark (**Issue with the infinite depth case**). Our proof in this paper also works for the case of infinite depth *provided that the equation of state for compressible gravity water wave system is reasonable in physics when* Ω *is the lower half space*. Indeed, if one assume $q = (\rho - 1)\lambda^{-2}$ for instance, then $\mathcal{F}_\lambda' = O(\lambda^2)$ and one can prove $\lambda\check{q} \in L^2(\Omega)$ in the L^2 estimates. Plugging the equation of state yields $\lambda^{-1}(\rho - 1) + \lambda g x_3 \in L^2(\Omega)$ and thus one may have to let $\lambda g x_3 \in L^2(\Omega)$, which requires some fast decay for λ near infinite depth, for example $\lambda = O(|x|^{-\frac{5}{2}-\delta})$ for some $\delta > 0$ (which also implies $(\rho - 1)\langle x\rangle^{2.5+\delta} \in L^2(\Omega)$). However, it is still unknown whether there is a physical equation of state such that the Mach number λ is related to the depth and has such fast decay toward infinite depth. Hence, the appearance of equation of state tells a crucial difference from the incompressible gravity water wave model, for which $\check{q}$ is a Lagrangian multiplier not related to the density.

1.4 An overview of previous results

The study of free-surface inviscid fluids has blossomed over the past two decades or so. Most of the previous studies focused on incompressible fluid models, i.e., the fluid velocity satisfies $\operatorname{div} u = 0$ and thus the density ρ is equal to a constant. In this case, the fluid pressure p is not determined by the equation of states but appears as a Lagrangian multiplier enforcing the divergence-free constraint. For the local well-posedness (LWP) for the free-boundary incompressible Euler equations, the first breakthrough came in Wu [66, 67] for the irrotational case and Christodoulou-Lindblad [10] and Lindblad [39, 42] for the case of nonzero vorticity. See also [52, 70, 29, 5, 37, 51] for the irrotational case and [13, 72, 44, 55, 56, 57, 3, 2, 63] for the case of nonzero vorticity. When the fluid velocity is irrotational (the vorticity $\operatorname{curl} u_0 = \mathbf{0}$, a condition that is preserved by the evolution), the problem is called the (incompressible and irrotational) water wave problem which has attracted great attention for the long time existence. Previous works mostly focused on the case of an unbounded domain diffeomorphic to lower half-space or $\mathbb{R}^{d-1} \times (-b, 0)$ and we refer to Wu [68, 69] for the first breakthrough and numerous related works [19, 20, 4, 30, 15, 23, 22, 24, 25, 64, 73][3]. See also [8] for the bounded domain case and [26, 58] for some special cases when the vorticity is nonzero.

The development for the free-boundary compressible Euler equations is much less, especially for the case of a liquid as opposed to a gas ($\rho|_\Sigma = 0$) in a physical vacuum. For the gas model, we refer to [32, 12, 14, 47, 33, 27] and references therein. For the liquid model, most previous works focus on the case of a bounded domain. We refer to Lindblad [40, 41] for the first result and related works [11, 43, 17, 21] for LWP or a priori estimates.

When the fluid domain is unbounded, that is, the compressible gravity water wave problem, the existing literature neglected the effect of surface tension. Trakhinin [59] first proved the LWP for the non-isentropic case by using Nash-Moser iteration which leads to a loss of regularity from initial data to solution. The first author proved the a priori estimates without loss of regularity and the incompressible limit for the isentropic case in [45], but it is still difficult to use the energy constructed in [45] to prove the local existence. Later, the authors [46] proved the LWP without using Nash-Moser, but the energy functional in [46] is not uniform in Mach number, and thus we cannot derive the incompressible limit (see the next paragraph) while constructing the solution. The second author refined and simplified the techniques in [59, 46] such that the LWP and the incompressible

[2]The Mach number is defined to be $M = u/c_s$. In the paper, the velocity is always of size $O(1)$ (in $L^2(\Omega)$) and thus $M = O(\lambda)$.

[3]It is well-known that one can reduce the incompressible Euler equations to a system of equations on the moving boundary when the velocity is irrotational. This method cannot be adapted to the study of compressible water waves with vorticity.

limit can be simultaneously proved in the study of compressible elastodynamics [71] which can be directly applied to Euler equations. However, the methods in these works do not apply to the case with nonzero surface tension.

Another topic in this paper concerns the incompressible limit of free-boundary compressible Euler equations. When the free-surface motion is neglected (that is, Euler equations in a fixed domain), there have been a lot of studies in this direction and we refer to [34, 35, 18, 54, 62, 7, 31, 28, 50, 1, 16]. However, much less is known about the incompressible limit of free-surface inviscid fluids. The first result was due to Lindblad and the first author [43] for the case of a bounded domain and zero surface tension. See also the first author's work [45] for compressible gravity water wave, the second author's work [71] for a simpler proof that works for both bounded and unbounded domain, and Disconzi and the first author [17] for the case $\sigma > 0$ in a bounded fluid domain.

In a nutshell, we develop a new method to prove the local-in-time solution of the motion of compressible gravity-capillary water waves with nonzero vorticity in this paper. The new method is expected to be:

- Unified: The LWP, the incompressible limit, and the zero surface tension limit can be simultaneously justified.
- Simple: A hyperbolic approach that does not lead to derivative loss or depends on the boundedness of the fluid domain. That is, we want to avoid using Nash-Moser iteration as in [59] and parabolic regularization as in [11]. In particular, the Galerkin method is needed when using parabolic regularization, but this is difficult to proceed in the case of an unbounded domain, as the spectrum of Laplacian is no longer discrete.
- Robust: The proof should not rely on the so-called irrotational assumption[4]. This is necessary to apply our method to free-boundary problems in complex fluids, such as magnetohydrodynamics (MHD), elastodynamics, and so on, for which the irrotational assumption no longer holds due to the strong coupling between the fluid motion and other physical quantities.

1.5 The main theorems

The first theorem concerns the local well-posedness for the motion of compressible gravity-capillary water waves modeled by (1.24), provided that the initial data satisfies certain compatibility conditions. Particularly, we say the data (ψ_0, v_0, q_0), where $q_0 = q(\rho_0)$, satisfies the zeroth compatibility condition if

$$q_0 = \sigma\mathcal{H}_0, \quad \text{on } \Sigma \tag{1.31}$$

holds. Moreover, the initial data satisfies the k-th ($k \geq 0$) compatibility condition if

$$(D_t^\varphi)^k q|_{t=0} = (D_t^\varphi)^k(\sigma\mathcal{H})|_{t=0}, \quad \text{on } \Sigma \tag{1.32}$$

holds.

Theorem 1.1 (**Local well-posedness**). Let $\sigma > 0$ be fixed. Let $(\psi_0, v_0, \rho_0 - 1) \in H^5(\Sigma) \times H^4(\Omega) \times H^4(\Omega)$ be the initial data of (1.24) that verifies the compatibility conditions (1.32) up to 3-rd order and $v_0^3|_{\Sigma_b} = 0$. Then there exists $T > 0$ depending only on the initial data, such that (1.24) admits a unique solution $(\psi(t), v(t), \rho(t))$ verifies the energy estimate expressed in terms of pressure

$$\sup_{0\leq t\leq T} E(t) \leq_{\sigma^{-1}} P(E(0)), \tag{1.33}$$

where $P(\cdots)$ is a generic polynomial in its arguments, and the energy $E(t)$ is defined to be

$$\begin{aligned} E(t) := \sum_{k=0}^{4} \left(\|\partial_t^k v(t)\|_{4-k}^2 + |\sqrt{\sigma}\overline{\nabla}\partial_t^k\psi(t)|_{4-k}^2\right) + \|\partial\check{q}(t)\|_3^2 + \sum_{k=1}^{3}\|\partial_t^k\check{q}(t)\|_{4-k}^2 + \|\sqrt{\mathcal{F}'(q)}\partial_t^4\check{q}(t)\|_0^2 \\ + \|\sqrt{\mathcal{F}'(q)}\check{q}(t)\|_0^2 + \|\rho(t) - 1\|_0^2 + g|\psi|_0^2. \end{aligned} \tag{1.34}$$

Here, $\|\cdot\|_s$ and $|\cdot|_s$ represents the interior Sobolev norm $\|\cdot\|_{H^s(\Omega)}$ and the boundary Sobolev norm $|\cdot|_{H^s(\Sigma)}$ on the fixed top Σ respectively. Also there exists a constant C, depending on ψ_0, v_0 and $\check{q}_0$, such that $E(0) \leq C$.

Remark. In Appendix B, we show that we can construct smooth initial data $(\psi_0, v_0, \check{q}_0)$ that satisfies the compatibility conditions up to order 3. These compatibility conditions are required so that we can show $E(0) \leq C$ by adapting the arguments in [17, Section 4.3].

[4]For Euler equations, if the initial vorticity is zero, then the vorticity is always zero. Based on this property, one can enhance the regularity of the flow map of fluid velocity to 1/2-order higher and thus the regularity of the vorticity is only 1/2-order lower than the velocity provided this holds for the initial vorticity. One can refer to [36] for the proof.

Remark. The second line in (1.34) is the L^2 part of the energy. Note that we do not have the control for $\|q(t)\|_0$ without the weight of the Mach number. That is why we write $\|\partial\check{q}\|_3$ instead of $\|\check{q}\|_4$ in the first line.

The next main theorem concerns the incompressible and zero-surface-tension double limits. We consider the Euler equations modeling the motion of incompressible gravity water waves satisfied by (ξ, w, q_{in}) with localized initial data (w_0, ξ_0) and $w_0^3|_{\Sigma_b} = 0$:

$$\begin{cases} D_t^{\varphi} w + \nabla^{\varphi} \mathfrak{p} = 0 & \text{in } [0,T]\times\Omega, \\ \nabla^{\varphi}\cdot w = 0 & \text{in } [0,T]\times\Omega, \\ \mathfrak{p} = q_{in} + g\varphi & \text{in } [0,T]\times\Omega, \\ \mathfrak{p} = g\xi & \text{on } [0,T]\times\Sigma, \\ \partial_t\xi = w\cdot\mathcal{N} & \text{on } [0,T]\times\Sigma, \\ w_3 = 0 & \text{on } [0,T]\times\Sigma_b, \\ (w,\xi)|_{t=0} = (w_0,\xi_0), \end{cases} \tag{1.35}$$

where we slightly abuse the notation by still setting $\varphi(t,x) = x_3 + \chi(x_3)\xi(t,x')$ to be the extension of ξ in Ω. Denote $(\psi^{\lambda,\sigma}, v^{\lambda,\sigma}, \rho^{\lambda,\sigma})$ to be the solution of (1.24) indexed by σ and λ, we prove that $(\psi^{\lambda,\sigma}, v^{\lambda,\sigma}, \rho^{\lambda,\sigma})$ converges to $(\xi, w, 1)$ as $\lambda,\sigma\to 0$ provided the convergence of initial datum. Note that the convergence of compressible initial datum already implies they are also localized datum.

Theorem 1.2 (**Incompressible and zero-surface-tension limits**)**.** Let $(\psi_0^{\lambda,\sigma}, v_0^{\lambda,\sigma}, \rho_0^{\lambda,\sigma}-1)$ be the initial data of (1.24) for each fixed $(\lambda,\sigma)\in\mathbb{R}^+\times\mathbb{R}^+$, verifying

a. The sequence of initial data $(\psi_0^{\lambda,\sigma}, v_0^{\lambda,\sigma}, \rho_0^{\lambda,\sigma}-1)\in H^5(\Sigma)\times H^4(\Omega)\times H^4(\Omega)$ satisfies (1.32) for $0\le k\le 3$.
b. $(\psi_0^{\lambda,\sigma}, v_0^{\lambda,\sigma}, \rho_0^{\lambda,\sigma}-1)\to(\xi_0, w_0, 0)$ in $C^2(\Sigma)\times C^2(\Omega)\times C^1(\Omega)$ as $\lambda,\sigma\to 0$.
c. *Both* incompressible and compressible pressures q and q_{in} satisfy the Rayleigh-Taylor sign condition

$$-\partial_3 q \ge c_0 > 0, \quad \text{on } \{t=0\}\times\Sigma, \tag{1.36}$$
$$-\partial_3 q_{in} \ge c_0 > 0, \quad \text{on } \{t=0\}\times\Sigma, \tag{1.37}$$

for some $c_0 > 0$.

Then it holds that

$$(\psi^{\lambda,\sigma}, v^{\lambda,\sigma}, \rho^{\lambda,\sigma}-1)\to(\xi, w, 0), \quad \text{in } C^0([0,T], C^2(\Sigma)\times C^2(\Omega)\times C^1(\Omega)),$$

after possibly passing to a subsequence.

Theorem 1.2 is a direct consequence of uniform-in-λ,σ estimates for the compressible gravity-capillary water wave system (1.24) and the compactness argument. Indeed, the energy estimate (1.33) established in Theorem 1.1 is already uniform in Mach number λ. The energy estimate (1.33) for $E(t)$ is also independent of the surface tension coefficient σ provided that the Rayleigh-Taylor sign condition (1.36) holds initially.

Remark. Although our energy functional $E(t)$ is expressed in terms of $\check{q}$, the incompressible limit process is only for $(\psi^{\lambda,\sigma}, v^{\lambda,\sigma}, \rho^{\lambda,\sigma})$ that converges to $(\zeta, w, 1)$. The compressible pressure q never converges to the incompressible pressure q_{in}, because the former one is the solution to a quasilinear symmetric hyperbolic system but the latter one appears as a Lagrangian multiplier. Indeed, as was indicated by [43, 45, 71], it is the enthalpy $h(\rho) := \int_1^{\rho} q'(r)/r\, dr$ of the compressible equations that converges to the incompressible pressure q_{in}. On the other hand, the convergence of $\|\rho^{\lambda,\sigma}-1\|_3$ can be easily proved if we write the continuity equation to be $D_t^{\varphi}(\rho-1) = -\rho(\nabla^{\varphi}\cdot v)$ and use Grönwall's inequality for its H^3 estimates.

Note that the energy estimate (1.33) requires $E(0) < +\infty$ and thus requires $\partial_t^3\check{q}(0) = O(1)$ and $\partial_t^4\check{q}(0) = O(\lambda^{-1})$. However, the propagation of the Rayleigh-Taylor sign condition only requires the boundedness of $\partial_t\partial_3 q$, not including higher-order time derivatives. We can also achieve this by adjusting the weight of the Mach number in the energy functional.

Theorem 1.3 (**Improved uniform estimates in** λ,σ)**.** Under the hypothesis of Theorem 1.1, if we further assume the Rayleigh-Taylor sign condition (1.36) holds for the initial data of (1.24), then we can establish a local-in-time estimate that is uniform in both Mach number λ and the surface tension coefficient σ for the following energy functional

$$
\begin{aligned}
E^{\lambda,\sigma}(t) = & \sum_{k=0}^{1} \|\partial_t^k v^{\lambda,\sigma}(t)\|_{4-k}^2 + \|\partial^{1-k}\partial_t^k \check{q}^{\lambda,\sigma}(t)\|_3^2 && \text{(Non-weighted interior norms)} \\
& + \sum_{k=0}^{1} |\sqrt{\sigma}\overline{\nabla}\partial_t^k \psi^{\lambda,\sigma}(t)|_{4-k}^2 + |\partial_t^k \psi^{\lambda,\sigma}(t)|_{4-k}^2 && \text{(Non-weighted boundary norms)} \\
& + \|(\mathcal{F}_\lambda')^{\frac{1}{2}} \check{q}^{\lambda,\sigma}(t)\|_0^2 + g|\psi^{\lambda,\sigma}(t)|_0^2 && (L^2 \text{ norms}) \\
& + \sum_{s=0}^{2} \|(\mathcal{F}_\lambda')^{\frac{s}{2}} \partial_t^{2+s} v^{\lambda,\sigma}(t)\|_{2-s}^2 + \|(\mathcal{F}_\lambda')^{\frac{s+1}{2}} \partial_t^{2+s} \check{q}^{\lambda,\sigma}(t)\|_{2-s}^2 && \text{(weighted interior norms)} \\
& + \sum_{s=0}^{2} |\sqrt{\sigma}(\mathcal{F}_\lambda')^{\frac{s}{2}} \overline{\nabla}\partial_t^{2+s} \psi^{\lambda,\sigma}(t)|_{2-s}^2 + |(\mathcal{F}_\lambda')^{\frac{s}{2}} \partial_t^{2+s} \psi^{\lambda,\sigma}(t)|_{2-s}^2 && \text{(weighted boundary norms).}
\end{aligned}
\tag{1.38}
$$

Here $(v^{\lambda,\sigma}, \check{q}^{\lambda,\sigma}, \psi^{\lambda,\sigma})$ is the solution to (1.24) with Mach number equal to λ and the weight $\mathcal{F}_\lambda' = O(\lambda^2)$.

Remark. The above estimate only requires $\partial_t q(0)$ to be bounded but not for $\partial_t^2 q(0) = O(\lambda^{-1})$. In this case, the continuity equation implies the initial divergence $\nabla^\varphi \cdot v_0 = O(\lambda^2)$, that is, the compressible data v_0 is a small perturbation of the incompressible data w_0, and this perturbation is *completely contributed by the compressibility*. Such compressible data are usually called "well-prepared initial data"[5]. On the other hand, the propagation of the Rayleigh-Taylor sign condition requires the boundedness of $\partial_t q$, so we have reached the minimal requirement for the initial data being "well-prepared".

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List of Notations

- (Fixed domain and its boundary) $\Omega := \{x \in \mathbb{R}^3 | -b < x_3 < 0\}$. $x = (x_1, x_2, x_3)$, and $x' = (x_1, x_2)$. $\Sigma := \{x \in \mathbb{R}^3 | x_3 = 0\}$, $\Sigma_b := \{x \in \mathbb{R}^3 | x_3 = -b\}$.
- (Tangential derivatives) $\mathcal{T}_0 = \partial_t$, $\mathcal{T}_1 = \overline{\partial}_1$, $\mathcal{T}_2 = \overline{\partial}_2$, $\mathcal{T}_3 = \omega(x_3)\partial_3$, where $\omega(x_3) \in C^\infty(-b, 0)$ is assumed to be bounded, comparable to $|x_3|$ in $[-2, 0]$ and vanishing on $\Sigma \cup \Sigma_b$.
- (L^∞-norm) $\|\cdot\|_\infty := \|\cdot\|_{L^\infty(\Omega)}$.
- (Sobolev norms) $\|\cdot\|_s := \|\cdot\|_{H^s(\Omega)}$, and $|\cdot|_s := \|\cdot\|_{H^s(\Sigma)}$.
- (Polynomials) $\mathcal{P}_0 := P(E(0))$, $\mathcal{P}_0^\kappa := P(E^\kappa(0))$. $P(\cdots)$ denotes a generic polynomial in its arguments.
- (Commutators) $[T, f]g = T(fg) - f(Tg)$, $[T, f, g] := T(fg) - T(f)g - fT(g)$ where T is a differential operator and f, g are functions.
- (Equality modulo lower order terms) $A \overset{L}{=} B$ means $A = B$ modulo lower order terms.

2 An overview of our methodology

Before going to the detailed proofs, we will briefly introduce our methodology for deriving energy estimates that are uniform in both surface tension and Mach number, and the construction of solutions to the linearized and the nonlinear problem via a carefully-designed approximation scheme.

2.1 Uniform estimates in Mach number and surface tension

Let us temporarily focus on the energy estimates of the original system (1.24) instead of the construction of solutions. Indeed, the strategies on the a priori estimates will illustrate why we need the approximation scheme defined in the next subsection.

[5] One can find the definitions of "well-prepared" and "ill-prepared" in [50, 1] for rescaled Euler system, which is equivalent to the statement in our paper.

2.1.1 Div-Curl analysis and reduction of pressure

We start with the control of $\|v\|_4$. Using div-curl decomposition, $\|v\|_4$ is bounded by $\|\nabla^\varphi\times v\|_3$, $\|\nabla^\varphi\cdot v\|_3$ and $\|\overline{\partial}^4 v\|_0$, where the curl part can be directly controlled by analyzing the evolution equation of $\nabla^\varphi\times v$. The continuity equation reduces the divergence to $\|\mathcal{F}'(q)D_t^\varphi q\|_3$. By using the definition of D_t^φ, it remains to control $\|\mathcal{F}'(q)\mathcal{T}q\|_3$ for a tangential derivative $\mathcal{T}=\partial_t,\overline{\partial}$ or $\omega(x_3)\partial_3$ where $\omega(x_3)\in C^\infty(-b,0)$ is assumed to be bounded, comparable to $|x_3|$ in $[-2,0]$ and vanishing on $\Sigma\cup\Sigma_b$. On the other hand, the momentum equation indicates that $-\nabla q\sim D_t^\varphi v$. So the control of $\|\check{q}\|_4$ is then reduced to $\|D_t^\varphi v\|_3$ and then to $\|\mathcal{T}v\|_3$. At this point, we have reduced one normal derivative to one tangential derivative $\mathcal{T}$.

We can similarly apply this reduction to the time derivatives of v and $\check{q}$. Finally, we need to control the $L^2(\Omega)$ norms of $\mathcal{T}^\alpha v$ and $\mathcal{T}^\alpha\check{q}$ with $|\alpha|=4$, where the tangential derivative $\mathcal{T}$ can be $\partial_t,\overline{\partial}$ and $\omega(x_3)\partial_3$.

2.1.2 Tangential estimates: Alinhac good unknowns

Reformulation in terms of Alinhac good unknowns. Define $\mathcal{T}^\alpha$ to be $\partial_t^{\alpha_0}\overline{\partial}_1^{\alpha_1}\overline{\partial}_2^{\alpha_2}(\omega\partial_3)^{\alpha_3}$ with $|\alpha|:=\alpha_0+\alpha_1+\alpha_2+\alpha_3=4$. In $\mathcal{T}^\alpha$-tangential estimates, we need to commute $\mathcal{T}^\alpha$ with ∇_i^φ. When $i=t,1,2$, the commutator $[\mathcal{T}^\alpha,\nabla_i^\varphi]f$ includes the term $(\partial_3\varphi)^{-1}\mathcal{T}^\alpha\partial_i\varphi\partial_3 f$, where the $L^2(\Omega)$ norm of $\mathcal{T}^\alpha\partial_i\varphi$ is controlled by $|\mathcal{T}^\alpha\partial_i\psi|_0$. However, the regularity of ψ obtained in $\mathcal{T}^\alpha$-estimates is $|\sqrt{\sigma}\mathcal{T}^\alpha\overline{\nabla}\psi|_0$. Using this to control the aforementioned commutator will fail taking zero surface tension limit.

To overcome this difficulty, we introduce the Alinhac good unknown method which reveals that the "essential" leading order term in $\mathcal{T}^\alpha(\nabla^\varphi f)$ is not $\nabla^\varphi(\mathcal{T}^\alpha f)$ but the covariant derivative ∇^φ of the "Alinhac good unknown" $\mathbf{F}$. Under our setting, the good unknown $\mathbf{F}$ for f with respect to $\mathcal{T}^\alpha$ is defined by $\mathbf{F}:=\mathcal{T}^\alpha f-\mathcal{T}^\alpha\varphi\partial_3^\varphi f$ and satisfies

$$\mathcal{T}^\alpha\nabla_i^\varphi f=\nabla_i^\varphi\mathbf{F}+\mathfrak{C}_i(f),\quad \mathcal{T}^\alpha D_t^\varphi f=D_t^\varphi\mathbf{F}+\mathfrak{D}(f),\tag{2.1}$$

where $\|\mathfrak{C}_i(f)\|_0$ and $\|\mathfrak{D}(f)\|_0$ can be directly controlled. Then we reformulate the $\mathcal{T}^\alpha$-differentiated system in terms of $\mathbf{V},\mathbf{Q}$ (the Alinhac good unknowns of $v,\check{q}$) as follows

$$\rho D_t^\varphi\mathbf{V}=-\nabla^\varphi\mathbf{Q}+\mathcal{R}^1\quad\text{in }\Omega,\tag{2.2}$$

$$\mathcal{F}'(q)D_t^\varphi\mathbf{Q}+\nabla^\varphi\cdot\mathbf{V}=\mathcal{R}^2-\mathfrak{C}_i(v^i)\quad\text{in }\Omega,\tag{2.3}$$

where $\mathcal{R}^1,\mathcal{R}^2$ are commutators that can be directly controlled. The boundary conditions now become

$$\mathbf{Q}=\sigma\mathcal{T}^\alpha\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}}\right)-\partial_3 q\mathcal{T}^\alpha\psi,\quad \mathbf{V}\cdot N=\partial_t\mathcal{T}^\alpha\psi+\overline{v}\cdot\overline{\nabla}\mathcal{T}^\alpha\psi-\mathcal{S}_1\quad\text{on }\Sigma,\tag{2.4}$$

$$\mathbf{V}_3=\mathcal{T}^\alpha v_3-\mathcal{T}^\alpha\varphi\partial_3 v_3=0\quad\text{on }\Sigma_b.\tag{2.5}$$

where $\mathcal{S}_1=\partial_3 v\cdot N\mathcal{T}^\alpha\psi+\sum_{\substack{|\beta_1|+|\beta_2|=4\\|\beta_1|,|\beta_2|>0}}\mathcal{T}^{\beta_1}v\cdot\mathcal{T}^{\beta_2}N$. $\mathbf{V}_3$ vanishes on Σ_b due to $v_3=0$ and $\varphi=-b$ on Σ_b.

In other words, the reformulation in Alinhac good unknowns takes into account the covariance under the change of coordinates such that we can proceed with the tangential estimates in the same way as L^2 estimates and avoid the regularity loss or the dependence on σ^{-1}. Such remarkable observation was due to Alinhac [6] and was first applied (implicitly) to the Q-tensor energy method to study free-surface inviscid fluids by Christodoulou-Lindblad [10]. See also [48, 65] for the explicit calculations for the inviscid limit of incompressible free-boundary Navier-Stokes equations.

Energy estimates. Testing (2.2) by $\mathbf{V}$, integrating by parts and invoking (2.3), we can easily obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\left(\int_\Omega\rho|\mathbf{V}|^2+\mathcal{F}'(q)|\mathbf{Q}|^2\mathrm{d}\mathcal{V}_t\right)=\int_{\Sigma_b}\mathbf{Q}\mathbf{V}_3\,\mathrm{d}x'-\int_\Sigma\mathbf{Q}(\mathbf{V}\cdot N)\,\mathrm{d}x'-\int_\Omega\mathbf{Q}\mathfrak{C}_i(v^i)\mathrm{d}\mathcal{V}_t+\text{ controllable terms},\tag{2.6}$$

where $\mathrm{d}\mathcal{V}_t:=\partial_3\varphi\,\mathrm{d}x$. Note that the first boundary integral is zero due to $\mathbf{V}_3|_{\Sigma_b}=0$ and the second boundary integral vanishes when $\alpha_3>0$ because the weight function $\omega(x_3)$ vanishes on Σ. Hence, it suffices to analyze the space-time derivatives. We take $\mathcal{T}^\alpha=\overline{\partial}^\alpha$ for an example to analyze the seond boundary integral.

$$\begin{aligned}-\int_\Sigma\mathbf{Q}(\mathbf{V}\cdot N)\,\mathrm{d}x'=&-\int_\Sigma\overline{\partial}^\alpha(\sigma\mathcal{H})\partial_t\overline{\partial}^\alpha\psi\,\mathrm{d}x'+\int_\Sigma\partial_3 q\overline{\partial}^\alpha\psi\partial_t\overline{\partial}^\alpha\psi\,\mathrm{d}x'\\&-\int_\Sigma\overline{\partial}^\alpha(\sigma\mathcal{H})(\overline{v}\cdot\overline{\nabla})\overline{\partial}^\alpha\psi\,\mathrm{d}x'+\int_\Sigma\partial_3 q\overline{\partial}^\alpha\psi(\overline{v}\cdot\overline{\nabla})\overline{\partial}^\alpha\psi\,\mathrm{d}x'+\int_\Sigma\mathbf{Q}\mathcal{S}_1\,\mathrm{d}x'.\end{aligned}\tag{2.7}$$

Invoking the explicit formula for the mean curvature and integrating $\overline{\nabla}\cdot$ by parts in the first integral in (2.7), we get

$$\mathrm{ST} := -\int_\Sigma \overline{\partial}^\alpha(\sigma\mathcal{H})\partial_t\overline{\partial}^\alpha\psi\,\mathrm{d}x' = -\frac{\sigma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma \frac{|\overline{\partial}^\alpha\overline{\nabla}\psi|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}} - \frac{|\overline{\nabla}\psi\cdot\overline{\partial}^\alpha\overline{\nabla}\psi|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}^3}\,\mathrm{d}x' + \cdots, \tag{2.8}$$

which together with the following inequality gives the boundary energy $|\sqrt{\sigma}\overline{\partial}^\alpha\overline{\nabla}\psi|_0^2$:

$$\forall \mathbf{a}\in\mathbb{R}^2,\quad \frac{|\mathbf{a}|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}} - \frac{|\overline{\nabla}\psi\cdot\mathbf{a}|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}^3} \geq \frac{|\mathbf{a}|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}^3}. \tag{2.9}$$

For the second term in (2.7), it produces the boundary energy without σ-weight *provided that the Rayleigh-Taylor sign condition*[6] $-\partial_3 q_0|_\Sigma \geq c_0 > 0$ *holds*. However, the Rayleigh-Taylor sign condition is only assumed when taking zero surface tension limit but not in the proof of local well-posedness for each *given* $\sigma > 0$. Therefore, we have to use the $\sqrt{\sigma}$-weighted energy to control this term when proving local well-posedness. Indeed, *it is the control of* $|\mathcal{T}^\alpha\psi|_0$ *that yields the only possibility that the energy estimates depend on* σ^{-1}. The third and fourth terms in (2.7) can be directly controlled after integrating $\overline{v}\cdot\overline{\nabla}$ by parts and using the symmetry. The fifth term can be controlled by invoking (2.4) and integrating $\overline{\nabla}$ by parts.

Crucial cancellation structure for the incompressible limit. We still need to analyze the interior integral $I_0 := -\int_\Omega \mathbf{Q}\mathfrak{C}_i(v^i)\mathrm{d}\mathcal{V}_t$. We note that $\mathcal{T}^\alpha\check{q}$ appears in this term, but the energy term obtained in tangential estimates is $\|\sqrt{\mathcal{F}'(q)}\mathbf{Q}\|_0^2$. Such a term introduces a possibility of failure in taking an incompressible limit. When there is at least one spatial derivative in $\mathcal{T}^\alpha$, one can reduce $\nabla\check{q}$ to $D_t^\varphi v$ to avoid losing the weight of sound speed. But in the estimates of full-time derivatives ∂_t^4, we no longer have a such reduction for $\check{q}$. Instead, we notice there is a cancellation structure when we combine I_0 with part of the boundary term $\int_\Sigma \mathbf{Q}\mathcal{S}_1\,\mathrm{d}x'$. Specifically, the boundary integral includes the following term

$$J_0 := 4\int_\Sigma \partial_t^4\check{q}\partial_t^3 v\cdot\partial_t N\,\mathrm{d}x',$$

while I_0 contains the following terms involving $\partial_t^4\check{q}$

$$\begin{aligned} I_{00} &:= -4\int_\Omega \partial_t^4\check{q}\partial_t\mathbf{N}\cdot\partial_3\partial_t^3 v\,\mathrm{d}x \\ I_{01} &:= -\int_\Omega \partial_t^4\check{q}\partial_3(\nabla^\varphi\cdot v)\partial_t^4\varphi\,\mathrm{d}x + 4\int_\Omega \partial_t^4\check{q}\partial_t^3(\nabla^\varphi\cdot v)\partial_3\partial_t\varphi\,\mathrm{d}x - 4\sum_{i=1}^2\int_\Omega \partial_t^4\check{q}\partial_3\partial_t\varphi\overline{\partial}_i\partial_t^3 v_i\,\mathrm{d}x. \end{aligned}$$

We first notice that I_{00} can be estimated together with J_0 by using divergence theorem (the boundary integral on Σ_b vanishes due to $\varphi|_{\Sigma_b} = -b$ being a constant.)

$$I_{00} + J_0 = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\partial_t^3\partial_3\check{q}\partial_t\mathbf{N} + \partial_t^3\check{q}\partial_t\partial_3\mathbf{N}\right)\cdot\partial_t^3 v\,\mathrm{d}x + \cdots \tag{2.10}$$

which can be controlled under time integral by using ε-Young's inequality. Next, we find that the first two terms in I_{01} contain $\nabla^\varphi\cdot v \sim -\mathcal{F}'(q)D_t^\varphi q$ which contributes to a weight $\mathcal{F}'(q)$ and thus avoids losing weight of sound speed on $\partial_t^4\check{q}$. Finally, the last term in I_{01} can be controlled under time integral if we integrate by parts in ∂_t and then $\overline{\partial}_i$.

Combining the steps above, we finish the control of Alinhac good unknowns $\mathbf{V}, \mathbf{Q}$. Then by using the definition of good unknowns, we know $\|\mathbf{F} - \mathcal{T}^\alpha f\|_0 \leq |\mathcal{T}^\alpha\psi|_0\|\partial f\|_\infty$ which is already controlled by the boundary energy of ψ. Therefore, the a priori estimates for system (1.24) are closed.

2.1.3 Incompressible limit and zero surface tension limit

The uniform estimates and the limit process. The limit process requires the energy estimates for (1.24) to be uniform in both σ and $\mathcal{F}'(q)$. Indeed, the energy estimates obtained above are already uniform in $\mathcal{F}'(q)$ thanks to the cancellation structure

[6] The Rayleigh-Taylor sign condition is just a constraint for the initial data. One can easily prove its short-time propagation by using the boundedness of $\partial_t\partial_3 q$. See [46, Section 3.7].

(2.10). The uniform-in-σ estimates require the Rayleigh-Taylor sign condition $-\partial_3 q_0 \geq c_0 > 0$ which propagates to a finite time interval. Once we have $-\partial_3 q \geq c_0/2 > 0$ for the solution to (1.24), the aforementioned problematic boundary term then becomes

$$\mathrm{RT} := \int_\Sigma \partial_3 q \mathcal{T}^\alpha \psi \partial_t \mathcal{T}^\alpha \psi \,\mathrm{d}x' = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Sigma (-\partial_3 q)|\mathcal{T}^\alpha \psi|^2 \,\mathrm{d}x' + \cdots \tag{2.11}$$

which gives the non-weighted boundary regularity $|\mathcal{T}^\alpha\psi|_0^2$ thanks to the Rayleigh-Taylor sign condition. The fourth term in (2.7) is also controlled by this non-weighted energy. Finally, the difference $\|\mathbf{F} - \mathcal{T}^\alpha f\|_0$ that contributes to $|\mathcal{T}^\alpha\psi|_0$ is also controlled by this non-weighted energy. Hence, we have excluded all the possibilities that make the energy estimates depend on σ^{-1}. It is natural to simultaneously take the incompressible limit and the vanishing surface tension limit.

As indicated in the remark after Theorem 1.2, the limit process is only for $(\psi^{\lambda,\sigma}, v^{\lambda,\sigma}, \rho^{\lambda,\sigma})$ that converges to $(\zeta, w, 1)$ where (ζ, w) is the solution to incompressible Euler equations without surface tension (1.35). The convergence of $\rho - 1$ is easily achieved by the H^3 estimates of the continuity equation $D_t^\varphi(\rho - 1) = \rho(\nabla^\varphi \cdot v)$.

Ideas of choosing weights of sound speed. We parametrize the sound speed as in (1.29) such that $\mathcal{F}_\lambda'(q) = O(\lambda^2)$. Our choices for the weights in Mach number depend on

- Reduction of pressure: $-\nabla\check{q} \sim D_t^\varphi v$ indicates that $\partial_t^k \partial\check{q}$ should have the same weight of Mach number as $\partial_t^{k+1} v$.
- Tangential estimates: $\mathcal{T}^\alpha v$ is controlled together with $\sqrt{\mathcal{F}_\lambda'}\mathcal{T}^\alpha\check{q}$. This indicates that $\partial_t^k v$ should have the same weight of Mach number as $\sqrt{\mathcal{F}_\lambda'}\partial_t^k\check{q}$.

Based on the above two factors, our energy functional should be designed as

$$E^{\lambda,\sigma}(t) := \underbrace{\sum_{0\leq k\leq m-1}\|\partial_t^k v\|_{N-k}^2 + \|\sqrt{\mathcal{F}_\lambda'}\check{q}\|_0^2 + \|\partial\check{q}\|_{N-1}^2 + \sum_{1\leq k\leq m-1}\|\partial_t^k\check{q}\|_{N-k}^2}_{\text{non-weighted part}} + \underbrace{\|\partial_t^m v\|_{N-m}^2 + \|\sqrt{\mathcal{F}_\lambda'}\partial_t^m\check{q}\|_{N-m}^2}_{\text{the first weighted term}}$$
$$+ \underbrace{\sum_{s=1}^{N-m}\|(\mathcal{F}_\lambda')^{\frac{s}{2}}\partial_t^{m+s} v\|_{N-m-s}^2 + \|(\mathcal{F}_\lambda')^{\frac{s+1}{2}}\partial_t^{m+s}\check{q}\|_{N-m-s}^2}_{\text{weighted part}} + \text{boundary energies.}$$

Note that the cancellation structure (2.10) is just a consequence of the divergence theorem but does not depend on whether $\mathcal{T}^\alpha$ contains ∂_t or not. If we follow the strategies above, it is not difficult to obtain the uniform estimates $E^{\lambda,\sigma}(t) \leq P(E^{\lambda,\sigma}(0))$ in some $[0, T]$ with T being independent of λ, σ. Therefore, it remains to determine the minimum value of $m \in \mathbb{N}^*$ such that $E^{\lambda,\sigma}(0) < +\infty$. Indeed, we could loosen the requirement to $m = 2$, that is, $\partial_t q(0)$ is bounded but $\partial_t^2 q(0) = O(\lambda^{-1})$ is not bounded.

Remark. If we only consider the incompressible limit, that is, either the "$\sigma = 0$" problem under Rayleigh-Taylor sign condition or the"$\sigma > 0$" problem for any *fixed* $\sigma > 0$, we believe the uniform-in-λ estimates can still be established even if the requirement for compressible initial data is further loosened to be "not well-prepared" in the sense that $\partial_t q(0)$ may not be bounded. However, the proof for that case is quite different from the strategies presented in this paper and there are also crucial differences between the $\sigma = 0$ case and $\sigma > 0$ case. So we would postpone this further problem to future work.

2.2 Approximation scheme: Tangential smoothing and artificial viscosity

2.2.1 Necessity of tangential smoothing

For free-surface inviscid fluids, the local existence is not a direct consequence of the a priori estimates. For example, if we try to do Picard iteration for the linearized system whose coefficient φ is replaced by a given function $\mathring{\varphi}$, then a crucial difference from the nonlinear system is that we may no longer obtain the boundary regularity from the analogue of ST term as in (2.8). Specifically, (2.8) now becomes

$$\mathrm{ST} = \sigma\int_\Sigma \overline{\partial}^\alpha\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\psi}{1 + |\overline{\nabla}\mathring{\psi}|^2}\right)\partial_t\overline{\partial}^\alpha\psi\,\mathrm{d}x' = -\frac{\sigma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma \frac{|\overline{\partial}^\alpha\overline{\nabla}\psi|^2}{\sqrt{1 + |\overline{\nabla}\psi|^2}} - \frac{(\overline{\nabla}\psi\cdot\overline{\partial}^\alpha\overline{\nabla}\psi)(\overline{\nabla}\mathring{\psi}\cdot\overline{\partial}^\alpha\overline{\nabla}\mathring{\psi})}{\sqrt{1 + |\overline{\nabla}\mathring{\psi}|^2}^3}\,\mathrm{d}x' + \cdots, \tag{2.12}$$

where the second term has no control because inequality (2.9) is not applicable here. Such a linearization yields the loss of a tangential derivative. Therefore, it is natural to mollify the coefficient φ in tangential directions to enhance its regularity.

2.2.2 Design of approximation system

For each $\kappa > 0$, we define Λ_κ to be the standard convolution mollifier on $\mathbb{R}^2$ with parameter $\kappa > 0$ and then define $\widetilde{\psi} := \Lambda_\kappa^2 \psi$ and $\widetilde{\varphi}(t,x) := x_3 + \chi(x_3)\widetilde{\psi}(t,x')$ to be the smoothed coefficients. We introduce the following nonlinear system with artificial viscosity whose coefficients are replaced by $\widetilde{\varphi}, \widetilde{\psi}$ to approximate the original system (1.24) as $\kappa \to 0_+$.

$$\begin{cases} \rho D_t^{\widetilde{\varphi}} v + \nabla^{\widetilde{\varphi}} \check{q} = -(\rho-1) g e_3, & \text{in } [0,T]\times\Omega, \\ \mathcal{F}'(q) D_t^{\widetilde{\varphi}} \check{q} + \nabla^{\widetilde{\varphi}} \cdot v = \mathcal{F}'(q) g v_3, & \text{in } [0,T]\times\Omega, \\ q = q(\rho), \check{q} = q + g\widetilde{\varphi} & \text{in } [0,T]\times\Omega, \\ \check{q} = g\widetilde{\psi} - \sigma \overline{\nabla} \cdot \left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}} \right) + \kappa^2 (1-\overline{\Delta})(v\cdot \widetilde{N}) & \text{on } [0,T]\times\Sigma, \\ \partial_t \psi = v \cdot \widetilde{N} & \text{on } [0,T]\times\Sigma, \\ v_3 = 0 & \text{on } [0,T]\times\Sigma_b, \\ (v,\rho,\psi)|_{t=0} = (v_0^\kappa, \rho_0^\kappa, \psi_0^\kappa). \end{cases} \tag{2.13}$$

Here,

$$\nabla_i^{\widetilde{\varphi}} = \partial_i^{\widetilde{\varphi}} = \partial_i - \frac{\partial_i \widetilde{\varphi}}{\partial_3 \widetilde{\varphi}} \partial_3, \ i = 1,2, \ \ \nabla_3^{\widetilde{\varphi}} = \ \partial_3^{\widetilde{\varphi}} = \frac{1}{\partial_3 \widetilde{\varphi}} \partial_3 \tag{2.14}$$

$$D_t^{\widetilde{\varphi}} = \partial_t + \overline{v} \cdot \overline{\nabla} + \frac{1}{\partial_3 \widetilde{\varphi}} (v \cdot \widetilde{\mathbf{N}} - \partial_t \varphi) \partial_3, \tag{2.15}$$

and $\overline{v} := (v_1, v_2)$, $\overline{\nabla} := (\partial_1, \partial_2)$ are the horizontal velocities and derivatives, $\overline{\Delta} := \overline{\nabla} \cdot \overline{\nabla} = \partial_1^2 + \partial_2^2$ is the flat tangential Laplacian, $\widetilde{N} := (-\partial_1 \widetilde{\psi}, -\partial_2 \widetilde{\psi}, 1)^\top$ is the smoothed Eulerian normal vector and $\widetilde{\mathbf{N}} := (-\partial_1 \widetilde{\varphi}, -\partial_2 \widetilde{\varphi}, 1)^\top$ is the extension of $\widetilde{N}$ into Ω.

The tangential smoothing method was first introduced in [13] to study incompressible Euler and then was generalized to study various free-surface inviscid fluids in Lagrangian coordinates. However, the free surface is assumed to be a graph, and the construction of a nonlinear approximate system is quite different from Lagrangian coordinates. The following issues are crucial and very technical.

- **Design the smoothed material derivative $D_t^{\widetilde{\varphi}}$.** We have to guarantee that the weight function in front of ∂_3 in $D_t^{\widetilde{\varphi}}$ should be the same as the kinematic boundary equation, otherwise there will be a *boundary mismatched term that cannot be controlled* in the Reynold transport formula. This explains why we do not mollify $\partial_t \varphi$ in $D_t^{\widetilde{\varphi}}$.

  Remark. One may alternatively set the kinematic boundary condition to be $\partial_t \widetilde{\psi} = v \cdot \widetilde{N}$ and thus define $D_t^{\widetilde{\varphi}} = \partial_t + \overline{v} \cdot \overline{\nabla} + \frac{1}{\partial_3 \widetilde{\varphi}} (v \cdot \widetilde{\mathbf{N}} - \partial_t \widetilde{\varphi}) \partial_3$. Under this setting, the surface tension term should be $-\sigma \overline{\nabla} \cdot (\overline{\nabla}\psi / |\widetilde{N}|)$. But the analogue of ST term in (2.8) becomes

  $$\mathrm{ST} = \sigma \int_\Sigma \overline{\partial}^\alpha \overline{\nabla} \cdot \left(\frac{\overline{\nabla}\psi}{1+|\overline{\nabla}\widetilde{\psi}|^2} \right) \partial_t \overline{\partial}^\alpha \psi \, \mathrm{d}x' = -\frac{\sigma}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Sigma \frac{|\overline{\partial}^\alpha \overline{\nabla} \Lambda_\kappa \psi|^2}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}} - \frac{(\overline{\nabla}\widetilde{\psi} \cdot \overline{\partial}^\alpha \overline{\nabla}\widetilde{\psi})(\overline{\nabla}\psi \cdot \overline{\partial}^\alpha \overline{\nabla}\widetilde{\psi})}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}^3} \, \mathrm{d}x' + \cdots, \tag{2.16}$$

  where the second term is not controllable because the inequality (2.9) is no longer applicable.

- **Introduce the artificial viscosity to control the mismatched terms.** The tangential mollification leads to some mismatched terms that should be controlled by the artificial viscosity term.

  a. The commutator $\mathfrak{D}(f)$ in (2.1) now involves a new term $\mathfrak{E}(f) = \partial_t \mathcal{T}^\alpha (\widetilde{\varphi} - \varphi) \partial_3^{\widetilde{\varphi}} f$ which should be bounded by $\kappa |\overline{\nabla} \partial_t \mathcal{T}^\alpha \psi|_0$ after using the mollifier property (3.6).

  b. The analysis of the ST term in (2.7) now introduces two extra commutators:

  $$\begin{aligned} \mathrm{ST} := & -\frac{\sigma}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Sigma \frac{|\overline{\partial}^\alpha \overline{\nabla} \Lambda_\kappa \psi|^2}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}} - \frac{|\overline{\nabla}\widetilde{\psi} \cdot \overline{\partial}^\alpha \overline{\nabla} \Lambda_\kappa \psi|^2}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}^3} \, \mathrm{d}x' \\ & - \sigma \int_\Sigma \overline{\partial}^\alpha \overline{\nabla} \Lambda_\kappa \psi \cdot \left(\left[\Lambda_\kappa, \frac{1}{|\widetilde{N}|} \right] \overline{\nabla} \partial_t \overline{\partial}^\alpha \psi \right) \mathrm{d}x' + \sigma \int_\Sigma \overline{\partial}^\alpha \overline{\nabla}_i \Lambda_\kappa \psi \cdot \left(\left[\Lambda_\kappa, \frac{\overline{\nabla}_i \widetilde{\psi} \overline{\nabla}_j \widetilde{\psi}}{|\widetilde{N}|^3} \right] \overline{\nabla}_j \partial_t \overline{\partial}^\alpha \psi \right) \mathrm{d}x', \end{aligned} \tag{2.17}$$

  and the control of the commutators requires the bound for $\kappa |\overline{\nabla} \partial_t \mathcal{T}^\alpha \psi|$ if we apply the inequality (3.10).

To control the above two crucial mismatched terms, we introduce the artificial viscosity term $-\kappa^2(1-\overline{\Delta})\partial_t\psi$ which gives the energy $|\kappa\langle\overline{\partial}\rangle\mathcal{T}^\alpha\partial_t\psi|_0$ to enhance the regularity of $\partial_t\psi$. It should also be noted that the coefficient must be κ^2 instead of any other power of κ, otherwise, the third term in (2.7) is not controllable.

Based on the strategies introduced in Section 2.1 and the above analysis of the mismatched terms, we can derive the uniform-in-κ a priori estimates for the nonlinear approximate system (2.13). We can also prove the initial data $(v_{0,\kappa},\rho_{0,\kappa},\psi_{0,\kappa})$ of (2.13) converges to the initial data of (1.24) as $\kappa\to 0$. Hence, it remains to solve (2.13) for each *fixed* $\kappa>0$.

2.3 Hyperbolic approach to solve the linearized system

With tangential mollification, the tangential derivative loss in the Picard iteration can be compensated by using the mollifier property $|\Lambda_\kappa f|_s\lesssim\kappa^{-s}|f|_0$ for $s\geq 0$. So, for each $\kappa>0$, the solvability of (2.13) is reduced to the solvability of its linearization.

2.3.1 Design of linearization

In the Picard iteration scheme, we start with $(v^{(0},\rho^{(0)},\psi^{(0)}) := (\mathbf{0},1,0)$ and $\psi^{(-1)} := \psi^{(0)}$. Inductively, for each $n\in\mathbb{N}$, given $(\mathring{v},\mathring{\rho},\mathring{\check{q}},\psi) := (v^{(n)},\rho^{(n)},\check{q}^{(n)},\psi^{(n)})$ and $\dot{\psi} := \psi^{(n-1)}$, we construct $(v^{(n+1)},\rho^{(n+1)},\check{q}^{(n+1)},\psi^{(n+1)})$ (denoted by $(v,\rho,\check{q},\psi)$) via the following linear system

$$\begin{cases}\mathring{\rho}D_t^{\tilde{\mathring{\varphi}}}v+\nabla^{\tilde{\mathring{\varphi}}}\check{q}=-(\mathring{\rho}-1)ge_3, & \text{in } [0,T]\times\Omega,\\ \mathring{\mathcal{F}}'(\mathring{q})D_t^{\tilde{\mathring{\varphi}}}\check{q}+\nabla^{\tilde{\mathring{\varphi}}}\cdot v=\mathring{\mathcal{F}}'(\mathring{q})g\mathring{v}_3, & \text{in } [0,T]\times\Omega,\\ q=q(\rho),\check{q}=q+g\tilde{\mathring{\varphi}} & \text{in } [0,T]\times\Omega,\\ \check{q}=g\tilde{\mathring{\psi}}-\sigma\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\tilde{\mathring{\psi}}}{\sqrt{1+|\overline{\nabla}\tilde{\mathring{\psi}}|^2}}\right)+\kappa^2(1-\overline{\Delta})(v\cdot\tilde{\mathring{N}}), & \text{on } [0,T]\times\Sigma,\\ \partial_t\psi=v\cdot\tilde{\mathring{N}}, & \text{on } [0,T]\times\Sigma,\\ v_3=0 & \text{on } [0,T]\times\Sigma_b,\\ (v,\rho,\psi)|_{t=0}=(v_0^\kappa,\rho_0^\kappa,\psi_0^\kappa). & \end{cases}\tag{2.18}$$

Here $\mathring{\mathcal{F}}:=\log\mathring{\rho}$. The linearized material derivative and covariant derivative are constructed as follows

$$D_t^{\tilde{\mathring{\varphi}}}:=\partial_t+\overline{\mathring{v}}\cdot\overline{\nabla}+\frac{1}{\partial_3\tilde{\mathring{\varphi}}}(\mathring{v}\cdot\dot{\tilde{\mathbf{N}}}-\partial_t\mathring{\varphi})\partial_3\tag{2.19}$$

$$\nabla_i^{\tilde{\mathring{\varphi}}}=\partial_i^{\tilde{\mathring{\varphi}}}=\partial_i-\frac{\partial_i\tilde{\mathring{\varphi}}}{\partial_3\tilde{\mathring{\varphi}}}\partial_3,\ i=1,2,\quad \nabla_3^{\tilde{\mathring{\varphi}}}=\partial_3^{\tilde{\mathring{\varphi}}}=\frac{1}{\partial_3\tilde{\mathring{\varphi}}}\partial_3.\tag{2.20}$$

Remark. Note that the weight in front of ∂_3 in $D_t^{\tilde{\mathring{\varphi}}}$ is $v^{(n)}\cdot\widetilde{\mathbf{N}}^{(n-1)}-\partial_t\varphi^{(n)}$ instead of $v^{(n+1)}\cdot\widetilde{\mathbf{N}}^{(n)}-\partial_t\varphi^{(n+1)}$ because we have to make sure (2.18) is a *linear* system of $(v^{(n+1)},\psi^{(n+1)})$. Also, this weight function should be compatible with the linearized kinematic boundary condition. That is why we design $D_t^{\tilde{\mathring{\varphi}}}$ in this way.

Remark. In (2.18), the surface tension term is assumed to be a given term instead of involving ψ. The boundary energy is no longer produced from the surface tension term but from the artificial viscosity. Note that we no longer require the estimates for (2.18) to be uniform in κ as we are solving it for each *fixed* κ. Furthermore, the boundary condition can be viewed as an elliptic equation $\kappa^2(1-\overline{\Delta})\partial_t\psi=\check{q}+\cdots$ and thus the regularity of ψ can be enhanced by using elliptic estimates.

2.3.2 Hyperbolic approach to solve the linearized system

Note that (2.18) is a first-order linear symmetric hyperbolic system with *characteristic boundary conditions*. Indeed, it can be written as $A_0(\mathring{U})\partial_t U+\sum_{i=1}^{3}A_i(\mathring{U})\partial_i U=\mathring{f}$ where $U:=(\underline{q},v_1,v_2,v_3)^\top$ and[7] the boundary matrix $A_3(\mathring{U})$ is a 4×4 matrix of rank 2.

One can use the duality argument introduced by Lax-Phillips [38] to prove the local existence in $L^2(\Omega)$. However, the $L^2(\Omega)$ estimates of the dual system of (2.18) cannot be closed. The reason is that the boundary condition for $\underline{q}^*$ (the dual variable of

[7] We introduce $\underline{q}:=\check{q}-\mathring{\mathfrak{h}}$ in order to homogenize the boundary conditions. Here $\mathring{\mathfrak{h}}$ is the harmonic extension of $g\tilde{\mathring{\psi}}-\sigma\overline{\nabla}\cdot(\overline{\nabla}\tilde{\mathring{\psi}}/|\tilde{\mathring{N}}|)$ into Ω. This is necessary, as the Lax-Phillips duality argument requires the boundary conditions to be homogeneous.

$\underline{q}$) has an extra minus sign $\underline{q}^* = -\kappa^2(1-\overline{\Delta})(w^* \cdot \mathring{\widetilde{N}})$ with w^* being the dual variable of v. Our idea to overcome this difficulty is to introduce another μ-regularization term in the boundary condition of (2.18):

$$\underline{q} = \kappa^2(1-\overline{\Delta})(v \cdot \mathring{\widetilde{N}}) + \mu(1-\overline{\Delta})\partial_t(v \cdot \mathring{\widetilde{N}}) \quad \text{on } \Sigma.$$

Then the $L^2(\Omega)$ estimate for the μ-regularized linear system is still easy to prove and is also uniform in μ. As for the dual system of the dual variables $W^* := (\underline{q}^*, w_1^*, w_2^*, w_3^*)^\top$, its boundary condition now reads

$$\underline{q}^* = -\kappa^2(1-\overline{\Delta})(w^* \cdot \mathring{\widetilde{N}}) + \mu(1-\overline{\Delta})\partial_t(w^* \cdot \mathring{\widetilde{N}}) \quad \text{on } \Sigma,$$

where the sign of the μ-term remains positive as we have to integrate by parts in t once more when deriving the dual system. The boundary integral arising from the L^2 estimates of the dual system now becomes

$$\int_\Sigma W^{*\top} \cdot A_3(\mathring{U}) W^* \, \mathrm{d}x' = -\mu \frac{\mathrm{d}}{\mathrm{d}t} |\langle\overline{\partial}\rangle(w^* \cdot \mathring{\widetilde{N}})|_0^2 - 2\kappa^2 |\langle\overline{\partial}\rangle(w^* \cdot \mathring{\widetilde{N}})|_0^2,$$

and thus can be closed for each *fixed* $\mu > 0$. Therefore, the existence of the (weak) solution of μ-regularized (2.18) in $L^2(\Omega)$ is proven for each fixed $\mu > 0$. Since the $L^2(\Omega)$ estimates for μ-regularized (2.18) is uniform in μ, we can take $\mu \to 0$ to obtain the $L^2(\Omega)$ weak solution to (2.18). This is actually a strong solution by the argument in [49, Section 2.2.3].

Remark. One cannot do such μ-regularization for the nonlinear κ-approximation system (2.13), otherwise, the third term in (2.7) cannot be controlled and thus the uniform-in-κ estimates cannot be established.

2.4 Some remarks

This paper extends the results of Trakhinin [59] and our previous work [46] to the case with nonzero surface tension and also avoids the regularity loss caused by Nash-Moser iteration in [59]. Moreover, we can simultaneously establish the local well-posedness and the incompressible limit, and the zero surface tension limit is also established if we assume the Rayleigh-Taylor sign condition holds initially. Thus, this paper gives a unified method to study the local well-posedness for the motion of water waves, compressible or incompressible, with or without surface tension.

Our proof is a hyperbolic approach, in the sense that we avoid using the interior-boundary parabolic regularization introduced in [11] to prove the local existence. Indeed, the parabolic regularization method relies on the enhanced regularity of the flow map and the boundedness of the fluid domain for the Galerkin approximation to solve the linearized parabolic problem.

Our method is robust, in the sense that we can try to apply the framework in this paper to study some free-surface complex fluid models, where the fluid motion is coupled with other physical quantities, such as MHD, elastodynamics, etc. One may have to apply hyperbolic approaches to these models due to the failure of the irrotational assumption. What's more, it is usually more difficult[8] to design the approximation scheme to construct the solutions to these models. To overcome these difficulties, our method provides an alternative idea: One can follow the method in this paper to prove the well-posedness for the $\sigma > 0$ case, and then consider the surface tension as a regularization of free surface to obtain the local-in-time solution for the $\sigma = 0$ case. Indeed, we believe this idea can be used to prove similar results in anisotropic Sobolev spaces for free-surface compressible ideal MHD with or without surface tension, which is a forthcoming paper by the second author that will appear soon.

3 Nonlinear approximate κ-problem

Now we come to the detailed proof. The first step is to introduce our approximation scheme. For each $\kappa > 0$, we construct a suitable approximate problem indexed by κ which is asymptotically consistent with (1.24).

3.1 The tangential mollification

Let $\zeta = \zeta(x') \in C_c^\infty(\mathbb{R}^2)$, satisfying $0 \le \zeta \le 1$ and $\int_{\mathbb{R}^2} \zeta \, dx' = 1$, be a standard cut-off function supported in the closed unit ball $\overline{B_1(\mathbf{0})}$. For each $\kappa > 0$, we set

$$\zeta_\kappa(x') = \kappa^{-2}\zeta(\kappa^{-1}x'),$$

[8] For example, the approximation scheme in [46, 71] works for Euler equations but may not work for compressible ideal MHD. An alternative way is to use Nash-Moser iteration as in [59, 60, 61], but that will introduce a big loss of regularity from initial data to solution.

and for each $f:\mathbb{R}^2\to\mathbb{R}$, we define

$$\Lambda_\kappa f(x') := \int_{\mathbb{R}^2} \zeta_\kappa(x'-z')f(z')\,dz'. \tag{3.1}$$

Also, for each $g:\mathbb{R}^3\to\mathbb{R}$, we set

$$\Lambda_\kappa g(x',z) := \int_{\mathbb{R}^2} \zeta_\kappa(x-z')g(z',x_3)\,dz'. \tag{3.2}$$

In other words, when acting on a function of three independent variables, Λ_κ becomes the smoothing operator in the tangential direction only. The next lemma records the properties that Λ_κ enjoys. This will be frequently used (sometimes silently) in the rest of this paper.

Lemma 3.1 ([46, Lemma 2.6])**.** Let $f:\mathbb{R}^2\to\mathbb{R}$ be a smooth function. For each $\kappa>0$, there hold

$$|\Lambda_\kappa f|_s \lesssim |f|_s, \quad \forall s\geq -0.5; \tag{3.3}$$

$$|\overline{\partial}\Lambda_\kappa f|_0 \lesssim \kappa^{-s}|f|_{1-s}, \quad \forall s\in[0,1]; \tag{3.4}$$

$$|f-\Lambda_\kappa f|_\infty \lesssim \sqrt{\kappa}|\overline{\partial} f|_{0.5} \tag{3.5}$$

$$|f-\Lambda_\kappa f|_{L^p} \lesssim \kappa|\overline{\partial} f|_{L^p}. \tag{3.6}$$

Also, for a smooth function $g:\mathbb{R}^3\to\mathbb{R}$, then

$$\|\Lambda_\kappa g\|_s \lesssim \|g\|_s, \quad \forall s\geq 0. \tag{3.7}$$

Moreover, let $h:\mathbb{R}^2\to\mathbb{R}$, and $[\Lambda_\kappa, f]h := \Lambda_\kappa(fh) - f\Lambda_\kappa(h)$. Then there hold

$$|[\Lambda_\kappa, f]g|_0 \lesssim |f|_{L^\infty}|g|_0, \tag{3.8}$$

$$|[\Lambda_\kappa, f]\overline{\partial} g|_0 \lesssim |f|_{W^{1,\infty}}|g|_0, \tag{3.9}$$

$$|[\Lambda_\kappa, f]\overline{\partial} g|_0 \lesssim \kappa|f|_{W^{1,\infty}}|\overline{\partial} g|_0. \tag{3.10}$$

3.2 Construction of the κ-problem

Let $\widetilde{\psi} := \Lambda_\kappa^2\psi$, $\varphi(t,x) = x_3 + \chi(x_3)\widetilde{\psi}(t,x') = \Lambda_\kappa^2\varphi(t,x)$, and $\widetilde{N} := (-\partial_1\widetilde{\psi}, -\partial_2\widetilde{\psi}, 1)^\top$. Then we set the approximate κ-problem of (1.24) to be

$$\begin{cases}
\rho D_t^{\widetilde{\varphi}} v + \nabla^{\widetilde{\varphi}}\check{q} = -(\rho-1)ge_3 & \text{in } [0,T]\times\Omega,\\
\mathcal{F}'(q)D_t^{\widetilde{\varphi}}\check{q} + \nabla^{\widetilde{\varphi}}\cdot v = \mathcal{F}'(q)gv_3 & \text{in } [0,T]\times\Omega,\\
q = q(\rho), \check{q} = q + g\widetilde{\varphi} & \text{in } [0,T]\times\Omega,\\
\check{q} = g\widetilde{\psi} - \sigma\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\widetilde{\psi}}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}}\right) + \kappa^2(1-\overline{\Delta})(v\cdot\widetilde{N}) & \text{on } [0,T]\times\Sigma,\\
\partial_t\psi = v\cdot\widetilde{N} & \text{on } [0,T]\times\Sigma,\\
v_3 = 0 & \text{on } [0,T]\times\Sigma_b,\\
(v,\rho,\psi)|_{t=0} = (v_{\kappa,0},\rho_{\kappa,0},\psi_{\kappa,0}).
\end{cases} \tag{3.11}$$

Here,

$$\partial_t^{\widetilde{\varphi}} = \partial_t - \frac{\partial_t\varphi}{\partial_3\widetilde{\varphi}}\partial_3, \tag{3.12}$$

$$\nabla_a^{\widetilde{\varphi}} = \partial_a^{\widetilde{\varphi}} = \partial_a - \frac{\partial_a\widetilde{\varphi}}{\partial_3\widetilde{\varphi}}\partial_3, \quad a=1,2, \tag{3.13}$$

$$\nabla_3^{\widetilde{\varphi}} = \partial_3^{\widetilde{\varphi}} = \frac{1}{\partial_3\widetilde{\varphi}}\partial_3, \tag{3.14}$$

$$D_t^{\widetilde{\varphi}} = \partial_t^{\widetilde{\varphi}} + v\cdot\nabla^{\widetilde{\varphi}}, \tag{3.15}$$

and $\overline{\Delta} = \partial_x^2 + \partial_y^2$ is the flat tangential Laplacian. Thanks to (3.12), the smoothed material derivative $D_t^{\widetilde{\varphi}}$ is equivalent to

$$D_t^{\widetilde{\varphi}} = \partial_t + \overline{v}\cdot\overline{\nabla} + \frac{1}{\partial_3\widetilde{\varphi}}(v\cdot\widetilde{\mathbf{N}} - \partial_t\varphi)\partial_3, \tag{3.16}$$

where $\widetilde{\mathbf{N}} := (-\partial_1\widetilde{\varphi}, -\partial_2\widetilde{\varphi}, 1)^\top$. Note that we do not replace $v \cdot \widetilde{\mathbf{N}} - \partial_t\varphi$ by $v \cdot \widetilde{\mathbf{N}} - \partial_t\widetilde{\varphi}$ in the last term, as this would generate a severe structural mismatch in the boundary estimates.

The approximate κ-system (3.11) is asymptotically consistent with (1.24) as $\kappa \to 0$. Furthermore, the artificial viscosity $\kappa(1-\overline{\Delta})(v \cdot \widetilde{N})$ in the modified boundary condition

$$\check{q} = g\widetilde{\psi} - \sigma\overline{\nabla} \cdot \left(\frac{\overline{\nabla}\widetilde{\psi}}{\sqrt{1 + |\overline{\nabla}\widetilde{\psi}|^2}} \right) + \kappa^2(1-\overline{\Delta})(v \cdot \widetilde{N}) \quad \text{on } \Sigma$$

is necessary to control the terms generated due to the loss of symmetry in (3.11). For each $\kappa > 0$, we denote the solution to (3.11) to be $(v^\kappa(t), \check{q}^\kappa(t), \rho^\kappa(t), \psi^\kappa(t))$. Our goal is to prove $\{v^\kappa(t), \check{q}^\kappa(t), \rho^\kappa(t), \psi^\kappa(t)\}_{\kappa>0}$ has a convergent subsequence that approximates the solution to the original system (1.24) as $\kappa \to 0$ in some time interval $[0, T]$ with T being independent of κ. From now on, we drop the index κ when analyzing the nonlinear κ-approximate system for the sake of clean notations.

4 Uniform-in-κ energy estimates for the nonlinear κ-problem

We shall establish the a priori energy estimate of (3.11) that is uniform for all $\kappa > 0$. For any solution $(v, \rho, \check{q}, \psi)$ to the nonlinear κ-system (3.11), we define the nonlinear energy $E^\kappa(t)$ to be

$$\begin{aligned} E^\kappa(t) =& \|\rho(t) - 1\|_0^2 + \sum_{k=0}^{4} \|\partial_t^k v(t)\|_{4-k}^2 + \sigma|\overline{\nabla}\partial_t^k \Lambda_\kappa \psi(t)|_{4-k}^2 + g|\Lambda_\kappa\psi|_0^2 + \int_0^t \left|\kappa\partial_t^{k+1}\psi(t)\right|_1^2 \mathrm{d}\tau \\ &+ \left\| \sqrt{\mathcal{F}'(q)}\check{q}(t) \right\|_0^2 + \|\partial\check{q}(t)\|_3^2 + \sum_{k=1}^{3} \|\partial_t^k \check{q}(t)\|_{4-k}^2 + \left\| \sqrt{\mathcal{F}'(q)}\partial_t^4\check{q}(t) \right\|_0^2. \end{aligned} \tag{4.1}$$

Specifically, we show

Proposition 4.1. There exists some $T > 0$, independent of κ and $\sqrt{\mathcal{F}'(q)}$, such that

$$\sup_{0\le t\le T} E^\kappa(t) \le P(E^\kappa(0)) =: \mathcal{P}_0^\kappa. \tag{4.2}$$

The key step of proving Proposition 4.1 is to show

$$\sup_{0\le t\le T} E^\kappa(t) \le \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t. \tag{4.3}$$

thanks to the Grönwall's inequality. The control of $E^\kappa(t)$ will be divided into 3 steps, i.e., the basic L^2 estimate, the div-curl analysis, and the interior tangential estimates. We remark here that the compatibility conditions have changed due to the artificial viscosity. The new compatibility conditions, expressed in terms of $\check{q}$, are

$$(D_t^{\widetilde{\varphi}})^k \check{q}|_{t=0} = (D_t^{\widetilde{\varphi}})^k(-g\widetilde{\psi} + \sigma\mathcal{H})|_{t=0} + (D_t^{\widetilde{\varphi}})^k \left(\kappa^2(1-\overline{\Delta})(v \cdot \widetilde{N})\right), \quad k = 0,1,2,3, \quad \text{on } \Sigma \tag{4.4}$$

We however are still able to construct initial data satisfying (4.4) in terms of $(\psi_{\kappa,0}, v_{\kappa,0}, \check{q}_{\kappa,0})$, that is uniformly bounded and converges to $(\psi_0, v_0, \check{q}_0)$ as $\kappa \to 0$. The details can be located in Appendix C.

4.1 L^2-estimate

First, we establish L^2-energy estimate for (3.11). Invoking Theorem (A.3), the identity $\nabla^{\widetilde{\varphi}}\widetilde{\varphi} = e_3$, and then integrating by parts, we have:

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho|v|^2\partial_3\widetilde{\varphi}\,\mathrm{d}x =& -\int_\Omega v \cdot \nabla^{\widetilde{\varphi}}\check{q}\partial_3\widetilde{\varphi}\,\mathrm{d}x - \int_\Omega (\rho-1)gv_3\partial_3\widetilde{\varphi}\,\mathrm{d}x + \frac{1}{2}\int_\Omega \rho|v|^2\partial_3\partial_t(\widetilde{\varphi}-\varphi)\,\mathrm{d}x \\ =& \int_\Omega \check{q}(\nabla^{\widetilde{\varphi}} \cdot v)\partial_3\widetilde{\varphi}\,\mathrm{d}x + \int_{\Sigma_b} v_3\check{q}\,\mathrm{d}x' - \int_\Sigma \partial_t\psi q\,\mathrm{d}x' - \int_\Sigma g\widetilde{\psi}\partial_t\psi\,\mathrm{d}x' \\ &- \int_\Omega (\rho-1)gv_3\partial_3\widetilde{\varphi}\,\mathrm{d}x + \frac{1}{2}\int_\Omega \rho|v|^2\partial_3\partial_t(\widetilde{\varphi}-\varphi)\,\mathrm{d}x. \end{aligned} \tag{4.5}$$

Plugging the continuity equation into the first integral, we get

$$\begin{aligned}\int_\Omega \check{q}(\nabla^{\widetilde{\varphi}}\cdot v)\partial_3\widetilde{\varphi}\,\mathrm{d}x =& -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathcal{F}'(q)|\check{q}|^2\partial_3\widetilde{\varphi}\,\mathrm{d}x + \frac{1}{2}\int_\Omega \rho D_t^{\widetilde{\varphi}}(\rho^{-1}\mathcal{F}'(q))|\check{q}|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega \mathcal{F}'(q)|\check{q}|^2\partial_3\partial_t(\widetilde{\varphi}-\varphi)\,\mathrm{d}x \\ &+ \int_\Omega \check{q}\mathcal{F}'(q)gv_3\partial_3\widetilde{\varphi}\,\mathrm{d}x \\ \lesssim& -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\sqrt{\mathcal{F}'(q)}q\right\|_0^2 + \left\|\sqrt{\mathcal{F}'(q)}q\right\|_0^2\left(\|D_t^{\widetilde{\varphi}}\rho\|_\infty + \|\partial_3\partial_t(\widetilde{\varphi}-\varphi)\|_\infty + \left\|\sqrt{\mathcal{F}'(q)}v_3\right\|_0\right).\end{aligned} \tag{4.6}$$

Here and in the sequel, we employ the notation $A \lesssim B$ to mean that $A \le CB$ for a universal constant C. The boundary integral on Σ_b vanishes due to $v_3|_{\Sigma_b} = 0$. Then we plug $q = -\sigma\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\widetilde{\psi}}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}}\right) + \kappa^2(1-\overline{\Delta})\partial_t\psi$ into the first boundary term in (4.5) and integrate by parts to get:

$$-\int_\Sigma \partial_t\psi q\,\mathrm{d}x' = -\sigma\int_\Sigma\left(\frac{\overline{\nabla}\widetilde{\psi}}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}}\right)\cdot\overline{\nabla}\partial_t\psi\,\mathrm{d}x' + \int_\Sigma\left|\kappa\langle\overline{\partial}\rangle\partial_t\psi\right|_0^2\mathrm{d}x', \tag{4.7}$$

where $\langle\cdot\rangle$ denotes the Japanese bracket. To treat the first term, we use the self-adjointness of Λ_κ in $L^2(\Sigma)$ to move one Λ_κ from $\overline{\nabla}\widetilde{\psi}$ to $\partial_t\psi$:

$$\begin{aligned}-\sigma\int_\Sigma\left(\frac{\overline{\nabla}\widetilde{\psi}}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}}\right)\cdot\overline{\nabla}\partial_t\psi\,\mathrm{d}x' =& -\sigma\int_\Sigma\frac{\overline{\nabla}\Lambda_\kappa\psi\cdot\partial_t\overline{\nabla}\Lambda_\kappa\psi}{|\widetilde{N}|}\,\mathrm{d}x' - \sigma\int_\Sigma\overline{\nabla}\Lambda_\kappa\psi\cdot\left([\Lambda_\kappa,|\widetilde{N}|^{-1}]\overline{\nabla}\partial_t\psi\right)\mathrm{d}x' \\ \lesssim -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left|\sqrt{\sigma}\frac{1}{|\widetilde{N}|^{\frac{1}{2}}}\overline{\nabla}\Lambda_\kappa\psi\right|_0^2 &+ \frac{1}{2}\int_\Sigma\partial_t(|\widetilde{N}|^{-1})\left|\sqrt{\sigma}\overline{\nabla}\Lambda_\kappa\psi\right|^2 + P(|\overline{\nabla}\widetilde{\psi}|_{W^{1,\infty}})\sigma\left|\sqrt{\sigma}\overline{\nabla}\Lambda_\kappa\psi\right|_0^2 + \varepsilon\left|\kappa\overline{\nabla}\partial_t\psi\right|_0^2.\end{aligned} \tag{4.8}$$

Now, we get the non-weighted L^2 boundary energy from the second boundary integral in (4.5)

$$-\int_\Sigma g\partial_t\psi\widetilde{\psi}\,\mathrm{d}x' = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega g|\Lambda_\kappa\psi|^2\,\mathrm{d}x'. \tag{4.9}$$

On the other hand, show the L^2 estimate for $\rho-1$ for the energy inequality. We use $D_t^{\widetilde{\varphi}}\rho = D_t^{\widetilde{\varphi}}(\rho-1)$ and $D_t^{\widetilde{\varphi}}\widetilde{\varphi} = v_3 + \partial_t(\widetilde{\varphi}-\varphi)$ to rewrite the continuity equation in terms of $\rho-1$:

$$D_t^{\widetilde{\varphi}}(\rho-1) + \rho(\nabla^{\widetilde{\varphi}}\cdot v) = -\partial_t(\widetilde{\varphi}-\varphi).$$

Testing this with $\rho-1$ in $L^2(\Omega)$ and using the mollifier property (3.6), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho-1\|_0^2 \lesssim \|\rho-1\|_0(\|\partial v\|_0 + \kappa|\overline{\partial}\partial_t\psi|_0). \tag{4.10}$$

Let

$$E_0^\kappa(t) = \|v\|_0^2 + \left\|\sqrt{\mathcal{F}'(q)}\check{q}\right\|_0^2 + \|\rho-1\|_0^2 + |\sqrt{g}\Lambda_\kappa\psi|_0^2 + \left|\sqrt{\sigma}\overline{\nabla}\Lambda_\kappa\psi\right|_0^2 + \int_0^T\int_\Sigma\left|\kappa\langle\overline{\partial}\rangle\partial_t\psi\right|_0^2\mathrm{d}x'\,\mathrm{d}t. \tag{4.11}$$

Since $1 \le |\widetilde{N}| = \sqrt{1+(\partial_1\widetilde{\psi})^2+(\partial_2\widetilde{\psi})^2}$, we combine (4.5)-(4.10) and obtain

$$E_0^\kappa(T) - E_0^\kappa(0) \lesssim \int_0^T P(|\overline{\nabla}\widetilde{\psi}|_{W^{1,\infty}}, \|\partial v\|_\infty, |\kappa\overline{\partial}\partial_t\psi|_{0.5})E_0^\kappa(t)\,\mathrm{d}t, \tag{4.12}$$

after choosing $\varepsilon > 0$ suitably small in (4.8). Here, we note that, using $\partial_3\partial_t(\widetilde{\varphi}-\varphi) = \chi'(x_3)\left(\partial_t\widetilde{\psi}(t,x') - \partial_t\psi(t,x')\right)$ together with (1.9) and (3.5) of Lemma 3.1, we have

$$\|\partial_3\partial_t(\widetilde{\varphi}-\varphi)\|_\infty \le |\partial_t\widetilde{\psi} - \partial_t\psi|_\infty \lesssim \sqrt{\kappa}|\overline{\partial}\partial_t\psi|_{0.5}, \tag{4.13}$$

where right side is directly controlled by invoking $\partial_t\psi = v\cdot\widetilde{\mathbf{N}} = -(\overline{v}\cdot\overline{\nabla})\widetilde{\psi} + v_3$ on Σ and the Sobolev trace lemma.

4.2 Reduction of pressure

We show how to reduce the control of the pressure to that of the velocity when there is at least one spatial derivative on q. This follows from using the momentum equation $\rho D_t^{\widetilde{\varphi}} v = -\nabla^{\widetilde{\varphi}} \check{q} - (\rho - 1) g e_3$. Particularly, by considering the third component of the momentum equation, we get

$$-(\partial_3\widetilde{\varphi})^{-1}\partial_3\check{q} - (\rho-1)ge_3 = \rho D_t^{\widetilde{\varphi}} v_3. \tag{4.14}$$

Since $\partial_3\widetilde{\varphi}$ is bounded from below, i.e., there exists $c_0 > 0$ such that $\partial_3\widetilde{\varphi} \geq c_0$, then

$$\|\partial_3\check{q}\|_0 \lesssim_{g,c_0} \|\rho - 1\|_0 + \|\rho\|_\infty \|D_t^{\widetilde{\varphi}} v_3\|_0, \tag{4.15}$$

where $D_t^{\widetilde{\varphi}} v_3 = \partial_t v_3 + \overline{v} \cdot \overline{\nabla} v_3 + \frac{1}{\partial_3\widetilde{\varphi}}(v \cdot \widetilde{\mathbf{N}} - \partial_t\varphi)\partial_3 v_3$. This implies that the L^2-norm of $\partial_3\check{q}$ is reduced to the L^2-norms of $\rho - 1$, $\partial_t v_3, \overline{\partial} v_3$ and $\omega(x_3)\partial_3 v_3$. Here $\omega(x_3) \in C^\infty(-b, 0)$ is assumed to be bounded, comparable to $|x_3|$ in $[-2, 0]$ and vanishing on Σ.

Let $\mathcal{T} = \partial_t$ or $\overline{\partial}$ or $\omega(x_3)\partial_3$ and $D = \partial$ or ∂_t. The above estimate yields the control of $\|D^k\partial_3\check{q}\|_0$ after taking D^k, $k \geq 1$ to (4.14). Specifically, at the leading order, $\|D^k\partial_3\check{q}\|_0$ is controlled by

$$C(g, c_0)\left(\|\mathcal{F}'(q)D^k\check{q}\|_0 + \|\mathcal{F}'(q)D^k\widetilde{\varphi}\|_0 + \|\rho\|_{L^\infty}\|D^k\mathcal{T} v_3\|_0\right). \tag{4.16}$$

In addition, by considering the first two components of the momentum equation, we have:

$$-\partial_i\check{q} = -(\partial_3\widetilde{\varphi})^{-1}\overline{\partial}_i\widetilde{\varphi}\partial_3\check{q} + \rho D_t^{\widetilde{\varphi}} v_i, \quad i = 1, 2. \tag{4.17}$$

and thus the control of $\overline{\partial}\check{q}$ is reduced to $\partial_3\check{q}$ and $D_t^{\widetilde{\varphi}} v_i = \partial_t v_i + (\overline{v} \cdot \overline{\nabla}) v_i + (\partial_3\widetilde{\varphi})^{-1}(v \cdot \widetilde{\mathbf{N}} - \partial_t\varphi)\partial_3 v_i$.

Lastly, using (4.14) and (4.17), we obtain

$$\|\partial_3\check{q}\|_\infty \lesssim_{g,c_0} \|\rho - 1\|_\infty + \|\rho\|_\infty\|D_t^{\widetilde{\varphi}} v_3\|_\infty, \tag{4.18}$$

$$\|\overline{\partial}\check{q}\|_\infty \lesssim_{g,c_0^{-1}} |\overline{\partial}\widetilde{\psi}|_\infty \|\partial_3\check{q}\|_\infty + \|\rho\|_\infty\|D_t^{\widetilde{\varphi}}\overline{v}\|_\infty. \tag{4.19}$$

Thus,

$$\|\partial q\|_\infty \lesssim_{g,c_0,c_0^{-1}} P(|\overline{\partial}\widetilde{\psi}|_\infty, \|\rho\|_\infty)\left(\|\rho - 1\|_\infty + \|D_t^{\widetilde{\varphi}} v\|_\infty\right). \tag{4.20}$$

Invoking the definition of $D_t^{\widetilde{\varphi}} v$, (4.20) implies that $\|\partial\check{q}\|_\infty$ is reduced to $\partial_t v$, $\overline{\partial} v$ and $\omega(x)\partial_3 v$ for some weight function $\omega(x)$ vanishing on Γ.

4.3 Div-Curl analysis

To estimate the Sobolev norm of v, we can use the div-curl analysis to convert the normal derivative to divergence and curl. First, we record the well-known div-curl decomposition lemma and refer to [21, Lemma B.2] for the proof.

Lemma 4.2 (Hodge elliptic estimates)**.** For any sufficiently smooth vector field X and $s \geq 1$, one has

$$\|X\|_s^2 \lesssim C(|\widetilde{\psi}|_s, |\overline{\nabla}\widetilde{\psi}|_{W^{1,\infty}})\left(\|X\|_0^2 + \|\nabla^{\widetilde{\varphi}} \cdot X\|_{s-1}^2 + \|\nabla^{\widetilde{\varphi}} \times X\|_{s-1}^2 + \|\overline{\partial}^\alpha X\|_0^2\right) \tag{4.21}$$

for any multi-index α with $|\alpha| = s$. The constant $C(|\widetilde{\psi}|_s, |\overline{\nabla}\widetilde{\psi}|_{W^{1,\infty}}) > 0$ depends linearly on $|\widetilde{\psi}|_s^2$.

We will apply Lemma 4.2 to $\|\partial_t^k v\|_{4-k}$ for $0 \leq k \leq 3$. Starting from $k = 0$, we have

$$\|v\|_4^2 \lesssim C(|\widetilde{\psi}|_4, |\overline{\nabla}\widetilde{\psi}|_{W^{1,\infty}})\left(\|v\|_0^2 + \|\nabla^{\widetilde{\varphi}} \cdot v\|_3^2 + \|\nabla^{\widetilde{\varphi}} \times v\|_3^2 + \|\overline{\partial}^4 v\|_0^2\right), \tag{4.22}$$

$$\|\partial_t^k v\|_{4-k}^2 \lesssim C(|\widetilde{\psi}|_{4-k}, |\overline{\nabla}\widetilde{\psi}|_{W^{1,\infty}})\left(\|\partial_t^k v\|_0^2 + \|\nabla^{\widetilde{\varphi}} \cdot \partial_t^k v\|_{3-k}^2 + \|\nabla^{\widetilde{\varphi}} \times \partial_t^k v\|_3^2 + \|\overline{\partial}^{4-k}\partial_t^k v\|_0^2\right) \tag{4.23}$$

where the L^2 norm has been controlled in (4.12) and the tangential derivatives will be studied in the next section by using Alinhac good unknowns. The divergence part is reduced to the estimates of q by using the continuity equation

$$\|\nabla^{\widetilde{\varphi}} \cdot v\|_3^2 = \left\|\mathcal{F}'(q)D_t^{\widetilde{\varphi}}\check{q}\right\|_3^2 + \left\|\mathcal{F}'(q)gv_3\right\|_3^2, \tag{4.24}$$

which will be further reduced to the tangential estimates of v by using the argument in Section 4.2. Similarly, when $k \geq 1$, we have

$$\nabla^{\widetilde{\varphi}} \cdot \partial_t^k v = -\partial_t^k(\mathcal{F}'(q) D_t^{\widetilde{\varphi}} \check{q}) - \partial_t^k(\mathcal{F}'(q) g v_3) + [\nabla^{\widetilde{\varphi}} \cdot, \partial_t^k] v \stackrel{L}{=} -\mathcal{F}'(q) \partial_t^k D_t^{\widetilde{\varphi}} \check{q} - \mathcal{F}'(q) g \partial_t^k v_3 + (\partial_3 \widetilde{\varphi})^{-1} \overline{\partial} \partial_t^k \widetilde{\varphi} \partial_3 v,$$

where $\stackrel{L}{=}$ means the omitted terms are of lower order. Also, since $D_t^{\widetilde{\varphi}} \check{q} = (\partial_t + \overline{v} \cdot \overline{\nabla}) \check{q} + (\partial_3 \widetilde{\varphi})^{-1} (v \cdot \widetilde{\mathbf{N}} - \partial_t \varphi) \partial_3 \check{q}$,

$$\mathcal{F}'(q) \partial_t^k D_t^{\widetilde{\varphi}} \check{q} \stackrel{L}{=} \mathcal{F}'(q)(\partial_t^{k+1} \check{q} + \overline{v} \cdot \overline{\nabla} \partial_t^k \check{q} + (\partial_3 \widetilde{\varphi})^{-1} (v \cdot \widetilde{\mathbf{N}} - \partial_t \varphi) \partial_3 \partial_t^k \check{q} + (\partial_3 \widetilde{\varphi})^{-1} \partial_t^k (v \cdot \widetilde{\mathbf{N}} - \partial_t \varphi) \partial_3 \check{q}),$$

where $\partial_t^k (v \cdot \widetilde{\mathbf{N}} - \partial_t \varphi) = (1 - \chi(x_3)) \partial_t^k (v \cdot \widetilde{\mathbf{N}})$ owing to $\partial_t \varphi = \chi(x_3) \partial_t \psi$, and $\|\partial_t^k (v \cdot \widetilde{\mathbf{N}})\|_{3-k}$ is of lower order. Thus the control of $\|\nabla^{\widetilde{\varphi}} \cdot \partial_t^k v\|_{3-k}$ is then reduced to the control of $\|\mathcal{F}'(q) \partial_t^{k+1} \check{q}\|_{3-k}, \|\mathcal{F}'(q) \partial_t^k \check{q}\|_{4-k}$ and $\|\overline{\partial} \partial_t^k \widetilde{\varphi}\|_{3-k} \lesssim |\overline{\partial} \partial_t^k \widetilde{\psi}|_{3-k}$ at the leading order. Specifically,

$$\begin{aligned} \|\nabla^{\widetilde{\varphi}} \cdot \partial_t^k v\|_{3-k}^2 \leq\ & C(c_0, g, \|v\|_\infty, \|\partial v\|_\infty, \|v \cdot \widetilde{\mathbf{N}} - \partial_t \varphi\|_\infty) \left(\left\|\mathcal{F}'(q) \partial_t^{k+1} \check{q}\right\|_{3-k}^2 + \left\|\mathcal{F}'(q) \partial_t^k \check{q}\right\|_{4-k}^2 + \left|\overline{\partial} \partial_t^k \widetilde{\psi}\right|_{3-k}^2 \right) \\ & + \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\, \mathrm{d}t, \end{aligned} \tag{4.25}$$

where the last two terms control all lower order terms generated above. The bound for $|\overline{\partial} \partial_t^k \widetilde{\psi}|_{3-k}$ will be obtained in $\partial_t^k \overline{\partial}^{4-k}$-tangential estimates. As for q, combining the reduction argument in Section 4.2 and (4.24), (4.25), we find that finally, we need to control the tangential derivatives of v (including time derivative) and $\|\mathcal{F}'(q) \partial_t^4 q\|_0$.

Next, we analyze the vorticity term. We take $\nabla^{\widetilde{\varphi}} \times$ in the momentum equation $\rho D_t^{\widetilde{\varphi}} v = -\nabla^{\widetilde{\varphi}} \check{q} + (\rho - 1) g e_3$ to get

$$\rho D_t^{\widetilde{\varphi}} (\nabla^{\widetilde{\varphi}} \times v) = -\nabla^{\widetilde{\varphi}} \times ((\rho - 1) g e_3) - (\nabla^{\widetilde{\varphi}} \rho) \times D_t^{\widetilde{\varphi}} v - \rho [\nabla^{\widetilde{\varphi}} \times, D_t^{\widetilde{\varphi}}] v,$$

where the first term on the right side is equal to $(-g \partial_2^{\widetilde{\varphi}} \rho, g \partial_1^{\widetilde{\varphi}} \rho, 0)^\top$ and the second term, using $D_t^{\widetilde{\varphi}} v = -\rho^{-1} \nabla^{\widetilde{\varphi}} q - g e_3$, is equal to

$$-(\nabla^{\widetilde{\varphi}} \rho) \times D_t^{\widetilde{\varphi}} v = \underbrace{\rho'(q)(\nabla^{\widetilde{\varphi}} q) \times (\nabla^{\widetilde{\varphi}} q)}_{=\vec{0}} + \nabla^{\widetilde{\varphi}} \rho \times g e_3 = (g \partial_2^{\widetilde{\varphi}} \rho, -g \partial_1^{\widetilde{\varphi}} \rho, 0)^\top$$

which exactly cancels the first term. Using $[\partial_i^{\widetilde{\varphi}}, D_t^{\widetilde{\varphi}}](\cdot) = \partial_i^{\widetilde{\varphi}} v^l \partial_l^{\widetilde{\varphi}}(\cdot) + \partial_i^{\widetilde{\varphi}} \partial_t (\widetilde{\varphi} - \varphi) \partial_3^{\widetilde{\varphi}}(\cdot)$, we get the evolution of the smoothed vorticity to be

$$\rho D_t^{\widetilde{\varphi}} (\nabla^{\widetilde{\varphi}} \times v)_i = -\rho \epsilon^{ijk} \partial_j^{\widetilde{\varphi}} v^l \partial_l^{\widetilde{\varphi}} v_k - \rho \epsilon^{ijk} \partial_j^{\widetilde{\varphi}} \partial_t (\widetilde{\varphi} - \varphi) \partial_3^{\widetilde{\varphi}} v_k, \tag{4.26}$$

where ϵ^{ijk} denotes the sign of the permutation $(ijk) \in S_3$.

To control $\|\nabla^{\widetilde{\varphi}} \times v\|_3$, we take ∂^3 in (4.26) to get

$$\rho D_t^{\widetilde{\varphi}} \left(\partial^3 (\nabla^{\widetilde{\varphi}} \times v)_i \right) = -\epsilon^{ijk} \partial^3 (\rho \partial_j^{\widetilde{\varphi}} v^l \partial_l^{\widetilde{\varphi}} v_k) - \epsilon^{ijk} \partial^3 (\rho \partial_j^{\widetilde{\varphi}} \partial_t (\widetilde{\varphi} - \varphi) \partial_3^{\widetilde{\varphi}} v_k) - [\partial^3, \rho D_t^{\widetilde{\varphi}}] (\nabla^{\widetilde{\varphi}} \times v)_i. \tag{4.27}$$

It is not necessary to write out the specific form of the right side of (4.27), but we just need to know the source terms in (4.27) contain ≤ 4 derivatives of v and $\widetilde{\varphi}$ except the mismatched term involving $\widetilde{\varphi} - \varphi$. This is easy to see because the only term containing 5 derivatives is the one on the left side of (4.27). Therefore, a straightforward L^2 estimate for (4.27) gives us the energy estimate

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \|\nabla^{\widetilde{\varphi}} \times v\|_3^2 \leq P(\|v\|_4, |\widetilde{\psi}|_4, \|\mathcal{F}'(q) \partial q\|_\infty, \|\mathcal{F}'(q) \partial^2 q\|_1, \kappa |\overline{\nabla} \partial_t \psi|_4), \tag{4.28}$$

where the mismatched term is controlled by using mollifier property (3.10) and $\varphi(t, x) = x_3 + \chi(x_3) \psi(t, x')$.

Similarly, we replace ∂^3 by $\partial_t^k \partial^{3-k}$ for $0 \leq k \leq 3$ to get

$$\rho D_t^{\widetilde{\varphi}} \left(\partial^{3-k} \partial_t^k (\nabla^{\widetilde{\varphi}} \times v)_i \right) = -\epsilon^{ijk} \partial_t^k \partial^{3-k} (\rho \partial_j^{\widetilde{\varphi}} v^l \partial_l^{\widetilde{\varphi}} v_k) - \epsilon^{ijk} \partial_t^k \partial^{3-k} (\rho \partial_j^{\widetilde{\varphi}} \partial_t (\widetilde{\varphi} - \varphi) \partial_3^{\widetilde{\varphi}} v_k) - [\partial_t^k \partial^{3-k}, \rho D_t^{\widetilde{\varphi}}] (\nabla^{\widetilde{\varphi}} \times v)_i, \tag{4.29}$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \|\partial_t^k (\nabla^{\widetilde{\varphi}} \times v)\|_{3-k}^2 \leq P(E^\kappa(t)). \tag{4.30}$$

Then we need to estimate the commutator $\|[\partial_t^k, \nabla^{\widetilde{\varphi}}\times]v\|_{3-k}$ to get the control of $\|\nabla^{\widetilde{\varphi}} \times \partial_t^k v\|_{3-k}$. Similarly, as in the control of divergence, we know the highest order term in the commutator should be $\|(-\partial_3\widetilde{\varphi})^{-1}\overline{\partial}\partial_t^k\widetilde{\varphi}\partial_3 v\|_{3-k} \lesssim \|\partial v\|_{3-k}\|\overline{\partial}\langle\overline{\partial}\rangle^{3-k}\partial_t^k\widetilde{\varphi}\|_0 \le \|\partial v\|_{3-k}|\overline{\partial}\partial_t^k\widetilde{\psi}|_{3-k}$. So we have the following conclusion

$$\|\nabla^{\widetilde{\varphi}} \times \partial_t^k v\|_{3-k}^2 \le |\overline{\partial}\partial_t^k\widetilde{\psi}|_{3-k}^2 + \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t. \tag{4.31}$$

Combining (4.22), (4.24), (4.25), (4.28), (4.31) and the argument in Section 4.2, it remains to control the tangential derivatives of v and full time derivatives of q, namely $\|\mathcal{F}'(q)\partial_t^4\check{q}\|_0$.

4.4 The $\mathcal{T}^\alpha$-differentiated equations

By the div-curl analysis, the crucial step is to study the higher order tangential energy estimate of (3.11). In particular, we define the following tangential derivatives

$$\mathcal{T}_0 = \partial_t, \quad \mathcal{T}_1 = \partial_1, \quad \mathcal{T}_2 = \partial_2, \quad \mathcal{T}_3 = \omega(x_3)\partial_3, \tag{4.32}$$

where $\omega \in C^\infty(-b, 0)$ is assumed to be bounded, comparable to $|x_3|$ when $-2 \le x_3 \le 0$ and vanishing on Σ. This requires us to commute $\mathcal{T}^\alpha$ with (3.11), where $\mathcal{T}^\alpha := \mathcal{T}_0^{\alpha_0}\mathcal{T}_1^{\alpha_1}\mathcal{T}_2^{\alpha_2}\mathcal{T}_3^{\alpha_3}$, and $|\alpha| \le 4$.

Remark. We need the tangential derivative $\mathcal{T}_3 = \omega(x_3)\partial_3$ to control the $(\partial_3\varphi)^{-1}(v \cdot \widetilde{\mathbf{N}} - \partial_t\varphi)\partial_3$ in the material derivative $D_t^{\widetilde{\varphi}}$. We do not include it in $E^\kappa(t)$ as ω is comparable to 1. However, we still need the estimates of $\mathcal{T}_3$ in the reduction of $\check{q}$.

We will not directly commute $\mathcal{T}^\alpha$ with $\nabla^{\widetilde{\varphi}}$. Instead, for $i = 1, 2, 3$, we observe that

$$\mathcal{T}^\alpha\partial_i^{\widetilde{\varphi}} f = \partial_i^{\widetilde{\varphi}}\mathcal{T}^\alpha f - \partial_3^{\widetilde{\varphi}} f \partial_i^{\widetilde{\varphi}}\mathcal{T}^\alpha\widetilde{\varphi} + \mathfrak{C}_i'(f), \tag{4.33}$$

where for $i = 1, 2$,

$$\mathfrak{C}_i'(f) = -\left[\mathcal{T}^\alpha, \frac{\partial_i\widetilde{\varphi}}{\partial_3\widetilde{\varphi}}, \partial_3 f\right] - \partial_3 f\left[\mathcal{T}^\alpha, \partial_i\widetilde{\varphi}, \frac{1}{\partial_3\widetilde{\varphi}}\right] + \partial_i\widetilde{\varphi}\partial_3 f\left[\mathcal{T}^{\alpha-\gamma}, \frac{1}{(\partial_3\widetilde{\varphi})^2}\right]\mathcal{T}^\gamma\partial_3\widetilde{\varphi} - \frac{\partial_i\widetilde{\varphi}}{\partial_3\widetilde{\varphi}}[\mathcal{T}^\alpha, \partial_3]f + \frac{\partial_i\widetilde{\varphi}}{(\partial_3\widetilde{\varphi})^2}\partial_3 f[\mathcal{T}^\alpha, \partial_3]\widetilde{\varphi}, \tag{4.34}$$

with $|\gamma| = 1$, and

$$\mathfrak{C}_3'(f) = \left[\mathcal{T}^\alpha, \frac{1}{\partial_3\widetilde{\varphi}}, \partial_3 f\right] + \partial_3 f\left[\mathcal{T}^{\alpha-\gamma}, \frac{1}{(\partial_3\widetilde{\varphi})^2}\right]\mathcal{T}^\gamma\partial_3\widetilde{\varphi} - \frac{1}{\partial_3\widetilde{\varphi}}[\mathcal{T}^\alpha, \partial_3]f + \frac{1}{(\partial_3\widetilde{\varphi})^2}\partial_3 f[\mathcal{T}^\alpha, \partial_3]\widetilde{\varphi}. \tag{4.35}$$

Since $\partial_i^{\widetilde{\varphi}}$ and $\partial_3^{\widetilde{\varphi}}$ commute, the identity (4.33) implies

$$\mathcal{T}^\alpha\partial_i^{\widetilde{\varphi}} f = \partial_i^{\widetilde{\varphi}}(\mathcal{T}^\alpha f - \partial_3^{\widetilde{\varphi}} f\mathcal{T}^\alpha\widetilde{\varphi}) + \underbrace{\partial_3^{\widetilde{\varphi}}\partial_i^{\widetilde{\varphi}} f\mathcal{T}^\alpha\widetilde{\varphi} + \mathfrak{C}_i'(f)}_{:=\mathfrak{C}_i(f)}. \tag{4.36}$$

The quantity $\mathcal{T}^\alpha f - \partial_3^{\widetilde{\varphi}} f\mathcal{T}^\alpha\widetilde{\varphi}$ is the so-called Alinhac good unknown associated with f. It was first observed by Alinhac [6] that the top order term of $\widetilde{\varphi}$ does not appear when we use the above good unknown. It is not hard to see that we can obtain the control of $\|\mathcal{T}^\alpha f\|_0$ from that of $\|\mathcal{T}^\alpha f - \partial_3^{\widetilde{\varphi}} f\mathcal{T}^\alpha\widetilde{\varphi}\|_0$. In particular,

$$\|\mathcal{T}^\alpha f\|_0 \le \|\mathcal{T}^\alpha f - \partial_3^{\widetilde{\varphi}} f\mathcal{T}^\alpha\widetilde{\varphi}\|_0 + \|\partial_3^{\widetilde{\varphi}} f\|_\infty\|\mathcal{T}^\alpha\widetilde{\varphi}\|_0. \tag{4.37}$$

In addition to this, we need to commute $\mathcal{T}^\alpha$ with

$$D_t^{\widetilde{\varphi}} = \partial_t + \overline{v} \cdot \overline{\nabla} + \frac{1}{\partial_3\widetilde{\varphi}}(v \cdot \widetilde{\mathbf{N}} - \partial_t\varphi)\partial_3.$$

A direct computation yields

$$\begin{aligned}\mathcal{T}^\alpha D_t^{\widetilde{\varphi}} f &= \mathcal{T}^\alpha\partial_t f + \mathcal{T}^\alpha(\overline{v} \cdot \overline{\partial} f) + \mathcal{T}^\alpha\left(\frac{1}{\partial_3\widetilde{\varphi}}(v \cdot \widetilde{\mathbf{N}} - \partial_t\varphi)\partial_3 f\right)\\ &= D_t^{\widetilde{\varphi}}\mathcal{T}^\alpha f + (v \cdot \mathcal{T}^\alpha\widetilde{\mathbf{N}} - \partial_t\mathcal{T}^\alpha\varphi)\partial_3^{\widetilde{\varphi}} f - \partial_3^{\widetilde{\varphi}}\mathcal{T}^\alpha\widetilde{\varphi}(v \cdot \widetilde{\mathbf{N}} - \partial_t\varphi)\partial_3^{\widetilde{\varphi}} f + \mathfrak{D}'(f),\end{aligned} \tag{4.38}$$

where

$$
\begin{aligned}
\mathfrak{D}'(f) &= [\mathcal{T}^\alpha, \overline{v}]\cdot\overline{\partial} f + \left[\mathcal{T}^\alpha, \frac{1}{\partial_3\widetilde{\varphi}}(v\cdot\widetilde{\mathbf{N}}-\partial_t\varphi), \partial_3 f\right] + \left[\mathcal{T}^\alpha, v\cdot\widetilde{\mathbf{N}}-\partial_t\varphi, \frac{1}{\partial_3\widetilde{\varphi}}\right]\partial_3 f + \frac{1}{\partial_3\widetilde{\varphi}}[\mathcal{T}^\alpha, v]\cdot\widetilde{\mathbf{N}}\partial_3 f \\
&-(v\cdot\widetilde{\mathbf{N}}-\partial_t\varphi)\partial_3 f\left[\overline{\partial}^{\alpha-\gamma}, \frac{1}{(\partial_3\widetilde{\varphi})^2}\right]\mathcal{T}^\gamma\partial_3\widetilde{\varphi} + \frac{1}{\partial_3\widetilde{\varphi}}(v\cdot\widetilde{\mathbf{N}}-\partial_t\varphi)[\mathcal{T}^\alpha,\partial_3]f + (v\cdot\widetilde{\mathbf{N}}-\partial_t\varphi)\frac{\partial_3 f}{(\partial_3\widetilde{\varphi})^2}[\mathcal{T}^\alpha,\partial_3]\widetilde{\varphi},
\end{aligned}
\tag{4.39}
$$

with $|\gamma| = 1$.

Since $v\cdot\mathcal{T}^\alpha\widetilde{\mathbf{N}} = -v_1\partial_1\mathcal{T}^\alpha\widetilde{\varphi} - v_2\partial_2\mathcal{T}^\alpha\widetilde{\varphi}$, then we must have

$$
\begin{aligned}
&(v\cdot\mathcal{T}^\alpha\widetilde{\mathbf{N}} - \partial_t\mathcal{T}^\alpha\varphi)\partial_3^{\widetilde{\varphi}} f - \partial_3^{\widetilde{\varphi}}\mathcal{T}^\alpha\widetilde{\varphi}(v\cdot\widetilde{\mathbf{N}}-\partial_t\varphi)\partial_3^{\widetilde{\varphi}} f \\
=&(v\cdot\mathcal{T}^\alpha\widetilde{\mathbf{N}} - \partial_t\mathcal{T}^\alpha\widetilde{\varphi})\partial_3^{\widetilde{\varphi}} f - \partial_3^{\widetilde{\varphi}}\mathcal{T}^\alpha\widetilde{\varphi}(v\cdot\widetilde{\mathbf{N}}-\partial_t\varphi)\partial_3^{\widetilde{\varphi}} f + \partial_t\mathcal{T}^\alpha(\widetilde{\varphi}-\varphi)\partial_3^{\widetilde{\varphi}} f \\
=&-\partial_3^{\widetilde{\varphi}} f\left(\partial_t + \overline{v}\cdot\overline{\nabla} + (v\cdot\widetilde{\mathbf{N}}-\partial_t\varphi)\partial_3^{\widetilde{\varphi}}\right)\mathcal{T}^\alpha\widetilde{\varphi} + \underbrace{\partial_t\mathcal{T}^\alpha(\widetilde{\varphi}-\varphi)\partial_3^{\widetilde{\varphi}} f}_{:=\mathfrak{E}(f)} \\
=&-\partial_3^{\widetilde{\varphi}} f D_t^{\widetilde{\varphi}}\mathcal{T}^\alpha\widetilde{\varphi} + \mathfrak{E}(f).
\end{aligned}
\tag{4.40}
$$

Thus,

$$
\begin{aligned}
\mathcal{T}^\alpha D_t^{\widetilde{\varphi}} f =& D_t^{\widetilde{\varphi}}\mathcal{T}^\alpha f - \partial_3^{\widetilde{\varphi}} f D_t^{\widetilde{\varphi}}\mathcal{T}^\alpha\widetilde{\varphi} + \mathfrak{D}'(f) + \mathfrak{E}(f) \\
=& D_t^{\widetilde{\varphi}}\left(\mathcal{T}^\alpha f - \partial_3^{\widetilde{\varphi}} f\mathcal{T}^\alpha\widetilde{\varphi}\right) + \mathfrak{D}(f) + \mathfrak{E}(f),
\end{aligned}
\tag{4.41}
$$

where $\mathfrak{D}(f) = (D_t^{\widetilde{\varphi}}\partial_3^{\widetilde{\varphi}} f)\mathcal{T}^\alpha\widetilde{\varphi} + \mathfrak{D}'(f)$.

Let

$$
\mathbf{V}_i := \mathcal{T}^\alpha v_i - \partial_3^{\widetilde{\varphi}} v_i\mathcal{T}^\alpha\widetilde{\varphi}, \quad \mathbf{Q} := \mathcal{T}^\alpha\check{q} - \partial_3^{\widetilde{\varphi}}\check{q}\mathcal{T}^\alpha\widetilde{\varphi}
\tag{4.42}
$$

respectively be the Alinhac good unknowns of v and $\check{q}$. Motivated by (4.36) and (4.41), we take $\mathcal{T}^\alpha$ to the first two equations of (1.5) to obtain

$$
\rho D_t^{\widetilde{\varphi}}\mathbf{V}_i + \partial_i^{\widetilde{\varphi}}\mathbf{Q} = \mathcal{R}_i^1,
\tag{4.43}
$$

$$
\mathcal{F}'(q)D_t^{\widetilde{\varphi}}\mathbf{Q} + \nabla^{\widetilde{\varphi}}\cdot\mathbf{V} = \mathcal{R}^2 - \mathfrak{C}_i(v^i),
\tag{4.44}
$$

where

$$
\mathcal{R}_i^1 := -[\mathcal{T}^\alpha,\rho]D_t^{\widetilde{\varphi}} v_i - \rho\left(\mathfrak{D}(v_i) + \mathfrak{E}(v_i)\right) - \mathfrak{C}_i(\check{q}),
\tag{4.45}
$$

$$
\mathcal{R}^2 := -[\mathcal{T}^\alpha,\mathcal{F}'(q)]D_t^{\widetilde{\varphi}}\check{q} - \mathcal{F}'(q)\left(\mathfrak{D}(\check{q}) + \mathfrak{E}(\check{q})\right) + \mathcal{T}^\alpha(\mathcal{F}'(q)gv_3).
\tag{4.46}
$$

In addition, since $\mathcal{T}^\alpha$ reduces to $\overline{\partial}^\alpha$ on Σ and $\overline{\partial}^\alpha\widetilde{N} = (-\partial_1\overline{\partial}^\alpha\widetilde{\psi}, -\partial_2\overline{\partial}^\alpha\widetilde{\psi}, 0)^\top$, the $\overline{\partial}^\alpha$-differentiated kinematic boundary condition then reads

$$
\partial_t\overline{\partial}^\alpha\psi + (\overline{v}\cdot\overline{\nabla})\overline{\partial}^\alpha\widetilde{\psi} - \mathbf{V}\cdot\widetilde{N} = \mathcal{S}_1 \quad \text{on } \Sigma, \quad \text{ and } \mathbf{V}_3 = 0 \quad \text{on } \Sigma_b,
\tag{4.47}
$$

where

$$
\mathcal{S}_1 := \partial_3 v\cdot\widetilde{N}\overline{\partial}^\alpha\widetilde{\psi} + \sum_{\substack{|\beta_1|+|\beta_2|=4\\|\beta_1|,|\beta_2|>0}}\overline{\partial}^{\beta_1}v\cdot\overline{\partial}^{\beta_2}\widetilde{N}.
\tag{4.48}
$$

Also, since $\check{q} = q + g\widetilde{\psi}$ and $\partial_3\widetilde{\psi} = 1$ on Σ, we have $\mathbf{Q}|_\Sigma = \overline{\partial}^\alpha\check{q} - \partial_3\check{q}\overline{\partial}^\alpha\widetilde{\psi} = \overline{\partial}^\alpha q + g\overline{\partial}^\alpha\widetilde{\psi} - \partial_3\check{q}\overline{\partial}^\alpha\widetilde{\psi} = \overline{\partial}^\alpha q - \partial_3 q\overline{\partial}^\alpha\widetilde{\psi}$, and thus the boundary condition of $\mathbf{Q}$ on Σ reads:

$$
\mathbf{Q} = -\sigma\overline{\partial}^\alpha\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\widetilde{\psi}}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}}\right) + \kappa^2(1-\overline{\Delta})\overline{\partial}^\alpha(v\cdot\widetilde{N}) - \partial_3 q\overline{\partial}^\alpha\widetilde{\psi}.
\tag{4.49}
$$

4.5 Tangential energy estimate with full spatial derivatives

In this subsection we study the spatially-differentiated equations, i.e., the equations obtained by commuting $\mathcal{T}^\alpha$, $\alpha_0 = 0$, and $|\alpha| = 4$, with (3.11). We aim to prove the following estimate

Proposition 4.3. For $\mathcal{T}^\alpha$ with multi-index α satisfying $\alpha_0 = 0$ and $|\alpha| = 4$, we have the energy inequality for $T > 0$ and $\kappa > 0$:

$$\|\mathcal{T}^\alpha v(T)\|_0^2 + \left\| \sqrt{\mathcal{F}'(q)}\mathcal{T}^\alpha \check{q}(T)\right\|_0^2 + \left| \sqrt{\sigma}\overline{\nabla}\overline{\partial}^\alpha \Lambda_\kappa \psi(T)\right|_0^2 + \int_0^T \left|\kappa\overline{\partial}^\alpha \partial_t \psi(t)\right|_1^2 \,\mathrm{d}t \lesssim \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t. \tag{4.50}$$

As mentioned in the last subsection, we will not directly consider the $\mathcal{T}^\alpha$-differentiated variables but use Alinhac good unknowns to get rid of higher order terms of $\widetilde{\psi}$. Invoking Lemma A.2 and Theorem A.3, testing (4.43) with $\mathbf{V}$ and then integrating over Ω with respect to the measure $\partial_3\widetilde{\varphi}\,\mathrm{d}x$, we get

$$\partial_t \frac{1}{2}\int_\Omega \rho|\mathbf{V}|^2\partial_3\widetilde{\varphi}\,\mathrm{d}x = \frac{1}{2}\int_\Omega \rho|\mathbf{V}|^2\partial_3\partial_t(\widetilde{\varphi}-\varphi)\,\mathrm{d}x + \int_\Omega \mathbf{Q}(\nabla^{\widetilde{\varphi}}\cdot\mathbf{V})\partial_3\widetilde{\varphi}\,\mathrm{d}x - \int_\Sigma \mathbf{Q}(\mathbf{V}\cdot\widetilde{N})\,\mathrm{d}x' + \int_\Omega \mathbf{V}\cdot\mathcal{R}^1\partial_3\widetilde{\varphi}\,\mathrm{d}x, \tag{4.51}$$

where the boundary integral on Σ_b vanishes thanks to $\mathbf{V}_3|_\Sigma = 0$. And from now on, we will no longer write any boundary integral on Σ_b due to the same reason. Before estimating the integrals in (4.51), we record some important properties that Alinhac good unknowns enjoy.

Lemma 4.4. Let $\mathbf{F} := \mathcal{T}^\alpha f - \partial_3^{\widetilde{\varphi}} f \mathcal{T}^\alpha\widetilde{\varphi}$ with $|\alpha| = 4$ and $\alpha_0 = 0$ be the Alinhac good unknowns associated with the smooth function f. Suppose that $\partial_3\widetilde{\varphi} \geq c_0 > 0$, then

$$\|\mathcal{T}^\alpha f\|_0 \leq \|\mathbf{F}\|_0 + P\left(c_0^{-1}, |\widetilde{\psi}|_4\right)\|\partial_3 f\|_\infty. \tag{4.52}$$

Furthermore, let $\mathfrak{C}(f)$, $\mathfrak{D}(f)$, and $\mathfrak{E}(f)$ be the remainder terms defined respectively in (4.36), (4.40), and (4.41). Then

$$\|\mathfrak{C}_i(f)\|_0 \leq P\left(c_0^{-1}, |\widetilde{\psi}|_4\right)\cdot\|f\|_4, \quad i = 1,2,3, \tag{4.53}$$

$$\|\mathfrak{D}(f)\|_0 \leq P\left(c_0^{-1}, |\widetilde{\psi}|_4, |\partial_t\widetilde{\psi}|_3\right)\cdot\left(\|f\|_4 + \|\partial_t f\|_3\right), \tag{4.54}$$

$$\|\mathfrak{E}(f)\|_0 \leq \kappa|\overline{\nabla}\mathcal{T}^\alpha\partial_t\psi|_0\|\partial f\|_\infty. \tag{4.55}$$

Proof. Since $\partial_3^{\widetilde{\varphi}} = (\partial_3\widetilde{\varphi})^{-1}\partial_3$, we have

$$\|\partial_3^{\widetilde{\varphi}} f\|_\infty\|\mathcal{T}^\alpha\widetilde{\varphi}\|_0 \leq P\left(c_0^{-1}, |\widetilde{\psi}|_4\right)\|\partial_3 f\|_\infty, \tag{4.56}$$

and so (4.52) follows from (4.37). Also, the estimates (4.53) and (4.54) follow from the definition of $\mathfrak{C}(f)$ and $\mathfrak{D}(f)$, (1.9), (3.7) in Lemma 3.1, and the Sobolev inequalities. To establish (4.55), we notice that

$$\|\mathfrak{E}(f)\|_0 \leq \|\partial_t\mathcal{T}^\alpha(\widetilde{\varphi}-\varphi)\|_0\|\partial_3^{\widetilde{\varphi}} f\|_\infty + \|\partial_3^{\widetilde{\varphi}}\mathcal{T}^\alpha\widetilde{\varphi}\|_0\|\partial_t(\widetilde{\varphi}-\varphi)\|_\infty\|\partial_3^{\widetilde{\varphi}} f\|_\infty.$$

Thus, it suffices to control the leading order terms $\|\partial_t\mathcal{T}^\alpha(\widetilde{\varphi}-\varphi)\|_0$ and $\|\partial_3^{\widetilde{\varphi}}\mathcal{T}^\alpha\widetilde{\varphi}\|_0$. We have

$$\begin{aligned}\partial_t\mathcal{T}^\alpha(\widetilde{\varphi}-\varphi) &= \partial_t\mathcal{T}^\alpha\left(\chi(x_3)\widetilde{\psi} - \chi(x_3)\psi\right)\\ &\leq \chi(x_3)\partial_t\overline{\partial}^\alpha(\widetilde{\psi}-\psi) + [\mathcal{T}^\alpha, \chi(x_3)]\,\partial_t(\widetilde{\psi}-\psi).\end{aligned}$$

The L^2-norm of the second term can be controlled by the RHS of (4.55) thanks to (1.9). By (3.6) in Lemma 3.1, we have

$$|\partial_t\overline{\partial}^\alpha(\widetilde{\psi}-\psi)|_0 \leq \kappa|\partial_t\psi|_5.$$

Also,

$$\partial_3^{\widetilde{\varphi}}\mathcal{T}^\alpha\widetilde{\varphi} = \partial_3^{\widetilde{\varphi}}\mathcal{T}^\alpha\left(\chi(x_3)\widetilde{\psi}\right) = \left(\partial_3^{\widetilde{\varphi}}\chi(x_3)\right)\mathcal{T}^\alpha\widetilde{\psi} + \left(\partial_3^{\widetilde{\varphi}}[\mathcal{T}^\alpha, \chi(x_3)]\right)\widetilde{\psi},$$

and so $\|\partial_3^{\widetilde{\varphi}}\mathcal{T}^\alpha\widetilde{\varphi}\|_0$ can be controlled by the RHS of (4.55). □

Remark. The appearance of $\mathfrak{E}(f)$ is a consequence of the tangential smoothing. This estimate of $\|\mathfrak{E}(f)\|_0$ yields a top order term $\kappa|\partial_t\psi|_5$, which can only be controlled by the energy contributed by the artificial viscosity. In other words, the artificial viscosity compensates for the loss of symmetry in the κ-equations.

4.5.1 Control of $\int_\Omega \rho|\mathbf{V}|^2\partial_3\partial_t(\widetilde{\varphi}-\varphi)\,\mathrm{d}x$: The integral contains the mismatched term.

We have

$$\int_\Omega \rho|\mathbf{V}|^2\partial_3\partial_t(\widetilde{\varphi}-\varphi)\,\mathrm{d}x \le \|\rho\|_\infty\|\mathbf{V}\|_0^2\|\partial_3\partial_t(\widetilde{\varphi}-\varphi)\|_\infty \lesssim \sqrt{\kappa}\|\mathbf{V}\|_0^2|\overline{\partial}\partial_t\psi|_{0.5}. \tag{4.57}$$

4.5.2 Control of $\int_\Omega \mathbf{V}\cdot\mathcal{R}^1\partial_3\widetilde{\varphi}\,\mathrm{d}x$: Error terms

We have

$$\int_\Omega \mathbf{V}\cdot\mathcal{R}^1\partial_3\widetilde{\varphi}\,\mathrm{d}x \le \|\mathbf{V}\|_0\|\mathcal{R}^1\|_0\|\partial_3\widetilde{\varphi}\|_\infty, \tag{4.58}$$

where the L^2 norm of $\mathcal{R}^1$ is directly controlled by using (4.46) and (4.53)-(4.55)

$$\|\mathcal{R}^1\|_0 \le P(\|\partial_3\widetilde{\varphi}\|_\infty, |\widetilde{\psi}|_4, |\partial_t\widetilde{\psi}|_3)\left(\kappa|\overline{\nabla}\mathcal{T}^\alpha\partial_t\psi|_0\|v\|_{\mathfrak{H}} + \|v\|_{\mathfrak{H}} + \|\partial_t v\|_3 + \|q\|_{\mathfrak{H}}\right), \tag{4.59}$$

where the term containing $\kappa|\overline{\nabla}\mathcal{T}^\alpha\partial_t\psi|_0$ should be controlled under time integral as we will get $L_t^2H_{x'}^1([0,T]\times\Sigma)$ bound for $\kappa\partial_t\mathcal{T}^\alpha\psi$ later.

4.5.3 Control of $\int_\Omega \mathbf{Q}(\nabla^{\widetilde{\varphi}}\cdot\mathbf{V})\partial_3\widetilde{\varphi}\,dx$: Tangential energy for Q

Equation (4.44) indicates

$$\int_\Omega \mathbf{Q}(\nabla^{\widetilde{\varphi}}\cdot\mathbf{V})\partial_3\widetilde{\varphi}\,\mathrm{d}x = -\int_\Omega \mathcal{F}'(q)\mathbf{Q}(D_t^{\widetilde{\varphi}}\mathbf{Q})\partial_3\widetilde{\varphi}\,\mathrm{d}x + \int_\Omega \mathbf{Q}(\mathcal{R}^2-\mathfrak{C}_i(v^i))\partial_3\widetilde{\varphi}\,\mathrm{d}x. \tag{4.60}$$

For the second term on the RHS of (4.60), we invoke the second inequality in (1.30) and then apply it to the definition of $\mathcal{R}^2$ in (4.46) to get:

$$\int_\Omega \mathbf{Q}\mathcal{R}^2\partial_3\widetilde{\varphi}\,\mathrm{d}x \le \|\sqrt{\mathcal{F}'(q)}\mathbf{Q}\|_0\|\mathcal{R}^2\|_0\|\partial_3\widetilde{\varphi}\|_\infty. \tag{4.61}$$

In other words, we "borrow" one $\sqrt{\mathcal{F}'(q)}$ from $\mathcal{R}^2$ and attach it to $\mathbf{Q}$. Thanks to (4.53)-(4.55), we control the L^2-norm of the rest of terms in $\mathcal{R}^2$ directly by

$$P(\|\partial_3\widetilde{\varphi}\|_\infty, |\widetilde{\psi}|_4, |\partial_t\widetilde{\psi}|_3)\left(\kappa|\overline{\nabla}\mathcal{T}^\alpha\partial_t\psi|_0\left\|\sqrt{\mathcal{F}'(q)}\check{q}\right\|_{\mathfrak{H}} + \left\|\sqrt{\mathcal{F}'(q)}\check{q}\right\|_{\mathfrak{H}} + \left\|\sqrt{\mathcal{F}'(q)}\partial_t\check{q}\right\|_3 + \left\|\sqrt{\mathcal{F}'(q)}gv_3\right\|_3\right), \tag{4.62}$$

where the term containing $\kappa|\overline{\nabla}\mathcal{T}^\alpha\partial_t\psi|_0$ should be controlled under time integral as we will get $L_t^2H_{x'}^1([0,T]\times\Sigma)$ bound for $\kappa\partial_t\mathcal{T}^\alpha\psi$ later. Then the contribution of $\mathfrak{C}_i(v^i)$ is controlled by

$$-\int_\Omega \mathbf{Q}(\mathfrak{C}_i(v^i))\partial_3\widetilde{\varphi}\,\mathrm{d}x \le P(|\widetilde{\psi}|_4, |\overline{\nabla}\widetilde{\psi}|_{W^{1,\infty}})|\mathcal{T}^\alpha\widetilde{\psi}|_0\|v\|_{\mathfrak{H}}\|\mathbf{Q}\|_0. \tag{4.63}$$

Here, $\|\mathbf{Q}\|_0$ contributes to $\|\overline{\partial}^\alpha\check{q}\|_0$ and $\|\partial_3^{\widetilde{\varphi}}\check{q}\overline{\partial}^\alpha\widetilde{\psi}\|_0$. The first term $\|\overline{\partial}^\alpha\check{q}\|_0$ is not weighted by $\sqrt{\mathcal{F}'(q)}$ and thus cannot be controlled directly by (4.50). Fortunately, we can overcome this issue by invoking (4.17). Similarly, $\|\partial_3^{\widetilde{\varphi}}\check{q}\overline{\partial}^\alpha\widetilde{\psi}\|_0 \le \|\partial_3^{\widetilde{\varphi}}\check{q}\|_\infty\|\overline{\partial}^\alpha\widetilde{\psi}\|_0$, where we use (4.20) to treat $\|\partial_3^{\widetilde{\varphi}}\check{q}\|_\infty$, and so this can be controlled uniformly as $\mathcal{F}'(q)\to 0$.

Furthermore, invoking the integration by parts formula (A.8), the first integral on the RHS of (4.60) becomes

$$\begin{aligned}\int_\Omega \mathcal{F}'(q)\mathbf{Q}(D_t^{\widetilde{\varphi}}\mathbf{Q})\partial_3\widetilde{\varphi}\,\mathrm{d}x =& -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathcal{F}'(q)|\mathbf{Q}|^2\partial_3\widetilde{\varphi}\,\mathrm{d}x + \frac{1}{2}\int_\Omega (D_t^{\widetilde{\varphi}}\mathcal{F}'(q))|\mathbf{Q}|^2\partial_3\widetilde{\varphi}\,\mathrm{d}x\\ &+ \frac{1}{2}\int_\Omega (\nabla^{\widetilde{\varphi}}\cdot v)\mathcal{F}'(q)|\mathbf{Q}|^2\partial_3\widetilde{\varphi}\,\mathrm{d}x + \frac{1}{2}\int_\Omega \mathcal{F}'(q)|\mathbf{Q}|^2\partial_3\partial_t(\widetilde{\varphi}-\varphi)\partial_3\widetilde{\varphi}\,\mathrm{d}x\\ \lesssim& -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\sqrt{\mathcal{F}'(q)}\mathbf{Q}\right\|_0^2 + \|\partial_3\widetilde{\varphi}\|_\infty\|\sqrt{\mathcal{F}'(q)}\mathbf{Q}\|_0^2\left(\|\partial v\|_\infty + \kappa|\overline{\nabla}\partial_t\psi|_{0.5}\right).\end{aligned} \tag{4.64}$$

4.5.4 Control of $-\int_\Sigma \mathbf{Q}(\mathbf{V}\cdot\widetilde{N})\,\mathrm{d}x'$: Boundary energy contributed by surface tension and artificial viscosity

Note that $\mathcal{T}_3=\vec{0}$ on Σ implies the corresponding good unknown $\mathbf{Q}=0$ on Σ, so it suffices to consider the case $\mathcal{T}^\alpha=\overline{\partial}^\alpha$ when analyzing the boundary integral. Using (4.47), we have

$$-\int_\Sigma \mathbf{Q}(\mathbf{V}\cdot\widetilde{N})\,\mathrm{d}x' = -\int_\Sigma \mathbf{Q}\left(\partial_t\overline{\partial}^\alpha\psi + (\overline{v}\cdot\overline{\nabla})\overline{\partial}^\alpha\widetilde{\psi} - \mathcal{S}_1\right)\mathrm{d}x'. \tag{4.65}$$

The first term is expected to give energy terms if we invoke the boundary condition (4.49) for $\mathbf{Q}$

$$\begin{aligned} I_1 := -\int_\Sigma \mathbf{Q}\partial_t\overline{\partial}^\alpha\psi\,\mathrm{d}x' &= \sigma\int_\Sigma \overline{\partial}^\alpha\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\widetilde{\psi}}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}}\right)\partial_t\overline{\partial}^\alpha\psi\,\mathrm{d}x' - \kappa^2\int_\Sigma \overline{\partial}^\alpha(1-\overline{\Delta})\partial_t\psi\cdot\overline{\partial}^\alpha\partial_t\psi\,\mathrm{d}x' + \int_\Sigma \partial_3 q\overline{\partial}^\alpha\widetilde{\psi}\partial_t\overline{\partial}^\alpha\psi\,\mathrm{d}x' \\ &=: \mathrm{ST}_1+\mathrm{ST}_2+\mathrm{RT}. \end{aligned} \tag{4.66}$$

Since $1-\overline{\Delta}=\langle\overline{\partial}\rangle^2$, where $\langle\cdot\rangle$ denotes the Japanese bracket, we find the term ST_2 gives us $\sqrt{\kappa}$-weighted enhanced energy after integration by parts :

$$\mathrm{ST}_2 = -\kappa^2\int_\Sigma \left|\overline{\partial}^\alpha\langle\overline{\partial}\rangle\partial_t\psi\right|^2\mathrm{d}x' = -\frac{\mathrm{d}}{\mathrm{d}t}\left|\kappa\overline{\partial}^\alpha\partial_t\psi\right|^2_{L^2_tH^1_{x'}}. \tag{4.67}$$

In the control of ST_1, we will repeatedly use

$$\overline{\partial}\left(\frac{1}{|\widetilde{N}|}\right) = \frac{\overline{\nabla}\widetilde{\psi}\cdot\overline{\partial}\overline{\nabla}\widetilde{\psi}}{|\widetilde{N}|^3}, \tag{4.68}$$

where $|\widetilde{N}|=\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}$ denotes the length of the smoothed normal vector $\widetilde{N}=(-\overline{\partial}_1\widetilde{\psi},-\overline{\partial}_2\widetilde{\psi},1)^\top$. Now we integrate $\overline{\nabla}\cdot$ by parts in ST_1 to get

$$\begin{aligned} \mathrm{ST}_1 =& -\sigma\int_\Sigma \frac{\overline{\partial}^\alpha\overline{\nabla}\widetilde{\psi}}{|\widetilde{N}|}\cdot\partial_t\overline{\partial}^\alpha\overline{\nabla}\psi\,\mathrm{d}x' + \sigma\int_\Sigma\frac{\overline{\nabla}\widetilde{\psi}\cdot\overline{\partial}^\alpha\overline{\nabla}\widetilde{\psi}}{|\widetilde{N}|^3}\overline{\nabla}\widetilde{\psi}\cdot\partial_t\overline{\partial}^\alpha\overline{\nabla}\psi\,\mathrm{d}x' \\ &-\sigma\int_\Sigma\left(\left[\overline{\partial}^{\alpha-\alpha'},\frac{1}{|\widetilde{N}|}\right]\overline{\partial}^{\alpha'}\overline{\nabla}\widetilde{\psi} + \left[\overline{\partial}^{\alpha-\alpha'},\frac{1}{|\widetilde{N}|^3}\right](\overline{\nabla}\psi\cdot\overline{\partial}^{\alpha'}\overline{\nabla}\widetilde{\psi}) - \frac{1}{|\widetilde{N}|^3}\left[\overline{\partial}^{\alpha-\alpha'},\overline{\nabla}\widetilde{\psi}\right]\overline{\partial}^{\alpha'}\overline{\nabla}\widetilde{\psi}\right)\cdot\partial_t\overline{\nabla}\overline{\partial}^\alpha\psi\,\mathrm{d}x' \\ =&: \mathrm{ST}_{11}+\mathrm{ST}_{12}+\mathrm{ST}_{13}, \end{aligned} \tag{4.69}$$

where α' is a multi-index with $|\alpha'|=1$.

The first two terms in (4.69) are expected to produce the energy contributed by surface tension. Before that, we need to move one mollifier from the top order term of $\widetilde{\psi}=\Lambda_\kappa^2\psi$ to the top order term of ψ by using the self-adjointness of Λ_κ in $L^2(\Sigma)$.

$$\begin{aligned} \mathrm{ST}_{11}+\mathrm{ST}_{12} =& -\sigma\int_\Sigma \frac{\overline{\partial}^\alpha\overline{\nabla}\Lambda_\kappa\psi\cdot\partial_t\overline{\partial}^\alpha\overline{\nabla}\Lambda_\kappa\psi}{|\widetilde{N}|} - \frac{(\overline{\nabla}\widetilde{\psi}\cdot\overline{\partial}^\alpha\overline{\nabla}\Lambda_\kappa\psi)(\overline{\nabla}\widetilde{\psi}\cdot\partial_t\overline{\partial}^\alpha\overline{\nabla}\Lambda_\kappa\psi)}{|\widetilde{N}|^3}\,\mathrm{d}x' \\ &-\sigma\int_\Sigma \overline{\partial}^\alpha\overline{\nabla}\Lambda_\kappa\psi\cdot\left(\left[\Lambda_\kappa,\frac{1}{|\widetilde{N}|}\right]\overline{\nabla}\partial_t\overline{\partial}^\alpha\psi\right)\mathrm{d}x' + \sigma\int_\Sigma\overline{\partial}^\alpha\overline{\nabla}_i\Lambda_\kappa\psi\cdot\left(\left[\Lambda_\kappa,\frac{\overline{\nabla}_i\widetilde{\psi}\overline{\nabla}_j\widetilde{\psi}}{|\widetilde{N}|^3}\right]\overline{\nabla}_j\partial_t\overline{\partial}^\alpha\psi\right)\mathrm{d}x' \\ =&: \mathrm{ST}_{10}+\mathrm{ST}_{11}^R+\mathrm{ST}_{12}^R. \end{aligned} \tag{4.70}$$

Then we find

$$\mathrm{ST}_{10} = -\frac{\sigma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma \frac{|\overline{\partial}^\alpha\overline{\nabla}\Lambda_\kappa\psi|^2}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}} - \frac{|\overline{\nabla}\widetilde{\psi}\cdot\overline{\partial}^\alpha\overline{\nabla}\Lambda_\kappa\psi|^2}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}^3}\,\mathrm{d}x' \tag{4.71}$$

$$+\frac{\sigma}{2}\int_\Sigma \partial_t\left(\frac{1}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}}\right)\left|\overline{\partial}^\alpha\overline{\nabla}\Lambda_\kappa\psi\right|^2 - \partial_t\left(\frac{1}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}^3}\right)\left|\overline{\nabla}\widetilde{\psi}\cdot\overline{\partial}^\alpha\overline{\nabla}\Lambda_\kappa\psi\right|^2\,\mathrm{d}x'. \tag{4.72}$$

To deal with the first term in ST_{10}, we plug $\mathbf{a} = \overline{\partial}^\alpha \overline{\nabla}\Lambda_\kappa\psi$ into the following inequality which can be proved by direct calculation

$$\frac{|\mathbf{a}|^2}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}} - \frac{|\overline{\nabla}\widetilde{\psi}\cdot\mathbf{a}|^2}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}^3} \geq \frac{|\mathbf{a}|^2}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}^3} \tag{4.73}$$

in order to get

$$\int_0^T \mathrm{ST}_{10}\,\mathrm{d}t + \frac{\sigma}{2}\int_\Sigma \frac{|\overline{\partial}^\alpha\overline{\nabla}\Lambda_\kappa\psi|^2}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}^3}\,\mathrm{d}x' \leq P(|\overline{\nabla}\widetilde{\psi}_0|_{L^\infty})\left|\sqrt{\sigma}\overline{\partial}^\alpha\overline{\nabla}\Lambda_\kappa\psi_0\right|_0^2 + \int_0^T (4.72)\,\mathrm{d}t, \tag{4.74}$$

where the term (4.72) is directly controlled by the energy term

$$(4.72) \leq P(|\overline{\nabla}\widetilde{\psi}|_{L^\infty})|\partial_t\overline{\nabla}\widetilde{\psi}|_{L^\infty}\left|\sqrt{\sigma}\overline{\partial}^\alpha\overline{\nabla}\Lambda_\kappa\psi\right|_0^2. \tag{4.75}$$

To finish the control of ST_1, it remains to control ST_{13} and ST_{11}^R, ST_{12}^R. The last two terms can be controlled by using the mollifier property (3.10) and the κ-weighted energy contributed by the artificial viscosity. We only list the detail of ST_{11}^R and then ST_{12}^R follows in the same way:

$$\begin{aligned}\int_0^T \mathrm{ST}_{11}^R \lesssim{}& \int_0^T \left|\sqrt{\sigma}\overline{\partial}^\alpha\Lambda_\kappa\psi\right|_0 P\left(|\overline{\nabla}\widetilde{\psi}|_\infty\right)|\overline{\nabla}\widetilde{\psi}|_{W^{1,\infty}}\left|\kappa\partial_t\overline{\partial}^\alpha\psi\right|_{\dot{H}^1}\,\mathrm{d}t\\ \lesssim{}& \varepsilon\left|\kappa\partial_t\overline{\partial}^\alpha\psi\right|_{L_t^2H_{x'}^1}^2 + \int_0^T P\left(|\overline{\nabla}\widetilde{\psi}|_\infty\right)|\overline{\nabla}\widetilde{\psi}|_{W^{1,\infty}}^2\left|\sqrt{\sigma}\overline{\partial}^\alpha\Lambda_\kappa\psi\right|_0^2\,\mathrm{d}t.\end{aligned} \tag{4.76}$$

As for ST_{13} in (4.69), we find three commutators have similar structures and the same highest order terms, so we only show the analysis of the first commutator. We notice that, the top order term in $[\overline{\partial}^{\alpha-\alpha'}, |\widetilde{N}|^{-1}]\overline{\partial}^{\alpha'}\overline{\nabla}\widetilde{\psi}$ appears when $\overline{\partial}^{\alpha-\alpha'}$ falls on $|\widetilde{N}|^{-1}$ or $\overline{\partial}^{\alpha''}$ falls on $|\widetilde{N}|^{-1}$ and $\overline{\partial}^{\alpha-\alpha'-\alpha''}$ falls on $\overline{\partial}^{\alpha'}\overline{\nabla}\widetilde{\psi}$ for some $|\alpha''| = 1$. In either of the two cases, the top order term contributes to the following integral in ST_{13}

$$-\sigma\int_\Sigma |\widetilde{N}|^{-3}\overline{\nabla}\widetilde{\psi}\;\overline{\partial}^{\alpha-\alpha'}\overline{\nabla}\widetilde{\psi}\;\overline{\partial}^{\alpha'}\overline{\nabla}\widetilde{\psi}\cdot\partial_t\overline{\nabla}\overline{\partial}^\alpha\psi\,\mathrm{d}x'. \tag{4.77}$$

We integrate one $\overline{\nabla}$ by parts to get

$$\sigma\int_\Sigma |\widetilde{N}|^{-3}\overline{\nabla}\widetilde{\psi}\;\overline{\partial}^{\alpha-\alpha'}\overline{\nabla}^2\widetilde{\psi}\;\overline{\partial}^{\alpha'}\overline{\nabla}\widetilde{\psi}\;\partial_t\overline{\partial}^\alpha\psi\,\mathrm{d}x'$$

modulo lower order terms, and then we move one mollifier from $\overline{\partial}^{\alpha-\alpha'}\overline{\nabla}^2\widetilde{\psi}$ to $\partial_t\overline{\partial}^\alpha\psi$ such that the main term is directly controlled

$$\sigma\int_\Sigma |\widetilde{N}|^{-3}\overline{\nabla}\widetilde{\psi}\;\overline{\partial}^{\alpha-\alpha'}\overline{\nabla}^2\Lambda_\kappa\psi\;\overline{\partial}^{\alpha'}\overline{\nabla}\widetilde{\psi}\;\partial_t\overline{\partial}^\alpha\Lambda_\kappa\psi\,\mathrm{d}x' \lesssim P(|\overline{\nabla}\widetilde{\psi}|_\infty)|\overline{\nabla}\widetilde{\psi}|_{W^{1,\infty}}\left|\sqrt{\sigma}\overline{\partial}^\alpha\overline{\nabla}\Lambda_\kappa\psi\right|_0\left|\sqrt{\sigma}\partial_t\overline{\partial}^\alpha\Lambda_\kappa\psi\right|_0, \tag{4.78}$$

where the last term will be controlled in $\partial_t\overline{\partial}^3$-estimates. Besides, we have to analyze the commutator involving Λ_κ:

$$\sigma\int_\Sigma \overline{\partial}^{\alpha-\alpha'}\overline{\nabla}^2\Lambda_\kappa\psi\left(\left[\Lambda_\kappa, P(\overline{\nabla}\widetilde{\psi})\overline{\partial}^{\alpha'}\overline{\nabla}\widetilde{\psi}\right]\partial_t\overline{\partial}^\alpha\psi\right)\mathrm{d}x', \tag{4.79}$$

which is controlled by the following terms after using (3.10)

$$\begin{aligned}\int_0^T (4.79)\,\mathrm{d}t \lesssim{}& \sqrt{\sigma}\int_0^T |\sqrt{\sigma}\overline{\nabla}\overline{\partial}^\alpha\Lambda_\kappa\psi|_0\cdot\kappa|\overline{\partial}\overline{\nabla}\widetilde{\psi}|_{W^{1,\infty}}P(|\overline{\nabla}\widetilde{\psi}|_{W^{1,\infty}})|\partial_t\overline{\partial}^\alpha\psi|_0\,\mathrm{d}t\\ \lesssim{}& \varepsilon\left|\kappa\partial_t\overline{\partial}^\alpha\psi\right|_{L_t^2L_{x'}^2}^2 + \int_0^T\left|\sqrt{\sigma}\overline{\nabla}\overline{\partial}^\alpha\Lambda_\kappa\psi\right|_0^2|\sqrt{\sigma}\overline{\nabla}\widetilde{\psi}|_{3.5}^2P(|\overline{\nabla}\widetilde{\psi}|_{2.5})\,\mathrm{d}t.\end{aligned} \tag{4.80}$$

Therefore, the terms ST_1, ST_2 have the following bound for any multi-index α with $|\alpha| = 4$

$$\int_0^T (\mathrm{ST}_1 + \mathrm{ST}_2)\,\mathrm{d}t + \left|\kappa\overline{\partial}^\alpha\partial_t\psi\right|_{L_t^2H_{x'}^1}^2 + \frac{\sigma}{2}\left|\overline{\nabla}\overline{\partial}^\alpha\Lambda_\kappa\psi(T)\right|_0^2 \lesssim \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t, \tag{4.81}$$

where we have chosen $\varepsilon > 0$ that appears above to be suitably small such that all ε-terms are absorbed by the κ-weighted energy.

To finish the control of I_1 defined in (4.66), it remains to control the term RT. Note that when we drop the mollifier and have the Rayleigh-Taylor sign condition $-\partial_3 q \geq \frac{c_0}{2} > 0$ on Σ, RT should directly give us the non-σ-weighted boundary energy. But we are now solving the gravity-capillary water wave system for fixed $\sigma > 0$ instead of taking vanishing surface tension limit, so we cannot assume $-\partial_3 q \geq \frac{c_0}{2} > 0$ on Σ. Thus this term is controlled by the surface tension energy after moving one mollifier

$$\begin{aligned}\int_0^T \mathrm{RT}\,\mathrm{d}t =& -\int_0^T\int_\Sigma \partial_3 q\overline{\partial}^\alpha\Lambda_\kappa\psi\cdot\partial_t\overline{\partial}^\alpha\Lambda_\kappa\psi\,\mathrm{d}x'\,\mathrm{d}t - \int_0^T\int_\Sigma \overline{\partial}^\alpha\Lambda_\kappa\psi\cdot\left([\Lambda_\kappa,\partial_3 q]\,\partial_t\overline{\partial}^\alpha\psi\right)\mathrm{d}x'\,\mathrm{d}t\\ \lesssim& \int_0^T |\partial q|_{L^\infty}\left|\overline{\partial}^\alpha\Lambda_\kappa\psi\right|_0\left|\partial_t\overline{\partial}^\alpha\Lambda_\kappa\psi\right|_0\mathrm{d}t + \varepsilon\left|\kappa\partial_t\overline{\partial}^\alpha\psi\right|^2_{L^2_tL^2_{x'}} + \int_0^T |\partial q|^2_{W^{1,\infty}}\left|\overline{\partial}^\alpha\Lambda_\kappa\psi\right|_0^2\mathrm{d}t\\ \lesssim& \,\varepsilon\left|\kappa\partial_t\overline{\partial}^\alpha\psi\right|^2_{L^2_tL^2_{x'}} + \int_0^T P\left(\|q\|_{\mathfrak{H}}, \left|\overline{\partial}^\alpha\Lambda_\kappa\psi\right|_0, \left|\partial_t\overline{\partial}^\alpha\Lambda_\kappa\psi\right|_0\right)\mathrm{d}t,\end{aligned} \tag{4.82}$$

where the term $\left|\partial_t\overline{\partial}^\alpha\Lambda_\kappa\psi\right|_0$ is the energy term obtained in $\overline{\partial}^{\alpha-\alpha'}\partial_t$-estimates for $|\alpha'| = 1$.

Remark. The RHS of (4.82) is not uniform in σ. However, as mentioned earlier, $-\int_0^T\int_\Sigma \partial_3 q\overline{\partial}^\alpha\Lambda_\kappa\psi\cdot\partial_t\overline{\partial}^\alpha\Lambda_\kappa\psi\,\mathrm{d}x'\,\mathrm{d}t$ contributes to a non-σ-weighted energy term $\int_\Sigma(-\partial_3 q)|\overline{\partial}^\alpha\Lambda_\kappa\psi|^2\,\mathrm{d}t$ provided the Rayleigh-Taylor sign condition holds. We shall revisit the control of RT in Section 7, where the zero surface tension limit is considered.

Combining this with (4.81), we get the estimate for I_1

$$\int_0^T I_1\,\mathrm{d}t + \left|\kappa\overline{\partial}^\alpha\partial_t\psi\right|^2_{L^2_tH^1_{x'}} + \frac{\sigma}{2}\left|\overline{\nabla}\overline{\partial}^\alpha\Lambda_\kappa\psi(T)\right|^2_0 \lesssim \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t, \tag{4.83}$$

after choosing $\varepsilon > 0$ that appears above to be suitably small.

The second term in (4.65) gives

$$\begin{aligned}I_2 :=& -\int_\Sigma \mathbf{Q}(\overline{v}\cdot\overline{\nabla})\overline{\partial}^\alpha\widetilde{\psi} = \sigma\int_\Sigma \overline{\partial}^\alpha\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\widetilde{\psi}}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}}\right)(\overline{v}\cdot\overline{\nabla})\overline{\partial}^\alpha\widetilde{\psi}\,\mathrm{d}x' - \kappa^2\int_\Sigma \overline{\partial}^\alpha(1-\overline{\Delta})\partial_t\psi\cdot(\overline{v}\cdot\overline{\nabla})\overline{\partial}^\alpha\widetilde{\psi}\,\mathrm{d}x'\\ &+ \int_\Sigma \partial_3 q\overline{\partial}^\alpha\widetilde{\psi}(\overline{v}\cdot\overline{\nabla})\overline{\partial}^\alpha\widetilde{\psi}\,\mathrm{d}x'\\ =&: I_{21} + I_{22} + I_{23},\end{aligned} \tag{4.84}$$

where we find that I_{22}, I_{23} can be directly controlled as follows:

$$\begin{aligned}\int_0^T I_{22}\,\mathrm{d}t \overset{\overline{\nabla}}{=}& -\kappa^2\int_0^T\int_\Sigma \overline{\partial}^\alpha\overline{\nabla}\partial_t\psi\cdot\overline{\nabla}\left((\overline{v}\cdot\overline{\nabla})\overline{\partial}^\alpha\widetilde{\psi}\right)\mathrm{d}x'\,\mathrm{d}t - \kappa^2\int_0^T\int_\Sigma \overline{\partial}^\alpha\partial_t\psi\cdot(\overline{v}\cdot\overline{\nabla})\overline{\partial}^\alpha\widetilde{\psi}\,\mathrm{d}x'\,\mathrm{d}t\\ \lesssim& \int_0^T \left|\kappa\overline{\partial}^\alpha\partial_t\psi\right|_0|\overline{\nabla}\overline{v}|_\infty\left|\kappa\overline{\nabla}^2\overline{\partial}^\alpha\widetilde{\psi}\right|_0\mathrm{d}t + \kappa\int_0^T\left|\kappa\overline{\partial}^\alpha\partial_t\psi\right|_0|\overline{v}|_\infty\left|\overline{\nabla}\overline{\partial}^\alpha\widetilde{\psi}\right|_0\mathrm{d}t\\ \lesssim& \,\varepsilon\left|\kappa\overline{\partial}^\alpha\partial_t\psi\right|^2_{L^2_tH^1_{x'}} + \int_0^T |\overline{v}|^2_{W^{1,\infty}}\left|\overline{\nabla}\overline{\partial}^\alpha\Lambda_\kappa\psi\right|^2_0\mathrm{d}t \lesssim \varepsilon\left|\kappa\overline{\partial}^\alpha\partial_t\psi\right|^2_{L^2_tH^1_{x'}} + \int_0^T P(E^\kappa(t))\,\mathrm{d}t,\end{aligned} \tag{4.85}$$

where we use the mollifier property (3.4) to control $|\kappa\overline{\nabla}^2\overline{\partial}^\alpha\widetilde{\psi}|_0 \lesssim \kappa\cdot\kappa^{-1}|\overline{\nabla}\overline{\partial}^\alpha\Lambda_\kappa\psi|_0$. This step also shows that why the power of κ must be 2 in the artificial viscosity, otherwise the control of I_{22} is not uniform in κ. For I_{23} we integrate $\overline{v}\cdot\overline{\nabla}$ by parts to get

$$I_{23} = \frac{1}{2}\int_\Sigma \overline{\nabla}\cdot(\overline{v}\,\partial_3 q)|\overline{\partial}^\alpha\widetilde{\psi}|^2\,\mathrm{d}x' \lesssim P(E^\kappa(t)). \tag{4.86}$$

The control of I_{21} is analogous to ST_1. Following (4.69) we have

$$\begin{aligned}I_{21} =& -\sigma\int_\Sigma\left(\frac{\overline{\partial}^\alpha\overline{\nabla}\widetilde{\psi}}{|\widetilde{N}|} - \frac{\overline{\nabla}\widetilde{\psi}\cdot\overline{\partial}^\alpha\overline{\nabla}\widetilde{\psi}}{|\widetilde{N}|^3}\overline{\nabla}\widetilde{\psi}\right)\cdot(\overline{v}\cdot\overline{\nabla})\overline{\partial}^\alpha\overline{\nabla}\widetilde{\psi}\,\mathrm{d}x'\\ &-\sigma\int_\Sigma\left(\left[\overline{\partial}^{\alpha-\alpha'}, \frac{1}{|\widetilde{N}|}\right]\overline{\partial}^{\alpha'}\overline{\nabla}\widetilde{\psi} + \left[\overline{\partial}^{\alpha-\alpha'}, \frac{1}{|\widetilde{N}|^3}\right](\overline{\nabla}\psi\cdot\overline{\partial}^{\alpha'}\overline{\nabla}\widetilde{\psi}) - \frac{1}{|\widetilde{N}|^3}\left[\overline{\partial}^{\alpha-\alpha'}, \overline{\nabla}\widetilde{\psi}\right]\overline{\partial}^{\alpha'}\overline{\nabla}\widetilde{\psi}\right)\cdot(\overline{v}\cdot\overline{\nabla})\overline{\nabla}\overline{\partial}^\alpha\widetilde{\psi}\,\mathrm{d}x'\\ =&: I_{211} + I_{212},\end{aligned} \tag{4.87}$$

where I_{212} can be directly controlled if we integrate $\overline{v} \cdot \overline{\nabla}$ by parts

$$I_{212} \lesssim P(|\widetilde{\psi}|_4)|\overline{v}|_{W^{1,\infty}} \left| \sqrt{\sigma}\overline{\nabla}\overline{\partial}^\alpha \widetilde{\psi} \right|_0^2 \leq P(E^\kappa(t)). \tag{4.88}$$

For I_{211}, we integrate $\overline{v} \cdot \overline{\nabla}$ by parts and use the symmetric structure to see

$$I_{211} \overset{L}{=} -\frac{\sigma}{2} \int_\Sigma (\overline{\nabla} \cdot \overline{v}) \left(\frac{|\overline{\partial}^\alpha \overline{\nabla} \widetilde{\psi}|^2}{|\widetilde{N}|} - \frac{|\overline{\nabla}\widetilde{\psi} \cdot \overline{\partial}^\alpha \overline{\nabla}\widetilde{\psi}|^2}{|\widetilde{N}|^3} \right) \mathrm{d}x' \lesssim P(|\overline{\nabla}\widetilde{\psi}|_\infty)|\overline{v}|_{W^{1,\infty}} \left| \sqrt{\sigma}\overline{\nabla}\overline{\partial}^\alpha \widetilde{\psi} \right|_0^2. \tag{4.89}$$

Therefore, plugging (4.85)-(4.89) into (4.84), we get the estimates for I_2

$$\int_0^T I_2 \,\mathrm{d}t \lesssim \varepsilon \left| \kappa \overline{\partial}^\alpha \partial_t \psi \right|_{L_t^2 H_{x'}^1}^2 + \int_0^T P(E^\kappa(t)). \tag{4.90}$$

It remains to control the term involving $\mathcal{S}_1$ which reads

$$\begin{aligned} I_3 &:= \int_\Sigma \mathbf{Q}\mathcal{S}_1 \,\mathrm{d}x' = \int_\Sigma \mathbf{Q} \left(\partial_3 v \cdot \widetilde{N} \overline{\partial}^\alpha \widetilde{\psi} + \sum_{\substack{|\beta_1|+|\beta_2|=4 \\ |\beta_1|,|\beta_2|>0}} \overline{\partial}^{\beta_1} v \cdot \overline{\partial}^{\beta_2} \widetilde{N} \right) \mathrm{d}x' \\ &= \int_\Sigma \left(\sigma \overline{\partial}^\alpha \mathcal{H} + \kappa^2 (1 - \overline{\Delta}) \overline{\partial}^\alpha \partial_t \psi - \partial_3 q \overline{\partial}^\alpha \widetilde{\psi} \right) \left(\partial_3 v \cdot \widetilde{N} \overline{\partial}^\alpha \widetilde{\psi} + \sum_{|\beta_1|=1, |\beta_2|=3} \overline{\partial}^{\beta_1} v \cdot \overline{\partial}^{\beta_2} \widetilde{N} \right) \mathrm{d}x' \\ &\quad + \sum_{\substack{|\beta_1|+|\beta_2|=4 \\ |\beta_1|\geq 1, 1\leq |\beta_2| \leq 2}} \int_\Sigma (\overline{\partial}^\alpha q - \overline{\partial}^\alpha \widetilde{\psi} \partial_3 q)(\overline{\partial}^{\beta_1} v \cdot \overline{\partial}^{\beta_2} \widetilde{N}) \,\mathrm{d}x' \\ &=: I_{31} + I_{32}, \end{aligned} \tag{4.91}$$

where we use the definition of $\mathbf{Q}$ in I_{32} and invoke the Dirichlet boundary condition (4.49) for $\mathbf{Q}$ in I_{31} such that the $L^2(\Sigma)$ bounds of $\overline{\partial}^\alpha v$ and non-weighted $\overline{\nabla}\overline{\partial}^\alpha \psi$ with $|\alpha| = 4$ can be avoided on Σ.

The term I_{32} can be directly controlled by

$$I_{32} \lesssim \sum_{\substack{|\beta_1|+|\beta_2|=4 \\ |\beta_1|\geq 1, 1\leq |\beta_2| \leq 2}} |\overline{\partial}^\alpha q|_{-\frac{1}{2}} \left| \overline{\partial}^{\beta_1} \overline{v} \cdot \overline{\nabla}^{\beta_2} \overline{\partial} \widetilde{\psi} \right|_{\frac{1}{2}} + \left| \overline{\partial}^\alpha \widetilde{\psi} \partial_3 q \right|_0 \left| \overline{\partial}^{\beta_1} \overline{v} \cdot \overline{\nabla}^{\beta_2} \overline{\partial} \widetilde{\psi} \right|_0 \lesssim \|q\|_4 \|v\|_4 |\widetilde{\psi}|_{3.5} + |\partial q|_{L^\infty} \|v\|_{3.5} |\widetilde{\psi}|_3 |\widetilde{\psi}|_4. \tag{4.92}$$

For I_{31}, we invoke $\mathcal{H} = -\overline{\nabla} \cdot (\overline{\nabla}\widetilde{\psi}/|\widetilde{N}|)$ and then integrate $\overline{\nabla}\cdot$ by parts in the mean curvature term and integrate one tangential derivative by parts in the viscosity term to get

$$I_{31} \lesssim P(|\overline{\nabla}\widetilde{\psi}|_\infty)|\partial v|_\infty \left(\left| \sqrt{\sigma}\overline{\nabla}\overline{\partial}^\alpha \widetilde{\psi} \right|_4^2 |\partial v|_\infty + \left| \kappa \overline{\partial}^\alpha \partial_t \psi \right|_1 \left| \kappa \overline{\nabla} \overline{\partial}^\alpha \widetilde{\psi} \right|_0 \right) + |\partial q|_{L^\infty} |\widetilde{\psi}|_4^2 |\overline{\partial} v|_\infty, \tag{4.93}$$

and thus yields

$$\int_0^T I_{31} \,\mathrm{d}t \lesssim \varepsilon \left| \kappa \overline{\partial}^\alpha \partial_t \psi \right|_{L_t^2 H_{x'}^1}^2 + \int_0^T P(E^\kappa(t)) \,\mathrm{d}t, \tag{4.94}$$

which together with (4.92) gives the bound for I_3

$$\int_0^T I_3 \,\mathrm{d}t \leq \varepsilon \left| \kappa \overline{\partial}^\alpha \partial_t \psi \right|_{L_t^2 H_{x'}^1}^2 + \int_0^T P(E^\kappa(t)) \,\mathrm{d}t. \tag{4.95}$$

Combining (4.65), (4.66), (4.83), (4.84), (4.90), (4.91), (4.95), we get the estimates for the boundary integral after choosing $\varepsilon > 0$ suitably small

$$-\int_0^T \int_\Sigma \mathbf{Q}(\mathbf{V} \cdot \widetilde{N}) \,\mathrm{d}x' + \left| \kappa \overline{\partial}^\alpha \partial_t \psi \right|_{L_t^2 H_{x'}^1([0,T]\times\Sigma)}^2 + \frac{\sigma}{2} \left| \overline{\nabla} \overline{\partial}^\alpha \Lambda_\kappa \psi(T) \right|_0^2 \lesssim \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t)) \,\mathrm{d}t. \tag{4.96}$$

Plugging the estimates (4.57)-(4.60), (4.64) and (4.96) into (4.51) and using $\rho \gtrsim 1, \partial_3\widetilde{\varphi} \gtrsim 1$, we get the estimates for the good unknowns

$$\|\mathbf{V}(T)\|_0^2 + \left\| \sqrt{\mathcal{F}'(q)}\mathbf{Q}(T)\right\|_0^2 + \left| \sqrt{\sigma}\overline{\nabla}\overline{\partial}^\alpha \Lambda_\kappa \psi(T)\right|_0^2 + \left|\kappa\overline{\partial}^\alpha \partial_t\psi\right|^2_{L_t^2 H_{x'}^1([0,T]\times\Sigma)} \lesssim \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t. \tag{4.97}$$

Finally, using the definition $\mathbf{V} = \mathcal{T}^\alpha v - \mathcal{T}^\alpha\widetilde{\varphi}\partial_3^{\widetilde{\varphi}} v$, we can replace $\|\mathbf{V}\|_0$ by $\|\mathcal{T}^\alpha v\|_0$ because their difference, namely $\mathcal{T}^\alpha\widetilde{\varphi}\partial_3^{\widetilde{\varphi}} v$, is bounded by $\mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t$. Indeed, using $\widetilde{\varphi}(t,x) = x_3 + \chi(x_3)\widetilde{\psi}(t,x')$ we only need to investigate the case $\mathcal{T} = \overline{\partial}$ because the weighted derivative $\mathcal{T} = \omega(x_3)\partial_3$ only falls on $\chi(x_3)$ and x_3 instead of $\widetilde{\psi}$. So we have $\|\overline{\partial}^\alpha\widetilde{\varphi}\|_0 \lesssim |\overline{\partial}^\alpha\widetilde{\psi}|_0$ which is already bounded by the surface tension energy and thus by $\mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t$ according to (4.97). Since $\|\partial_3^{\widetilde{\varphi}} v\|_\infty \leq \|v\|_3\|\partial_3\widetilde{\varphi}\|_\infty \leq \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t$, we have

$$\|\mathcal{T}^\alpha v(T)\|_0^2 + \left\| \sqrt{\mathcal{F}'(q)}\mathcal{T}^\alpha \check{q}(T)\right\|_0^2 + \left| \sqrt{\sigma}\overline{\nabla}\overline{\partial}^\alpha \Lambda_\kappa \psi(T)\right|_0^2 + \int_0^T \left|\kappa\overline{\partial}^\alpha \partial_t\psi(t)\right|_1^2 \,\mathrm{d}t \lesssim \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t. \tag{4.98}$$

We remark here that we can employ the same analysis to prove the tangential estimates with mixed spatial-time derivatives.

Proposition 4.5. Let α be the multi-index satisfying $1 \leq \alpha_0 \leq 3$ and $|\alpha| = 4$, we have the energy inequality for $T > 0$ and $\kappa > 0$:

$$\|\mathcal{T}^\alpha v(T)\|_0^2 + \left\| \sqrt{\mathcal{F}'(q)}\mathcal{T}^\alpha \check{q}(T)\right\|_0^2 + \left| \sqrt{\sigma}\overline{\nabla}\overline{\partial}^\alpha \Lambda_\kappa \psi(T)\right|_0^2 + \int_0^T |\kappa\mathcal{T}^\alpha \partial_t\psi(t)|_1^2 \,\mathrm{d}t \lesssim \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t. \tag{4.99}$$

4.6 Tangential energy estimate with time derivatives

In this subsection, we study the time-differentiated equations, i.e., the equations obtained by commuting ∂_t^4 with (3.11). We aim to prove the following estimates

Proposition 4.6. We have the energy inequality for $T > 0$ and $\kappa > 0$

$$\|\partial_t^4 v(T)\|_0^2 + \left\| \sqrt{\mathcal{F}'(q)}\partial_t^4 \check{q}(T)\right\|_0^2 + \left| \sqrt{\sigma}\overline{\nabla}\partial_t^4 \Lambda_\kappa \psi(T)\right|_0^2 + \int_0^T \left|\kappa\partial_t^5 \psi(t)\right|_1^2 \,\mathrm{d}t \lesssim \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t. \tag{4.100}$$

Although the proof appears to be similar to what has been done in the previous subsection, it should be mentioned that we only have $L^2(\Omega)$ regularity for the full-time derivatives of v and q, and thus we do not have any information about their boundary regularity. When the full-time derivatives of v and q appear on the boundary, we use either the artificial viscosity or Euler equations to reduce a time derivative to a spatial derivative.

4.6.1 Alinhac good unknowns for full-time derivatives

To begin with, we still introduce the Alinhac good unknowns of v, q with respect to ∂_t^4. Using the same notation as before, we define

$$\mathbf{V}_i := \partial_t^4 v_i - \partial_3^{\widetilde{\varphi}} v_i \partial_t^4\widetilde{\varphi}, \quad \mathbf{Q} := \partial_t^4 \check{q} - \partial_3^{\widetilde{\varphi}}\check{q}\partial_t^4\widetilde{\varphi}. \tag{4.101}$$

Similarly, as Section 4.4, we have

$$\partial_t^4(\nabla_i^{\widetilde{\varphi}} f) = \nabla_i^{\widetilde{\varphi}}\mathbf{F} + \mathfrak{C}_i(f), \tag{4.102}$$

where $\mathfrak{C}_i(f) := \partial_3^{\widetilde{\varphi}}\partial_i^{\widetilde{\varphi}} f\partial_t^4\widetilde{\varphi} + \mathfrak{C}_i'(f)$ and

$$\mathfrak{C}_i'(f) = -\left[\partial_t^4, \frac{\partial_i\widetilde{\varphi}}{\partial_3\widetilde{\varphi}}, \partial_3 f\right] - \partial_3 f\left[\partial_t^4, \partial_i\widetilde{\varphi}, \frac{1}{\partial_3\widetilde{\varphi}}\right] + \partial_i\widetilde{\varphi}\partial_3 f\left[\partial_t^3, \frac{1}{(\partial_3\widetilde{\varphi})^2}\right]\partial_t\partial_3\widetilde{\varphi}, \quad i = 1,2 \tag{4.103}$$

$$\mathfrak{C}_3'(f) = \left[\partial_t^4, \frac{1}{\partial_3\widetilde{\varphi}}, \partial_3 f\right] + \partial_3 f\left[\partial_t^3, \frac{1}{(\partial_3\widetilde{\varphi})^2}\right]\partial_t\partial_3\widetilde{\varphi}. \tag{4.104}$$

Then we take ∂_t^4 to the first two equations of (1.5) to obtain

$$\rho D_t^{\widetilde{\varphi}} \mathbf{V}_i + \nabla_i^{\widetilde{\varphi}} \mathbf{Q} = \mathcal{R}_i^1, \tag{4.105}$$
$$\mathcal{F}'(q) D_t^{\widetilde{\varphi}} \mathbf{Q} + \nabla^{\widetilde{\varphi}} \cdot \mathbf{V} = \mathcal{R}^2 - \mathfrak{C}^i(v_i), \tag{4.106}$$

where

$$\mathcal{R}_i^1 := -[\partial_t^4, \rho] D_t^{\widetilde{\varphi}} v_i - \rho\left(\mathfrak{D}(v_i) + \mathfrak{E}(v_i)\right) - \mathfrak{C}_i(\check{q}), \tag{4.107}$$
$$\mathcal{R}^2 := -[\partial_t^4, \mathcal{F}'(q)] D_t^{\widetilde{\varphi}} \check{q} - \mathcal{F}'(q)\left(\mathfrak{D}(\check{q}) + \mathfrak{E}(\check{q})\right) + \partial_t^4(\mathcal{F}'(q) g v_3), \tag{4.108}$$

and the commutators $\mathfrak{D}(f), \mathfrak{E}(f)$ are defined in the same way as in (4.39) and (4.40) by replacing $\mathcal{T}^\alpha$ with ∂_t^4 and replacing $\overline{\partial}$ with ∂_t. The last two terms in (4.39) vanish because ∂_t^4 directly commutes with ∂_3. Analogous to Lemma 4.4, we list the estimates for commutators $\mathfrak{C}, \mathfrak{D}, \mathfrak{E}$.

Lemma 4.7. Let $\mathbf{F} := \partial_t^4 f - \partial_3^{\widetilde{\varphi}} f \partial_t^4 \widetilde{\varphi}$ be the Alinhac good unknowns of f with respect to ∂_t^4. Assuming that $\partial_3 \widetilde{\varphi} \geq c_0 > 0$, then

$$\|\partial_t^4 f\|_0 \leq \|\mathbf{F}\|_0 + c_0^{-1} \|\partial_3 f\|_\infty |\partial_t^4 \widetilde{\psi}|_0, \tag{4.109}$$
$$\|\mathfrak{C}_i(f)\|_0 \leq P\left(c_0^{-1}, |\overline{\nabla}\widetilde{\psi}|_\infty, \sum_{k=1}^3 |\overline{\nabla}\partial_t^k \widetilde{\psi}|_{3-k}\right) \cdot \left(\|\partial f\|_\infty + \sum_{k=1}^3 \|\partial_t^k f\|_{4-k}\right), \quad i = 1,2,3, \tag{4.110}$$
$$\|\mathfrak{D}(f)\|_0 \leq P\left(c_0^{-1}, |\overline{\nabla}\widetilde{\psi}|_\infty, \sum_{k=1}^3 |\overline{\nabla}\partial_t^k \widetilde{\psi}|_{3-k}\right) \cdot \left(\|\partial f\|_\infty + \sum_{k=1}^3 \|\partial_t^k f\|_{4-k}\right), \tag{4.111}$$
$$\|\mathfrak{E}(f)\|_0 \leq \kappa |\overline{\nabla}\partial_t^5 \psi|_0 \|\partial f\|_\infty. \tag{4.112}$$

The ∂_t^4-differentiated kinematic boundary condition now reads

$$\partial_t^5 \psi + (\overline{v} \cdot \overline{\nabla}) \overline{\partial}^4 \widetilde{\psi} - \mathbf{V} \cdot \widetilde{N} = \mathcal{S}_1^*, \quad \text{on } \Sigma, \tag{4.113}$$

where

$$\mathcal{S}_1^* := \partial_3 v \cdot \widetilde{N} \partial_t^4 \widetilde{\psi} + \sum_{1 \leq \beta \leq 3} \binom{4}{\beta} \partial_t^\beta v \cdot \partial_t^{4-\beta} \widetilde{N}. \tag{4.114}$$

Also, since $\mathbf{Q}|_\Sigma = \partial_t^4 q - \partial_3^{\widetilde{\varphi}} q \partial_t^4 \widetilde{\psi}$, the boundary condition of $\mathbf{Q}$ on Σ reads

$$\mathbf{Q} = -\sigma \partial_t^4 \overline{\nabla} \cdot \left(\frac{\overline{\nabla}\widetilde{\psi}}{\sqrt{1 + |\overline{\nabla}\widetilde{\psi}|^2}} \right) + \kappa^2 (1 - \overline{\Delta}) \partial_t^5 \psi - \partial_3 q \partial_t^4 \widetilde{\psi}. \tag{4.115}$$

4.6.2 Energy estimates for the full-time derivatives

Replacing $\mathcal{T}^\alpha$ by ∂_t^4 in (4.51), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_\Omega \rho |\mathbf{V}|^2 \partial_3 \widetilde{\varphi} \, \mathrm{d}x = \frac{1}{2} \int_\Omega \rho |\mathbf{V}|^2 \partial_3 \partial_t (\widetilde{\varphi} - \varphi) \, \mathrm{d}x + \int_\Omega \mathbf{Q} (\nabla^{\widetilde{\varphi}} \cdot \mathbf{V}) \partial_3 \widetilde{\varphi} \, \mathrm{d}x - \int_\Sigma \mathbf{Q} (\mathbf{V} \cdot \widetilde{N}) \, \mathrm{d}x' + \int_\Omega \mathbf{V} \cdot \mathcal{R}^1 \partial_3 \widetilde{\varphi} \, \mathrm{d}x., \tag{4.116}$$

where the first term and the last term are controlled in the same way as (4.57)-(4.59), so we omit the details. As for the second term, we follow (4.60)-(4.64) to get

$$
\begin{aligned}
&\int_\Omega \mathbf{Q}(\nabla^{\widetilde{\varphi}}\cdot\mathbf{V})\partial_3\widetilde{\varphi}\,\mathrm{d}x\\
=&\underbrace{-\int_\Omega \partial_t^4\check{q}\,\mathfrak{C}_i(v^i)\partial_3\widetilde{\varphi}\,\mathrm{d}x}_{=:I_0^*}+\int_\Omega \partial_t^4\widetilde{\varphi}\,\partial_3^{\widetilde{\varphi}}\check{q}\,\mathfrak{C}_i(v^i)\partial_3\widetilde{\varphi}\,\mathrm{d}x-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\sqrt{\mathcal{F}'(q)}\mathbf{Q}\right\|_0^2\\
&+\left\|\sqrt{\mathcal{F}'(q)}\mathbf{Q}\right\|_0^2(\|\partial v\|_\infty+\kappa|\overline{\nabla}\partial_t\psi|_{0.5})+\left\|\sqrt{\mathcal{F}'(q)}\mathbf{Q}\right\|_0\|\mathcal{R}^2\|_0\\
\lesssim& I_0^*-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\sqrt{\mathcal{F}'(q)}\mathbf{Q}\right\|_0^2+\left\|\sqrt{\mathcal{F}'(q)}\partial_t^4\check{q}\right\|_0^2\left(\|\partial v\|_\infty+\kappa|\overline{\nabla}\partial_t\psi|_{0.5}+|\kappa\overline{\nabla}\partial_t^5\psi|_0\right)\\
&+P\left(c_0^{-1},|\overline{\nabla}\widetilde{\psi}|_\infty,\sum_{k=1}^3|\overline{\nabla}\partial_t^k\widetilde{\psi}|_{3-k}\right)|\partial_t^4\widetilde{\psi}|_0\left\|\sqrt{\mathcal{F}'(q)}\partial_t^4\check{q}\right\|_0\left(\|\partial v,\partial q\|_\infty+\sum_{k=1}^3\|\partial_t^k\check{q},\partial_t^k v\|_{4-k}+\|\mathcal{F}'(q)\partial_t^4 v_3\|_0\right).
\end{aligned}
\tag{4.117}
$$

At this point, we do not control $I_0^* := -\int_\Omega \partial_t^4 q\,\mathfrak{C}_i(v^i)\partial_3\widetilde{\varphi}\,\mathrm{d}x$ as in (4.63), because this requires the bound for $\|\partial_t^4 q\|_0$. We can obtain the estimate of $\|\sqrt{\mathcal{F}'(q)}\partial_t^4 q\|_0$ only as we can no longer use the momentum equation to reduce $\partial_t^4 q$ due to lack of spatial derivatives. The method in (4.63) is still valid here when we prove the well-posedness while $\mathcal{F}'(q)$ is bounded from below. However, we would like to show that our estimate can be adjusted to be uniform in $\mathcal{F}'(q)$. To achieve this, we find that the problematic terms in $\mathfrak{C}_i(v^i)$ can be exactly canceled by the boundary error term $\mathcal{S}_1$ defined in (4.114). Therefore, this term should be controlled together with the boundary integral if we want our energy estimates to be uniform in sound speed.

Next, we analyze the boundary integral. Most of the steps are parallel to Section 4.5.4 if we replace $\overline{\partial}^\alpha$ by ∂_t^4, so we will omit the details of those repeated steps but only list the different steps. Plugging the boundary conditions (4.113) and (4.115) into $-\int_\Sigma \mathbf{Q}(\mathbf{V}\cdot\widetilde{N})\,\mathrm{d}x'$, we get

$$
-\int_\Sigma \mathbf{Q}(\mathbf{V}\cdot\widetilde{N})\,\mathrm{d}x'=-\int_\Sigma \mathbf{Q}\partial_t^5\psi\,\mathrm{d}x'-\int_\Sigma \mathbf{Q}(\overline{v}\cdot\overline{\nabla})\partial_t^4\widetilde{\psi}\,\mathrm{d}x'+\int_\Sigma \mathbf{Q}\mathcal{S}_1^*\,\mathrm{d}x'=:I_1^*+I_2^*+I_3^*,
\tag{4.118}
$$

and I_1^* is further divided into three parts

$$
\begin{aligned}
I_1^*:=-\int_\Sigma \mathbf{Q}\partial_t^5\psi\,\mathrm{d}x'=&\,\sigma\int_\Sigma \partial_t^4\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\widetilde{\psi}}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}}\right)\partial_t^5\psi\,\mathrm{d}x'-\kappa^2\int_\Sigma \partial_t^4(1-\overline{\Delta})\partial_t\psi\cdot\partial_t^5\psi\,\mathrm{d}x'+\int_\Sigma \partial_3 q\,\partial_t^4\widetilde{\psi}\,\partial_t^5\psi\,\mathrm{d}x'\\
=:&\ \mathrm{ST}_1^*+\ \mathrm{ST}_2^*+\ \mathrm{RT}^*.
\end{aligned}
\tag{4.119}
$$

Mimicing the steps (4.67)-(4.81), we can get the bounds for $\mathrm{ST}_1^*,\ \mathrm{ST}_2^*$

$$
\int_0^T \mathrm{ST}_1^*+\ \mathrm{ST}_2^*\,\mathrm{d}t+\left|\kappa\partial_t^5\psi\right|^2_{L_t^2H_{x'}^1}+\frac{\sigma}{2}\left|\overline{\nabla}\partial_t^4\Lambda_\kappa\psi(T)\right|_0^2\lesssim\mathcal{P}_0^\kappa+\int_0^T P(E^\kappa(t))\,\mathrm{d}t.
\tag{4.120}
$$

Remark. Parallel to the remark after (4.82), $-\int_0^T \mathrm{RT}^*\,\mathrm{d}t$ would contribute to the non-σ-weighted energy $\int_\Sigma(-\partial_3 q)|\partial_t^4\Lambda_\kappa\psi|^2\,\mathrm{d}t$ if the Rayleigh-Taylor sign condition holds. This will be revisited in Section 7.

As for RT^*, if we still follow (4.82) to get:

$$
\int_0^T \mathrm{RT}^*\,\mathrm{d}t\lesssim\varepsilon\left|\kappa\partial_t^5\psi\right|^2_{L_t^2L_{x'}^2}+\int_0^T P\left(\|q\|_{\mathfrak{H}},\left|\partial_t^4\Lambda_\kappa\psi\right|_0,\left|\partial_t^5\Lambda_\kappa\psi\right|_0\right)\mathrm{d}t,
$$

then we find that the term $|\partial_t^5\Lambda_\kappa\psi|_0$ is not included in $E^\kappa(t)$ because there is no spatial derivative here. To overcome this, we invoke the kinematic boundary condition $\partial_t\psi=-\overline{v}\cdot\overline{\nabla}\widetilde{\psi}+v_3$ and take ∂_t^4 to get

$$
\partial_t^5\psi=-(\overline{v}\cdot\overline{\nabla})\partial_t^4\widetilde{\psi}+\partial_t^4 v_3-[\partial_t^4,\overline{v}\cdot]\overline{\nabla}\widetilde{\psi}=-(\overline{v}\cdot\overline{\nabla})\partial_t^4\widetilde{\psi}+\partial_t^4 v\cdot\widetilde{N}-[\partial_t^4,\overline{v}\cdot,\overline{\nabla}\widetilde{\psi}],
\tag{4.121}
$$

and thus

$$\begin{aligned}\mathrm{RT}^* &= -\int_\Sigma \partial_3 q \partial_t^4 \widetilde{\psi} (\overline{v}\cdot\overline{\nabla})\partial_t^4\widetilde{\psi}\,\mathrm{d}x' + \int_\Sigma \partial_3 q\partial_t^4\widetilde{\psi}\partial_t^4 v\cdot\widetilde{N}\,\mathrm{d}x' - \int_\Sigma \partial_3 q\partial_t^4\widetilde{\psi}[\partial_t^4,\overline{v}\cdot,\overline{\nabla}\widetilde{\psi}]\,\mathrm{d}x' \\ &=:\ \mathrm{RT}_1^* +\ \mathrm{RT}_2^* +\ \mathrm{RT}_3^*.\end{aligned} \tag{4.122}$$

Note that we only need to analyze the contribution of RT_2^* because the contribution of the other two terms will be canceled by part of I_2^* and I_3^*. To do this, we need to derive the equation for $\partial_t^4\cdot\widetilde{N}$ on Σ. Recall that

$$D_t^{\widetilde{\varphi}}\big|_\Sigma = \partial_t + (\overline{v}\cdot\overline{\nabla}) + (\partial_3\widetilde{\psi})^{-1}\underbrace{(v\cdot\widetilde{\mathbf{N}}-\partial_t\varphi)}_{=0\text{ on }\Sigma}\partial_3 = \partial_t + (\overline{v}\cdot\overline{\nabla}),$$

we have the following identity by projecting the momentum equation onto the direction of $\widetilde{N}$ on Σ

$$\rho\partial_t v\cdot\widetilde{N} = -(\rho-1)g - \rho(\overline{v}\cdot\overline{\nabla})v\cdot\widetilde{N} + \overline{\nabla}\widetilde{\psi}\cdot\overline{\nabla}\check{q} - (1+|\overline{\nabla}\widetilde{\psi}|^2)\partial_3\check{q},$$

and thus

$$\rho\partial_t^4 v\cdot\widetilde{N} \overset{L}{=} -\partial_t^3\rho g - \rho(\overline{v}\cdot\overline{\nabla})\partial_t^3 v\cdot\widetilde{N} + \overline{\nabla}\widetilde{\psi}\cdot\overline{\nabla}\partial_t^3\check{q} - |\widetilde{N}|^2\partial_3\partial_t^3\check{q}, \tag{4.123}$$

where $\overset{L}{=}$ means the omitted terms are of lower order. The contribution of the first three terms in (4.123) can be directly controlled after integrating $\overline{\nabla}$ by parts and using the Sobolev trace lemma

$$\begin{aligned}&\int_\Sigma \rho^{-1}\partial_3 q\partial_t^4\widetilde{\psi}\left(\rho(\overline{v}\cdot\overline{\nabla})\partial_t^3 v\cdot\widetilde{N} + \overline{\nabla}\widetilde{\psi}\cdot\overline{\nabla}\partial_t^3\check{q} - \partial_t^3\rho g\right)\,\mathrm{d}x' \\ \overset{L}{=}& -\int_\Sigma \rho^{-1}\overline{\nabla}\partial_t^4\widetilde{\psi}\cdot\left(\partial_3 q\left(\rho\overline{v}\partial_t^3 v\cdot\widetilde{N} + \overline{\nabla}\widetilde{\psi}\partial_t^3\check{q}\right)\right)\,\mathrm{d}x' - \int_\Sigma \rho^{-1}\partial_3 q\partial_t^4\widetilde{\psi}\partial_t^3\rho g\,\mathrm{d}x' \\ \lesssim& \|\partial q\|_2\left(\left|\overline{\nabla}\partial_t^4\widetilde{\psi}\right|_0 P\left(\|\partial_t^3 v\|_1, \|\partial_t^3\check{q}\|_1, |\overline{\nabla}\widetilde{\psi}|_\infty\right) + |\partial_t^4\widetilde{\psi}|_0\|\mathcal{F}'(q)\partial_t^3 q\|_1|\rho|_\infty\right).\end{aligned} \tag{4.124}$$

Remark. Note that the right side of (4.124) involves $|\overline{\nabla}\partial_t^4\widetilde{\psi}|_0$ whose control relies on σ^{-1}. This is due to the lacking of the Rayleigh-Taylor sign condition. When taking the zero surface tension limit, the Rayleigh-Taylor sign condition is assumed and thus the RT term can be directly controlled.

Then for the last term, we need to do the same reduction for $\partial_t^4\psi$

$$\partial_t^4\psi = -(\overline{v}\cdot\overline{\nabla})\partial_t^3\widetilde{\psi} + \partial_t^3 v_3 - [\partial_t^3,\overline{v}\cdot]\overline{\nabla}\widetilde{\psi} = -(\overline{v}\cdot\overline{\nabla})\partial_t^3\widetilde{\psi} + \partial_t^3 v\cdot\widetilde{N} - [\partial_t^3,\overline{v}\cdot,\overline{\nabla}\widetilde{\psi}]. \tag{4.125}$$

Using (4.125) and Sobolev trace lemma, it is controlled by

$$|\partial_t^4\widetilde{\psi}|_0 \lesssim P(|\overline{\nabla}\widetilde{\psi}|_\infty, |\overline{\nabla}\partial_t\widetilde{\psi}|_\infty)\left(|\overline{\nabla}\partial_t^3\widetilde{\psi}|_0 + \|\partial_t^3 v\|_1 + \|\partial_t^2 v\|_2 + |\overline{\nabla}\partial_t^2\widetilde{\psi}|_0\right). \tag{4.126}$$

Now we plug the equality above into the boundary integral $-\int_\Sigma \rho^{-1}\partial_3 q|\widetilde{N}|^2\partial_t^4\widetilde{\psi}\partial_3\partial_t^3\check{q}\,\mathrm{d}x'$. Note that the unit exterior normal vector to Σ is $(0,0,1)^\top$ (not the Eulerian normal vector $\widetilde{N}$!), we can use the divergence theorem to rewrite the boundary integral into the interior, and integrate by parts in ∂_t to get the following estimate

$$\begin{aligned}&-\int_0^T\int_\Sigma \rho^{-1}\partial_3 q|\widetilde{N}|^2\partial_t^4\widetilde{\psi}\partial_3\partial_t^3\check{q}\,\mathrm{d}x'\,\mathrm{d}t \overset{L}{=} \int_0^T\int_\Sigma \rho^{-1}\partial_3 q|\widetilde{N}|^2\Lambda_\kappa^2\left((\overline{v}\cdot\overline{\nabla})\partial_t^3\widetilde{\psi} - \partial_t^3 v\cdot\widetilde{N}\right)\partial_3\partial_t^3\check{q}\,\mathrm{d}x'\,\mathrm{d}t \\ =& \int_0^T\int_\Omega \partial_3\left(\rho^{-1}\partial_3 q|\widetilde{\mathbf{N}}|^2\Lambda_\kappa^2\left((\overline{v}\cdot\overline{\nabla})\partial_t^3\widetilde{\varphi} - \partial_t^3 v\cdot\widetilde{\mathbf{N}}\right)\partial_3\partial_t^3\check{q}\right)\,\mathrm{d}x\,\mathrm{d}t \\ \overset{L}{=}& \int_0^T\int_\Omega \rho^{-1}\partial_3 q|\widetilde{\mathbf{N}}|^2\Lambda_\kappa^2\left((\overline{v}\cdot\overline{\nabla})\partial_t^3\widetilde{\varphi} - \partial_t^3 v\cdot\widetilde{\mathbf{N}}\right)\cdot\partial_3^2\partial_t^3\check{q}\,\mathrm{d}x\,\mathrm{d}t \\ \overset{\partial_t}{=}& -\int_\Omega \rho^{-1}\partial_3 q|\widetilde{\mathbf{N}}|^2\Lambda_\kappa^2\left((\overline{v}\cdot\overline{\nabla})\partial_t^3\widetilde{\varphi} - \partial_t^3 v\cdot\widetilde{\mathbf{N}}\right)\cdot\partial_3^2\partial_t^2\check{q}\,\mathrm{d}x \\ &+ \int_0^T\int_\Omega \rho^{-1}\partial_3 q|\widetilde{\mathbf{N}}|^2\partial_t\Lambda_\kappa^2\left((\overline{v}\cdot\overline{\nabla})\partial_t^3\widetilde{\varphi} - \partial_t^3 v\cdot\widetilde{\mathbf{N}}\right)\cdot\partial_3^2\partial_t^2\check{q}\,\mathrm{d}x\,\mathrm{d}t \\ \lesssim& \varepsilon\|\partial_t^2\partial^2\check{q}\|_0^2 + \mathcal{P}_0^\kappa + \int_0^T P\left(\|\partial_t^4 v\|_0, \|\partial_t^3 v\|_1, \|\partial_t v\|_\infty, \|\partial_t^2\check{q}\|_2, |\overline{\nabla}\widetilde{\psi}|_\infty, |\overline{\nabla}\partial_t^3\widetilde{\psi}|_0, |\overline{\nabla}\partial_t^4\widetilde{\psi}|_0\right)\,\mathrm{d}t.\end{aligned} \tag{4.127}$$

Combining this with (4.120), (4.122), (4.124) and (4.127), we get the estimate for I_1^*

$$\int_0^T I_1^*\,\mathrm{d}t + \left|\kappa\partial_t^5\psi\right|^2_{L_t^2H_{x'}^1} + \frac{\sigma}{2}\left|\overline{\nabla}\partial_t^4\Lambda_\kappa\psi(T)\right|_0^2 \lesssim \varepsilon\|\partial_t^2\partial^2\check{q}\|_0^2 + \int_0^T \mathrm{RT}_1^* + \mathrm{RT}_3^*\,\mathrm{d}t + \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t, \tag{4.128}$$

after choosing $\varepsilon > 0$ that appears above to be suitably small.

Next we expand I_2^*, I_3^* defined in (4.118)

$$\begin{aligned} I_2^* + I_3^* = &-\int_\Sigma \partial_t^4\check{q}(\overline{v}\cdot\overline{\nabla})\partial_t^4\widetilde{\psi}\,\mathrm{d}x' + \int_\Sigma \partial_t^4\widetilde{\psi}\partial_3 q(\overline{v}\cdot\overline{\nabla})\partial_t^4\widetilde{\psi}\,\mathrm{d}x' \\ &+ \int_\Sigma \partial_t^4\check{q}\mathcal{S}_1\,\mathrm{d}x' - \int_\Sigma \partial_t^4\widetilde{\psi}\partial_3 q\partial_3 v\cdot\widetilde{N}\partial_t^4\widetilde{\psi}\,\mathrm{d}x' - \int_\Sigma \partial_t^4\widetilde{\psi}\partial_3 q\left(\sum_{1\leq\beta\leq 3}\binom{4}{\beta}\partial_t^\beta v\cdot\partial_t^{4-\beta}\widetilde{N}\right)\mathrm{d}x' \end{aligned} \tag{4.129}$$

and we find that the second term exactly cancels RT_1^* and the fifth term exactly cancels RT_3^* defined in (4.122). The first term can be controlled in the same way as I_{21}, I_{22} defined in (4.84) after replacing $\overline{\partial}^4$ by ∂_t^4. The fourth term is directly controlled by $P(E^\kappa(t))$ by using the Sobolev trace lemma.

Hence, it suffices to analyze the third term. Using the definition of $\mathcal{S}_1^*$, we have

$$\int_\Sigma \partial_t^4\check{q}\mathcal{S}_1^*\,\mathrm{d}x' = \int_\Sigma \partial_t^4\check{q}\left(\partial_3 v\cdot\widetilde{N}\partial_t^4\widetilde{\psi}\right)\mathrm{d}x' - 4\int_\Sigma \partial_t^4\check{q}\partial_t^3 v\cdot\partial_t\widetilde{N}\,\mathrm{d}x' + \sum_{1\leq\beta\leq 2}\binom{4}{\beta}\int_\Sigma \partial_t^4\check{q}\partial_t^\beta v\cdot\overline{\nabla}\partial_t^{4-\beta}\widetilde{\psi}\,\mathrm{d}x', \tag{4.130}$$

where the first term can be controlled by the surface tension energy after invoking (4.115) and integrating $\overline{\nabla}$ by parts; and the last term can be controlled after integrating by part in ∂_t under time integral. But for the remaining term

$$I_{30}^* := 4\int_\Sigma \partial_t^4\check{q}\partial_t^3 v\cdot\partial_t\widetilde{N}\,\mathrm{d}x', \tag{4.131}$$

we neither have anything about the boundary regularity of $\partial_t^4 q$ nor try to integrate $\langle\overline{\partial}\rangle^{-1/2}$ by parts as in the control of (4.91).

We can still control I_{30} *together with the interior term* $I_0^* := -\int_\Omega \partial_t^4 q\mathfrak{C}_i(v^i)\partial_3\widetilde{\varphi}\,\mathrm{d}x$ defined in (4.117). In fact, invoking (4.103) and (4.104), we know $\mathfrak{C}_i(v^i)$ includes the following terms involving ≥ 3 time derivatives of v^i and ≥ 4 derivatives of $\widetilde{\varphi}$:

$$\partial_3^{\widetilde{\varphi}}\partial_i^{\widetilde{\varphi}}v^i\partial_t^4\widetilde{\varphi} = \mathfrak{C}_i(v^i) - \mathfrak{C}_i'(v^i),\quad i = 1,2,3, \tag{4.132}$$

$$-4\partial_t\left(\frac{\overline{\partial}_i\widetilde{\varphi}}{\partial_3\widetilde{\varphi}}\right)\partial_t^3\partial_3 v^i = 4\partial_t\widetilde{\mathbf{N}}_i\partial_3^{\widetilde{\varphi}}\partial_t^3 v^i + 4\frac{\partial_3\partial_t\widetilde{\varphi}\overline{\partial}_i\widetilde{\varphi}}{\partial_3\widetilde{\varphi}}\partial_3^{\widetilde{\varphi}}\partial_t^3 v^i \text{ from the first commutator in } \mathfrak{C}_i'(v^i)\ \ i = 1,2, \tag{4.133}$$

$$4\partial_t\left(\frac{1}{\partial_3\widetilde{\varphi}}\right)\partial_t^3\partial_3 v^3 = -4\frac{\partial_3\partial_t\widetilde{\varphi}}{\partial_3\widetilde{\varphi}}\partial_3^{\widetilde{\varphi}}\partial_t^3 v^3 \text{ from the first commutator in } \mathfrak{C}_3'(v^3), \tag{4.134}$$

while the terms in $\mathfrak{C}_i'(v^i)$ containing only ≤ 2 time derivatives of v^i and ≤ 3 time derivatives of $\widetilde{\varphi}$ are controlled directly after integrating ∂_t by parts under time integral.

The contribution of the above four terms in I_0^* is divided into three parts

$$I_{00}^* := -4\int_\Omega \partial_t^4\check{q}\partial_t\widetilde{\mathbf{N}}_i\partial_3\partial_t^3 v^i\,\mathrm{d}x \tag{4.135}$$

$$I_{01}^* := -\int_\Omega \partial_t^4\check{q}\partial_3(\nabla^{\widetilde{\varphi}}\cdot v)\partial_t^4\widetilde{\varphi}\,\mathrm{d}x \tag{4.136}$$

$$I_{02}^* := -4\sum_{i=1}^2\int_\Omega \partial_t^4\check{q}\left(\frac{\partial_3\partial_t\widetilde{\varphi}\overline{\partial}_i\widetilde{\varphi}}{\partial_3\widetilde{\varphi}}\right)\partial_3^{\widetilde{\varphi}}\partial_t^3 v^i\partial_3\widetilde{\varphi}\,\mathrm{d}x + 4\int_\Omega \partial_t^4\check{q}\left(\frac{\partial_3\partial_t\widetilde{\varphi}}{\partial_3\widetilde{\varphi}}\right)\partial_3^{\widetilde{\varphi}}\partial_t^3 v^3\partial_3\widetilde{\varphi}\,\mathrm{d}x. \tag{4.137}$$

Integrating ∂_3 by parts in I_{00}^* and using $N_3 = 1$, we find the boundary term exactly cancels with I_{30}^*, so we have

$$\begin{aligned} I_{30}^* + I_{00}^* &= 4\int_\Omega\left(\partial_t^4\partial_3\check{q}\partial_t\widetilde{\mathbf{N}} + \partial_t^4\check{q}\partial_t\partial_3\widetilde{\mathbf{N}}\right)\cdot\partial_t^3 v\,\mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\partial_t^3\partial_3\check{q}\partial_t\widetilde{\mathbf{N}} + \partial_t^3 q\partial_t\partial_3\widetilde{\mathbf{N}}\right)\cdot\partial_t^3 v\,\mathrm{d}x + \int_\Omega \partial_t^3\partial_3\check{q}\partial_t(\partial_t\widetilde{\mathbf{N}}\cdot\partial_t^3 v) + \partial_t^3\check{q}\partial_t(\partial_t\partial_3\widetilde{\mathbf{N}}\cdot\partial_t^3 v)\,\mathrm{d}x, \end{aligned} \tag{4.138}$$

and thus under the time integral, we have the bounds after using ε-Young's inequality

$$\int_0^T I_{00}^* + I_{30}^* \,\mathrm{d}t \lesssim \varepsilon\|\partial_t^3\partial\check{q}\|_0^2 + \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t. \tag{4.139}$$

Next, the term I_{01}^* can be directly controlled if we insert the continuity equation $\nabla^{\widetilde{\varphi}}\cdot v = -\mathcal{F}'(q)D_t^{\widetilde{\varphi}}q$

$$I_{01}^* \lesssim \left\|\sqrt{\mathcal{F}'(q)}\partial_t^4\check{q}\right\|_0 \left\|\sqrt{\mathcal{F}'(q)}\partial_t q,\ \sqrt{\mathcal{F}'(q)}\partial q\right\|_{W^{1,\infty}} |\partial_t^4\widetilde{\psi}|_0. \tag{4.140}$$

As for I_{02}^*, we note that $-\overline{\partial}_i\widetilde{\varphi}\partial_3^{\widetilde{\varphi}}\partial_t^3 v^i = \partial_i^{\varphi}\partial_t^3 v^i - \overline{\partial}_i\partial_t^3 v_i$ for $i=1,2$. So it becomes

$$\begin{aligned} I_{02}^* &= 4\int_\Omega \partial_t^4\check{q}\partial_3\partial_t\widetilde{\varphi}(\nabla^{\widetilde{\varphi}}\cdot\partial_t^3 v)\,\mathrm{d}x - 4\sum_{i=1}^2\int_\Omega \partial_t^4\check{q}\partial_3\partial_t\widetilde{\varphi}\overline{\partial}_i\partial_t^3 v_i\,\mathrm{d}x \\ &\overset{L}{=} 4\int_\Omega \partial_t^4\check{q}\partial_t^3(\nabla^{\widetilde{\varphi}}\cdot v)\partial_3\partial_t\widetilde{\varphi}\,\mathrm{d}x - 4\sum_{i=1}^2\int_\Omega \partial_t^4\check{q}\partial_3\partial_t\widetilde{\varphi}\overline{\partial}_i\partial_t^3 v_i\,\mathrm{d}x, \end{aligned} \tag{4.141}$$

where the first term is controlled by $\left\|\sqrt{\mathcal{F}'(q)}\partial_t^4\check{q}\right\|_0\left(\left\|\sqrt{\mathcal{F}'(q)}\partial_t^4\check{q}\right\|_0 + \left\|\sqrt{\mathcal{F}'(q)}\partial_t^3\partial\check{q}\right\|_0 + \left\|\sqrt{\mathcal{F}'(q)}\partial_t^3 v_3\right\|_0\right)|\partial_t\widetilde{\psi}|_\infty$ after invoking the continuity equation; and the second term is controlled under time integral after integrating by parts ∂_t and then $\overline{\partial}_i$. So we have

$$\int_0^T I_{02}^*\,\mathrm{d}t \lesssim \varepsilon\|\partial_t^3\overline{\partial}\check{q}\|_0^2 + \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t. \tag{4.142}$$

Summarizing (4.116), (4.117), (4.120), (4.122), (4.128)-(4.131), (4.139), (4.140) and (4.142), we finally get the control of the Alinhac good unknowns $\mathbf{V}$ and $\mathbf{Q}$ with respect to ∂_t^4

$$\|\mathbf{V}(T)\|_0^2 + \left\|\sqrt{\mathcal{F}'(q)}\mathbf{Q}(T)\right\|_0^2 + \left|\sqrt{\sigma}\overline{\nabla}\partial_t^4\Lambda_\kappa\psi(T)\right|_0^2 + \int_0^T \left|\kappa\partial_t^5\psi\right|_1^2\,\mathrm{d}t \lesssim \varepsilon\left(\|\partial_t^2\partial^2\check{q}\|_0^2 + \|\partial_t^3\partial\check{q}\|_0^2\right) + \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t. \tag{4.143}$$

To recover the energy for $\|\partial_t^4 v\|_0^2$ and $\|\sqrt{\mathcal{F}'(q)}\partial_t^4\check{q}\|_0^2$, it suffices to invoke (4.109) and use the estimate of $|\partial_t^4\widetilde{\psi}|_0$ in (4.126). Note that the right side of (4.126) has been controlled in $\overline{\partial}^{4-k}\partial_t^k$-estimates for $k\le 3$, so we already have $|\partial_t^4\widetilde{\psi}|_0 \le \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t$ and thus

$$\|\partial_t^4 v(T)\|_0^2 + \left\|\sqrt{\mathcal{F}'(q)}\partial_t^4\check{q}(T)\right\|_0^2 + \left|\sqrt{\sigma}\overline{\nabla}\partial_t^4\Lambda_\kappa\psi(T)\right|_0^2 + \int_0^T \left|\kappa\partial_t^5\psi\right|_1^2\,\mathrm{d}t \lesssim \varepsilon\left(\|\partial_t^2\partial^2\check{q}\|_0^2 + \|\partial_t^3\partial\check{q}\|_0^2\right) + \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t. \tag{4.144}$$

4.7 A priori estimates for the nonlinear κ-approximate problem

Now we choose $\varepsilon > 0$ suitably small and then combine the tangential estimates (4.98) and (4.144) with div-curl analysis, reduction of pressure and L^2 estimates in Section 4.1 ~ Section 4.3 to get the following energy inequality

$$E^\kappa(T) \le E^\kappa(0) + \int_0^T P(E^\kappa(t))\,\mathrm{d}t. \tag{4.145}$$

Since the right side of the energy inequality does not rely on κ^{-1}, we can use Grönwall's inequality to prove that there exists some $T_0 > 0$ independent of $\kappa > 0$ such that

$$\sup_{0\le t\le T_0} E^\kappa(t) \le P(E^\kappa(0)). \tag{4.146}$$

We also note that the above energy estimate does not rely on $\mathcal{F}'(q)^{-1}$, as a special cancellation structure enjoyed by the Alinhac good unknowns and delicate analysis (4.130)-(4.142) exclude the only possibility that might make the energy estimates not uniform in Mach number. Therefore, our a priori bound is also uniform in Mach number.

5 Well-posedness of the nonlinear κ-approximate system

For the nonlinear κ-approximate problem (3.11), we have established the uniform-in-κ estimates. Once we prove the well-posedness of (3.11) for each *fixed* $\kappa > 0$, we can take the limit $\kappa \to 0$ to prove the local existence of the original system (1.24). We would use Picard iteration to construct the solution to (3.11) for each fixed $\kappa > 0$. Fix a $\kappa > 0$, we start with $(v^{(0)}, \rho^{(0)}, \psi^{(0)}) := (\mathbf{0}, 1, 0)$ and also define $\psi^{(-1)} := \psi^{(0)}$. Then we construct the solution by the following iteration scheme: For any $n \geq 0$, given $\{(v^{(k)}, \rho^{(k)}, \psi^{(k)})\}_{k\leq n}$, we define $(v^{(n+1)}, \rho^{(n+1)}, \psi^{(n+1)})$ to be the solution to the following linear system whose coefficients depend on $(v^{(n)}, \rho^{(n)}, \psi^{(n)})$ and $\psi^{(n-1)}$:

$$\begin{cases} \rho^{(n)} D_t^{\widetilde{\varphi}^{(n)}} v^{(n+1)} + \nabla^{\widetilde{\varphi}^{(n)}} \check{q}^{(n+1)} = -(\rho^{(n)} - 1) g e_3 & \text{in } [0,T]\times\Omega, \\ \mathcal{F}^{(n)\prime}(q^{(n)}) D_t^{\widetilde{\varphi}^{(n)}} \check{q}^{(n+1)} + \nabla^{\widetilde{\varphi}^{(n)}} \cdot v^{(n+1)} = \mathcal{F}^{(n)\prime}(q^{(n)}) g v_3^{(n)} & \text{in } [0,T]\times\Omega, \\ q^{(n+1)} = q^{(n+1)}(\rho^{(n+1)}), \check{q}^{(n+1)} = q^{(n+1)} + g\widetilde{\varphi}^{(n)} & \text{in } [0,T]\times\Omega, \\ \check{q}^{(n+1)} = g\widetilde{\psi}^{(n)} - \sigma\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\widetilde{\psi}^{(n)}}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}^{(n)}|^2}}\right) + \kappa^2(1-\overline{\Delta})(v^{(n+1)}\cdot\widetilde{N}^{(n)}) & \text{on } [0,T]\times\Sigma, \\ \partial_t\psi^{(n+1)} = v^{(n+1)}\cdot\widetilde{N}^{(n)} & \text{on } [0,T]\times\Sigma, \\ v_3^{(n+1)} = 0 & \text{on } [0,T]\times\Sigma_b, \\ (v^{(n+1)}, \rho^{(n+1)}, \psi^{(n+1)})|_{t=0} = (v_{0,\kappa}, \rho_{0,\kappa}, \psi_{0,\kappa}), \end{cases} \tag{5.1}$$

where for any $k \leq n+1$, $\varphi^{(k)}(t,x)$ is the extension of $\psi^{(k)}$ defined by $\varphi^{(k)}(t,x) := x_3 + \chi(x_3)\psi^{(k)}$ and $\widetilde{\varphi}^{(k)} := x_3 + \chi(x_3)\widetilde{\psi}^{(k)}$ is the smoothed version of $\varphi^{(k)}$. The linearized material derivative is defined to be the following linear operator

$$D_t^{\widetilde{\varphi}^{(n)}} := \partial_t + \overline{v}^{(n)}\cdot\overline{\nabla} + \frac{1}{\partial_3\widetilde{\varphi}^{(n)}}(v^{(n)}\cdot\widetilde{\mathbf{N}}^{(n-1)} - \partial_t\varphi^{(n)})\partial_3, \tag{5.2}$$

and the covariant derivatives are defined to be

$$\partial_t^{\widetilde{\varphi}^{(n)}} = \partial_t - \frac{\partial_t\varphi^{(n)}}{\partial_3\widetilde{\varphi}^{(n)}}\partial_3, \tag{5.3}$$

$$\nabla_a^{\widetilde{\varphi}^{(n)}} = \partial_a^{\widetilde{\varphi}^{(n)}} = \partial_a - \frac{\partial_a\widetilde{\varphi}^{(n)}}{\partial_3\widetilde{\varphi}^{(n)}}\partial_3, \quad a = 1,2, \tag{5.4}$$

$$\nabla_3^{\widetilde{\varphi}^{(n)}} = \partial_3^{\widetilde{\varphi}^{(n)}} = \frac{1}{\partial_3\widetilde{\varphi}^{(n)}}\partial_3. \tag{5.5}$$

Remark. Note that the linearized material derivative is no longer equal to $\partial_t^{\widetilde{\varphi}^{(n)}} + v^{(n)}\cdot\nabla^{\widetilde{\varphi}^{(n)}}$. Indeed, one has to set the weight of ∂_3 to be $v^{(n)}\cdot\widetilde{\mathbf{N}}^{(n-1)} - \partial_t\varphi^{(n)}$ to guarantee both the linearity of this operator and the consistency with the linearized kinematic boundary condition $\partial_t\psi^{(n+1)} = v^{(n+1)}\cdot\widetilde{N}^{(n)}$.

Remark. Note that the surface tension term in (5.1) is completely a given term instead of being $-\sigma\overline{\nabla}\cdot(\overline{\nabla}\widetilde{\psi}^{(n+1)}/|\widetilde{N}^{(n)})$. Under this setting, we can still do energy estimates for $\psi^{(n+1)}$ by using the kinematic boundary condition and the viscosity term.

For simplicity of notations, for any $n \geq 0$, we denote $(v^{(n+1)}, \rho^{(n+1)}, q^{(n+1)}, \psi^{(n+1)})$, $(v^{(n)}, \rho^{(n)}, q^{(n)}, \psi^{(n)})$ and $\psi^{(n-1)}$ by (v, ρ, q, ψ), $(\mathring{v}, \mathring{\rho}, \mathring{q}, \mathring{\psi})$ and $\dot{\psi}$. Hence, we need to solve the following linearized version of system (3.11) for each fixed $\kappa > 0$ and then establish an energy estimate to proceed with the iteration scheme.

$$\begin{cases} \mathring{\rho} D_t^{\mathring{\widetilde{\varphi}}} v + \nabla^{\mathring{\widetilde{\varphi}}}\check{q} = -(\mathring{\rho} - 1) g e_3, & \text{in } [0,T]\times\Omega, \\ \mathring{\mathcal{F}}'(\mathring{q}) D_t^{\mathring{\widetilde{\varphi}}}\check{q} + \nabla^{\mathring{\widetilde{\varphi}}}\cdot v = \mathring{\mathcal{F}}'(\mathring{q}) g \mathring{v}_3, & \text{in } [0,T]\times\Omega, \\ q = q(\rho), \check{q} = q + g\mathring{\widetilde{\varphi}} & \text{in } [0,T]\times\Omega, \\ \check{q} = g\mathring{\widetilde{\psi}} - \sigma\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\mathring{\widetilde{\psi}}}{\sqrt{1+|\overline{\nabla}\mathring{\widetilde{\psi}}|^2}}\right) + \kappa^2(1-\overline{\Delta})(v\cdot\mathring{\widetilde{N}}), & \text{on } [0,T]\times\Sigma, \\ \partial_t\psi = v\cdot\mathring{\widetilde{N}}, & \text{on } [0,T]\times\Sigma, \\ v_3 = 0 & \text{on } [0,T]\times\Sigma_b, \\ (v,\rho,\psi)|_{t=0} = (v_0^\kappa, \rho_0^\kappa, \psi_0^\kappa). \end{cases} \tag{5.6}$$

Here $\mathring{\mathcal{F}} := \log\mathring{\rho}$. The linearized material derivative now becomes

$$D_t^{\tilde{\mathring{\varphi}}} := \partial_t + \bar{\mathring{v}} \cdot \overline{\nabla} + \frac{1}{\partial_3\tilde{\mathring{\varphi}}}(\mathring{v} \cdot \tilde{\mathring{\mathbf{N}}} - \partial_t\mathring{\varphi})\partial_3 \tag{5.7}$$

and the covariant derivatives with respect to $\tilde{\mathring{\varphi}}$ are defined to be

$$\partial_t^{\tilde{\mathring{\varphi}}} := \partial_t - \frac{\partial_t\mathring{\varphi}}{\partial_3\tilde{\mathring{\varphi}}}\partial_3, \tag{5.8}$$

$$\nabla_a^{\tilde{\mathring{\varphi}}} = \partial_a^{\tilde{\mathring{\varphi}}} = \partial_a - \frac{\partial_a\tilde{\mathring{\varphi}}}{\partial_3\tilde{\mathring{\varphi}}}\partial_3, \quad a = 1, 2, \tag{5.9}$$

$$\nabla_3^{\tilde{\mathring{\varphi}}} = \partial_3^{\tilde{\mathring{\varphi}}} = \frac{1}{\partial_3\tilde{\mathring{\varphi}}}\partial_3, \tag{5.10}$$

where $\bar{\mathring{v}} \cdot \overline{\nabla} := \mathring{v}_1\partial_1 + \mathring{v}_2\partial_2$. Note that, by the kinematic boundary condition, the normal component in $D_t^{\tilde{\mathring{\varphi}}}$, namely $(\partial_3\tilde{\mathring{\varphi}})^{-1}(\mathring{v} \cdot \tilde{\mathring{\mathbf{N}}} - \partial_t\mathring{\varphi})\partial_3$ vanishes on Σ.

From now on, we assume the following given quantities are bounded in some time interval $t \in [0, T^\kappa]$. This also works as the induction hypothesis for the uniform-in-n estimates for (5.6):

$$\begin{aligned} \|\mathring{\rho} - 1\|_0^2 + \sum_{k=0}^{4} \|\partial_t^k \mathring{v}\|_{4-k}^2 + \|\mathring{\mathcal{F}}'(\mathring{q})\check{q}\|_0^2 + \|\partial\mathring{\check{q}}\|_3^2 + \sum_{k=1}^{3} \|\partial_t^k \mathring{\check{q}}\|_{4-k}^2 + \|\mathring{\mathcal{F}}'(\mathring{q})\partial_t^4\mathring{\check{q}}\|_0^2 \\ +\kappa^4|\mathring{\psi}|_{5.5}^2 + \sum_{k=0}^{3} \kappa^4|\partial_t^{k+1}\mathring{\psi}, \partial_t^{k+1}\mathring{\psi}|_{5.5-k}^2 + \kappa^2 \int_0^t |\partial_t^5\mathring{\psi}|_1^2 \mathrm{d}\tau < \mathring{K}_0. \end{aligned} \tag{5.11}$$

5.1 Construction of solution to the linearized approximate system

We can prove that system (5.6) is a symmetric hyperbolic system with characteristic boundary conditions. Therefore, we want to use the duality argument developed by Lax-Phillips [38] to prove the local existence. Before doing this, we have to make sure the boundary conditions are homogeneous.

5.1.1 The homogeneous linearized approximate system

We introduce the variable $\mathring{\mathfrak{h}}$ defined by the harmonic extension

$$\begin{cases} -\Delta\mathring{\mathfrak{h}} = 0 & \text{in } \Omega, \\ \mathring{\mathfrak{h}} = g\tilde{\mathring{\psi}} - \sigma\overline{\nabla} \cdot \left(\frac{\overline{\nabla}\tilde{\mathring{\psi}}}{\sqrt{1+|\overline{\nabla}\tilde{\mathring{\psi}}|^2}}\right) & \text{on } \Sigma, \\ \partial_3\mathring{\mathfrak{h}} = 0 & \text{on } \Sigma_b, \end{cases} \tag{5.12}$$

and define $\underline{q} = \check{q} - \mathring{\mathfrak{h}}$. Then (5.6) becomes the following linear hyperbolic system with *homogeneous* boundary condition

$$\begin{cases} \mathring{\rho}D_t^{\tilde{\mathring{\varphi}}}v + \nabla^{\tilde{\mathring{\varphi}}}\underline{q} = -\nabla^{\tilde{\mathring{\varphi}}}\mathring{\mathfrak{h}} - (\mathring{\rho} - 1)ge_3, & \text{in } [0, T] \times \Omega, \\ \mathring{\mathcal{F}}'(\mathring{q})D_t^{\tilde{\mathring{\varphi}}}\underline{q} + \nabla^{\tilde{\mathring{\varphi}}} \cdot v = \mathring{\mathcal{F}}'(\mathring{q})(\mathring{v}_3 - D_t^{\tilde{\mathring{\varphi}}}\mathring{\mathfrak{h}}), & \text{in } [0, T] \times \Omega, \\ q = q(\rho),\ \underline{q} = q + g\tilde{\mathring{\varphi}} - \mathring{\mathfrak{h}} & \text{in } [0, T] \times \Omega, \\ \underline{q} = \kappa^2(1 - \overline{\Delta})(v \cdot \tilde{\mathring{N}}), & \text{on } [0, T] \times \Sigma, \\ v_3 = 0 & \text{on } [0, T] \times \Sigma_b, \\ (v, \rho, \psi)|_{t=0} = (v_0, \rho_0, \psi_0). \end{cases} \tag{5.13}$$

Note that the coefficients in (5.13) only rely on $\dot{\psi}, \tilde{\mathring{\psi}}, \mathring{v}$, and $\mathring{\rho}$ which is already given. The kinematic boundary condition, namely $\partial_t\psi = v \cdot \tilde{\mathring{N}} = -(\bar{v} \cdot \overline{\nabla})\tilde{\mathring{\psi}} + v_3$ on Σ, is used to define ψ after solving (v, q) from (5.13).

We define $U := (\underline{q}, v_1, v_2, v_3)^\top$, then (5.13) can be expressed in terms of U by

$$A_0(\mathring{U})\partial_t U + \sum_{i=1}^{3} A_i(\mathring{U})\partial_i U = \mathring{f}, \tag{5.14}$$

where $\mathring{f} := \left(\mathring{\mathcal{F}}'(\mathring{q})(\mathring{v}_3 - D_t^{\mathring{\widetilde{\varphi}}}\mathring{\mathfrak{h}}), -\partial_1^{\mathring{\widetilde{\varphi}}}\mathring{\mathfrak{h}}, -\partial_2^{\mathring{\widetilde{\varphi}}}\mathring{\mathfrak{h}}, -\partial_3^{\mathring{\widetilde{\varphi}}}\mathring{\mathfrak{h}} - (\mathring{\rho} - 1)g\right)^\top$, $A_0(\mathring{U}) = \operatorname{diag}\left[\mathring{\mathcal{F}}'(\mathring{q}), \mathring{\rho}, \mathring{\rho}, \mathring{\rho}\right]$ and

$$A_i(\mathring{U}) = \begin{bmatrix} \mathring{\mathcal{F}}'(\mathring{q})\mathring{v}_i & e_i^\top \\ e_i & \mathring{\rho}\mathring{v}_i\mathbf{I}_3 \end{bmatrix} \text{ for } i = 1, 2, \quad A_3(\mathring{U}) = \frac{1}{\partial_3\mathring{\widetilde{\varphi}}} \begin{bmatrix} \mathring{\mathcal{F}}'(\mathring{q})(\mathring{v} \cdot \dot{\widetilde{\mathbf{N}}} - \partial_t\mathring{\varphi}) & \mathring{\widetilde{\mathbf{N}}}^\top \\ \mathring{\widetilde{\mathbf{N}}} & \mathring{\rho}(\mathring{v} \cdot \dot{\widetilde{\mathbf{N}}} - \partial_t\mathring{\varphi})\mathbf{I}_3 \end{bmatrix}.$$

Since $(\partial_t\mathring{\varphi} - \mathring{v} \cdot \dot{\widetilde{\mathbf{N}}},)|_\Sigma = 0$ and $e_3 = (0, 0, 1)^\top$ is the unit exterior normal vector to Σ, we know that the boundary matrix, namely the normal projection of the coefficient matrices onto Σ, is

$$\sum_{i=1}^{3} A_i(\mathring{U})e_{3i} = A_3(\mathring{U}) = \begin{bmatrix} 0 & \mathring{\widetilde{N}}^\top \\ \mathring{\widetilde{N}} & \mathbf{0}_3 \end{bmatrix} \quad \text{on } \Sigma$$

which is a 4×4, rank 2 (constant rank but not full rank) matrix having one negative eigenvalue, one positive eigenvalue, and two zero eigenvalues. So we know system (5.13) is a first-order symmetric hyperbolic system with characteristic boundary conditions. The number of boundary conditions should be equal to the number of negative eigenvalues. Therefore, the correct number of boundary conditions for (5.13) is indeed equal to 1 which means (5.13) is solvable. After solving (5.13), we use the kinematic boundary condition to define ψ for the next step of the iteration.

5.1.2 Well-posedness in L^2 via μ-regularization

From the duality argument by Lax-Phillips [38], we need to prove the following two things for the well-posedness of (5.13) in some function space X

- Establish a priori estimate (without loss of regularity from source term to solution) for (5.13) in X,
- Establish a priori estimate (without loss of regularity from source term to solution) for the dual system of (5.13) in X'.

We choose $X = L^2(\Omega)$ as we don't know what exactly the dual space of $H^s(\Omega)$ $(s > 0)$ is. We define $W^* = (\underline{q}^*, w_1^*, w_2^*, w_3^*)^\top$ to be the dual variables of $U = (\underline{q}, v_1, v_2, v_3)^\top$. By testing (5.13) with W^* in $L^2(\Omega)$, one can derive the system of W^* which reads

$$A_0(\mathring{U})\partial_t W^* + \sum_{i=1}^{3} A_i(\mathring{U})\partial_i W^* + A_4(\mathring{U})W^* = \mathring{f}^*$$

with boundary condition $\underline{q}^*|_\Sigma = -\kappa^2(1 - \overline{\Delta})(w \cdot \mathring{\widetilde{N}})$, where $A_4 := -\partial_t A_0^\top - \sum_{i=1}^{3} \partial_i A_i^\top - (\mathring{\rho} - 1)g\mathbf{E}_{44}$ with $\mathbf{E}_{44} = \operatorname{diag}[0, 0, 0, 1]$. Note that we do not have the dual variable for ψ because ψ is completely determined by the original linearized system. That is why we only have one boundary condition for the dual system.

We notice that there is an extra minus sign in the boundary condition for $\underline{q}^*$. So, one cannot close the L^2-type a priori estimate for the dual system even if we can derive that L^2-type a priori estimate for (5.13). To avoid this difficulty, we introduce another viscosity term in the boundary for $\underline{q}$ in (5.13). That is, we alternatively consider the μ-regularized linear problem for $U = (\underline{q}, v_1, v_2, v_3)^\top$

$$A_0(\mathring{U})\partial_t U + \sum_{i=1}^{3} A_i(\mathring{U})\partial_i U = \mathring{f}, \tag{5.15}$$

with boundary conditions

$$\underline{q} = \kappa^2(1 - \overline{\Delta})(v \cdot \mathring{\widetilde{N}}) + \mu(1 - \overline{\Delta})\partial_t(v \cdot \mathring{\widetilde{N}}) \quad \text{on } \Sigma. \tag{5.16}$$

Then the dual system of (5.15)-(5.16) reads

$$A_0(\mathring{U})\partial_t W^* + \sum_{i=1}^{3} A_i(\mathring{U})\partial_i W^* + A_4(\mathring{U})W^* = \mathring{f}^* \tag{5.17}$$

with boundary condition

$$\underline{q}^* = -\kappa^2(1-\overline{\Delta})(w^*\cdot\mathring{\tilde{N}}) + \mu(1-\overline{\Delta})\partial_t(w^*\cdot\mathring{\tilde{N}}) \quad \text{on } \Sigma, \tag{5.18}$$

where $A_4 := -\partial_t A_0^\top - \sum_{i=1}^{3}\partial_i A_i^\top - (\mathring{\rho}-1)g\mathbf{E}_{44}$ with $\mathbf{E}_{44} = \text{diag}[0,0,0,1]$. Note that we have to integrate by parts once more in t variable when deriving the boundary condition for $\underline{q}^*$. That is why there is a minus sign in front of $\mu(1-\overline{\Delta})(w\cdot\mathring{\tilde{N}})$.

Now we are going to derive the L^2 a priori estimates for both (5.15) and (5.17). For linear system (5.15), we test it with U in $L^2(\Omega)$ and use the symmetry of the coefficient matrices to get

$$\int_\Omega U^\top\cdot A_0(\mathring{U})U\,\mathrm{d}x = \int_\Omega U^\top\cdot\mathring{f} - \sum_{i=1}^{3}\int_\Omega U^\top\cdot\partial_i A_i(\mathring{U})U\,\mathrm{d}x - \int_\Sigma U^\top\cdot A_3(\mathring{U})U\,\mathrm{d}x, \tag{5.19}$$

where the interior integrals are directly controlled by $C(\mathring{K}_0)\|U\|_0^2$ and the boundary integral reads

$$\begin{aligned} &-\int_\Sigma U^\top\cdot A_3(\mathring{U})U\,\mathrm{d}x' = -2\int_\Sigma(v\cdot\mathring{\tilde{N}})\underline{q}\,\mathrm{d}x' \\ =&\, 2\kappa^2\int_\Omega\Big((1-\overline{\Delta})(w^*\cdot\mathring{\tilde{N}})\Big)(w^*\cdot\mathring{\tilde{N}})\,\mathrm{d}x' + 2\mu\int_\Sigma\partial_t\Big((1-\overline{\Delta})(w^*\cdot\mathring{\tilde{N}})\Big)(w^*\cdot\mathring{\tilde{N}})\,\mathrm{d}x' \\ =&-\mu\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma\Big|\langle\overline{\partial}\rangle(w^*\cdot\mathring{\tilde{N}})\Big|_0^2\,\mathrm{d}x' - 2\kappa^2\Big|\langle\overline{\partial}\rangle(w^*\cdot\mathring{\tilde{N}})\Big|_0^2. \end{aligned} \tag{5.20}$$

We define $\mathring{E}_0(t) := \|v(t)\|_0^2 + \left\|\sqrt{\mathring{\mathcal{F}}'(\mathring{q})}\underline{q}(t)\right\|_0^2 + \int_0^t|\kappa(v\cdot\mathring{\tilde{N}})(\tau)|_1^2\mathrm{d}\tau + |\sqrt{\mu}(v\cdot\mathring{\tilde{N}})(t)|_1^2$, then the above analysis shows that

$$\mathring{E}_0(T) - \mathring{E}_0(0) \le C(\mathring{K}_0)\int_0^T \mathring{E}_0(t) + \sqrt{\mathring{E}_0(t)}\|\mathring{f}(t)\|_0\,\mathrm{d}t. \tag{5.21}$$

and thus by Grönwall's inequality we finish the L^2 control for (5.15). Note that this a priori bound is also uniform in μ.

Next, we show the L^2 estimates for the dual system (5.17). Note that the matrix $A_4(\mathring{U})$ is still in $L^\infty(\Omega)$, so we test (5.17) by W^* and take L^2 inner product to get

$$\int_\Omega W^{*\top}\cdot A_0(\mathring{U})W^*\,\mathrm{d}x = \int_\Omega W^{*\top}\cdot\mathring{f}^* - W^{*\top}\cdot\left(\sum_{i=1}^{3}\partial_i A_i(\mathring{U}) + A_4(\mathring{U}) + \mathring{\rho}g\mathbf{E}_{44}\right)W^*\,\mathrm{d}x - \int_\Sigma(W^*)^\top\cdot A_3(\mathring{U})W^*\,\mathrm{d}x', \tag{5.22}$$

where the interior integral is directly controlled by $C(\mathring{K}_0)\|W^*\|_0^2$, but now there is a sign change in the boundary integral, which reads

$$\begin{aligned} &-\int_\Sigma(W^*)^\top\cdot A_3(\mathring{U})W^*\,\mathrm{d}x' = -2\int_\Sigma(w^*\cdot\mathring{\tilde{N}})\underline{q}^*\,\mathrm{d}x' \\ =&\, 2\kappa^2\int_\Omega\Big((1-\overline{\Delta})(w^*\cdot\mathring{\tilde{N}})\Big)(w^*\cdot\mathring{\tilde{N}})\,\mathrm{d}x' - 2\mu\int_\Sigma\partial_t\Big((1-\overline{\Delta})(w^*\cdot\mathring{\tilde{N}})\Big)(w^*\cdot\mathring{\tilde{N}})\,\mathrm{d}x' \\ \lesssim&-\mu\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma\Big|\langle\overline{\partial}\rangle(w^*\cdot\mathring{\tilde{N}})\Big|_0^2\,\mathrm{d}x' + 2\kappa^2\Big|\langle\overline{\partial}\rangle(w^*\cdot\mathring{\tilde{N}})\Big|_0^2. \end{aligned} \tag{5.23}$$

One can see that the new viscosity term involving μ is used to control the term $+2\kappa^2|\langle\overline{\partial}\rangle(w^*\cdot\mathring{\tilde{N}})|_0^2$ due to the change of sign.

So, if we define $\mathring{E}_0^*(t) = \|w^*(t)\|_0^2 + \left\|\sqrt{\mathring{\mathcal{F}}'(\mathring{q})}\underline{q}^*(t)\right\|_0^2 + \mu\left|(w^*\cdot\mathring{\tilde{N}})(t)\right|_1^2$, then we have

$$\mathring{E}_0^*(T) - \mathring{E}_0^*(0) \lesssim_{\mu^{-1}} C(\mathring{K}_0)\int_0^T \mathring{E}_0^*(t) + \sqrt{\mathring{E}_0^*(t)}\|\mathring{f}^*(t)\|_0\,\mathrm{d}t, \tag{5.24}$$

and thus Grönwall's inequality helps us close the L^2 estimates.

Combining (5.21) and (5.24), we close the $L^2(\Omega)$ a priori bounds for both linear system (5.15)-(5.16) and its dual system (5.17)-(5.18). And such energy bounds have no regularity loss from their source terms to solutions. Therefore, by the argument in Lax-Phillips [38](see also [53, Theorem 5,9]), for each fixed $\mu > 0$, system (5.17)-(5.18) admits a unique solution $U \in L^2(\Omega)$. Since the energy bound (5.21) for (5.15)-(5.16) is uniform in μ, we can take the limit $\mu \to 0_+$ to obtain a local-in-time solution of the homogeneous linearized problem (5.13). Finally, the modification $\mathring{\mathfrak{h}}$ is easily controlled by using the property of the harmonic function

$$\forall s > -\frac{1}{2}, \quad \|\mathring{\mathfrak{h}}\|_{s+\frac{1}{2}} \lesssim |\mathring{\mathfrak{h}}|_s \le g|\mathring{\tilde{\psi}}|_s + P(|\overline{\nabla}\mathring{\tilde{\psi}}|_s)|\overline{\nabla}^2\mathring{\tilde{\psi}}|_s,$$

which implies the local existence for L^2 (weak) solution to the linearized κ-approximate system (5.6). By the argument in [49, Section 2.2.3](see also [53, Theorem 4, 8]), the weak solution U is actually a strong solution.

5.2 Higher-order estimates for the linearized system

Now we prove higher-order energy estimates for the linearized system (5.6). We define the following energy functional

$$\begin{aligned}\mathring{E}^\kappa(t) :=& \|\rho(t)-1\|_0^2 + \sum_{k=0}^4 \|\partial_t^k v(t)\|_{4-k}^2 + \kappa^2 \int_0^t \left|\partial_t^{k+1}\psi(\tau)\right|_{5-k}^2 \mathrm{d}\tau \\ &+ \|\sqrt{\mathring{\mathcal{F}}'(\mathring{q})}\check{q}(t)\|_0^2 + \|\partial\check{q}(t)\|_3^2 + \sum_{k=1}^3 \|\partial_t^k\check{q}(t)\|_{4-k}^2 + \|\sqrt{\mathring{\mathcal{F}}'(\mathring{q})}\partial_t^4\check{q}(t)\|_0^2,\end{aligned} \tag{5.25}$$

Proposition 5.1. There exists some $T^\kappa > 0$ depending on κ and a constant $C(\kappa^{-1}, \mathring{K}_0) > 0$, such that

$$\sup_{0\le t\le T^\kappa} \mathring{E}^\kappa(t) \le C(\kappa^{-1}, \mathring{K}_0)\mathring{E}^\kappa(0), \tag{5.26}$$

and ψ and its time derivatives have the following bounds in $t \in [0, T^\kappa]$

$$|\psi(t)|_{5.5}^2 + \sum_{k=0}^3 |\partial_t^{k+1}\psi(t)|_{5.5-k}^2 \le C(\kappa^{-1}, \mathring{K}_0)\mathring{E}^\kappa(t). \tag{5.27}$$

5.2.1 L^2 estimates

We define the L^2 energy for the linearized system (5.6) to be

$$\mathring{E}_0^\kappa(t) := \|\rho(t)-1\|_0^2 + \|v(t)\|_0^2 + \|\sqrt{\mathring{\mathcal{F}}'(\mathring{q})}\check{q}(t)\|_0^2 + \kappa^2\int_0^t |\partial_t\psi(\tau)|_1^2 \mathrm{d}\tau. \tag{5.28}$$

The control of $\mathring{E}_0$ follows in the same way as the L^2 a priori estimates for (5.15) when $\mu = 0$. Note that the control of $\|\rho-1\|_0^2$ follows from testing the linearized continuity equation $\mathring{\mathcal{F}}'(\mathring{q})\mathcal{F}'(q)^{-1}D_t^{\mathring{\tilde{\varphi}}}(\rho-1) + \rho(\nabla^{\mathring{\tilde{\varphi}}}\cdot v) = 0$ by $\rho - 1$ in $L^2(\Omega)$. Also one can control the $L^2(\Sigma)$ norm of ψ via $\psi(t) = \psi_{0,\kappa} + \int_0^t \partial_t\psi(\tau)\mathrm{d}\tau$.

5.2.2 Div-Curl analysis

To estimate the Sobolev norms of v, we invoke the following Hodge decomposition lemma which is exactly from [9, Theorem 1.1].

Lemma 5.2 (Hodge elliptic estimates)**.** For any sufficiently smooth vector field X and $s \ge 1$, one has

$$\|X\|_s^2 \lesssim C(|\mathring{\tilde{\psi}}|_{s+\frac{1}{2}}, |\overline{\nabla}\mathring{\tilde{\psi}}|_{W^{1,\infty}})\left(\|X\|_0^2 + \|\nabla^{\mathring{\tilde{\varphi}}}\cdot X\|_{s-1}^2 + \|\nabla^{\mathring{\tilde{\varphi}}}\times X\|_{s-1}^2 + |X\cdot\mathring{\tilde{N}}|_{s-\frac{1}{2}}^2 + |X_3|^2_{H^{s-\frac{1}{2}}(\Sigma_b)}\right), \tag{5.29}$$

where the constant $C(|\mathring{\tilde{\psi}}|_{s+\frac{1}{2}}, |\overline{\nabla}\mathring{\tilde{\psi}}|_{W^{1,\infty}}) > 0$ depends linearly on $|\mathring{\tilde{\psi}}|_{s+\frac{1}{2}}^2$.

Applying this lemma to v with $s = 4$, one has

$$\|v\|_4^2 \lesssim C(|\mathring{\tilde{\psi}}|_{4.5}, |\overline{\nabla}\mathring{\tilde{\psi}}|_{W^{1,\infty}})\left(\|v\|_0^2 + \|\nabla^{\mathring{\tilde{\varphi}}}\cdot v\|_3^2 + \|\nabla^{\mathring{\tilde{\varphi}}}\times v\|_3^2 + |v\cdot\mathring{\tilde{N}}|_{3.5}^2\right). \tag{5.30}$$

Now we control the curl term. Taking $\nabla^{\mathring{\widetilde{\varphi}}}\times$ in the first equation of (5.6), we get the evolution equation of $\nabla^{\mathring{\widetilde{\varphi}}}\times v$

$$\mathring{\rho} D_t^{\mathring{\widetilde{\varphi}}}(\nabla^{\mathring{\widetilde{\varphi}}}\times v) = \mathring{\rho}[\nabla^{\mathring{\widetilde{\varphi}}}\times, D_t^{\mathring{\widetilde{\varphi}}}]v + \nabla^{\mathring{\widetilde{\varphi}}}\mathring{\rho}\times(\mathring{\rho}^{-1}\nabla^{\mathring{\widetilde{\varphi}}}\check{q}), \tag{5.31}$$

and taking three derivatives we get

$$\mathring{\rho} D_t^{\mathring{\widetilde{\varphi}}}\partial^3(\nabla^{\mathring{\widetilde{\varphi}}}\times v) = \partial^3\left(\mathring{\rho}[\nabla^{\mathring{\widetilde{\varphi}}}\times, D_t^{\mathring{\widetilde{\varphi}}}]v) + \nabla^{\mathring{\widetilde{\varphi}}}\mathring{\rho}\times(\mathring{\rho}^{-1}\nabla^{\mathring{\widetilde{\varphi}}}\check{q})\right) - [\partial^3, \mathring{\rho} D_t^{\mathring{\widetilde{\varphi}}}](\nabla^{\mathring{\widetilde{\varphi}}}\times v). \tag{5.32}$$

We expect that the source terms in (5.32) only contain ≤ 4 derivatives of $v, \check{q}$ and quantities marked with a ring, but there still exists a mismatched term in $([\nabla^{\mathring{\widetilde{\varphi}}}\times, D_t^{\mathring{\widetilde{\varphi}}}]v)^i = \epsilon^{ijk}\nabla_j^{\mathring{\widetilde{\varphi}}}\mathring{v}^l\nabla_l^{\mathring{\widetilde{\varphi}}}v_k + \epsilon^{ijk}\nabla_j^{\mathring{\widetilde{\varphi}}}\partial_t(\varphi - \dot{\widetilde{\varphi}})\partial_3^{\mathring{\widetilde{\varphi}}}v_k$. The contribution of $\dot{\widetilde{\psi}}$ is controlled by $\mathring{K}_0$. So, standard L^2 estimates for the ∂^3-differentiated evolution equation of $\nabla^{\mathring{\widetilde{\varphi}}}\times v$ and Reynold transport formula (A.9) gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^{\mathring{\widetilde{\varphi}}}\times v\|_3^2 \leq P(\mathring{K}_0)(\|v\|_4^2 + \|\check{q}\|_4\|v\|_4 + |\partial_t\psi|_4\|\partial v\|_\infty). \tag{5.33}$$

Finally, using the linearized continuity equation, we can control the divergence

$$\|\nabla^{\mathring{\widetilde{\varphi}}}\cdot v\|_3^2 \leq \left\|\mathring{\mathcal{F}}'(\mathring{q})D_t^{\mathring{\widetilde{\varphi}}}\check{q}\right\|_3^2 + \left\|\mathring{\mathcal{F}}'(\mathring{q})g\mathring{v}_3\right\|_3^2. \tag{5.34}$$

The div-curl analysis for the time derivatives proceeded similarly. We first do the div-curl decomposition for $1 \leq k \leq 3$

$$\|\partial_t^k v\|_{4-k}^2 \lesssim C(|\mathring{\widetilde{\psi}}|_{4.5-k}, |\overline{\nabla}\mathring{\widetilde{\psi}}|_{W^{1,\infty}})\left(\|\partial_t^k v\|_0^2 + \|\nabla^{\mathring{\widetilde{\varphi}}}\cdot\partial_t^k v\|_{3-k}^2 + \|\nabla^{\mathring{\widetilde{\varphi}}}\times\partial_t^k v\|_{3-k}^2 + |\partial_t^k v\cdot\mathring{\widetilde{N}}|_{3.5-k}^2\right). \tag{5.35}$$

We replace ∂^3 by $\partial_t^k\partial^{3-k}$ for $0 \leq k \leq 3$ in (5.32) to get the evolution equation for curl

$$\mathring{\rho} D_t^{\mathring{\widetilde{\varphi}}}\left(\partial^{3-k}\partial_t^k(\nabla^{\widetilde{\varphi}}\times v)\right) = \partial_t^k\partial^{3-k}\left(\mathring{\rho}[\nabla^{\mathring{\widetilde{\varphi}}}\times, D_t^{\mathring{\widetilde{\varphi}}}]v) + \nabla^{\mathring{\widetilde{\varphi}}}\mathring{\rho}\times(\mathring{\rho}^{-1}\nabla^{\mathring{\widetilde{\varphi}}}\check{q})\right) - [\partial_t^k\partial^{3-k}, \mathring{\rho} D_t^{\mathring{\widetilde{\varphi}}}](\nabla^{\mathring{\widetilde{\varphi}}}\times v), \tag{5.36}$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\partial_t^k(\nabla^{\mathring{\widetilde{\varphi}}}\times v)\|_{3-k}^2 \leq P(\mathring{E}^\kappa(t)). \tag{5.37}$$

Then we find the highest order term in the commutator $[\partial_t^k, \nabla^{\mathring{\widetilde{\varphi}}}\times]v$ should be $\overline{\partial}\partial_t\mathring{\widetilde{\varphi}}\partial_t^{k-1}\partial_3 v$, so we have

$$\|\nabla^{\mathring{\widetilde{\varphi}}}\times\partial_t^k v\|_{3-k}^2 \leq C(\mathring{K}_0)\left(\mathring{E}^\kappa(0) + \int_0^T \mathring{E}^\kappa(t)\,\mathrm{d}t\right). \tag{5.38}$$

As for divergence, taking time derivatives in the continuity equation, we get

$$\nabla^{\mathring{\widetilde{\varphi}}}\cdot\partial_t^k v = -\partial_t^k(\mathring{\mathcal{F}}'(\mathring{q})D_t^{\mathring{\widetilde{\varphi}}}\check{q} + \mathring{\mathcal{F}}'(\mathring{q})g\mathring{v}_3) + [\nabla^{\mathring{\widetilde{\varphi}}}\cdot, \partial_t^k]v \overset{L}{=} -\mathring{\mathcal{F}}'(\mathring{q})(\partial_t^k D_t^{\mathring{\widetilde{\varphi}}}\check{q} + g\partial_t^k\mathring{v}_3) + (\partial_3\mathring{\widetilde{\varphi}})^{-1}\overline{\partial}\partial_t^k\mathring{\widetilde{\varphi}}\partial_3 v.$$

Parallel to the analysis for (4.25), since $\|\overline{\partial}\partial_t^k\mathring{\widetilde{\varphi}}\|_{3-k} \leq \mathring{K}_0$ thanks to (5.11), we have $\|\nabla^{\mathring{\widetilde{\varphi}}}\cdot\partial_t^k v\|_{3-k}$ is reduced to the control of $\|\mathring{\mathcal{F}}'(\mathring{q})\partial_t^{k+1}\check{q}\|_{3-k}$ and $\|\mathring{\mathcal{F}}'(\mathring{q})\partial_t^k\check{q}\|_{4-k}$ at the top order. Thus,

$$\|\nabla^{\mathring{\widetilde{\varphi}}}\cdot\partial_t^k v\|_{3-k}^2 \leq (C(\mathring{K}_0) + 1)\left(\left\|\mathring{\mathcal{F}}'(q)\partial_t^{k+1}\check{q}\right\|_{3-k}^2 + \left\|\mathring{\mathcal{F}}'(q)\partial_t^k\check{q}\right\|_{4-k}^2\right). \tag{5.39}$$

5.2.3 Estimates for ψ and normal traces

The normal trace terms in (5.30) and (5.35) can be directly controlled by applying boundary elliptic estimates to the linearized viscous surface tension equation $\kappa^2(1-\overline{\Delta})(v\cdot\mathring{\widetilde{N}}) = q - \sigma\mathcal{H}(\overline{\nabla}\mathring{\widetilde{\psi}}, \overline{\nabla}^2\mathring{\widetilde{\psi}})$. We start with $|v\cdot\mathring{\widetilde{N}}|_{3.5}$

$$|v\cdot\mathring{\widetilde{N}}|_{3.5}^2 \leq \kappa^{-2}\left(|q|_{1.5}^2 + \sigma|\overline{\nabla}^2\mathring{\widetilde{\psi}}|_{1.5}^2 P(|\overline{\nabla}\mathring{\widetilde{\psi}}|_{1.5})\right) \leq \kappa^{-2}P(\mathring{K}_0)\|\check{q}\|_2^2. \tag{5.40}$$

Taking time derivatives in the kinematic boundary condition, we know

$$\partial_t^k v\cdot\mathring{\widetilde{N}} = \partial_t^{k+1}\psi - \sum_{j=1}^k\binom{k}{j}\partial_t^{k-j}\overline{v}\cdot\partial_t^j\overline{\nabla}\mathring{\widetilde{\psi}},$$

and thus

$$|\partial_t v\cdot \mathring{\widetilde{N}}|_{2.5}\leq |\partial_t^2\psi|_{2.5}+|\overline{v}\cdot\overline{\nabla}\partial_t\mathring{\widetilde{\psi}}|_{2.5}\leq |\partial_t^2\psi|_{2.5}+\|v_{\kappa,0}\|_3^2+P(\mathring{K}_0)\int_0^T\|\partial_t\overline{v}(t)\|_3\,\mathrm{d}t. \tag{5.41}$$

Then we take a time derivative in the linearized viscous surface tension equation to get

$$\kappa(1-\overline{\Delta})\partial_t^2\psi=\partial_t q-\sigma\partial_t\mathcal{H}(\overline{\nabla}\mathring{\widetilde{\psi}},\overline{\nabla}^2\mathring{\widetilde{\psi}}),$$

which implies $|\partial_t^2\psi|_{2.5}\leq\|\partial_t q\|_1+P(\mathring{K}_0)$. Repeatedly, we can take more time derivatives to see

$$|\partial_t^k v\cdot\mathring{\widetilde{N}}|_{3.5-k}^2\leq|\partial_t^{k+1}\psi|_{3.5-k}^2+\mathcal{P}_0^\kappa+P(\mathring{K}_0)\int_0^T\mathring{E}^\kappa(t)\,\mathrm{d}t, \tag{5.42}$$

and then $|\partial_t^{k+1}\psi|_{3.5-k}$ is controlled via boundary elliptic estimates

$$|\partial_t^3\psi|_{1.5}^2\approx|\langle\overline{\partial}\rangle^{-\frac12}\partial_t^3\psi|_2^2\leq|\langle\overline{\partial}\rangle^{-\frac12}\partial_t^2\check{q}|_0^2+P(\mathring{K}_0)\leq\|\partial_t^2\check{q}\|_1^2+P(\mathring{K}_0)\leq\|\partial_t^2\check{q}(0)\|_1^2+P(\mathring{K}_0)+\int_0^T\mathring{E}^\kappa(t)\,\mathrm{d}t, \tag{5.43}$$

$$|\partial_t^4\psi|_{0.5}\approx|\langle\overline{\partial}\rangle^{-\frac32}\partial_t^4\psi|_2\leq|\langle\overline{\partial}\rangle^{-\frac32}\partial_t^3\check{q}|_0+P(\mathring{K}_0)\leq\|\partial_t^3\check{q}\|_1+P(\mathring{K}_0), \tag{5.44}$$

where the term $\|\partial_t^3\check{q}\|_1$ will be further reduced to $\|\partial_t^3\mathcal{T}v\|_0$.

5.2.4 Reduction of pressure to tangential derivatives of v

Now we show the reduction of pressure. We start with $\|\check{q}\|_4$. From the linearized momentum equation, we know

$$\begin{aligned}-(\partial_3\mathring{\widetilde{\varphi}})^{-1}\partial_3 q&=(\mathring{\rho}-1)g+\mathring{\rho}D_t^{\mathring{\widetilde{\varphi}}}v_3,\\ -\partial_i q&=(\partial_3\mathring{\widetilde{\varphi}})^{-1}\overline{\partial}_i\mathring{\widetilde{\varphi}}\partial_3 q+\mathring{\rho}D_t^{\mathring{\widetilde{\varphi}}}v_i,\quad i=1,2,\end{aligned}$$

and thus we have the following estimates after taking ∂^3 and using $D_t^{\mathring{\widetilde{\varphi}}}=(\partial_t+\overline{\mathring{v}}\cdot\overline{\nabla})+(\partial_3\mathring{\widetilde{\varphi}})^{-1}(\mathring{v}\cdot\mathring{\widetilde{\mathbf{N}}}-\partial_t\mathring{\varphi})\partial_3$ to get

$$\|\check{q}\|_4\lesssim_{\mathring{K}_0}\|\check{q}\|_0+\|\mathcal{T}v\|_3+\|\mathring{\rho}-1\|_3, \tag{5.45}$$

where $\mathcal{T}$ denotes a tangential derivative, including $\partial_t,\overline{\partial}$ and $\omega(x)\partial_3$ for some weight function ω that vanishes on Σ and is approximately equal to $|x_3|$ near Σ. Replacing ∂^3 by $\partial^{3-k}\partial_t^k$, we know the estimates of $\partial_t^k\check{q}$ is reduced to the estimates of $\partial_t^{k+1}v$ and $\partial_t^k\mathcal{T}v$.

5.2.5 Control of full time derivatives

From the reduction procedures for $\check{q}$ and the div-curl analysis for v, we know a spatial derivative of $\check{q}$ is reduced to a tangential derivative of v, and the divergence of v is reduced to $\mathring{\mathcal{F}}'(\mathring{q})\partial_t\check{q}$. Repeatedly, it remains to control $\partial_t^4 q$ and $\mathcal{T}^\alpha v$ with $|\alpha|=4$. Here we only present the proof for the estimates of full-time derivatives which is parallel to Section 4.6, and the other $\mathcal{T}^\alpha$-estimates are easier. We introduce the Alinhac good unknowns $\mathring{\mathbf{V}},\mathring{\mathbf{Q}}$ for the ∂_t^4-differentiated linearized system (5.6)

$$\mathring{\mathbf{V}}:=\partial_t^4 v-\partial_t^4\mathring{\widetilde{\varphi}}\partial_3^{\mathring{\widetilde{\varphi}}}v,\quad \mathring{\mathbf{Q}}:=\partial_t^4\check{q}-\partial_t^4\mathring{\widetilde{\varphi}}\partial_3^{\mathring{\widetilde{\varphi}}}\check{q} \tag{5.46}$$

Similarly as Section 4.6, we have the following identity for $f=v_i$ and $\check{q}$

$$\partial_t^4(\nabla_i^{\mathring{\widetilde{\varphi}}}f)=\nabla_i^{\mathring{\widetilde{\varphi}}}\mathring{\mathbf{F}}+\mathring{\mathfrak{C}}_i(f), \tag{5.47}$$

where $\mathring{\mathfrak{C}}_i(f):=\partial_3^{\mathring{\widetilde{\varphi}}}\partial_i^{\mathring{\widetilde{\varphi}}}f\partial_t^4\mathring{\widetilde{\varphi}}+\mathring{\mathfrak{C}}_i'(f)$ and

$$\mathring{\mathfrak{C}}_i'(f)=-\left[\partial_t^4,\frac{\partial_i\mathring{\widetilde{\varphi}}}{\partial_3\mathring{\widetilde{\varphi}}},\partial_3 f\right]-\partial_3 f\left[\partial_t^4,\partial_i\mathring{\widetilde{\varphi}},\frac{1}{\partial_3\mathring{\widetilde{\varphi}}}\right]+\partial_i\mathring{\widetilde{\varphi}}\partial_3 f\left[\partial_t^3,\frac{1}{(\partial_3\mathring{\widetilde{\varphi}})^2}\right]\partial_t\partial_3\mathring{\widetilde{\varphi}},\quad i=1,2 \tag{5.48}$$

$$\mathring{\mathfrak{C}}_3'(f)=\left[\partial_t^4,\frac{1}{\partial_3\mathring{\widetilde{\varphi}}},\partial_3 f\right]+\partial_3 f\left[\partial_t^3,\frac{1}{(\partial_3\mathring{\widetilde{\varphi}})^2}\right]\partial_t\partial_3\mathring{\widetilde{\varphi}}. \tag{5.49}$$

Then we take ∂_t^4 to the first two equations of (5.6) to obtain

$$\mathring{\rho} D_t^{\mathring{\widetilde{\varphi}}} \mathring{\mathbf{V}}_i + \nabla_i^{\mathring{\widetilde{\varphi}}} \mathring{\mathbf{Q}} = \mathring{\mathcal{R}}_i^1, \tag{5.50}$$

$$\mathring{\mathcal{F}}'(\mathring{q}) D_t^{\mathring{\widetilde{\varphi}}} \mathring{\mathbf{Q}} + \nabla^{\mathring{\widetilde{\varphi}}} \cdot \mathring{\mathbf{V}} = \mathring{\mathcal{R}}^2 - \mathring{\mathfrak{C}}_i(v^i), \tag{5.51}$$

where

$$\mathring{\mathcal{R}}_i^1 := -\,[\partial_t^4, \mathring{\rho}] D_t^{\mathring{\widetilde{\varphi}}} v_i - \mathring{\rho}\left(\mathring{\mathfrak{D}}(v_i) + \mathring{\mathfrak{E}}(v_i)\right) - \mathring{\mathfrak{C}}_i(\check{q}) - \partial_t^4 \mathring{\rho} g \delta_{3i}, \tag{5.52}$$

$$\mathring{\mathcal{R}}^2 := -\,[\partial_t^4, \mathring{\mathcal{F}}'(\mathring{q})] D_t^{\mathring{\widetilde{\varphi}}} \check{q} - \mathring{\mathcal{F}}'(\mathring{q})\left(\mathring{\mathfrak{D}}(\check{q}) + \mathring{\mathfrak{E}}(\check{q})\right) + \partial_t^4(\mathring{\mathcal{F}}'(\mathring{q}) g \mathring{v}_3), \tag{5.53}$$

and the commutators $\mathring{\mathfrak{D}}(f), \mathring{\mathfrak{E}}(f)$ are defined in the same way as in (4.39) and (4.40) by replacing $\mathcal{T}^\alpha$ with ∂_t^4, replacing $\overline{\partial}$ with ∂_t, replacing $\widetilde{\varphi}$ by $\mathring{\widetilde{\varphi}}$. The last two terms in (4.39) vanish because ∂_t^4 commutes with ∂_3. Specifically, we have

$$\partial_t^4 D_t^{\mathring{\widetilde{\varphi}}} f = D_t^{\mathring{\widetilde{\varphi}}} \mathring{\mathbf{F}} + \mathring{\mathfrak{D}}(f) + \mathring{\mathfrak{E}}(f) \tag{5.54}$$

where $\mathring{\mathfrak{D}}(f) := (D_t^{\mathring{\widetilde{\varphi}}} \partial_3^{\mathring{\widetilde{\varphi}}} f)\partial_t^4 \mathring{\widetilde{\varphi}} + \mathring{\mathfrak{D}}'(f)$ and

$$\begin{aligned}\mathring{\mathfrak{D}}'(f) =& [\partial_t^4, \mathring{\overline{v}}] \cdot \overline{\partial} f + \left[\partial_t^4, \frac{1}{\partial_3 \mathring{\widetilde{\varphi}}}(\mathring{v} \cdot \dot{\widetilde{\mathbf{N}}} - \partial_t \mathring{\varphi}), \partial_3 f\right] + \left[\partial_t^4, \mathring{v} \cdot \dot{\widetilde{\mathbf{N}}} - \partial_t \mathring{\varphi}, \frac{1}{\partial_3 \mathring{\widetilde{\varphi}}}\right] \partial_3 f + \frac{1}{\partial_3 \mathring{\widetilde{\varphi}}}[\partial_t^4, \mathring{v}] \cdot \dot{\widetilde{\mathbf{N}}} \partial_3 f \\ & - 4(\mathring{v} \cdot \dot{\widetilde{\mathbf{N}}} - \partial_t \mathring{\varphi})\partial_3 f \left[\partial_t^3, \frac{1}{(\partial_3 \mathring{\widetilde{\varphi}})^2}\right] \partial_t \partial_3 \mathring{\widetilde{\varphi}},\end{aligned} \tag{5.55}$$

and

$$\mathring{\mathfrak{E}}(f) := \partial_t^5(\dot{\widetilde{\varphi}} - \mathring{\varphi})\partial_3^{\widetilde{\varphi}} f. \tag{5.56}$$

Analogous to Lemma 4.4, we list the estimates for commutators $\mathring{\mathfrak{C}}, \mathring{\mathfrak{D}}, \mathring{\mathfrak{E}}$.

Lemma 5.3. Let $\mathring{\mathbf{F}} := \partial_t^4 f - \partial_3^{\mathring{\widetilde{\varphi}}} f \partial_t^4 \mathring{\widetilde{\varphi}}$ be the Alinhac good unknowns associated with the smooth function f. Assume $\partial_3 \widetilde{\varphi} \geq c_0 > 0$ and let $\mathring{\mathfrak{C}}(f)$, $\mathring{\mathfrak{D}}(f)$, and $\mathring{\mathfrak{E}}(f)$ be the remainder terms defined as mentioned above. Then

$$\|\partial_t^4 f\|_0 \leq \|\mathring{\mathbf{F}}\|_0 + c_0^{-1}\|\partial_3 f\|_\infty |\partial_t^4 \widetilde{\psi}|_0, \tag{5.57}$$

$$\|\mathring{\mathfrak{C}}_i(f)\|_0 \leq P\left(c_0^{-1}, |\overline{\nabla} \mathring{\widetilde{\psi}}|_\infty, \sum_{k=1}^3 |\overline{\nabla} \partial_t^k \mathring{\widetilde{\psi}}|_{3-k}\right) \cdot \left(\|\partial f\|_\infty + \sum_{k=1}^3 \|\partial_t^k f\|_{4-k}\right), \quad i = 1, 2, 3, \tag{5.58}$$

$$\|\mathring{\mathfrak{D}}(f)\|_0 \leq P\left(c_0^{-1}, |\overline{\nabla} \mathring{\widetilde{\psi}}|_\infty, \sum_{k=1}^3 |\overline{\nabla} \partial_t^k \mathring{\widetilde{\psi}}, \overline{\nabla} \partial_t^k \dot{\widetilde{\psi}}|_{3-k}\right) \cdot \left(\|\partial f\|_\infty + \sum_{k=1}^3 \|\partial_t^k f\|_{4-k}\right), \tag{5.59}$$

$$\|\mathring{\mathfrak{E}}(f)\|_0 \leq (|\partial_t^5 \psi|_0 + |\partial_t^5 \widetilde{\psi}|_0)\|\partial f\|_\infty. \tag{5.60}$$

We introduce the boundary conditions for $\mathring{\mathbf{V}}, \mathring{\mathbf{Q}}$. The ∂_t^4-differentiated linearized kinematic boundary condition now reads

$$\partial_t^5 \psi + (\overline{v} \cdot \overline{\nabla})\overline{\partial}^4 \mathring{\widetilde{\psi}} - \mathring{\mathbf{V}} \cdot \mathring{\widetilde{N}} = \mathring{\mathcal{S}}_1, \quad \text{on } \Sigma, \tag{5.61}$$

where

$$\mathring{\mathcal{S}}_1 := \partial_3 v \cdot \mathring{\widetilde{N}} \partial_t^4 \mathring{\widetilde{\psi}} + \sum_{1 \leq j \leq 3} \binom{4}{j} \partial_t^j v \cdot \partial_t^{4-j} \mathring{\widetilde{N}}. \tag{5.62}$$

Also, since $\mathring{\mathbf{Q}}|_\Sigma = \partial_t^4 \check{q} - \partial_3^{\mathring{\widetilde{\varphi}}} \check{q} \partial_t^4 \mathring{\widetilde{\psi}}$, the boundary condition of $\mathring{\mathbf{Q}}$ on Σ reads

$$\mathring{\mathbf{Q}} = -\sigma \partial_t^4 \overline{\nabla} \cdot \left(\frac{\overline{\nabla} \mathring{\widetilde{\psi}}}{\sqrt{1 + |\overline{\nabla} \mathring{\widetilde{\psi}}|^2}}\right) + \kappa^2(1 - \overline{\Delta})\partial_t^5 \psi - \partial_3 \check{q} \partial_t^4 \mathring{\widetilde{\psi}} + g \partial_t^4 \mathring{\widetilde{\psi}}. \tag{5.63}$$

Using Reynold transport formula (A.9), we have the following equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega \mathring{\rho}|\mathring{\mathbf{V}}|^2\partial_3\widetilde{\mathring{\varphi}}\,\mathrm{d}x = \frac{1}{2}\int_\Omega |\mathring{\mathbf{V}}|^2\left((D_t^{\widetilde{\mathring{\varphi}}}\mathring{\rho} + \mathring{\rho}\nabla^{\widetilde{\mathring{\varphi}}}\cdot\mathring{v})\partial_3\widetilde{\mathring{\varphi}} + \mathring{\rho}\mathring{M}\right)\mathrm{d}x + \int_\Omega \mathring{\mathbf{Q}}(\nabla^{\widetilde{\mathring{\varphi}}}\cdot\mathring{\mathbf{V}})\partial_3\widetilde{\mathring{\varphi}}\,\mathrm{d}x - \int_\Sigma \mathring{\mathbf{Q}}(\mathring{\mathbf{V}}\cdot\widetilde{\mathring{N}})\,\mathrm{d}x' + \int_\Omega \mathring{\mathbf{V}}\cdot\mathring{\mathcal{R}}^1\partial_3\widetilde{\varphi}\,\mathrm{d}x, \tag{5.64}$$

where $\mathring{M} := \partial_t\partial_3(\dot{\widetilde{\varphi}} - \mathring{\varphi}) + \partial_3(\partial_t + \overline{\mathring{v}}\cdot\overline{\nabla})(\widetilde{\mathring{\varphi}} - \dot{\widetilde{\varphi}})$ represents the mismatched terms involving tangential smoothing in the linearized Reynold transport formula. The contribution of the first integral can be directly controlled by $\|\mathring{M}\|_\infty \leq P(\mathring{K}_0)$ because all these quantities are already given. The last integral is also directly controlled by $P(\mathring{K}_0)\|\mathring{\mathbf{V}}\|_0\sqrt{\mathring{E}^\kappa(t)}$. Then for the second term in (5.64), we invoke (5.51) to get the estimates parallel to (4.117)

$$\begin{aligned}
&\int_\Omega \mathring{\mathbf{Q}}(\nabla^{\widetilde{\mathring{\varphi}}}\cdot\mathring{\mathbf{V}})\partial_3\widetilde{\mathring{\varphi}}\,\mathrm{d}x\\
=&\underbrace{-\int_\Omega \partial_t^4\check{q}\mathring{\mathfrak{C}}_i(v^i)\partial_3\widetilde{\mathring{\varphi}}\,\mathrm{d}x}_{=:\mathring{I}_0} + \int_\Omega \partial_t^4\widetilde{\mathring{\varphi}}\partial_3^{\widetilde{\mathring{\varphi}}}\check{q}\mathring{\mathfrak{C}}_i(v^i)\partial_3\widetilde{\mathring{\varphi}}\,\mathrm{d}x - \int_\Omega \mathring{\mathcal{F}}'(\mathring{q})D_t^{\widetilde{\mathring{\varphi}}}\mathring{\mathbf{Q}}\,\mathring{\mathbf{Q}}\partial_3\widetilde{\mathring{\varphi}}\,\mathrm{d}x + \int_\Omega \mathring{\mathcal{R}}^2\mathring{\mathbf{Q}}\partial_3\widetilde{\mathring{\varphi}}\,\mathrm{d}x\\
\lesssim&\ \mathring{I}_0 - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\sqrt{\mathring{\mathcal{F}}'(\mathring{q})}\mathbf{Q}\right\|_0^2 + \left\|\sqrt{\mathring{\mathcal{F}}'(\mathring{q})}\partial_t^4\check{q}\right\|_0^2(\|\nabla^{\widetilde{\mathring{\varphi}}}\cdot\mathring{v}\|_\infty + \|\mathring{M}\|_\infty)\\
&+\|\mathring{\mathfrak{C}}_i(v^i)\|_0\|\partial\check{q}\|_\infty|\partial_t^4\widetilde{\mathring{\psi}}|_0 + \left\|\sqrt{\mathring{\mathcal{F}}'(\mathring{q})}\mathbf{Q}\right\|_0\left\|\sqrt{\mathring{\mathcal{F}}'(\mathring{q})^{-1}}\mathring{\mathcal{R}}^2\right\|_0\\
\lesssim&\ \mathring{I}_0 - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\sqrt{\mathring{\mathcal{F}}'(\mathring{q})}\mathbf{Q}\right\|_0^2 + P(\mathring{K}_0)\mathring{E}^\kappa(t),
\end{aligned}\tag{5.65}$$

where we note that all terms in $\mathring{\mathcal{R}}^2$ contains at least linear weight $\mathring{\mathcal{F}}'(\mathring{q})$ and thus the control of $\sqrt{\mathring{\mathcal{F}}'(\mathring{q})^{-1}}\mathring{\mathcal{R}}^2$ is still uniform in $\mathring{\mathcal{F}}'(\mathring{q})$.

Now it remains to control the boundary integral. Compared with the nonlinear system, the estimate for the linearized system is easier as the surface tension term now becomes a given term. Plugging (5.61) and (5.63) into the boundary integral, we get

$$\begin{aligned}
-\int_\Sigma \mathring{\mathbf{Q}}(\mathring{\mathbf{V}}\cdot\widetilde{\mathring{N}})\,\mathrm{d}x' =& -\int_\Sigma \partial_t^4\overline{\nabla}\cdot(\overline{\nabla}\widetilde{\mathring{\psi}}/|\widetilde{\mathring{N}}|)\partial_t^5\psi\,\mathrm{d}x' - \kappa^2\int_\Sigma \partial_t^4(1-\overline{\Delta})\partial_t\psi\cdot\partial_t^5\psi\,\mathrm{d}x'\\
&-\int_\Sigma g\partial_t^4\widetilde{\mathring{\psi}}\partial_t^5\psi\,\mathrm{d}x' + \int_\Sigma \partial_3\check{q}\partial_t^4\widetilde{\mathring{\psi}}\partial_t^5\psi\,\mathrm{d}x'\\
&-\int_\Sigma \mathbf{Q}(\overline{\mathring{v}}\cdot\overline{\nabla})\partial_t^4\widetilde{\mathring{\psi}}\,\mathrm{d}x' + \int_\Sigma \mathring{\mathbf{Q}}\mathring{\mathcal{S}}_1\,\mathrm{d}x',
\end{aligned}\tag{5.66}$$

where the second term gives us the boundary energy

$$-\kappa^2\int_\Sigma \partial_t^4(1-\overline{\Delta})\partial_t\psi\cdot\partial_t^5\psi\,\mathrm{d}x' = \kappa^2\int_\Sigma \left|\langle\overline{\partial}\rangle\partial_t^5\psi\right|^2\,\mathrm{d}x' \tag{5.67}$$

We note that the first, the third, and the fourth terms in (5.66) can all be directly controlled under the time integral

$$-\int_0^T\int_\Sigma \partial_t^4\overline{\nabla}\cdot(\overline{\nabla}\widetilde{\mathring{\psi}}/|\widetilde{\mathring{N}}|)\partial_t^5\psi\,\mathrm{d}x'\,\mathrm{d}t \lesssim \varepsilon|\partial_t^5\psi|^2_{L_t^2H_{x'}^1} + P(|\overline{\nabla}\widetilde{\mathring{\psi}}|_\infty)|\overline{\nabla}\partial_t^4\widetilde{\mathring{\psi}}|_0^2 \leq_{\kappa^{-1}} \varepsilon\mathring{E}^\kappa(T) + P(\mathring{K}_0) \tag{5.68}$$

$$-\int_0^T\int_\Sigma (g-\partial_3\check{q})\partial_t^4\widetilde{\mathring{\psi}}\partial_t^5\psi\,\mathrm{d}x'\,\mathrm{d}t \leq \varepsilon|\partial_t^5\psi|^2_{L_t^2L_{x'}^2} + |\partial_t^4\widetilde{\mathring{\psi}}|_0^2(1+\|\partial\check{q}\|^2_{L_t^2L_x^\infty}) \leq_{\kappa^{-1}} \varepsilon\mathring{E}^\kappa(T) + P(\mathring{K}_0)\int_0^T \mathring{E}^\kappa(t)\,\mathrm{d}t. \tag{5.69}$$

The fifth term is also controlled directly by using the mollifier property (3.5):

$$\begin{aligned}
-\int_0^T\int_\Sigma \mathbf{Q}(\overline{\mathring{v}}\cdot\overline{\nabla})\partial_t^4\widetilde{\mathring{\psi}}\,\mathrm{d}x'\,\mathrm{d}t =& -\int_0^T \sigma\int_\Sigma \partial_t^4\overline{\nabla}\cdot(\overline{\nabla}\widetilde{\mathring{\psi}}/|\widetilde{\mathring{N}}|)(\overline{\mathring{v}}\cdot\overline{\nabla})\partial_t^4\widetilde{\mathring{\psi}}\,\mathrm{d}x'\,\mathrm{d}t + \kappa^2\int_0^T\int_\Sigma (1-\overline{\Delta})\partial_t^5\psi(\overline{\mathring{v}}\cdot\overline{\nabla})\partial_t^4\widetilde{\mathring{\psi}}\,\mathrm{d}x'\,\mathrm{d}t\\
&+\int_0^T\int_\Sigma (g-\partial_3\check{q})\partial_t^4\widetilde{\mathring{\psi}}(\overline{\mathring{v}}\cdot\overline{\nabla})\partial_t^4\widetilde{\mathring{\psi}}\,\mathrm{d}x'\,\mathrm{d}t\\
\lesssim_{\kappa^{-1}}&\ \varepsilon|\partial_t^5\psi|^2_{L_t^2H_{x'}^1} + P(\mathring{K}_0)\int_0^T \mathring{E}^\kappa(t)\,\mathrm{d}t.
\end{aligned}\tag{5.70}$$

It remains to analyze the last integral which will be canceled with $\mathring{I}_0$ defined in (5.65). Following the analysis in (4.130)-(4.139), we have

$$\int_\Sigma \mathring{\mathbf{Q}}\mathring{\mathcal{S}}_1 \,\mathrm{d}x' = 4\int_\Sigma \partial_t^4\check{q}\partial_t^3 v\cdot\partial_t\mathring{\widetilde{N}}\,\mathrm{d}x' + \text{ controllable terms,} \tag{5.71}$$

$$\mathring{I}_0 = -4\int_\Omega \partial_t^4\check{q}\partial_t\mathring{\widetilde{\mathbf{N}}}_i\partial_3\partial_t^3 v^i\,\mathrm{d}x + \text{ controllable terms,} \tag{5.72}$$

and then we add them together and use the divergence theorem to get

$$\begin{aligned}&4\int_\Sigma \partial_t^4\check{q}\partial_t^3 v\cdot\partial_t\mathring{\widetilde{N}}\,\mathrm{d}x' - 4\int_\Omega \partial_t^4\check{q}\partial_t\mathring{\widetilde{\mathbf{N}}}_i\partial_3\partial_t^3 v^i\,\mathrm{d}x\\ =&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(\partial_t^3\partial_3\check{q}\partial_t\mathring{\widetilde{\mathbf{N}}}+\partial_t^3\check{q}\partial_t\partial_3\mathring{\widetilde{\mathbf{N}}})\cdot\partial_t^3 v\,\mathrm{d}x + \int_\Omega \partial_t^3\partial_3\check{q}\partial_t(\partial_t\mathring{\widetilde{\mathbf{N}}}\cdot\partial_t^3 v) + \partial_t^3\check{q}\partial_t(\partial_t\partial_3\mathring{\widetilde{\mathbf{N}}}\cdot\partial_3 v)\,\mathrm{d}x,\end{aligned} \tag{5.73}$$

whose time integral can be easily bounded by $\varepsilon\|\partial_t^3\partial_3\check{q}\|_0^2 + \mathring{E}^\kappa(0) + P(\mathring{K}_0)\int_0^T \mathring{E}^\kappa(t)\,\mathrm{d}t$. Hence, we get the control of boundary integral

$$-\int_0^T\int_\Sigma \mathring{\mathbf{Q}}(\mathring{\mathbf{V}}\cdot\mathring{\widetilde{N}})\,\mathrm{d}x'\,\mathrm{d}t + \kappa^2\int_0^T\int_\Sigma \left|\langle\overline{\partial}\rangle\partial_t^5\widetilde{\psi}\right|_0^2\mathrm{d}t \le \varepsilon\|\partial_t^3\partial_3\check{q}\|_0^2 + \mathring{E}^\kappa(0) + P(\mathring{K}_0)\int_0^T \mathring{E}^\kappa(t)\,\mathrm{d}t. \tag{5.74}$$

Combining this with (5.64), (5.65) and the definition of Alinhac good unknowns we get the estimates for the full-time derivatives

$$\|\partial_t^4 v(t)\|_0^2 + \left\|\sqrt{\mathring{\mathcal{F}}'(\mathring{q})}\partial_t^4\check{q}\right\|_0^2 + \kappa^2\int_0^T\int_\Sigma\left|\langle\overline{\partial}\rangle\partial_t^5\widetilde{\psi}\right|_0^2\mathrm{d}t \le \varepsilon\|\partial_t^3\partial_3\check{q}\|_0^2 + \mathring{E}^\kappa(0) + P(\mathring{K}_0)\int_0^T \mathring{E}^\kappa(t)\,\mathrm{d}t. \tag{5.75}$$

This together with div-curl analysis gives us the energy inequality of $\mathring{E}^\kappa(t)$ after choosing $\varepsilon > 0$ suitably small

$$\mathring{E}^\kappa(t) \le_{\kappa^{-1}} \mathring{E}^\kappa(0) + P(\mathring{K}_0)\int_0^t \mathring{E}^\kappa(\tau)\mathrm{d}\tau, \tag{5.76}$$

which together with Grönwall's inequality implies that there exists some $T^\kappa > 0$ such that

$$\sup_{0\le t\le T^\kappa}\mathring{E}^\kappa(t) \le C(\kappa^{-1},\mathring{K}_0)\mathring{E}^\kappa(0).$$

5.2.6 Regularity of ψ and its time derivatives

The regularity of $\partial_t^{k+1}\psi$ $(0\le k\le 3)$ can be enhanced to $H^{5.5-k}$ by the boundary elliptic estimates once we close the energy estimates for $\mathring{E}^\kappa(t)$. Since the boundary condition gives

$$\kappa^2(1-\overline{\Delta})\partial_t\psi = \check{q} - g\mathring{\widetilde{\psi}} + \sigma\mathcal{H}(\overline{\nabla}\mathring{\widetilde{\psi}}, \overline{\nabla}^2\mathring{\widetilde{\psi}}).$$

Hence, by virtue of (5.11) and the elltipic estimate, it holds that

$$|\partial_t^{k+1}\psi|_{5.5-k} \le \kappa^{-2}\left(\sigma P(|\overline{\nabla}\mathring{\widetilde{\psi}}|_\infty)|\partial_t^k\overline{\nabla}^2\mathring{\widetilde{\psi}}|_{3.5-k} + |\partial_t^k q|_{3.5-k} + P(\mathring{K}_0)\right) \le C(\kappa^{-1},\mathring{K}_0)\mathring{E}^\kappa. \tag{5.77}$$

Moreover, $|\psi|_{5.5}$ is controlled by

$$|\psi|_{5.5} \le |\psi_{0,\kappa}|_{5.5} + \int_0^T |\partial_t\psi(t)|_{5.5}\,\mathrm{d}t. \tag{5.78}$$

Therefore, the uniform-in-n estimates for (5.6) are proven by induction.

5.3 Picard iteration

So far, we have established the local existence and the uniform-in-n estimates for the linearized system (5.1) for each fixed $\kappa > 0$, namely

$$\begin{cases} \rho^{(n)} D_t^{\widetilde{\varphi}^{(n)}} v^{(n+1)} + \nabla^{\widetilde{\varphi}^{(n)}} \check{q}^{(n+1)} = -(\rho^{(n)} - 1) g e_3 & \text{in } [0,T]\times\Omega, \\ \mathcal{F}^{(n)\prime}(q^{(n)}) D_t^{\widetilde{\varphi}^{(n)}} \check{q}^{(n+1)} + \nabla^{\widetilde{\varphi}^{(n)}} \cdot v^{(n+1)} = \mathcal{F}^{(n)\prime}(q^{(n)}) g v_3^{(n)} & \text{in } [0,T]\times\Omega, \\ q^{(n+1)} = q^{(n+1)}(\rho^{(n+1)}), \check{q}^{(n+1)} = q^{(n+1)} + g\widetilde{\varphi}^{(n)} & \text{in } [0,T]\times\Omega, \\ \check{q}^{(n+1)} = g\widetilde{\psi}^{(n)} - \sigma\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\widetilde{\psi}^{(n)}}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}^{(n)}|^2}}\right) + \kappa^2(1-\overline{\Delta})(v^{(n+1)}\cdot\widetilde{N}^{(n)}) & \text{on } [0,T]\times\Sigma, \\ \partial_t\psi^{(n+1)} = v^{(n+1)}\cdot\widetilde{N}^{(n)} & \text{on } [0,T]\times\Sigma, \\ v_3^{(n+1)} = 0 & \text{on } [0,T]\times\Sigma_b, \\ (v^{(n+1)}, \rho^{(n+1)}, \psi^{(n+1)})|_{t=0} = (v_0^\kappa, \rho_0^\kappa, \psi_0^\kappa), \end{cases} \tag{5.79}$$

where $\psi^{(n)}, \varphi^{(n)}, D_t^{\widetilde{\varphi}^{(n)}}, \nabla^{\widetilde{\varphi}^{(n)}}$ are defined in (5.2)-(5.5). Now it suffices to prove that, for each fixed $\kappa > 0$, the sequence $\{(v^{(n)}, \check{q}^{(n)}, \psi^{(n)})\}_{n\in\mathbb{N}^*}$ has a strongly convergent subsequence. Once we prove this, the limit of that subsequence becomes the solution to the nonlinear κ-approximate system (3.11) for this chosen κ.

For a function sequence $\{f^{(n)}\}$ we define $[f]^{(n)} := f^{(n+1)} - f^{(n)}$ and then we find that $\{([v]^{(n)}, [\check{q}]^{(n)}, [\psi]^{(n)})\}$ satisfies the following linear system

$$\begin{cases} \rho^{(n)} D_t^{\widetilde{\varphi}^{(n)}} [v]^{(n)} + \nabla^{\widetilde{\varphi}^{(n)}} [\check{q}]^{(n)} = -\mathring{f}_v^{(n)} & \text{in } [0,T]\times\Omega, \\ \mathcal{F}^{(n)\prime}(q^{(n)}) D_t^{\widetilde{\varphi}^{(n)}} [\check{q}]^{(n)} + \nabla^{\widetilde{\varphi}^{(n)}} \cdot [v]^{(n)} = -\mathring{f}_q^{(n)} & \text{in } [0,T]\times\Omega, \\ [\check{q}]^{(n)} = [q]^{(n)} + g[\widetilde{\varphi}]^{(n-1)} & \text{in } [0,T]\times\Omega, \\ [\check{q}]^{(n)} = g[\widetilde{\psi}]^{(n-1)} - \sigma[\mathcal{H}]^{(n-1)} + \kappa^2(1-\overline{\Delta})([v]^{(n)}\cdot\widetilde{N}^{(n)}) + \kappa^2(1-\overline{\Delta})(v^{(n)}\cdot[\widetilde{N}]^{(n-1)}), & \text{on } [0,T]\times\Sigma, \\ \partial_t[\psi]^{(n)} = [v]^{(n)}\cdot\widetilde{N}^{(n)} + (v^{(n)}\cdot[\widetilde{N}]^{(n-1)}), & \text{on } [0,T]\times\Sigma, \\ [v_3^{(n)}] = 0 & \text{on } [0,T]\times\Sigma_b, \\ ([v]^{(n)}, [\rho]^{(n)}, [\psi]^{(n)})|_{t=0} = (\mathbf{0}, 0, 0), \end{cases} \tag{5.80}$$

where $\mathring{f}_v^{(n)}$ and $\mathring{f}_q^{(n)}$ are defined by

$$\mathring{f}_v^{(n)} := [\rho]^{(n-1)}\partial_t v^{(n)} + [\rho\overline{v}]^{(n-1)}\cdot\overline{\nabla}v^{(n)} + [\rho V_{\mathring{\mathbf{N}}}]^{(n-1)}\partial_3 v^{(n)} + [\rho]^{(n-1)} g e_3 + \partial_3\check{q}^{(n)}[A_{i3}]^{(n-1)}, \tag{5.81}$$

$$\begin{aligned} \mathring{f}_q^{(n)} := {} & [\mathcal{F}'(q)]^{(n-1)}(\partial_t\check{q}^{(n)} - g v_3^{(n-1)}) + [\mathcal{F}'(q)\overline{v}]^{(n-1)}\cdot\overline{\nabla}\check{q}^{(n)} + [\mathcal{F}'(q)V_{\mathring{\mathbf{N}}}]^{(n-1)}\partial_3\check{q}^{(n)} \\ & - \mathcal{F}^{(n)\prime}(q^{(n)}) g[v_3]^{(n-1)} + \partial_3 v_i^{(n)}[A_{i3}]^{(n-1)}, \end{aligned} \tag{5.82}$$

and

$$V_{\mathring{\mathbf{N}}}^{(n)} := \frac{1}{\partial_3\widetilde{\varphi}^{(n)}}(v^{(n)}\cdot\widetilde{\mathbf{N}}^{(n-1)} - \partial_t\varphi^{(n)}),\ A_{13}^{(n)} := -\frac{\partial_1\widetilde{\varphi}^{(n)}}{\partial_3\widetilde{\varphi}^{(n)}},\ A_{23}^{(n)} := -\frac{\partial_2\widetilde{\varphi}^{(n)}}{\partial_3\widetilde{\varphi}^{(n)}},\ A_{33}^{(n)} := \frac{1}{\partial_3\widetilde{\varphi}^{(n)}},$$

$$[\mathcal{H}]^{(n-1)} := \mathcal{H}(\overline{\nabla}\widetilde{\psi}^{(n)}) - \mathcal{H}(\overline{\nabla}\widetilde{\psi}^{(n-1)}),\ \mathcal{H}(\overline{\nabla}\widetilde{\psi}) := -\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\widetilde{\psi}}{1+|\overline{\nabla}\widetilde{\psi}|^2}\right).$$

For $n \geq 1$, we define the energy of (5.80) $[E]^{(n)}$ to be the following quantity

$$[E]^{(n)}(t) := \sum_{k=0}^{3} \|\partial_t^k[v]^{(n)}(t)\|_{3-k}^2 + \|\partial_t^k[\check{q}]^{(n)}(t)\|_{3-k}^2 + \int_0^t \left|\partial_t^{k+1}[\psi]^{(n)}(\tau)\right|_{4-k}^2 \mathrm{d}\tau + |[\psi]^{(n)}(t)|_4^2 \tag{5.83}$$

It suffices to control $[E]^{(n)}(t)$ and use $([v]^{(n)}, [\rho]^{(n)}, [\psi]^{(n)})|_{t=0} = (\mathbf{0}, 0, 0)$ to show that $[E]^{(n)}(t) \leq \frac{1}{4}([E]^{(n-1)}(t) + [E]^{(n-2)}(t))$ in some time interval $[0, T_1^\kappa]$. The estimates for $[E]^{(n)}(t)$ are parallel to Section 5.2, so we will not go into every detail but only list the sketch of the proof.

5.3.1 Div-curl analysis for $[v]^{(n)}$

By Lemma 5.2, we have the following inequalities for $k = 0, 1, 2$

$$\|\partial_t^k[v]^{(n)}\|_{3-k}^2 \leq C(\mathring{K}_0)\left(\|\partial_t^k[v]^{(n)}\|_0^2 + \|\nabla^{\widetilde{\varphi}^{(n)}}\times\partial_t^k[v]^{(n)}\|_{2-k}^2 + \|\nabla^{\widetilde{\varphi}^{(n)}}\cdot\partial_t^k[v]^{(n)}\|_{2-k}^2 + |\partial_t^k[v]^{(n)}\cdot\widetilde{N}^{(n)}|_{2.5-k}^2\right). \tag{5.84}$$

The estimates for $L^2(\Omega)$ norms follow in the same way as Section 5.2.1 so we do not repeat here. For the curl part, we take $\nabla^{\widetilde{\varphi}^{(n)}}\times$ in the first equation of (5.80) to get

$$\rho^{(n)} D_t^{\widetilde{\varphi}^{(n)}}(\nabla^{\widetilde{\varphi}^{(n)}} \times [v]^{(n)}) = -\nabla^{\widetilde{\varphi}^{(n)}} \times \mathring{f}_v^{(n)} - \nabla^{\widetilde{\varphi}^{(n)}} \rho^{(n)} \times D_t^{\widetilde{\varphi}^{(n)}}[v]^{(n)} + \rho^{(n)}[\nabla^{\widetilde{\varphi}^{(n)}}\times, D_t^{\widetilde{\varphi}^{(n)}}][v]^{(n)}, \tag{5.85}$$

where $([\nabla^{\widetilde{\varphi}^{(n)}}\times, D_t^{\widetilde{\varphi}^{(n)}}][v]^{(n)})^i = \epsilon^{ijk}\nabla_j^{\widetilde{\varphi}^{(n)}} v_l^{(n)} \nabla_l^{\widetilde{\varphi}^{(n)}}[v]_k^{(n)} + \epsilon^{ijk}\nabla_j^{\widetilde{\varphi}^{(n)}}\partial_t(\widetilde{\varphi}^{(n)} - \widetilde{\varphi}^{(n-1)})\partial_3[v]_k^{(n)}$ and $\nabla^{\widetilde{\varphi}^{(n)}} \times \mathring{f}_v^{(n)}$ contains at most two derivatives of $v^{(n)}, \varphi^{(n)}, \varphi^{(n-1)}, \varphi^{(n-2)}$. Taking ∂^2, we have

$$\rho^{(n)} D_t^{\widetilde{\varphi}^{(n)}}(\partial^2\nabla^{\widetilde{\varphi}^{(n)}} \times [v]^{(n)}) = \partial^2(\text{RHS of (5.85)}) - [\partial^2, \rho^{(n)} D_t^{\widetilde{\varphi}^{(n)}}](\nabla^{\widetilde{\varphi}^{(n)}} \times [v]^{(n)}). \tag{5.86}$$

Based on the analysis above, we find that the leading-order terms of $[v]^{(n)}, [v]^{(n-1)}$ must be linear in $[v]^{(n)}, [v]^{(n-1)}$ respectively. Using Reynold transport formula (A.9) for the linearized system, the curl part can be directly controlled as in (5.33)

$$\begin{aligned}\|\nabla^{\widetilde{\varphi}^{(n)}} \times [v]^{(n)}(T)\|_2^2 &\leq C(\mathring{K}_0)\left(\|\underbrace{\nabla^{\widetilde{\varphi}^{(n)}} \times [v]^{(n)}(0)}_{=0}\|_2^2 + \int_0^T P(\mathring{E}^{(n)}, \mathring{E}^{(n-1)}, \mathring{E}^{(n-2)})[E]^{(n)}(t)\,\mathrm{d}t\right) \\ &\leq C(\mathring{K}_0)\int_0^T [E]^{(n)}(t) + [\mathring{E}]^{(n-1)}(t) + [\mathring{E}]^{(n-2)}(t)\,\mathrm{d}t.\end{aligned} \tag{5.87}$$

Similarly, replacing ∂^2 by $\partial^{2-k}\partial_t^k$ for $k = 1, 2$, we get similar results

$$\|\nabla^{\widetilde{\varphi}^{(n)}} \times \partial_t^k[v]^{(n)}(T)\|_{2-k}^2 \leq C(\mathring{K}_0)\int_0^T [E]^{(n)}(t) + [\mathring{E}]^{(n-1)}(t) + [\mathring{E}]^{(n-2)}(t)\,\mathrm{d}t. \tag{5.88}$$

As for the divergence, the second equation in (5.80) gives

$$\|\nabla^{\widetilde{\varphi}^{(n)}} \cdot [v]^{(n)}\|_2^2 \leq \|\mathcal{F}^{(n)\prime}(q^{(n)}) D_t^{\widetilde{\varphi}^{(n)}}[\check{q}]^{(n)}\|_2^2 + \|\mathring{f}_q\|_2^2 \leq P(\mathring{K}_0)\|\mathcal{F}^{(n)\prime}(q^{(n)})\mathcal{T}[\check{q}]^{(n)}\|_2^2, \tag{5.89}$$

where $\mathcal{T} = \partial_t$ or $\overline{\partial}$ or $\omega\partial_3$ for a bounded weight function ω vanishing on Σ. Therefore, the divergence is then reduced to the tangential derivatives of $[\check{q}]$. Similarly, the divergence of $\partial_t^k[v]^{(n)}$ is reduced to $\partial_t^k\mathcal{T}\check{q}$ so we omit this reduce step.

Next, the normal traces are still controlled by using boundary elliptic estimates. Note that the Dirichlet boundary condition for $[\check{q}]^{(n)}$ can be written as

$$-\kappa^2(1-\overline{\Delta})([v]^{(n)} \cdot \widetilde{N}^{(n)}) = -[q]^{(n)} - \sigma\left(\mathcal{H}(\overline{\nabla}\widetilde{\psi}^{(n)}) - \mathcal{H}(\overline{\nabla}\widetilde{\psi}^{(n-1)})\right) + \kappa^2(1-\overline{\Delta})(v^{(n)} \cdot [\widetilde{N}]^{(n-1)}), \tag{5.90}$$

and thus

$$|[v]^{(n)} \cdot \widetilde{N}^{(n)}|_{2.5}^2 \leq_{\kappa^{-1}} \|[q]^{(n)}\|_1^2 + P(\mathring{K}_0) + |\overline{v}^{(n)} \cdot \overline{\nabla}\widetilde{\psi}^{(n-1)}|_{2.5}^2 + |v_3^{(n)}|_{2.5}^2 \leq \|[q]^{(n)}\|_1^2 + P(\mathring{K}_0). \tag{5.91}$$

Similarly, we have for $k = 1, 2$

$$|\partial_t^k[v]^{(n)} \cdot \widetilde{N}^{(n)}|_{2.5-k}^2 \leq_{\kappa^{-1}} \|\partial_t^k[q]^{(n)}\|_1^2 + P(\mathring{K}_0). \tag{5.92}$$

5.3.2 Reduction of pressure $[\check{q}]^{(n)}$

This step is also quite similar to Section 5.2.4. We first consider the third component of the first equation in (5.80)

$$(\partial_3\widetilde{\varphi}^{(n)})^{-1}\partial_3[\check{q}]^{(n)} = -\rho^{(n)} D_t^{\widetilde{\varphi}^{(n)}}[v]^{(n)} + \mathring{f}_v^{(n)}, \tag{5.93}$$

which means the control of $\partial_3[\check{q}]^{(n)}$ is reduced to $\mathcal{T}[v]^{(n)}$. Then considering the first and second components, we can further reduce the control of $\overline{\partial}_i\check{q}$ $(i = 1, 2)$ to $\partial_3\check{q}$ and $\mathcal{T}v$ due to $\nabla_i^{\mathring{\widetilde{\varphi}}} = \overline{\partial}_i - \overline{\partial}_i\mathring{\widetilde{\varphi}}\partial_3^{\mathring{\widetilde{\varphi}}}$. Therefore, combining the div-curl analysis and reduction procedures for $[\check{q}]^{(n)}$, it suffices to control $\partial_t^2\overline{\partial}[\check{q}]^{(n)}$ and $\partial_t^3[\check{q}]^{(n)}$.

5.3.3 Tangential estimates for full-time derivatives

Again we only show the control of $\partial_t^3[\check{q}]^{(n)}$ and introduce the Alinhac good unknowns

$$[\mathbf{V}]^{(n)} := \partial_t^3[v]^{(n)} - \partial_t^3\widetilde{\varphi}^{(n)}\partial_3^{\widetilde{\varphi}^{(n)}}[v]^{(n)}, \quad [\mathbf{Q}]^{(n)} := \partial_t^3[\check{q}]^{(n)} - \partial_t^3\widetilde{\varphi}^{(n)}\partial_3^{\widetilde{\varphi}^{(n)}}[\check{q}]^{(n)}. \tag{5.94}$$

The Alinhac good unknowns $[\mathbf{V}]^{(n)}, [\mathbf{Q}]^{(n)}$ satisfy the following linear system

$$\rho^{(n)}D_t^{\widetilde{\varphi}^{(n)}}[\mathbf{V}_i]^{(n)} + \nabla_i^{\widetilde{\varphi}^{(n)}}[\mathbf{Q}]^{(n)} = \mathring{\mathcal{R}}_i^{1(n)}, \tag{5.95}$$

$$\mathcal{F}^{(n)'}(q^{(n)})D_t^{\widetilde{\varphi}^{(n)}}[\mathbf{Q}]^{(n)} + \nabla^{\widetilde{\varphi}^{(n)}}\cdot\mathbf{V}^{(n)} = \mathring{\mathcal{R}}^{2(n)} - \mathring{\mathfrak{C}}_i([v^i]^{(n)}), \tag{5.96}$$

where

$$\mathring{\mathcal{R}}_i^{1(n)} := -\partial_t^3\mathring{f}_{v_i}^{(n)} - [\partial_t^3, \rho^{(n)}]D_t^{\widetilde{\varphi}^{(n)}}[v_i]^{(n)} - \rho^{(n)}\left(\mathring{\mathfrak{D}}([v_i]^{(n)}) + \mathring{\mathfrak{E}}([v_i]^{(n)})\right) - \mathring{\mathfrak{C}}_i([\check{q}]) \tag{5.97}$$

$$\mathring{\mathcal{R}}^{2(n)} := -\partial_t^3\mathring{f}_q^{(n)} - [\partial_t^3, \mathcal{F}^{(n)'}(q^{(n)})]D_t^{\widetilde{\varphi}^{(n)}}[\check{q}]^{(n)} - \mathcal{F}^{(n)}(q^{(n)})\left(\mathring{\mathfrak{D}}([\check{q}]^{(n)}) + \mathring{\mathfrak{E}}([\check{q}]^{(n)})\right), \tag{5.98}$$

where $\mathring{\mathfrak{C}}, \mathring{\mathfrak{D}}, \mathring{\mathfrak{E}}$ are defined in a parallel way to (5.48)-(5.56), after replacing ∂_t^4 by ∂_t^3 and replacing $v, \check{q}, \mathring{\varphi}, \dot{\varphi}$ by $v^{(n)}, \check{q}^{(n)}, \varphi^{(n)}, \varphi^{(n-1)}$ respectively. The boundary conditions now become

$$[\mathbf{Q}]^{(n)} = g\partial_t^3[\widetilde{\psi}]^{(n-1)} - \sigma\partial_t^3\left(\mathcal{H}(\overline{\nabla}\widetilde{\psi}^{(n)}) - \mathcal{H}(\overline{\nabla}\widetilde{\psi}^{(n-1)})\right) + \kappa^2(1-\overline{\Delta})\partial_t^4[\psi]^{(n)} \tag{5.99}$$

$$[\mathbf{V}]^{(n)}\cdot\widetilde{N}^{(n)} = \partial_t^4[\psi]^{(n)} - \sum_{k=1}^{3}\binom{3}{k}\partial_t^{3-k}[v]^{(n)}\cdot\partial_t^k\widetilde{N}^{(n)} - \partial_t^3\widetilde{\psi}^{(n)}\partial_3[v]^{(n)}\cdot\widetilde{N}^{(n)} - \partial_t^3(\overline{v}^{(n)}\cdot[\widetilde{N}]^{(n-1)}) \tag{5.100}$$

Following (5.64)-(5.65), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \rho^{(n)}|[\mathbf{V}]^{(n)}|^2\partial_3\widetilde{\varphi}^{(n)}\,\mathrm{d}x + \int_\Omega \mathcal{F}^{(n)'}(q^{(n)})|[\mathbf{Q}]^{(n)}|^2\partial_3\widetilde{\varphi}^{(n)}\,\mathrm{d}x\right)\\
\le\,& C(\mathring{K}_0)\left(\int_0^T [\mathring{E}]^{(n)}(t) + [\mathring{E}]^{(n-1)}(t) + [\mathring{E}]^{(n-2)}(t)\,\mathrm{d}t\right) - \int_\Sigma [\mathbf{Q}]^{(n)}([\mathbf{V}]^{(n)}\cdot\widetilde{N}^{(n)})\,\mathrm{d}x' - \int_\Omega \partial_t^3[\check{q}]^{(n)}\mathring{\mathfrak{C}}_i([v^i]^{(n)})\partial_3\widetilde{\varphi}^{(n)}\,\mathrm{d}x.
\end{aligned} \tag{5.101}$$

Again, following (5.66)-(5.71), the artificial viscosity term gives the boundary energy

$$\begin{aligned}
&-\int_\Sigma [\mathbf{Q}]^{(n)}([\mathbf{V}]^{(n)}\cdot\widetilde{N}^{(n)})\,\mathrm{d}x' + \kappa^2\int_0^T \left|\partial_t^4[\psi]^{(n)}\right|_1^2\,\mathrm{d}t\\
\le\,& 3\int_\Sigma \partial_t^3[\check{q}]^{(n)}\partial_t^2[v]^{(n)}\cdot\partial_t\widetilde{N}^{(n)}\,\mathrm{d}x' + C(\mathring{K}_0)\left([\mathring{E}]^{(n)}(0) + \int_0^T [\mathring{E}]^{(n)}(t) + [\mathring{E}]^{(n-1)}(t) + [\mathring{E}]^{(n-2)}(t)\,\mathrm{d}t\right),
\end{aligned} \tag{5.102}$$

where we note that the contribution of $\partial_t^3(\overline{v}^{(n)}\cdot[\widetilde{N}]^{(n-1)})$ is controlled in a similar way as (4.127), that is, one can use divergence theorem to rewrite this term to be an interior integral.

Following (5.72), we have

$$-\int_\Omega \partial_t^3[\check{q}]^{(n)}\mathring{\mathfrak{C}}_i([v^i]^{(n)})\partial_3\widetilde{\varphi}^{(n)}\,\mathrm{d}x = -3\int_\Omega \partial_t^3[\check{q}]^{(n)}\partial_t\widetilde{\mathbf{N}}_i^{(n)}\partial_3\partial_t^2[v^i]\,\mathrm{d}x + \text{ controllable terms,} \tag{5.103}$$

and thus it can be controlled together with the remaining boundary integral by using divergence theorem and integration by parts in t

$$\begin{aligned}
&3\int_\Sigma \partial_t^3[\check{q}]^{(n)}\partial_t^2[v]^{(n)}\cdot\partial_t\widetilde{N}^{(n)}\,\mathrm{d}x' - 3\int_\Omega \partial_t^3[\check{q}]^{(n)}\partial_t\widetilde{\mathbf{N}}_i^{(n)}\partial_3\partial_t^2[v^i]\,\mathrm{d}x = 3\int_\Omega \partial_3\left(\partial_t^3[\check{q}]^{(n)}\partial_t\widetilde{\mathbf{N}}_i^{(n)}\right)\cdot\partial_t^2[v]^{(n)}\,\mathrm{d}x\\
\overset{\partial_t}{=}& -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \partial_3\left(\partial_t^2[\check{q}]^{(n)}\partial_t\widetilde{\mathbf{N}}_i^{(n)}\right)\cdot\partial_t^2[v]^{(n)}\,\mathrm{d}x\\
&+\int_\Omega \partial_t^2\partial_3[\check{q}]^{(n)}\partial_t(\partial_t\widetilde{\mathbf{N}}^{(n)}\cdot\partial_t^2[v]^{(n)}) + \partial_t^2[\check{q}]\partial_t(\partial_t\partial_3\widetilde{\mathbf{N}}^{(n)}\cdot\partial_t^2[v]^{(n)})\,\mathrm{d}x\\
\lesssim\,& \varepsilon\|\partial_t^2[\check{q}]^{(n)}\|_1^2 + C(\mathring{K}_0)\left([\mathring{E}]^{(n)}(0) + \int_0^T [\mathring{E}]^{(n)}(t) + [\mathring{E}]^{(n-1)}(t)\,\mathrm{d}t\right).
\end{aligned} \tag{5.104}$$

Combining the above analysis and using the definition of Alinhac good unknowns, we get the control of full-time derivatives

$$\begin{aligned}&\|\partial_t^3[v]^{(n)}(t)\|_0^2+\|\sqrt{\mathcal{F}^{(n)\prime}}(q^{(n)})\partial_t^3[\check{q}]^{(n)}(t)\|_0^2+\kappa^2\int_0^t|\partial_t^4\psi(\tau)|_1^2\mathrm{d}\tau\\&\leq\varepsilon\|\partial_t^2[\check{q}]^{(n)}\|_1^2+C(\mathring{K}_0,\kappa^{-1})\left([\mathring{E}]^{(n)}(0)+\int_0^T[\mathring{E}]^{(n)}(t)+[\mathring{E}]^{(n-1)}(t)+[\mathring{E}]^{(n-2)}(t)\,\mathrm{d}t\right).\end{aligned}\tag{5.105}$$

5.4 Well-posedness of the nonlinear κ-approximate problem

Combining the div-curl analysis, the control of the normal traces, the reduction of $[\check{q}]$ and the analysis of full-time derivatives for the linear system (5.80) for $[v]^{(n)},[\check{q}]^{(n)},[\psi]^{(n)}$, we finally get the following energy estimates

$$[\mathring{E}]^{(n)}(t)\leq C(\mathring{K}_0,\kappa^{-1})\left([\mathring{E}]^{(n)}(0)+\int_0^T[\mathring{E}]^{(n)}(t)+[\mathring{E}]^{(n-1)}(t)+[\mathring{E}]^{(n-2)}(t)\,\mathrm{d}t\right).\tag{5.106}$$

Since $[v]^{(n)},[\check{q}]^{(n)},[\psi]^{(n)}$ have zero initial data, one can repeatedly use (5.80) to show that their time derivatives also vanish on $\{t=0\}$, as one can observe that every term in the first two equations of (5.80) contains exactly one term involving $[f]^{(n)}$ or $[f]^{(n-1)}$ whose initial value is zero. This implies $[\mathring{E}]^{(n)}(0)=0$, and thus there exists some $T_1^\kappa>0$ independent of n, such that

$$\sup_{0\leq t\leq T_1^\kappa}[\mathring{E}]^{(n)}(t)\leq\frac{1}{4}\left(\sup_{0\leq t\leq T_1^\kappa}[\mathring{E}]^{(n-1)}(t)+\sup_{0\leq t\leq T_1^\kappa}[\mathring{E}]^{(n-2)}(t)\right),\tag{5.107}$$

and thus we know by induction that

$$\sup_{0\leq t\leq T_1^\kappa}[\mathring{E}]^{(n)}(t)\leq C(\mathring{K}_0,\kappa^{-1})/2^{n-1}\to0\text{ as }n\to+\infty.\tag{5.108}$$

Hence, for any fixed $\kappa>0$, the sequence of approximate solutions $\{(v^{(n)},\check{q}^{(n)},\rho^{(n)},\psi^{(n)})\}_{n\in\mathbb{N}^*}$ has a strongly convergent subsequence, whose limit $(v^\kappa,\check{q}^\kappa,\rho^\kappa,\psi^\kappa)$ is exactly the solution to the nonlinear κ-problem (3.11). The uniqueness follows from a parallel argument.

6 Well-posedness of the gravity-capillary water wave system

For any fixed $\sigma>0$, we can prove the local existence of the original system by the following steps. In Section 5, we prove the local well-posedness and higher-order energy estimates of the linearized system (5.6) for each *fixed* $\kappa>0$ and use Picard iteration to construct the unique strong solution to the nonlinear κ-approximate problem (3.11) defined in Section 3.2. To pass the limit $\kappa\to0_+$ to the original system (1.24), we prove the uniform-in-κ estimates for (3.11) in Section 4. Therefore, we prove the local-in-time existence for the stronger solution to the compressible gravity-capillary water wave system (1.24), that is, given initial data (v_0,ρ_0,ψ_0), there exists $T'>0$ only depending on the initial data, such that the original system (1.24) has a solution (v,ρ,ψ) satisfying the energy estimates

$$\sup_{0\leq t\leq T'}E(t)\leq P(E(0)).\tag{6.1}$$

To prove the well-posedness, it suffices to prove the uniqueness of the solution to (1.24). We assume $\{(v^{(n)},\check{q}^{(n)},\rho^{(n)},\psi^{(n)})\}_{n=1,2}$ to be two solutions to (1.24) and define $[f]=f^{(1)}-f^{(2)}$ for any function f. Then it suffices to prove $([v],[\check{q}],[\rho],[\psi])=(\mathbf{0},0,0,0)$. We find that $([v],[\check{q}],[\rho],[\psi])=(\mathbf{0},0,0,0)$ satisfies the following system

$$\begin{cases}\rho^{(1)}D_t^{\varphi^{(1)}}[v]+\nabla^{\varphi^{(1)}}[\check{q}]=-f_v&\text{in }[0,T]\times\Omega,\\\mathcal{F}'(q^{(1)})D_t^{\varphi^{(1)}}[\check{q}]+\nabla^{\varphi^{(1)}}\cdot[v]=-f_q&\text{in }[0,T]\times\Omega,\\{}[\check{q}]=[q]+g[\varphi]&\text{in }[0,T]\times\Omega,\\{}[\check{q}]=g[\psi]-\sigma\left(\mathcal{H}(\overline{\nabla}\psi^{(1)})-\mathcal{H}(\overline{\nabla}\psi^{(2)})\right)&\text{on }[0,T]\times\Sigma,\\\partial_t[\psi]=[v]\cdot N^{(1)}+v^{(2)}\cdot[N]&\text{on }[0,T]\times\Sigma,\\{}[v_3]=0&\text{on }[0,T]\times\Sigma_b,\\([v],[\check{q}],[\psi])|_{t=0}=(\mathbf{0},0,0)\end{cases}\tag{6.2}$$

where the functions f_v, f_q are defined by

$$f_v := [\rho]\partial_t v^{(2)} + [\rho\overline{v}] \cdot \overline{\nabla} v^{(2)} + [\rho V_{\mathbf{N}}]\partial_3 v^{(2)} + \rho^{(2)} g e_3 + \partial_3 \check{q}^{(2)}[A_{i3}] \tag{6.3}$$

$$\begin{aligned} f_q :=& [\mathcal{F}'(q)](\partial_t \check{q}^{(2)} - g v_3^{(2)}) + [\mathcal{F}'(q)\overline{v}] \cdot \overline{\nabla}\check{q}^{(2)} + [\mathcal{F}'(q)V_{\mathbf{N}}]\partial_3 q^{(2)} \\ &- \mathcal{F}'(q^{(2)})g[v_3] + \partial_3 v_i^{(2)}[A_{i3}], \end{aligned} \tag{6.4}$$

and

$$V_{\mathbf{N}} := \frac{1}{\partial_3\varphi}(v\cdot\mathbf{N} - \partial_t\varphi),\ A_{13} := -\frac{\partial_1\varphi}{\partial_3\varphi},\ A_{23} := -\frac{\partial_2\varphi}{\partial_3\varphi},\ A_{33} := \frac{1}{\partial_3\varphi},$$
$$\mathcal{H}(\overline{\nabla}\psi) := \overline{\nabla}\cdot\left(\frac{\overline{\nabla}\psi}{|N|}\right),\ \ \mathcal{H}(\overline{\nabla}\psi^{(1)}) - \mathcal{H}(\overline{\nabla}\psi^{(2)}) = \overline{\nabla}\cdot\left(\frac{\overline{\nabla}[\psi]}{|N^{(1)}|} - \left(\frac{1}{|N^{(1)}|} - \frac{1}{|N^{(2)}|}\right)\overline{\nabla}\psi^{(2)}\right).$$

Define the energy functional $[E](t)$ for (6.2) to be

$$[E](t) := \sum_{k=0}^{3} \|\partial_t^k[v]\|_{3-k}^2 + \sigma|\overline{\nabla}\partial_t^k[\psi]|_{3-k}^2 + g|[\psi]|_0^2 + \sum_{k=0}^{2}\|\partial_t^k[\check{q}]\|_{3-k}^2 + \|\sqrt{\mathcal{F}'(q^{(1)})}\partial_t^3[\check{q}]\|_0^2, \tag{6.5}$$

and then we can mimic the proof for the uniform-in-κ estimates (setting $\kappa = 0$) in Section 4 to show that $[E](0) = 0$ and $[E](t)$ satisfies the following energy inequality

$$[E](T) \le \int_0^T P(E(t))[E](t)\,\mathrm{d}t. \tag{6.6}$$

The only difference is that the boundary integral produces some extra terms that are controlled using mollification before. The main contribution of the boundary integral arising from $\overline{\partial}^3$-tangential estimates is

$$-\int_\Sigma [\mathbf{Q}][\mathbf{V}]\cdot N^{(1)}\,\mathrm{d}x' \overset{L}{=} -\int_\Sigma \overline{\partial}^3[q]\partial_t\overline{\partial}^3[\psi]\,\mathrm{d}x' + \int_\Sigma \overline{\partial}^3[q]\overline{\partial}^3(v^{(2)}\cdot[N])\,\mathrm{d}x', \tag{6.7}$$

where $[\mathbf{Q}], [\mathbf{V}]$ are the Alinhac good unknowns of $\check{q}, v$ with respect to $\overline{\partial}^3$, namely $[\mathbf{F}] := \overline{\partial}^3 f - \overline{\partial}^3\varphi\partial_3^\varphi f$.

For the first integral, we use the boundary condition for $[\check{q}]$ and $[\check{q}] = q + g\varphi$ to see

$$-\int_\Sigma \overline{\partial}^3[q]\partial_t\overline{\partial}^3[\psi]\,\mathrm{d}x' \overset{L}{=} -\frac{\sigma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma |N^{(1)}|^{-1}\left|\overline{\partial}^3\overline{\nabla}[\psi]\right|_0^2\mathrm{d}x' - \sigma\int_\Sigma \frac{\overline{\partial}^3\overline{\nabla}[\psi]\cdot\overline{\nabla}(\psi^{(1)}+\psi^{(2)})}{|N^{(1)}||N^{(2)}|(|N^{(1)}|+|N^{(2)}|)}\overline{\nabla}\psi^{(2)}\cdot\partial_t\overline{\nabla}\overline{\partial}^3[\psi]\,\mathrm{d}x', \tag{6.8}$$

where the first term gives the boundary energy in $[E](t)$, and the second term appears when $\overline{\partial}^3$ falls on

$$|N^{(1)}|^{-1} - |N^{(2)}|^{-1} = \frac{|N^{(2)}|^2 - |N^{(1)}|^2}{|N^{(1)}||N^{(2)}|(|N^{(1)} + |N^{(2)}|)}.$$

This part is controlled by rewriting $\partial_t\overline{\nabla\partial}^3[\psi] = \partial_t\overline{\nabla\partial}^3(\psi^{(1)} + \psi^{(2)})$

$$\begin{aligned} &-\sigma\int_\Sigma \frac{\overline{\partial}^3\overline{\nabla}[\psi]\cdot\overline{\nabla}(\psi^{(1)}+\psi^{(2)})}{|N^{(1)}||N^{(2)}|(|N^{(1)}|+|N^{(2)}|)}\overline{\nabla}\psi^{(2)}\cdot\partial_t\overline{\nabla\partial}^3[\psi]\,\mathrm{d}x' \\ &\le P(|\overline{\nabla}\psi^{(1)},\overline{\nabla}\psi^{(2)}|_\infty)|\sqrt{\sigma}\overline{\nabla\partial}^3[\psi]|_0(|\sqrt{\sigma}\partial_t\psi^{(1)}|_4 + |\sqrt{\sigma}\partial_t\psi^{(2)}|_4) \\ &\le \varepsilon|\sqrt{\sigma}\overline{\nabla\partial}^3[\psi]|_0^2 + P(|\overline{\nabla}\psi^{(1)},\overline{\nabla}\psi^{(2)}|_\infty)E(t) \le \varepsilon[E](t) + P(E(t)). \end{aligned}$$

The energy inequality for $[E](t)$ together with Grönwall's inequality and the energy bounds for $E(t)$ implies that there exists some $T \in [0, T']$ only depending on the initial data of (1.24), such that $\sup_{0\le t\le T}[E](t) \le [E](0) = 0$. Therefore, the solution to (6.2) must be zero. The uniqueness is proven.

7 Incompressible limit and zero surface tension limit

This section is devoted to showing that we can pass the solution of (1.24) to the incompressible and zero surface tension double limits. In other words, we study the behavior of the solution of (1.24) as both the Mach number λ and surface tension coefficient σ tend to 0. Recall that the Mach number λ is defined in Section 1.3.

We study the incompressible Euler equations modeling the motion of incompressible gravity water waves without surface tension satisfied by (ξ, w, q_{in}) with initial data (w_0, ξ_0) and $w_0^3|_{\Sigma_b} = 0$:

$$\begin{cases} D_t^{\varphi} w + \nabla^{\varphi} \mathfrak{p} = 0 & \text{in } [0,T]\times\Omega, \\ \nabla^{\varphi}\cdot w = 0 & \text{in } [0,T]\times\Omega, \\ \mathfrak{p} = q_{in} + g\varphi & \text{in } [0,T]\times\Omega, \\ \mathfrak{p} = g\xi & \text{on } [0,T]\times\Sigma, \\ \partial_t \xi = w\cdot \mathcal{N} & \text{on } [0,T]\times\Sigma, \\ w_3 = 0 & \text{on } [0,T]\times\Sigma_b, \\ (w,\xi)|_{t=0} = (w_0, \xi_0), \end{cases} \tag{7.1}$$

where we define $\varphi(t,x) = x_3 + \chi(x_3)\xi(t,x')$ to be the extension of ξ in Ω after slightly abuse of notations. Denote by $(\psi^{\lambda,\sigma}, v^{\lambda,\sigma}, \rho^{\lambda,\sigma})$ the solution of (1.24) indexed by λ and σ, our goal is to show

$$(\psi^{\lambda,\sigma}, v^{\lambda,\sigma}, \rho^{\lambda,\sigma}) \to (\xi, w, 1) \quad \text{in } C^0([0,T], C^2(\Sigma)\times C^2(\Omega)\times C^1(\Omega)), \quad \text{as } \lambda,\sigma\to 0, \tag{7.2}$$

provided that:

1. The sequence of initial data $(\psi_0^{\lambda,\sigma}, v_0^{\lambda,\sigma}, \rho_0^{\lambda,\sigma} - 1) \in H^5(\Sigma)\times H^4(\Omega)\times H^4(\Omega)$ satisfies the compatibility conditions up to order 3 and $v_0^{3;\lambda,\sigma}|_{\Sigma_b} = 0$. The compatibility condition of order k ($k\geq 0$), expressed in terms of the modified pressure, reads
$$(D_t^{\varphi})^k \check{q}|_{\{t=0\}\times\Sigma} = \sigma (D_t^{\varphi})^k \left(\mathcal{H} + g\psi\right)|_{\{t=0\}\times\Sigma}. \tag{7.3}$$
Since $D_t^{\varphi} = \partial_t + \overline{v}\cdot\overline{\partial}$ on Σ, we can rewrite (7.3) as:
$$(\partial_t + \overline{v}\cdot\overline{\partial})^k \check{q}|_{t=0} = \sigma(\partial_t + \overline{v}\cdot\overline{\partial})^k \left(\mathcal{H} + g\psi\right)|_{t=0} \quad \text{on } \Sigma. \tag{7.4}$$
The existence of such data is discussed in Appendix B.
2. $(\psi_0^{\lambda,\sigma}, v_0^{\lambda,\sigma}, \rho_0^{\lambda,\sigma}) \to (\xi, w, 1)$ in $C^2(\Sigma)\times C^1(\Omega)\times C^1(\Omega)$ as $\lambda,\sigma\to 0$.
3. The compressible pressure q and the incompressible pressure q_{in} satisfy the Rayleigh-Taylor sign condition:
$$-\partial_3 q \geq c_0 > 0, \quad \text{on } \{t=0\}\times\Sigma, \tag{7.5}$$
$$-\partial_3 q_{in} \geq c_0 > 0, \quad \text{on } \{t=0\}\times\Sigma. \tag{7.6}$$

The key step of showing the λ,σ double limit is to prove an energy estimate of (1.24) that is uniform in both λ and σ. In fact, the analysis in Section 4 indicates that the energy estimate for (4.1) is already uniform in λ. In particular, one can see that the tangential energy estimates in Sections 4.5-4.6 are uniform in $\mathcal{F}'_\lambda$, which is $\approx \lambda^2$ by (1.29).

Nevertheless, in this section, we want to use a weaker energy functional that does not require $\|\partial_t^k q\|_{4-k}$, $k\geq 2$ to be uniformly bounded in λ. This being said, denoting $\mathcal{F}'_\lambda = \mathcal{F}'_\lambda(q)$, we study

$$\begin{aligned} E^{\lambda,\sigma}(t) = &\sum_{k=0}^{1} \|\partial_t^k v^{\lambda,\sigma}(t)\|_{4-k}^2 + \|\partial^{1-k}\partial_t^k \check{q}^{\lambda,\sigma}(t)\|_3^2 \qquad \text{(Non-weighted interior norms)} \\ &+ \sum_{k=0}^{1} |\sqrt{\sigma}\overline{\nabla}\partial_t^k \psi^{\lambda,\sigma}(t)|_{4-k}^2 + |\partial_t^k \psi^{\lambda,\sigma}(t)|_{4-k}^2 \qquad \text{(Non-weighted boundary norms)} \\ &+ \|\sqrt{\mathcal{F}'_\lambda}\check{q}^{\lambda,\sigma}(t)\|_0^2 + g|\psi^{\lambda,\sigma}(t)|_0^2 \qquad (L^2 \text{ norms}) \\ &+ \sum_{s=0}^{2} \|(\mathcal{F}'_\lambda)^{\frac{s}{2}}\partial_t^{2+s} v^{\lambda,\sigma}(t)\|_{2-s}^2 + \|(\mathcal{F}'_\lambda)^{\frac{s+1}{2}}\partial_t^{2+s}\check{q}^{\lambda,\sigma}(t)\|_{2-s}^2 \qquad \text{(weighted interior norms)} \\ &+ \sum_{s=0}^{2} |\sqrt{\sigma}(\mathcal{F}'_\lambda)^{\frac{s}{2}}\overline{\nabla}\partial_t^{2+s}\psi^{\lambda,\sigma}(t)|_{2-s}^2 + |(\mathcal{F}'_\lambda)^{\frac{s}{2}}\partial_t^{2+s}\psi^{\lambda,\sigma}(t)|_{2-s}^2 \qquad \text{(weighted boundary norms).} \end{aligned} \tag{7.7}$$

Note that we need the data to verify compatibility conditions (7.4) up to order 3 to guarantee $E^{\lambda,\sigma}(0) < \infty$. The $\mathcal{F}_\lambda'$-weights attached to $v^{\lambda,\sigma}$, $\check{q}^{\lambda,\sigma}$ and their time derivatives are determined as follows: First, the momentum equation in (1.24) implies that the control of $\|\check{q}^{\lambda,\sigma}\|_4$ requires that of $\|\partial_t v^{\lambda,\sigma}\|_3$. Similarly, the control of $\|\partial_t \check{q}^{\lambda,\sigma}\|_3$ requires that of $\|\partial_t^2 v^{\lambda,\sigma}\|_2$, and this procedure continuous until controlling $\|\mathcal{F}_\lambda'\partial_t^3\check{q}^{\lambda,\sigma}\|_1$ by $\|\mathcal{F}_\lambda'\partial_t^4 v^{\lambda,\sigma}\|_0$. Second, our energy estimate indicates that we need to control $\|\partial_t^2 v^{\lambda,\sigma}\|_2$ and $\|\sqrt{\mathcal{F}_\lambda'}\partial_t^2\check{q}^{\lambda,\sigma}\|_2$ together. Analogously, $\|\sqrt{\mathcal{F}_\lambda'}\partial_t^3 v^{\lambda,\sigma}\|_1$ and $\|\mathcal{F}_\lambda'\partial_t^3\check{q}^{\lambda,\sigma}\|_1$ have to be controlled simultaneously, the same holds for $\|\mathcal{F}_\lambda'\partial_t^4 v^{\lambda,\sigma}\|_0$ and $\|(\mathcal{F}_\lambda')^{\frac32}\partial_t^4\check{q}^{\lambda,\sigma}\|_0$. One can summary the $\mathcal{F}_\lambda'$-weight determination procedure (starting from $\partial_t\check{q}^{\lambda,\sigma}$) as:

$$\begin{tikzcd} & \|\partial_t^2 v^{\lambda,\sigma}\|_2 \arrow[d, "\text{energy}"] & \|(\mathcal{F}_\lambda')^{\frac12}\partial_t^3 v^{\lambda,\sigma}\|_1 \arrow[d, "\text{energy}"] & \|\mathcal{F}_\lambda'\partial_t^4 v^{\lambda,\sigma}\|_0 \arrow[d, "\text{energy}"] \\ \|\partial_t \check{q}^{\lambda,\sigma}\|_3 \arrow[ur, "\text{equation}"] & \|(\mathcal{F}_\lambda')^{\frac12}\partial_t^2\check{q}^{\lambda,\sigma}\|_2 \arrow[ur, "\text{equation}"] & \|\mathcal{F}_\lambda'\partial_t^3\check{q}^{\lambda,\sigma}\|_1 \arrow[ur, "\text{equation}"] & \|(\mathcal{F}_\lambda')^{\frac32}\partial_t^4\check{q}^{\lambda,\sigma}\|_0 \end{tikzcd} \tag{7.8}$$

On the other hand, we point out that the energy estimate in Proposition 4.1 is almost uniform in σ except for the control of $|\mathcal{T}^\alpha|_0$ ($|\mathcal{T}^\alpha\widetilde{\psi}|_0$ for the nonlinear κ-approximate problem). Indeed, as mentioned in the paragraph above (4.82), we cannot control $\int_0^T \mathrm{RT}\,\mathrm{d}t$ uniformly in σ unless the Rayleigh-Taylor sign condition holds

$$-\partial_3 q \geq \frac{c_0}{2} > 0, \quad \text{on } \Sigma. \tag{7.9}$$

7.1 Discussion on the estimate of $E^{\lambda,\sigma}(t)$

As mentioned above, the energy estimate of $E^{\lambda,\sigma}(t)$ is almost identical to what has been done in Section 4, and thus we shall only point out the key differences. First, the estimate of $E^{\lambda,\sigma}(t)$ concerns the original system (1.24) and so $\kappa = 0$. This means $\widetilde{\psi}, \widetilde{\varphi}$ agree with ψ, φ, respectively. Hence, all the mismatched terms introduced by tangential smoothing and artificial viscosity vanish, such as the mismatch $\mathfrak{C}(f)$ in good unknowns (4.40) and the boundary terms in (4.70), (4.84) involving mismatches, artificial viscosity. Second, we take the $\mathcal{F}_\lambda'$-weighted tangential derivatives $\mathfrak{D}^\alpha$ to (1.24) to produce the tangential estimates. Let $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ with $|\alpha| = 4$ and we define

$$\mathfrak{D}^\alpha := \begin{cases} \partial_t^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}\mathcal{T}_3^{\alpha_3}, & \text{when } \alpha_0 = 0, 1, 2, \\ \sqrt{\mathcal{F}_\lambda'}\partial_t^3\partial_1^{\alpha_1}\partial_2^{\alpha_2}\mathcal{T}_3^{\alpha_3}, & \\ \mathcal{F}_\lambda'\partial_t^4. & \end{cases} \tag{7.10}$$

Note that the $\mathcal{F}_\lambda'$-weight is chosen based on the energy (7.7).

The energy estimate for $E^{\lambda,\sigma}(t)$ is uniform-in-λ by adapting the proof of Proposition 4.1. On the other hand, to obtain a uniform-in-σ estimate, we need to control

$$\mathrm{RT} = \int_\Sigma (-\partial_3 q)\overline{\mathfrak{D}}^\alpha\psi\,\partial_t\overline{\mathfrak{D}}^\alpha\psi\,\mathrm{d}x', \tag{7.11}$$

where $\overline{\mathfrak{D}}^\alpha = \mathfrak{D}^\alpha$ with $|\alpha| = 4$ and $\alpha_3 = 0$. Owing to (7.9), we have

$$\mathrm{RT} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Sigma (-\partial_3 q)|\overline{\mathfrak{D}}^\alpha\psi|^2\,\mathrm{d}x' + \int_\Sigma (\partial_3\partial_t q)|\overline{\mathfrak{D}}^\alpha\psi|^2\,\mathrm{d}x'. \tag{7.12}$$

The first term is positive and thus contributes to the energy, while the second term $\leq \|\partial_3\partial_t q\|_\infty E^{\lambda,\sigma}$. Therefore, the terms involving $|\overline{\mathfrak{D}}^\alpha\psi|_0$, such as (4.82), (4.98) and RT^* in (4.121)-(4.127), can be controlled without any quantities depending on σ^{-1}.

7.2 The incompressible and zero surface tension double limits

The energy bound on (7.7) implies the boundedness of $\|v^{\lambda,\sigma}(t)\|_4^2 + |\psi^{\lambda,\sigma}(t)|_4^2$ uniformly in both λ and σ within the time interval $[0, T]$. Thus, for each fixed t, the Morrey-type embeddings $H^4(\Omega) \hookrightarrow C^{2,\frac12}(\Omega)$ and $H^4(\Sigma) \hookrightarrow C^{2,\alpha}(\Sigma)$ ($\forall 0 < \alpha < 1$) imply that $v^{\lambda,\sigma}(t)$ is equicontinuous and uniformly bounded in $C^2(\Omega)$ and $\psi^{\lambda,\sigma}(t)$ is equicontinuous and uniformly bounded in $C^2(\Sigma)$. So we have $(v^{\lambda,\sigma}, \psi^{\lambda,\sigma}) \to (w, \xi)$ in $C^0([0,T]; C^2(\Omega) \times C^2(\Sigma))$ as $\lambda, \sigma \to 0$. Moreover, as $D_t^\varphi = \partial_t + (\overline{v}\cdot\overline{\nabla}) + (\partial_3\varphi)^{-1}(v\cdot\mathbf{N} - \partial_t\varphi)\partial_3$, invoking the continuity equation

$$\mathcal{F}_\lambda'(q)D_t^\varphi\check{q}^{\lambda,\sigma} + \nabla^\varphi\cdot v^{\lambda,\sigma} = g\mathcal{F}_\lambda'(q)D_t^\varphi v_3^{\lambda,\sigma},$$

and because $\|\partial_{t,x}\check{q}^{\lambda,\sigma}(t)\|_3, \|\partial_{t,x}v^{\lambda,\sigma}(t)\|_3$ are uniformly bounded in $[0,T]$, we have

$$\nabla^{\varphi}\cdot v^{\lambda,\sigma}\to\nabla^{\varphi}\cdot w=0,\quad \text{in } L^{\infty}(\Omega). \tag{7.13}$$

Similarly one can prove $(\partial_t^k v^{\lambda,\sigma},\partial_t^k\psi^{\lambda,\sigma})\to(\partial_t^k w,\partial_t^k\xi)$ in $C([0,T];C^{2-k}(\Omega)\times C^{2-k}(\Sigma))$ for $0\le k\le 2$.

Lastly, since the continuity can be written as

$$D_t^{\varphi}(\rho^{\lambda,\sigma}-1)+\rho^{\lambda,\sigma}(\nabla^{\varphi}\cdot v^{\lambda,\sigma})=0,$$

we can derive the energy estimate for $\rho^{\lambda,\sigma}-1$ in $H^3(\Omega)$ as:

$$\frac{d}{dt}\frac{1}{2}\|\rho^{\lambda,\sigma}-1\|_3^2\le\|\rho^{\lambda,\sigma}-1\|_0\Big(\|v^{\lambda,\sigma}\|_4+|\overline{\partial}\psi^{\lambda,\sigma}|_3\Big), \tag{7.14}$$

where $\|v^{\lambda,\sigma}\|_4$, $|\overline{\partial}\psi^{\lambda,\sigma}|_3$ are bounded by $E^{\lambda,\sigma}$. Therefore, the Sobolev embedding suggests that $\rho^{\lambda,\sigma}(t)-1$ is equicontinuous and uniformly bounded in $C^1(\Omega)$. This implies $\rho^{\lambda,\sigma}-1\to 0$ in $C^0([0,T],C^1(\Omega))$.

A The Reynold transport theorems

Below, the formulas involving $\widetilde{\varphi},\widetilde{\psi}$ are used for the nonlinear κ-problem (3.11) and the formulas involving $\mathring{\widetilde{\varphi}},\mathring{\widetilde{\psi}}$ are used for the linearized κ-problem (5.6).

Lemma A.1. Let f,g be smooth functions defined on $[0,T]\times\Omega$. Then there hold

$$\partial_t\int_{\Omega}fg\partial_3\widetilde{\varphi}\,dx=\int_{\Omega}(\partial_t^{\widetilde{\varphi}}f)g\partial_3\widetilde{\varphi}\,dx+\int_{\Omega}f(\partial_t^{\widetilde{\varphi}}g)\partial_3\widetilde{\varphi}\,dx+\int_{x_3=0}fg\partial_t\psi\,dx'+\int_{\Omega}fg\partial_3\partial_t(\widetilde{\varphi}-\varphi)\,dx, \tag{A.1}$$

$$\partial_t\int_{\Omega}fg\partial_3\mathring{\widetilde{\varphi}}\,dx=\int_{\Omega}(\partial_t^{\mathring{\widetilde{\varphi}}}f)g\partial_3\mathring{\widetilde{\varphi}}\,dx+\int_{\Omega}f(\partial_t^{\mathring{\widetilde{\varphi}}}g)\partial_3\mathring{\widetilde{\varphi}}\,dx+\int_{x_3=0}fg\partial_t\mathring{\psi}\,dx'+\int_{\Omega}fg\partial_3\partial_t(\mathring{\widetilde{\varphi}}-\mathring{\varphi})\,dx. \tag{A.2}$$

Proof. In view of (3.12),

$$\begin{aligned}\text{LHS of (A.1)}&=\int_{\Omega}(\partial_t f)g\partial_3\widetilde{\varphi}\,\mathrm{d}x+\int_{\Omega}f(\partial_t g)\partial_3\widetilde{\varphi}\,\mathrm{d}x+\int_{\Omega}fg\partial_3\partial_t\widetilde{\varphi}\,\mathrm{d}x\\&=\int_{\Omega}fg\partial_3\partial_t\widetilde{\varphi}\,\mathrm{d}x+\int_{\Omega}(\partial_t^{\widetilde{\varphi}}f)g\partial_3\widetilde{\varphi}\,\mathrm{d}x+\int_{\Omega}f(\partial_t^{\widetilde{\varphi}}g)\partial_3\widetilde{\varphi}\,\mathrm{d}x+\overbrace{\int_{\Omega}\partial_t\varphi(\partial_3 f)g\,\mathrm{d}x}^{i}+\overbrace{\int_{\Omega}\partial_t\varphi(\partial_3 g)f\,\mathrm{d}x}^{ii}.\end{aligned}$$

Integrating ∂_3 in ii by parts, we have

$$ii=\int_{x_3=0}fg\partial_t\psi\,dx'-\int_{x_3=-b}fg\underbrace{\partial_t\varphi}_{=\partial_t(-b)=0}\,\mathrm{d}x'-\int_{\Omega}fg\partial_3\partial_t\varphi\,\mathrm{d}x-i.$$

This concludes the proof of (A.1). Moreover, in light of (5.8),

$$\begin{aligned}\text{LHS of (A.2)}&=\int_{\Omega}(\partial_t f)g\partial_3\mathring{\widetilde{\varphi}}\,\mathrm{d}x+\int_{\Omega}f(\partial_t g)\partial_3\mathring{\widetilde{\varphi}}\,\mathrm{d}x+\int_{\Omega}fg\partial_3\partial_t\mathring{\widetilde{\varphi}}\,\mathrm{d}x\\&=\int_{\Omega}fg\partial_3\partial_t\mathring{\widetilde{\varphi}}\,\mathrm{d}x+\int_{\Omega}(\partial_t^{\mathring{\widetilde{\varphi}}}f)g\partial_3\mathring{\widetilde{\varphi}}\,\mathrm{d}x+\int_{\Omega}f(\partial_t^{\mathring{\widetilde{\varphi}}}g)\partial_3\mathring{\widetilde{\varphi}}\,\mathrm{d}x+\overbrace{\int_{\Omega}\partial_t\mathring{\varphi}(\partial_3 f)g\,\mathrm{d}x}^{\mathring{i}}+\overbrace{\int_{\Omega}\partial_t\mathring{\varphi}(\partial_3 g)f\,\mathrm{d}x}^{\mathring{ii}}.\end{aligned}$$

Integrating ∂_3 in $\mathring{ii}$ by parts, we have

$$\mathring{ii}=\int_{x_3=0}fg\partial_t\mathring{\psi}\,dx'-\int_{\Omega}fg\partial_3\partial_t\mathring{\varphi}\,\mathrm{d}x-\mathring{i},$$

and thus (A.2) follows. ◻

Lemma A.2 (**Integration by parts for covariant derivatives**). Let f, g be defined as in Lemma A.1. Then there hold

$$\int_\Omega (\partial_i^{\widetilde{\varphi}} f) g \partial_3 \widetilde{\varphi} \, \mathrm{d}x = -\int_\Omega f(\partial_i^{\widetilde{\varphi}} g) \partial_3 \widetilde{\varphi} \, \mathrm{d}x + \int_{x_3=0} f g \widetilde{N}_i \, dx', \tag{A.3}$$

$$\int_\Omega (\partial_i^{\mathring{\widetilde{\varphi}}} f) g \partial_3 \mathring{\widetilde{\varphi}} \, \mathrm{d}x = -\int_\Omega f(\partial_i^{\mathring{\widetilde{\varphi}}} g) \partial_3 \mathring{\widetilde{\varphi}} \, \mathrm{d}x + \int_{x_3=0} f g \mathring{\widetilde{N}}_i \, dx'. \tag{A.4}$$

Proof. (A.3) follows from the fact that $\partial_i^{\widetilde{\varphi}}$ is the covariant spatial derivative and $\partial_3 \widetilde{\psi} \, \mathrm{d}x$ is the associated volume element. (A.4) follows from a parallel argument. □

Let $D_t^{\widetilde{\varphi}}$ be the smoothed material derivative defined in (3.15). Then the following theorem holds.

Theorem A.3 (**Reynold transport theorem for nonlinear κ-problem**). Let f be a smooth function defined on $[0, T] \times \Omega$. Then there holds

$$\frac{1}{2} \partial_t \int_\Omega \rho |f|^2 \partial_3 \widetilde{\varphi} \, \mathrm{d}x = \int_\Omega \rho (D_t^{\widetilde{\varphi}} f) f \partial_3 \widetilde{\varphi} \, \mathrm{d}x + \frac{1}{2} \int_\Omega \rho |f|^2 \partial_3 \partial_t (\widetilde{\varphi} - \varphi) \, \mathrm{d}x. \tag{A.5}$$

Proof. First, we express

$$\int_\Omega \rho (D_t^{\widetilde{\varphi}} f) f \partial_3 \widetilde{\varphi} \, \mathrm{d}x = \int_\Omega \rho (\partial_t^{\widetilde{\varphi}} f) f \partial_3 \widetilde{\varphi} \, \mathrm{d}x + \int_\Omega \rho (v \cdot \nabla^{\widetilde{\varphi}} f) f \partial_3 \widetilde{\varphi} \, \mathrm{d}x.$$

Invoking (A.1), we have

$$\int_\Omega \rho (\partial_t^{\widetilde{\varphi}} f) f \partial_3 \widetilde{\varphi} \, \mathrm{d}x = \partial_t \int_\Omega \rho |f|^2 \partial_3 \widetilde{\varphi} \, \mathrm{d}x - \int_\Omega \partial_t^{\widetilde{\varphi}} (\rho f) f \partial_3 \widetilde{\varphi} \, \mathrm{d}x - \int_{x_3=0} \rho |f|^2 \partial_t \psi \, dx' - \int_\Omega \rho |f|^2 \partial_3 \partial_t (\widetilde{\varphi} - \varphi) \, \mathrm{d}x,$$

and this indicates that

$$\int_\Omega \rho (\partial_t^{\widetilde{\varphi}} f) f \partial_3 \widetilde{\varphi} \, \mathrm{d}x = \frac{1}{2} \partial_t \int_\Omega \rho |f|^2 \partial_3 \widetilde{\varphi} \, \mathrm{d}x \overbrace{- \frac{1}{2} \int_\Omega (\partial_t^{\widetilde{\varphi}} \rho) |f|^2 \partial_3 \widetilde{\varphi} \, \mathrm{d}x}^{A} \overbrace{- \frac{1}{2} \int_{x_3=0} \rho |f|^2 \partial_t \psi \, dx'}^{C} - \frac{1}{2} \int_\Omega \rho |f|^2 \partial_3 \partial_t (\widetilde{\varphi} - \varphi) \, \mathrm{d}x. \tag{A.6}$$

Furthermore, invoking (A.3), we have

$$\begin{aligned} \int_\Omega \rho (v \cdot \nabla^{\widetilde{\varphi}} f) f \partial_3 \widetilde{\varphi} \, \mathrm{d}x &= \int_\Omega \nabla^{\widetilde{\varphi}} \cdot (\rho v f) f \partial_3 \widetilde{\varphi} \, \mathrm{d}x - \int_\Omega \nabla^{\widetilde{\varphi}} \cdot (\rho v) |f|^2 \partial_3 \widetilde{\varphi} \, \mathrm{d}x \\ &= -\int_\Omega \rho f (v \cdot \nabla^{\widetilde{\varphi}} f) \partial_3 \widetilde{\varphi} \, \mathrm{d}x + \overbrace{\int_{x_3=0} \rho |f|^2 v \cdot \widetilde{N} \, dx'}^{D} - \int_\Omega \nabla^{\widetilde{\varphi}} \cdot (\rho v) |f|^2 \partial_3 \widetilde{\varphi} \, \mathrm{d}x, \end{aligned}$$

and thus

$$\int_\Omega \rho (v \cdot \nabla^{\widetilde{\varphi}} f) f \partial_3 \widetilde{\varphi} \, \mathrm{d}x = \frac{1}{2} \int_{x_3=0} \rho |f|^2 v \cdot \widetilde{N} \, dx' \overbrace{- \frac{1}{2} \int_\Omega \nabla^{\widetilde{\varphi}} \cdot (\rho v) |f|^2 \partial_3 \widetilde{\varphi} \, \mathrm{d}x}^{B}. \tag{A.7}$$

We have $A + B = C + D = 0$ thanks to the second and fifth equations of (3.11), respectively. Hence, (A.5) follows after adding (A.6) and (A.7) up. □

Theorem A.3 leads to the following two corollaries. The first one records the integration by parts formula for $D_t^{\widetilde{\varphi}}$.

Corollary A.4 (**Reynold transport theorem for nonlinear κ-problem**). It holds that

$$\partial_t \int_\Omega f g \partial_3 \widetilde{\varphi} \, \mathrm{d}x = \int_\Omega (D_t^{\widetilde{\varphi}} f) g \partial_3 \widetilde{\varphi} \, \mathrm{d}x + \int_\Omega f (D_t^{\widetilde{\varphi}} g) \partial_3 \widetilde{\varphi} \, \mathrm{d}x + \int_\Omega (\nabla^{\widetilde{\varphi}} \cdot v) f g \partial_3 \widetilde{\varphi} \, \mathrm{d}x + \int_\Omega f g \partial_3 \partial_t (\widetilde{\varphi} - \varphi) \, \mathrm{d}x. \tag{A.8}$$

Proof. Given (A.1), we have

$$\int_\Omega (\partial_t^{\widetilde{\varphi}} f) g \partial_3 \widetilde{\varphi} \, \mathrm{d}x = \partial_t \int_\Omega f g \partial_3 \widetilde{\varphi} \, \mathrm{d}x - \int_\Omega f (\partial_t^{\widetilde{\varphi}} g) \partial_3 \widetilde{\varphi} \, \mathrm{d}x - \int_{x_3=0} f g \partial_t \psi \, dx' - \int_\Omega f g \partial_3 \partial_t (\widetilde{\varphi} - \varphi) \, \mathrm{d}x,$$

Also, (A.3) yields

$$\begin{aligned}
\int_\Omega (v \cdot \nabla^{\widetilde{\varphi}} f) g \partial_3 \widetilde{\varphi} \, \mathrm{d}x &= \int_\Omega \nabla^{\widetilde{\varphi}} \cdot (v f) g \partial_3 \widetilde{\varphi} \, \mathrm{d}x - \int_\Omega (\nabla^{\widetilde{\varphi}} \cdot v) f g \partial_3 \widetilde{\varphi} \, \mathrm{d}x \\
= -\int_\Omega f (v \cdot \nabla^{\widetilde{\varphi}} g) \partial_3 \widetilde{\varphi} \, \mathrm{d}x &+ \int_{x_3=0} f g (v \cdot \widetilde{N}) \, dx' - \int_\Omega (\nabla^{\widetilde{\varphi}} \cdot v) f g \partial_3 \widetilde{\varphi} \, \mathrm{d}x.
\end{aligned}$$

Then we obtain (A.8) by adding these up. □

The second corollary concerns the transport theorem as well as the integration by parts formula for the linearized material derivative $D_t^{\mathring{\widetilde{\varphi}}}$, defined in (5.7).

Corollary A.5 (Reynold transport theorem for linearized κ-problem). Let $D_t^{\mathring{\widetilde{\varphi}}} := \partial_t + (\bar{\mathring{v}} \cdot \overline{\nabla}) + \frac{1}{\partial_3 \mathring{\widetilde{\varphi}}} (\mathring{v} \cdot \dot{\widetilde{\mathbf{N}}} - \partial_t \mathring{\varphi}) \partial_3$ be the linearized material derivative defined in (5.7). Then there holds

$$\begin{aligned}
\frac{1}{2} \partial_t \int_\Omega \mathring{\rho} |f|^2 \partial_3 \mathring{\widetilde{\varphi}} \, \mathrm{d}x = &\int_\Omega \mathring{\rho} (D_t^{\mathring{\widetilde{\varphi}}} f) f \partial_3 \mathring{\widetilde{\varphi}} \, \mathrm{d}x + \frac{1}{2} \int_\Omega \left(D_t^{\mathring{\widetilde{\varphi}}} \mathring{\rho} + \mathring{\rho} \nabla^{\mathring{\widetilde{\varphi}}} \cdot \mathring{v} \right) |f|^2 \partial_3 \mathring{\widetilde{\varphi}} \, dx \\
&+ \frac{1}{2} \int_\Omega \mathring{\rho} |f|^2 \left(\partial_3 \partial_t (\dot{\widetilde{\varphi}} - \mathring{\varphi}) + \partial_3 (\partial_t + \bar{\mathring{v}} \cdot \overline{\nabla}) (\mathring{\widetilde{\varphi}} - \dot{\widetilde{\varphi}}) \right) \mathrm{d}x.
\end{aligned} \tag{A.9}$$

$$\begin{aligned}
\frac{1}{2} \partial_t \int_\Omega |f|^2 \partial_3 \mathring{\widetilde{\varphi}} \, \mathrm{d}x = &\int_\Omega (D_t^{\mathring{\widetilde{\varphi}}} f) f \partial_3 \mathring{\widetilde{\varphi}} \, \mathrm{d}x + \frac{1}{2} \int_\Omega \nabla^{\mathring{\widetilde{\varphi}}} \cdot \mathring{v} |f|^2 \partial_3 \mathring{\widetilde{\varphi}} \, \mathrm{d}x \\
&+ \frac{1}{2} \int_\Omega |f|^2 \left(\partial_3 \partial_t (\dot{\widetilde{\varphi}} - \mathring{\varphi}) + \partial_3 (\partial_t + \bar{\mathring{v}} \cdot \overline{\nabla}) (\mathring{\widetilde{\varphi}} - \dot{\widetilde{\varphi}}) \right) \mathrm{d}x.
\end{aligned} \tag{A.10}$$

Proof. It suffices to show (A.9) only since the proof of (A.10) follows by setting $\mathring{\rho} = 1$. We write the first term on the RHS of (A.9) as

$$\int_\Omega \mathring{\rho} (D_t^{\mathring{\widetilde{\varphi}}} f) f \partial_3 \mathring{\widetilde{\varphi}} \, \mathrm{d}x = \int_\Omega \mathring{\rho} (\partial_t f) f \partial_3 \mathring{\widetilde{\varphi}} \, \mathrm{d}x + \int_\Omega \mathring{\rho} \left((\bar{\mathring{v}} \cdot \overline{\nabla}) f \right) f \partial_3 \mathring{\widetilde{\varphi}} \, \mathrm{d}x + \int_\Omega \mathring{\rho} \left((\mathring{v} \cdot \dot{\widetilde{\mathbf{N}}} - \partial_t \mathring{\varphi}) \partial_3 f \right) f \, \mathrm{d}x, \tag{A.11}$$

and then integrate ∂_t, $\overline{\nabla}$ and ∂_3 by parts respectively in these terms to get:

$$\begin{aligned}
\int_\Omega \mathring{\rho} (D_t^{\mathring{\widetilde{\varphi}}} f) f \partial_3 \mathring{\widetilde{\varphi}} \, \mathrm{d}x = &\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_\Omega \mathring{\rho} |f|^2 \partial_3 \mathring{\widetilde{\varphi}} \, \mathrm{d}x - \frac{1}{2} \int_\Omega \left(\partial_t \mathring{\rho} + \bar{\mathring{v}} \cdot \overline{\nabla} \mathring{\rho} + \frac{1}{\partial_3 \mathring{\widetilde{\varphi}}} (\mathring{v} \cdot \dot{\widetilde{\mathbf{N}}} - \partial_t \mathring{\varphi}) \partial_3 \mathring{\rho} \right) |f|^2 \partial_3 \mathring{\widetilde{\varphi}} \, \mathrm{d}x \\
&- \frac{1}{2} \int_\Omega \mathring{\rho} (\overline{\nabla} \cdot \bar{\mathring{v}}) |f|^2 \partial_3 \mathring{\widetilde{\varphi}} \, \mathrm{d}x - \frac{1}{2} \int_\Omega \mathring{\rho} |f|^2 (\partial_t + \bar{\mathring{v}} \cdot \overline{\nabla}) \partial_3 \mathring{\widetilde{\varphi}} \, \mathrm{d}x \\
&- \frac{1}{2} \int_\Omega \mathring{\rho} \partial_3 (-(\bar{\mathring{v}} \cdot \overline{\nabla}) \dot{\widetilde{\varphi}} + \mathring{v}_3 - \partial_t \mathring{\varphi}) |f|^2 \, \mathrm{d}x,
\end{aligned} \tag{A.12}$$

where we used $\mathring{v} \cdot \dot{\widetilde{\mathbf{N}}} = -(\bar{\mathring{v}} \cdot \overline{\nabla}) \dot{\widetilde{\varphi}} + \mathring{v}_3$ in the last line. We find that the second integral in the first line is $\int_\Omega D_t^{\mathring{\widetilde{\varphi}}} \mathring{\rho} |f|^2 \partial_3 \mathring{\widetilde{\varphi}} \, \mathrm{d}x$. Also, the term in the last line can be written as

$$\begin{aligned}
&- \frac{1}{2} \int_\Omega \mathring{\rho} \partial_3 (-(\bar{\mathring{v}} \cdot \overline{\nabla}) \dot{\widetilde{\varphi}} + \mathring{v}_3 - \partial_t \mathring{\varphi}) |f|^2 \, \mathrm{d}x \\
= &- \frac{1}{2} \int_\Omega \mathring{\rho} |f|^2 \left(\frac{1}{\partial_3 \mathring{\widetilde{\varphi}}} \partial_3 v_3 - \frac{\overline{\partial}_1 \mathring{\widetilde{\varphi}}}{\partial_3 \mathring{\widetilde{\varphi}}} \partial_3 \mathring{v}_1 - \frac{\overline{\partial}_2 \mathring{\widetilde{\varphi}}}{\partial_3 \mathring{\widetilde{\varphi}}} \partial_3 \mathring{v}_2 \right) \partial_3 \mathring{\widetilde{\varphi}} \, \mathrm{d}x \\
&+ \frac{1}{2} \int_\Omega \mathring{\rho} |f|^2 \partial_3 \bar{\mathring{v}} \cdot \overline{\nabla} (\dot{\widetilde{\varphi}} - \mathring{\widetilde{\varphi}}) \, \mathrm{d}x + \frac{1}{2} \int_\Omega \mathring{\rho} |f|^2 (\partial_t \partial_3 \mathring{\varphi} + (\bar{\mathring{v}} \cdot \overline{\nabla}) \partial_3 \dot{\widetilde{\varphi}}) \, \mathrm{d}x.
\end{aligned} \tag{A.13}$$

The first term on the RHS together with the third term in (A.12) contributes to

$$\frac{1}{2}\int_{\Omega} \mathring{\rho}(\nabla^{\mathring{\widetilde{\varphi}}}\cdot\hat{v})|f|^2\partial_3\mathring{\widetilde{\varphi}}\,\mathrm{d}x$$

in (A.9). Meanwhile, the terms in the last line of (A.13) together with the fourth term in (A.12) give the terms in (A.9) with mismatches. □

B Construction of initial data for the original system

This section aims to construct the initial data for Theorem 1.2 and Theorem 1.3 satisfying the compatibility conditions

$$(D_t^{\varphi})^j q|_{\{t=0\}\times\Sigma} = (D_t^{\varphi})^j(\sigma\mathcal{H})|_{\{t=0\}\times\Sigma}, \quad j=0,1,2,3.$$

Since $D_t^{\varphi}|_{\Sigma} = \partial_t + \overline{v}\cdot\overline{\partial}$ and $\mathcal{H} = -\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}}\right)$, we rewrite the compatibility conditions in terms of $\check{q}$ as

$$(\partial_t+\overline{v}\cdot\overline{\partial})^j\check{q}|_{\{t=0\}\times\Sigma} = -\sigma(\partial_t+\overline{v}\cdot\overline{\partial})^j\left(\overline{\nabla}\cdot\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}}+g\psi\right)\Bigg|_{\{t=0\}\times\Sigma}, \quad j=0,1,2,3. \tag{B.1}$$

Here, we use the modified pressure $\check{q}$ since we want $\partial\check{q}_0\in L^2(\Omega)$ for the sake of convenience. Such compatibility conditions are required to show that $E(t)$ (defined as (1.34)), and $E^{\lambda,\sigma}(t)$ (defined as (1.38)) are bounded at $t=0$ by adapting the arguments in [17, Section 4.3].

B.1 Formal construction

We shall adapt the method developed in [17] to construct smooth data $(\psi_0, v_0, \check{q}_0)$ that satisfies (B.1). We first describe the method formally which serves as a good guideline. The key difference, however, is that in [17] we constructed the initial data in Lagrangian coordinates, where (B.1) has a different formulation.

By identifying $\mathcal{F}_\lambda'(q)=\lambda^2$ without loss of generality, and since $\partial_1\varphi|_{\Sigma}=\partial_1\psi$, $\partial_2\varphi|_{\Sigma}=\partial_2\psi$, $\partial_3\varphi|_{\Sigma}=1$, the momentum and continuity equations reduce respectively to

$$\rho(\partial_t+\overline{v}\cdot\overline{\partial})v+\nabla^{\varphi}\check{q} = -g(\rho-1)e_3,\ \text{ on }\Sigma \tag{B.2}$$

$$\lambda^2(\partial_t+\overline{v}\cdot\overline{\partial})\check{q}+\operatorname{div} v = \partial_1\psi\partial_3 v^1+\partial_2\psi\partial_3 v^2+\lambda^2 g v^3,\ \text{ on }\Sigma, \tag{B.3}$$

where $\nabla^{\varphi}q = (\partial_1 q-\partial_1\psi\partial_3 q, \partial_2 q-\partial_2\psi\partial_3 q, \partial_3 q)^\top$ and $\operatorname{div} v = \partial\cdot v$. By ignoring the terms contributed by the denominator, we have $\mathcal{H}\sim -\overline{\Delta}\psi$. Invoking the kinematic boundary condition $\partial_t\psi = v\cdot N$, we have

$$(\partial_t+\overline{v}\cdot\overline{\partial})\psi = v^3, \quad \text{on }\Sigma,$$

we obtain from the zeroth compatibility condition $\check{q}\sim-\sigma\overline{\Delta}\psi$ that

$$(\partial_t+\overline{v}\cdot\overline{\partial})\check{q}\sim -\sigma\overline{\Delta}v^3, \quad \text{on }\Sigma, \tag{B.4}$$

which is the first compatibility condition. Since the continuity equation (B.3) implies $\lambda^2(\partial_t+\overline{v}\cdot\overline{\partial})\check{q}\sim-\operatorname{div} v$, we can deduce from (B.4) that:

$$\operatorname{div} v\sim\sigma\lambda^2\overline{\Delta}v^3, \quad \text{on }\Sigma. \tag{B.5}$$

Furthermore, the momentum equation (B.2) implies $(\partial_t+\overline{v}\cdot\overline{\partial})v^3\sim-\partial_3\check{q}$, and thus the second compatibility condition becomes:

$$(\partial_t+\overline{v}\cdot\overline{\partial})^2\check{q}\sim-\sigma(\partial_t+\overline{v}\cdot\overline{\partial})\overline{\Delta}v^3\sim\sigma\overline{\Delta}\partial_3\check{q}, \quad \text{on }\Sigma. \tag{B.6}$$

Taking $\partial_t+\overline{v}\cdot\overline{\partial}$ to the continuity equation to obtain $\lambda^2(\partial_t+\overline{v}\cdot\overline{\partial})^2\check{q}\sim-\operatorname{div}(\partial_t+\overline{v}\cdot\overline{\partial})v\sim\Delta\check{q}$, and this gives

$$\partial_3^2\check{q}\sim\sigma\lambda^2\overline{\Delta}\partial_3\check{q}-\overline{\Delta}\check{q}, \quad \text{on }\Sigma. \tag{B.7}$$

Finally, we derive from the third compatibility condition

$$(\partial_t + \overline{v}\cdot\overline{\partial})^3\check{q} \sim \sigma\overline{\Delta}\partial_3(\partial_t + \overline{v}\cdot\overline{\partial})\check{q} \sim \sigma\lambda^{-2}\overline{\Delta}\partial_3 \mathrm{div}\, v, \quad \text{on } \Sigma, \tag{B.8}$$

together with the relation $\lambda^2(\partial_t + \overline{v}\cdot\overline{\partial})^3\check{q} \sim \Delta(\partial_t + \overline{v}\cdot\overline{\partial})\check{q} \sim \lambda^{-2}\Delta\mathrm{div}\, v$ obtained by taking $(\partial_t + \overline{v}\cdot\overline{\partial})^2$ to the continuity equation that

$$\Delta\mathrm{div}\, v \sim \sigma\lambda^2\overline{\Delta}\partial_3\mathrm{div}\, v, \quad \text{on } \Sigma. \tag{B.9}$$

In other words,

$$\partial_3^3 v \sim \sigma\lambda^2\overline{\Delta}\partial_3\mathrm{div}\, v - \Delta\partial_1 v - \Delta\partial_2 v - \overline{\Delta}\partial_3 v, \quad \text{on } \Sigma. \tag{B.10}$$

Therefore, the first order compatibility condition yields an "identity in terms of v" (B.5), the second order compatibility condition yields an "identity in terms of q" (B.7), and lastly, the third order compatibility condition yields an "identity in terms of v" again (B.10).

We construct our data by the following iterative procedure. To begin with, let $(\xi_0, \mathbf{w}_0, p_0)$ be the generic smooth *localized* incompressible data that verifies the zeroth order compatibility condition $\check{p}_0 = -\sigma\overline{\nabla}\cdot\frac{\overline{\nabla}\xi_0}{\sqrt{1+|\overline{\nabla}\xi_0|^2}} + g\xi_0$ on Σ. In the first step, we fixed a smooth function ψ_0 which represents the moving interface, and construct the data satisfying the first compatibility condition. Given (B.5), we shall need to construct the appropriate velocity vector field denoted by $\mathbf{u}_0 = (\mathbf{u}_0^1, \mathbf{u}_0^2, \mathbf{u}_0^3)$. We achieve this by setting $\mathbf{u}_0^1 = \mathbf{w}_0^1$, $\mathbf{u}_0^2 = \mathbf{w}_0^2$, and construct $\mathbf{u}_0^3$ by solving a poly-harmonic equation of order 2:

$$\begin{cases} \Delta^2\mathbf{u}_0^3 = \Delta^2\mathbf{w}_0^3, & \text{in } \Omega,\\ \mathbf{u}_0^3 = \mathbf{w}_0^3, \quad \partial_3\mathbf{u}_0^3 \sim -\partial_1\mathbf{w}_0^1 - \partial_2\mathbf{w}_0^2 + \sigma\lambda^2\overline{\Delta}\mathbf{w}_0^3, & \text{on } \Sigma,\\ \mathbf{u}_0^3 = \mathbf{w}_0^3, \quad \partial_3\mathbf{u}_0^3 = \partial_3\mathbf{w}_0^3 & \text{on } \Sigma_b. \end{cases} \tag{B.11}$$

In particular, the boundary condition $\partial_3\mathbf{u}_0^3 \sim -\partial_1\mathbf{w}_0^1 - \partial_2\mathbf{w}_0^2 + \sigma\lambda^2\overline{\Delta}\mathbf{w}_0^3$ is derived from (B.5).

In the second step, we construct the data verifying the second compatibility condition. We shall construct $\check{q}_0$ here because of (B.7). This is achieved by solving a poly-harmonic equation of order 3:

$$\begin{cases} \Delta^3\check{q}_0 = \Delta^3\check{p}_0, & \text{in } \Omega,\\ \check{q}_0 = \check{p}_0, \quad \partial_3\check{q}_0 = \partial_3\check{p}_0, & \text{on } \Sigma,\\ \partial_3^2\check{q}_0 \sim \sigma\lambda^2\overline{\Delta}\partial_3\check{p}_0 - \overline{\Delta}\check{p}_0, & \text{on } \Sigma,\\ \partial_3^j\check{q}_0 = 0 \ \ (0 \le j \le 2), & \text{on } \Sigma_b. \end{cases} \tag{B.12}$$

It can be seen that the boundary condition $\partial_3^2\check{q}_0 \sim \sigma\lambda^2\overline{\Delta}\partial_3\check{p}_0$ is a consequence of (B.7).

In the third (and final) step, we construct the data verifying the compatibility conditions up to order 3 with a fixed smooth function representing the moving interface still denoted by ψ_0. Since q_0 has been constructed, we need only to construct $v_0 = (v_0^1, v_0^2, v_0^3)$ by setting $\mathbf{w}_0^1 = v_0^1$, $\mathbf{w}_0^2 = v_0^2$, and solving the following order 4 poly-harmonic equation for v_0^3:

$$\begin{cases} \Delta^4 v_0^3 = \Delta^4\mathbf{u}_0^3, & \text{in } \Omega,\\ v_0^3 = \mathbf{u}_0^3, \quad \partial_3 v_0^3 \sim -\partial_1\mathbf{u}_0^1 - \partial_2\mathbf{u}_0^2 + \sigma\lambda^2\overline{\Delta}\mathbf{u}_0^3 & \text{on } \Sigma,\\ \partial_3^2 v_0^3 \sim -\partial_3\partial_1\mathbf{u}_0^1 - \partial_3\partial_2\mathbf{u}_0^2 + \sigma\lambda^2\overline{\Delta}\partial_3\mathbf{u}_0^3, & \text{on } \Sigma,\\ \partial_3^3 v_0^3 = -\Delta\partial_1\mathbf{u}_0^1 - \Delta\partial_2\mathbf{u}_0^2 + \sigma\lambda^2\overline{\Delta}\partial_3\mathrm{div}\,\mathbf{u}_0 - \overline{\Delta}\partial_3\mathbf{u}_0^3, & \text{on } \Sigma,\\ \partial_3^j v_0^3 = \partial_3^j\mathbf{u}_0^3 \ \ (0 \le j \le 3) & \text{on } \Sigma_b. \end{cases} \tag{B.13}$$

The second and third boundary conditions arise from (B.5), whereas the fourth boundary condition is derived from (B.10).

B.2 The full construction procedure

We shall repeat the method introduced in Subsection B.1 with detailed boundary conditions generated by the compatibility conditions. We will use $\mathcal{P}, \mathcal{Q}$ to denote generic polynomials. Apart from this, we will set

$$0 \le k' \le 1, \quad 0 \le k \le 2, \quad 0 \le l \le 3,$$

throughout.

By invoking the commutator

$$[\overline{\partial}^s, \partial_t + \overline{v} \cdot \overline{\partial}] = [\overline{\partial}^s, \overline{v}] \cdot \overline{\partial}, \tag{B.14}$$

and since it holds on Σ that

$$(\partial_t + \overline{v} \cdot \overline{\partial})\psi = v^3, \quad \check{q} = -\sigma\left(\frac{\overline{\Delta}\psi}{|N|} - \frac{\overline{\partial}\psi \cdot \overline{\partial}\overline{\nabla}\psi}{|N|^3}\right) + g\psi, \quad |N| = \sqrt{1 + |\overline{\nabla}\psi|^2},$$

the first compatibility condition reads

$$(\partial_t + \overline{v} \cdot \overline{\partial})\check{q} = \sigma\mathcal{P}(\frac{1}{|N|}, \overline{\partial}^k\psi, \overline{\partial}^k\overline{v}, \overline{\partial}^k v^3), \quad \text{on } \Sigma. \tag{B.15}$$

In addition, the continuity equation (B.3) gives

$$\lambda^2(\partial_t + \overline{v} \cdot \overline{\partial})\check{q} = -\mathrm{div}\, v + \overline{\partial}\psi \cdot \partial_3\overline{v} + \lambda^2 g v^3, \quad \text{on } \Sigma. \tag{B.16}$$

Hence, we combine (B.15) and (B.16) to get

$$\mathrm{div}\, v = \sigma\lambda^2\mathcal{P}(|N|^{-1}, \overline{\partial}^k\psi, \overline{\partial}^k\overline{v}, \overline{\partial}^k v^3, \partial_3\overline{v}), \quad \text{on } \Sigma. \tag{B.17}$$

and the equation used to determine $\mathbf{u}_0^3$ is

$$\begin{cases} \Delta^2\mathbf{u}_0^3 = \Delta^2\mathbf{w}_0^3, & \text{in } \Omega, \\ \mathbf{u}_0^3 = \mathbf{w}_0^3, & \text{on } \Sigma \cup \Sigma_b, \\ \partial_3\mathbf{u}_0^3 = -\partial_1\mathbf{w}_0^1 - \partial_2\mathbf{w}_0^2 + \sigma\lambda^2\mathcal{P}(|N_0|^{-1}, \overline{\partial}^k\psi_0, \overline{\partial}^k\overline{\mathbf{w}}_0, \overline{\partial}^k\mathbf{w}_0^3, \partial_3\overline{\mathbf{w}}_0), & \text{on } \Sigma, \\ \partial_3\mathbf{u}_0^3 = \partial_3\mathbf{w}_0^3 & \text{on } \Sigma_b. \end{cases} \tag{B.18}$$

whose rough version is given by (B.11). Let $s_0 \geq 8$. The poly-harmonic estimate yields

$$\|\mathbf{u}_0^3 - \mathbf{w}_0^3\|_{s_0} \lesssim \underbrace{\|\Delta^2(\mathbf{u}_0^3 - \mathbf{w}_0^3)\|_{s_0-4}}_{=0} + \underbrace{|\mathbf{u}_0^3 - \mathbf{w}_0^3|_{s_0-0.5}}_{=0} + |\partial_3(\mathbf{u}_0^3 - \mathbf{w}_0^3)|_{s_0-1.5} \leq \lambda^2 C(|\psi_0|_s, \|\mathbf{w}_0\|_s), \tag{B.19}$$

for some $s > s_0$, and hence $\|\mathbf{u}_0^3 - \mathbf{w}_0^3\|_{s_0} \to 0$ as $\lambda \to 0$.

We construct $\check{q}_0$ using the second-order compatibility condition in the next stage. Owing to (B.2), the identities

$$\rho(\partial_t + \overline{v} \cdot \overline{\partial})\overline{v} + \overline{\partial}\check{q} = \overline{\partial}\psi\partial_3\check{q}, \quad \text{and } \rho(\partial_t + \overline{v} \cdot \overline{\partial})v_3 + \partial_3\check{q} = -g(\rho - 1), \tag{B.20}$$

hold on Σ, and we view $\rho = \rho(\check{q})$ here and throughout. Taking $\partial_t + \overline{v} \cdot \overline{\partial}$ to (B.15) and invoking (B.14), we have

$$(\partial_t + \overline{v} \cdot \overline{\partial})^2\check{q} = \sigma\mathcal{P}(\rho^{-1}, |N|^{-1}, \overline{\partial}^l\psi, \overline{\partial}^k\overline{v}, \overline{\partial}^k v^3, \overline{\partial}^l\check{q}, \overline{\partial}^k\partial_3\check{q}), \quad \text{on } \Sigma. \tag{B.21}$$

Moreover, by taking $\partial_t + \overline{v} \cdot \overline{\partial}$ to the continuity equation (B.3), we get

$$\lambda^2(\partial_t + \overline{v} \cdot \overline{\partial})^2\check{q} = -\mathrm{div}\,(\partial_t + \overline{v} \cdot \overline{\partial})v + [\mathrm{div}\,, (\partial_t + \overline{v} \cdot \overline{\partial})]v + (\partial_t + \overline{v} \cdot \overline{\partial})(\overline{\partial}\psi \cdot \partial_3\overline{v} + \lambda^2 g v^3), \tag{B.22}$$

where $[\mathrm{div}\,, (\partial_t + \overline{v} \cdot \overline{\partial})]v = \partial_i\overline{v} \cdot \overline{\partial}v^i$,

$$-\mathrm{div}\,(\partial_t + \overline{v} \cdot \overline{\partial})\overline{v} = \partial^\tau(\rho^{-1}\partial_\tau\check{q}) - \partial^\tau(\rho^{-1}\partial_\tau\psi\partial_3\check{q}) + \underbrace{\partial_3(\rho^{-1}\partial_3\check{q})}_{=\rho^{-1}\partial_3^2\check{q} + \partial_3\rho^{-1}\partial_3\check{q}} + g\partial_3\Big(\rho^{-1}(\rho - 1)\Big), \quad \tau = 1, 2, \tag{B.23}$$

and

$$\begin{aligned}(\partial_t + \overline{v} \cdot \overline{\partial})(\overline{\partial}\psi \cdot \partial_3\overline{v} + \lambda^2 g v^3) = {} & \overline{\partial}v^3 \cdot \partial_3\overline{v} + \overline{\partial}\psi \cdot \partial_3(-\rho^{-1}\overline{\partial}\check{q} + \rho^{-1}\overline{\partial}\psi\partial_3\check{q}) \\ & -\overline{\partial}\overline{v} \cdot \overline{\partial}\psi \cdot \partial_3\overline{v} - \overline{\partial}\psi \cdot \partial_3\overline{v} \cdot \partial_3\overline{v} + \lambda^2 g(-\rho^{-1}\partial_3\check{q} - g\rho^{-1}(\rho - 1)).\end{aligned} \tag{B.24}$$

Since the third term on the RHS of (B.24) contributes to $\rho^{-1}|\overline{\partial}\psi|^2\partial_3^2\check{q}$, it holds that

$$\lambda^2(\partial_t+\overline{v}\cdot\overline{\partial})^2\check{q}=\rho^{-1}(1+|\overline{\partial}\psi|^2)\partial_3^2\check{q}+Q(\rho^{-1},|N|^{-1},\overline{\partial}^k\psi,\overline{\partial}^{k'}\partial_3 v,\overline{\partial}^{k'}\partial_3\check{q}),\quad \text{on } \Sigma. \tag{B.25}$$

Therefore, we combine (B.21) and (B.25) to get

$$\rho^{-1}(1+|\overline{\partial}\psi|^2)\partial_3^2 q=\sigma\lambda^2\mathcal{P}(\rho^{-1},|N|^{-1},\overline{\partial}^l\psi,\overline{\partial}^k\overline{v},\overline{\partial}^k v^3,\overline{\partial}^l q,\overline{\partial}^k\partial_3 q)+Q(\rho^{-1},|N|^{-1},\overline{\partial}^k\psi,\overline{\partial}^{k'}\partial_3 v,\overline{\partial}^{k'}\partial_3 q),\quad \text{on } \Sigma, \tag{B.26}$$

and we set $\check{q}_0$ by solving

$$\begin{cases}\Delta^3\check{q}_0=\Delta^3\check{p}_0, & \text{in } \Omega,\\ \check{q}_0=\check{p}_0,\quad \partial_3\check{q}_0=\partial_3\check{p}_0, & \text{on } \Sigma,\\ \partial_3^2\check{q}_0=\rho_0(1+|\overline{\partial}\psi_0|^2)^{-1}\Big(\sigma\lambda^2\mathcal{P}(\rho_0^{-1},|N_0|^{-1},\overline{\partial}^l\psi_0,\overline{\partial}^k\overline{\mathbf{u}}_0,\overline{\partial}^k\mathbf{u}_0^3,\overline{\partial}^l\check{p}_0,\overline{\partial}^k\partial_3\check{p}_0) & \\ +Q(\rho_0^{-1},|N_0|^{-1},\overline{\partial}^k\psi_0,\overline{\partial}^{k'}\partial_3\mathbf{u}_0,\overline{\partial}^{k'}\partial_3\check{p}_0)\Big), & \text{on } \Sigma,\\ \partial_3^j\check{q}_0=0\ \ (0\le j\le 2) & \text{on } \Sigma_b.\end{cases} \tag{B.27}$$

whose rough version is (B.12). Also, the poly-harmonic estimate implies

$$\|\check{q}_0\|_{s_0}\lesssim\|\Delta^3\check{p}_0\|_{s_0-6}+|\check{p}_0|_{s_0-0.5}+|\partial_3\check{p}_0|_{s_0-1.5}+|\partial_3^2\check{q}_0|_{s-2.5}\le\lambda^2C_1(|\psi_0|_s,\|\mathbf{u}_0\|_s,\|\check{p}_0\|_s)+C_2(|\psi_0|_s,\|\mathbf{u}_0\|_s,\|\check{p}_0\|_s), \tag{B.28}$$

for some $s>s_0$.

Finally, we construct v_0^3 using the third-order compatibility condition in the last stage. We obtain

$$(\partial_t+\overline{v}\cdot\overline{\partial})^3\check{q}=\sigma\mathcal{P}(\rho^{-1},|N|^{-1},\overline{\partial}^l\psi,\overline{\partial}^l\overline{v},\overline{\partial}^l v^3,\partial^l\check{q})\left(\lambda^{-2}\overline{\partial}^4\psi+\lambda^{-2}\overline{\partial}^4 v+\lambda^{-2}\overline{\partial}^l\partial_3 v+\lambda^{-2}\overline{\partial}^k\partial_3^2 v\right),\quad \text{on } \Sigma, \tag{B.29}$$

by taking $(\partial_t+\overline{v}\cdot\overline{\partial})$ to (B.21). Further, taking $(\partial_t+\overline{v}\cdot\overline{\partial})$ to (B.25) to get

$$\lambda^2(\partial_t+\overline{v}\cdot\overline{\partial})^3\check{q}=-\lambda^{-2}\rho^{-1}(1+|\overline{\partial}\psi|^2)\partial_3^2\mathrm{div}\, v+Q(\rho^{-1},|N|^{-1},\overline{\partial}^l\psi,\overline{\partial}^l v,\overline{\partial}^k\partial_3 v,\overline{\partial}^{k'}\partial_3^2 v,\overline{\partial}^l\check{q}),\quad \text{on } \Sigma. \tag{B.30}$$

Therefore, we combine (B.29) and (B.30) to obtain

$$\begin{aligned}\rho^{-1}(1+|\overline{\partial}\psi|^2)\partial_3^2\mathrm{div}\, v=&\ \sigma\lambda^2\mathcal{P}(\rho^{-1},|N|^{-1},\overline{\partial}^l\psi,\overline{\partial}^l\overline{v},\overline{\partial}^l v^3,\partial^l\check{q})\left(\overline{\partial}^4\psi+\overline{\partial}^4 v+\overline{\partial}^l\partial_3 v+\overline{\partial}^k\partial_3^2 v\right)\\ &+\lambda^2 Q(\rho^{-1},|N|^{-1},\overline{\partial}^l\psi,\overline{\partial}^l\psi,\overline{\partial}^l v,\overline{\partial}^k\partial_3 v,\overline{\partial}^{k'}\partial_3^2 v,\overline{\partial}^l\check{q}),\quad \text{on } \Sigma,\end{aligned} \tag{B.31}$$

and we set v_0^3 by solving

$$\begin{cases}\Delta^4 v_0^3=\Delta^4\mathbf{u}_0^3, & \text{in } \Omega,\\ v_0^3=\mathbf{u}_0^3, & \text{on } \Sigma,\\ \partial_3 v_0^3=-\partial_1\mathbf{u}_0^1-\partial_2\mathbf{u}_0^2+\sigma\lambda^2\mathcal{P}(|N_0|^{-1},\overline{\partial}^k\psi_0,\overline{\partial}^k\mathbf{u}_0,\overline{\partial}^k\mathbf{u}_0^3,\partial_3\overline{\mathbf{u}}_0),\quad 0\le k\le 2, & \text{on } \Sigma,\\ \partial_3^2 v_0^3=-\partial_1\partial_3\mathbf{u}_0^1-\partial_2\partial_3\mathbf{u}_0^2+\sigma\lambda^2\partial_3\mathcal{P}(|N_0|^{-1},\overline{\partial}^k\psi_0,\overline{\partial}^k\mathbf{u}_0,\overline{\partial}^k\mathbf{u}_0^3,\partial_3\overline{\mathbf{u}}_0), & \text{on } \Sigma,\\ \partial_3^3 v_0^3=\rho_0(1+|\overline{\partial}\psi_0|^2)^{-1}\Big(\sigma\lambda^2\mathcal{P}(\rho_0^{-1},|N_0|^{-1},\overline{\partial}^l\psi_0,\overline{\partial}^l\overline{\mathbf{u}}_0,\overline{\partial}^l\mathbf{u}_0^3,\partial^l\check{q}_0)\left(\overline{\partial}^4\psi_0+\overline{\partial}^4\mathbf{u}_0+\overline{\partial}^l\partial_3\mathbf{u}_0+\overline{\partial}^k\partial_3^2\mathbf{u}_0\right) & \\ +\lambda^2 Q(\rho_0^{-1},|N_0|^{-1},\overline{\partial}^l\psi_0,\overline{\partial}^l\psi,\overline{\partial}^l\mathbf{u}_0,\overline{\partial}^k\partial_3\mathbf{u}_0,\overline{\partial}^{k'}\partial_3^2\mathbf{u}_0,\overline{\partial}^l\check{q}_0)\Big)-\rho_0^{-1}(1+|\overline{\partial}\psi_0|^2)\partial_3^2(\partial_1\mathbf{u}_0^1+\partial_2\mathbf{u}_0^2), & \text{on } \Sigma,\\ \partial_3^j v_0^3=\partial_3^j\mathbf{u}_0^3\ \ (0\le j\le 3) & \text{on } \Sigma_b.\end{cases} \tag{B.32}$$

whose rough version is (B.13). By the poly-harmonic estimate, we have

$$\|v_0^3-\mathbf{u}_0^3\|_{s_0}\lesssim\|\Delta^4(v_0^3-\mathbf{u}_0^3)\|_{s_0-8}+|v_0^3-\mathbf{u}_0^3|_{s_0-0.5}+|\partial^3(v_0^3-\mathbf{u}_0^3)|_{s_0-1.5}+|\partial_3^2(v_0^3-\mathbf{u}_0^3)|_{s_0-2.5}+|\partial^3(v_0^3-\mathbf{u}_0^3)|_{s_0-3.5}. \tag{B.33}$$

The first two terms on the RHS are 0. Invoking (B.19), (B.28), we have, for some s,s' satisfying $s>s'>s_0$, that

$$|v_0^3-\mathbf{u}_0^3|_{s_0-0.5}+|\partial^3(v_0^3-\mathbf{u}_0^3)|_{s_0-1.5}\le\lambda^2C(|\psi_0|_{s'},\|\mathbf{u}_0\|_{s'})\le\lambda^2C(|\psi_0|_s,\|\mathbf{w}_0\|_s),$$

and

$$|\partial_3^2(v_0^3-\mathbf{u}_0^3)|_{s_0-2.5}\le\lambda^2C(|\psi_0|_{s'},\|\mathbf{u}_0\|_{s'},\|\check{q}_0\|_{s'})\le\lambda^2C(|\psi_0|_s,\|\mathbf{w}_0\|_s,\|\check{p}_0\|_s).$$

Thus,

$$\|v_0^3 - \mathbf{u}_0^3\|_{s_0} \le \lambda^2 C(|\psi_0|_s, \|\mathbf{w}_0\|_s, \|\check{p}_0\|_s). \tag{B.34}$$

In particular, since we have set $\mathbf{w}_0^\tau = \mathbf{u}_0^\tau = v_0^\tau$, $\tau = 1, 2$, we deduce from (B.19) and (B.34) that

$$\|v_0 - \mathbf{w}_0\|_{s_0} \le \|v_0^3 - \mathbf{u}_0^3\|_{s_0} + \|\mathbf{u}_0^3 - \mathbf{w}_0^3\|_{s_0} = O(\lambda^2). \tag{B.35}$$

In addition, we deduce from $\nabla^\varphi \cdot \mathbf{w}_0 = 0$ and (B.35) that

$$\|\nabla^\varphi \cdot v_0\|_{C^1} = O(\lambda^2). \tag{B.36}$$

Apart from these, it can be seen from (B.27) and (B.32) that $\|v_0\|_{s_0}$ and $\|\check{q}_0\|_{s_0}$ are uniform in both σ and λ. This ensures us take the zero surface tension and incompressible limits at the same time.

C Construction of initial data for the nonlinear κ-approximate system

The construction of smooth initial data for the κ-problem (3.11) is parallel to what has been done in the previous section and thus we shall only sketch the details. We will set

$$0 \le k' \le 1, \quad 0 \le k \le 2, \quad 0 \le l \le 3, \quad 0 \le m \le 4, \quad 0 \le n \le 5$$

in the sequel.

Let (ψ_0, v_0, q_0) be the smooth initial data constructed in the previous section. Our goal is to construct $(\psi_{\kappa,0}, v_{\kappa,0}, q_{\kappa,0})$ that satisfies the κ-compatibility conditions up to the third order:

$$(\partial_t + \overline{v} \cdot \overline{\partial})^j q|_{t=0} = \sigma(\partial_t + \overline{v} \cdot \overline{\partial})^j \mathcal{H}|_{t=0} + \kappa^2 (\partial_t + \overline{v} \cdot \overline{\partial})^j \left((1 - \overline{\Delta})(-\partial_1 \widetilde{\psi} v^1 - \partial_2 \widetilde{\psi} v^2 + v^3)\right)|_{t=0}, \quad j = 0, 1, 2, 3. \tag{C.1}$$

Setting $\psi_{\kappa,0} = \psi_0$, we need only to compute the last term on the RHS to formulate the poly-harmonic equations for $q_{\kappa,0}$ and $v_{\kappa,0}$. Since

$$[(1 - \overline{\Delta}), \partial_t + \overline{v} \cdot \overline{\partial}] = -[\overline{\Delta}, \overline{v}] \cdot \overline{\partial},$$

we have, when $j = 1$:

$$(\partial_t + \overline{v} \cdot \overline{\partial})\Big((1 - \overline{\Delta})(-\overline{\partial} \widetilde{\psi} \cdot \overline{v} + v^3)\Big) = \mathcal{R}(\overline{\partial}^l \psi, \overline{\partial}^l v, \overline{\partial}^l \widetilde{\psi}, \overline{\partial}^l \widetilde{v}^3, \overline{\partial}^l \check{q}, \overline{\partial}^k \partial_3 \check{q}), \text{ on } \Sigma. \tag{C.2}$$

This implies that the equation used to determine $\mathbf{u}_{\kappa,0}^3$ is

$$\begin{cases} \Delta^2 \mathbf{u}_{\kappa,0}^3 = \Delta^2 v_0^3, & \text{in } \Omega, \\ \mathbf{u}_{\kappa,0}^3 = v_0^3, & \text{on } \Sigma, \\ \partial_3 \mathbf{u}_{\kappa,0}^3 = -\partial_1 v_0^1 - \partial_2 v_0^2 + \sigma \lambda^2 \mathcal{P}(|N_0|^{-1}, \overline{\partial}^k \psi_0, \overline{\partial}^k v_0, \overline{\partial}^k v_0^3, \partial_3 \overline{v}_0) & \\ +\kappa^2 \lambda^2 \mathcal{R}(\overline{\partial}^l \psi_0, \overline{\partial}^l v_0, \overline{\partial}^l \widetilde{\psi}_0, \overline{\partial}^l \widetilde{v}_0^3, \overline{\partial}^l \check{q}_0, \overline{\partial}^k \partial_3 \check{q}_0), & \text{on } \Sigma, \\ \partial_3^j v_{\kappa,0}^3 = \partial_3^j \mathbf{u}_0^3 \ \ (0 \le j \le 1) & \text{on } \Sigma_b. \end{cases} \tag{C.3}$$

which is parallel to (B.18).

Then, when $j = 2$, we have

$$\begin{aligned} &(\partial_t + \overline{v} \cdot \overline{\partial})^2 \Big((1 - \overline{\Delta})(-\overline{\partial} \widetilde{\psi} \cdot \overline{v} + v^3)\Big) = (\partial_t + \overline{v} \cdot \overline{\partial}) \mathcal{R}(\overline{\partial}^l \psi, \overline{\partial}^l v, \overline{\partial}^l \widetilde{\psi}, \overline{\partial}^l \widetilde{v}^3, \overline{\partial}^l \check{q}, \overline{\partial}^k \partial_3 \check{q}) \\ &= \mathcal{R}(\overline{\partial}^l \psi, \overline{\partial}^l \widetilde{\psi}, \overline{\partial}^l v^3, \overline{\partial}^l \widetilde{v}^3, \overline{\partial}^m \check{q}, \overline{\partial}^l \partial_3 \check{q}, \lambda^{-2} \overline{\partial}^4 v, \lambda^{-2} \overline{\partial}^l \partial_3 v, \lambda^{-2} \overline{\partial}^k \partial_3^2 v, \lambda^{-2} \overline{\partial}^4 \psi), \quad \text{on } \Sigma, \end{aligned} \tag{C.4}$$

where the power of λ^{-1} does not exceed 2. Thus, we determine $q_{\kappa,0}$ by solving

$$\begin{cases} \Delta^3 \check{q}_{\kappa,0} = \Delta^3 \check{q}_0, & \text{in } \Omega, \\ \check{q}_{\kappa,0} = \check{q}_0, \quad \partial_3 \check{q}_{\kappa,0} = \partial_3 \check{q}_0, & \text{on } \Sigma, \\ \partial_3^2 \check{q}_{\kappa,0} = \rho_0 (1 + |\overline{\partial} \psi_0|^2)^{-1} \Big(\sigma \lambda^2 \mathcal{P}(\rho_0^{-1}, |N_0|^{-1}, \overline{\partial}^l \psi_0, \overline{\partial}^k \overline{\mathbf{u}}_{\kappa,0}, \overline{\partial}^k \mathbf{u}_{\kappa,0}^3, \overline{\partial}^l \check{q}_0, \overline{\partial}^k \partial_3 \check{q}_0) & \\ +Q(\rho_0^{-1}, |N_0|^{-1}, \overline{\partial}^k \psi_0, \overline{\partial}^{k'} \partial_3 \mathbf{u}_{\kappa,0}, \overline{\partial}^{k'} \partial_3 \check{q}_0) & \\ +\kappa^2 \lambda^2 \mathcal{R}(\overline{\partial}^l \psi_0, \overline{\partial}^l \widetilde{\psi}_0, \overline{\partial}^l \mathbf{u}_{\kappa,0}^3, \overline{\partial}^l \widetilde{\mathbf{u}}_{\kappa,0}^3, \overline{\partial}^m \check{q}_0, \overline{\partial}^l \partial_3 \check{q}_0, \overline{\partial}^4 \mathbf{u}_{\kappa,0}, \overline{\partial}^l \partial_3 \mathbf{u}_{\kappa,0}, \overline{\partial}^k \partial_3^2 \mathbf{u}_{\kappa,0}, \overline{\partial}^4 \psi_0)\Big), & \text{on } \Sigma, \\ \partial_3^j \check{q}_0^3 = 0 \ \ (0 \le j \le 2) & \text{on } \Sigma_b. \end{cases} \tag{C.5}$$

Finally, when $j=3$, we have

$$(\partial_t+\overline{v}\cdot\overline{\partial})^3\Big((1-\overline{\Delta})(-\overline{\partial}\widetilde{\psi}\cdot\overline{v}+v^3)\Big)=\mathcal{R}(\overline{\partial}^m\psi,\overline{\partial}^m\widetilde{\psi},\overline{\partial}^m v^3,\overline{\partial}^m\widetilde{v}^3,\overline{\partial}^n\check{q},\overline{\partial}^m\partial_3\check{q},\lambda^{-2}\overline{\partial}^5 v,\lambda^{-2}\overline{\partial}^m\partial_3 v,\lambda^{-2}\overline{\partial}^l\partial_3^2 v,\lambda^{-2}\overline{\partial}^5\psi),\quad \text{on }\Sigma, \tag{C.6}$$

where the power of λ^{-1} does not exceed 4. Therefore, we construct $v^3_{\kappa,0}$ by solving

$$\begin{cases}\Delta^4 v^3_{\kappa,0}=\Delta^4\mathbf{u}^3_{\kappa,0}, & \text{in }\Omega,\\ v^3_{\kappa,0}=\mathbf{u}^3_{\kappa,0}, & \text{on }\Sigma,\\ \partial_3 v^3_{\kappa,0}=-\partial_1\mathbf{u}^1_{\kappa,0}-\partial_2\mathbf{u}^2_{\kappa,0}+\sigma\lambda^2\mathcal{P}(|N_0|^{-1},\overline{\partial}^k\psi_0,\overline{\partial}^k\mathbf{u}_{\kappa,0},\overline{\partial}^k\mathbf{u}^3_{\kappa,0},\partial_3\overline{\mathbf{u}}_{\kappa,0}) & \\ +\kappa^2\lambda^2\mathcal{R}(\overline{\partial}^l\psi_0,\overline{\partial}^l\mathbf{u}_{\kappa,0},\overline{\partial}^l\widetilde{\psi}_0,\overline{\partial}^l\widetilde{\mathbf{u}}^3_{\kappa,0},\overline{\partial}^l\check{q}_{\kappa,0},\overline{\partial}^k\partial_3\check{q}_{\kappa,0}), & \text{on }\Sigma,\\ \partial_3^2 v^3_{\kappa,0}=-\partial_1\partial_3\mathbf{u}^1_{\kappa,0}-\partial_2\partial_3\mathbf{u}^2_{\kappa,0}+\sigma\lambda^2\partial_3\mathcal{P}(|N_0|^{-1},\overline{\partial}^k\psi_0,\overline{\partial}^k\mathbf{u}_{\kappa,0},\overline{\partial}^k\mathbf{u}^3_{\kappa,0},\partial_3\overline{\mathbf{u}}_{\kappa,0}) & \\ +\kappa^2\lambda^2\partial_3\mathcal{R}(\overline{\partial}^l\psi_0,\overline{\partial}^l\mathbf{u}_{\kappa,0},\overline{\partial}^l\widetilde{\psi}_0,\overline{\partial}^l\widetilde{\mathbf{u}}^3_{\kappa,0},\overline{\partial}^l\check{q}_{\kappa,0},\overline{\partial}^k\partial_3\check{q}_{\kappa,0}), & \text{on }\Sigma,\\ \partial_3^3 v^3_{\kappa,0}=\rho_0(1+|\overline{\partial}\psi_0|^2)^{-1}\Big(\sigma\lambda^2\mathcal{P}(\rho_0^{-1},|N_0|^{-1},\overline{\partial}^l\psi_0,\overline{\partial}^l\overline{\mathbf{u}}_{\kappa,0},\overline{\partial}^l\mathbf{u}^3_{\kappa,0},\partial^l\check{q}_0)\left(\overline{\partial}^4\psi_0+\overline{\partial}^4\mathbf{u}_{\kappa,0}+\overline{\partial}^3\partial_3\mathbf{u}_{\kappa,0}+\overline{\partial}^2\partial_3^2\mathbf{u}_{\kappa,0}\right) & \\ +\lambda^2\mathcal{Q}(\rho_0^{-1},|N_0|^{-1},\overline{\partial}^l\psi_0,\overline{\partial}^l\mathbf{u}_0,\overline{\partial}^k\partial_3\mathbf{u}_0,\overline{\partial}^{k'}\partial_3^2\mathbf{u}_0,\overline{\partial}^l\check{q}_{\kappa,0}) & \\ +\mathcal{R}(\overline{\partial}^m\psi_0,\overline{\partial}^m\widetilde{\psi}_0,\overline{\partial}^m\mathbf{u}^3_{\kappa,0},\overline{\partial}^m\widetilde{\mathbf{u}}^3_{\kappa,0},\overline{\partial}^n\check{q}_{\kappa,0},\overline{\partial}^m\partial_3\check{q}_{\kappa,0},\overline{\partial}^5\mathbf{u}_{\kappa,0},\overline{\partial}^m\partial_3\mathbf{u}_{\kappa,0},\overline{\partial}^l\partial_3^2\mathbf{u}_{\kappa,0},\overline{\partial}^5\psi_0)\Big) & \\ -\rho_0^{-1}(1+|\overline{\partial}\psi_0|^2)\partial_3^2(\partial_1\mathbf{u}^1_{\kappa,0}+\partial_2\mathbf{u}^2_{\kappa,0}), & \text{on }\Sigma,\\ \partial_3^j v^3_{\kappa,0}=\partial_3^j\mathbf{u}^3_0\ \ (0\le j\le 3) & \text{on }\Sigma_b.\end{cases} \tag{C.7}$$

Let $\lambda>0$ be fixed. Invoking the poly-harmonic estimate subsequently to (C.3), (C.5), and (C.7), we obtain that $\|v_{\kappa,0}\|_{s_0}$ and $\|\check{q}_{\kappa,0}\|_{s_0}$ are bounded for some $s_0\ge 8$. Thus, the energy $E^\kappa(t)$ (defined as (4.1)) is bounded at $t=0$. In addition,

$$\|v_{\kappa,0}-v_0\|_{s_0},\ \text{and}\ \|\check{q}_{\kappa,0}-\check{q}\|_{s_0}\to 0,\quad \text{as }\ \kappa\to 0.$$

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