Compressible Gravity-Capillary Water Waves with Vorticity: Local Well-Posedness, Incompressible and Zero-Surface-Tension Limits

CHENYUN LUO* JUNYAN ZHANG †

Abstract

We consider 3D compressible isentropic Euler equations describing the motion of a liquid in an unbounded initial domain with a moving boundary and a fixed flat bottom at finite depth. The liquid is under the influence of gravity and surface tension and it is not assumed to be irrotational. We prove local well-posedness by combining a carefully designed approximate system and a hyperbolic approach which allows us to avoid using Nash-Moser iteration. The energy estimates yield no regularity loss and are uniform in Mach number, and they are uniform in surface tension coefficient under the Rayleigh-Taylor sign condition. We thus simultaneously obtain incompressible and zero surface tension limits. Moreover, we can drop the uniform boundedness (with respect to Mach number) on high-order time derivatives by applying the paradifferential calculus to the analysis of the free-surface evolution.

Mathematics Subject Classification (2020): 35L65, 35Q35, 76N10.
Keywords: compressible water waves, free-boundary problem, well-posedness, incompressible limit, paradifferential calculus.

Contents

*Department of Mathematics, The Chinese University of Hong Kong, Shatin, NT, Hong Kong. CL is supported in part by the Hong Kong RGC grant CUHK-24304621. Email: `cluo@math.cuhk.edu.hk`

†Department of Mathematics, National University of Singapore, Singapore. Email: `zhangjy@nus.edu.sg`

1 Introduction

In this paper, we study the motion of water waves in $\mathbb{R}^3$ described by the compressible Euler equations:

$$\begin{cases} \rho(\partial_t + u\cdot\nabla)u = -\nabla p - \rho g e_3, & \text{in } \mathcal{D} \\ \partial_t\rho + \nabla\cdot(\rho u) = 0 & \text{in } \mathcal{D} \\ p = p(\rho) & \text{in } \mathcal{D} \end{cases} \tag{1.1}$$

where $\mathcal{D} = \bigcup_{0\le t\le T}\{t\}\times\mathcal{D}_t$ with $\mathcal{D}_t := \{(x_1,x_2,x_3)\in\mathbb{R}^3 : -b < x_3 < \psi(t,x_1,x_2)\}$ with $b > 10$ a given constant representing the unbounded domain with finite depth occupied by the fluid at each fixed time t, whose boundary $\partial\mathcal{D}_t$ is determined by a moving surface represented via the graph $\Sigma_t := \{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_3 = \psi(t,x_1,x_2)\}$ and a flat bottom $\Sigma_b := \{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_3 = -b\}$. We will consider the case when $\Sigma_t\cap\Sigma_b = \emptyset$. This is easy to achieve in a short interval by assuming $\|\psi(0,\cdot)\|_{L^\infty(\mathbb{R}^2)} \le 1$.

In the first two equations of (1.1), u, ρ, p represent the fluid's velocity, density, and pressure, respectively. Also, we assume that the fluid is under the influence of the gravity $\rho g e_3$, with $g > 0$ and $e_3 = (0,0,1)^\top$. The third equation of (1.1) is known to be the equation of states which satisfies

$$p'(\rho) > 0, \quad \text{for } \rho \ge \bar\rho_0, \tag{1.2}$$

where $\bar\rho_0$ is a positive constant (we set $\bar\rho_0 = 1$ for simplicity), which is in the case of an isentropic *liquid*[1]. The equation of states is required to close the system of compressible Euler equations. We mention here that in the case of a *gas* $\bar\rho_0 = 0$, and we shall not discuss this in the paper.

The initial and boundary conditions of the system (1.1) are

$$\mathcal{D}_0 = \{x : (0,x)\in\mathcal{D}\}, \quad \text{and } u = u_0, \rho = \rho_0 \quad \text{on } \{t=0\}\times\mathcal{D}_0, \tag{1.3}$$

$$D_t|_{\partial\mathcal{D}} \in T(\partial\mathcal{D}), \quad u_3|_{\Sigma_b} = 0, \quad p|_{\Sigma_t} = \sigma\mathcal{H}, \tag{1.4}$$

where $T(\partial\mathcal{D})$ stands for the tangent bundle of $\partial\mathcal{D}$. The first condition in (1.4) is the kinematic boundary condition, which indicates that the free surface boundary moves with the normal component of the velocity (see (1.16) for an explicit illustration). The second condition is the slip condition imposed on the flat bottom Σ_b. The last condition in (1.4) shows that the pressure is balanced by surface tension on the moving surface Σ_t. Here, $\sigma > 0$ is called the surface tension constant, and $\mathcal{H}$ denotes the mean curvature of the free boundary of the fluid domain. Note that $\mathcal{H}$, $T(\partial\mathcal{D})$ and p are functions of the unknowns u, ρ and $\mathcal{D}$. So these quantities are not known a priori and hence have to be determined alongside a solution to the problem. Let $D_t := \partial_t + u\cdot\nabla$ be the material derivative. The equations modeling the motion of compressible gravity-capillary water waves read

$$\begin{cases} \rho D_t u = -\nabla p - \rho g e_3, & \text{in } \mathcal{D}, \\ \partial_t\rho + \nabla\cdot(\rho u) = 0, & \text{in } \mathcal{D}, \\ p = p(\rho), & \text{in } \mathcal{D}, \\ (u,\rho,\mathcal{D})|_{t=0} = (u_0,\rho_0,\mathcal{D}_0), \end{cases} \tag{1.5}$$

equipped with the boundary conditions

$$\begin{cases} p = \sigma\mathcal{H} & \text{on } \bigcup_{0\le t\le T}\{t\}\times\Sigma_t, \\ u_3 = 0 & \text{on } [0,T]\times\Sigma_b, \\ D_t|_{\partial\mathcal{D}} \in T(\partial\mathcal{D}). \end{cases} \tag{1.6}$$

System (1.5) together with (1.6) admits a conserved quantity

$$\mathsf{E}_0(t) := \frac{1}{2}\int_{\mathcal{D}_t}\rho|u|^2\,\mathrm{d}x + \int_{\mathcal{D}_t}\rho\mathsf{Q}(\rho)\,\mathrm{d}x + \int_{\mathcal{D}_t}(\rho-1)gx_3\,\mathrm{d}x + \int_{\Sigma_t} g|\psi|^2 + \sigma\left(\sqrt{1+|\overline{\nabla}\psi|^2}-1\right)\mathrm{d}x',$$

where $\mathsf{Q}(\rho) := \int_1^\rho p(r)r^{-2}\,\mathrm{d}r$ and $\mathrm{d}x' := \mathrm{d}x_1\,\mathrm{d}x_2$. A direct calculation (cf. [70, Section 6.1]) shows $\mathsf{E}_0'(t) = 0$. Note that we need a *localized* initial data such that $\mathsf{E}_0(0) < +\infty$ which can be achieved similarly as in [47, Section 7].

[1] In general, the equation of state is $p = p(\rho, S)$ where S denotes the entropy of the fluid and satisfies $(\partial_t + u\cdot\nabla)S = 0$. It is required to have $\partial p/\partial\rho > 0$. When S is a constant, we say the fluid is isentropic. Also, the assumptions $p'(\rho) > 0$ and $\rho \ge \bar\rho_0$ ensure the hyperbolicity of (1.1).

1.1 Fixing the fluid domain

We shall convert (1.5)-(1.6) into a system of equations defined on the fixed domain

$$\Omega = \{(x_1, x_2, x_3) : -b < x_3 < 0\}.$$

One way to achieve this would be to consider the Lagrangian coordinates. Nevertheless, here, we consider a family of diffeomorphism $\Phi(t,\cdot) : \Omega \to \mathcal{D}_t$ characterized by the moving surface boundary. In particular, let

$$\Phi(t, x_1, x_2, x_3) = (x_1, x_2, \varphi(t, x_1, x_2, x_3)), \tag{1.7}$$

where

$$\varphi(t, x_1, x_2, x_3) = x_3 + \chi(x_3)\psi(t, x_1, x_2). \tag{1.8}$$

Here, $\chi \in C_c^\infty(-b, 0]$ is a smooth cut-off function satisfying the following bound for some small constant $\delta_0 > 0$:

$$\|\chi'\|_{L^\infty(-b,0]} \le \frac{1}{\|\psi_0\|_\infty + 1}, \quad \sum_{j=1}^{5} \|\chi^{(j)}\|_{L^\infty(-b,0]} \le C, \quad \chi = 1 \ \text{ on } (-\delta_0, 0], \tag{1.9}$$

for some generic constant $C > 0$.

We will write $x' = (x_1, x_2)$ throughout the rest of this paper. It can be seen that

$$\partial_3\varphi(t, x', x_3) = 1 + \chi'(x_3)\psi(t, x') > 0, \quad t \in [0, T],$$

for some small $T > 0$, which ensures that $\Phi(t)$ is a diffeomorphism.

Let $x = (x', x_3) \in \Omega$. We denote respectively by

$$v(t, x) = u(t, \Phi(t, x)), \quad \rho(t, x) = \mathrm{\rho}(t, \Phi(t, x)), \quad q(t, x) = p(t, \Phi(t, x)) \tag{1.10}$$

the velocity, density, and pressure defined on the fixed domain Ω. Also, we introduce the differential operators

$$\partial_t^\varphi = \partial_t - \frac{\partial_t\varphi}{\partial_3\varphi}\partial_3, \tag{1.11}$$

$$\nabla_a^\varphi = \partial_a^\varphi = \partial_a - \frac{\partial_a\varphi}{\partial_3\varphi}\partial_3, \quad a = 1, 2, \tag{1.12}$$

$$\nabla_3^\varphi = \partial_3^\varphi = \frac{1}{\partial_3\varphi}\partial_3, \tag{1.13}$$

and thus the following identities hold:

$$\partial_\alpha u \circ \Phi = \partial_\alpha^\varphi v, \quad \partial_\alpha \mathrm{\rho} \circ \Phi = \partial_\alpha^\varphi \rho, \quad \partial_\alpha p \circ \Phi = \partial_\alpha^\varphi q, \quad \alpha = t, 1, 2, 3. \tag{1.14}$$

Moreover, setting

$$\overline{\nabla} = \overline{\partial} := (\partial_1, \partial_2),$$

the boundary condition (1.6) is turned into

$$q = -\sigma\overline{\nabla} \cdot \left(\frac{\overline{\nabla}\psi}{\sqrt{1 + |\overline{\nabla}\psi|^2}}\right), \quad \text{on } [0, T] \times \Sigma, \tag{1.15}$$

$$\partial_t\psi = v \cdot N, \quad N = (-\partial_1\psi, -\partial_2\psi, 1)^\top, \quad \text{on } [0, T] \times \Sigma, \tag{1.16}$$

$$v_3 = 0, \quad \text{on } [0, T] \times \Sigma_b, \tag{1.17}$$

respectively, where $\Sigma = \{x_3 = 0\}$ and $\Sigma_b = \{x_3 = -b\}$. Let $D_t^\varphi := \partial_t^\varphi + v \cdot \nabla^\varphi$. Then the system (1.5) and (1.6) are converted into

$$
\begin{cases}
\rho D_t^\varphi v + \nabla^\varphi q = -\rho g e_3 & \text{in } [0,T]\times\Omega, \\
\partial_t^\varphi \rho + \nabla^\varphi \cdot (\rho v) = 0 & \text{in } [0,T]\times\Omega, \\
q = q(\rho) & \text{in } [0,T]\times\Omega, \\
q = -\sigma \overline{\nabla} \cdot \left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}} \right) & \text{on } [0,T]\times\Sigma, \\
\partial_t \psi = v \cdot N & \text{on } [0,T]\times\Sigma, \\
v_3 = 0 & \text{on } [0,T]\times\Sigma_b, \\
(v,\rho,\psi)|_{t=0} = (v_0,\rho_0,\psi_0).
\end{cases}
\tag{1.18}
$$

The second equation of (1.18), i.e., the continuity equation, can be re-expressed as

$$
D_t^\varphi \rho + \rho \nabla^\varphi \cdot v = 0. \tag{1.19}
$$

Let $\mathcal{F} = \mathcal{F}(q) := \log \rho(q)$. Since $q'(\rho) > 0$ indicates $\mathcal{F}'(q) > 0$, then (1.19) is equivalent to

$$
\mathcal{F}'(q) D_t^\varphi q + \nabla^\varphi \cdot v = 0. \tag{1.20}
$$

Also, by invoking (1.11)-(1.13), we can alternatively write the material derivative D_t^φ as

$$
D_t^\varphi = \partial_t + \overline{v} \cdot \overline{\nabla} + \frac{1}{\partial_3 \varphi}(v \cdot \mathbf{N} - \partial_t \varphi)\partial_3, \tag{1.21}
$$

where $\overline{v} \cdot \overline{\nabla} = v_1\partial_1 + v_2\partial_2$, and $\mathbf{N} := (-\partial_1\varphi, -\partial_2\varphi, 1)$. This formulation provides a good motivation to define the smoothed material derivative in Section 3 and the linearized material derivative in Section 5.

1.2 The new formulation with modified pressure

Since the gravity term $\rho g e_3 \notin L^2(\Omega)$, we then use $\partial_i^\varphi \varphi = \delta_{i3}$ to rewrite the momentum equation as

$$
\rho D_t^\varphi v + \nabla^\varphi \check{q} = -(\rho - 1) g e_3,
$$

where

$$
\check{q} := q + g\varphi, \tag{1.22}
$$

is the "modified" pressure balanced by gravity. Under this setting, the fluid pressure gradient $\nabla^\varphi \check{q}$ becomes an $L^2(\Omega)$ function and the source term becomes $(\rho - 1)ge_3$ which is also in $L^2(\Omega)$ if we assume the initial data $\rho_0 - 1 \in L^2(\Omega)$. We then directly calculate that $D_t^\varphi \varphi = v_3$, so the continuity equation (1.20) now becomes

$$
\mathcal{F}'(q) D_t^\varphi \check{q} + \nabla^\varphi \cdot v = \mathcal{F}'(q) g D_t^\varphi \varphi = \mathcal{F}'(q) g v_3, \tag{1.23}
$$

and thus the compressible gravity-capillary water wave system is now reformulated as follows:

$$
\begin{cases}
\rho D_t^\varphi v + \nabla^\varphi \check{q} = -(\rho - 1) g e_3 & \text{in } [0,T]\times\Omega, \\
\mathcal{F}'(q) D_t^\varphi \check{q} + \nabla^\varphi \cdot v = \mathcal{F}'(q) g v_3 & \text{in } [0,T]\times\Omega, \\
q = q(\rho), \check{q} = q + g\varphi & \text{in } [0,T]\times\Omega, \\
\check{q} = g\psi - \sigma \overline{\nabla} \cdot \left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}} \right) & \text{on } [0,T]\times\Sigma, \\
\partial_t \psi = v \cdot N & \text{on } [0,T]\times\Sigma, \\
v_3 = 0 & \text{on } [0,T]\times\Sigma_b, \\
(v,\rho,\psi)|_{t=0} = (v_0,\rho_0,\psi_0).
\end{cases}
\tag{1.24}
$$

1.3 The equation of states and sound speed

Part of this paper is devoted to studying the behavior of the solution of (1.24) as either the sound speed goes to infinity or the surface tension σ coefficient goes to 0. The former is known to be the incompressible limit, and the latter is known to be the zero surface tension limit. Mathematically, it is convenient to view the sound speed $c_s := \sqrt{q'(\rho)}$ as a family of parameters. As in [17, 18, 19, 45, 47], we consider a family $\{q_{\lambda'}(\rho)\}$ parametrized by $\lambda' \in (0,\infty)$, where

$$(\lambda')^2 := q'_{\lambda'}(\rho)|_{\rho=1}. \tag{1.25}$$

Here and in the sequel, we slightly abuse the terminology and call λ' the sound speed. A typical choice of the equation of states $q_{\lambda'}(\rho)$ would be the Tait type equation

$$q_{\lambda'}(\rho) = \gamma^{-1}(\lambda')^2(\rho^\gamma - 1), \quad \gamma \geq 1. \tag{1.26}$$

When viewing the density as a function of the pressure, this indicates

$$\rho_{\lambda'}(q) = \left(\frac{\gamma}{(\lambda')^2}q + 1\right)^{\frac{1}{\gamma}}, \quad \text{and } \log(\rho_{\lambda'}(q)) = \gamma^{-1}\log\left(\frac{\gamma}{(\lambda')^2}q + 1\right). \tag{1.27}$$

Hence, we can view $\mathcal{F}(q)$ as a parametrized family $\{\mathcal{F}_\lambda(q)\}$ as well, where $\lambda = \frac{1}{\lambda'}$. Indeed, we have

$$\mathcal{F}_\lambda(q) = \gamma^{-1}\log(\lambda^2\gamma q + 1). \tag{1.28}$$

We again slightly abuse the terminology and call λ the Mach number[2]. Furthermore, there exists $C > 0$ such that

$$C^{-1}\lambda^2 \leq \mathcal{F}'_\lambda(q) \leq C\lambda^2. \tag{1.29}$$

Also, we assume

$$|\mathcal{F}_\lambda^{(s)}(q)| \leq C, \quad |\mathcal{F}_\lambda^{(s)}(q)| \leq C|\mathcal{F}'_\lambda(q)|^s \leq C\mathcal{F}'_\lambda(q) \tag{1.30}$$

holds for $0 \leq s \leq 4$.

Remark 1.1 (Issue with the infinite depth case). Our proof in this paper also works for the case of infinite depth, that is, $\mathcal{D}_t = \{(x', x_3) : -\infty < x_3 < \psi(t, x')\}$. Nevertheless, the equation of state should be modified such that the pressure also depends on the depth. An example of this is to assume p satisfies $\frac{\partial p}{\partial x_3}|_{\rho=1} = -g$ (cf. Jang-Tice-Wang [38]). Otherwise, the Mach number λ may also be x_3-dependent. It should also be noted that there is no such issue for the incompressible gravity water wave model, in which $\check{q}$ is a Lagrangian multiplier not related to the density.

1.4 An overview of previous results

The study of free-surface inviscid fluids has blossomed over the past two decades or so. Most of the previous studies focused on incompressible fluid models, i.e., the fluid velocity satisfies $\operatorname{div} u = 0$ and thus the density ρ is equal to a constant. In this case, the fluid pressure p is not determined by the equation of states but appears as a Lagrangian multiplier enforcing the divergence-free constraint. For the local well-posedness (LWP) for the free-boundary incompressible Euler equations, the first breakthrough came in Wu [71, 72] for the irrotational case[3] and Christodoulou-Lindblad [11] and Lindblad [41, 44] for the case of nonzero vorticity. We also refer to [55, 75, 30, 6, 39, 54] for the irrotational flows and [14, 78, 46, 58, 59, 60, 3, 2, 68] for the case of nonzero vorticity. In addition to the LWP theory, the incompressible and irrotational water waves have attracted great attention for their long-time existence. We refer to Wu [73, 74] for the first breakthrough and numerous related works [20, 21, 4, 31, 16, 24, 23, 25, 26, 69, 79] See also [9] for the bounded domain case and [27, 62] for some special cases when the vorticity is nonzero.

It is well-known that one can reduce the incompressible Euler equations to a system of equations on the moving boundary when the velocity is irrotational. This method cannot be adapted to the study of compressible water waves. The development for free-boundary compressible Euler equations is much less, especially for the case of a liquid as opposed to a gas in a physical vacuum satisfying $\rho|_\Sigma = 0$. For the gas model, we refer to [33, 13, 15, 49, 34, 28] and references therein. For the liquid

[2]The Mach number is defined to be $M = u/c_s$. In the paper, the velocity is always of size $O(1)$ (in $L^2(\Omega)$) and thus $M = O(\lambda)$.

[3]The vorticity curl $u_0 = \mathbf{0}$, a condition that is preserved by the evolution

model, most previous works focus on the case of a bounded domain, and we refer to Lindblad [42, 43] and related works [12, 45, 18, 22]. When the fluid domain is unbounded, that is, the compressible gravity water waves problem, the existing literature neglects the effect of surface tension. Trakhinin [64] first proved the LWP for the non-isentropic case by using Nash-Moser iteration which leads to a loss of regularity from initial data to solution. The a priori estimate without loss of regularity is shown in Luo [47], but it is still difficult to use the energy constructed there to prove the local existence. Recently, in [48], the authors proved the LWP for compressible gravity water waves without using Nash-Moser iteration.

About the incompressible limit of inviscid fluids, that is, the singular limit as Mach number goes to 0, there have been a lot of studies for the Cauchy problem or the fixed-domain problems. We refer to [35, 36, 19, 57, 17] for "well-prepared initial data" (div $u_0 = O(\lambda)$ and $\partial_t u|_{t=0} = O(1)$, wehre λ is the Mach number) and [67, 8, 32, 29, 53, 1] for "ill-prepared (general) initial data" (div $u_0 = O(1)$ and $\partial_t u|_{t=0} = O(\lambda^{-1})$). However, much less is known about the incompressible limit of free-surface inviscid fluids: Lindblad and the first author [45], the first author [47] and Disconzi and the first author [18] established incompressible limit results for free-surface Euler equations with zero or nonzero surface tension.

It should be noted that the uniform energy estimates are not consistent with the ones obtained by the local existence result. Moreover, *the uniform boundedness (with respect to Mach number λ) of top-order time derivatives of the velocity is necessary* in [45, 47, 18], which is more restrictive than the commonly-used definition of "well-prepared initial data". Very recently, the second author [76] established LWP and the incompressible limit simultaneously with the same energy functional for compressible elastodynamics, which can be directly applied to Euler equations without surface tension. Also, only $\partial_t^2 u|_{t=0} = O(1)$ is required in [76] which is an essential improvement of [45, 47, 18] and is also an optimal requirement of well-prepared data for free-surface inviscid fluids without surface tension, as the propagation of Rayleigh-Taylor sign condition already requires the uniform boundedness of $\nabla\partial_t q \sim \partial_t^2 u$. However, the method and observations in [76] heavily rely on the vanishing boundary condition for the pressure on the free surface, which cannot be generalized to the case of nonzero surface tension or two-phase vortex-sheet problems.

In this paper, we study the system of compressible gravity-capillary water waves. Specifically, we are further interested in developing a "unified framework" in order to simultaneously establish LWP and the incompressible limit for compressible inviscid fluids (not just Euler equations) with or without surface tension. These two limit processes are expected to be mutually independent, that is, no extra relation between the Mach number and the surface-tension coefficient is required. Besides, we manage to drop the boundedness assumption on high-order time derivatives by combining the pressure decomposition, inspired by Shatah-Zeng [59, 60], with the paradifferential approach used in Alazard-Burq-Zuily [2, 3].

1.5 The main theorems

The first theorem concerns the local well-posedness for the motion of compressible gravity-capillary water waves modeled by (1.24), provided that the initial data satisfies certain compatibility conditions. Particularly, we say the data (ψ_0, v_0, q_0), where $q_0 = q(\rho_0)$, satisfies the k-th ($k = 0, 1, 2, 3, \cdots$) compatibility conditions if

$$\begin{aligned} &(D_t^{\varphi})^k q|_{t=0} = (D_t^{\varphi})^k(\sigma\mathcal{H})|_{t=0}, &&\text{on } \Sigma, \\ &\partial_t^k v_3|_{t=0} = 0, &&\text{on } \Sigma_b, \end{aligned} \tag{1.31}$$

hold.

Theorem 1.1 (**Local well-posedness**). Let $b > 10$, and $\sigma > 0$ be fixed. Let $(\psi_0, v_0, \rho_0 - 1) \in H^5(\Sigma) \times H^4(\Omega) \times H^4(\Omega)$ be the initial data of (1.24) that verifies the compatibility conditions (1.31) up to the third order, and $|\psi_0|_\infty \leq 1$. Then there exists $T > 0$ depending only on the initial data, such that (1.24) admits a unique solution $(\psi(t), v(t), \rho(t))$ verifies the energy estimate:

$$\sup_{0\leq t\leq T} E(t) \leq C(\sigma^{-1})P(E(0)), \tag{1.32}$$

where

$$\begin{aligned} E(t) &:= E_0(t) + E_4(t), \\ E_0(t) &:= \|\rho(t) - 1\|_0^2 + g|\psi|_0^2 + \|\sqrt{\mathcal{F}'(q)}\check{q}(t)\|_0^2 + \sum_{k=1}^{3} \|\sqrt{\mathcal{F}'(q)}\partial_t^k \check{q}(t)\|_0^2, \\ E_4(t) &:= \sum_{k=0}^{4}\left(\|\partial_t^k v(t)\|_{4-k}^2 + |\sqrt{\sigma}\overline{\nabla}\partial_t^k\psi(t)|_{4-k}^2\right) + \|\partial\check{q}(t)\|_3^2 + \sum_{k=1}^{3}\|\partial_t^k\partial\check{q}(t)\|_{3-k}^2 + \|\sqrt{\mathcal{F}'(q)}\partial_t^4\check{q}(t)\|_0^2. \end{aligned} \tag{1.33}$$

is the energy of (1.24) expressed in terms of $(\psi, v, \check{q})$, and $P(\cdot)$ is a generic non-negative continuous function in its arguments. In addition to this, we have

$$\sup_{t\in[0,T]} |\psi(t)|_\infty \leq 10. \tag{1.34}$$

Lastly, there exists a constant C, depending on ψ_0, v_0 and $\check{q}_0$, such that $E(0) \leq C$.

In above and throughout, we use $\|\cdot\|_s$ and $|\cdot|_s$ to represent respectively the interior Sobolev norm $\|\cdot\|_{H^s(\Omega)}$ and the boundary Sobolev norm $\|\cdot\|_{H^s(\Sigma)}$.

Remark 1.2. In Appendix B, we show that we can construct smooth initial data $(\psi_0, v_0, \check{q}_0)$ that satisfies the compatibility conditions up to order 3. These compatibility conditions are required so that we can show $E(0) \leq C$ by adapting the arguments in [18, Section 4.3].

Remark 1.3. The second line in (1.33) is the L^2-part of the energy, where $\|\check{q}(t)\|_0^2$ is $\mathcal{F}'(q)$-weighted, which ties to the Mach number. That is why we write $\|\partial\check{q}\|_3$ instead of $\|\check{q}\|_4$ in the first line.

The next main theorem concerns the incompressible and zero surface tension limits. We consider the Euler equations modeling the motion of incompressible gravity water waves satisfied by (ξ, w, q_{in}) with localized initial data (w_0, ξ_0):

$$\begin{cases} D_t^\varphi w + \nabla^\varphi \mathfrak{p} = 0 & \text{in } [0,T]\times\Omega, \\ \nabla^\varphi \cdot w = 0 & \text{in } [0,T]\times\Omega, \\ \mathfrak{p} = q_{in} + g\varphi & \text{in } [0,T]\times\Omega, \\ \mathfrak{p} = g\xi & \text{on } [0,T]\times\Sigma, \\ \partial_t\xi = w\cdot\mathcal{N} & \text{on } [0,T]\times\Sigma, \\ w_3 = 0 & \text{on } [0,T]\times\Sigma_b, \\ (w,\xi)|_{t=0} = (w_0,\xi_0), \end{cases} \tag{1.35}$$

where we slightly abuse the notation by still setting $\varphi(t,x) = x_3 + \chi(x_3)\xi(t,x')$ to be the extension of ξ in Ω. Denote by $(\psi^{\lambda,\sigma}, v^{\lambda,\sigma}, \rho^{\lambda,\sigma})$ the solution of (1.24) indexed by σ and λ, we prove that $(\psi^{\lambda,\sigma}, v^{\lambda,\sigma}, \rho^{\lambda,\sigma})$ converges to $(\xi, w, 1)$ as $\lambda, \sigma \to 0$ provided the convergence of the initial data in a suitable sense. Note that the convergence of the compressible initial data implies that it is also localized.

Theorem 1.2 (Incompressible and zero surface tension limits). Let $(\psi_0^{\lambda,\sigma}, v_0^{\lambda,\sigma}, \rho_0^{\lambda,\sigma} - 1)$ be the initial data of (1.24) for each fixed $(\lambda,\sigma) \in \mathbb{R}^+ \times \mathbb{R}^+$, verifying:

a. The sequence of initial data $(\psi_0^{\lambda,\sigma}, v_0^{\lambda,\sigma}, \rho_0^{\lambda,\sigma} - 1) \in H^5(\Sigma)\times H^4(\Omega)\times H^4(\Omega)$ satisfies (1.31) for $0 \leq k \leq 3$, and $|\psi_0^{\lambda,\sigma}|_\infty \leq 1$.
b. $(\psi_0^{\lambda,\sigma}, v_0^{\lambda,\sigma}, \rho_0^{\lambda,\sigma} - 1) \to (\xi_0, w_0, 0)$ in $H^4(\Sigma)\times H^4(\Omega)\times H^3(\Omega)$ as $\lambda,\sigma \to 0$.
c. *Both* incompressible and compressible pressures $q^{\lambda,\sigma}$ and q_{in} satisfy the Rayleigh-Taylor sign condition

$$-\partial_3 q^{\lambda,\sigma} \geq c_0 > 0, \quad \text{on } \{t=0\}\times\Sigma, \tag{1.36}$$
$$-\partial_3 q_{in} \geq c_0 > 0, \quad \text{on } \{t=0\}\times\Sigma, \tag{1.37}$$

for some $c_0 > 0$.

Then it holds that

$$(\psi^{\lambda,\sigma}, v^{\lambda,\sigma}, \rho^{\lambda,\sigma} - 1) \to (\xi, w, 0),$$

weakly* in $L^\infty([0,T]; H^4(\Sigma)\times H^4(\Omega)\times H^3(\Omega))$, and strongly in $C^0([0,T]; H^{4-\delta}_{\text{loc}}(\Sigma)\times H^{4-\delta}_{\text{loc}}(\Omega)\times H^{3-\delta}_{\text{loc}}(\Omega))$ for any $\delta \in (0,1]$.

Theorem 1.2 is a direct consequence of uniform-in-λ,σ estimates for the compressible gravity-capillary water wave system (1.24) and the Aubin-Lions lemma. Indeed, the energy estimate (1.32) established in Theorem 1.1 is already uniform in Mach number λ. In addition to this, one can show that (1.32) is uniform in the surface tension coefficient σ provided that the Rayleigh-Taylor sign condition (1.36) holds initially.

Remark 1.4. Although our energy functional $E(t)$ is expressed in terms of $\check{q}$, the incompressible limit is given in $(\psi^{\lambda,\sigma}, v^{\lambda,\sigma}, \rho^{\lambda,\sigma})$ which converges to $(\zeta, w, 1)$. We do not expect that the compressible pressure q converges to the incompressible pressure q_{in} as $\lambda \to 0$, because the former is the solution to a quasilinear symmetric hyperbolic system but the latter appears as a Lagrangian multiplier. Indeed, as was indicated by [45, 47, 76], it is the enthalpy $h(\rho) := \int_1^\rho q'(r)/r\,dr$ of the compressible equations that converges to the incompressible pressure q_{in}. On the other hand, the convergence of $\|\rho^{\lambda,\sigma} - 1\|_3$ can be easily proved if we write the continuity equation to be $D_t^\varphi(\rho - 1) = -\rho(\nabla^\varphi \cdot v)$ and use Grönwall's inequality for its H^3-estimate.

It should be noted that the energy (1.33) requires that the time derivatives up to at least order 3 are bounded initially, i.e., $\partial_t^k \check{q}(0) = O(1)$, $0 \le k \le 3$, while $\partial_t^4 \check{q}(0) = O(\lambda^{-1})$, or equivalently the uniform boundedness for the top-order time derivatives of the velocity $\partial_t^4 v = O(1)$. *This condition can certainly be weakened.* In fact, the propagation of the Rayleigh-Taylor sign condition only requires the boundedness of $\partial_t \partial_3 q$, or equivalently $\partial_t^2 v = O(1)$, not including higher-order time derivatives. Motivated by this, we prove the following improved estimates.

Theorem 1.3 (**Improved uniform estimates in** λ, σ**).** Under the hypothesis of Theorem 1.1, if we further assume $(\psi_0, v_0, \rho_0 - 1) \in H^6(\Sigma) \times H^5(\Omega) \times H^5(\Omega)$ satisfying the compatibility conditions up to 4-th order and assume the Rayleigh-Taylor sign condition (1.36) holds for the initial data of (1.24), then

$$\sup_{0 \le t \le T} \mathfrak{E}(t) \le P(\mathfrak{E}(0)), \tag{1.38}$$

holds uniform in both λ and σ, where

$$\begin{aligned}
\mathfrak{E}(t) &:= E_0(t) + \mathfrak{E}_4(t) + E_5(t), \\
\mathfrak{E}_4(t) &:= \|v\|_4^2 + \|\partial \check{q}\|_3^2 + |\sqrt{\sigma}\psi|_5^2 + |\psi|_4^2 + \|\partial_t v, \partial_t \check{q}\|_3^2 + |\sqrt{\sigma}\partial_t \psi|_4^2 + |\partial_t \psi|_{3.5}^2 \\
&\quad + \|\partial_t^2 v, \lambda \partial_t^2 \check{q}\|_2^2 + |\sqrt{\sigma}\partial_t^2 \psi|_3^2 + |\partial_t^2 \psi|_{2.5}^2 + |\partial_t^3 \psi|_{1.5}^2 \\
&\quad + \sum_{k=3}^{4} \|\lambda \partial_t^k (v, \check{q})\|_{4-k}^2 + |\sqrt{\sigma}\lambda \partial_t^k \psi|_{5-k}^2 + |\lambda \partial_t^4 \psi|_{0.5}^2 \\
E_5(t) &:= \sum_{k=0}^{5} \left\| \lambda^2 \partial_t^k (v, (\mathcal{F}'(q))^{\frac{(k-4)_+}{2}} \check{q}) \right\|_{5-k}^2 + \left| \sqrt{\sigma}\lambda^2 \partial_t^k \psi \right|_{6-k}^2 + \left| \lambda^2 \partial_t^k \psi \right|_{5-k}^2,
\end{aligned} \tag{1.39}$$

and $(k-4)_+ := \max\{0, k-4\}$.

Remark 1.5. The above estimate only requires $\nabla \partial_t q(0) \sim \partial_t^2 u(0)$ to be bounded (with respect to λ) because we need to control the evolution of the Rayleigh-Taylor sign, namely $\|\partial_t \partial_3 q\|_{L^\infty(\Sigma)}$, when taking the *incompressible and zero surface tension limits simultaneously*. However, we do not require $\partial_t^k v(0)$ to be uniformly bounded for $k > 2$. On the other hand, the propagation of the Rayleigh-Taylor sign condition requires the boundedness of $\partial_t q$, so we have reached the minimal requirement for the initial data being "well-prepared".

List of Notations

- (Fixed domain and its boundary) $\Omega := \{x \in \mathbb{R}^3 : -b < x_3 < 0\}$. $x = (x_1, x_2, x_3)$, and $x' = (x_1, x_2)$. $\Sigma := \{x \in \mathbb{R}^3 : x_3 = 0\}$, $\Sigma_b := \{x \in \mathbb{R}^3 : x_3 = -b\}$.
- (Tangential derivatives) $\mathcal{T}_0 = \partial_t$, $\mathcal{T}_1 = \overline{\partial}_1$, $\mathcal{T}_2 = \overline{\partial}_2$, $\mathcal{T}_3 = \omega(x_3)\partial_3$, where $\omega(x_3) \in C^\infty(-b, 0)$ is assumed to be bounded, comparable to $|x_3|$ in $[-2, 0]$ and vanishing on $\Sigma \cup \Sigma_b$.
- (L^∞-norm) $\|\cdot\|_\infty := \|\cdot\|_{L^\infty(\Omega)}$, $|\cdot|_\infty := \|\cdot\|_{L^\infty(\Sigma)}$.
- (Sobolev norms) $\|\cdot\|_s := \|\cdot\|_{H^s(\Omega)}$, and $|\cdot|_s := \|\cdot\|_{H^s(\Sigma)}$.
- (Continuous functions) $\mathcal{P}_0 := P(E(0))$, $\mathcal{P}_0^\kappa := P(E^\kappa(0))$. $P(\cdots)$ denotes a generic non-negative continuous function in its arguments.
- (Commutators) $[T, f]g = T(fg) - f(Tg)$, $[T, f, g] := T(fg) - T(f)g - fT(g)$ where T is a differential operator and f, g are functions.
- (Equality modulo lower order terms) $A \stackrel{L}{=} B$ means $A = B$ modulo lower order terms.

2 An overview of our methodology

Before going to the detailed proofs, we will briefly introduce our methodology for deriving energy estimates that are uniform in both surface tension and Mach number, and the construction of solutions to the linearized and the nonlinear problem via a carefully-designed approximation scheme.

2.1 Uniform estimates in Mach number and surface tension

Let us temporarily focus on the a priori energy estimate of the original system (1.24) instead of the construction of solutions. Indeed, the strategies on the a priori estimate will illustrate why we need the approximation scheme defined in the next subsection.

2.1.1 Div-Curl analysis and reduction of pressure

The first step is to reduce the normal derivatives for (1.24) and we start with the control of $\|v\|_4$. Using the div-curl decomposition, $\|v\|_4$ is bounded by $\|\nabla^\varphi \times v\|_3$, $\|\nabla^\varphi \cdot v\|_3$ and $\|\overline{\partial}{}^4 v\|_0$, where the curl part can be directly controlled by analyzing its evolution equation. The continuity equation reduces the divergence to $\|\mathcal{F}'(q)D_t^\varphi q\|_3$ which is a tangential derivative and includes a time derivative. As for the pressure $\check{q}$, the momentum equation indicates that $-\nabla\check{q} \sim D_t^\varphi v$, which again converts a normal derivative to a tangential derivative. This reduction can also be applied to the time derivatives of v and $\check{q}$ up to the third order. As a consequence, the control of the full Sobolev norms of v and $\check{q}$ (and their time derivatives) is reduced to the control of $\mathcal{T}^\alpha v$ and $\mathcal{T}^\alpha \check{q}$ ($|\alpha| = 4$) in $L^2(\Omega)$ with appropriate weights in Mach number where $\mathcal{T}$ represents any of the tangential derivatives $\partial_t, \overline{\partial}$ or $\omega(x_3)\partial_3$ where $\omega(x_3) \in C^\infty(-b, 0)$ is bounded, comparable to $|x_3|$ in $x_3 \in (-2, 0)$ and vanishing on $\Sigma \cup \Sigma_b$.

2.1.2 Tangential estimates: Alinhac good unknowns

Define $\mathcal{T}^\alpha$ to be $\partial_t^{\alpha_0}\overline{\partial}_1^{\alpha_1}\overline{\partial}_2^{\alpha_2}(\omega\partial_3)^{\alpha_3}$ with $|\alpha| := \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 4$. In $\mathcal{T}^\alpha$-tangential estimates, we need to commute $\mathcal{T}^\alpha$ with ∇_i^φ. When $i = t, 1, 2$, the commutator $[\mathcal{T}^\alpha, \nabla_i^\varphi]f$ includes the term $(\partial_3\varphi)^{-1}\mathcal{T}^\alpha\partial_i\varphi\partial_3 f$, where the $L^2(\Omega)$-norm of $\mathcal{T}^\alpha\partial_i\varphi$ is controlled by $|\mathcal{T}^\alpha\partial_i\psi|_0$. However, the regularity of ψ obtained in $\mathcal{T}^\alpha$-estimates is $|\sqrt{\sigma}\mathcal{T}^\alpha\overline{\nabla}\psi|_0$. Thus, the direct control of the aforementioned commutator fails to be uniform in σ. To overcome this difficulty, we introduce the Alinhac's method which reveals that the "essential" leading order term in $\mathcal{T}^\alpha(\nabla^\varphi f)$ is not $\nabla^\varphi(\mathcal{T}^\alpha f)$ but the covariant derivative of $\mathbf{F}$ (i.e., $\nabla^\varphi\mathbf{F}$), where $\mathbf{F} := \mathcal{T}^\alpha f - \mathcal{T}^\alpha\varphi\partial_3^\varphi f$. Here, $\mathbf{F}$ is the so-called Alinhac good unknown associated with f, which satisfies

$$\mathcal{T}^\alpha\nabla_i^\varphi f = \nabla_i^\varphi\mathbf{F} + \mathfrak{C}_i(f), \quad \mathcal{T}^\alpha D_t^\varphi f = D_t^\varphi\mathbf{F} + \mathfrak{D}(f), \tag{2.1}$$

where $\|\mathfrak{C}_i(f)\|_0$ and $\|\mathfrak{D}(f)\|_0$ can be directly controlled. In other words, the reformulation in Alinhac good unknowns takes into account the covariance under the change of coordinates such that we can proceed with the tangential estimates in the same way as the L^2-estimate and avoid the additional regularity on the nonlinear coefficients that cannot be controlled in a σ-uniform fashion. Such remarkable observation was due to Alinhac [7] and was first applied (implicitly) to free-surface inviscid fluids by Christodoulou-Lindblad [11]. See also [50, 70] for the explicit calculations for the inviscid limit of incompressible free-boundary Navier-Stokes equations.

Let $\mathbf{V}, \mathbf{Q}$ be the Alinhac good unknowns of $v, \check{q}$ associated with $\mathcal{T}^\alpha$ and then we obtain several major terms from the tangential estimates

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\left(\int_\Omega \rho|\mathbf{V}|^2 + \mathcal{F}'(q)|\mathbf{Q}|^2\mathrm{d}\mathcal{V}_t\right)\\ =&\ \mathrm{ST} + \mathrm{RT} + \int_\Sigma \mathcal{T}^\alpha\check{q}[\mathcal{T}^\alpha, v\cdot, N]\,\mathrm{d}x' - \int_\Omega \mathcal{T}^\alpha\check{q}[\mathcal{T}^\alpha, \partial_3 v\cdot, \mathbf{N}]\,\mathrm{d}\mathcal{V}_t + \text{ controllable terms,}\end{aligned} \tag{2.2}$$

where $\mathrm{d}\mathcal{V}_t := \partial_3\varphi\,\mathrm{d}x$ and

$$\mathrm{ST} := -\int_\Sigma \mathcal{T}^\alpha(\sigma\mathcal{H})\partial_t\mathcal{T}^\alpha\psi\,\mathrm{d}x', \qquad \mathrm{RT} := -\int_\Sigma (-\partial_3 q)\mathcal{T}^\alpha\psi\partial_t\mathcal{T}^\alpha\psi\,\mathrm{d}x. \tag{2.3}$$

Also note that $\mathcal{T}^\alpha$ only contains ∂_t and $\overline{\partial}$ on $\Sigma \cup \Sigma_b$ as the weight function $\omega(x_3)$ vanishes on the boundary.

For the term ST, invoking the explicit formula for the mean curvature and integrating $\overline{\nabla}\cdot$ by parts, we obtain

$$\mathrm{ST} = -\frac{\sigma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma \frac{|\mathcal{T}^\alpha\overline{\nabla}\psi|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}} - \frac{|\overline{\nabla}\psi\cdot\mathcal{T}^\alpha\overline{\nabla}\psi|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}^3}\,\mathrm{d}x' + \cdots, \tag{2.4}$$

which together with the following inequality gives the boundary energy $|\sqrt{\sigma}\overline{\partial}{}^\alpha\overline{\nabla}\psi|_0^2$:

$$\forall \mathbf{a} \in \mathbb{R}^2, \quad \frac{|\mathbf{a}|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}} - \frac{|\overline{\nabla}\psi\cdot\mathbf{a}|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}^3} \geq \frac{|\mathbf{a}|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}^3}. \tag{2.5}$$

For the term RT, it produces the boundary energy without σ-weight *provided that the Rayleigh-Taylor sign condition*[4] $-\partial_3 q_0|_\Sigma \geq c_0 > 0$ *holds*. However, the Rayleigh-Taylor sign condition is only assumed when taking zero surface tension limit but not in the proof of local well-posedness for each *given* $\sigma > 0$. Therefore, we have to use the $\sqrt{\sigma}$-weighted energy to control this term when proving local well-posedness. Indeed, *it is the direct control of* $|\mathcal{T}^\alpha\psi|_0$ *and* $|\partial_t\mathcal{T}^\alpha\psi|_0$ *that yields the only possibility that the energy estimate depends on* σ^{-1}.

The remaining two terms contributes to a crucial structure for the incompressible limit. When $\mathcal{T}^\alpha = \partial_t^4$ is a full time derivative, we cannot control them individually due to a loss of Mach number weight. Instead, we shall combine them together and use the divergence theorem to reduce a time derivative on $\check{q}$. The leading-order terms are

$$4\int_\Sigma \partial_t^4\check{q}\partial_t^3 v\cdot\partial_t N\,\mathrm{d}x' - 4\int_\Omega \partial_t^4\check{q}\partial_t\mathbf{N}\cdot\partial_3\partial_t^3 v\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\partial_t^3\partial_3\check{q}\partial_t\mathbf{N} + \partial_t^3\check{q}\partial_t\partial_3\mathbf{N}\right)\cdot\partial_t^3 v\,\mathrm{d}x + \cdots,$$

which can be directly controlled under time integral.

Combining the steps above, we finish the control of Alinhac good unknowns $\mathbf{V},\mathbf{Q}$. Then by using the definition of good unknowns, we know $\|\mathbf{F}-\mathcal{T}^\alpha f\|_0 \leq |\mathcal{T}^\alpha\psi|_0\|\partial f\|_\infty$ which is already controlled by the boundary energy of ψ. Therefore, the a priori estimate for the system (1.24) is closed, which is uniform in Mach number and also uniform in σ under the Rayleigh-Taylor sign condition.

2.2 Improved incompressible limit

The uniform estimates obtained above require the uniform boundedness of top-order time derivatives of v, which is far more restrictive than the usual definition of "well-prepared initial data" ($\nabla^\varphi\cdot v|_{t=0} = O(\lambda),\ \partial_t v|_{t=0} = O(1)$). A natural question is whether we can remove such boundedness assumption on high-order time derivatives, which is a necessary step to find a possible way to study the case of "ill-prepared data" ($\nabla^\varphi\cdot v|_{t=0} = O(1),\ \partial_t v|_{t=0} = O(\lambda^{-1})$).

2.2.1 Difficulties in free-boundary problems

There have been numerous results for fixed-domain problems or the Cauchy problem [35, 36, 19, 57], but this is rather nontrivial under the free-surface setting due to the interaction between the free-surface motion and the interior pressure waves. Indeed, when commuting $\mathcal{T}^\alpha$ with ∇^φ when $\mathcal{T}^\alpha$ contains both spatial derivatives and time derivatives, the usage of $\nabla q \sim \partial_t v$ actually produces an extra time derivative without λ-weight. When $\partial_t^k v$ is assigned with a different λ-weight from that of $\partial_t^k\check{q}$ in the energy functional, there exhibits a loss of λ-weight due to the substitution $\nabla\check{q} \sim \partial_t v$, which is actually *caused by the free-surface motion*. The second author [76] dropped such assumption for the case of zero surface tension, but this result heavily relies on the vanishing boundary value of q as stated at the end of Section 1.4.

The above analysis indicates us to avoid the interior tangential estimates. Instead, when treating the time derivatives, we shall use another div-curl inequality

$$\|X\|_s^2 \lesssim C(|\psi|_{s+\frac12}, |\overline{\nabla}\psi|_{W^{1,\infty}})\left(\|X\|_0^2 + \|\nabla^\varphi\cdot X\|_{s-1}^2 + \|\nabla^\varphi\times X\|_{s-1}^2 + |X\cdot N|_{s-\frac12}^2\right),\quad \forall s\geq 1, \tag{2.6}$$

in order to directly analyze the evolution of the free surface. In view of the new energy $\mathfrak{E}_4(t)$ defined in (1.39), we shall apply this inequality to $X = \partial_t^2 v$ and the kinematic boundary condition indicates us to control $|\partial_t^3\psi|_{1.5}$ without any weights of λ,σ.

2.2.2 The evolution equation of the free surface and its paralinearization

The evolution equation of the free surface is derived by time-differentiating the kinematic boundary condition and invoking the momentum equation, which leads to $\rho\overline{D_t}^2\psi = -\partial_3\check{q} - (\rho-1)g$ with $\overline{D_t} := D_t^\varphi|_\Sigma = \partial_t + \overline{v}\cdot\overline{\nabla}$. We shall further differentiate this equation with ∂_t^2 and *convert the Neumann boundary value of* $\check{q}$ *to a Dirichlet-type condition* in order to utilize the boundary condition $\check{q} = \sigma\mathcal{H}$. We introduce the Alinhac good unknown $Q := \partial_t^2\check{q} - \partial_t^2\varphi\partial_3^\varphi\check{q}$ to obtain

$$\rho\overline{D_t}^2\partial_t^2\psi = -N\cdot\nabla^\varphi Q + \cdots$$

The next step is to separate the contribution of $\check{q}$ on the boundary from that in the interior. We notice that Q satisfies a wave equation

$$\rho\lambda^2(D_t^\varphi)^2 Q - \Delta^\varphi Q = \cdots\ \text{in}\ \Omega,\quad Q|_\Sigma = \sigma\partial_t^2\mathcal{H} - \partial_3 q\partial_t^2\psi,\quad \partial_3 Q|_{\Sigma_b} = -\partial_t^2\rho g.$$

[4] The Rayleigh-Taylor sign condition is just a constraint for the initial data. One can easily prove its short-time propagation by using the boundedness of $\partial_t\partial_3 q$. See [48, Section 3.7].

Inspired by Shatah-Zeng [59, 60], we define $Q = Q_h + Q_w$ where

$$-\Delta^{\varphi} Q_h = 0 \text{ in } \Omega, \quad Q_h = Q \text{ on } \Sigma, \quad \partial_3 Q_h = 0 \text{ on } \Sigma_b,$$
$$-\Delta^{\varphi} Q_w = -\rho\lambda^2 (D_t^{\varphi})^2 Q + \cdots \text{ in } \Omega, \quad Q_w = 0 \text{ on } \Sigma, \quad \partial_3 Q_w = \partial_3 Q \text{ on } \Sigma_b.$$

Under this setting, we obtain the following evolution equation

$$\rho \overline{D_t}^2 \partial_t^2 \psi + \sigma \mathfrak{N}_\psi (\partial_t^2 \mathcal{H}) - \mathfrak{N}_\psi (\partial_3 q \partial_t^2 \psi) = -N \cdot \nabla^{\varphi} Q_w + \cdots \quad \text{on } \Sigma \tag{2.7}$$

where $\mathfrak{N}_\psi$ is the Dirichlet-to-Neumann (DtN) operator associated to (Ω, ψ) and we refer to Definition 7.1 for details. Since DtN operator is a first-order operator with positive principal symbol and the mean curvature operator is a second-order elliptic operator, we formally have

$$\rho \overline{D_t}^2 (\partial_t^2 \psi) + \sigma \underbrace{C_1(\cdots)}_{>0} \langle \overline{\partial} \rangle^3 (\partial_t^2 \psi) + (-\partial_3 q) \underbrace{C_2(\cdots)}_{>0} \langle \overline{\partial} \rangle (\partial_t^2 \psi) = -N \cdot \nabla^{\varphi} Q_w + \cdots \quad \text{on } \Sigma$$

Thus, we can adopt the paralinearization used in Alazard-Burq-Zuily [2, 3] to calculate the principal symbol of their composition in order for an explicit uniform-in-λ energy estimate of $|\partial_t^3 \psi|_{1.5}$ and $|\sqrt{\sigma} \partial_t^2 \psi|_3$ (and also $|\partial_t^2 \psi|_2$, uniformly in σ, under the Rayleigh-Taylor sign condition). We refer to Section 7.3-7.5 for detailed computations.

2.2.3 Necessity of the weighted fifth-order energy

Note that the new energy $\mathfrak{E}(t)$ defined in (1.39) also includes a λ^2-weighted fifth-order energy. This is actually necessary to control the contribution of pressure wave, namely the term $|N \cdot \nabla^{\varphi} Q_w|_{1.5}$. Since Q_w has zero boundary value on Σ and its Neumann boundary value on Σ_b is easy to control, we can convert it to the control of $\|\Delta^{\varphi} Q_w\|_1$ which further requires the bound for $\|\lambda^2 \partial_t^4 \check{q}\|_1$, which is exactly a λ^2-weighted fifth-order term.

Note that the control of $E_5(t)$ in (1.39) is completely parallel to that of $E_4(t)$ defined in (1.33), as the structure of these two energy functionals are exactly same except that each term of E_5 is assigned with a λ^2-weight. One can check that the control of all commutators arising from tangential estimates leads to no loss of λ weight and we refer to Section 7.5.2 for details.

Remark 2.1. The combination of the pressure decomposition and the paralinearization of the free-surface motion allows us to "separate" the contribution of free-surface motion (in particular, the surface tension) and interior pressure waves, and these two parts are related via the term $N \cdot \nabla^{\varphi} Q_w$ which naturally leads to the fifth-order energy. This method essentially improves the previous results [45, 47, 18] where the uniform boundedness of top-order time derivatives of v is necessary. Also, our method no longer relies on the vanishing boundary value of pressure as in [76]. Thus, we believe that the approach developed in this paper can be applied to other "coupled" fluid models or the vortex-sheet problems[5]. Furthermore, our method may open the possibility to study the incompressible limit of free-surface fluids with ill-prepared initial data.

2.3 The approximation scheme to prove the existence

2.3.1 Motivation to design the approximation

For free-surface inviscid fluids, the local existence is not a direct consequence of the a priori estimate. For example, if we try to do Picard iteration for the linearized system whose coefficient φ is replaced by a given function $\mathring{\varphi}$, then a crucial difference from the nonlinear system is that we may no longer obtain the boundary regularity from the analogue of ST term as in (2.4). Specifically, we consider (2.4) with full spatial tangential derivatives:

$$\mathrm{ST} = \sigma \int_\Sigma \overline{\partial}^\alpha \overline{\nabla} \cdot \left(\frac{\overline{\nabla} \psi}{1 + |\overline{\nabla} \mathring{\psi}|^2} \right) \partial_t \overline{\partial}^\alpha \psi \, \mathrm{d}x' = -\frac{\sigma}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Sigma \frac{|\overline{\partial}^\alpha \overline{\nabla} \psi|^2}{\sqrt{1 + |\overline{\nabla} \mathring{\psi}|^2}} - \frac{(\overline{\nabla} \psi \cdot \overline{\partial}^\alpha \overline{\nabla} \psi)(\overline{\nabla} \mathring{\psi} \cdot \overline{\partial}^\alpha \overline{\nabla} \mathring{\psi})}{\sqrt{1 + |\overline{\nabla} \mathring{\psi}|^2}^3} \, \mathrm{d}x' + \cdots, \tag{2.8}$$

where the second term has no control because inequality (2.5) is not applicable here. Such a linearization yields the loss of a tangential derivative. Besides, the unknowns with full time derivatives only have $L^2(\Omega)$ integrability and thus have no boundary regularity. Some crucial cancellations no longer hold after linearization. Therefore, it is natural to regularize the coefficient φ in both t and x' variables.

[5] The second author recently applied this method to the incompressible limit for current-vortex sheets in ideal compressible MHD. See [77]. This is, to our knowledge, the first result about the incompressible limit of inviscid vortex sheets.

2.3.2 The approximation system: important steps of its construction

For each $\kappa > 0$, we define Λ_κ to be the standard convolution mollifier on $\mathbb{R}^2$ with parameter $\kappa > 0$ and then define $\widetilde{\psi} := \Lambda_\kappa^2 \psi$ and $\widetilde{\varphi}(t,x) := x_3 + \chi(x_3)\widetilde{\psi}(t,x')$ to be the smoothed coefficients. We introduce the following nonlinear system with artificial viscosity whose coefficients are replaced by $\widetilde{\varphi}, \widetilde{\psi}$ that is asymptotically consistent with the original system (1.24) as $\kappa \to 0_+$.

$$\begin{cases} \rho D_t^{\widetilde{\varphi}} v + \nabla^{\widetilde{\varphi}} \check{q} = -(\rho - 1) g e_3, & \text{in } [0,T]\times\Omega, \\ \mathcal{F}'(q) D_t^{\widetilde{\varphi}} \check{q} + \nabla^{\widetilde{\varphi}} \cdot v = \mathcal{F}'(q) g v_3, & \text{in } [0,T]\times\Omega, \\ q = q(\rho), \check{q} = q + g\widetilde{\varphi} & \text{in } [0,T]\times\Omega, \\ \check{q} = g\widetilde{\psi} - \sigma \overline{\nabla} \cdot \left(\frac{\overline{\nabla}\widetilde{\psi}}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}} \right) + \kappa^2 (1 - \overline{\Delta})(v \cdot \widetilde{N}) & \text{on } [0,T]\times\Sigma, \\ \partial_t \psi = v \cdot \widetilde{N} & \text{on } [0,T]\times\Sigma, \\ v_3 = 0 & \text{on } [0,T]\times\Sigma_b, \\ (v,\rho,\psi)|_{t=0} = (v_0^\kappa, \rho_0^\kappa, \psi_0^\kappa). \end{cases} \tag{2.9}$$

Here,

$$\nabla_i^{\widetilde{\varphi}} = \partial_i^{\widetilde{\varphi}} = \partial_i - \frac{\partial_i \widetilde{\varphi}}{\partial_3 \widetilde{\varphi}} \partial_3,\ i = 1,2,\ \ \nabla_3^{\widetilde{\varphi}} = \partial_3^{\widetilde{\varphi}} = \frac{1}{\partial_3 \widetilde{\varphi}} \partial_3, \tag{2.10}$$

$$D_t^{\widetilde{\varphi}} = \partial_t + \overline{v} \cdot \overline{\nabla} + \frac{1}{\partial_3 \widetilde{\varphi}} (v \cdot \widetilde{\mathbf{N}} - \partial_t \varphi) \partial_3, \tag{2.11}$$

and $\overline{v} := (v_1, v_2)$, $\overline{\nabla} := (\partial_1, \partial_2)$ are the horizontal velocities and derivatives, $\overline{\Delta} := \overline{\nabla} \cdot \overline{\nabla} = \partial_1^2 + \partial_2^2$ is the flat tangential Laplacian, $\widetilde{N} := (-\partial_1 \widetilde{\psi}, -\partial_2 \widetilde{\psi}, 1)^\top$ is the smoothed Eulerian normal vector and $\widetilde{\mathbf{N}} := (-\partial_1 \widetilde{\varphi}, -\partial_2 \widetilde{\varphi}, 1)^\top$ is the extension of $\widetilde{N}$ into Ω.

The tangential smoothing method was first introduced in [14] to study incompressible Euler and then was generalized to study various free-surface inviscid fluids in Lagrangian coordinates. However, the free surface is now assumed to be a graph, and the construction of a nonlinear approximate system is quite different from Lagrangian coordinates. The following issues are crucial and very technical.

- **Design the smoothed material derivative $D_t^{\widetilde{\varphi}}$.** When restricted on Σ, the weight function in front of ∂_3 in $D_t^{\widetilde{\varphi}}$ should agree with the kinematic boundary condition. Otherwise, there will be a *boundary mismatched term that cannot be controlled* when studying $\frac{d}{dt}E(t)$. Therefore, we cannot mollify $\partial_t \varphi$ in $D_t^{\widetilde{\varphi}}$.
- **Introduce the artificial viscosity to control the mismatched terms.** The tangential mollification leads to some mismatched terms that should be controlled by the artificial viscosity term.
  a. The commutator $\mathfrak{D}(f)$ in (2.1) now involves a new term $\mathfrak{e}(f) = \partial_t \mathcal{T}^\alpha (\widetilde{\varphi} - \varphi) \partial_3^{\widetilde{\varphi}} f$ which should be bounded by $\kappa |\overline{\nabla} \partial_t \mathcal{T}^\alpha \psi|_0$ after using the mollifier property (3.6).
  b. The analysis of the ST term introduces two extra commutators, whose control requires the bound for $\kappa |\overline{\nabla} \partial_t \mathcal{T}^\alpha \psi|$.

  To control the above two crucial mismatched terms, we introduce the artificial viscosity term $-\kappa^2 (1 - \overline{\Delta}) \partial_t \psi$ which gives the energy $|\kappa \langle \overline{\partial} \rangle \mathcal{T}^\alpha \partial_t \psi|_0$ to enhance the regularity of $\partial_t \psi$. Due to technical reasons, it should be noted that the coefficient must be κ^2 instead of any other power of κ in the artificial viscosity. The details are explained in Section 4 below (4.85).

It should also be noted that the design of the linearized κ-regularized problem is also crucial and technically complicated, as we must define the "new free surface" in each step of iteration and the boundary conditions must keep consistent with the nonlinear problem. We refer to those rather technical constructions to the beginning of Section 5.

Now, once the coefficients involving φ are regularized in both t and x' variables, the loss of derivatives can be compensated by such regularization for each fixed $\kappa > 0$. That is, the existence of nonlinear approximate problem (2.9) is resolved for each fixed $\kappa > 0$. Based on the strategies introduced in Section 2.1 and the above analysis of the mismatched terms, we can derive the uniform-in-κ a priori estimates for the nonlinear approximate system (2.9). We can also prove the initial data $(v_{0,\kappa}, \rho_{0,\kappa}, \psi_{0,\kappa})$ of (2.9) converges to the initial data of (1.24) as $\kappa \to 0$. This completes the proof of the existence of the original system (1.24).

3 Nonlinear approximate κ-problem

The first step to prove the local well-posedness is to introduce our approximation scheme. For each $\kappa > 0$, we construct a suitable approximate problem indexed by κ which is asymptotically consistent with (1.24).

3.1 The tangential mollification

Let $\zeta = \zeta(x') \in C_c^\infty(\mathbb{R}^2)$, satisfying $0 \le \zeta \le 1$ and $\int_{\mathbb{R}^2} \zeta\, dx' = 1$, be a standard cut-off function supported in the closed unit ball $\overline{B_1(\mathbf{0})}$. For each $\kappa > 0$, we set

$$\zeta_\kappa(x') = \kappa^{-2}\zeta(\kappa^{-1}x'),$$

and for each $f : \mathbb{R}^2 \to \mathbb{R}$, we define

$$\Lambda_\kappa f(x') := \int_{\mathbb{R}^2} \zeta_\kappa(x'-z')f(z')\,dz'. \tag{3.1}$$

Also, for each $g : \mathbb{R}^3 \to \mathbb{R}$, we set

$$\Lambda_\kappa g(x', z) := \int_{\mathbb{R}^2} \zeta_\kappa(x-z')g(z', x_3)\,dz'. \tag{3.2}$$

In other words, when acting on a function of three independent variables, Λ_κ becomes the smoothing operator in the tangential direction only. The next lemma records the properties that Λ_κ enjoys. This will be frequently used (sometimes silently) in the rest of this paper.

Lemma 3.1 ([48, Lemma 2.6])**.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. For each $\kappa > 0$, we have:

$$|\Lambda_\kappa f|_s \lesssim |f|_s, \quad \forall s \ge -0.5; \tag{3.3}$$

$$|\overline{\partial}\Lambda_\kappa f|_0 \lesssim \kappa^{-s}|f|_{1-s}, \quad \forall s \in [0,1]; \tag{3.4}$$

$$|f - \Lambda_\kappa f|_\infty \lesssim \sqrt{\kappa}|\overline{\partial} f|_{0.5} \tag{3.5}$$

$$|f - \Lambda_\kappa f|_{L^p} \lesssim \kappa|\overline{\partial} f|_{L^p}. \tag{3.6}$$

Also, for a smooth function $g : \mathbb{R}^3 \to \mathbb{R}$, then

$$\|\Lambda_\kappa g\|_s \lesssim \|g\|_s, \quad \forall s \ge 0. \tag{3.7}$$

Moreover, let $h : \mathbb{R}^2 \to \mathbb{R}$, and $[\Lambda_\kappa, f]h := \Lambda_\kappa(fh) - f\Lambda_\kappa(h)$. Then we have:

$$|[\Lambda_\kappa, f]g|_0 \lesssim |f|_{L^\infty}|g|_0, \tag{3.8}$$

$$|[\Lambda_\kappa, f]\overline{\partial} g|_0 \lesssim |f|_{W^{1,\infty}}|g|_0, \tag{3.9}$$

$$|[\Lambda_\kappa, f]\overline{\partial} g|_0 \lesssim \kappa|f|_{W^{1,\infty}}|\overline{\partial} g|_0. \tag{3.10}$$

3.2 Construction of the κ-problem

Let $\widetilde{\psi} := \Lambda_\kappa^2\psi$, $\varphi(t,x) = x_3 + \chi(x_3)\widetilde{\psi}(t,x') = \Lambda_\kappa^2\varphi(t,x)$, and $\widetilde{N} := (-\partial_1\widetilde{\psi}, -\partial_2\widetilde{\psi}, 1)^\top$. Then we set the approximate κ-problem of (1.24) to be

$$\begin{cases} \rho D_t^{\widetilde{\varphi}} v + \nabla^{\widetilde{\varphi}}\check{q} = -(\rho-1)ge_3 & \text{in } [0,T]\times\Omega, \\ \mathcal{F}'(q)D_t^{\widetilde{\varphi}}\check{q} + \nabla^{\widetilde{\varphi}}\cdot v = \mathcal{F}'(q)gv_3 & \text{in } [0,T]\times\Omega, \\ q = q(\rho), \check{q} = q + g\widetilde{\varphi} & \text{in } [0,T]\times\Omega, \\ \check{q} = g\widetilde{\psi} - \sigma\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\widetilde{\psi}}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}}\right) + \kappa^2(1-\overline{\Delta})(v\cdot\widetilde{N}) & \text{on } [0,T]\times\Sigma, \\ \partial_t\psi = v\cdot\widetilde{N} & \text{on } [0,T]\times\Sigma, \\ v_3 = 0 & \text{on } [0,T]\times\Sigma_b, \\ (v,\rho,\psi)|_{t=0} = (v_{\kappa,0}, \rho_{\kappa,0}, \psi_{\kappa,0}). \end{cases} \tag{3.11}$$

Here,

$$\partial_t^{\widetilde{\varphi}} = \partial_t - \frac{\partial_t\varphi}{\partial_3\widetilde{\varphi}}\partial_3, \tag{3.12}$$

$$\nabla_a^{\widetilde{\varphi}} = \partial_a^{\widetilde{\varphi}} = \partial_a - \frac{\partial_a\widetilde{\varphi}}{\partial_3\widetilde{\varphi}}\partial_3, \quad a = 1,2, \tag{3.13}$$

$$\nabla_3^{\widetilde{\varphi}} = \partial_3^{\widetilde{\varphi}} = \frac{1}{\partial_3\widetilde{\varphi}}\partial_3, \tag{3.14}$$

$$D_t^{\widetilde{\varphi}} = \partial_t^{\widetilde{\varphi}} + v\cdot\nabla^{\widetilde{\varphi}}, \tag{3.15}$$

and $\overline{\Delta} = \partial_x^2 + \partial_y^2$ is the flat tangential Laplacian. Thanks to (3.12), the smoothed material derivative $D_t^{\widetilde{\varphi}}$ is equivalent to

$$D_t^{\widetilde{\varphi}} = \partial_t + \overline{v} \cdot \overline{\nabla} + \frac{1}{\partial_3 \widetilde{\varphi}}(v \cdot \widetilde{\mathbf{N}} - \partial_t \varphi)\partial_3, \tag{3.16}$$

where $\widetilde{\mathbf{N}} := (-\partial_1 \widetilde{\varphi}, -\partial_2 \widetilde{\varphi}, 1)^\top$. Note that we do not replace $v \cdot \widetilde{\mathbf{N}} - \partial_t \varphi$ by $v \cdot \widetilde{\mathbf{N}} - \partial_t \widetilde{\varphi}$ in the last term, as this would generate a severe structural mismatch in the boundary estimates.

The approximate κ-system (3.11) is asymptotically consistent with (1.24) as $\kappa \to 0$. Furthermore, the artificial viscosity $\kappa(1 - \overline{\Delta})(v \cdot \widetilde{N})$ in the modified boundary condition

$$\check{q} = g\widetilde{\psi} - \sigma\overline{\nabla} \cdot \left(\frac{\overline{\nabla}\widetilde{\psi}}{\sqrt{1 + |\overline{\nabla}\widetilde{\psi}|^2}} \right) + \kappa^2(1 - \overline{\Delta})(v \cdot \widetilde{N}) \quad \text{on } \Sigma$$

is necessary to control the terms generated due to the loss of symmetry in (3.11).

4 Uniform energy estimates for the nonlinear κ-problem

For each fixed $\kappa > 0$, we denote by $(v^\kappa(t), \rho^\kappa(t), \check{q}^\kappa(t), \psi^\kappa(t))$ the solution of the nonlinear κ-system (3.11). Let $\sigma > 0$ be fixed. We aim to show that $\{v^\kappa(t), \check{q}^\kappa(t), \rho^\kappa(t), \psi^\kappa(t)\}_{\kappa>0}$ has a convergent subsequence that approximates the solution to the original system (1.24) as $\kappa \to 0$ in some time interval $[0, T]$ with T being independent of κ. From now on, we drop the superscript κ when analyzing the nonlinear κ-approximate system for the sake of clean notations. Let

$$\begin{aligned}
E^\kappa(t) =& E_0^\kappa(t) + E_4^\kappa(t),\\
E_0^\kappa(t) =& \|\rho(t) - 1\|_0^2 + g|\Lambda_\kappa \psi|_0^2 + \|\sqrt{\mathcal{F}'(q)}\check{q}(t)\|_0^2 + \sum_{k=1}^{3} \|\sqrt{\mathcal{F}'(q)}\partial_t^k \check{q}(t)\|_0^2,\\
E_4^\kappa(t) =& \sum_{k=0}^{4} \|\partial_t^k v(t)\|_{4-k}^2 + \sigma|\overline{\nabla}\partial_t^k \Lambda_\kappa \psi(t)|_{4-k}^2 + \left\| \sqrt{\mathcal{F}'(q)}\check{q}(t) \right\|_0^2 + \|\partial\check{q}(t)\|_3^2 + \sum_{k=1}^{3} \|\partial_t^k \partial\check{q}(t)\|_{3-k}^2\\
&+ \left\| \sqrt{\mathcal{F}'(q)}\partial_t^4 \check{q}(t) \right\|_0^2 + \sum_{k=0}^{4} \int_0^t \left|\kappa\partial_t^{k+1}\psi(\tau)\right|_{5-k}^2 \mathrm{d}\tau.
\end{aligned} \tag{4.1}$$

Theorem 4.1. For each fixed $\sigma > 0$, there exists some $T_\sigma > 0$, independent of κ and $\sqrt{\mathcal{F}'(q)}$, such that

$$E^\kappa(t) \le P(E^\kappa(0)) =: \mathcal{P}_0^\kappa, \quad \text{for every } 0 \le t \le T_\sigma. \tag{4.2}$$

Thanks to the Grönwall's inequality, the key step of proving Theorem 4.1 is to show that

$$\sup_{0\le t\le T} E^\kappa(t) \le \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t, \tag{4.3}$$

for some $T > 0$ chosen sufficiently small. The control of $E^\kappa(t)$ will be divided into 3 steps, i.e., the basic L^2 estimate, the div-curl analysis, and the interior tangential estimates. We remark that the compatibility conditions on Σ have changed due to the artificial viscosity. The new compatibility conditions, expressed in terms of $\check{q}$, are

$$(D_t^{\widetilde{\varphi}})^k \check{q}|_{t=0} = (D_t^{\widetilde{\varphi}})^k(-g\widetilde{\psi} + \sigma\mathcal{H})|_{t=0} + (D_t^{\widetilde{\varphi}})^k \left(\kappa^2(1 - \overline{\Delta})(v \cdot \widetilde{N})\right)|_{t=0}, \quad k = 0, 1, 2, 3, \quad \text{on } \Sigma \tag{4.4}$$

We however are still able to construct initial data satisfying (4.4) in terms of $(\psi_{\kappa,0}, v_{\kappa,0}, \check{q}_{\kappa,0})$, that is uniformly bounded and converges to $(\psi_0, v_0, \check{q}_0)$ as $\kappa \to 0$. The details can be located in Appendix C.

4.1 L^2-estimate

First, we establish L^2-energy estimate for (3.11). Invoking Theorem (A.3), the identity $\nabla^{\widetilde{\varphi}}\widetilde{\varphi} = e_3$, and then integrating by parts, we have:

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho|v|^2\partial_3\widetilde{\varphi}\,\mathrm{d}x &= -\int_\Omega v\cdot\nabla^{\widetilde{\varphi}}\check{q}\partial_3\widetilde{\varphi}\,\mathrm{d}x - \int_\Omega(\rho-1)gv_3\partial_3\widetilde{\varphi}\,\mathrm{d}x + \frac{1}{2}\int_\Omega \rho|v|^2\partial_3\partial_t(\widetilde{\varphi}-\varphi)\,\mathrm{d}x\\
&= \int_\Omega \check{q}(\nabla^{\widetilde{\varphi}}\cdot v)\partial_3\widetilde{\varphi}\,\mathrm{d}x + \int_{\Sigma_b} v_3\check{q}\,\mathrm{d}x' - \int_\Sigma \partial_t\psi q\,\mathrm{d}x' - \int_\Sigma g\widetilde{\psi}\partial_t\psi\,\mathrm{d}x'\\
&\quad - \int_\Omega(\rho-1)gv_3\partial_3\widetilde{\varphi}\,\mathrm{d}x + \frac{1}{2}\int_\Omega \rho|v|^2\partial_3\partial_t(\widetilde{\varphi}-\varphi)\,\mathrm{d}x.
\end{aligned} \tag{4.5}
$$

Plugging the continuity equation into the first integral, we get

$$
\begin{aligned}
\int_\Omega \check{q}(\nabla^{\widetilde{\varphi}}\cdot v)\partial_3\widetilde{\varphi}\,\mathrm{d}x &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathcal{F}'(q)|\check{q}|^2\partial_3\widetilde{\varphi}\,\mathrm{d}x + \frac{1}{2}\int_\Omega \rho D_t^{\widetilde{\varphi}}(\rho^{-1}\mathcal{F}'(q))|\check{q}|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega \mathcal{F}'(q)|\check{q}|^2\partial_3\partial_t(\widetilde{\varphi}-\varphi)\,\mathrm{d}x\\
&\quad + \int_\Omega \check{q}\mathcal{F}'(q)gv_3\partial_3\widetilde{\varphi}\,\mathrm{d}x\\
&\lesssim -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\sqrt{\mathcal{F}'(q)}q\right\|_0^2 + \left\|\sqrt{\mathcal{F}'(q)}q\right\|_0^2\left(\|D_t^{\widetilde{\varphi}}\rho\|_\infty + \|\partial_3\partial_t(\widetilde{\varphi}-\varphi)\|_\infty + \left\|\sqrt{\mathcal{F}'(q)}v_3\right\|_0\right).
\end{aligned} \tag{4.6}
$$

Here and in the sequel, we employ the notation $A \lesssim B$ to mean that $A \le CB$ for a universal constant C. The boundary integral on Σ_b vanishes due to $v_3|_{\Sigma_b} = 0$. Then we plug $q = -\sigma\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}}\right) + \kappa^2(1-\overline{\Delta})\partial_t\psi$ into the first boundary term in (4.5) and integrate by parts to get:

$$
-\int_\Sigma \partial_t\psi q\,\mathrm{d}x' = -\sigma\int_\Sigma\left(\frac{\overline{\nabla}\widetilde{\psi}}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}}\right)\cdot\overline{\nabla}\partial_t\psi\,\mathrm{d}x' + \int_\Sigma\left|\kappa\langle\overline{\partial}\rangle\partial_t\psi\right|_0^2\,\mathrm{d}x', \tag{4.7}
$$

where $\langle\cdot\rangle$ denotes the Japanese bracket. To treat the first term, we use the self-adjointness of Λ_κ in $L^2(\Sigma)$ to move one Λ_κ from $\overline{\nabla}\widetilde{\psi}$ to $\partial_t\psi$:

$$
\begin{aligned}
-\sigma\int_\Sigma\left(\frac{\overline{\nabla}\widetilde{\psi}}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}}\right)\cdot\overline{\nabla}\partial_t\psi\,\mathrm{d}x' &= -\sigma\int_\Sigma\frac{\overline{\nabla}\Lambda_\kappa\psi\cdot\partial_t\overline{\nabla}\Lambda_\kappa\psi}{|\widetilde{N}|}\,\mathrm{d}x' - \sigma\int_\Sigma\overline{\nabla}\Lambda_\kappa\psi\cdot\left([\Lambda_\kappa,|\widetilde{N}|^{-1}]\overline{\nabla}\partial_t\psi\right)\mathrm{d}x'\\
&\lesssim -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left|\sqrt{\sigma}\frac{1}{|\widetilde{N}|^{\frac{1}{2}}}\overline{\nabla}\Lambda_\kappa\psi\right|_0^2 + \frac{1}{2}\int_\Sigma\partial_t(|\widetilde{N}|^{-1})\left|\sqrt{\sigma}\overline{\nabla}\Lambda_\kappa\psi\right|^2 + P(|\overline{\nabla}\widetilde{\psi}|_{W^{1,\infty}})\sigma\left|\sqrt{\sigma}\overline{\nabla}\Lambda_\kappa\psi\right|_0^2 + \varepsilon\left|\kappa\overline{\nabla}\partial_t\psi\right|_0^2.
\end{aligned} \tag{4.8}
$$

Now, we get the non-weighted L^2-boundary energy from the second boundary integral in (4.5):

$$
-\int_\Sigma g\partial_t\psi\widetilde{\psi}\,\mathrm{d}x' = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega g|\Lambda_\kappa\psi|^2\,\mathrm{d}x'. \tag{4.9}
$$

On the other hand, show the L^2-estimate for $\rho-1$ for the energy inequality. We use $D_t^{\widetilde{\varphi}}\rho = D_t^{\widetilde{\varphi}}(\rho-1)$ and $D_t^{\widetilde{\varphi}}\widetilde{\varphi} = v_3 + \partial_t(\widetilde{\varphi}-\varphi)$ to rewrite the continuity equation in terms of $\rho-1$:

$$
D_t^{\widetilde{\varphi}}(\rho-1) + \rho(\nabla^{\widetilde{\varphi}}\cdot v) = -\partial_t(\widetilde{\varphi}-\varphi).
$$

Testing this with $\rho-1$ in $L^2(\Omega)$ and using the mollifier property (3.6), we get

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho-1\|_0^2 \lesssim \|\rho-1\|_0(\|\partial v\|_0 + \kappa|\overline{\partial}\partial_t\psi|_0). \tag{4.10}
$$

Let

$$
E_0^\kappa(t) = \|v\|_0^2 + \left\|\sqrt{\mathcal{F}'(q)}\check{q}\right\|_0^2 + \|\rho-1\|_0^2 + |\sqrt{g}\Lambda_\kappa\psi|_0^2 + \left|\sqrt{\sigma}\overline{\nabla}\Lambda_\kappa\psi\right|_0^2 + \int_0^T\int_\Sigma\left|\kappa\langle\overline{\partial}\rangle\partial_t\psi\right|_0^2\,\mathrm{d}x'\,\mathrm{d}t. \tag{4.11}
$$

Since $1 \le |\widetilde{N}| = \sqrt{1 + (\partial_1\widetilde{\psi})^2 + (\partial_2\widetilde{\psi})^2}$, we combine (4.5)-(4.10) and obtain

$$E_0^\kappa(T) - E_0^\kappa(0) \lesssim \int_0^T P(|\overline{\nabla}\widetilde{\psi}|_{W^{1,\infty}}, \|\partial v\|_\infty, |\kappa\overline{\partial}\partial_t\psi|_{0.5}) E_0^\kappa(t)\,\mathrm{d}t, \tag{4.12}$$

after choosing $\varepsilon > 0$ suitably small in (4.8). Here, we note that, using $\partial_3\partial_t(\widetilde{\varphi} - \varphi) = \chi'(x_3)\left(\partial_t\widetilde{\psi}(t, x') - \partial_t\psi(t, x')\right)$ together with (1.9) and (3.5) of Lemma 3.1, we have

$$\|\partial_3\partial_t(\widetilde{\varphi} - \varphi)\|_\infty \le |\partial_t\widetilde{\psi} - \partial_t\psi|_\infty \lesssim \sqrt{\kappa}|\overline{\partial}\partial_t\psi|_{0.5}, \tag{4.13}$$

where right side is directly controlled by invoking $\partial_t\psi = v \cdot \widetilde{\mathbf{N}} = -(\overline{v} \cdot \overline{\nabla})\widetilde{\psi} + v_3$ on Σ and the Sobolev trace lemma.

4.2 Reduction of pressure

We show how to reduce the control of the pressure to that of the velocity when there is at least one spatial derivative on q. This follows from using the momentum equation $\rho D_t^{\widetilde{\varphi}} v = -\nabla^{\widetilde{\varphi}}\check{q} - (\rho - 1)ge_3$. Particularly, by considering the third component of the momentum equation, we get

$$-(\partial_3\widetilde{\varphi})^{-1}\partial_3\check{q} - (\rho - 1)ge_3 = \rho D_t^{\widetilde{\varphi}} v_3. \tag{4.14}$$

Since $\partial_3\widetilde{\varphi}$ is bounded from below, i.e., there exists $c_0 > 0$ such that $\partial_3\widetilde{\varphi} \ge c_0$, then

$$\|\partial_3\check{q}\|_0 \lesssim_{g,c_0} \|\rho - 1\|_0 + \|\rho\|_\infty \|D_t^{\widetilde{\varphi}} v_3\|_0, \tag{4.15}$$

where $D_t^{\widetilde{\varphi}} v_3 = \partial_t v_3 + \overline{v} \cdot \overline{\nabla} v_3 + \frac{1}{\partial_3\widetilde{\varphi}}(v \cdot \widetilde{\mathbf{N}} - \partial_t\varphi)\partial_3 v_3$. This implies that the L^2-norm of $\partial_3\check{q}$ is reduced to the L^2-norms of $\rho - 1$, $\partial_t v_3, \overline{\partial} v_3$ and $\omega(x_3)\partial_3 v_3$. Here $\omega(x_3) \in C^\infty(-b, 0)$ is assumed to be bounded, comparable to $|x_3|$ in $[-2, 0]$ and vanishing on Σ.

Let $\mathcal{T} = \partial_t$ or $\overline{\partial}$ or $\omega(x_3)\partial_3$ and $D = \partial$ or ∂_t. The above estimate yields the control of $\|D^k\partial_3\check{q}\|_0$ after taking D^k, $k \ge 1$ to (4.14). Specifically, at the leading order, $\|D^k\partial_3\check{q}\|_0$ is controlled by

$$C(g, c_0)\left(\|\mathcal{F}'(q)D^k\check{q}\|_0 + \|\mathcal{F}'(q)D^k\widetilde{\varphi}\|_0 + \|\rho\|_{L^\infty}\|D^k\mathcal{T} v_3\|_0\right). \tag{4.16}$$

In addition, by considering the first two components of the momentum equation, we have:

$$-\partial_i\check{q} = -(\partial_3\widetilde{\varphi})^{-1}\overline{\partial}_i\widetilde{\varphi}\partial_3\check{q} + \rho D_t^{\widetilde{\varphi}} v_i, \quad i = 1, 2. \tag{4.17}$$

and thus the control of $\overline{\partial}\check{q}$ is reduced to $\partial_3\check{q}$ and $D_t^{\widetilde{\varphi}} v_i = \partial_t v_i + (\overline{v} \cdot \overline{\nabla})v_i + (\partial_3\widetilde{\varphi})^{-1}(v \cdot \widetilde{\mathbf{N}} - \partial_t\varphi)\partial_3 v_i$.

Lastly, using (4.14) and (4.17), we obtain

$$\|\partial_3\check{q}\|_\infty \lesssim_{g,c_0} \|\rho - 1\|_\infty + \|\rho\|_\infty\|D_t^{\widetilde{\varphi}} v_3\|_\infty, \tag{4.18}$$

$$\|\overline{\partial}\check{q}\|_\infty \lesssim_{g,c_0^{-1}} |\overline{\partial}\widetilde{\psi}|_\infty\|\partial_3\check{q}\|_\infty + \|\rho\|_\infty\|D_t^{\widetilde{\varphi}}\overline{v}\|_\infty. \tag{4.19}$$

Thus,

$$\|\partial q\|_\infty \lesssim_{g,c_0,c_0^{-1}} P(|\overline{\partial}\widetilde{\psi}|_\infty, \|\rho\|_\infty)\left(\|\rho - 1\|_\infty + \|D_t^{\widetilde{\varphi}} v\|_\infty\right). \tag{4.20}$$

Invoking the definition of $D_t^{\widetilde{\varphi}} v$, (4.20) implies that $\|\partial\check{q}\|_\infty$ is reduced to $\partial_t v$, $\overline{\partial} v$ and $\omega(x)\partial_3 v$ for some weight function $\omega(x)$ vanishing on Γ.

4.3 Div-Curl analysis

To estimate the Sobolev norm of v, we can use the div-curl analysis to convert one normal derivative to the divergence and curl. First, we record the well-known div-curl decomposition lemma and refer to [22, Lemma B.2] for the proof.

Lemma 4.2 (Hodge-type elliptic estimates)**.** For any sufficiently smooth vector field X and $s \ge 1$, one has

$$\|X\|_s^2 \lesssim C(|\widetilde{\psi}|_s, |\overline{\nabla}\widetilde{\psi}|_{W^{1,\infty}})\left(\|X\|_0^2 + \|\nabla^{\widetilde{\varphi}} \cdot X\|_{s-1}^2 + \|\nabla^{\widetilde{\varphi}} \times X\|_{s-1}^2 + \|\overline{\partial}^\alpha X\|_0^2\right), \tag{4.21}$$

for any multi-index α with $|\alpha| = s$. The constant $C(|\widetilde{\psi}|_s, |\overline{\nabla}\widetilde{\psi}|_{W^{1,\infty}}) > 0$ depends linearly on $|\widetilde{\psi}|_s^2$.

We will apply Lemma 4.2 to $\|\partial_t^k v\|_{4-k}$ for $0 \le k \le 3$. Starting from $k=0$, we have:

$$\|v\|_4^2 \lesssim C(|\widetilde{\psi}|_4, |\overline{\nabla}\widetilde{\psi}|_{W^{1,\infty}}) \left(\|v\|_0^2 + \|\nabla^{\widetilde{\varphi}} \cdot v\|_3^2 + \|\nabla^{\widetilde{\varphi}} \times v\|_3^2 + \|\overline{\partial}^4 v\|_0^2\right), \tag{4.22}$$

$$\|\partial_t^k v\|_{4-k}^2 \lesssim C(|\widetilde{\psi}|_{4-k}, |\overline{\nabla}\widetilde{\psi}|_{W^{1,\infty}}) \left(\|\partial_t^k v\|_0^2 + \|\nabla^{\widetilde{\varphi}} \cdot \partial_t^k v\|_{3-k}^2 + \|\nabla^{\widetilde{\varphi}} \times \partial_t^k v\|_3^2 + \|\overline{\partial}^{4-k}\partial_t^k v\|_0^2\right), \tag{4.23}$$

where the L^2-norm has been controlled in (4.12) and the tangential derivatives will be studied in the next section by using Alinhac good unknowns. The divergence part is reduced to the estimates of q by using the continuity equation

$$\|\nabla^{\widetilde{\varphi}} \cdot v\|_3^2 = \left\|\mathcal{F}'(q) D_t^{\widetilde{\varphi}} \check{q}\right\|_3^2 + \left\|\mathcal{F}'(q) g v_3\right\|_3^2, \tag{4.24}$$

which will be further reduced to the tangential estimates of v by using the argument in Section 4.2. Similarly, when $k = 1,2,3$, we have

$$\nabla^{\widetilde{\varphi}} \cdot \partial_t^k v = -\partial_t^k(\mathcal{F}'(q) D_t^{\widetilde{\varphi}} \check{q}) - \partial_t^k(\mathcal{F}'(q) g v_3) + [\nabla^{\widetilde{\varphi}} \cdot, \partial_t^k] v \overset{L}{=} -\mathcal{F}'(q)\partial_t^k D_t^{\widetilde{\varphi}} \check{q} - \mathcal{F}'(q) g \partial_t^k v_3 + (\partial_3 \widetilde{\varphi})^{-1} \overline{\partial} \partial_t^k \widetilde{\varphi} \partial_3 v,$$

where $\overset{L}{=}$ means equality modulo lower-order terms. This implies

$$\|\nabla^{\widetilde{\varphi}} \cdot \partial_t^k v\|_{3-k}^2 \le C(c_0, g, \|v\|_{W^{1,\infty}}) \left(\left\|\mathcal{F}'(q)\partial_t^k D_t^{\varphi} \check{q}\right\|_{3-k}^2 + \left|\overline{\partial}\partial_t^k \widetilde{\psi}\right|_{3-k}^2 + \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t \right), \tag{4.25}$$

where the last two terms control all lower-order terms generated above. Since the material derivative $D_t^{\varphi} = \partial_t + \overline{v} \cdot \overline{\nabla}$ on Σ, the term $\mathcal{F}'(q)\partial_t^k D_t^{\varphi} \check{q}$ involves only tangential derivatives with appropriate $\mathcal{F}'$-weight. By combining this div-curl analysis and the reduction of pressure in Section 4.2, we eventually only need to control the mixed space-time tangential derivatives of v, ψ, and $\check{q}$. We refer to Propositions 4.3, 4.5, and 4.6 for the details.

Next, we analyze the vorticity term. We take $\nabla^{\widetilde{\varphi}} \times$ in the momentum equation $\rho D_t^{\widetilde{\varphi}} v = -\nabla^{\widetilde{\varphi}} \check{q} + (\rho - 1) g e_3$ to get

$$\rho D_t^{\widetilde{\varphi}} (\nabla^{\widetilde{\varphi}} \times v) = -\nabla^{\widetilde{\varphi}} \times ((\rho-1) g e_3) - (\nabla^{\widetilde{\varphi}} \rho) \times D_t^{\widetilde{\varphi}} v - \rho [\nabla^{\widetilde{\varphi}} \times, D_t^{\widetilde{\varphi}}] v,$$

where the first term on the right side is equal to $(-g\partial_2^{\widetilde{\varphi}}\rho, g\partial_1^{\widetilde{\varphi}}\rho, 0)^\top$ and the second term, using $D_t^{\widetilde{\varphi}} v = -\rho^{-1}\nabla^{\widetilde{\varphi}} q - g e_3$, is equal to

$$-(\nabla^{\widetilde{\varphi}}\rho) \times D_t^{\widetilde{\varphi}} v = \underbrace{\rho'(q)(\nabla^{\widetilde{\varphi}} q) \times (\nabla^{\widetilde{\varphi}} q)}_{=\vec{0}} + \nabla^{\widetilde{\varphi}}\rho \times g e_3 = (g\partial_2^{\widetilde{\varphi}}\rho, -g\partial_1^{\widetilde{\varphi}}\rho, 0)^\top$$

which exactly cancels the first term. Using $[\partial_i^{\widetilde{\varphi}}, D_t^{\widetilde{\varphi}}](\cdot) = \partial_i^{\widetilde{\varphi}} v^l \partial_l^{\widetilde{\varphi}}(\cdot) + \partial_i^{\widetilde{\varphi}} \partial_t(\widetilde{\varphi} - \varphi)\partial_3^{\widetilde{\varphi}}(\cdot)$, we get the evolution of the smoothed vorticity to be

$$\rho D_t^{\widetilde{\varphi}} (\nabla^{\widetilde{\varphi}} \times v)_i = -\rho \epsilon^{ijk} \partial_j^{\widetilde{\varphi}} v^l \partial_l^{\widetilde{\varphi}} v_k - \rho \epsilon^{ijk} \partial_j^{\widetilde{\varphi}} \partial_t(\widetilde{\varphi} - \varphi)\partial_3^{\widetilde{\varphi}} v_k, \tag{4.26}$$

where ϵ^{ijk} denotes the sign of the permutation $(ijk) \in S_3$.

To control $\|\nabla^{\widetilde{\varphi}} \times v\|_3$, we take ∂^3 in (4.26) to get

$$\rho D_t^{\widetilde{\varphi}} \left(\partial^3 (\nabla^{\widetilde{\varphi}} \times v)_i\right) = -\epsilon^{ijk}\partial^3(\rho \partial_j^{\widetilde{\varphi}} v^l \partial_l^{\widetilde{\varphi}} v_k) - \epsilon^{ijk}\partial^3(\rho \partial_j^{\widetilde{\varphi}} \partial_t(\widetilde{\varphi} - \varphi)\partial_3^{\widetilde{\varphi}} v_k) - [\partial^3, \rho D_t^{\widetilde{\varphi}}](\nabla^{\widetilde{\varphi}} \times v)_i. \tag{4.27}$$

It is not necessary to write out the specific form of the right side of (4.27), but we just need to know the source terms in (4.27) contain ≤ 4 derivatives of v and $\widetilde{\varphi}$ except the mismatched term involving $\widetilde{\varphi} - \varphi$. This is easy to see because the only term containing 5 derivatives is the one on the left side of (4.27). Therefore, a straightforward L^2 estimate for (4.27) gives us the energy estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\nabla^{\widetilde{\varphi}} \times v\|_3^2 \le P(\|v\|_4, |\widetilde{\psi}|_4, \|\mathcal{F}'(q)\partial q\|_\infty, \|\mathcal{F}'(q)\partial^2 q\|_1, \kappa|\overline{\nabla}\partial_t \psi|_4), \tag{4.28}$$

where the mismatched term is controlled by using mollifier property (3.10) and $\varphi(t,x) = x_3 + \chi(x_3)\psi(t,x')$.

Similarly, we replace ∂^3 by $\partial_t^k \partial^{3-k}$ for $0 \le k \le 3$ to get

$$\rho D_t^{\widetilde{\varphi}} \left(\partial^{3-k}\partial_t^k (\nabla^{\widetilde{\varphi}} \times v)_i\right) = -\epsilon^{ijk}\partial_t^k\partial^{3-k}(\rho \partial_j^{\widetilde{\varphi}} v^l \partial_l^{\widetilde{\varphi}} v_k) - \epsilon^{ijk}\partial_t^k\partial^{3-k}(\rho \partial_j^{\widetilde{\varphi}} \partial_t(\widetilde{\varphi} - \varphi)\partial_3^{\widetilde{\varphi}} v_k) - [\partial_t^k\partial^{3-k}, \rho D_t^{\widetilde{\varphi}}](\nabla^{\widetilde{\varphi}} \times v)_i, \tag{4.29}$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\partial_t^k(\nabla^{\widetilde{\varphi}} \times v)\|_{3-k}^2 \le P(E^\kappa(t)). \tag{4.30}$$

Then we need to estimate the commutator $\|[\partial_t^k, \nabla^{\widetilde{\varphi}}\times]v\|_{3-k}$ to get the control of $\|\nabla^{\widetilde{\varphi}} \times \partial_t^k v\|_{3-k}$. Similarly, as in the control of divergence, we know the highest order term in the commutator should be $\|(-\partial_3\widetilde{\varphi})^{-1}\overline{\partial}\partial_t^k\widetilde{\varphi}\partial_3 v\|_{3-k} \lesssim \|\partial v\|_{3-k}\|\overline{\partial}\langle\overline{\partial}\rangle^{3-k}\partial_t^k\widetilde{\varphi}\|_0 \leq \|\partial v\|_{3-k}|\overline{\partial}\partial_t^k\widetilde{\psi}|_{3-k}$. So we have the following conclusion

$$\|\nabla^{\widetilde{\varphi}} \times \partial_t^k v\|_{3-k}^2 \leq |\overline{\partial}\partial_t^k\widetilde{\psi}|_{3-k}^2 + \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t. \tag{4.31}$$

Combining (4.22), (4.24), (4.25), (4.28), (4.31) and the argument in Section 4.2, it remains to control the tangential derivatives of v and full time derivatives of q, namely $\|\mathcal{F}'(q)\partial_t^4\check{q}\|_0$.

4.4 The $\mathcal{T}^\alpha$-differentiated equations

By the div-curl analysis, the crucial step is to study the higher order tangential energy estimate of (3.11). In particular, we define the following tangential derivatives

$$\mathcal{T}_0 = \partial_t, \quad \mathcal{T}_1 = \partial_1, \quad \mathcal{T}_2 = \partial_2, \quad \mathcal{T}_3 = \omega(x_3)\partial_3, \tag{4.32}$$

where $\omega \in C^\infty(-b, 0)$ is assumed to be bounded, comparable to $|x_3|$ when $-2 \leq x_3 \leq 0$ and vanishing on Σ. This requires us to commute $\mathcal{T}^\alpha$ with (3.11), where $\mathcal{T}^\alpha := \mathcal{T}_0^{\alpha_0}\mathcal{T}_1^{\alpha_1}\mathcal{T}_2^{\alpha_2}\mathcal{T}_3^{\alpha_3}$, and $|\alpha| \leq 4$.

Remark 4.1. We need the tangential derivative $\mathcal{T}_3 = \omega(x_3)\partial_3$ to control the $(\partial_3\varphi)^{-1}(v \cdot \widetilde{\mathbf{N}} - \partial_t\varphi)\partial_3$ in the material derivative $D_t^{\widetilde{\varphi}}$. We do not include it in $E^\kappa(t)$ as ω is comparable to 1. However, we still need the estimates of $\mathcal{T}_3$ in the reduction of $\check{q}$.

We will not directly commute $\mathcal{T}^\alpha$ with $\nabla^{\widetilde{\varphi}}$. Instead, for $i = 1, 2, 3$, we observe that

$$\mathcal{T}^\alpha\partial_i^{\widetilde{\varphi}} f = \partial_i^{\widetilde{\varphi}}\mathcal{T}^\alpha f - \partial_3^{\widetilde{\varphi}} f\partial_i^{\widetilde{\varphi}}\mathcal{T}^\alpha\widetilde{\varphi} + \mathfrak{C}_i'(f), \tag{4.33}$$

where for $i = 1, 2$,

$$\mathfrak{C}_i'(f) = -\left[\mathcal{T}^\alpha, \frac{\partial_i\widetilde{\varphi}}{\partial_3\widetilde{\varphi}}, \partial_3 f\right] - \partial_3 f\left[\mathcal{T}^\alpha, \partial_i\widetilde{\varphi}, \frac{1}{\partial_3\widetilde{\varphi}}\right] - \partial_i\widetilde{\varphi}\partial_3 f\left[\mathcal{T}^{\alpha-\gamma}, \frac{1}{(\partial_3\widetilde{\varphi})^2}\right]\mathcal{T}^\gamma\partial_3\widetilde{\varphi} - \frac{\partial_i\widetilde{\varphi}}{\partial_3\widetilde{\varphi}}[\mathcal{T}^\alpha, \partial_3]f + \frac{\partial_i\widetilde{\varphi}}{(\partial_3\widetilde{\varphi})^2}\partial_3 f[\mathcal{T}^\alpha, \partial_3]\widetilde{\varphi}, \tag{4.34}$$

with $|\gamma| = 1$, and

$$\mathfrak{C}_3'(f) = \left[\mathcal{T}^\alpha, \frac{1}{\partial_3\widetilde{\varphi}}, \partial_3 f\right] + \partial_3 f\left[\mathcal{T}^{\alpha-\gamma}, \frac{1}{(\partial_3\widetilde{\varphi})^2}\right]\mathcal{T}^\gamma\partial_3\widetilde{\varphi} + \frac{1}{\partial_3\widetilde{\varphi}}[\mathcal{T}^\alpha, \partial_3]f - \frac{1}{(\partial_3\widetilde{\varphi})^2}\partial_3 f[\mathcal{T}^\alpha, \partial_3]\widetilde{\varphi}. \tag{4.35}$$

Since $\partial_i^{\widetilde{\varphi}}$ and $\partial_3^{\widetilde{\varphi}}$ commute, the identity (4.33) implies

$$\mathcal{T}^\alpha\partial_i^{\widetilde{\varphi}} f = \partial_i^{\widetilde{\varphi}}(\mathcal{T}^\alpha f - \partial_3^{\widetilde{\varphi}} f\mathcal{T}^\alpha\widetilde{\varphi}) + \underbrace{\partial_3^{\widetilde{\varphi}}\partial_i^{\widetilde{\varphi}} f\mathcal{T}^\alpha\widetilde{\varphi} + \mathfrak{C}_i'(f)}_{:=\mathfrak{C}_i(f)}. \tag{4.36}$$

The quantity $\mathcal{T}^\alpha f - \partial_3^{\widetilde{\varphi}} f\mathcal{T}^\alpha\widetilde{\varphi}$ is the so-called Alinhac good unknown associated with f. It was first observed by Alinhac [7] that the top order term of $\widetilde{\varphi}$ does not appear when we use the above good unknown. It is not hard to see that we can obtain the control of $\|\mathcal{T}^\alpha f\|_0$ from that of $\|\mathcal{T}^\alpha f - \partial_3^{\widetilde{\varphi}} f\mathcal{T}^\alpha\widetilde{\varphi}\|_0$. In particular,

$$\|\mathcal{T}^\alpha f\|_0 \leq \|\mathcal{T}^\alpha f - \partial_3^{\widetilde{\varphi}} f\mathcal{T}^\alpha\widetilde{\varphi}\|_0 + \|\partial_3^{\widetilde{\varphi}} f\|_\infty\|\mathcal{T}^\alpha\widetilde{\varphi}\|_0. \tag{4.37}$$

In addition to this, we need to commute $\mathcal{T}^\alpha$ with

$$D_t^{\widetilde{\varphi}} = \partial_t + \overline{v} \cdot \overline{\nabla} + \frac{1}{\partial_3\widetilde{\varphi}}(v \cdot \widetilde{\mathbf{N}} - \partial_t\varphi)\partial_3.$$

A direct computation yields:

$$\begin{aligned}\mathcal{T}^\alpha D_t^{\widetilde{\varphi}} f &= \mathcal{T}^\alpha\partial_t f + \mathcal{T}^\alpha(\overline{v} \cdot \overline{\partial} f) + \mathcal{T}^\alpha\left(\frac{1}{\partial_3\widetilde{\varphi}}(v \cdot \widetilde{\mathbf{N}} - \partial_t\varphi)\partial_3 f\right)\\ &= D_t^{\widetilde{\varphi}}\mathcal{T}^\alpha f + (v \cdot \mathcal{T}^\alpha\widetilde{\mathbf{N}} - \partial_t\mathcal{T}^\alpha\varphi)\partial_3^{\widetilde{\varphi}} f - \partial_3^{\widetilde{\varphi}}\mathcal{T}^\alpha\widetilde{\varphi}(v \cdot \widetilde{\mathbf{N}} - \partial_t\varphi)\partial_3^{\widetilde{\varphi}} f + \mathfrak{D}'(f),\end{aligned} \tag{4.38}$$

where

$$\mathfrak{D}'(f) = [\mathcal{T}^\alpha, \overline{v}]\cdot\overline{\partial} f + \left[\mathcal{T}^\alpha, \frac{1}{\partial_3\widetilde{\varphi}}(v\cdot\widetilde{\mathbf{N}}-\partial_t\varphi), \partial_3 f\right] + \left[\mathcal{T}^\alpha, v\cdot\widetilde{\mathbf{N}}-\partial_t\varphi, \frac{1}{\partial_3\widetilde{\varphi}}\right]\partial_3 f + \frac{1}{\partial_3\widetilde{\varphi}}[\mathcal{T}^\alpha, v]\cdot\widetilde{\mathbf{N}}\partial_3 f$$
$$-(v\cdot\widetilde{\mathbf{N}}-\partial_t\varphi)\partial_3 f\left[\overline{\partial}^{\alpha-\gamma}, \frac{1}{(\partial_3\widetilde{\varphi})^2}\right]\mathcal{T}^\gamma\partial_3\widetilde{\varphi} + \frac{1}{\partial_3\widetilde{\varphi}}(v\cdot\widetilde{\mathbf{N}}-\partial_t\varphi)[\mathcal{T}^\alpha,\partial_3]f + (v\cdot\widetilde{\mathbf{N}}-\partial_t\varphi)\frac{\partial_3 f}{(\partial_3\widetilde{\varphi})^2}[\mathcal{T}^\alpha,\partial_3]\widetilde{\varphi}, \tag{4.39}$$

with $|\gamma| = 1$.

Since $v\cdot\mathcal{T}^\alpha\widetilde{\mathbf{N}} = -v_1\partial_1\mathcal{T}^\alpha\widetilde{\varphi} - v_2\partial_2\mathcal{T}^\alpha\widetilde{\varphi}$, then we must have

$$\begin{aligned}
&(v\cdot\mathcal{T}^\alpha\widetilde{\mathbf{N}} - \partial_t\mathcal{T}^\alpha\varphi)\partial_3^{\widetilde{\varphi}} f - \partial_3^{\widetilde{\varphi}}\mathcal{T}^\alpha\widetilde{\varphi}(v\cdot\widetilde{\mathbf{N}}-\partial_t\varphi)\partial_3^{\widetilde{\varphi}} f\\
=&(v\cdot\mathcal{T}^\alpha\widetilde{\mathbf{N}} - \partial_t\mathcal{T}^\alpha\widetilde{\varphi})\partial_3^{\widetilde{\varphi}} f - \partial_3^{\widetilde{\varphi}}\mathcal{T}^\alpha\widetilde{\varphi}(v\cdot\widetilde{\mathbf{N}}-\partial_t\varphi)\partial_3^{\widetilde{\varphi}} f + \partial_t\mathcal{T}^\alpha(\widetilde{\varphi}-\varphi)\partial_3^{\widetilde{\varphi}} f\\
=&-\partial_3^{\widetilde{\varphi}} f\left(\partial_t + \overline{v}\cdot\overline{\nabla} + (v\cdot\widetilde{\mathbf{N}}-\partial_t\varphi)\partial_3^{\widetilde{\varphi}}\right)\mathcal{T}^\alpha\widetilde{\varphi} + \underbrace{\partial_t\mathcal{T}^\alpha(\widetilde{\varphi}-\varphi)\partial_3^{\widetilde{\varphi}} f}_{:=\mathfrak{E}(f)}\\
=&-\partial_3^{\widetilde{\varphi}} f D_t^{\widetilde{\varphi}}\mathcal{T}^\alpha\widetilde{\varphi} + \mathfrak{E}(f).
\end{aligned}\tag{4.40}$$

Thus,

$$\begin{aligned}
\mathcal{T}^\alpha D_t^{\widetilde{\varphi}} f =& D_t^{\widetilde{\varphi}}\mathcal{T}^\alpha f - \partial_3^{\widetilde{\varphi}} f D_t^{\widetilde{\varphi}}\mathcal{T}^\alpha\widetilde{\varphi} + \mathfrak{D}'(f) + \mathfrak{E}(f)\\
=&D_t^{\widetilde{\varphi}}\left(\mathcal{T}^\alpha f - \partial_3^{\widetilde{\varphi}} f\mathcal{T}^\alpha\widetilde{\varphi}\right) + \mathfrak{D}(f) + \mathfrak{E}(f),
\end{aligned}\tag{4.41}$$

where $\mathfrak{D}(f) = (D_t^{\widetilde{\varphi}}\partial_3^{\widetilde{\varphi}} f)\mathcal{T}^\alpha\widetilde{\varphi} + \mathfrak{D}'(f)$.

Let

$$\mathbf{V}_i := \mathcal{T}^\alpha v_i - \partial_3^{\widetilde{\varphi}} v_i\mathcal{T}^\alpha\widetilde{\varphi}, \quad \mathbf{Q} := \mathcal{T}^\alpha\check{q} - \partial_3^{\widetilde{\varphi}}\check{q}\mathcal{T}^\alpha\widetilde{\varphi} \tag{4.42}$$

respectively be the Alinhac good unknowns of v and $\check{q}$. Motivated by (4.36) and (4.41), we take $\mathcal{T}^\alpha$ to the first two equations of (1.5) to obtain

$$\rho D_t^{\widetilde{\varphi}}\mathbf{V}_i + \partial_i^{\widetilde{\varphi}}\mathbf{Q} = \mathcal{R}_i^1, \tag{4.43}$$
$$\mathcal{F}'(q)D_t^{\widetilde{\varphi}}\mathbf{Q} + \nabla^{\widetilde{\varphi}}\cdot\mathbf{V} = \mathcal{R}^2 - \mathfrak{C}_i(v^i), \tag{4.44}$$

where

$$\mathcal{R}_i^1 := -[\mathcal{T}^\alpha,\rho]D_t^{\widetilde{\varphi}} v_i - \rho\left(\mathfrak{D}(v_i) + \mathfrak{E}(v_i)\right) - \mathfrak{C}_i(\check{q}), \tag{4.45}$$
$$\mathcal{R}^2 := -[\mathcal{T}^\alpha,\mathcal{F}'(q)]D_t^{\widetilde{\varphi}}\check{q} - \mathcal{F}'(q)\left(\mathfrak{D}(\check{q}) + \mathfrak{E}(\check{q})\right) + \mathcal{T}^\alpha(\mathcal{F}'(q)gv_3). \tag{4.46}$$

In addition, since $\mathcal{T}^\alpha$ reduces to $\overline{\partial}^\alpha$ on Σ and $\overline{\partial}^\alpha\widetilde{N} = (-\partial_1\overline{\partial}^\alpha\widetilde{\psi}, -\partial_2\overline{\partial}^\alpha\widetilde{\psi}, 0)^\top$, the $\overline{\partial}^\alpha$-differentiated kinematic boundary condition then reads

$$\partial_t\overline{\partial}^\alpha\psi + (\overline{v}\cdot\overline{\nabla})\overline{\partial}^\alpha\widetilde{\psi} - \mathbf{V}\cdot\widetilde{N} = \mathcal{S}_1 \quad \text{on } \Sigma, \quad \text{and } \mathbf{V}_3 = 0 \quad \text{on } \Sigma_b, \tag{4.47}$$

where

$$\mathcal{S}_1 := \partial_3 v\cdot\widetilde{N}\overline{\partial}^\alpha\widetilde{\psi} + [\overline{\partial}^\alpha, v\cdot, N]. \tag{4.48}$$

Also, since $\check{q} = q + g\widetilde{\psi}$ and $\partial_3\widetilde{\varphi}|_\Sigma = 1$, we have $\mathbf{Q}|_\Sigma = (\overline{\partial}^\alpha\check{q} - \partial_3\check{q}\overline{\partial}^\alpha\widetilde{\psi})|_\Sigma = \overline{\partial}^\alpha q + g\overline{\partial}^\alpha\widetilde{\psi} - (\partial_3 q + g)\overline{\partial}^\alpha\widetilde{\psi} = \overline{\partial}^\alpha q - \partial_3 q\overline{\partial}^\alpha\widetilde{\psi}$, and thus the boundary condition of $\mathbf{Q}$ on Σ reads:

$$\mathbf{Q} = -\sigma\overline{\partial}^\alpha\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\widetilde{\psi}}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}}\right) + \kappa^2(1-\overline{\Delta})\overline{\partial}^\alpha(v\cdot\widetilde{N}) - \partial_3 q\overline{\partial}^\alpha\widetilde{\psi}. \tag{4.49}$$

4.5 Tangential energy estimate with full spatial derivatives

In this subsection, we study the spatially-differentiated equations, i.e., the equations obtained by commuting $\mathcal{T}^\alpha$, $\alpha_0 = 0$, and $|\alpha| = 4$, with (3.11). We aim to prove the following estimate:

Proposition 4.3. For $\mathcal{T}^\alpha$ with multi-index α satisfying $\alpha_0 = 0$ and $|\alpha| = 4$, we have

$$\|\mathcal{T}^\alpha v(T)\|_0^2 + \left\|\sqrt{\mathcal{F}'(q)}\mathcal{T}^\alpha \check{q}(T)\right\|_0^2 + \left|\sqrt{\sigma}\overline{\nabla}\overline{\partial}^\alpha \Lambda_\kappa \psi(T)\right|_0^2 + \int_0^T \left|\kappa\overline{\partial}^\alpha \partial_t \psi(t)\right|_1^2 \,\mathrm{d}t \lesssim \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t. \tag{4.50}$$

We will not directly consider the $\mathcal{T}^\alpha$-differentiated variables but use Alinhac good unknowns to get rid of higher order terms of $\widetilde{\psi}$. Invoking Lemma A.2 and Theorem A.3, testing (4.43) with $\mathbf{V}$ and then integrating over Ω with respect to the measure $\partial_3\widetilde{\varphi}\,\mathrm{d}x$, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|\mathbf{V}|^2\partial_3\widetilde{\varphi}\,\mathrm{d}x = \frac{1}{2}\int_\Omega \rho|\mathbf{V}|^2\partial_3\partial_t(\widetilde{\varphi}-\varphi)\,\mathrm{d}x + \int_\Omega \mathbf{Q}(\nabla^{\widetilde{\varphi}}\cdot\mathbf{V})\partial_3\widetilde{\varphi}\,\mathrm{d}x - \int_\Sigma \mathbf{Q}(\mathbf{V}\cdot\widetilde{N})\,\mathrm{d}x' + \int_\Omega \mathbf{V}\cdot\mathcal{R}^1\partial_3\widetilde{\varphi}\,\mathrm{d}x, \tag{4.51}$$

where the boundary integral on Σ_b vanishes thanks to $\mathbf{V}_3|_\Sigma = 0$. From now on, we will no longer write any boundary integral on Σ_b due to the same reason. Before estimating the integrals in (4.51), we record some important properties that Alinhac good unknowns enjoy.

Lemma 4.4. Let $\mathbf{F} := \mathcal{T}^\alpha f - \partial_3^{\widetilde{\varphi}} f \mathcal{T}^\alpha \widetilde{\varphi}$ with $|\alpha| = 4$ and $\alpha_0 = 0$ be the Alinhac good unknowns associated with the smooth function f. Suppose that $\partial_3\widetilde{\varphi} \geq c_0 > 0$, then

$$\|\mathcal{T}^\alpha f\|_0 \leq \|\mathbf{F}\|_0 + P\left(c_0^{-1}, |\widetilde{\psi}|_4\right)\|\partial_3 f\|_\infty. \tag{4.52}$$

Furthermore, let $\mathfrak{C}(f)$, $\mathfrak{D}(f)$, and $\mathfrak{E}(f)$ be the remainder terms defined respectively in (4.36), (4.40), and (4.41). Then

$$\|\mathfrak{C}_i(f)\|_0 \leq P\left(c_0^{-1}, |\widetilde{\psi}|_4\right)\cdot\|f\|_4, \quad i = 1,2,3, \tag{4.53}$$

$$\|\mathfrak{D}(f)\|_0 \leq P\left(c_0^{-1}, |\widetilde{\psi}|_4, |\partial_t\widetilde{\psi}|_3\right)\cdot(\|f\|_4 + \|\partial_t f\|_3), \tag{4.54}$$

$$\|\mathfrak{E}(f)\|_0 \leq \kappa|\overline{\nabla}\mathcal{T}^\alpha\partial_t\psi|_0\|\partial f\|_\infty. \tag{4.55}$$

Proof. Since $\partial_3^{\widetilde{\varphi}} = (\partial_3\widetilde{\varphi})^{-1}\partial_3$, we have

$$\|\partial_3^{\widetilde{\varphi}} f\|_\infty\|\mathcal{T}^\alpha\widetilde{\varphi}\|_0 \leq P\left(c_0^{-1}, |\widetilde{\psi}|_4\right)\|\partial_3 f\|_\infty, \tag{4.56}$$

and so (4.52) follows from (4.37). Also, the estimates (4.53) and (4.54) follow from the definition of $\mathfrak{C}(f)$ and $\mathfrak{D}(f)$, (1.9), (3.7) in Lemma 3.1, and the Sobolev inequalities. To establish (4.55), we notice that

$$\|\mathfrak{E}(f)\|_0 \leq \|\partial_t\mathcal{T}^\alpha(\widetilde{\varphi}-\varphi)\|_0\|\partial_3^{\widetilde{\varphi}} f\|_\infty + \|\partial_3^{\widetilde{\varphi}}\mathcal{T}^\alpha\widetilde{\varphi}\|_0\|\partial_t(\widetilde{\varphi}-\varphi)\|_\infty\|\partial_3^{\widetilde{\varphi}} f\|_\infty.$$

Thus, it suffices to control the leading order terms $\|\partial_t\mathcal{T}^\alpha(\widetilde{\varphi}-\varphi)\|_0$ and $\|\partial_3^{\widetilde{\varphi}}\mathcal{T}^\alpha\widetilde{\varphi}\|_0$. We have

$$\begin{aligned}\partial_t\mathcal{T}^\alpha(\widetilde{\varphi}-\varphi) &= \partial_t\mathcal{T}^\alpha\left(\chi(x_3)\widetilde{\psi} - \chi(x_3)\psi\right)\\ &\leq \chi(x_3)\partial_t\overline{\partial}^\alpha(\widetilde{\psi}-\psi) + [\mathcal{T}^\alpha, \chi(x_3)]\,\partial_t(\widetilde{\psi}-\psi).\end{aligned}$$

The L^2-norm of the second term can be controlled by the RHS of (4.55) thanks to (1.9). By (3.6) in Lemma 3.1, we have

$$|\partial_t\overline{\partial}^\alpha(\widetilde{\psi}-\psi)|_0 \leq \kappa|\partial_t\psi|_5.$$

Also,

$$\partial_3^{\widetilde{\varphi}}\mathcal{T}^\alpha\widetilde{\varphi} = \partial_3^{\widetilde{\varphi}}\mathcal{T}^\alpha\left(\chi(x_3)\widetilde{\psi}\right) = \left(\partial_3^{\widetilde{\varphi}}\chi(x_3)\right)\mathcal{T}^\alpha\widetilde{\psi} + \left(\partial_3^{\widetilde{\varphi}}[\mathcal{T}^\alpha, \chi(x_3)]\right)\widetilde{\psi},$$

and so $\|\partial_3^{\widetilde{\varphi}}\mathcal{T}^\alpha\widetilde{\varphi}\|_0$ can be controlled by the RHS of (4.55). □

Remark 4.2. The appearance of $\mathfrak{E}(f)$ is a consequence of the tangential smoothing. This estimate of $\|\mathfrak{E}(f)\|_0$ yields a top order term $\kappa|\partial_t\psi|_5$, which can only be controlled by the energy contributed by the artificial viscosity. In other words, the artificial viscosity compensates for the loss of symmetry in the κ-equations.

4.5.1 Control of $\int_\Omega \rho|\mathbf{V}|^2\partial_3\partial_t(\widetilde{\varphi}-\varphi)\,\mathrm{d}x$: The integral contains the mismatched term.

We have

$$\int_\Omega \rho|\mathbf{V}|^2\partial_3\partial_t(\widetilde{\varphi}-\varphi)\,\mathrm{d}x \le \|\rho\|_\infty\|\mathbf{V}\|_0^2\|\partial_3\partial_t(\widetilde{\varphi}-\varphi)\|_\infty \lesssim \sqrt{\kappa}\|\mathbf{V}\|_0^2|\overline{\partial}\partial_t\psi|_{0.5}. \tag{4.57}$$

4.5.2 Control of $\int_\Omega \mathbf{V}\cdot\mathcal{R}^1\partial_3\widetilde{\varphi}\,\mathrm{d}x$: Error terms

We have

$$\int_\Omega \mathbf{V}\cdot\mathcal{R}^1\partial_3\widetilde{\varphi}\,\mathrm{d}x \le \|\mathbf{V}\|_0\|\mathcal{R}^1\|_0\|\partial_3\widetilde{\varphi}\|_\infty, \tag{4.58}$$

where the L^2-norm of $\mathcal{R}^1$ is directly controlled by using (4.45) and (4.53)–(4.55):

$$\|\mathcal{R}^1\|_0 \le P(\|\partial_3\widetilde{\varphi}\|_\infty, |\widetilde{\psi}|_4, |\partial_t\widetilde{\psi}|_3)\left(\kappa|\overline{\nabla}\mathcal{T}^\alpha\partial_t\psi|_0\|v\|_4 + \|v\|_4 + \|\partial_t v\|_3 + \|\check{q}\|_4\right), \tag{4.59}$$

where the term containing $\kappa|\overline{\nabla}\mathcal{T}^\alpha\partial_t\psi|_0$ should be controlled under time integral as we will get $L_t^2H_{x'}^1([0,T]\times\Sigma)$ bound for $\kappa\partial_t\mathcal{T}^\alpha\psi$ later.

4.5.3 Control of $\int_\Omega \mathbf{Q}(\nabla^{\widetilde{\varphi}}\cdot\mathbf{V})\partial_3\widetilde{\varphi}\,dx$: Tangential energy for Q

Equation (4.44) indicates that

$$\int_\Omega \mathbf{Q}(\nabla^{\widetilde{\varphi}}\cdot\mathbf{V})\partial_3\widetilde{\varphi}\,\mathrm{d}x = -\int_\Omega \mathcal{F}'(q)\mathbf{Q}(D_t^{\widetilde{\varphi}}\mathbf{Q})\partial_3\widetilde{\varphi}\,\mathrm{d}x + \int_\Omega \mathbf{Q}(\mathcal{R}^2 - \mathfrak{C}_i(v^i))\partial_3\widetilde{\varphi}\,\mathrm{d}x. \tag{4.60}$$

For the second term on the RHS of (4.60), we invoke the second inequality in (1.30) and then apply it to the definition of $\mathcal{R}^2$ in (4.46) to get:

$$\int_\Omega \mathbf{Q}\mathcal{R}^2\partial_3\widetilde{\varphi}\,\mathrm{d}x \le \|\sqrt{\mathcal{F}'(q)}\mathbf{Q}\|_0\|\mathcal{R}^2\|_0\|\partial_3\widetilde{\varphi}\|_\infty. \tag{4.61}$$

In other words, we "borrow" one $\sqrt{\mathcal{F}'(q)}$ from $\mathcal{R}^2$ and attach it to $\mathbf{Q}$. Thanks to (4.53)-(4.55), we control the L^2-norm of the rest of terms in $\mathcal{R}^2$ directly by

$$P(\|\partial_3\widetilde{\varphi}\|_\infty, |\widetilde{\psi}|_4, |\partial_t\widetilde{\psi}|_3)\left(\kappa|\overline{\nabla}\mathcal{T}^\alpha\partial_t\psi|_0\left\|\sqrt{\mathcal{F}'(q)}\check{q}\right\|_4 + \left\|\sqrt{\mathcal{F}'(q)}\check{q}\right\|_4 + \left\|\sqrt{\mathcal{F}'(q)}\partial_t\check{q}\right\|_3 + \left\|\sqrt{\mathcal{F}'(q)}gv_3\right\|_3\right), \tag{4.62}$$

where the term containing $\kappa|\overline{\nabla}\mathcal{T}^\alpha\partial_t\psi|_0$ should be controlled under time integral as we will get $L_t^2H_{x'}^1([0,T]\times\Sigma)$ bound for $\kappa\partial_t\mathcal{T}^\alpha\psi$ later. Then the contribution of $\mathfrak{C}_i(v^i)$ is controlled by

$$-\int_\Omega \mathbf{Q}(\mathfrak{C}_i(v^i))\partial_3\widetilde{\varphi}\,\mathrm{d}x \le P(|\widetilde{\psi}|_4, |\overline{\nabla}\widetilde{\psi}|_{W^{1,\infty}})|\mathcal{T}^\alpha\widetilde{\psi}|_0\|v\|_4\|\mathbf{Q}\|_0. \tag{4.63}$$

Here, $\|\mathbf{Q}\|_0$ contributes to $\|\overline{\partial}^\alpha\check{q}\|_0$ and $\|\partial_3^{\widetilde{\varphi}}\check{q}\overline{\partial}^\alpha\widetilde{\psi}\|_0$. The first term $\|\overline{\partial}^\alpha\check{q}\|_0$ is not weighted by $\sqrt{\mathcal{F}'(q)}$ and thus cannot be controlled directly by (4.50). Fortunately, we can overcome this issue by invoking (4.17). Similarly, $\|\partial_3^{\widetilde{\varphi}}\check{q}\overline{\partial}^\alpha\widetilde{\psi}\|_0 \le \|\partial_3^{\widetilde{\varphi}}\check{q}\|_\infty\|\overline{\partial}^\alpha\widetilde{\psi}\|_0$, where we use (4.20) to treat $\|\partial_3^{\widetilde{\varphi}}\check{q}\|_\infty$, and so this can be controlled uniformly as $\mathcal{F}'(q)\to 0$.

Furthermore, invoking the integration by parts formula (A.8), the first integral on the RHS of (4.60) becomes

$$\begin{aligned}\int_\Omega \mathcal{F}'(q)\mathbf{Q}(D_t^{\widetilde{\varphi}}\mathbf{Q})\partial_3\widetilde{\varphi}\,\mathrm{d}x =& -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathcal{F}'(q)|\mathbf{Q}|^2\partial_3\widetilde{\varphi}\,\mathrm{d}x + \frac{1}{2}\int_\Omega (D_t^{\widetilde{\varphi}}\mathcal{F}'(q))|\mathbf{Q}|^2\partial_3\widetilde{\varphi}\,\mathrm{d}x \\ &+ \frac{1}{2}\int_\Omega (\nabla^{\widetilde{\varphi}}\cdot v)\mathcal{F}'(q)|\mathbf{Q}|^2\partial_3\widetilde{\varphi}\,\mathrm{d}x + \frac{1}{2}\int_\Omega \mathcal{F}'(q)|\mathbf{Q}|^2\partial_3\partial_t(\widetilde{\varphi}-\varphi)\partial_3\widetilde{\varphi}\,\mathrm{d}x \\ \lesssim& -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\sqrt{\mathcal{F}'(q)}\mathbf{Q}\right\|_0^2 + \|\partial_3\widetilde{\varphi}\|_\infty\|\sqrt{\mathcal{F}'(q)}\mathbf{Q}\|_0^2\left(\|\partial v\|_\infty + \kappa|\overline{\nabla}\partial_t\psi|_{0.5}\right).\end{aligned} \tag{4.64}$$

4.5.4 Control of $-\int_\Sigma \mathbf{Q}(\mathbf{V}\cdot\widetilde{N})\,\mathrm{d}x'$: Boundary energy contributed by surface tension and artificial viscosity

Note that $\mathcal{T}_3=\vec{0}$ on Σ implies the corresponding good unknown $\mathbf{Q}=0$ on Σ, so it suffices to consider the case $\mathcal{T}^\alpha=\overline{\partial}^\alpha$ when analyzing the boundary integral. Using (4.47), we have

$$-\int_\Sigma \mathbf{Q}(\mathbf{V}\cdot\widetilde{N})\,\mathrm{d}x'=-\int_\Sigma \mathbf{Q}\left(\partial_t\overline{\partial}^\alpha\psi+(\overline{v}\cdot\overline{\nabla})\overline{\partial}^\alpha\widetilde{\psi}-\mathcal{S}_1\right)\mathrm{d}x'. \tag{4.65}$$

The first term is expected to contribute to two coercive terms if we invoke the boundary condition (4.49) of $\mathbf{Q}$:

$$\begin{aligned}I_1:=-\int_\Sigma \mathbf{Q}\partial_t\overline{\partial}^\alpha\psi\,\mathrm{d}x'&=\sigma\int_\Sigma\overline{\partial}^\alpha\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\widetilde{\psi}}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}}\right)\partial_t\overline{\partial}^\alpha\psi\,\mathrm{d}x'-\kappa^2\int_\Sigma\overline{\partial}^\alpha(1-\overline{\Delta})\partial_t\psi\cdot\overline{\partial}^\alpha\partial_t\psi\,\mathrm{d}x'+\int_\Sigma\partial_3 q\overline{\partial}^\alpha\widetilde{\psi}\partial_t\overline{\partial}^\alpha\psi\,\mathrm{d}x'\\&=:\ \mathrm{ST}_1+\ \mathrm{ST}_2+\ \mathrm{RT}.\end{aligned} \tag{4.66}$$

Since $1-\overline{\Delta}=\langle\overline{\partial}\rangle^2$, where $\langle\cdot\rangle$ denotes the Japanese bracket, we find the term ST_2 gives us $\sqrt{\kappa}$-weighted enhanced energy after integration by parts :

$$\mathrm{ST}_2=-\kappa^2\int_\Sigma\left|\overline{\partial}^\alpha\langle\overline{\partial}\rangle\partial_t\psi\right|^2\,\mathrm{d}x'=-\frac{\mathrm{d}}{\mathrm{d}t}\left|\kappa\overline{\partial}^\alpha\partial_t\psi\right|^2_{L^2_tH^1_{x'}}. \tag{4.67}$$

In the control of ST_1, we will repeatedly use

$$\overline{\partial}\left(\frac{1}{|\widetilde{N}|}\right)=\frac{\overline{\nabla}\widetilde{\psi}\cdot\overline{\partial}\overline{\nabla}\widetilde{\psi}}{|\widetilde{N}|^3}, \tag{4.68}$$

where $|\widetilde{N}|=\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}$ denotes the length of the smoothed normal vector $\widetilde{N}=(-\overline{\partial}_1\widetilde{\psi},-\overline{\partial}_2\widetilde{\psi},1)^\top$. Now we integrate $\overline{\nabla}\cdot$ by parts in ST_1 to get

$$\begin{aligned}\mathrm{ST}_1=&-\sigma\int_\Sigma\frac{\overline{\partial}^\alpha\overline{\nabla}\widetilde{\psi}}{|\widetilde{N}|}\cdot\partial_t\overline{\partial}^\alpha\overline{\nabla}\psi\,\mathrm{d}x'+\sigma\int_\Sigma\frac{\overline{\nabla}\widetilde{\psi}\cdot\overline{\partial}^\alpha\overline{\nabla}\widetilde{\psi}}{|\widetilde{N}|^3}\overline{\nabla}\widetilde{\psi}\cdot\partial_t\overline{\partial}^\alpha\overline{\nabla}\psi\,\mathrm{d}x'\\&-\sigma\int_\Sigma\left(\left[\overline{\partial}^{\alpha-\alpha'},\frac{1}{|\widetilde{N}|}\right]\overline{\partial}^{\alpha'}\overline{\nabla}\widetilde{\psi}+\overline{\nabla}\widetilde{\psi}\left[\overline{\partial}^{\alpha-\alpha'},\frac{1}{|\widetilde{N}|^3}\right](\overline{\nabla}\widetilde{\psi}\cdot\overline{\partial}^{\alpha'}\overline{\nabla}\widetilde{\psi})-\frac{1}{|\widetilde{N}|^3}\left[\overline{\partial}^{\alpha-\alpha'},\overline{\nabla}\widetilde{\psi}\right]\overline{\partial}^{\alpha'}\overline{\nabla}\widetilde{\psi}\right)\cdot\partial_t\overline{\nabla}\overline{\partial}^\alpha\psi\,\mathrm{d}x'\\=:&\ \mathrm{ST}_{11}+\ \mathrm{ST}_{12}+\ \mathrm{ST}_{13},\end{aligned} \tag{4.69}$$

where α' is a multi-index with $|\alpha'|=1$.

The first two terms in (4.69) are expected to produce the energy contributed by the surface tension. Before that, we need to move one mollifier from the top order term of $\widetilde{\psi}=\Lambda_\kappa^2\psi$ to the top order term of ψ by using the self-adjointness of Λ_κ in $L^2(\Sigma)$:

$$\begin{aligned}\mathrm{ST}_{11}+\ \mathrm{ST}_{12}=&-\sigma\int_\Sigma\frac{\overline{\partial}^\alpha\overline{\nabla}\Lambda_\kappa\psi\cdot\partial_t\overline{\partial}^\alpha\overline{\nabla}\Lambda_\kappa\psi}{|\widetilde{N}|}-\frac{(\overline{\nabla}\widetilde{\psi}\cdot\overline{\partial}^\alpha\overline{\nabla}\Lambda_\kappa\psi)(\overline{\nabla}\widetilde{\psi}\cdot\partial_t\overline{\partial}^\alpha\overline{\nabla}\Lambda_\kappa\psi)}{|\widetilde{N}|^3}\,\mathrm{d}x'\\&-\sigma\int_\Sigma\overline{\partial}^\alpha\overline{\nabla}\Lambda_\kappa\psi\cdot\left(\left[\Lambda_\kappa,\frac{1}{|\widetilde{N}|}\right]\overline{\nabla}\partial_t\overline{\partial}^\alpha\psi\right)\mathrm{d}x'+\sigma\int_\Sigma\overline{\partial}^\alpha\overline{\nabla}_i\Lambda_\kappa\psi\cdot\left(\left[\Lambda_\kappa,\frac{\overline{\nabla}_i\widetilde{\psi}\overline{\nabla}_j\widetilde{\psi}}{|\widetilde{N}|^3}\right]\overline{\nabla}_j\partial_t\overline{\partial}^\alpha\psi\right)\mathrm{d}x'\\=:&\ \mathrm{ST}_{10}+\ \mathrm{ST}_{11}^R+\ \mathrm{ST}_{12}^R.\end{aligned} \tag{4.70}$$

Then we find

$$\mathrm{ST}_{10}=-\frac{\sigma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma\frac{|\overline{\partial}^\alpha\overline{\nabla}\Lambda_\kappa\psi|^2}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}}-\frac{|\overline{\nabla}\widetilde{\psi}\cdot\overline{\partial}^\alpha\overline{\nabla}\Lambda_\kappa\psi|^2}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}^3}\,\mathrm{d}x' \tag{4.71}$$

$$+\frac{\sigma}{2}\int_\Sigma\partial_t\left(\frac{1}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}}\right)\left|\overline{\partial}^\alpha\overline{\nabla}\Lambda_\kappa\psi\right|^2-\partial_t\left(\frac{1}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}^3}\right)\left|\overline{\nabla}\widetilde{\psi}\cdot\overline{\partial}^\alpha\overline{\nabla}\Lambda_\kappa\psi\right|^2\,\mathrm{d}x'. \tag{4.72}$$

To deal with the first term in ST_{10}, we plug $\mathbf{a} = \overline{\partial}^\alpha\overline{\nabla}\Lambda_\kappa\psi$ into the following inequality which can be proved by direct calculation:

$$\frac{|\mathbf{a}|^2}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}} - \frac{|\overline{\nabla}\widetilde{\psi}\cdot\mathbf{a}|^2}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}^3} \geq \frac{|\mathbf{a}|^2}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}^3}, \tag{4.73}$$

in order to get

$$\int_0^T \mathrm{ST}_{10}\,\mathrm{d}t + \frac{\sigma}{2}\int_\Sigma \frac{|\overline{\partial}^\alpha\overline{\nabla}\Lambda_\kappa\psi|^2}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}^3}\,\mathrm{d}x' \leq P(|\overline{\nabla}\widetilde{\psi}_0|_{L^\infty})\left|\sqrt{\sigma}\overline{\partial}^\alpha\overline{\nabla}\Lambda_\kappa\psi_0\right|_0^2 + \int_0^T (4.72)\,\mathrm{d}t, \tag{4.74}$$

where the terms in (4.72) can be controlled directly:

$$(4.72) \leq P(|\overline{\nabla}\widetilde{\psi}|_{L^\infty})|\partial_t\overline{\nabla}\widetilde{\psi}|_{L^\infty}\left|\sqrt{\sigma}\overline{\partial}^\alpha\overline{\nabla}\Lambda_\kappa\psi\right|_0^2. \tag{4.75}$$

To finish the control of ST_1, it remains to control ST_{13} and ST_{11}^R, ST_{12}^R. The last two terms can be controlled by using the mollifier property (3.10) and the κ-weighted energy contributed by the artificial viscosity. For ST_{11}^R, we have

$$\begin{aligned}\int_0^T \mathrm{ST}_{11}^R &\lesssim \int_0^T \left|\sqrt{\sigma}\overline{\partial}^\alpha\Lambda_\kappa\psi\right|_0 P\left(|\overline{\nabla}\widetilde{\psi}|_\infty\right)|\overline{\nabla}\widetilde{\psi}|_{W^{1,\infty}}\left|\kappa\partial_t\overline{\partial}^\alpha\psi\right|_{\dot{H}^1}\,\mathrm{d}t\\ &\lesssim \varepsilon\left|\kappa\partial_t\overline{\partial}^\alpha\psi\right|^2_{L_t^2H_{x'}^1} + \int_0^T P\left(|\overline{\nabla}\widetilde{\psi}|_\infty\right)|\overline{\nabla}\widetilde{\psi}|^2_{W^{1,\infty}}\left|\sqrt{\sigma}\overline{\partial}^\alpha\Lambda_\kappa\psi\right|_0^2\,\mathrm{d}t.\end{aligned} \tag{4.76}$$

Also, ST_{12}^R can be controlled similarly.

As for ST_{13} in (4.69), we find that all three commutators have similar structures and the same leading order terms, so we only show the analysis of the first commutator. Note that the leading order term in $[\overline{\partial}^{\alpha-\alpha'}, |\widetilde{N}|^{-1}]\overline{\partial}^{\alpha'}\overline{\nabla}\widetilde{\psi}$ appears when $\overline{\partial}^{\alpha-\alpha'}$ falls on $|\widetilde{N}|^{-1}$ or $\overline{\partial}^{\alpha''}$ falls on $|\widetilde{N}|^{-1}$ and $\overline{\partial}^{\alpha-\alpha'-\alpha''}$ falls on $\overline{\partial}^{\alpha'}\overline{\nabla}\widetilde{\psi}$ for some $|\alpha''|=1$. In either of the two cases, the top-order term contributes to the following integral:

$$-\sigma\int_\Sigma |\widetilde{N}|^{-3}\overline{\nabla}\widetilde{\psi}\,\overline{\partial}^{\alpha-\alpha'}\overline{\nabla}\widetilde{\psi}\,\overline{\partial}^{\alpha'}\overline{\nabla}\widetilde{\psi}\cdot\partial_t\overline{\nabla}\overline{\partial}^\alpha\psi\,\mathrm{d}x'. \tag{4.77}$$

We integrate one $\overline{\nabla}$ by parts to get

$$\sigma\int_\Sigma |\widetilde{N}|^{-3}\overline{\nabla}\widetilde{\psi}\,\overline{\partial}^{\alpha-\alpha'}\overline{\nabla}^2\widetilde{\psi}\,\overline{\partial}^{\alpha'}\overline{\nabla}\widetilde{\psi}\,\partial_t\overline{\partial}^\alpha\psi\,\mathrm{d}x'$$

modulo lower order terms, and then we move one Λ_κ from $\overline{\partial}^{\alpha-\alpha'}\overline{\nabla}^2\widetilde{\psi}$ to $\partial_t\overline{\partial}^\alpha\psi$ such that the main term is directly controlled as:

$$\sigma\int_\Sigma |\widetilde{N}|^{-3}\overline{\nabla}\widetilde{\psi}\,\overline{\partial}^{\alpha-\alpha'}\overline{\nabla}^2\Lambda_\kappa\psi\,\overline{\partial}^{\alpha'}\overline{\nabla}\widetilde{\psi}\,\partial_t\overline{\partial}^\alpha\Lambda_\kappa\psi\,\mathrm{d}x' \lesssim P(|\overline{\nabla}\widetilde{\psi}|_\infty)|\overline{\nabla}\widetilde{\psi}|_{W^{1,\infty}}\left|\sqrt{\sigma}\overline{\partial}^\alpha\overline{\nabla}\Lambda_\kappa\psi\right|_0\left|\sqrt{\sigma}\partial_t\overline{\partial}^\alpha\Lambda_\kappa\psi\right|_0, \tag{4.78}$$

where the last term will be controlled in $\partial_t\overline{\partial}^3$-estimates. Besides, we have to analyze the commutator involving Λ_κ:

$$\sigma\int_\Sigma \overline{\partial}^{\alpha-\alpha'}\overline{\nabla}^2\Lambda_\kappa\psi\left(\left[\Lambda_\kappa, P(\overline{\nabla}\widetilde{\psi})\overline{\partial}^{\alpha'}\overline{\nabla}\widetilde{\psi}\right]\partial_t\overline{\partial}^\alpha\psi\right)\mathrm{d}x', \tag{4.79}$$

which is controlled under the time integral:

$$\begin{aligned}\int_0^T (4.79)\,\mathrm{d}t &\lesssim \sqrt{\sigma}\int_0^T |\sqrt{\sigma}\overline{\nabla}\overline{\partial}^\alpha\Lambda_\kappa\psi|_0\cdot\kappa|\overline{\partial}\overline{\nabla}\widetilde{\psi}|_{W^{1,\infty}}P(|\overline{\nabla}\widetilde{\psi}|_{W^{1,\infty}})|\partial_t\overline{\partial}^\alpha\psi|_0\,\mathrm{d}t\\ &\lesssim \varepsilon\left|\kappa\partial_t\overline{\partial}^\alpha\psi\right|^2_{L_t^2L_{x'}^2} + \int_0^T\left|\sqrt{\sigma}\overline{\nabla}\overline{\partial}^\alpha\Lambda_\kappa\psi\right|_0^2|\sqrt{\sigma}\overline{\nabla}\widetilde{\psi}|^2_{3.5}P(|\overline{\nabla}\widetilde{\psi}|_{2.5})\,\mathrm{d}t.\end{aligned} \tag{4.80}$$

Therefore,

$$\int_0^T(\mathrm{ST}_1 + \mathrm{ST}_2)\,\mathrm{d}t + \left|\kappa\overline{\partial}^\alpha\partial_t\psi\right|^2_{L_t^2H_{x'}^1} + \frac{\sigma}{2}\left|\overline{\nabla}\overline{\partial}^\alpha\Lambda_\kappa\psi(T)\right|_0^2 \lesssim \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t, \tag{4.81}$$

where we have chosen $\varepsilon > 0$ that appears above to be suitably small such that all ε-terms are absorbed by the κ-weighted energy.

To finish the control of I_1 defined in (4.66), it remains to control the term RT. Note that when we drop the mollifier and have the Rayleigh-Taylor sign condition $-\partial_3 q \geq \frac{c_0}{2} > 0$ assumed on Σ, RT should directly give us the non-σ-weighted boundary energy. But since we are now solving the gravity-capillary water wave system for fixed $\sigma > 0$ instead of taking vanishing surface tension limit, we cannot assume $-\partial_3 q \geq \frac{c_0}{2} > 0$ on Σ. Thus this term is controlled by the surface tension energy after moving one Λ_κ:

$$\begin{aligned}\int_0^T \mathrm{RT}\,\mathrm{d}t =& -\int_0^T \int_\Sigma \partial_3 q \overline{\partial}^\alpha \Lambda_\kappa \psi \cdot \partial_t \overline{\partial}^\alpha \Lambda_\kappa \psi \,\mathrm{d}x'\,\mathrm{d}t - \int_0^T \int_\Sigma \overline{\partial}^\alpha \Lambda_\kappa \psi \cdot \left([\Lambda_\kappa, \partial_3 q]\, \partial_t \overline{\partial}^\alpha \psi \right) \mathrm{d}x'\,\mathrm{d}t \\ \lesssim& \int_0^T |\partial q|_{L^\infty} \left| \overline{\partial}^\alpha \Lambda_\kappa \psi \right|_0 \left| \partial_t \overline{\partial}^\alpha \Lambda_\kappa \psi \right|_0 \mathrm{d}t + \varepsilon \left| \kappa \partial_t \overline{\partial}^\alpha \psi \right|^2_{L^2_t L^2_{x'}} + \int_0^T |\partial q|^2_{W^{1,\infty}} \left| \overline{\partial}^\alpha \Lambda_\kappa \psi \right|_0^2 \mathrm{d}t \\ \lesssim& \,\varepsilon \left| \kappa \partial_t \overline{\partial}^\alpha \psi \right|^2_{L^2_t L^2_{x'}} + \int_0^T P\left(\|\check{q}\|_4, \left| \overline{\partial}^\alpha \Lambda_\kappa \psi \right|_0, \left| \partial_t \overline{\partial}^\alpha \Lambda_\kappa \psi \right|_0 \right) \mathrm{d}t,\end{aligned} \tag{4.82}$$

where the term $\left|\partial_t \overline{\partial}^\alpha \Lambda_\kappa \psi\right|_0$ is the energy term obtained in $\overline{\partial}^{\alpha-\alpha'} \partial_t$-estimates for $|\alpha'| = 1$.

Remark 4.3. The RHS of (4.82) is not uniform in σ. However, as mentioned earlier, $-\int_0^T \int_\Sigma \partial_3 q \overline{\partial}^\alpha \Lambda_\kappa \psi \cdot \partial_t \overline{\partial}^\alpha \Lambda_\kappa \psi \,\mathrm{d}x'\,\mathrm{d}t$ contributes to a non-σ-weighted energy term $\int_\Sigma (-\partial_3 q) |\overline{\partial}^\alpha \Lambda_\kappa \psi|^2 \,\mathrm{d}t$ provided the Rayleigh-Taylor sign condition holds. We shall revisit the control of RT in Section 7, where the zero surface tension limit is considered.

Combining this with (4.81), we get the estimate for I_1

$$\int_0^T I_1 \,\mathrm{d}t + \left| \kappa \overline{\partial}^\alpha \partial_t \psi \right|^2_{L^2_t H^1_{x'}} + \frac{\sigma}{2} \left| \overline{\nabla} \overline{\partial}^\alpha \Lambda_\kappa \psi(T) \right|_0^2 \lesssim \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t, \tag{4.83}$$

after choosing $\varepsilon > 0$ that appears above to be suitably small.

The second term in (4.65) gives

$$\begin{aligned} I_2 :=& -\int_\Sigma \mathbf{Q} (\overline{v} \cdot \overline{\nabla}) \overline{\partial}^\alpha \widetilde{\psi} = \sigma \int_\Sigma \overline{\partial}^\alpha \overline{\nabla} \cdot \left(\frac{\overline{\nabla} \widetilde{\psi}}{\sqrt{1 + |\overline{\nabla} \widetilde{\psi}|^2}} \right) (\overline{v} \cdot \overline{\nabla}) \overline{\partial}^\alpha \widetilde{\psi} \,\mathrm{d}x' - \kappa^2 \int_\Sigma \overline{\partial}^\alpha (1 - \overline{\Delta}) \partial_t \psi \cdot (\overline{v} \cdot \overline{\nabla}) \overline{\partial}^\alpha \widetilde{\psi} \,\mathrm{d}x' \\ &+ \int_\Sigma \partial_3 q \overline{\partial}^\alpha \widetilde{\psi} (\overline{v} \cdot \overline{\nabla}) \overline{\partial}^\alpha \widetilde{\psi} \,\mathrm{d}x' \\ =:& \, I_{21} + I_{22} + I_{23}, \end{aligned} \tag{4.84}$$

where we find that I_{22}, I_{23} can be directly controlled as follows:

$$\begin{aligned} \int_0^T I_{22}\,\mathrm{d}t \overset{\overline{\nabla}}{=}& -\kappa^2 \int_0^T \int_\Sigma \overline{\partial}^\alpha \overline{\nabla} \partial_t \psi \cdot \overline{\nabla} \left((\overline{v} \cdot \overline{\nabla}) \overline{\partial}^\alpha \widetilde{\psi} \right) \mathrm{d}x'\,\mathrm{d}t - \kappa^2 \int_0^T \int_\Sigma \overline{\partial}^\alpha \partial_t \psi \cdot (\overline{v} \cdot \overline{\nabla}) \overline{\partial}^\alpha \widetilde{\psi} \,\mathrm{d}x'\,\mathrm{d}t \\ \lesssim& \int_0^T \left| \kappa \overline{\partial}^\alpha \partial_t \psi \right|_0 |\overline{\nabla} \overline{v}|_\infty \left| \kappa \overline{\nabla}^2 \overline{\partial}^\alpha \widetilde{\psi} \right|_0 \mathrm{d}t + \kappa \int_0^T \left| \kappa \overline{\partial}^\alpha \partial_t \psi \right|_0 |\overline{v}|_\infty \left| \overline{\nabla} \overline{\partial}^\alpha \widetilde{\psi} \right|_0 \mathrm{d}t \\ \lesssim& \,\varepsilon \left| \kappa \overline{\partial}^\alpha \partial_t \psi \right|^2_{L^2_t H^1_{x'}} + \int_0^T |\overline{v}|^2_{W^{1,\infty}} \left| \overline{\nabla} \overline{\partial}^\alpha \Lambda_\kappa \psi \right|_0^2 \mathrm{d}t \lesssim \varepsilon \left| \kappa \overline{\partial}^\alpha \partial_t \psi \right|^2_{L^2_t H^1_{x'}} + \int_0^T P(E^\kappa(t))\,\mathrm{d}t, \end{aligned} \tag{4.85}$$

where we use the mollifier property (3.4) to control $|\kappa \overline{\nabla}^2 \overline{\partial}^\alpha \widetilde{\psi}|_0 \lesssim \kappa \cdot \kappa^{-1} |\overline{\nabla} \overline{\partial}^\alpha \Lambda_\kappa \psi|_0$. This step also shows why the power of κ must be 2 in the artificial viscosity, otherwise, the control of I_{22} is not uniform in κ. For I_{23} we integrate $\overline{v} \cdot \overline{\nabla}$ by parts to get

$$I_{23} = \frac{1}{2} \int_\Sigma \overline{\nabla} \cdot (\overline{v}\, \partial_3 q) |\overline{\partial}^\alpha \widetilde{\psi}|^2 \,\mathrm{d}x' \lesssim P(E^\kappa(t)). \tag{4.86}$$

The control of I_{21} is analogous to ST_1. Following (4.69), we have

$$\begin{aligned} I_{21} =& -\sigma \int_\Sigma \left(\frac{\overline{\partial}^\alpha \overline{\nabla} \widetilde{\psi}}{|\widetilde{N}|} - \frac{\overline{\nabla} \widetilde{\psi} \cdot \overline{\partial}^\alpha \overline{\nabla} \widetilde{\psi}}{|\widetilde{N}|^3} \overline{\nabla} \widetilde{\psi} \right) \cdot (\overline{v} \cdot \overline{\nabla}) \overline{\partial}^\alpha \overline{\nabla} \widetilde{\psi} \,\mathrm{d}x' \\ & -\sigma \int_\Sigma \left(\left[\overline{\partial}^{\alpha-\alpha'}, \frac{1}{|\widetilde{N}|} \right] \overline{\partial}^{\alpha'} \overline{\nabla} \widetilde{\psi} + \left[\overline{\partial}^{\alpha-\alpha'}, \frac{1}{|\widetilde{N}|^3} \right] (\overline{\nabla} \psi \cdot \overline{\partial}^{\alpha'} \overline{\nabla} \widetilde{\psi}) - \frac{1}{|\widetilde{N}|^3} \left[\overline{\partial}^{\alpha-\alpha'}, \overline{\nabla} \widetilde{\psi} \right] \overline{\partial}^{\alpha'} \overline{\nabla} \widetilde{\psi} \right) \cdot (\overline{v} \cdot \overline{\nabla}) \overline{\nabla} \overline{\partial}^\alpha \widetilde{\psi} \,\mathrm{d}x' \\ =:& \, I_{211} + I_{212}, \end{aligned} \tag{4.87}$$

where I_{212} can be directly controlled if we integrate $\overline{v}\cdot\overline{\nabla}$ by parts:

$$I_{212} \lesssim P(|\widetilde{\psi}|_4)|\overline{v}|_{W^{1,\infty}}\left|\sqrt{\sigma}\overline{\nabla}\overline{\partial}^{\alpha}\widetilde{\psi}\right|_0^2 \leq P(E^{\kappa}(t)). \tag{4.88}$$

For I_{211}, we integrate $\overline{v}\cdot\overline{\nabla}$ by parts and use the symmetric structure to see

$$I_{211} \overset{L}{=} -\frac{\sigma}{2}\int_{\Sigma}(\overline{\nabla}\cdot\overline{v})\left(\frac{|\overline{\partial}^{\alpha}\overline{\nabla}\widetilde{\psi}|^2}{|\widetilde{N}|} - \frac{|\overline{\nabla}\widetilde{\psi}\cdot\overline{\partial}^{\alpha}\overline{\nabla}\widetilde{\psi}|^2}{|\widetilde{N}|^3}\right)\mathrm{d}x' \lesssim P(|\overline{\nabla}\widetilde{\psi}|_{\infty})|\overline{v}|_{W^{1,\infty}}\left|\sqrt{\sigma}\overline{\nabla}\overline{\partial}^{\alpha}\widetilde{\psi}\right|_0^2. \tag{4.89}$$

Therefore, plugging (4.85)-(4.89) into (4.84), we get the estimates for I_2:

$$\int_0^T I_2\,\mathrm{d}t \lesssim \varepsilon\left|\kappa\overline{\partial}^{\alpha}\partial_t\psi\right|^2_{L^2_tH^1_{x'}} + \int_0^T P(E^{\kappa}(t)). \tag{4.90}$$

It remains to control the term involving $\mathcal{S}_1$ which reads

$$\begin{aligned}
I_3 :=& \int_{\Sigma}\mathbf{Q}\mathcal{S}_1\,\mathrm{d}x' = \int_{\Sigma}\mathbf{Q}\left(\partial_3 v\cdot\widetilde{N}\overline{\partial}^{\alpha}\widetilde{\psi} + \sum_{\substack{|\beta_1|+|\beta_2|=4\\|\beta_1|,|\beta_2|>0}}\overline{\partial}^{\beta_1}v\cdot\overline{\partial}^{\beta_2}\widetilde{N}\right)\mathrm{d}x'\\
=& \int_{\Sigma}\left(\sigma\overline{\partial}^{\alpha}\mathcal{H} + \kappa^2(1-\overline{\Delta})\overline{\partial}^{\alpha}\partial_t\psi - \partial_3 q\overline{\partial}^{\alpha}\widetilde{\psi}\right)\left(\partial_3 v\cdot\widetilde{N}\overline{\partial}^{\alpha}\widetilde{\psi} + \sum_{|\beta_1|=1,|\beta_2|=3}\overline{\partial}^{\beta_1}v\cdot\overline{\partial}^{\beta_2}\widetilde{N}\right)\mathrm{d}x'\\
&+ \sum_{\substack{|\beta_1|+|\beta_2|=4\\|\beta_1|\geq 1,1\leq|\beta_2|\leq 2}}\int_{\Sigma}(\overline{\partial}^{\alpha}q - \overline{\partial}^{\alpha}\widetilde{\psi}\partial_3 q)(\overline{\partial}^{\beta_1}v\cdot\overline{\partial}^{\beta_2}\widetilde{N})\,\mathrm{d}x'\\
=&:I_{31}+I_{32},
\end{aligned} \tag{4.91}$$

where we use the definition of $\mathbf{Q}$ in I_{32} and invoke the Dirichlet boundary condition (4.49) for $\mathbf{Q}$ in I_{31} such that the $L^2(\Sigma)$ bound of $\overline{\partial}^{\alpha}v$ and non-σ-weighted $\overline{\nabla}\overline{\partial}^{\alpha}\psi$ with $|\alpha|=4$ can be avoided on Σ.

The term I_{32} can be directly controlled as:

$$I_{32} \lesssim \sum_{\substack{|\beta_1|+|\beta_2|=4\\|\beta_1|\geq 1,1\leq|\beta_2|\leq 2}}|\overline{\partial}^{\alpha}q|_{-\frac12}\left|\overline{\partial}^{\beta_1}\overline{v}\cdot\overline{\nabla}^{\beta_2}\overline{\partial}\widetilde{\psi}\right|_{\frac12} + \left|\overline{\partial}^{\alpha}\widetilde{\psi}\partial_3 q\right|_0\left|\overline{\partial}^{\beta_1}\overline{v}\cdot\overline{\nabla}^{\beta_2}\overline{\partial}\widetilde{\psi}\right|_0 \lesssim \|q\|_4\|v\|_4|\widetilde{\psi}|_{3.5} + |\partial q|_{L^{\infty}}\|v\|_{3.5}|\widetilde{\psi}|_3|\widetilde{\psi}|_4. \tag{4.92}$$

For I_{31}, we invoke $\mathcal{H} = -\overline{\nabla}\cdot(\overline{\nabla}\widetilde{\psi}/|\widetilde{N}|)$ and then integrate $\overline{\nabla}\cdot$ by parts in the mean curvature term and integrate one tangential derivative by parts in the viscosity term to get:

$$I_{31} \lesssim P(|\overline{\nabla}\widetilde{\psi}|_{\infty})|\partial v|_{\infty}\left(\left|\sqrt{\sigma}\overline{\nabla}\overline{\partial}^{\alpha}\widetilde{\psi}\right|_4^2|\partial v|_{\infty} + \left|\kappa\overline{\partial}^{\alpha}\partial_t\psi\right|_1\left|\kappa\overline{\nabla}\overline{\partial}^{\alpha}\widetilde{\psi}\right|_0\right) + |\partial q|_{L^{\infty}}|\widetilde{\psi}|_4^2|\overline{\partial}v|_{\infty}, \tag{4.93}$$

and thus yields

$$\int_0^T I_{31}\,\mathrm{d}t \lesssim \varepsilon\left|\kappa\overline{\partial}^{\alpha}\partial_t\psi\right|^2_{L^2_tH^1_{x'}} + \int_0^T P(E^{\kappa}(t))\,\mathrm{d}t, \tag{4.94}$$

which together with (4.92) gives the bound for I_3:

$$\int_0^T I_3\,\mathrm{d}t \leq \varepsilon\left|\kappa\overline{\partial}^{\alpha}\partial_t\psi\right|^2_{L^2_tH^1_{x'}} + \int_0^T P(E^{\kappa}(t))\,\mathrm{d}t. \tag{4.95}$$

Combining (4.65), (4.66), (4.83), (4.84), (4.90), (4.91), (4.95), we get the estimates for the boundary integral after choosing $\varepsilon>0$ suitably small:

$$-\int_0^T\int_{\Sigma}\mathbf{Q}(\mathbf{V}\cdot\widetilde{N})\,\mathrm{d}x' + \left|\kappa\overline{\partial}^{\alpha}\partial_t\psi\right|^2_{L^2_tH^1_{x'}([0,T]\times\Sigma)} + \frac{\sigma}{2}\left|\overline{\nabla}\overline{\partial}^{\alpha}\Lambda_{\kappa}\psi(T)\right|_0^2 \lesssim \mathcal{P}_0^{\kappa} + \int_0^T P(E^{\kappa}(t))\,\mathrm{d}t. \tag{4.96}$$

Plugging the estimates (4.57)-(4.60), (4.64) and (4.96) into (4.51) and using $\rho \gtrsim 1, \partial_3\widetilde{\varphi} \gtrsim 1$, we get the estimates for the good unknowns:

$$\|\mathbf{V}(T)\|_0^2 + \left\| \sqrt{\mathcal{F}'(q)}\mathbf{Q}(T)\right\|_0^2 + \left| \sqrt{\sigma}\overline{\nabla}\overline{\partial}^\alpha \Lambda_\kappa \psi(T)\right|_0^2 + \left|\kappa\overline{\partial}^\alpha\partial_t\psi\right|^2_{L_t^2 H_{x'}^1([0,T]\times\Sigma)} \lesssim \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t. \tag{4.97}$$

Finally, using the definition $\mathbf{V} = \mathcal{T}^\alpha v - \mathcal{T}^\alpha\widetilde{\varphi}\partial_3^{\widetilde{\varphi}} v$, we can replace $\|\mathbf{V}\|_0$ by $\|\mathcal{T}^\alpha v\|_0$ because their difference, namely $\mathcal{T}^\alpha\widetilde{\varphi}\partial_3^{\widetilde{\varphi}} v$, is bounded by $\mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t$. Indeed, using $\widetilde{\varphi}(t,x) = x_3 + \chi(x_3)\widetilde{\psi}(t,x')$ we only need to investigate the case $\mathcal{T} = \overline{\partial}$ because the weighted derivative $\mathcal{T} = \omega(x_3)\partial_3$ only falls on $\chi(x_3)$ and x_3 instead of $\widetilde{\psi}$. So we have $\|\overline{\partial}^\alpha\widetilde{\varphi}\|_0 \lesssim |\overline{\partial}^\alpha\widetilde{\psi}|_0$ which is already bounded by the surface tension energy and thus by $\mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t$ according to (4.97). Since $\|\partial_3^{\widetilde{\varphi}} v\|_\infty \le \|v\|_3\|\partial_3\widetilde{\varphi}\|_\infty \le \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t$, we have

$$\|\mathcal{T}^\alpha v(T)\|_0^2 + \left\| \sqrt{\mathcal{F}'(q)}\mathcal{T}^\alpha \check{q}(T)\right\|_0^2 + \left| \sqrt{\sigma}\overline{\nabla}\overline{\partial}^\alpha \Lambda_\kappa \psi(T)\right|_0^2 + \int_0^T \left|\kappa\overline{\partial}^\alpha\partial_t\psi(t)\right|_1^2\,\mathrm{d}t \lesssim \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t. \tag{4.98}$$

We remark here that we can employ the same analysis to prove the tangential estimates with mixed spatial-time derivatives.

Proposition 4.5. Let α be the multi-index satisfying $1 \le \alpha_0 \le 3$ and $|\alpha| = 4$, we have:

$$\|\mathcal{T}^\alpha v(T)\|_0^2 + \left\| \sqrt{\mathcal{F}'(q)}\mathcal{T}^\alpha \check{q}(T)\right\|_0^2 + \left| \sqrt{\sigma}\overline{\nabla}\overline{\partial}^\alpha \Lambda_\kappa \psi(T)\right|_0^2 + \int_0^T |\kappa\mathcal{T}^\alpha\partial_t\psi(t)|_1^2\,\mathrm{d}t \lesssim \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t. \tag{4.99}$$

4.6 Tangential energy estimate with time derivatives

In this subsection, we study the time-differentiated equations, i.e., the equations obtained by commuting ∂_t^4 with (3.11). We aim to prove:

Proposition 4.6. We have

$$\|\partial_t^4 v(T)\|_0^2 + \left\| \sqrt{\mathcal{F}'(q)}\partial_t^4 \check{q}(T)\right\|_0^2 + \left| \sqrt{\sigma}\overline{\nabla}\partial_t^4 \Lambda_\kappa \psi(T)\right|_0^2 + \int_0^T \left|\kappa\partial_t^5\psi(t)\right|_1^2\,\mathrm{d}t \lesssim \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t. \tag{4.100}$$

Although the proof appears to be similar to what has been done in the previous subsection, it should be mentioned that we only have $L^2(\Omega)$-regularity for the full-time derivatives of v and q, and thus we do not have any information about their boundary regularity. When the full-time derivatives of v and q appear on the boundary, we use either the artificial viscosity or Euler equations to reduce a time derivative to a spatial derivative.

4.6.1 Alinhac good unknowns for full-time derivatives

To begin with, we still introduce the Alinhac good unknowns of v, q with respect to ∂_t^4. Using the same notation as before, we define

$$\mathbf{V}_i := \partial_t^4 v_i - \partial_3^{\widetilde{\varphi}} v_i \partial_t^4\widetilde{\varphi}, \quad \mathbf{Q} := \partial_t^4\check{q} - \partial_3^{\widetilde{\varphi}}\check{q}\partial_t^4\widetilde{\varphi}. \tag{4.101}$$

Parallel to (4.36) , we have

$$\partial_t^4(\nabla_i^{\widetilde{\varphi}} f) = \nabla_i^{\widetilde{\varphi}}\mathbf{F} + \mathfrak{C}_i(f), \tag{4.102}$$

where $\mathfrak{C}_i(f) := \partial_3^{\widetilde{\varphi}}\partial_i^{\widetilde{\varphi}} f \partial_t^4\widetilde{\varphi} + \mathfrak{C}_i'(f)$ and

$$\mathfrak{C}_i'(f) = -\left[\partial_t^4, \frac{\partial_i\widetilde{\varphi}}{\partial_3\widetilde{\varphi}}, \partial_3 f\right] - \partial_3 f\left[\partial_t^4, \partial_i\widetilde{\varphi}, \frac{1}{\partial_3\widetilde{\varphi}}\right] + \partial_i\widetilde{\varphi}\partial_3 f\left[\partial_t^3, \frac{1}{(\partial_3\widetilde{\varphi})^2}\right]\partial_t\partial_3\widetilde{\varphi}, \quad i = 1,2 \tag{4.103}$$

$$\mathfrak{C}_3'(f) = \left[\partial_t^4, \frac{1}{\partial_3\widetilde{\varphi}}, \partial_3 f\right] + \partial_3 f\left[\partial_t^3, \frac{1}{(\partial_3\widetilde{\varphi})^2}\right]\partial_t\partial_3\widetilde{\varphi}. \tag{4.104}$$

Then we take ∂_t^4 to the first two equations of (1.5) to obtain

$$\rho D_t^{\widetilde{\varphi}}\mathbf{V}_i + \nabla_i^{\widetilde{\varphi}}\mathbf{Q} = \mathcal{R}_i^1, \tag{4.105}$$

$$\mathcal{F}'(q) D_t^{\widetilde{\varphi}}\mathbf{Q} + \nabla^{\widetilde{\varphi}}\cdot\mathbf{V} = \mathcal{R}^2 - \mathfrak{C}^i(v_i), \tag{4.106}$$

where

$$\mathcal{R}_i^1 := -[\partial_t^4, \rho] D_t^{\widetilde{\varphi}} v_i - \rho\,(\mathfrak{D}(v_i) + \mathfrak{E}(v_i)) - \mathfrak{C}_i(\check{q}), \tag{4.107}$$

$$\mathcal{R}^2 := -[\partial_t^4, \mathcal{F}'(q)] D_t^{\widetilde{\varphi}} \check{q} - \mathcal{F}'(q)\,(\mathfrak{D}(\check{q}) + \mathfrak{E}(\check{q})) + \partial_t^4(\mathcal{F}'(q) g v_3), \tag{4.108}$$

and the commutators $\mathfrak{D}(f), \mathfrak{e}(f)$ are defined in the same way as in (4.39) and (4.40) by replacing $\mathcal{T}^\alpha$ with ∂_t^4 and replacing $\overline{\partial}$ with ∂_t. The last two terms in (4.39) vanish because ∂_t^4 directly commutes with ∂_3. Analogous to Lemma 4.4, we list the estimates for commutators $\mathfrak{C}, \mathfrak{D}, \mathfrak{e}$.

Lemma 4.7. Let $\mathbf{F} := \partial_t^4 f - \partial_3^{\widetilde{\varphi}} f \partial_t^4 \widetilde{\varphi}$ be the Alinhac good unknowns of f with respect to ∂_t^4. Assuming that $\partial_3 \widetilde{\varphi} \geq c_0 > 0$, then

$$\|\partial_t^4 f\|_0 \leq \|\mathbf{F}\|_0 + c_0^{-1} \|\partial_3 f\|_\infty |\partial_t^4 \widetilde{\psi}|_0, \tag{4.109}$$

$$\|\mathfrak{C}_i(f)\|_0 \leq P\left(c_0^{-1}, |\overline{\nabla}\widetilde{\psi}|_\infty, \sum_{k=1}^{3} |\overline{\nabla}\partial_t^k \widetilde{\psi}|_{3-k}\right) \cdot \left(\|\partial f\|_\infty + \sum_{k=1}^{3} \|\partial_t^k f\|_{4-k}\right), \quad i = 1, 2, 3, \tag{4.110}$$

$$\|\mathfrak{D}(f)\|_0 \leq P\left(c_0^{-1}, |\overline{\nabla}\widetilde{\psi}|_\infty, \sum_{k=1}^{3} |\overline{\nabla}\partial_t^k \widetilde{\psi}|_{3-k}\right) \cdot \left(\|\partial f\|_\infty + \sum_{k=1}^{3} \|\partial_t^k f\|_{4-k}\right), \tag{4.111}$$

$$\|\mathfrak{E}(f)\|_0 \leq \kappa |\overline{\nabla}\partial_t^5 \psi|_0 \|\partial f\|_\infty. \tag{4.112}$$

The ∂_t^4-differentiated kinematic boundary condition now reads

$$\partial_t^5 \psi + (\overline{v} \cdot \overline{\nabla}) \overline{\partial}^4 \widetilde{\psi} - \mathbf{V} \cdot \widetilde{N} = \mathcal{S}_1^*, \quad \text{on } \Sigma, \tag{4.113}$$

where

$$\mathcal{S}_1^* := \partial_3 v \cdot \widetilde{N} \partial_t^4 \widetilde{\psi} + \sum_{1 \leq \beta \leq 3} \binom{4}{\beta} \partial_t^\beta v \cdot \partial_t^{4-\beta} \widetilde{N}. \tag{4.114}$$

Also, since $\mathbf{Q}|_\Sigma = \partial_t^4 q - \partial_3^{\widetilde{\varphi}} q \partial_t^4 \widetilde{\psi}$, the boundary condition of $\mathbf{Q}$ on Σ reads

$$\mathbf{Q} = -\sigma \partial_t^4 \overline{\nabla} \cdot \left(\frac{\overline{\nabla}\widetilde{\psi}}{\sqrt{1 + |\overline{\nabla}\widetilde{\psi}|^2}} \right) + \kappa^2 (1 - \overline{\Delta}) \partial_t^5 \psi - \partial_3 q \partial_t^4 \widetilde{\psi}. \tag{4.115}$$

4.6.2 Energy estimates for the full-time derivatives

Replacing $\mathcal{T}^\alpha$ by ∂_t^4 in (4.51), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_\Omega \rho |\mathbf{V}|^2 \partial_3 \widetilde{\varphi}\, \mathrm{d}x = \frac{1}{2} \int_\Omega \rho |\mathbf{V}|^2 \partial_3 \partial_t (\widetilde{\varphi} - \varphi)\, \mathrm{d}x + \int_\Omega \mathbf{Q} (\nabla^{\widetilde{\varphi}} \cdot \mathbf{V}) \partial_3 \widetilde{\varphi}\, \mathrm{d}x - \int_\Sigma \mathbf{Q} (\mathbf{V} \cdot \widetilde{N})\, \mathrm{d}x' + \int_\Omega \mathbf{V} \cdot \mathcal{R}^1 \partial_3 \widetilde{\varphi}\, \mathrm{d}x., \tag{4.116}$$

where the first term and the last term are controlled in the same way as (4.57)-(4.59), so we omit the details. As for the second term, we follow (4.60)-(4.64) to get

$$\begin{aligned}
&\int_\Omega \mathbf{Q} (\nabla^{\widetilde{\varphi}} \cdot \mathbf{V}) \partial_3 \widetilde{\varphi}\, \mathrm{d}x \\
= &\underbrace{-\int_\Omega \partial_t^4 \check{q} \mathfrak{C}_i(v^i) \partial_3 \widetilde{\varphi}\, \mathrm{d}x}_{=: I_0^*} + \int_\Omega \partial_t^4 \widetilde{\varphi} \partial_3^{\widetilde{\varphi}} \check{q} \mathfrak{C}_i(v^i) \partial_3 \widetilde{\varphi}\, \mathrm{d}x - \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left\| \sqrt{\mathcal{F}'(q)} \mathbf{Q} \right\|_0^2 \\
&+ \left\| \sqrt{\mathcal{F}'(q)} \mathbf{Q} \right\|_0^2 (\|\partial v\|_\infty + \kappa |\overline{\nabla}\partial_t \psi|_{0.5}) + \left\| \sqrt{\mathcal{F}'(q)} \mathbf{Q} \right\|_0 \|\mathcal{R}^2\|_0 \\
\lesssim\ & I_0^* - \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left\| \sqrt{\mathcal{F}'(q)} \mathbf{Q} \right\|_0^2 + \left\| \sqrt{\mathcal{F}'(q)} \partial_t^4 \check{q} \right\|_0^2 \left(\|\partial v\|_\infty + \kappa |\overline{\nabla}\partial_t \psi|_{0.5} + |\kappa \overline{\nabla} \partial_t^5 \psi|_0 \right) \\
&+ P\left(c_0^{-1}, |\overline{\nabla}\widetilde{\psi}|_\infty, \sum_{k=1}^{3} |\overline{\nabla}\partial_t^k \widetilde{\psi}|_{3-k}\right) |\partial_t^4 \widetilde{\psi}|_0 | \left\| \sqrt{\mathcal{F}'(q)} \partial_t^4 \check{q} \right\|_0 \left(\|\partial v, \partial q\|_\infty + \sum_{k=1}^{3} \|\partial_t^k \check{q}, \partial_t^k v\|_{4-k} + \|\mathcal{F}'(q) \partial_t^4 v_3\|_0 \right).
\end{aligned} \tag{4.117}$$

At this point, we are not able to control $I_0^* := -\int_\Omega \partial_t^4 q \mathfrak{C}_i(v^i)\partial_3\widetilde{\varphi}\,\mathrm{d}x$ as in (4.63) since this requires the bound for $\|\partial_t^4 q\|_0$. We can only obtain the control of $\|\sqrt{\mathcal{F}'(q)}\partial_t^4 q\|_0$ from the energy estimate because we can no longer use the momentum equation to reduce $\partial_t^4 q$ due to lack of spatial derivatives. *Although the method in* (4.63) *is still valid here when we prove the well-posedness provided that $\mathcal{F}'(q)$ is bounded from below, we would like to show that our estimate can be adjusted to be uniform in $\mathcal{F}'(q)$.* To achieve this, we find that the problematic terms in $\mathfrak{C}_i(v^i)$ can be exactly canceled by the boundary error term $\mathcal{S}_1$ defined in (4.114). Therefore, this term should be controlled together with the boundary integral if we want our energy estimates to be uniform in the Mach number.

Next, we analyze the boundary integral. Most of the steps are parallel to Section 4.5.4 if we replace $\overline{\partial}^\alpha$ by ∂_t^4, so we will omit the details of those repeated steps but only list the different steps. Plugging the boundary conditions (4.113) and (4.115) into $-\int_\Sigma \mathbf{Q}(\mathbf{V}\cdot\widetilde{N})\,\mathrm{d}x'$, we get

$$-\int_\Sigma \mathbf{Q}(\mathbf{V}\cdot\widetilde{N})\,\mathrm{d}x' = -\int_\Sigma \mathbf{Q}\partial_t^5\psi\,\mathrm{d}x' - \int_\Sigma \mathbf{Q}(\overline{v}\cdot\overline{\nabla})\partial_t^4\widetilde{\psi}\,\mathrm{d}x' + \int_\Sigma \mathbf{Q}\mathcal{S}_1^*\,\mathrm{d}x' =: I_1^* + I_2^* + I_3^*, \tag{4.118}$$

and I_1^* is further divided into three parts:

$$\begin{aligned} I_1^* := -\int_\Sigma \mathbf{Q}\partial_t^5\psi\,\mathrm{d}x' &= \sigma\int_\Sigma \partial_t^4\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\widetilde{\psi}}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}}\right)\partial_t^5\psi\,\mathrm{d}x' - \kappa^2\int_\Sigma \partial_t^4(1-\overline{\Delta})\partial_t\psi\cdot\partial_t^5\psi\,\mathrm{d}x' + \int_\Sigma \partial_3 q\partial_t^4\widetilde{\psi}\partial_t^5\psi\,\mathrm{d}x' \\ &=:\ \mathrm{ST}_1^* +\ \mathrm{ST}_2^* +\ \mathrm{RT}^*. \end{aligned} \tag{4.119}$$

Mimicing the steps (4.67)-(4.81), we can get the bounds for $\mathrm{ST}_1^*,\ \mathrm{ST}_2^*$:

$$\int_0^T \mathrm{ST}_1^* +\ \mathrm{ST}_2^*\,\mathrm{d}t + \left|\kappa\partial_t^5\psi\right|^2_{L_t^2H_{x'}^1} + \frac{\sigma}{2}\left|\overline{\nabla}\partial_t^4\Lambda_\kappa\psi(T)\right|_0^2 \lesssim \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t. \tag{4.120}$$

Remark 4.4. Parallel to the remark after (4.82), $-\int_0^T \mathrm{RT}^*\,\mathrm{d}t$ would contribute to the non-σ-weighted energy $\int_\Sigma(-\partial_3 q)|\partial_t^4\Lambda_\kappa\psi|^2\,\mathrm{d}t$ if the Rayleigh-Taylor sign condition holds. This will be revisited in Section 7.

As for RT^*, if we still follow (4.82) to get:

$$\int_0^T \mathrm{RT}^*\,\mathrm{d}t \lesssim \varepsilon\left|\kappa\partial_t^5\psi\right|^2_{L_t^2L_{x'}^2} + \int_0^T P\left(\|\check{q}\|_4, \left|\partial_t^4\Lambda_\kappa\psi\right|_0, \left|\partial_t^5\Lambda_\kappa\psi\right|_0\right)\,\mathrm{d}t,$$

then we find that the term $|\partial_t^5\Lambda_\kappa\psi|_0$ is not included in $E^\kappa(t)$ because there is no spatial derivative here. To overcome this, we invoke the kinematic boundary condition $\partial_t\psi = -\overline{v}\cdot\overline{\nabla}\widetilde{\psi} + v_3$ and take ∂_t^4 to get

$$\partial_t^5\psi = -(\overline{v}\cdot\overline{\nabla})\partial_t^4\widetilde{\psi} + \partial_t^4 v_3 - [\partial_t^4, \overline{v}\cdot]\overline{\nabla}\widetilde{\psi} = -(\overline{v}\cdot\overline{\nabla})\partial_t^4\widetilde{\psi} + \partial_t^4 v\cdot\widetilde{N} - [\partial_t^4, \overline{v}\cdot, \overline{\nabla}\widetilde{\psi}], \tag{4.121}$$

and thus

$$\begin{aligned} \mathrm{RT}^* &= -\int_\Sigma \partial_3 q\partial_t^4\widetilde{\psi}(\overline{v}\cdot\overline{\nabla})\partial_t^4\widetilde{\psi}\,\mathrm{d}x' + \int_\Sigma \partial_3 q\partial_t^4\widetilde{\psi}\partial_t^4 v\cdot\widetilde{N}\,\mathrm{d}x' - \int_\Sigma \partial_3 q\partial_t^4\widetilde{\psi}[\partial_t^4, \overline{v}\cdot, \overline{\nabla}\widetilde{\psi}]\,\mathrm{d}x' \\ &=:\ \mathrm{RT}_1^* +\ \mathrm{RT}_2^* +\ \mathrm{RT}_3^*. \end{aligned} \tag{4.122}$$

Note that we only need to analyze the contribution of RT_2^* because the contribution of the other two terms will be canceled by part of I_2^* and I_3^*. To do this, we need to derive the equation for $\partial_t^4\cdot\widetilde{N}$ on Σ. Recall that

$$D_t^{\widetilde{\varphi}}\big|_\Sigma = \partial_t + (\overline{v}\cdot\overline{\nabla}) + (\partial_3\widetilde{\psi})^{-1}\underbrace{(v\cdot\widetilde{\mathbf{N}} - \partial_t\varphi)}_{=0\text{ on }\Sigma}\partial_3 = \partial_t + (\overline{v}\cdot\overline{\nabla}),$$

we have the following identity by projecting the momentum equation onto the direction of $\widetilde{N}$ on Σ:

$$\rho\partial_t v\cdot\widetilde{N} = -(\rho-1)g - \rho(\overline{v}\cdot\overline{\nabla})v\cdot\widetilde{N} + \overline{\nabla}\widetilde{\psi}\cdot\overline{\nabla}\check{q} - (1+|\overline{\nabla}\widetilde{\psi}|^2)\partial_3\check{q},$$

and thus

$$\rho\partial_t^4 v\cdot\widetilde{N} \overset{L}{=} -\partial_t^3\rho g - \rho(\overline{v}\cdot\overline{\nabla})\partial_t^3 v\cdot\widetilde{N} + \overline{\nabla}\widetilde{\psi}\cdot\overline{\nabla}\partial_t^3\check{q} - |\widetilde{N}|^2\partial_3\partial_t^3\check{q}. \tag{4.123}$$

The contribution of the first three terms in (4.123) can be directly controlled after integrating $\overline{\nabla}$ by parts and using the Sobolev trace lemma:

$$
\begin{aligned}
&\int_\Sigma \rho^{-1}\partial_3 q \partial_t^4 \widetilde{\psi}\left(\rho(\overline{v}\cdot\overline{\nabla})\partial_t^3 v\cdot \widetilde{N} + \overline{\nabla}\widetilde{\psi}\cdot\overline{\nabla}\partial_t^3 \check{q} - \partial_t^3\rho g\right) \mathrm{d}x' \\
\stackrel{L}{=} & -\int_\Sigma \rho^{-1}\overline{\nabla}\partial_t^4\widetilde{\psi}\cdot\left(\partial_3 q\left(\rho\overline{v}\partial_t^3 v\cdot\widetilde{N} + \overline{\nabla}\widetilde{\psi}\partial_t^3\check{q}\right)\right)\mathrm{d}x' - \int_\Sigma \rho^{-1}\partial_3 q\partial_t^4\widetilde{\psi}\partial_t^3\rho g\,\mathrm{d}x' \\
\lesssim & \|\partial q\|_2\left(\left|\overline{\nabla}\partial_t^4\widetilde{\psi}\right|_0 P\left(\|\partial_t^3 v\|_1, \|\partial_t^3\check{q}\|_1, |\overline{\nabla}\widetilde{\psi}|_\infty\right) + |\partial_t^4\widetilde{\psi}|_0\|\mathscr{F}'(q)\partial_t^3 q\|_1|\rho|_\infty\right).
\end{aligned}
\tag{4.124}
$$

Remark 4.5. Note that the right side of (4.124) involves $|\overline{\nabla}\partial_t^4\widetilde{\psi}|_0$ whose control relies on σ^{-1}. This is due to the lack of the Rayleigh-Taylor sign condition. When taking the zero surface tension limit, the Rayleigh-Taylor sign condition is assumed and thus the RT term can be directly controlled.

Then for the last term, we need to do the same reduction for $\partial_t^4\psi$:

$$
\partial_t^4\psi = -(\overline{v}\cdot\overline{\nabla})\partial_t^3\widetilde{\psi} + \partial_t^3 v_3 - [\partial_t^3, \overline{v}\cdot]\overline{\nabla}\widetilde{\psi} = -(\overline{v}\cdot\overline{\nabla})\partial_t^3\widetilde{\psi} + \partial_t^3 v\cdot\widetilde{N} - [\partial_t^3, \overline{v}\cdot, \overline{\nabla}\widetilde{\psi}]. \tag{4.125}
$$

Using (4.125) and Sobolev trace lemma, it is controlled by

$$
|\partial_t^4\widetilde{\psi}|_0 \lesssim P(|\overline{\nabla}\widetilde{\psi}|_\infty, |\overline{\nabla}\partial_t\widetilde{\psi}|_\infty)\left(|\overline{\nabla}\partial_t^3\widetilde{\psi}|_0 + \|\partial_t^3 v\|_1 + \|\partial_t^2 v\|_2 + |\overline{\nabla}\partial_t^2\widetilde{\psi}|_0\right). \tag{4.126}
$$

Now we plug the equality above into the boundary integral $-\int_\Sigma \rho^{-1}\partial_3 q|\widetilde{N}|^2\partial_t^4\widetilde{\psi}\partial_3\partial_t^3\check{q}\,\mathrm{d}x'$. Note that the unit exterior normal vector to Σ is $(0,0,1)^\top$ (not the Eulerian normal vector $\widetilde{N}$!), we can use the divergence theorem to rewrite the boundary integral into the interior, and integrate by parts in ∂_t to get the following estimate:

$$
\begin{aligned}
&-\int_0^T\int_\Sigma \rho^{-1}\partial_3 q|\widetilde{N}|^2\partial_t^4\widetilde{\psi}\partial_3\partial_t^3\check{q}\,\mathrm{d}x'\,\mathrm{d}t \stackrel{L}{=} \int_0^T\int_\Sigma \rho^{-1}\partial_3 q|\widetilde{N}|^2\Lambda_\kappa^2\left((\overline{v}\cdot\overline{\nabla})\partial_t^3\widetilde{\psi} - \partial_t^3 v\cdot\widetilde{N}\right)\partial_3\partial_t^3\check{q}\,\mathrm{d}x'\,\mathrm{d}t \\
=& \int_0^T\int_\Omega \partial_3\left(\rho^{-1}\partial_3 q|\widetilde{\mathbf{N}}|^2\Lambda_\kappa^2\left((\overline{v}\cdot\overline{\nabla})\partial_t^3\widetilde{\varphi} - \partial_t^3 v\cdot\widetilde{\mathbf{N}}\right)\partial_3\partial_t^3\check{q}\right)\mathrm{d}x\,\mathrm{d}t \\
\stackrel{L}{=}& \int_0^T\int_\Omega \rho^{-1}\partial_3 q|\widetilde{\mathbf{N}}|^2\Lambda_\kappa^2\left((\overline{v}\cdot\overline{\nabla})\partial_t^3\widetilde{\varphi} - \partial_t^3 v\cdot\widetilde{\mathbf{N}}\right)\cdot\partial_3^2\partial_t^3\check{q}\,\mathrm{d}x\,\mathrm{d}t \\
\stackrel{\partial_t}{=}& -\int_\Omega \rho^{-1}\partial_3 q|\widetilde{\mathbf{N}}|^2\Lambda_\kappa^2\left((\overline{v}\cdot\overline{\nabla})\partial_t^3\widetilde{\varphi} - \partial_t^3 v\cdot\widetilde{\mathbf{N}}\right)\cdot\partial_3^2\partial_t^2\check{q}\,\mathrm{d}x \\
&+\int_0^T\int_\Omega \rho^{-1}\partial_3 q|\widetilde{\mathbf{N}}|^2\partial_t\Lambda_\kappa^2\left((\overline{v}\cdot\overline{\nabla})\partial_t^3\widetilde{\varphi} - \partial_t^3 v\cdot\widetilde{\mathbf{N}}\right)\cdot\partial_3^2\partial_t^2\check{q}\,\mathrm{d}x\,\mathrm{d}t \\
\lesssim& \varepsilon\|\partial_t^2\partial^2\check{q}\|_0^2 + \mathcal{P}_0^\kappa + \int_0^T P\left(\|\partial_t^4 v\|_0, \|\partial_t^3 v\|_1, \|\partial_t v\|_\infty, \|\partial_t^2\check{q}\|_2, |\overline{\nabla}\widetilde{\psi}|_\infty, |\overline{\nabla}\partial_t^3\widetilde{\psi}|_0, |\overline{\nabla}\partial_t^4\widetilde{\psi}|_0\right)\mathrm{d}t.
\end{aligned}
\tag{4.127}
$$

Combining this with (4.120), (4.122), (4.124) and (4.127), we get the estimate for I_1^*:

$$
\int_0^T I_1^*\,\mathrm{d}t + \left|\kappa\partial_t^5\psi\right|^2_{L_t^2 H_{x'}^1} + \frac{\sigma}{2}\left|\overline{\nabla}\partial_t^4\Lambda_\kappa\psi(T)\right|_0^2 \lesssim \varepsilon\|\partial_t^2\partial^2\check{q}\|_0^2 + \int_0^T \mathrm{RT}_1^* + \mathrm{RT}_3^*\,\mathrm{d}t + \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t, \tag{4.128}
$$

after choosing $\varepsilon > 0$ that appears above to be suitably small.

Next we expand I_2^*, I_3^* defined in (4.118)

$$
\begin{aligned}
I_2^* + I_3^* =& -\int_\Sigma \partial_t^4\check{q}(\overline{v}\cdot\overline{\nabla})\partial_t^4\widetilde{\psi}\,\mathrm{d}x' + \int_\Sigma \partial_t^4\widetilde{\psi}\partial_3 q(\overline{v}\cdot\overline{\nabla})\partial_t^4\widetilde{\psi}\,\mathrm{d}x' \\
&+\int_\Sigma \partial_t^4\check{q}\mathcal{S}_1\,\mathrm{d}x' - \int_\Sigma \partial_t^4\widetilde{\psi}\partial_3 q\partial_3 v\cdot\widetilde{N}\partial_t^4\widetilde{\psi}\,\mathrm{d}x' - \int_\Sigma \partial_t^4\widetilde{\psi}\partial_3 q\left(\sum_{1\le\beta\le3}\binom{4}{\beta}\partial_t^\beta v\cdot\partial_t^{4-\beta}\widetilde{N}\right)\mathrm{d}x'
\end{aligned}
\tag{4.129}
$$

and we find that the second term exactly cancels RT_1^* and the fifth term exactly cancels RT_3^* defined in (4.122). The first term can be controlled in the same way as I_{21}, I_{22} defined in (4.84) after replacing $\overline{\partial}^4$ by ∂_t^4. The fourth term is directly controlled by $P(E^\kappa(t))$ by using the Sobolev trace lemma.

Hence, it suffices to analyze the third term. Using the definition of $\mathcal{S}_1^*$, we have

$$\int_\Sigma \partial_t^4 \check{q} \mathcal{S}_1^* \,\mathrm{d}x' = \int_\Sigma \partial_t^4 \check{q} \left(\partial_3 v \cdot \widetilde{N} \partial_t^4 \widetilde{\psi}\right) \mathrm{d}x' - 4 \int_\Sigma \partial_t^4 \check{q} \partial_t^3 v \cdot \partial_t \widetilde{N} \,\mathrm{d}x' + \sum_{1\leq\beta\leq 2} \binom{4}{\beta} \int_\Sigma \partial_t^4 \check{q} \partial_t^\beta v \cdot \overline{\nabla} \partial_t^{4-\beta} \widetilde{\psi} \,\mathrm{d}x', \tag{4.130}$$

where the first term can be controlled by the surface tension energy after invoking (4.115) and integrating $\overline{\nabla}$ by parts; and the last term can be controlled after integrating by part in ∂_t under time integral. But for the remaining term

$$I_{30}^* := 4 \int_\Sigma \partial_t^4 \check{q} \partial_t^3 v \cdot \partial_t \widetilde{N} \,\mathrm{d}x', \tag{4.131}$$

we have neither the $L^2(\Sigma)$-regularity of $\partial_t^4 q$ nor the possibility of integrating $\frac{1}{2}$-time derivatives by parts as in the control of (4.91).

Fortunately, we can still control I_{30} *together with the interior term* $I_0^* := -\int_\Omega \partial_t^4 q \mathfrak{C}_i(v^i) \partial_3 \widetilde{\varphi} \,\mathrm{d}x$ defined in (4.117). In fact, invoking (4.103) and (4.104), we know $\mathfrak{C}_i(v^i)$ includes the following terms involving ≥ 3 time derivatives of v^i and ≥ 4 derivatives of $\widetilde{\varphi}$:

$$\partial_3^{\widetilde{\varphi}} \partial_i^{\widetilde{\varphi}} v^i \partial_t^4 \widetilde{\varphi} = \mathfrak{C}_i(v^i) - \mathfrak{C}_i'(v^i), \quad i = 1,2,3, \tag{4.132}$$

$$-4\partial_t \left(\frac{\overline{\partial}_i \widetilde{\varphi}}{\partial_3 \widetilde{\varphi}}\right) \partial_t^3 \partial_3 v^i = 4\partial_t \widetilde{\mathbf{N}}_i \partial_3^{\widetilde{\varphi}} \partial_t^3 v^i + 4 \frac{\partial_3 \partial_t \widetilde{\varphi} \overline{\partial}_i \widetilde{\varphi}}{\partial_3 \widetilde{\varphi}} \partial_3^{\widetilde{\varphi}} \partial_t^3 v^i \text{ from the first commutator in } \mathfrak{C}_i'(v^i) \;\; i = 1,2, \tag{4.133}$$

$$4\partial_t \left(\frac{1}{\partial_3 \widetilde{\varphi}}\right) \partial_t^3 \partial_3 v^3 = -4 \frac{\partial_3 \partial_t \widetilde{\varphi}}{\partial_3 \widetilde{\varphi}} \partial_3^{\widetilde{\varphi}} \partial_t^3 v^3 \text{ from the first commutator in } \mathfrak{C}_3'(v^3), \tag{4.134}$$

while the terms in $\mathfrak{C}_i'(v^i)$ containing only ≤ 2 time derivatives of v^i and ≤ 3 time derivatives of $\widetilde{\varphi}$ are controlled directly after integrating ∂_t by parts under time integral.

The contribution of the above four terms in I_0^* is divided into three parts:

$$I_{00}^* := -4 \int_\Omega \partial_t^4 \check{q} \partial_t \widetilde{\mathbf{N}}_i \partial_3 \partial_t^3 v^i \,\mathrm{d}x \tag{4.135}$$

$$I_{01}^* := -\int_\Omega \partial_t^4 \check{q} \partial_3 (\nabla^{\widetilde{\varphi}} \cdot v) \partial_t^4 \widetilde{\varphi} \,\mathrm{d}x \tag{4.136}$$

$$I_{02}^* := -4 \sum_{i=1}^2 \int_\Omega \partial_t^4 \check{q} \left(\frac{\partial_3 \partial_t \widetilde{\varphi} \overline{\partial}_i \widetilde{\varphi}}{\partial_3 \widetilde{\varphi}}\right) \partial_3^{\widetilde{\varphi}} \partial_t^3 v^i \partial_3 \widetilde{\varphi} \,\mathrm{d}x + 4 \int_\Omega \partial_t^4 \check{q} \left(\frac{\partial_3 \partial_t \widetilde{\varphi}}{\partial_3 \widetilde{\varphi}}\right) \partial_3^{\widetilde{\varphi}} \partial_t^3 v^3 \partial_3 \widetilde{\varphi} \,\mathrm{d}x. \tag{4.137}$$

Integrating ∂_3 by parts in I_{00}^* and using $N_3 = 1$, we find the boundary term exactly cancels with I_{30}^*, so we have:

$$\begin{aligned} I_{30}^* + I_{00}^* &= 4 \int_\Omega \left(\partial_t^4 \partial_3 \check{q} \partial_t \widetilde{\mathbf{N}} + \partial_t^4 \check{q} \partial_t \partial_3 \widetilde{\mathbf{N}}\right) \cdot \partial_t^3 v \,\mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\partial_t^3 \partial_3 \check{q} \partial_t \widetilde{\mathbf{N}} + \partial_t^3 \check{q} \partial_t \partial_3 \widetilde{\mathbf{N}}\right) \cdot \partial_t^3 v \,\mathrm{d}x + \int_\Omega \partial_t^3 \partial_3 \check{q} \partial_t (\partial_t \widetilde{\mathbf{N}} \cdot \partial_t^3 v) + \partial_t^3 \check{q} \partial_t (\partial_t \partial_3 \widetilde{\mathbf{N}} \cdot \partial_t^3 v) \,\mathrm{d}x. \end{aligned} \tag{4.138}$$

Under the time integral, we have the following bounds after using ε-Young's inequality:

$$\begin{aligned} \int_0^T I_{00}^* + I_{30}^* \,\mathrm{d}t &\lesssim \varepsilon \|\partial_t^3 \partial \check{q}\|_0^2 + \mathcal{P}_0^\kappa + \int_0^T (\|\partial_t^3 \check{q}(t)\|_0 + 1) P(E^\kappa(t)) \,\mathrm{d}t \\ &\lesssim \varepsilon \|\partial_t^3 \partial \check{q}\|_0^2 + \varepsilon \int_0^T \|\partial_t^3 \check{q}(t)\|_0^2 \,\mathrm{d}t + \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t)) \,\mathrm{d}t. \end{aligned} \tag{4.139}$$

Here, we still need to study $\int_0^T \|\partial_t^3 \check{q}(t)\|_0^2 \,\mathrm{d}t$, as the reduction scheme does not apply to $\partial_t^3 \check{q}(t)$ due to lack of spatial derivatives. We control this term through the fundamental theorem of calculus: For each $x_3 \in (-b, 0)$, we write

$$\partial_t^3 \check{q}(t, x', x_3) = \partial_t^3 \check{q}(t, x', 0) + \int_0^{x_3} \partial_3 \partial_t^3 \check{q}(t, x', z) \, dz,$$

and so

$$\left(\partial_t^3\check{q}(t,x',x_3)\right)^2 \lesssim_b \left(\partial_t^3\check{q}(t,x',0)\right)^2 + \int_0^{x_3} \left(\partial_3\partial_t^3\check{q}(t,x',z)\right)^2 \, dz.$$

Integrating both sides with respect to the spatial variables, we obtain

$$\|\partial_t^3\check{q}(t)\|_0^2 \lesssim_b |\partial_t^3\check{q}(t)|_0^2 + \|\partial_3\partial_t^3\check{q}(t)\|_0^2.$$

The second term on the RHS is bounded by $E^\kappa(t)$. For the first term, note that

$$\partial_t^3\check{q} = -\sigma\partial_t^3\left(\frac{\overline{\nabla}\widetilde{\psi}}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}} + g\widetilde{\psi}\right) + \kappa^2(1-\overline{\Delta})\partial_t^4\psi, \quad \text{on } \Sigma,$$

where

$$-\sigma\partial_t^3\left(\frac{\overline{\nabla}\widetilde{\psi}}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}|^2}}\right) \overset{L}{=} -\sigma\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\partial_t^3\widetilde{\psi}}{\sqrt{(1+|\overline{\nabla}\widetilde{\psi}|^2)}} - \frac{\overline{\nabla}\widetilde{\psi}\cdot\overline{\nabla}\partial_t^3\widetilde{\psi}}{\sqrt{(1+|\overline{\nabla}\widetilde{\psi}|^2)}^{3}}\overline{\nabla}\widetilde{\psi}\right),$$

which indicates that $\partial_t^3\check{q}|_\Sigma$ consists of non-κ-weighted terms with at most 5 derivatives on $\widetilde{\psi}$ with at most 3 times derivatives, and a κ-weighted term $\kappa^2(1-\overline{\Delta})\partial_t^4\psi$. Therefore,

$$\int_0^T \|\partial_t^3\check{q}(t)\|_0^2\,\mathrm{d}t \lesssim \int_0^T P\left(\sum_{k=0}^3 |\sqrt{\sigma}\overline{\nabla}\partial_t^k\Lambda_\kappa\psi(t)|_{4-k}, \sum_{k=0}^3 |\partial_t^k\Lambda_\kappa\psi(t)|_{4-k}\right)\mathrm{d}t + \int_0^T |\kappa\partial_t^4\psi(t)|_2^2\,\mathrm{d}t$$

By combining this with (4.139), we conclude:

$$\int_0^T I_{00}^* + I_{30}^*\,\mathrm{d}t \lesssim \varepsilon\|\partial_t^3\partial\check{q}\|_0^2 + \varepsilon\int_0^T \|\kappa\partial_t^4\psi(t)\|_2^2\,\mathrm{d}t + \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t. \tag{4.140}$$

Next, the term I_{01}^* can be directly controlled if we insert the continuity equation $\nabla^{\widetilde{\varphi}}\cdot v = -\mathcal{F}'(q)D_t^{\widetilde{\varphi}}q$

$$I_{01}^* \lesssim \left\|\sqrt{\mathcal{F}'(q)}\partial_t^4\check{q}\right\|_0 \left\|\sqrt{\mathcal{F}'(q)}\partial_t q,\ \sqrt{\mathcal{F}'(q)}\partial q\right\|_{W^{1,\infty}} |\partial_t^4\widetilde{\psi}|_0. \tag{4.141}$$

As for I_{02}^*, we note that $-\overline{\partial}_i\widetilde{\varphi}\partial_3^{\widetilde{\varphi}}\partial_t^3 v^i = \partial_i^{\varphi}\partial_t^3 v^i - \overline{\partial}_i\partial_t^3 v_i$ for $i=1,2$. So it becomes

$$\begin{aligned} I_{02}^* &= 4\int_\Omega \partial_t^4\check{q}\partial_3\partial_t\widetilde{\varphi}(\nabla^{\widetilde{\varphi}}\cdot\partial_t^3 v)\,\mathrm{d}x - 4\sum_{i=1}^2\int_\Omega \partial_t^4\check{q}\partial_3\partial_t\widetilde{\varphi}\overline{\partial}_i\partial_t^3 v_i\,\mathrm{d}x \\ &\overset{L}{=} 4\int_\Omega \partial_t^4\check{q}\partial_t^3(\nabla^{\widetilde{\varphi}}\cdot v)\partial_3\partial_t\widetilde{\varphi}\,\mathrm{d}x - 4\sum_{i=1}^2\int_\Omega \partial_t^4\check{q}\partial_3\partial_t\widetilde{\varphi}\overline{\partial}_i\partial_t^3 v_i\,\mathrm{d}x, \end{aligned} \tag{4.142}$$

where the first term is controlled by

$$\left\|\sqrt{\mathcal{F}'(q)}\partial_t^4\check{q}\right\|_0\left(\left\|\sqrt{\mathcal{F}'(q)}\partial_t^4\check{q}\right\|_0 + \left\|\sqrt{\mathcal{F}'(q)}\partial_t^3\partial\check{q}\right\|_0 + \left\|\sqrt{\mathcal{F}'(q)}\partial_t^3 v_3\right\|_0\right)|\partial_t\widetilde{\psi}|_\infty$$

after invoking the continuity equation, and the second term is controlled under time integral after integrating by parts first in ∂_t and then in $\overline{\partial}_i$. So we have:

$$\int_0^T I_{02}^*\,\mathrm{d}t \lesssim \varepsilon\|\partial_t^3\overline{\partial}\check{q}\|_0^2 + \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t. \tag{4.143}$$

Summarizing (4.116), (4.117), (4.120), (4.122), (4.128)-(4.131), (4.140), (4.141) and (4.143), we finally get the control of the Alinhac good unknowns $\mathbf{V}$ and $\mathbf{Q}$ with respect to ∂_t^4:

$$\|\mathbf{V}(T)\|_0^2 + \left\|\sqrt{\mathcal{F}'(q)}\mathbf{Q}(T)\right\|_0^2 + \left|\sqrt{\sigma}\overline{\nabla}\partial_t^4\Lambda_\kappa\psi(T)\right|_0^2 + \int_0^T \left|\kappa\partial_t^5\psi\right|_1^2\,\mathrm{d}t \lesssim \varepsilon\left(\|\partial_t^2\partial^2\check{q}\|_0^2 + \|\partial_t^3\partial\check{q}\|_0^2\right) + \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t. \tag{4.144}$$

To recover the energy for $\|\partial_t^4 v\|_0^2$ and $\|\sqrt{\mathcal{F}'(q)}\partial_t^4\check{q}\|_0^2$, it suffices to invoke (4.109) and use the estimate of $|\partial_t^4\widetilde{\psi}|_0$ in (4.126). Note that the right side of (4.126) has been controlled in $\overline{\partial}^{4-k}\partial_t^k$-estimates for $k \le 3$, so we already have $|\partial_t^4\widetilde{\psi}|_0 \le \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t$ and thus

$$\|\partial_t^4 v(T)\|_0^2 + \left\|\sqrt{\mathcal{F}'(q)}\partial_t^4\check{q}(T)\right\|_0^2 + \left|\sqrt{\sigma}\overline{\nabla}\partial_t^4\Lambda_\kappa\psi(T)\right|_0^2 + \int_0^T \left|\kappa\partial_t^5\psi\right|_1^2\,\mathrm{d}t \lesssim \varepsilon\left(\|\partial_t^2\partial^2\check{q}\|_0^2 + \|\partial_t^3\partial\check{q}\|_0^2\right) + \mathcal{P}_0^\kappa + \int_0^T P(E^\kappa(t))\,\mathrm{d}t. \quad (4.145)$$

4.7 A priori estimates for the nonlinear κ-approximate problem

Now we choose $\varepsilon > 0$ suitably small and then combine the tangential estimates (4.98) and (4.145) with div-curl analysis, reduction of pressure and L^2-estimates in Section 4.1–Section 4.3 to get the following energy inequality:

$$E^\kappa(T) \le E^\kappa(0) + \int_0^T P(E^\kappa(t))\,\mathrm{d}t. \quad (4.146)$$

Since the right-hand side of the energy inequality does not rely on κ^{-1}, we can use Grönwall's inequality to prove that there exists some $T_0 > 0$ independent of $\kappa > 0$ such that

$$\sup_{0\le t\le T_0} E^\kappa(t) \le P(E^\kappa(0)). \quad (4.147)$$

We also note that the above energy estimate does not rely on $\mathcal{F}'(q)^{-1}$, as a special cancellation structure enjoyed by the Alinhac good unknowns and delicate analysis (4.130)-(4.143) exclude the only possibility that might make the energy estimates not uniform in Mach number. Therefore, our a priori bound is also uniform in Mach number.

5 Well-posedness of the nonlinear κ-approximate system

For the nonlinear κ-approximate problem (3.11), we have established the uniform-in-κ estimates. Once we prove the well-posedness of (3.11) for each *fixed* $\kappa > 0$, we can take the limit $\kappa \to 0$ to prove the local existence of the original system (1.24). We would use Picard iteration to construct the solution to (3.11) for each fixed $\kappa > 0$. We start with $(v^{(0)}, \rho^{(0)}, \psi^{(0)}) := (\mathbf{0}, 1, 0)$ and also define $\psi^{(-1)} := \psi^{(0)}$. Then we construct the solution by the following iteration scheme: For any $n \ge 0$, given $\{(v^{(k)}, \rho^{(k)}, \psi^{(k)})\}_{k\le n}$, we define $(v^{(n+1)}, \rho^{(n+1)}, \psi^{(n+1)})$ to be the solution to the following linear system whose coefficients depend on $(v^{(n)}, \rho^{(n)}, \psi^{(n)})$ and $\psi^{(n-1)}$:

$$\begin{cases}
\rho^{(n)} D_t^{\widetilde{\varphi}^{(n)}} v^{(n+1)} + \nabla^{\widetilde{\varphi}^{(n)}}\check{q}^{(n+1)} = -(\rho^{(n)} - 1)ge_3 & \text{in } [0,T]\times\Omega,\\
\mathcal{F}^{(n)\prime}(q^{(n)}) D_t^{\widetilde{\varphi}^{(n)}}\check{q}^{(n+1)} + \nabla^{\widetilde{\varphi}^{(n)}}\cdot v^{(n+1)} = \mathcal{F}^{(n)\prime}(q^{(n)}) g v_3^{(n)} & \text{in } [0,T]\times\Omega,\\
q^{(n+1)} = q^{(n+1)}(\rho^{(n+1)}),\ \check{q}^{(n+1)} = q^{(n+1)} + g\widetilde{\varphi}^{(n)} & \text{in } [0,T]\times\Omega,\\
\check{q}^{(n+1)} = g\widetilde{\psi}^{(n)} - \sigma\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\widetilde{\psi}^{(n)}}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}^{(n)}|^2}}\right) + \kappa^2(1-\overline{\Delta})(v^{(n+1)}\cdot\widetilde{N}^{(n)}) & \text{on } [0,T]\times\Sigma,\\
\partial_t\psi^{(n+1)} = v^{(n+1)}\cdot\widetilde{N}^{(n)} & \text{on } [0,T]\times\Sigma,\\
v_3^{(n+1)} = 0 & \text{on } [0,T]\times\Sigma_b,\\
(v^{(n+1)}, \rho^{(n+1)}, \psi^{(n+1)})|_{t=0} = (v_{0,\kappa}, \rho_{0,\kappa}, \psi_{0,\kappa}),
\end{cases} \quad (5.1)$$

where for any $k \le n+1$, $\varphi^{(k)}(t,x)$ is the extension of $\psi^{(k)}$ defined by $\varphi^{(k)}(t,x) := x_3 + \chi(x_3)\psi^{(k)}$ and $\widetilde{\varphi}^{(k)} := x_3 + \chi(x_3)\widetilde{\psi}^{(k)}$ is the smoothed version of $\varphi^{(k)}$. The linearized material derivative is defined to be the following linear operator:

$$D_t^{\widetilde{\varphi}^{(n)}} := \partial_t + \overline{v}^{(n)}\cdot\overline{\nabla} + \frac{1}{\partial_3\widetilde{\varphi}^{(n)}}(v^{(n)}\cdot\widetilde{\mathbf{N}}^{(n-1)} - \partial_t\varphi^{(n)})\partial_3, \quad (5.2)$$

and the covariant derivatives are defined to be

$$\partial_t^{\widetilde{\varphi}^{(n)}} = \partial_t - \frac{\partial_t\varphi^{(n)}}{\partial_3\widetilde{\varphi}^{(n)}}\partial_3, \quad (5.3)$$

$$\nabla_a^{\widetilde{\varphi}^{(n)}} = \partial_a^{\widetilde{\varphi}^{(n)}} = \partial_a - \frac{\partial_a\widetilde{\varphi}^{(n)}}{\partial_3\widetilde{\varphi}^{(n)}}\partial_3, \quad a = 1,2, \quad (5.4)$$

$$\nabla_3^{\widetilde{\varphi}^{(n)}} = \partial_3^{\widetilde{\varphi}^{(n)}} = \frac{1}{\partial_3\widetilde{\varphi}^{(n)}}\partial_3. \quad (5.5)$$

Remark 5.1. Note that the linearized material derivative is no longer equal to $\partial_t^{\widetilde{\varphi}^{(n)}} + v^{(n)} \cdot \nabla^{\widetilde{\varphi}^{(n)}}$. Indeed, one has to set the weight of ∂_3 to be $v^{(n)} \cdot \widetilde{\mathbf{N}}^{(n-1)} - \partial_t \varphi^{(n)}$ to guarantee both the linearity of this operator and the consistency with the linearized kinematic boundary condition $\partial_t \psi^{(n+1)} = v^{(n+1)} \cdot \widetilde{N}^{(n)}$.

Remark 5.2. Note that the surface tension term in (5.1) is now replaced by a given term instead of being $-\sigma\overline{\nabla} \cdot (\overline{\nabla}\widetilde{\psi}^{(n+1)}/|\widetilde{N}^{(n)}|)$. Under this setting, we can still do energy estimates for $\psi^{(n+1)}$ by using the kinematic boundary condition and the viscosity term.

For simplicity of notations, for any $n \geq 0$, we denote $(v^{(n+1)}, \rho^{(n+1)}, q^{(n+1)}, \psi^{(n+1)})$, $(v^{(n)}, \rho^{(n)}, q^{(n)}, \psi^{(n)})$ and $\psi^{(n-1)}$ by (v, ρ, q, ψ), $(\mathring{v}, \mathring{\rho}, \mathring{q}, \mathring{\psi})$ and $\dot{\psi}$. Hence, we need to solve the following linearized version of system (3.11) for each fixed $\kappa > 0$ and then establish an energy estimate to proceed with the iteration scheme.

$$
\begin{cases}
\mathring{\rho} D_t^{\mathring{\widetilde{\varphi}}} v + \nabla^{\mathring{\widetilde{\varphi}}} \check{q} = -(\mathring{\rho} - 1) g e_3, & \text{in } [0,T] \times \Omega, \\
\mathring{\mathcal{F}}'(\mathring{q}) D_t^{\mathring{\widetilde{\varphi}}} \check{q} + \nabla^{\mathring{\widetilde{\varphi}}} \cdot v = \mathring{\mathcal{F}}'(\mathring{q}) g \mathring{v}_3, & \text{in } [0,T] \times \Omega, \\
q = q(\rho), \check{q} = q + g\mathring{\widetilde{\varphi}} & \text{in } [0,T] \times \Omega, \\
\check{q} = g\mathring{\widetilde{\psi}} - \sigma\overline{\nabla} \cdot \left(\frac{\overline{\nabla}\mathring{\widetilde{\psi}}}{\sqrt{1+|\overline{\nabla}\mathring{\widetilde{\psi}}|^2}} \right) + \kappa^2 (1 - \overline{\Delta})(v \cdot \mathring{\widetilde{N}}), & \text{on } [0,T] \times \Sigma, \\
\partial_t \psi = v \cdot \mathring{\widetilde{N}}, & \text{on } [0,T] \times \Sigma, \\
v_3 = 0 & \text{on } [0,T] \times \Sigma_b, \\
(v, \rho, \psi)|_{t=0} = (v_0^\kappa, \rho_0^\kappa, \psi_0^\kappa).
\end{cases}
\tag{5.6}
$$

Here $\mathring{\mathcal{F}} := \log \mathring{\rho}$. The linearized material derivative now becomes:

$$
D_t^{\mathring{\widetilde{\varphi}}} := \partial_t + \overline{\mathring{v}} \cdot \overline{\nabla} + \frac{1}{\partial_3 \mathring{\widetilde{\varphi}}} (\mathring{v} \cdot \dot{\widetilde{\mathbf{N}}} - \partial_t \mathring{\varphi}) \partial_3 \tag{5.7}
$$

and the covariant derivatives with respect to $\mathring{\widetilde{\varphi}}$ are defined to be

$$
\partial_t^{\mathring{\widetilde{\varphi}}} := \partial_t - \frac{\partial_t \mathring{\varphi}}{\partial_3 \mathring{\widetilde{\varphi}}} \partial_3, \tag{5.8}
$$

$$
\nabla_a^{\mathring{\widetilde{\varphi}}} = \partial_a^{\mathring{\widetilde{\varphi}}} = \partial_a - \frac{\partial_a \mathring{\widetilde{\varphi}}}{\partial_3 \mathring{\widetilde{\varphi}}} \partial_3, \quad a = 1, 2, \tag{5.9}
$$

$$
\nabla_3^{\mathring{\widetilde{\varphi}}} = \partial_3^{\mathring{\widetilde{\varphi}}} = \frac{1}{\partial_3 \mathring{\widetilde{\varphi}}} \partial_3, \tag{5.10}
$$

where $\overline{\mathring{v}} \cdot \overline{\nabla} := \mathring{v}_1 \partial_1 + \mathring{v}_2 \partial_2$. Note that, by the kinematic boundary condition, the normal component in $D_t^{\mathring{\widetilde{\varphi}}}$, namely $(\partial_3 \mathring{\widetilde{\varphi}})^{-1} (\mathring{v} \cdot \dot{\widetilde{\mathbf{N}}} - \partial_t \mathring{\varphi}) \partial_3$ vanishes on Σ.

From now on, we assume the following given quantities are bounded in some time interval $t \in [0, T^\kappa]$. This also works as the induction hypothesis for the uniform-in-n estimates for (5.6):

$$
\begin{aligned}
\|\mathring{\rho} - 1\|_0^2 + \sum_{k=0}^{4} \|\partial_t^k \mathring{v}\|_{4-k}^2 + \|\dot{\mathcal{F}}'(\dot{q}) \check{q}\|_0^2 + \|\partial \mathring{\check{q}}\|_3^2 + \sum_{k=1}^{3} \|\partial_t^k \mathring{\check{q}}\|_{4-k}^2 + \|\dot{\mathcal{F}}'(\dot{q}) \partial_t^4 \mathring{\check{q}}\|_0^2 \\
+ \kappa^4 |\mathring{\psi}|_{5.5}^2 + \sum_{k=0}^{3} \kappa^4 |\partial_t^{k+1} \mathring{\psi}, \partial_t^{k+1} \dot{\psi}|_{5.5-k}^2 + \kappa^2 \int_0^t |\partial_t^5 \mathring{\psi}|_1^2 \mathrm{d}\tau < \mathring{K}_0.
\end{aligned}
\tag{5.11}
$$

Here, the additional $\frac{1}{2}$-regularity for $\partial_t^j \mathring{\psi}$ and $\partial_t^j \dot{\psi}$, $j = 0, 1, 2, 3, 4$ is contributed by the artificial viscosity whenever $\kappa > 0$ is fixed.

5.1 Construction of solution to the linearized approximate system

We can prove that system (5.6) is a symmetric hyperbolic system with characteristic boundary conditions. Therefore, we want to use the duality argument developed by Lax-Phillips [40] to prove the local existence. Before doing this, we have to make sure the boundary conditions are homogeneous.

5.1.1 The homogeneous linearized approximate system

We introduce the variable $\mathring{\mathfrak{h}}$ defined by the harmonic extension

$$\begin{cases} -\Delta\mathring{\mathfrak{h}} = 0 & \text{in } \Omega, \\ \mathring{\mathfrak{h}} = g\mathring{\widetilde{\psi}} - \sigma\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\mathring{\widetilde{\psi}}}{\sqrt{1+|\overline{\nabla}\mathring{\widetilde{\psi}}|^2}}\right) & \text{on } \Sigma, \\ \partial_3\mathring{\mathfrak{h}} = 0 & \text{on } \Sigma_b, \end{cases} \tag{5.12}$$

and define $\underline{q} = \check{q} - \mathring{\mathfrak{h}}$. Then (5.6) becomes the following linear hyperbolic system with *homogeneous* boundary conditions:

$$\begin{cases} \mathring{\rho}D_t^{\mathring{\widetilde{\varphi}}}v + \nabla^{\mathring{\widetilde{\varphi}}}\underline{q} = -\nabla^{\mathring{\widetilde{\varphi}}}\mathring{\mathfrak{h}} - (\mathring{\rho}-1)ge_3, & \text{in } [0,T]\times\Omega, \\ \mathring{\mathcal{F}}'(\mathring{q})D_t^{\mathring{\widetilde{\varphi}}}\underline{q} + \nabla^{\mathring{\widetilde{\varphi}}}\cdot v = \mathring{\mathcal{F}}'(\mathring{q})(g\mathring{v}_3 - D_t^{\mathring{\widetilde{\varphi}}}\mathring{\mathfrak{h}}), & \text{in } [0,T]\times\Omega, \\ q = q(\rho),\ \underline{q} = q + g\mathring{\widetilde{\varphi}} - \mathring{\mathfrak{h}} & \text{in } [0,T]\times\Omega, \\ \underline{q} = \kappa^2(1-\overline{\Delta})(v\cdot\mathring{\widetilde{N}}), & \text{on } [0,T]\times\Sigma, \\ v_3 = 0 & \text{on } [0,T]\times\Sigma_b, \\ (v,\rho)|_{t=0} = (v_0,\rho_0). \end{cases} \tag{5.13}$$

Note that the coefficients in (5.13) rely on $\dot{\psi}, \mathring{\psi}, \mathring{v}$, and $\mathring{\rho}$ only, all of which are already given. The kinematic boundary condition, namely $\partial_t\psi = v\cdot\mathring{\widetilde{N}} = -(\overline{v}\cdot\overline{\nabla})\mathring{\widetilde{\psi}} + v_3$ on Σ, is used to define ψ after solving (v,q) from (5.13).

We define $U := (\underline{q}, v_1, v_2, v_3)^\top$, then (5.13) can be expressed in terms of U by

$$A_0(\mathring{U})\partial_t U + \sum_{i=1}^3 A_i(\mathring{U})\partial_i U = \mathring{f}, \tag{5.14}$$

where $\mathring{f} := \left(\mathring{\mathcal{F}}'(\mathring{q})(g\mathring{v}_3 - D_t^{\mathring{\widetilde{\varphi}}}\mathring{\mathfrak{h}}), -\partial_1^{\mathring{\widetilde{\varphi}}}\mathring{\mathfrak{h}}, -\partial_2^{\mathring{\widetilde{\varphi}}}\mathring{\mathfrak{h}}, -\partial_3^{\mathring{\widetilde{\varphi}}}\mathring{\mathfrak{h}} - (\mathring{\rho}-1)g\right)^\top$, $A_0(\mathring{U}) = \operatorname{diag}\left[\mathring{\mathcal{F}}'(\mathring{q}), \mathring{\rho}, \mathring{\rho}, \mathring{\rho}\right]$, and

$$A_i(\mathring{U}) = \begin{bmatrix} \mathring{\mathcal{F}}'(\mathring{q})\mathring{v}_i & e_i^\top \\ e_i & \mathring{\rho}\mathring{v}_i\mathbf{I}_3 \end{bmatrix} \text{ for } i = 1,2,\quad A_3(\mathring{U}) = \frac{1}{\partial_3\mathring{\widetilde{\varphi}}}\begin{bmatrix} \mathring{\mathcal{F}}'(\mathring{q})(\mathring{v}\cdot\dot{\widetilde{\mathbf{N}}} - \partial_t\mathring{\varphi}) & \mathring{\widetilde{\mathbf{N}}}^\top \\ \mathring{\widetilde{\mathbf{N}}} & \mathring{\rho}(\mathring{v}\cdot\dot{\widetilde{\mathbf{N}}} - \partial_t\mathring{\varphi})\mathbf{I}_3 \end{bmatrix}.$$

Since $(\partial_t\mathring{\varphi} - \mathring{v}\cdot\dot{\widetilde{\mathbf{N}}},)|_\Sigma = 0$ and $e_3 = (0,0,1)^\top$ is the unit exterior normal vector to Σ, we know that the boundary matrix, namely the normal projection of the coefficient matrices onto Σ, is

$$\sum_{i=1}^3 A_i(\mathring{U})e_{3i} = A_3(\mathring{U}) = \begin{bmatrix} 0 & \mathring{\widetilde{N}}^\top \\ \mathring{\widetilde{N}} & \mathbf{0}_3 \end{bmatrix} \quad \text{on } \Sigma$$

which is a 4×4 matrix of rank 2 (constant rank but not full rank) with one negative eigenvalue, one positive eigenvalue, and two zero eigenvalues. This being said, the system (5.13) is a first-order symmetric hyperbolic system with characteristic boundary conditions. The number of boundary conditions should be equal to the number of negative eigenvalues. Therefore, the correct number of boundary conditions for (5.13) is indeed equal to 1 which means (5.13) is solvable. After solving (5.13), we use the kinematic boundary condition to define ψ for the next step of the iteration.

5.1.2 Well-posedness in L^2 via μ-regularization

From the duality argument by Lax-Phillips [40], we need to prove the following in order to get the well-posedness of (5.13) in some function space X:

- We need to establish a priori estimate (without loss of regularity from the source term) for (5.13) in X.
- We need to establish a priori estimate (without loss of regularity from the source term) for the dual system of (5.13) in X'.

We choose $X = L^2([0,T]; L^2(\Omega))$ whose dual space X' is just itself. We define $W^* = (\underline{q}^*, w_1^*, w_2^*, w_3^*)^\top$ to be the dual variables of $U = (\underline{q}, v_1, v_2, v_3)^\top$. By testing (5.13) with W^* in $L^2([0,T]; L^2(\Omega))$, one can derive the system of W^* which reads

$$A_0(\mathring{U})\partial_t W^* + \sum_{i=1}^3 A_i(\mathring{U})\partial_i W^* + A_4(\mathring{U})W^* = \mathring{f}^*$$

with boundary condition $\underline{q}^*|_\Sigma = -\kappa^2(1-\overline{\Delta})(w^*\cdot\mathring{\widetilde{N}})$, where $A_4 := -\partial_t A_0^\top - \sum_{i=1}^3 \partial_i A_i^\top - (\mathring{\rho}-1)g\mathbf{E}_{44}$ with $\mathbf{E}_{44} = \text{diag}[0,0,0,1]$. Note that we do not have the dual variable for ψ because ψ is completely determined by the original linearized system. That is why we only have one boundary condition for the dual system.

We notice that there is an extra minus sign in the boundary condition for $\underline{q}^*$. So, one cannot close the L^2-type a priori estimate for the dual system even if we can derive that L^2-type a priori estimate for (5.13). To avoid this difficulty, we introduce another viscosity term in the boundary for $\underline{q}$ in (5.13). That is, we alternatively consider the μ-regularized linear problem for $U = (\underline{q}, v_1, v_2, v_3)^\top$, which reads

$$A_0(\mathring{U})\partial_t U + \sum_{i=1}^3 A_i(\mathring{U})\partial_i U = \mathring{f}, \tag{5.15}$$

with boundary condition

$$\underline{q} = \kappa^2(1-\overline{\Delta})(v\cdot\mathring{\widetilde{N}}) + \mu(1-\overline{\Delta})\partial_t(v\cdot\mathring{\widetilde{N}}) \quad \text{on } \Sigma. \tag{5.16}$$

Then the dual system of (5.15)-(5.16) reads

$$A_0(\mathring{U})\partial_t W^* + \sum_{i=1}^3 A_i(\mathring{U})\partial_i W^* + A_4(\mathring{U})W^* = \mathring{f}^* \tag{5.17}$$

with boundary condition

$$\underline{q}^* = -\kappa^2(1-\overline{\Delta})(w^*\cdot\mathring{\widetilde{N}}) + \mu(1-\overline{\Delta})\partial_t(w^*\cdot\mathring{\widetilde{N}}) \quad \text{on } \Sigma, \tag{5.18}$$

where $A_4 := -\partial_t A_0^\top - \sum_{i=1}^3 \partial_i A_i^\top - (\mathring{\rho}-1)g\mathbf{E}_{44}$ with $\mathbf{E}_{44} = \text{diag}[0,0,0,1]$. Note that we have to integrate by parts once more in t variable when deriving the boundary condition for $\underline{q}^*$. This is the reason that an extra minus sign appears in front of $\kappa^2(1-\overline{\Delta})(w^*\cdot\mathring{\widetilde{N}})$.

Now we are going to derive the a priori estimates for both (5.15) and (5.17). For linear system (5.15), we test it with U in $L^2(\Omega)$ and use the symmetry of the coefficient matrices to get:

$$\int_\Omega U^\top \cdot A_0(\mathring{U})U\,\mathrm{d}x = \int_\Omega U^\top\cdot\mathring{f} - \sum_{i=1}^3\int_\Omega U^\top\cdot\partial_i A_i(\mathring{U})U\,\mathrm{d}x - \int_\Sigma U^\top\cdot A_3(\mathring{U})U\,\mathrm{d}x, \tag{5.19}$$

where the interior integrals are directly controlled by $C(\mathring{K}_0)\|U\|_0^2$ and the boundary integral reads:

$$\begin{aligned} &-\int_\Sigma U^\top\cdot A_3(\mathring{U})U\,\mathrm{d}x' = -2\int_\Sigma (v\cdot\mathring{\widetilde{N}})\underline{q}\,\mathrm{d}x' \\ =& -2\kappa^2\int_\Omega \left((1-\overline{\Delta})(v\cdot\mathring{\widetilde{N}})\right)(v\cdot\mathring{\widetilde{N}})\,\mathrm{d}x' - 2\mu\int_\Sigma \partial_t\left((1-\overline{\Delta})(v\cdot\mathring{\widetilde{N}})\right)(v\cdot\mathring{\widetilde{N}})\,\mathrm{d}x' \\ =& -\mu\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma \left|\langle\overline{\partial}\rangle(v\cdot\mathring{\widetilde{N}})\right|_0^2\mathrm{d}x' - 2\kappa^2\left|\langle\overline{\partial}\rangle(v\cdot\mathring{\widetilde{N}})\right|_0^2. \end{aligned} \tag{5.20}$$

We define

$$\mathring{E}_0(t) := \|v(t)\|_0^2 + \left\|\sqrt{\mathring{\mathscr{F}}'(\mathring{q})}\underline{q}(t)\right\|_0^2 + \int_0^t |\kappa(v\cdot\mathring{\widetilde{N}})(\tau)|_1^2\mathrm{d}\tau + |\sqrt{\mu}(v\cdot\mathring{\widetilde{N}})(t)|_1^2,$$

then the above analysis shows that

$$\mathring{E}_0(T) - \mathring{E}_0(0) \leq C(\mathring{K}_0) \int_0^T \mathring{E}_0(t) + \sqrt{\mathring{E}_0(t)} \|\mathring{f}(t)\|_0 \, \mathrm{d}t. \tag{5.21}$$

and thus by Grönwall's inequality we finish the L^2-estimate for (5.15). Note that this a priori bound is also uniform in μ.

Next, we show the L^2-estimate for the dual system (5.17). Note that the matrix $A_4(\mathring{U})$ is still in $L^\infty(\Omega)$, so we test (5.17) by W^* and take L^2-inner product to get

$$\int_\Omega W^{*\top} \cdot A_0(\mathring{U}) W^* \, \mathrm{d}x = \int_\Omega W^{*\top} \cdot \mathring{f}^* - W^{*\top} \cdot \left(\sum_{i=1}^3 \partial_i A_i(\mathring{U}) + A_4(\mathring{U}) + \mathring{\rho} g \mathbf{E}_{44} \right) W^* \, \mathrm{d}x - \int_\Sigma (W^*)^\top \cdot A_3(\mathring{U}) W^* \, \mathrm{d}x', \tag{5.22}$$

where the interior integral is directly controlled by $C(\mathring{K}_0)\|W^*\|_0^2$, but now there is a sign change in the boundary integral, which reads:

$$\begin{aligned} & - \int_\Sigma (W^*)^\top \cdot A_3(\mathring{U}) W^* \, \mathrm{d}x' = -2 \int_\Sigma (w^* \cdot \mathring{\tilde{N}}) \underline{q}^* \, \mathrm{d}x' \\ = & 2\kappa^2 \int_\Omega \left((1 - \overline{\Delta})(w^* \cdot \mathring{\tilde{N}}) \right) (w^* \cdot \mathring{\tilde{N}}) \, \mathrm{d}x' - 2\mu \int_\Sigma \partial_t \left((1 - \overline{\Delta})(w^* \cdot \mathring{\tilde{N}}) \right) (w^* \cdot \mathring{\tilde{N}}) \, \mathrm{d}x' \\ \lesssim & -\mu \frac{\mathrm{d}}{\mathrm{d}t} \int_\Sigma \left| \langle \overline{\partial} \rangle (w^* \cdot \mathring{\tilde{N}}) \right|_0^2 \mathrm{d}x' + 2\kappa^2 \left| \langle \overline{\partial} \rangle (w^* \cdot \mathring{\tilde{N}}) \right|_0^2 . \end{aligned} \tag{5.23}$$

One can see that the new viscosity term involving μ controls the term $2\kappa^2 |\langle \overline{\partial} \rangle (w^* \cdot \mathring{\tilde{N}})|_0^2$ due to the change of sign.

So, if we define

$$\mathring{E}_0^*(t) = \|w^*(t)\|_0^2 + \left\| \sqrt{\mathcal{F}'(\mathring{q})} \underline{q}^*(t) \right\|_0^2 + \mu \left| (w^* \cdot \mathring{\tilde{N}})(t) \right|_1^2 ,$$

then we have

$$\mathring{E}_0^*(T) - \mathring{E}_0^*(0) \lesssim_{\mu^{-1}} C(\mathring{K}_0) \int_0^T \mathring{E}_0^*(t) + \sqrt{\mathring{E}_0^*(t)} \|\mathring{f}^*(t)\|_0 \, \mathrm{d}t, \tag{5.24}$$

and thus Grönwall's inequality helps us close the L^2-estimate.

Combining (5.21) and (5.24), we close the a priori bounds for both linear systems (5.15)-(5.16) and its dual system (5.17)-(5.18). Such energy bounds have no regularity loss from their source terms to solutions. Therefore, by the argument in Lax-Phillips [40](see also [56, Theorem 5,9]), for each fixed $\mu > 0$, system (5.17)-(5.18) admits a unique solution $U \in L^2([0,T]; L^2(\Omega))$. Since the energy bound (5.21) for (5.15)-(5.16) is uniform in μ, we can take the limit $\mu \to 0_+$ to obtain a local-in-time solution of the homogeneous linearized problem (5.13). Finally, the modification $\mathring{\mathfrak{h}}$ is easily controlled by using the property of the harmonic function

$$\forall s > -\frac{1}{2}, \ \ \|\mathring{\mathfrak{h}}\|_{s+\frac{1}{2}} \lesssim |\mathring{\mathfrak{h}}|_s \leq g|\mathring{\tilde{\psi}}|_s + P(|\overline{\nabla} \mathring{\tilde{\psi}}|_s) |\overline{\nabla}^2 \mathring{\tilde{\psi}}|_s,$$

which implies the local existence for L^2-(weak)-solution to the linearized κ-approximate system (5.6). By the argument in [51, Section 2.2.3](see also [56, Theorem 4, 8]), the weak solution U is actually a strong solution.

5.2 Higher-order estimates for the linearized system

Now we prove higher-order energy estimates for the linearized system (5.6).

Proposition 5.1. Let

$$\begin{aligned} \mathring{E}^\kappa(t) := & \|\rho(t) - 1\|_0^2 + \sum_{k=0}^4 \|\partial_t^k v(t)\|_{4-k}^2 + \kappa^2 \int_0^t \left| \partial_t^{k+1} \psi(\tau) \right|_{5-k}^2 \mathrm{d}\tau \\ & + \| \sqrt{\mathcal{F}'(\mathring{q})} \check{q}(t)\|_0^2 + \|\partial \check{q}(t)\|_3^2 + \sum_{k=1}^3 \|\partial_t^k \check{q}(t)\|_{4-k}^2 + \| \sqrt{\mathcal{F}'(\mathring{q})} \partial_t^4 \check{q}(t)\|_0^2. \end{aligned} \tag{5.25}$$

Then there exists some $T^\kappa > 0$ depending on κ and a constant $C(\kappa^{-1}, \mathring{K}_0) > 0$, such that

$$\sup_{0\le t\le T^\kappa} \mathring{E}^\kappa(t) \le C(\kappa^{-1}, \mathring{K}_0)\mathring{E}^\kappa(0). \tag{5.26}$$

Apart from that, we have

$$|\psi(t)|_{5.5}^2 + \sum_{k=0}^{3} |\partial_t^{k+1}\psi(t)|_{5.5-k}^2 \le C(\kappa^{-1}, \mathring{K}_0)\mathring{E}^\kappa(t), \quad \text{for all } t \in [0, T^\kappa]. \tag{5.27}$$

5.2.1 L^2-estimate

We define the L^2-energy for the linearized system (5.6) to be

$$\mathring{E}_0^\kappa(t) := \|\rho(t) - 1\|_0^2 + \|v(t)\|_0^2 + \|\sqrt{\mathring{\mathcal{F}}'(\mathring{q})}\check{q}(t)\|_0^2 + \kappa^2 \int_0^t |\partial_t\psi(\tau)|_1^2 \mathrm{d}\tau. \tag{5.28}$$

The control of $\mathring{E}_0$ is identical to the a priori estimate for (5.15) when $\mu = 0$. Note that the control of $\|\rho - 1\|_0^2$ follows from testing the linearized continuity equation $\mathring{\mathcal{F}}'(\mathring{q})\mathcal{F}'(q)^{-1}D_t^{\mathring{\tilde{\varphi}}}(\rho - 1) + \rho(\nabla^{\mathring{\tilde{\varphi}}} \cdot v) = 0$ by $\rho - 1$ in $L^2(\Omega)$. Also one can control the $L^2(\Sigma)$ norm of ψ through $\psi(t) = \psi_{0,\kappa} + \int_0^t \partial_t\psi(\tau)\mathrm{d}\tau$.

5.2.2 Div-Curl analysis

To estimate the Sobolev norms of v, we invoke the following Hodge decomposition lemma which is exactly from [10, Theorem 1.1].

Lemma 5.2 (Hodge elliptic estimates)**.** For any sufficiently smooth vector field X and $s \ge 1$, one has

$$\|X\|_s^2 \lesssim C(|\mathring{\tilde{\psi}}|_{s+\frac12}, |\overline{\nabla}\mathring{\tilde{\psi}}|_{W^{1,\infty}})\left(\|X\|_0^2 + \|\nabla^{\mathring{\tilde{\varphi}}} \cdot X\|_{s-1}^2 + \|\nabla^{\mathring{\tilde{\varphi}}} \times X\|_{s-1}^2 + |X \cdot \mathring{\tilde{N}}|_{s-\frac12}^2 + |X_3|^2_{H^{s-\frac12}(\Sigma_b)}\right), \tag{5.29}$$

where the constant $C(|\mathring{\tilde{\psi}}|_{s+\frac12}, |\overline{\nabla}\mathring{\tilde{\psi}}|_{W^{1,\infty}}) > 0$ depends linearly on $|\mathring{\tilde{\psi}}|_{s+\frac12}^2$.

Applying this lemma to v with $s = 4$, one has

$$\|v\|_4^2 \lesssim C(|\mathring{\tilde{\psi}}|_{4.5}, |\overline{\nabla}\mathring{\tilde{\psi}}|_{W^{1,\infty}})\left(\|v\|_0^2 + \|\nabla^{\mathring{\tilde{\varphi}}} \cdot v\|_3^2 + \|\nabla^{\mathring{\tilde{\varphi}}} \times v\|_3^2 + |v \cdot \mathring{\tilde{N}}|_{3.5}^2\right). \tag{5.30}$$

Now we control the curl term. Taking $\nabla^{\mathring{\tilde{\varphi}}}\times$ in the first equation of (5.6), we get the evolution equation satisfied by $\nabla^{\mathring{\tilde{\varphi}}} \times v$:

$$\mathring{\rho}D_t^{\mathring{\tilde{\varphi}}}(\nabla^{\mathring{\tilde{\varphi}}} \times v) = \mathring{\rho}[\nabla^{\mathring{\tilde{\varphi}}}\times, D_t^{\mathring{\tilde{\varphi}}}]v + \nabla^{\mathring{\tilde{\varphi}}}\mathring{\rho} \times (\mathring{\rho}^{-1}\nabla^{\mathring{\tilde{\varphi}}}\check{q}), \tag{5.31}$$

and taking three derivatives we get

$$\mathring{\rho}D_t^{\mathring{\tilde{\varphi}}}\partial^3(\nabla^{\mathring{\tilde{\varphi}}} \times v) = \partial^3\left(\mathring{\rho}[\nabla^{\mathring{\tilde{\varphi}}}\times, D_t^{\mathring{\tilde{\varphi}}}]v) + \nabla^{\mathring{\tilde{\varphi}}}\mathring{\rho} \times (\mathring{\rho}^{-1}\nabla^{\mathring{\tilde{\varphi}}}\check{q})\right) - [\partial^3, \mathring{\rho}D_t^{\mathring{\tilde{\varphi}}}](\nabla^{\mathring{\tilde{\varphi}}} \times v). \tag{5.32}$$

We expect that the source terms in (5.32) only contain ≤ 4 derivatives of $v, \check{q}$ and quantities marked with a ring, but there still exists a mismatched term in $([\nabla^{\mathring{\tilde{\varphi}}}\times, D_t^{\mathring{\tilde{\varphi}}}]v)^i = \epsilon^{ijk}\nabla_j^{\mathring{\tilde{\varphi}}}\mathring{v}^l\nabla_l^{\mathring{\tilde{\varphi}}}v_k + \epsilon^{ijk}\nabla_j^{\mathring{\tilde{\varphi}}}\partial_t(\varphi - \dot{\tilde{\varphi}})\partial_3^{\mathring{\tilde{\varphi}}}v_k$. The contribution of $\dot{\tilde{\psi}}$ is controlled by $\mathring{K}_0$. So, the standard L^2-estimate for the ∂^3-differentiated evolution equation of $\nabla^{\mathring{\tilde{\varphi}}} \times v$ and Reynold transport formula (A.9) gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^{\mathring{\tilde{\varphi}}} \times v\|_3^2 \le P(\mathring{K}_0)(\|v\|_4^2 + \|\check{q}\|_4\|v\|_4 + |\partial_t\psi|_4\|\partial v\|_\infty). \tag{5.33}$$

Finally, using the linearized continuity equation, we can control the divergence

$$\|\nabla^{\mathring{\tilde{\varphi}}} \cdot v\|_3^2 \le \left\|\mathring{\mathcal{F}}'(\mathring{q})D_t^{\mathring{\tilde{\varphi}}}\check{q}\right\|_3^2 + \left\|\mathring{\mathcal{F}}'(\mathring{q})g\mathring{v}_3\right\|_3^2. \tag{5.34}$$

The div-curl analysis for the time derivatives is proceeded similarly. First, the div-curl analysis for $\|\partial_t^k v\|_{4-k}^2$, $1 \le k \le 3$ yields

$$\|\partial_t^k v\|_{4-k}^2 \lesssim C(|\tilde{\mathring{\psi}}|_{4.5-k}, |\overline{\nabla}\tilde{\mathring{\psi}}|_{W^{1,\infty}}) \left(\|\partial_t^k v\|_0^2 + \|\nabla^{\tilde{\mathring{\varphi}}} \cdot \partial_t^k v\|_{3-k}^2 + \|\nabla^{\tilde{\mathring{\varphi}}} \times \partial_t^k v\|_{3-k}^2 + |\partial_t^k v \cdot \tilde{\mathring{N}}|_{3.5-k}^2 \right). \tag{5.35}$$

We replace ∂^3 by $\partial_t^k \partial^{3-k}$ for $0 \le k \le 3$ in (5.32) to get the evolution equation:

$$\mathring{\rho} D_t^{\tilde{\mathring{\varphi}}} \left(\partial^{3-k} \partial_t^k (\nabla^{\tilde{\varphi}} \times v) \right) = \partial_t^k \partial^{3-k} \left(\mathring{\rho} [\nabla^{\tilde{\mathring{\varphi}}} \times, D_t^{\tilde{\mathring{\varphi}}}] v) + \nabla^{\tilde{\mathring{\varphi}}} \mathring{\rho} \times (\mathring{\rho}^{-1} \nabla^{\tilde{\mathring{\varphi}}} \check{q}) \right) - [\partial_t^k \partial^{3-k}, \mathring{\rho} D_t^{\tilde{\mathring{\varphi}}}] (\nabla^{\tilde{\mathring{\varphi}}} \times v), \tag{5.36}$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \|\partial_t^k (\nabla^{\tilde{\mathring{\varphi}}} \times v)\|_{3-k}^2 \le P(\mathring{E}^\kappa(t)). \tag{5.37}$$

Now, since the leading order term in the commutator $[\partial_t^k, \nabla^{\tilde{\mathring{\varphi}}} \times] v$ should be $\overline{\partial} \partial_t \tilde{\mathring{\varphi}} \partial_t^{k-1} \partial_3 v$, we have

$$\|\nabla^{\tilde{\mathring{\varphi}}} \times \partial_t^k v\|_{3-k}^2 \le C(\mathring{K}_0) \left(\mathring{E}^\kappa(0) + \int_0^T \mathring{E}^\kappa(t) \,\mathrm{d}t \right). \tag{5.38}$$

As for divergence, by taking ∂_t^k, $1 \le k \le 3$ in the continuity equation, we get

$$\nabla^{\tilde{\mathring{\varphi}}} \cdot \partial_t^k v = -\partial_t^k (\mathring{\mathcal{F}}'(\mathring{q}) D_t^{\tilde{\mathring{\varphi}}} \check{q} + \mathring{\mathcal{F}}'(\mathring{q}) g \mathring{v}_3) + [\nabla^{\tilde{\mathring{\varphi}}} \cdot, \partial_t^k] v \stackrel{L}{=} -\mathring{\mathcal{F}}'(\mathring{q}) (\partial_t^k D_t^{\tilde{\mathring{\varphi}}} \check{q} + g \partial_t^k \mathring{v}_3) + (\partial_3 \tilde{\mathring{\varphi}})^{-1} \overline{\partial} \partial_t^k \tilde{\mathring{\varphi}} \partial_3 v.$$

Parallel to the analysis for (4.25), since $\|\overline{\partial} \partial_t^k \tilde{\mathring{\varphi}}\|_{3-k} \le \mathring{K}_0$ thanks to (5.11), we have $\|\nabla^{\tilde{\mathring{\varphi}}} \cdot \partial_t^k v\|_{3-k}$ is reduced to the control of $\|\mathring{\mathcal{F}}'(\mathring{q}) \partial_t^{k+1} \check{q}\|_{3-k}$ and $\|\mathring{\mathcal{F}}'(\mathring{q}) \partial_t^k \check{q}\|_{4-k}$ at the top order. Thus,

$$\|\nabla^{\tilde{\mathring{\varphi}}} \cdot \partial_t^k v\|_{3-k}^2 \le (C(\mathring{K}_0) + 1) \left(\left\| \mathring{\mathcal{F}}'(q) \partial_t^{k+1} \check{q} \right\|_{3-k}^2 + \left\| \mathring{\mathcal{F}}'(q) \partial_t^k \check{q} \right\|_{4-k}^2 \right). \tag{5.39}$$

5.2.3 Estimates for ψ and normal traces

The normal trace terms in (5.30) and (5.35) can be directly controlled by applying boundary elliptic estimates to the linearized viscous surface tension equation $\kappa^2 (1 - \overline{\Delta})(v \cdot \tilde{\mathring{N}}) = q - \sigma \mathcal{H}(\overline{\nabla} \tilde{\mathring{\psi}}, \overline{\nabla}^2 \tilde{\mathring{\psi}})$. We start with controlling $|v \cdot \tilde{\mathring{N}}|_{3.5}$:

$$|v \cdot \tilde{\mathring{N}}|_{3.5}^2 \le \kappa^{-2} \left(|q|_{1.5}^2 + \sigma |\overline{\nabla}^2 \tilde{\mathring{\psi}}|_{1.5}^2 P(|\overline{\nabla} \tilde{\mathring{\psi}}|_{1.5}) \right) \le \kappa^{-2} P(\mathring{K}_0) \|\check{q}\|_2^2. \tag{5.40}$$

Taking time derivatives in the kinematic boundary condition, we obtain:

$$\partial_t^k v \cdot \tilde{\mathring{N}} = \partial_t^{k+1} \psi - \sum_{j=1}^k \binom{k}{j} \partial_t^{k-j} \overline{v} \cdot \partial_t^j \overline{\nabla} \tilde{\mathring{\psi}},$$

and thus

$$|\partial_t v \cdot \tilde{\mathring{N}}|_{2.5} \le |\partial_t^2 \psi|_{2.5} + |\overline{v} \cdot \overline{\nabla} \partial_t \tilde{\mathring{\psi}}|_{2.5} \le |\partial_t^2 \psi|_{2.5} + \|v_{\kappa,0}\|_3^2 + P(\mathring{K}_0) \int_0^T \|\partial_t \overline{v}(t)\|_3 \,\mathrm{d}t. \tag{5.41}$$

Then we take a time derivative in the linearized viscous surface tension equation to get

$$\kappa (1 - \overline{\Delta}) \partial_t^2 \psi = \partial_t q - \sigma \partial_t \mathcal{H}(\overline{\nabla} \tilde{\mathring{\psi}}, \overline{\nabla}^2 \tilde{\mathring{\psi}}),$$

which implies $|\partial_t^2 \psi|_{2.5} \le \|\partial_t q\|_1 + P(\mathring{K}_0)$. Repeatedly, we can take more time derivatives to obtain

$$|\partial_t^k v \cdot \tilde{\mathring{N}}|_{3.5-k}^2 \le |\partial_t^{k+1} \psi|_{3.5-k}^2 + \mathcal{P}_0^\kappa + P(\mathring{K}_0) \int_0^T \mathring{E}^\kappa(t) \,\mathrm{d}t, \tag{5.42}$$

and then $|\partial_t^{k+1} \psi|_{3.5-k}$ is controlled via boundary elliptic estimates:

$$|\partial_t^3 \psi|_{1.5}^2 \approx |\langle \overline{\partial} \rangle^{-\frac{1}{2}} \partial_t^3 \psi|_2^2 \le |\langle \overline{\partial} \rangle^{-\frac{1}{2}} \partial_t^2 \check{q}|_0^2 + P(\mathring{K}_0) \le \|\partial_t^2 \check{q}\|_1^2 + P(\mathring{K}_0) \le \|\partial_t^2 \check{q}(0)\|_1^2 + P(\mathring{K}_0) + \int_0^T \mathring{E}^\kappa(t) \,\mathrm{d}t, \tag{5.43}$$

$$|\partial_t^4 \psi|_{0.5} \approx |\langle \overline{\partial} \rangle^{-\frac{3}{2}} \partial_t^4 \psi|_2 \le |\langle \overline{\partial} \rangle^{-\frac{3}{2}} \partial_t^3 \check{q}|_0 + P(\mathring{K}_0) \le \|\partial_t^3 \check{q}\|_1 + P(\mathring{K}_0), \tag{5.44}$$

where the leading order term $\|\partial \partial_t^3 \check{q}\|_0$ on the RHS of (5.44) will be further reduced through the reduction scheme shown in the upcoming subsection.

5.2.4 Reduction of pressure

We start with $\|\check{q}\|_4$. From the linearized momentum equation, we know

$$-(\partial_3\mathring{\widetilde{\varphi}})^{-1}\partial_3 q = (\mathring{\rho}-1)g + \mathring{\rho}D_t^{\mathring{\widetilde{\varphi}}}v_3,$$
$$-\partial_i q = (\partial_3\mathring{\widetilde{\varphi}})^{-1}\overline{\partial}_i\mathring{\widetilde{\varphi}}\partial_3 q + \mathring{\rho}D_t^{\mathring{\widetilde{\varphi}}}v_i,\quad i=1,2,$$

and thus we have the following estimates after taking ∂^3 and using $D_t^{\mathring{\widetilde{\varphi}}} = (\partial_t + \overline{\mathring{v}}\cdot\overline{\nabla}) + (\partial_3\mathring{\widetilde{\varphi}})^{-1}(\mathring{v}\cdot\mathring{\widetilde{\mathbf{N}}} - \partial_t\mathring{\varphi})\partial_3$ to get

$$\|\check{q}\|_4 \lesssim_{\mathring{K}_0} \|\check{q}\|_0 + \|\mathcal{T}v\|_3 + \|\mathring{\rho}-1\|_3, \tag{5.45}$$

where $\mathcal{T}$ denotes a tangential derivative, including $\partial_t, \overline{\partial}$ and $\omega(x)\partial_3$ for some weight function ω that vanishes on Σ and is approximately equal to $|x_3|$ near Σ. Replacing ∂^3 by $\partial^{3-k}\partial_t^k$, we know the estimate of $\|\partial_t^k\partial^{4-k}\check{q}\|_0$ is reduced to the estimate of $\|\partial_t^k\mathcal{T}v\|_{3-k}$. Combining this with the div-curl analysis in Subsection 5.2.2 we can reduce the top order mixed norms $\|\partial_t^k\partial^{4-k}v\|_0$ and $\|\partial_t^k\partial^{4-k}\check{q}\|_0$ to $\|\mathcal{T}^\alpha v\|_0$, $|\alpha|=4$, and $\left\|\sqrt{\mathring{\mathcal{F}}'(\mathring{q})}\partial_t^4\check{q}\right\|_0$, all of which are part of the tangential energy.

5.2.5 Control of full time derivatives

From the reduction procedures for $\check{q}$ and the div-curl analysis for v, we know a spatial derivative of $\check{q}$ is reduced to a tangential derivative of v, and the divergence of v is reduced to $\mathring{\mathcal{F}}'(\mathring{q})\partial_t\check{q}$. Repeatedly, it remains to control $\sqrt{\mathring{\mathcal{F}}'(\mathring{q})}\partial_t^4\check{q}$ and $\mathcal{T}^\alpha v$ with $|\alpha|=4$ in $L^2(\Omega)$. Here we only present the proof for the estimate with full-time derivatives which is parallel to Section 4.6, and the mixed space-time tangential estimates are easier. We introduce the Alinhac good unknowns $\mathring{\mathbf{V}}, \mathring{\mathbf{Q}}$ for the ∂_t^4-differentiated linearized system (5.6):

$$\mathring{\mathbf{V}} := \partial_t^4 v - \partial_t^4\mathring{\widetilde{\varphi}}\partial_3^{\mathring{\widetilde{\varphi}}}v,\quad \mathring{\mathbf{Q}} := \partial_t^4\check{q} - \partial_t^4\mathring{\widetilde{\varphi}}\partial_3^{\mathring{\widetilde{\varphi}}}\check{q} \tag{5.46}$$

Similar to the arguments in Section 4.6, when $f=v_i$ and $\check{q}$, the following identity holds:

$$\partial_t^4(\nabla_i^{\mathring{\widetilde{\varphi}}}f) = \nabla_i^{\mathring{\widetilde{\varphi}}}\mathring{\mathbf{F}} + \mathring{\mathfrak{C}}_i(f), \tag{5.47}$$

where $\mathring{\mathfrak{C}}_i(f) := \partial_3^{\mathring{\widetilde{\varphi}}}\partial_i^{\mathring{\widetilde{\varphi}}}f\,\partial_t^4\mathring{\widetilde{\varphi}} + \mathring{\mathfrak{C}}_i'(f)$. Also,

$$\mathring{\mathfrak{C}}_i'(f) = -\left[\partial_t^4, \frac{\partial_i\mathring{\widetilde{\varphi}}}{\partial_3\mathring{\widetilde{\varphi}}}, \partial_3 f\right] - \partial_3 f\left[\partial_t^4, \partial_i\mathring{\widetilde{\varphi}}, \frac{1}{\partial_3\mathring{\widetilde{\varphi}}}\right] - \partial_i\mathring{\widetilde{\varphi}}\partial_3 f\left[\partial_t^3, \frac{1}{(\partial_3\mathring{\widetilde{\varphi}})^2}\right]\partial_t\partial_3\mathring{\widetilde{\varphi}},\quad i=1,2 \tag{5.48}$$

$$\mathring{\mathfrak{C}}_3'(f) = \left[\partial_t^4, \frac{1}{\partial_3\mathring{\widetilde{\varphi}}}, \partial_3 f\right] + \partial_3 f\left[\partial_t^3, \frac{1}{(\partial_3\mathring{\widetilde{\varphi}})^2}\right]\partial_t\partial_3\mathring{\widetilde{\varphi}}. \tag{5.49}$$

Then we take ∂_t^4 to the first two equations of (5.6) to obtain:

$$\mathring{\rho}D_t^{\mathring{\widetilde{\varphi}}}\mathring{\mathbf{V}}_i + \nabla_i^{\mathring{\widetilde{\varphi}}}\mathring{\mathbf{Q}} = \mathring{\mathcal{R}}_i^1, \tag{5.50}$$

$$\mathring{\mathcal{F}}'(\mathring{q})D_t^{\mathring{\widetilde{\varphi}}}\mathring{\mathbf{Q}} + \nabla^{\mathring{\widetilde{\varphi}}}\cdot\mathring{\mathbf{V}} = \mathring{\mathcal{R}}^2 - \mathring{\mathfrak{C}}_i(v^i), \tag{5.51}$$

where

$$\mathring{\mathcal{R}}_i^1 := -[\partial_t^4, \mathring{\rho}]D_t^{\mathring{\widetilde{\varphi}}}v_i - \mathring{\rho}\left(\mathring{\mathfrak{D}}(v_i) + \mathring{\mathfrak{e}}(v_i)\right) - \mathring{\mathfrak{C}}_i(\check{q}) - \partial_t^4\mathring{\rho}g\delta_{3i}, \tag{5.52}$$

$$\mathring{\mathcal{R}}^2 := -[\partial_t^4, \mathring{\mathcal{F}}'(\mathring{q})]D_t^{\mathring{\widetilde{\varphi}}}\check{q} - \mathring{\mathcal{F}}'(\mathring{q})\left(\mathring{\mathfrak{D}}(\check{q}) + \mathring{\mathfrak{e}}(\check{q})\right) + \partial_t^4(\mathring{\mathcal{F}}'(\mathring{q})g\mathring{v}_3), \tag{5.53}$$

and the commutators $\mathring{\mathfrak{D}}(f), \mathring{\mathfrak{e}}(f)$ are defined in the same way as in (4.39) and (4.40) by replacing $\mathcal{T}^\alpha, \overline{\partial}, \widetilde{\varphi}$ respectively with $\partial_t^4, \partial_t, \mathring{\widetilde{\varphi}}$. The last two terms in (4.39) vanish because ∂_t^4 commutes with ∂_3. Specifically, we have:

$$\partial_t^4 D_t^{\mathring{\widetilde{\varphi}}}f = D_t^{\mathring{\widetilde{\varphi}}}\mathring{\mathbf{F}} + \mathring{\mathfrak{D}}(f) + \mathring{\mathfrak{e}}(f), \tag{5.54}$$

where $\mathring{\mathfrak{D}}(f) := (D_t^{\tilde{\mathring{\varphi}}} \partial_3^{\tilde{\mathring{\varphi}}} f)\partial_t^4 \tilde{\mathring{\varphi}} + \mathring{\mathfrak{D}}'(f)$, and

$$\mathring{\mathfrak{D}}'(f) = [\partial_t^4, \bar{\mathring{v}}] \cdot \bar{\partial} f + \left[\partial_t^4, \frac{1}{\partial_3 \tilde{\mathring{\varphi}}}(\mathring{v} \cdot \dot{\tilde{\mathbf{N}}} - \partial_t \dot{\varphi}), \partial_3 f\right] + \left[\partial_t^4, \mathring{v} \cdot \dot{\tilde{\mathbf{N}}} - \partial_t \dot{\varphi}, \frac{1}{\partial_3 \tilde{\mathring{\varphi}}}\right] \partial_3 f + \frac{1}{\partial_3 \tilde{\mathring{\varphi}}} [\partial_t^4, \mathring{v}] \cdot \dot{\tilde{\mathbf{N}}} \partial_3 f$$
$$- 4(\mathring{v} \cdot \dot{\tilde{\mathbf{N}}} - \partial_t \dot{\varphi}) \partial_3 f \left[\partial_t^3, \frac{1}{(\partial_3 \tilde{\mathring{\varphi}})^2}\right] \partial_t \partial_3 \tilde{\mathring{\varphi}}, \tag{5.55}$$

$$\mathring{\mathfrak{e}}(f) := \partial_t^5 (\dot{\tilde{\varphi}} - \dot{\varphi}) \partial_3^{\tilde{\varphi}} f. \tag{5.56}$$

Analogous to Lemma 4.4, the following estimates hold.

Lemma 5.3. Let $\mathring{\mathbf{F}} := \partial_t^4 f - \partial_3^{\tilde{\mathring{\varphi}}} f \partial_t^4 \tilde{\mathring{\varphi}}$ be the Alinhac good unknowns associated with the smooth function f. Assume $\partial_3 \tilde{\varphi} \geq c_0 > 0$, and let $\mathring{\mathfrak{C}}(f)$, $\mathring{\mathfrak{D}}(f)$, and $\mathring{\mathfrak{e}}(f)$ be the remainder terms defined as above. Then

$$\|\partial_t^4 f\|_0 \leq \|\mathring{\mathbf{F}}\|_0 + c_0^{-1} \|\partial_3 f\|_\infty |\partial_t^4 \tilde{\psi}|_0, \tag{5.57}$$

$$\|\mathring{\mathfrak{C}}_i(f)\|_0 \leq P\left(c_0^{-1}, |\bar{\nabla}\tilde{\mathring{\psi}}|_\infty, \sum_{k=1}^3 |\bar{\nabla}\partial_t^k \tilde{\mathring{\psi}}|_{3-k}\right) \cdot \left(\|\partial f\|_\infty + \sum_{k=1}^3 \|\partial_t^k f\|_{4-k}\right), \quad i = 1, 2, 3, \tag{5.58}$$

$$\|\mathring{\mathfrak{D}}(f)\|_0 \leq P\left(c_0^{-1}, |\bar{\nabla}\tilde{\mathring{\psi}}|_\infty, \sum_{k=1}^3 |\bar{\nabla}\partial_t^k \tilde{\mathring{\psi}}, \bar{\nabla}\partial_t^k \dot{\tilde{\psi}}|_{3-k}\right) \cdot \left(\|\partial f\|_\infty + \sum_{k=1}^3 \|\partial_t^k f\|_{4-k}\right), \tag{5.59}$$

$$\|\mathring{\mathfrak{e}}(f)\|_0 \leq (|\partial_t^5 \psi|_0 + |\partial_t^5 \dot{\tilde{\psi}}|_0) \|\partial f\|_\infty. \tag{5.60}$$

Next, we introduce the boundary conditions for $\mathring{\mathbf{V}}, \mathring{\mathbf{Q}}$. The ∂_t^4-differentiated linearized kinematic boundary condition now reads:

$$\partial_t^5 \psi + (\bar{v} \cdot \bar{\nabla}) \bar{\partial}^4 \tilde{\mathring{\psi}} - \mathring{\mathbf{V}} \cdot \tilde{\mathring{N}} = \mathring{\mathcal{S}}_1, \quad \text{on } \Sigma, \tag{5.61}$$

where

$$\mathring{\mathcal{S}}_1 := \partial_3 v \cdot \tilde{\mathring{N}} \partial_t^4 \tilde{\mathring{\psi}} + \sum_{1 \leq j \leq 3} \binom{4}{j} \partial_t^j v \cdot \partial_t^{4-j} \tilde{\mathring{N}}. \tag{5.62}$$

Also, since $\mathring{\mathbf{Q}}|_\Sigma = \partial_t^4 \check{q} - \partial_3^{\tilde{\mathring{\varphi}}} \check{q} \partial_t^4 \tilde{\mathring{\psi}}$, the boundary condition of $\mathring{\mathbf{Q}}$ on Σ reads:

$$\mathring{\mathbf{Q}} = -\sigma \partial_t^4 \bar{\nabla} \cdot \left(\frac{\bar{\nabla}\tilde{\mathring{\psi}}}{\sqrt{1 + |\bar{\nabla}\tilde{\mathring{\psi}}|^2}}\right) + \kappa^2 (1 - \bar{\Delta}) \partial_t^5 \psi - \partial_3 \check{q} \partial_t^4 \tilde{\mathring{\psi}} + g \partial_t^4 \tilde{\mathring{\psi}}. \tag{5.63}$$

Invoking (A.9), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_\Omega \mathring{\rho} |\mathring{\mathbf{V}}|^2 \partial_3 \tilde{\mathring{\varphi}} \, \mathrm{d}x &= \frac{1}{2} \int_\Omega |\mathring{\mathbf{V}}|^2 \left((D_t^{\tilde{\mathring{\varphi}}} \mathring{\rho} + \mathring{\rho} \nabla^{\tilde{\mathring{\varphi}}} \cdot \mathring{v}) \partial_3 \tilde{\mathring{\varphi}} + \mathring{\rho} \mathring{M} \right) \mathrm{d}x \\ &+ \int_\Omega \mathring{\mathbf{Q}} (\nabla^{\tilde{\mathring{\varphi}}} \cdot \mathring{\mathbf{V}}) \partial_3 \tilde{\mathring{\varphi}} \, \mathrm{d}x - \int_\Sigma \mathring{\mathbf{Q}} (\mathring{\mathbf{V}} \cdot \tilde{\mathring{N}}) \, \mathrm{d}x' + \int_\Omega \mathring{\mathbf{V}} \cdot \mathring{\mathcal{R}}^1 \partial_3 \tilde{\varphi} \, \mathrm{d}x, \end{aligned} \tag{5.64}$$

where $\mathring{M} := \partial_t \partial_3 (\dot{\tilde{\varphi}} - \dot{\varphi}) + \partial_3 (\partial_t + \bar{\mathring{v}} \cdot \bar{\nabla})(\tilde{\mathring{\varphi}} - \dot{\tilde{\varphi}})$ represents the mismatched terms involving tangential smoothing in (A.9). The first integral on the RHS can be directly controlled by $P(\mathring{K}_0)$ because all these quantities are already given. Moreover, the last integral is directly controlled by $P(\mathring{K}_0) \|\mathring{\mathbf{V}}\|_0 \sqrt{\mathring{E}^\kappa(t)}$. For the second term in (5.64), we invoke (5.51) to get the estimates parallel to (4.117):

$$
\begin{aligned}
&\int_\Omega \mathring{\mathbf{Q}}(\nabla^{\mathring{\tilde{\varphi}}}\cdot\mathring{\mathbf{V}})\partial_3\mathring{\tilde{\varphi}}\,\mathrm{d}x\\
=&\underbrace{-\int_\Omega \partial_t^4\check{q}\mathring{\mathfrak{C}}_i(v^i)\partial_3\mathring{\tilde{\varphi}}\,\mathrm{d}x}_{=:\mathring{I}_0}+\int_\Omega \partial_t^4\mathring{\tilde{\varphi}}\partial_3^{\mathring{\tilde{\varphi}}}\check{q}\mathring{\mathfrak{C}}_i(v^i)\partial_3\mathring{\tilde{\varphi}}\,\mathrm{d}x-\int_\Omega \mathring{\mathcal{F}}'(\mathring{q})D_t^{\mathring{\tilde{\varphi}}}\mathring{\mathbf{Q}}\;\mathring{\mathbf{Q}}\partial_3\mathring{\tilde{\varphi}}\,\mathrm{d}x+\int_\Omega \mathring{\mathcal{R}}^2\mathring{\mathbf{Q}}\partial_3\mathring{\tilde{\varphi}}\,\mathrm{d}x\\
\lesssim&\,\mathring{I}_0-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\sqrt{\mathring{\mathcal{F}}'(\mathring{q})}\mathbf{Q}\right\|_0^2+\left\|\sqrt{\mathring{\mathcal{F}}'(\mathring{q})}\partial_t^4\check{q}\right\|_0^2(\|\nabla^{\mathring{\tilde{\varphi}}}\cdot\mathring{v}\|_\infty+\|\mathring{M}\|_\infty)\\
&+\|\mathring{\mathfrak{C}}_i(v^i)\|_0\|\partial\check{q}\|_\infty|\partial_t^4\mathring{\tilde{\psi}}|_0+\left\|\sqrt{\mathring{\mathcal{F}}'(\mathring{q})}\mathbf{Q}\right\|_0\left\|\sqrt{\mathring{\mathcal{F}}'(\mathring{q})}^{-1}\mathring{\mathcal{R}}^2\right\|_0\\
\lesssim&\,\mathring{I}_0-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\sqrt{\mathring{\mathcal{F}}'(\mathring{q})}\mathbf{Q}\right\|_0^2+P(\mathring{K}_0)\mathring{E}^\kappa(t),
\end{aligned}
\tag{5.65}
$$

where we note that all terms in $\mathring{\mathcal{R}}^2$ come with $\mathring{\mathcal{F}}'(\mathring{q})$ and thus the control of $\sqrt{\mathring{\mathcal{F}}'(\mathring{q})}^{-1}\mathring{\mathcal{R}}^2$ is still uniform in $\mathring{\mathcal{F}}'(\mathring{q})$.

Now it remains to control the boundary integral. Compared with the nonlinear system, the estimate for the linearized system is easier as the surface tension term now becomes a given term. Plugging (5.61) and (5.63) into the boundary integral, we get

$$
\begin{aligned}
-\int_\Sigma \mathring{\mathbf{Q}}(\mathring{\mathbf{V}}\cdot\mathring{\tilde{N}})\,\mathrm{d}x'=&-\int_\Sigma \partial_t^4\overline{\nabla}\cdot(\overline{\nabla}\mathring{\tilde{\psi}}/|\mathring{\tilde{N}}|)\partial_t^5\psi\,\mathrm{d}x'-\kappa^2\int_\Sigma \partial_t^4(1-\overline{\Delta})\partial_t\psi\cdot\partial_t^5\psi\,\mathrm{d}x'\\
&-\int_\Sigma g\partial_t^4\mathring{\tilde{\psi}}\partial_t^5\psi\,\mathrm{d}x'+\int_\Sigma \partial_3\check{q}\partial_t^4\mathring{\tilde{\psi}}\partial_t^5\psi\,\mathrm{d}x'\\
&-\int_\Sigma \mathbf{Q}(\overline{\mathring{v}}\cdot\overline{\nabla})\partial_t^4\mathring{\tilde{\psi}}\,\mathrm{d}x'+\int_\Sigma \mathring{\mathbf{Q}}\mathring{\mathcal{S}}_1\,\mathrm{d}x',
\end{aligned}
\tag{5.66}
$$

where the second term gives us the boundary energy

$$
-\kappa^2\int_\Sigma \partial_t^4(1-\overline{\Delta})\partial_t\psi\cdot\partial_t^5\psi\,\mathrm{d}x'=\kappa^2\int_\Sigma \left|\langle\overline{\partial}\rangle\partial_t^5\psi\right|^2\,\mathrm{d}x'. \tag{5.67}
$$

We note that the first, the third, and the fourth terms in (5.66) can all be directly controlled under the time integral, i.e.,

$$
-\int_0^T\int_\Sigma \partial_t^4\overline{\nabla}\cdot(\overline{\nabla}\mathring{\tilde{\psi}}/|\mathring{\tilde{N}}|)\partial_t^5\psi\,\mathrm{d}x'\,\mathrm{d}t\lesssim\varepsilon|\partial_t^5\psi|^2_{L_t^2H_{x'}^1}+P(|\overline{\nabla}\mathring{\tilde{\psi}}|_\infty)|\overline{\nabla}\partial_t^4\mathring{\tilde{\psi}}|_0^2\le_{\kappa^{-1}}\varepsilon\mathring{E}^\kappa(T)+P(\mathring{K}_0) \tag{5.68}
$$

$$
-\int_0^T\int_\Sigma (g-\partial_3\check{q})\partial_t^4\mathring{\tilde{\psi}}\partial_t^5\psi\,\mathrm{d}x'\,\mathrm{d}t\le\varepsilon|\partial_t^5\psi|^2_{L_t^2L_{x'}^2}+|\partial_t^4\mathring{\tilde{\psi}}|_0^2(1+\|\partial\check{q}\|^2_{L_t^2L_x^\infty})\le_{\kappa^{-1}}\varepsilon\mathring{E}^\kappa(T)+P(\mathring{K}_0)\int_0^T\mathring{E}^\kappa(t)\,\mathrm{d}t. \tag{5.69}
$$

Further, the fifth term is controlled directly by invoking (3.5):

$$
\begin{aligned}
-\int_0^T\int_\Sigma \mathbf{Q}(\overline{\mathring{v}}\cdot\overline{\nabla})\partial_t^4\mathring{\tilde{\psi}}\,\mathrm{d}x'\,\mathrm{d}t=&-\int_0^T\sigma\int_\Sigma \partial_t^4\overline{\nabla}\cdot(\overline{\nabla}\mathring{\tilde{\psi}}/|\mathring{\tilde{N}}|)(\overline{\mathring{v}}\cdot\overline{\nabla})\partial_t^4\mathring{\tilde{\psi}}\,\mathrm{d}x'\,\mathrm{d}t+\kappa^2\int_0^T\int_\Sigma(1-\overline{\Delta})\partial_t^5\psi(\overline{\mathring{v}}\cdot\overline{\nabla})\partial_t^4\mathring{\tilde{\psi}}\,\mathrm{d}x'\,\mathrm{d}t\\
&+\int_0^T\int_\Sigma (g-\partial_3\check{q})\partial_t^4\mathring{\tilde{\psi}}(\overline{\mathring{v}}\cdot\overline{\nabla})\partial_t^4\mathring{\tilde{\psi}}\,\mathrm{d}x'\,\mathrm{d}t\\
\lesssim_{\kappa^{-1}}&\,\varepsilon|\partial_t^5\psi|^2_{L_t^2H_{x'}^1}+P(\mathring{K}_0)\int_0^T\mathring{E}^\kappa(t)\,\mathrm{d}t.
\end{aligned}
\tag{5.70}
$$

It remains to analyze the last integral in (5.66) which will be canceled with $\mathring{I}_0$ defined in (5.65). Following the analysis in (4.130)–(4.140), we have

$$
\int_\Sigma \mathring{\mathbf{Q}}\mathring{\mathcal{S}}_1\,\mathrm{d}x'=4\int_\Sigma \partial_t^4\check{q}\partial_t^3v\cdot\partial_t\mathring{\tilde{N}}\,\mathrm{d}x'+\text{ controllable terms}, \tag{5.71}
$$

$$
\mathring{I}_0=-4\int_\Omega \partial_t^4\check{q}\partial_t\mathring{\tilde{\mathbf{N}}}_i\partial_3\partial_t^3v^i\,\mathrm{d}x+\text{ controllable terms}, \tag{5.72}
$$

and then we add them together and use the divergence theorem to get

$$\begin{aligned}&4\int_\Sigma \partial_t^4\check{q}\partial_t^3 v\cdot\partial_t\mathring{\widetilde{N}}\,\mathrm{d}x' - 4\int_\Omega \partial_t^4\check{q}\partial_t\mathring{\widetilde{\mathbf{N}}}_i\partial_3\partial_t^3 v^i\,\mathrm{d}x\\ =&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(\partial_t^3\partial_3\check{q}\partial_t\mathring{\widetilde{\mathbf{N}}}+\partial_t^3\check{q}\partial_t\partial_3\mathring{\widetilde{\mathbf{N}}})\cdot\partial_t^3 v\,\mathrm{d}x+\int_\Omega\partial_t^3\partial_3\check{q}\partial_t(\partial_t\mathring{\widetilde{\mathbf{N}}}\cdot\partial_t^3 v)+\partial_t^3\check{q}\partial_t(\partial_t\partial_3\mathring{\widetilde{\mathbf{N}}}\cdot\partial_3 v)\,\mathrm{d}x,\end{aligned}\tag{5.73}$$

whose time integral can be easily bounded by $\varepsilon\|\partial_t^3\partial_3\check{q}\|_0^2+\mathring{E}^\kappa(0)+P(\mathring{K}_0)\int_0^T\mathring{E}^\kappa(t)\,\mathrm{d}t$. Hence, we get the control of boundary integral

$$-\int_0^T\int_\Sigma\mathring{\mathbf{Q}}(\mathring{\mathbf{V}}\cdot\mathring{\widetilde{N}})\,\mathrm{d}x'\,\mathrm{d}t+\kappa^2\int_0^T\int_\Sigma\left|\langle\overline{\partial}\rangle\partial_t^5\widetilde{\psi}\right|_0^2\mathrm{d}t\le\varepsilon\|\partial_t^3\partial_3\check{q}\|_0^2+\mathring{E}^\kappa(0)+P(\mathring{K}_0)\int_0^T\mathring{E}^\kappa(t)\,\mathrm{d}t.\tag{5.74}$$

Combining this with (5.64), (5.65) and the definition of Alinhac good unknowns we get the estimates for the full-time derivatives

$$\|\partial_t^4 v(t)\|_0^2+\left\|\sqrt{\mathring{\mathcal{F}}'(\mathring{q})}\partial_t^4\check{q}\right\|_0^2+\kappa^2\int_0^t\int_\Sigma\left|\langle\overline{\partial}\rangle\partial_t^5\widetilde{\psi}\right|_0^2\mathrm{d}\tau\le\varepsilon\|\partial_t^3\partial_3\check{q}\|_0^2+\mathring{E}^\kappa(0)+P(\mathring{K}_0)\int_0^t\mathring{E}^\kappa(t)\mathrm{d}\tau.\tag{5.75}$$

This, together with div-curl analysis gives us the energy inequality of $\mathring{E}^\kappa(t)$ after choosing $\varepsilon>0$ suitably small:

$$\mathring{E}^\kappa(t)\le_{\kappa^{-1}}\mathring{E}^\kappa(0)+P(\mathring{K}_0)\int_0^t\mathring{E}^\kappa(\tau)\mathrm{d}\tau,\tag{5.76}$$

which implies that there exists some $T^\kappa>0$ such that

$$\sup_{0\le t\le T^\kappa}\mathring{E}^\kappa(t)\le C(\kappa^{-1},\mathring{K}_0)\mathring{E}^\kappa(0).$$

Therefore, the uniform-in-n estimates for (5.6) are proven by induction.

5.2.6 Regularity of ψ and its time derivatives

The regularity of $\partial_t^{k+1}\psi$ ($0\le k\le 3$) can be enhanced to $H^{5.5-k}$ by the boundary elliptic estimates once we close the energy estimates for $\mathring{E}^\kappa(t)$. Note that the boundary condition gives

$$\kappa^2(1-\overline{\Delta})\partial_t\psi=\check{q}-g\mathring{\widetilde{\psi}}+\sigma\mathcal{H}(\overline{\nabla}\mathring{\widetilde{\psi}},\overline{\nabla}^2\mathring{\widetilde{\psi}}),$$

thus, by (5.11) and the elliptic estimate, it holds that

$$|\partial_t^{k+1}\psi|_{5.5-k}\le\kappa^{-2}\left(\sigma P(|\overline{\nabla}\mathring{\widetilde{\psi}}|_\infty)|\partial_t^k\overline{\nabla}^2\mathring{\widetilde{\psi}}|_{3.5-k}+|\partial_t^k q|_{3.5-k}+P(\mathring{K}_0)\right)\le C(\kappa^{-1},\mathring{K}_0)\mathring{E}^\kappa.\tag{5.77}$$

Moreover, $|\psi|_{5.5}$ is controlled by

$$|\psi|_{5.5}\le|\psi_{0,\kappa}|_{5.5}+\int_0^T|\partial_t\psi(t)|_{5.5}\,\mathrm{d}t.\tag{5.78}$$

5.3 Picard iteration

So far, we have established the local existence and the uniform-in-n estimates for the linearized system (5.1) for each fixed $\kappa>0$, namely

$$\begin{cases}\rho^{(n)}D_t^{\widetilde{\varphi}^{(n)}}v^{(n+1)}+\nabla^{\widetilde{\varphi}^{(n)}}\check{q}^{(n+1)}=-(\rho^{(n)}-1)ge_3 & \text{in }[0,T]\times\Omega,\\ \mathcal{F}^{(n)\prime}(q^{(n)})D_t^{\widetilde{\varphi}^{(n)}}\check{q}^{(n+1)}+\nabla^{\widetilde{\varphi}^{(n)}}\cdot v^{(n+1)}=\mathcal{F}^{(n)\prime}(q^{(n)})gv_3^{(n)} & \text{in }[0,T]\times\Omega,\\ q^{(n+1)}=q^{(n+1)}(\rho^{(n+1)}),\ \check{q}^{(n+1)}=q^{(n+1)}+g\widetilde{\varphi}^{(n)} & \text{in }[0,T]\times\Omega,\\ \check{q}^{(n+1)}=g\widetilde{\psi}^{(n)}-\sigma\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\widetilde{\psi}^{(n)}}{\sqrt{1+|\overline{\nabla}\widetilde{\psi}^{(n)}|^2}}\right)+\kappa^2(1-\overline{\Delta})(v^{(n+1)}\cdot\widetilde{N}^{(n)}) & \text{on }[0,T]\times\Sigma,\\ \partial_t\psi^{(n+1)}=v^{(n+1)}\cdot\widetilde{N}^{(n)} & \text{on }[0,T]\times\Sigma,\\ v_3^{(n+1)}=0 & \text{on }[0,T]\times\Sigma_b,\\ (v^{(n+1)},\rho^{(n+1)},\psi^{(n+1)})|_{t=0}=(v_0^\kappa,\rho_0^\kappa,\psi_0^\kappa), \end{cases}\tag{5.79}$$

where $\psi^{(n)}, \varphi^{(n)}, D_t^{\widetilde{\varphi}^{(n)}}, \nabla^{\widetilde{\varphi}^{(n)}}$ are defined in (5.2)-(5.5). Now it suffices to prove that, for each fixed $\kappa > 0$, the sequence $\{(v^{(n)}, \check{q}^{(n)}, \psi^{(n)})\}_{n\in\mathbb{N}^*}$ has a strongly convergent subsequence. Once we prove this, the limit of that subsequence becomes the solution to the nonlinear κ-approximate system (3.11) for this chosen κ.

For a function sequence $\{f^{(n)}\}$ we define $[f]^{(n)} := f^{(n+1)} - f^{(n)}$ and then we find that $\{([v]^{(n)}, [\check{q}]^{(n)}, [\psi]^{(n)})\}$ satisfies the following linear system

$$\begin{cases} \rho^{(n)} D_t^{\widetilde{\varphi}^{(n)}} [v]^{(n)} + \nabla^{\widetilde{\varphi}^{(n)}} [\check{q}]^{(n)} = -\mathring{f}_v^{(n)} & \text{in } [0,T]\times\Omega, \\ \mathcal{F}^{(n)\prime}(q^{(n)}) D_t^{\widetilde{\varphi}^{(n)}} [\check{q}]^{(n)} + \nabla^{\widetilde{\varphi}^{(n)}} \cdot [v]^{(n)} = -\mathring{f}_q^{(n)} & \text{in } [0,T]\times\Omega, \\ [\check{q}]^{(n)} = [q]^{(n)} + g[\widetilde{\varphi}]^{(n-1)} & \text{in } [0,T]\times\Omega, \\ [\check{q}]^{(n)} = g[\widetilde{\psi}]^{(n-1)} - \sigma[\mathcal{H}]^{(n-1)} + \kappa^2(1-\overline{\Delta})([v]^{(n)}\cdot \widetilde{N}^{(n)}) + \kappa^2(1-\overline{\Delta})(v^{(n)}\cdot[\widetilde{N}]^{(n-1)}), & \text{on } [0,T]\times\Sigma, \\ \partial_t[\psi]^{(n)} = [v]^{(n)}\cdot \widetilde{N}^{(n)} + (v^{(n)}\cdot[\widetilde{N}]^{(n-1)}), & \text{on } [0,T]\times\Sigma, \\ [v_3^{(n)}] = 0 & \text{on } [0,T]\times\Sigma_b, \\ ([v]^{(n)}, [\rho]^{(n)}, [\psi]^{(n)})|_{t=0} = (\mathbf{0},0,0), \end{cases} \tag{5.80}$$

where $\mathring{f}_v^{(n)}$ and $\mathring{f}_q^{(n)}$ are defined by

$$\mathring{f}_v^{(n)} := [\rho]^{(n-1)}\partial_t v^{(n)} + [\rho\overline{v}]^{(n-1)}\cdot\overline{\nabla} v^{(n)} + [\rho V_{\mathring{\mathbf{N}}}]^{(n-1)}\partial_3 v^{(n)} + [\rho]^{(n-1)} g e_3 + \partial_3 \check{q}^{(n)}[A_{i3}]^{(n-1)}, \tag{5.81}$$

$$\begin{aligned} \mathring{f}_q^{(n)} :=& [\mathcal{F}'(q)]^{(n-1)}(\partial_t\check{q}^{(n)} - g v_3^{(n-1)}) + [\mathcal{F}'(q)\overline{v}]^{(n-1)}\cdot\overline{\nabla}\check{q}^{(n)} + [\mathcal{F}'(q)V_{\mathring{\mathbf{N}}}]^{(n-1)}\partial_3\check{q}^{(n)} \\ &- \mathcal{F}^{(n)\prime}(q^{(n)}) g[v_3]^{(n-1)} + \partial_3 v_i^{(n)}[A_{i3}]^{(n-1)}, \end{aligned} \tag{5.82}$$

and

$$V_{\mathring{\mathbf{N}}}^{(n)} := \frac{1}{\partial_3\widetilde{\varphi}^{(n)}}(v^{(n)}\cdot\widetilde{\mathbf{N}}^{(n-1)} - \partial_t\varphi^{(n)}),\ A_{13}^{(n)} := -\frac{\partial_1\widetilde{\varphi}^{(n)}}{\partial_3\widetilde{\varphi}^{(n)}},\ A_{23}^{(n)} := -\frac{\partial_2\widetilde{\varphi}^{(n)}}{\partial_3\widetilde{\varphi}^{(n)}},\ A_{33}^{(n)} := \frac{1}{\partial_3\widetilde{\varphi}^{(n)}},$$

$$[\mathcal{H}]^{(n-1)} := \mathcal{H}(\overline{\nabla}\widetilde{\psi}^{(n)}) - \mathcal{H}(\overline{\nabla}\widetilde{\psi}^{(n-1)}),\ \mathcal{H}(\overline{\nabla}\widetilde{\psi}) := -\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\widetilde{\psi}}{1+|\overline{\nabla}\widetilde{\psi}|^2}\right).$$

For $n \geq 1$, we define the energy of (5.80) $[E]^{(n)}$ to be the following quantity

$$[E]^{(n)}(t) := \sum_{k=0}^{3} \|\partial_t^k [v]^{(n)}(t)\|_{3-k}^2 + \|\partial_t^k[\check{q}]^{(n)}(t)\|_{3-k}^2 + \int_0^t \left|\partial_t^{k+1}[\psi]^{(n)}(\tau)\right|_{4-k}^2 \mathrm{d}\tau + |[\psi]^{(n)}(t)|_4^2 \tag{5.83}$$

It suffices to control $[E]^{(n)}(t)$ and use $([v]^{(n)}, [\rho]^{(n)}, [\psi]^{(n)})|_{t=0} = (\mathbf{0},0,0)$ to show that $[E]^{(n)}(t) \leq \frac{1}{4}([E]^{(n-1)}(t) + [E]^{(n-2)}(t))$ in some time interval $[0, T_1^\kappa]$. The estimates for $[E]^{(n)}(t)$ are parallel to Section 5.2, so we will not go into every detail but only list the sketch of the proof.

5.3.1 Div-curl analysis for $[v]^{(n)}$

By Lemma 5.2, we have the following inequalities for $k = 0, 1, 2$:

$$\|\partial_t^k[v]^{(n)}\|_{3-k}^2 \leq C(\mathring{K}_0)\left(\|\partial_t^k[v]^{(n)}\|_0^2 + \|\nabla^{\widetilde{\varphi}^{(n)}}\times\partial_t^k[v]^{(n)}\|_{2-k}^2 + \|\nabla^{\widetilde{\varphi}^{(n)}}\cdot\partial_t^k[v]^{(n)}\|_{2-k}^2 + |\partial_t^k[v]^{(n)}\cdot\widetilde{N}^{(n)}|_{2.5-k}^2\right). \tag{5.84}$$

The estimates for $L^2(\Omega)$ norms follow in the same way as Section 5.2.1 so we do not repeat here. For the curl part, we take $\nabla^{\widetilde{\varphi}^{(n)}}\times$ in the first equation of (5.80) to get

$$\rho^{(n)} D_t^{\widetilde{\varphi}^{(n)}}(\nabla^{\widetilde{\varphi}^{(n)}}\times[v]^{(n)}) = -\nabla^{\widetilde{\varphi}^{(n)}}\times\mathring{f}_v^{(n)} - \nabla^{\widetilde{\varphi}^{(n)}}\rho^{(n)}\times D_t^{\widetilde{\varphi}^{(n)}}[v]^{(n)} + \rho^{(n)}[\nabla^{\widetilde{\varphi}^{(n)}}\times, D_t^{\widetilde{\varphi}^{(n)}}][v]^{(n)}, \tag{5.85}$$

where $([\nabla^{\widetilde{\varphi}^{(n)}}\times, D_t^{\widetilde{\varphi}^{(n)}}][v]^{(n)})^i = \epsilon^{ijk}\nabla_j^{\widetilde{\varphi}^{(n)}} v_l^{(n)}\nabla_l^{\widetilde{\varphi}^{(n)}}[v]_k^{(n)} + \epsilon^{ijk}\nabla_j^{\widetilde{\varphi}^{(n)}}\partial_t(\widetilde{\varphi}^{(n)} - \widetilde{\varphi}^{(n-1)})\partial_3[v]_k^{(n)}$ and $\nabla^{\widetilde{\varphi}^{(n)}}\times\mathring{f}_v^{(n)}$ contains at most two derivatives of $v^{(n)}, \varphi^{(n)}, \varphi^{(n-1)}, \varphi^{(n-2)}$. Taking ∂^2, we have

$$\rho^{(n)} D_t^{\widetilde{\varphi}^{(n)}}(\partial^2\nabla^{\widetilde{\varphi}^{(n)}}\times[v]^{(n)}) = \partial^2(\text{RHS of (5.85)}) - [\partial^2, \rho^{(n)} D_t^{\widetilde{\varphi}^{(n)}}](\nabla^{\widetilde{\varphi}^{(n)}}\times[v]^{(n)}). \tag{5.86}$$

Based on the analysis above, we find that the leading-order terms of $[v]^{(n)}, [v]^{(n-1)}$ must be linear in $[v]^{(n)}, [v]^{(n-1)}$ respectively. Using Reynold transport formula (A.9) for the linearized system, the curl part can be directly controlled as in (5.33):

$$\|\nabla^{\widetilde{\varphi}^{(n)}} \times [v]^{(n)}(T)\|_2^2 \le C(\mathring{K}_0)\left(\|\underbrace{\nabla^{\widetilde{\varphi}^{(n)}} \times [v]^{(n)}(0)}_{=0}\|_2^2 + \int_0^T P(\mathring{E}^{(n)}, \mathring{E}^{(n-1)}, \mathring{E}^{(n-2)})[E]^{(n)}(t)\,\mathrm{d}t\right) \tag{5.87}$$
$$\le C(\mathring{K}_0)\int_0^T [E]^{(n)}(t) + [\mathring{E}]^{(n-1)}(t) + [\mathring{E}]^{(n-2)}(t)\,\mathrm{d}t.$$

Similarly, replacing ∂^2 by $\partial^{2-k}\partial_t^k$ for $k = 1, 2$, we get

$$\|\nabla^{\widetilde{\varphi}^{(n)}} \times \partial_t^k[v]^{(n)}(T)\|_{2-k}^2 \le\ C(\mathring{K}_0)\int_0^T [E]^{(n)}(t) + [\mathring{E}]^{(n-1)}(t) + [\mathring{E}]^{(n-2)}(t)\,\mathrm{d}t. \tag{5.88}$$

As for the divergence, the second equation in (5.80) gives

$$\|\nabla^{\widetilde{\varphi}^{(n)}} \cdot [v]^{(n)}\|_2^2 \le \|\mathcal{F}^{(n)\prime}(q^{(n)})D_t^{\widetilde{\varphi}^{(n)}}[\check{q}]^{(n)}\|_2^2 + \|\mathring{f}_q\|_2^2 \le P(\mathring{K}_0)\|\mathcal{F}^{(n)\prime}(q^{(n)})\mathcal{T}[\check{q}]^{(n)}\|_2^2, \tag{5.89}$$

where $\mathcal{T} = \partial_t$ or $\overline{\partial}$ or $\omega\partial_3$ for a bounded weight function ω vanishing on Σ. Therefore, the divergence is then reduced to the tangential derivatives of $[\check{q}]$. Similarly, the divergence of $\partial_t^k[v]^{(n)}$ is reduced to $\partial_t^k\mathcal{T}\check{q}$.

Next, the normal traces are still controlled by using boundary elliptic estimates. Note that the Dirichlet boundary condition for $[\check{q}]^{(n)}$ can be written as

$$-\kappa^2(1-\overline{\Delta})([v]^{(n)} \cdot \widetilde{N}^{(n)}) = -[q]^{(n)} - \sigma\left(\mathcal{H}(\overline{\nabla}\widetilde{\psi}^{(n)}) - \mathcal{H}(\overline{\nabla}\widetilde{\psi}^{(n-1)})\right) + \kappa^2(1-\overline{\Delta})(v^{(n)} \cdot [\widetilde{N}]^{(n-1)}), \tag{5.90}$$

and thus

$$|[v]^{(n)} \cdot \widetilde{N}^{(n)}|_{2.5}^2 \le_{\kappa^{-1}} \|[q]^{(n)}\|_1^2 + P(\mathring{K}_0) + |\overline{v}^{(n)} \cdot \overline{\nabla}\widetilde{\psi}^{(n-1)}|_{2.5}^2 + |v_3^{(n)}|_{2.5}^2 \le \|[q]^{(n)}\|_1^2 + P(\mathring{K}_0). \tag{5.91}$$

Similarly, we have for $k = 1, 2$

$$|\partial_t^k[v]^{(n)} \cdot \widetilde{N}^{(n)}|_{2.5-k}^2 \le_{\kappa^{-1}} \|\partial_t^k[q]^{(n)}\|_1^2 + P(\mathring{K}_0). \tag{5.92}$$

5.3.2 Reduction of pressure $[\check{q}]^{(n)}$

This is similar to the arguments in Section 5.2.4. We first consider the third component of the first equation in (5.80):

$$(\partial_3\widetilde{\varphi}^{(n)})^{-1}\partial_3[\check{q}]^{(n)} = -\rho^{(n)}D_t^{\widetilde{\varphi}^{(n)}}[v]^{(n)} + \mathring{f}_v^{(n)}, \tag{5.93}$$

which means the control of $\partial_3[\check{q}]^{(n)}$ is reduced to $\mathcal{T}[v]^{(n)}$. Then by considering the first and second components, we can further reduce the control of $\overline{\partial}_i\check{q}$ $(i = 1, 2)$ to $\partial_3\check{q}$ and $\mathcal{T}v$ since $\nabla_i^{\mathring{\widetilde{\varphi}}} = \overline{\partial}_i - \overline{\partial}_i\mathring{\widetilde{\varphi}}\partial_3^{\mathring{\widetilde{\varphi}}}$. Therefore, combining the div-curl analysis and reduction procedures for $[\check{q}]^{(n)}$, it suffices to control $\partial_t^2\overline{\partial}[\check{q}]^{(n)}$ and $\partial_t^3[\check{q}]^{(n)}$.

5.3.3 Tangential estimates for full-time derivatives

Again we only show the control of $\partial_t^3[\check{q}]^{(n)}$ by introducing the Alinhac good unknowns:

$$[\mathbf{V}]^{(n)} := \mathring{\mathbf{V}}^{(n+1)} - \mathring{\mathbf{V}}^{(n)} = \partial_t^3[v]^{(n)} - \partial_t^3\widetilde{\varphi}^{(n)}\partial_3^{\widetilde{\varphi}^{(n)}}[v]^{(n)} - \partial_t^3\widetilde{\varphi}^{(n)}\partial_3^{[\widetilde{\varphi}]^{(n-1)}}v^{(n)} - \partial_t^3[\widetilde{\varphi}]^{(n-1)}\partial_3^{\widetilde{\varphi}^{(n-1)}}v^{(n)}, \tag{5.94}$$
$$[\mathbf{Q}]^{(n)} := \mathring{\mathbf{Q}}^{(n+1)} - \mathring{\mathbf{Q}}^{(n)} = \partial_t^3[\check{q}]^{(n)} - \partial_t^3\widetilde{\varphi}^{(n)}\partial_3^{\widetilde{\varphi}^{(n)}}[\check{q}]^{(n)} - \partial_t^3\widetilde{\varphi}^{(n)}\partial_3^{[\widetilde{\varphi}]^{(n-1)}}\check{q}^{(n)} - \partial_t^3[\widetilde{\varphi}]^{(n-1)}\partial_3^{\widetilde{\varphi}^{(n-1)}}\check{q}^{(n)}. \tag{5.95}$$

For a function f and its associated Alinhac good unknown $\mathbf{F}$, we have

$$\partial_t^3(\partial_i^{\widetilde{\varphi}^{(n)}}[f]^{(n)} + \partial_i^{[\widetilde{\varphi}]^{(n-1)}}f^{(n)}) = \partial_i^{\widetilde{\varphi}^{(n)}}[\mathbf{F}]^{(n)} + [\mathfrak{C}]_i^{(n)}(f),$$
$$\partial_t^3(D_t^{\widetilde{\varphi}^{(n)}}[f]^{(n)} + D_t^{[\widetilde{\varphi}]^{(n-1)}}f^{(n)}) = D_t^{\widetilde{\varphi}^{(n)}}[\mathbf{F}]^{(n)} + [\mathfrak{D}]^{(n)}(f) + [\mathfrak{e}]^{(n)}(f)$$

with

$$[\mathfrak{C}]_i^{(n)}(f) = \mathfrak{C}_i^{(n)}(f^{(n+1)}) - \mathfrak{C}_i^{(n-1)}(f^{(n)}) + \text{lower-order controllable terms},$$
$$[\mathfrak{D}]_i^{(n)}(f) = \mathfrak{D}^{(n)}(f^{(n+1)}) - \mathfrak{D}^{(n-1)}(f^{(n)}) + \text{lower-order controllable terms},$$
$$[\mathfrak{e}]_i^{(n)}(f) = \mathfrak{e}^{(n)}(f^{(n+1)}) - \mathfrak{e}^{(n-1)}(f^{(n)}) + \text{lower-order controllable terms},$$

where $\mathfrak{C}_i^{(n)}(f^{(m)}), \mathfrak{D}^{(n)}(f^{(m)})$ and $\mathfrak{e}^{(n)}(f^{(m)})$ are defined by replacing $\partial_t^4, \partial_t^5$ by $\partial_t^3, \partial_t^4$, replacing the coefficient 4 in $\mathring{\mathfrak{D}}$ by 3 and setting $\mathring{\varphi} = \varphi^{(n)}$, $\dot{\varphi} = \varphi^{(n-1)}$, $f^{(n+1)} = f$ and $f^{(n)} = \mathring{f}$ in (5.48)-(5.56).

The Alinhac good unknowns $[\mathbf{V}]^{(n)}, [\mathbf{Q}]^{(n)}$ satisfy the following linear system:

$$\rho^{(n)} D_t^{\widetilde{\varphi}^{(n)}} [\mathbf{V}]^{(n)} + \nabla^{\widetilde{\varphi}^{(n)}} [\mathbf{Q}]^{(n)} = -\mathfrak{C}^{(n)}(\check{q}^{(n+1)}) + \mathfrak{C}^{(n-1)}(\check{q}^{(n)}) + [\mathring{\mathcal{R}}]_v, \tag{5.96}$$

$$\mathcal{F}^{(n)\prime}(q^{(n)}) D_t^{\widetilde{\varphi}^{(n)}} [\mathbf{Q}]^{(n)} + \nabla^{\widetilde{\varphi}^{(n)}} \cdot \mathbf{V}^{(n)} = -\mathfrak{C}_i^{(n)}(v_i^{(n+1)}) + \mathfrak{C}_i^{(n-1)}(v_i^{(n)}) + [\mathring{\mathcal{R}}]_q, \tag{5.97}$$

where $[R]$ terms consist of $\partial_t^3 \mathring{f}$ terms in (5.80) and the omitted commutator terms in the definition of Alinhac good unknowns $[\mathbf{V}], [\mathbf{Q}]$ and they are controllable in $L^2(\Omega)$.

$$\|[\mathring{\mathcal{R}}]\|_0^2 \le C(\mathring{K}_0)([E]^{(n)}(t) + [E]^{(n-1)}(t) + [E]^{(n-2)}(t)). \tag{5.98}$$

The boundary conditions now become:

$$\begin{aligned} [\mathbf{Q}]^{(n)} &= g\partial_t^3[\widetilde{\psi}]^{(n-1)} + \sigma\partial_t^3(\mathcal{H}(\overline{\nabla}\widetilde{\psi}^{(n)}) - \mathcal{H}(\overline{\nabla}\widetilde{\psi}^{(n-1)})) - \kappa^2(1-\overline{\Delta})\partial_t^4[\psi]^{(n)} \\ &\quad + \partial_t^3\widetilde{\psi}^{(n)}\partial_3[\check{q}]^{(n)} + \partial_t^3[\widetilde{\psi}]^{(n-1)}\partial_3\check{q}^{(n)} \end{aligned} \tag{5.99}$$

$$\begin{aligned} [\mathbf{V}]^{(n)} \cdot \widetilde{N}^{(n)} &= \partial_t^4[\psi]^{(n)} + [\overline{v}]^{(n)} \cdot \overline{\nabla}\partial_t^3\widetilde{\psi}^{(n)} + (\overline{v} \cdot \overline{\nabla})\partial_t^3[\widetilde{\psi}]^{(n-1)} + \partial_t^3\overline{v}^{(n)} \cdot \overline{\nabla}[\widetilde{\psi}]^{(n-1)} \\ &\quad - (\partial_3[v]^{(n)} \cdot \widetilde{N}^{(n)})\partial_t^3\widetilde{\psi}^{(n)} + (\partial_3 v^{(n)} \cdot \widetilde{N}^{(n)})\partial_t^3[\widetilde{\psi}]^{(n-1)} + [\partial_t^3, \widetilde{N}^{(n)}\cdot, v^{(n+1)}] - [\partial_t^3, \widetilde{N}^{(n-1)}\cdot, v^{(n)}]. \end{aligned} \tag{5.100}$$

Following the analysis in Section 5.2.5, we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \rho^{(n)}|[\mathbf{V}]^{(n)}|^2\partial_3\widetilde{\varphi}^{(n)}\,\mathrm{d}x + \int_\Omega \mathcal{F}^{(n)\prime}(q^{(n)})|[\mathbf{Q}]^{(n)}|^2\partial_3\widetilde{\varphi}^{(n)}\,\mathrm{d}x\right) + \kappa^2\int_0^T \left|\partial_t^4[\psi]^{(n)}\right|_1^2\,\mathrm{d}t \\ &\le C(\mathring{K}_0)\left([\mathring{E}]^{(n)}(0) + \int_0^T [\mathring{E}]^{(n)}(t) + [\mathring{E}]^{(n-1)}(t) + [\mathring{E}]^{(n-2)}(t)\,\mathrm{d}t\right) \\ &\quad - \int_\Sigma [\mathring{\mathbf{Q}}]^{(n)}[\partial_t^3, \widetilde{\mathbf{N}}\cdot, v^{(n-1)}]\,\mathrm{d}x' + \int_\Omega [\mathring{\mathbf{Q}}]^{(n)}\mathfrak{C}_i^{(n-1)}(v_i^{(n)})\mathrm{d}\mathcal{V}_t^{(n)} \end{aligned} \tag{5.101}$$

where the last line is analyzed in the same way as in (5.72) (by using divergence theorem and integration by parts in time variable). Here we only list the highest-order terms. We have

$$\begin{aligned} &- \int_\Sigma [\mathring{\mathbf{Q}}]^{(n)}[\partial_t^3, \widetilde{\mathbf{N}}\cdot, v^{(n-1)}]\,\mathrm{d}x' + \int_\Omega [\mathring{\mathbf{Q}}]^{(n)}\mathfrak{C}_i^{(n-1)}(v_i^{(n)})\mathrm{d}\mathcal{V}_t^{(n)} \\ &= \int_\Omega \partial_3^{\widetilde{\varphi}^{(n)}}[\mathring{\mathbf{Q}}]^{(n)}[\partial_t^3, \widetilde{\mathbf{N}}\cdot, v^{(n-1)}]\mathrm{d}\mathcal{V}_t^{(n)} + \text{ controllable terms,} \end{aligned} \tag{5.102}$$

and thus it can be controlled under time integral:

$$\int_0^T \int_\Omega \partial_3^{\widetilde{\varphi}^{(n)}}[\mathring{\mathbf{Q}}]^{(n)}[\partial_t^3, \widetilde{\mathbf{N}}\cdot, v^{(n-1)}]\mathrm{d}\mathcal{V}_t^{(n)}\,\mathrm{d}t \lesssim \quad \varepsilon\|\partial_t^2[\check{q}]^{(n)}\|_1^2 + C(\mathring{K}_0)\left([\mathring{E}]^{(n)}(0) + \int_0^T [\mathring{E}]^{(n)}(t) + [\mathring{E}]^{(n-1)}(t)\,\mathrm{d}t\right). \tag{5.103}$$

Combining the above analysis and using the definition of Alinhac good unknowns, we get

$$\begin{aligned} &\|\partial_t^3[v]^{(n)}(t)\|_0^2 + \|\sqrt{\mathcal{F}^{(n)\prime}}(q^{(n)})\partial_t^3[\check{q}]^{(n)}(t)\|_0^2 + \kappa^2\int_0^t |\partial_t^4\psi(\tau)|_1^2\mathrm{d}\tau \\ &\le \varepsilon\|\partial_t^2[\check{q}]^{(n)}\|_1^2 + C(\mathring{K}_0, \kappa^{-1})\left([\mathring{E}]^{(n)}(0) + \int_0^T [\mathring{E}]^{(n)}(t) + [\mathring{E}]^{(n-1)}(t) + [\mathring{E}]^{(n-2)}(t)\,\mathrm{d}t\right). \end{aligned} \tag{5.104}$$

5.4 Well-posedness of the nonlinear κ-approximate problem

Combining the div-curl analysis, the control of the normal traces, the reduction of $[\check{q}]$ and the analysis of full-time derivatives for the linear system (5.80) for $[v]^{(n)}, [\check{q}]^{(n)}, [\psi]^{(n)}$, we arrive at the energy estimate:

$$[\mathring{E}]^{(n)}(t) \le C(\mathring{K}_0, \kappa^{-1})\left([\mathring{E}]^{(n)}(0) + \int_0^T [\mathring{E}]^{(n)}(t) + [\mathring{E}]^{(n-1)}(t) + [\mathring{E}]^{(n-2)}(t)\,\mathrm{d}t\right). \tag{5.105}$$

Since $[v]^{(n)}, [\check{q}]^{(n)}, [\psi]^{(n)}$ have zero initial data, one can repeatedly use (5.80) to show that their time derivatives also vanish on $\{t=0\}$, as one can observe that every term in the first two equations of (5.80) contains exactly one term involving $[f]^{(n)}$ or $[f]^{(n-1)}$ whose initial value is zero. This implies $[\mathring{E}]^{(n)}(0)=0$, and thus there exists some $T_1^\kappa>0$ independent of n, such that

$$\sup_{0\leq t\leq T_1^\kappa}[\mathring{E}]^{(n)}(t)\leq\frac{1}{4}\left(\sup_{0\leq t\leq T_1^\kappa}[\mathring{E}]^{(n-1)}(t)+\sup_{0\leq t\leq T_1^\kappa}[\mathring{E}]^{(n-2)}(t)\right), \tag{5.106}$$

and thus we know by induction that

$$\sup_{0\leq t\leq T_1^\kappa}[\mathring{E}]^{(n)}(t)\leq C(\mathring{K}_0,\kappa^{-1})/2^{n-1}\to 0 \text{ as } n\to+\infty. \tag{5.107}$$

Hence, for any fixed $\kappa>0$, the sequence of approximate solutions $\{(v^{(n)},\check{q}^{(n)},\rho^{(n)},\psi^{(n)})\}_{n\in\mathbb{N}^*}$ has a strongly convergent subsequence, whose limit $(v^\kappa,\check{q}^\kappa,\rho^\kappa,\psi^\kappa)$ is exactly the solution to the nonlinear κ-problem (3.11). The uniqueness follows from a parallel argument.

6 Well-posedness and incompressible limit of the gravity(-capillary) water wave system

We are ready to prove the local existence of the original water wave system (1.24) for each fixed $\sigma>0$. In Section 5, we prove the local well-posedness and higher-order energy estimates of the linearized system (5.6) for each *fixed* $\kappa>0$ and use Picard iteration to construct the unique strong solution to the nonlinear κ-approximate problem (3.11) defined in Section 3.2. To pass the limit $\kappa\to 0_+$ to the original system (1.24), we prove the uniform-in-κ estimates for (3.11) in Section 4. Therefore, we prove the local-in-time existence for the stronger solution to the compressible gravity-capillary water wave system (1.24), that is, given initial data (v_0,ρ_0,ψ_0), there exists $T'>0$ only depending on the initial data, such that the original system (1.24) has a solution (v,ρ,ψ) satisfying the energy estimates

$$\sup_{0\leq t\leq T'}E(t)\leq P(E(0)). \tag{6.1}$$

6.1 Uniqueness

To prove the well-posedness, it suffices to prove the uniqueness of the solution to (1.24). We assume

$$\{(v^{(n)},\check{q}^{(n)},\rho^{(n)},\psi^{(n)})\}_{n=1,2}$$

to be two solutions to (1.24) and define $[f]=f^{(1)}-f^{(2)}$ for any function f. Then it suffices to prove $([v],[\check{q}],[\rho],[\psi])=(\mathbf{0},0,0,0)$. We find that $([v],[\check{q}],[\rho],[\psi])=(\mathbf{0},0,0,0)$ satisfies the following system:

$$\begin{cases}\rho^{(1)}D_t^{\varphi^{(1)}}[v]+\nabla^{\varphi^{(1)}}[\check{q}]=-f_v & \text{in } [0,T]\times\Omega,\\ \mathcal{F}'(q^{(1)})D_t^{\varphi^{(1)}}[\check{q}]+\nabla^{\varphi^{(1)}}\cdot[v]=-f_q & \text{in } [0,T]\times\Omega,\\ [\check{q}]=[q]+g[\varphi] & \text{in } [0,T]\times\Omega,\\ [\check{q}]=g[\psi]-\sigma\left(\mathcal{H}(\overline{\nabla}\psi^{(1)})-\mathcal{H}(\overline{\nabla}\psi^{(2)})\right) & \text{on } [0,T]\times\Sigma,\\ \partial_t[\psi]=[v]\cdot N^{(1)}+v^{(2)}\cdot[N] & \text{on } [0,T]\times\Sigma,\\ [v_3]=0 & \text{on } [0,T]\times\Sigma_b,\\ ([v],[\check{q}],[\psi])|_{t=0}=(\mathbf{0},0,0)\end{cases} \tag{6.2}$$

where the functions f_v,f_q are defined by

$$f_v:=[\rho]\partial_t v^{(2)}+[\rho\overline{v}]\cdot\overline{\nabla}v^{(2)}+[\rho V_{\mathbf{N}}]\partial_3 v^{(2)}+\rho^{(2)}ge_3+\partial_3\check{q}^{(2)}[A_{i3}] \tag{6.3}$$

$$\begin{aligned}f_q:=&[\mathcal{F}'(q)](\partial_t\check{q}^{(2)}-gv_3^{(2)})+[\mathcal{F}'(q)\overline{v}]\cdot\overline{\nabla}\check{q}^{(2)}+[\mathcal{F}'(q)V_{\mathbf{N}}]\partial_3 q^{(2)}\\&-\mathcal{F}'(q^{(2)})g[v_3]+\partial_3 v_i^{(2)}[A_{i3}],\end{aligned} \tag{6.4}$$

and

$$V_{\mathbf{N}} := \frac{1}{\partial_3\varphi}(v\cdot\mathbf{N}-\partial_t\varphi),\ A_{13} := -\frac{\partial_1\varphi}{\partial_3\varphi},\ A_{23} := -\frac{\partial_2\varphi}{\partial_3\varphi},\ A_{33} := \frac{1}{\partial_3\varphi},$$
$$\mathcal{H}(\overline{\nabla}\psi) := \overline{\nabla}\cdot\left(\frac{\overline{\nabla}\psi}{|N|}\right),\ \ \mathcal{H}(\overline{\nabla}\psi^{(1)})-\mathcal{H}(\overline{\nabla}\psi^{(2)}) = \overline{\nabla}\cdot\left(\frac{\overline{\nabla}[\psi]}{|N^{(1)}|}-\left(\frac{1}{|N^{(1)}|}-\frac{1}{|N^{(2)}|}\right)\overline{\nabla}\psi^{(2)}\right).$$

Let

$$[E](t) := \sum_{k=0}^{3}\|\partial_t^k[v]\|_{3-k}^2+\sigma|\overline{\nabla}\partial_t^k[\psi]|_{3-k}^2+g|[\psi]|_0^2+\sum_{k=0}^{2}\|\partial_t^k[\check{q}]\|_{3-k}^2+\|\sqrt{\mathcal{F}'(q^{(1)})}\partial_t^3[\check{q}]\|_0^2. \tag{6.5}$$

We can then mimic the proof for the uniform-in-κ estimates (setting $\kappa = 0$) in Section 4 to show that $[E](0) = 0$ and $[E](t)$ satisfies the following energy inequality

$$[E](T) \leq \int_0^T P(E(t))[E](t)\,\mathrm{d}t. \tag{6.6}$$

Here, compared with the process of Picard iteration, the only difference is that the boundary integral produces some extra terms that are controlled using mollification before, and we must use the surface tension instead of the artificial viscosity term to produce the boundary regularity. Following the analysis in Section 5.3.3, the main contribution of the boundary integral arising from $\overline{\partial}^3$-tangential estimates is

$$-\int_\Sigma [\mathbf{Q}][\mathbf{V}]\cdot N^{(1)}\,\mathrm{d}x' \overset{L}{=} -\int_\Sigma \overline{\partial}^3[q]\partial_t\overline{\partial}^3[\psi]\,\mathrm{d}x' + \int_\Sigma \overline{\partial}^3[q]\overline{\partial}^3(v^{(2)}\cdot[N])\,\mathrm{d}x', \tag{6.7}$$

where $[\mathbf{Q}],[\mathbf{V}]$ are the Alinhac good unknowns of $[\check{q}],[v]$ with respect to $\overline{\partial}^3$ and $\varphi^{(1)}$, that is, $[\mathbf{F}] := \mathbf{F}^{(1)}-\mathbf{F}^{(2)}$. For the first integral, we have

$$-\int_\Sigma \overline{\partial}^3[q]\partial_t\overline{\partial}^3[\psi]\,\mathrm{d}x' \overset{L}{=} -\frac{\sigma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma |N^{(1)}|^{-1}\left|\overline{\partial}^3\overline{\nabla}[\psi]\right|_0^2\mathrm{d}x' - \sigma\int_\Sigma \frac{\overline{\partial}^3\overline{\nabla}[\psi]\cdot\overline{\nabla}(\psi^{(1)}+\psi^{(2)})}{|N^{(1)}||N^{(2)}|(|N^{(1)}|+|N^{(2)}|)}\overline{\nabla}\psi^{(2)}\cdot\partial_t\overline{\nabla}\overline{\partial}^3[\psi]\,\mathrm{d}x', \tag{6.8}$$

where the first term gives the boundary energy in $[E](t)$, and the second term appears when $\overline{\partial}^3$ falls on

$$|N^{(1)}|^{-1}-|N^{(2)}|^{-1} = \frac{|N^{(2)}|^2-|N^{(1)}|^2}{|N^{(1)}||N^{(2)}|(|N^{(1)}+|N^{(2)}|)}.$$

This term is controlled by

$$\begin{aligned}
&-\sigma\int_\Sigma \frac{\overline{\partial}^3\overline{\nabla}[\psi]\cdot\overline{\nabla}(\psi^{(1)}+\psi^{(2)})}{|N^{(1)}||N^{(2)}|(|N^{(1)}|+|N^{(2)}|)}\overline{\nabla}\psi^{(2)}\cdot\partial_t\overline{\nabla}\overline{\partial}^3[\psi]\,\mathrm{d}x'\\
&\leq P(|\overline{\nabla}\psi^{(1)},\overline{\nabla}\psi^{(2)}|_\infty)|\sqrt{\sigma}\overline{\nabla}\overline{\partial}^3[\psi]|_0(|\sqrt{\sigma}\partial_t\psi^{(1)}|_4+|\sqrt{\sigma}\partial_t\psi^{(2)}|_4)\\
&\leq \varepsilon|\sqrt{\sigma}\overline{\nabla}\overline{\partial}^3[\psi]|_0^2+P(|\overline{\nabla}\psi^{(1)},\overline{\nabla}\psi^{(2)}|_\infty)E(t) \leq \varepsilon[E](t)+P(E(t)).
\end{aligned}$$

The energy inequality for $[E](t)$ together with Grönwall's inequality and the energy bounds for $E(t)$ implies that there exists some $T\in[0,T']$ only depending on the initial data of (1.24), such that $\sup_{0\leq t\leq T}[E](t)\leq 2[E](0)=0$. Therefore, the solution to (6.2) must be zero. The uniqueness is proven, and the continuous dependence on initial data in $H^3(\Omega)$ for $v,\check{q}$ and in $H^4(\Sigma)$ for ψ is similarly proven.

6.2 Incompressible and zero-surface-tension limits

This section is devoted to showing that we can pass the solution of (1.24) to the incompressible and zero surface tension double limits. In other words, we study the behavior of the solution of (1.24) as both the Mach number λ and surface tension coefficient σ tend to 0. Recall that the Mach number λ is defined in Section 1.3.

We study the incompressible Euler equations modeling the motion of incompressible gravity water waves without surface tension satisfied by (ξ, w, q_{in}) with initial data (w_0, ξ_0) and $w_0^3|_{\Sigma_b} = 0$:

$$\begin{cases} D_t^\varphi w + \nabla^\varphi \mathfrak{p} = 0 & \text{in } [0,T]\times\Omega, \\ \nabla^\varphi \cdot w = 0 & \text{in } [0,T]\times\Omega, \\ \mathfrak{p} = q_{in} + g\varphi & \text{in } [0,T]\times\Omega, \\ \mathfrak{p} = g\xi & \text{on } [0,T]\times\Sigma, \\ \partial_t \xi = w\cdot\mathcal{N} & \text{on } [0,T]\times\Sigma, \\ w_3 = 0 & \text{on } [0,T]\times\Sigma_b, \\ (w,\xi)|_{t=0} = (w_0,\xi_0), \end{cases} \tag{6.9}$$

where we define $\varphi(t,x) = x_3 + \chi(x_3)\xi(t,x')$ to be the extension of ξ in Ω after slightly abuse of notations. Denote by $(\psi^{\lambda,\sigma}, v^{\lambda,\sigma}, \rho^{\lambda,\sigma})$ the solution of (1.24) indexed by λ and σ, our goal is to show:

$$(\psi^{\lambda,\sigma}, v^{\lambda,\sigma}, \rho^{\lambda,\sigma}) \to (\xi, w, 1) \quad \text{in } C^0([0,T]; H^{4-\delta}_{\text{loc}}(\Sigma)\times H^{4-\delta}_{\text{loc}}(\Omega)\times H^{3-\delta}_{\text{loc}}(\Omega)), \quad \text{for any } \delta\in(0,1], \tag{6.10}$$

provided that:

1. The sequence of initial data $(\psi_0^{\lambda,\sigma}, v_0^{\lambda,\sigma}, \rho_0^{\lambda,\sigma}-1) \in H^5(\Sigma)\times H^4(\Omega)\times H^4(\Omega)$ satisfies the compatibility conditions up to order 3, $|\psi_0^{\lambda,\sigma}|_\infty \le 1$, and $v_0^{3;\lambda,\sigma}|_{\Sigma_b} = 0$. The compatibility condition of order k ($k \ge 0$), expressed in terms of the modified pressure, reads

$$(D_t^\varphi)^k \check{q}^{\lambda,\sigma}|_{\{t=0\}\times\Sigma} = \sigma (D_t^\varphi)^k \left(\mathcal{H}^{\lambda,\sigma} + g\psi^{\lambda,\sigma}\right)|_{\{t=0\}\times\Sigma}. \tag{6.11}$$

   Since $D_t^\varphi = \partial_t + \overline{v}^{\lambda,\sigma}\cdot\overline{\partial}$ on Σ, we can rewrite (6.11) as:

$$(\partial_t + \overline{v}^{\lambda,\sigma}\cdot\overline{\partial})^k \check{q}^{\lambda,\sigma}|_{t=0} = \sigma(\partial_t + \overline{v}^{\lambda,\sigma}\cdot\overline{\partial})^k \left(\mathcal{H}^{\lambda,\sigma} + g\psi^{\lambda,\sigma}\right)|_{t=0} \quad \text{on } \Sigma. \tag{6.12}$$

   Apart from this, we require

$$\partial_t^k v^{3;\lambda,\sigma}|_{\{t=0\}\times\Sigma_b} = 0, \quad k = 0,1,2,3, \tag{6.13}$$

   The existence of such data is discussed in Appendix B.
2. $(\psi_0^{\lambda,\sigma}, v_0^{\lambda,\sigma}, \rho_0^{\lambda,\sigma}) \to (\xi, w, 1)$ in $H^4(\Sigma)\times H^4(\Omega)\times H^3(\Omega)$ as $\lambda,\sigma\to 0$.
3. The compressible pressure $q^{\lambda,\sigma}$ and the incompressible pressure q_{in} satisfy the Rayleigh-Taylor sign condition:

$$-\partial_3 q^{\lambda,\sigma} \ge c_0 > 0, \quad \text{on } \{t=0\}\times\Sigma, \tag{6.14}$$
$$-\partial_3 q_{in} \ge c_0 > 0, \quad \text{on } \{t=0\}\times\Sigma. \tag{6.15}$$

The key step of showing the λ,σ-double limits is to prove an energy estimate of (1.24) that is uniform in both λ and σ. The analysis in Section 4 indicates that the energy estimate for (4.1) is already uniform in λ. In particular, one can see that the tangential energy estimates in Sections 4.5-4.6 are uniform in $\mathcal{F}_\lambda'$, which is of size $O(\lambda^2)$ by (1.29).

The energy bound that we obtained from the local existence implies the boundedness of $\|\partial_t^k v^{\lambda,\sigma}(t)\|_{4-k}^2 + |\partial_t^k \psi^{\lambda,\sigma}(t)|_{4-k}^2$ ($k \le 4$) uniformly in both λ and σ within the time interval $[0,T]$. Thus,

$$(v^{\lambda,\sigma}, \psi^{\lambda,\sigma}) \to (w,\xi), \quad \text{as } \lambda,\sigma\to 0, \tag{6.16}$$

weakly-* in $L^\infty([0,T]; H^4(\Omega)\times H^4(\Sigma))$, and strongly in $C^0([0,T]; H^{4-\delta}_{\text{loc}}(\Omega)\times H^{4-\delta}_{\text{loc}}(\Sigma))$ for any $0<\delta\le 1$. Here, the strong convergence is a direct consequence of the Aubin-Lions lemma, and the uniqueness of the limit function implies the convergence without squeezing a subsequence.

Moreover, as $D_t^\varphi = \partial_t + (\overline{v}\cdot\overline{\nabla}) + (\partial_3\varphi)^{-1}(v\cdot\mathbf{N} - \partial_t\varphi)\partial_3$, invoking the continuity equation

$$\mathcal{F}_\lambda'(q) D_t^\varphi \check{q}^{\lambda,\sigma} + \nabla^\varphi\cdot v^{\lambda,\sigma} = g\mathcal{F}_\lambda'(q) D_t^\varphi v_3^{\lambda,\sigma},$$

and because $\|D_t^\varphi \check{q}^{\lambda,\sigma}(t)\|_3, \|D_t^\varphi v^{\lambda,\sigma}(t)\|_3$ are uniformly bounded in $[0,T]$, we have

$$\nabla^\varphi\cdot v^{\lambda,\sigma} \to \nabla^\varphi\cdot w = 0, \tag{6.17}$$

weakly-* in $L^\infty([0,T];H^3(\Omega))$, and strongly in $C^0([0,T];H^{3-\delta}_{\text{loc}}(\Omega))$. Once again, the strong convergence is obtained thanks to the Aubin-Lions lemma.

Finally, since the continuity equation can be expressed as

$$D_t^\varphi(\rho^{\lambda,\sigma}-1)+\rho^{\lambda,\sigma}(\nabla^\varphi\cdot v^{\lambda,\sigma})=0,$$

we can derive the energy estimate for $\rho^{\lambda,\sigma}-1$ in $H^3(\Omega)$ as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\rho^{\lambda,\sigma}-1\|_3^2\leq\|\rho^{\lambda,\sigma}-1\|_0\Big(\|v^{\lambda,\sigma}\|_4+|\overline{\partial}\psi^{\lambda,\sigma}|_3\Big),\tag{6.18}$$

where $\|v^{\lambda,\sigma}\|_4$, $|\overline{\partial}\psi^{\lambda,\sigma}|_3$ are bounded by $E^{\lambda,\sigma}$. Similarly, we can prove the uniform bound also for $\|\partial_t(\rho^{\lambda,\sigma}-1)\|_2^2$. Therefore, $\rho^{\lambda,\sigma}\to 1$ weakly* in $L^\infty([0,T];H^3(\Omega)$, and strongly in $C^0([0,T];H^{3-\delta}_{\text{loc}}(\Omega))$.

7 Improved incompressible limit for well-prepared initial data

Recall that the uniform boundedness (with respect to Mach number) of top-order time derivatives is required to establish the uniform-in-(λ,σ) estimates in Theorem 1.2. However, only the boundedness of first-order time derivatives is required, namely $\operatorname{div} v=O(\lambda)$ and $\partial_t v=O(1)$ if the initial data is well-prepared. In this section, we aim to drop the boundedness assumption for high-order time derivatives. Since we also need to guarantee the propagation of the Rayleigh-Taylor sign condition, the uniform boundedness of $\partial_t\partial_3 q\sim\partial_t^2 v$ is still required.

It should be noted that there is a new difficulty in the control of the "weaker" energy $\mathfrak{E}(t)$: There exhibits a loss of weight of Mach number in $\overline{\partial}^2\partial_t^2$-tangential estimates when analyzing $\mathfrak{E}_4(t)$. In particular, we have to control the following quantity in the cancellation structure used at the end of Section 4.6:

$$\int_\Omega(\overline{\partial}\partial_3\partial_t^2 v_i)(\overline{\partial}\mathbf{N}_i)(\overline{\partial}\partial_3\partial_t^2 q)\,\mathrm{d}x,$$

in which $\partial_t^2 q$ has to be uniformly bounded with respect to Mach number. However, now we only have $\partial_t^2 q=O(1/\lambda)$, which leads to a loss of λ-weight. Besides, similar difficulty also appears in the control of $-\int_\Omega\mathbf{V}^\pm\cdot\mathfrak{C}(q^\pm)\mathrm{d}\mathcal{V}_t$. Indeed, such loss of λ-weight necessarily happens in $\overline{\partial}^2\partial_t^2$-tangential estimates because of the following two reasons

1. $\overline{\partial}^2\partial_t^2 q$ needs one more λ-weight than $\overline{\partial}^2\partial_t^2 v$, and
2. The (extension of) normal vector $\mathbf{N}$, which arises from the commutator $[\overline{\partial}^2\partial_t^2,\mathbf{N}_i/\partial_3\varphi,\partial_3 f]$ in $\mathfrak{C}_i(f)$, may NOT absorb a time derivative.

Such loss of weights of Mach number is completely caused by the free-surface motion because the commutator $\mathfrak{C}(f)$ is not needed in the fixed-domain setting. In the second author's previous work [76] considering compressible inviscid fluids *without surface tension*, such essential difficulty can be avoided thanks to the vanishing Dirichlet boundary condition $q|_\Sigma=0$, but that framework is no longer applicable here. To get rid of the loss of Mach number, we have to find a new way to control $\partial_t^2 v$. We also need to introduce a new energy functional:

$$\mathfrak{E}(t):=\mathfrak{E}_4(t)+E_5(t),\tag{7.1}$$

$$\begin{aligned}\mathfrak{E}_4(t):=&\|v\|_4^2+\|\check{q}\|_4^2+|\sqrt{\sigma}\psi|_5^2+|\psi|_4^2+\|\partial_t v,\partial_t\check{q}\|_3^2+|\sqrt{\sigma}\partial_t\psi|_4^2+|\partial_t\psi|_{3.5}^2\\&+\|\partial_t^2 v,\lambda\partial_t^2\check{q}\|_2^2+|\sqrt{\sigma}\partial_t^2\psi|_3^2+|\partial_t^2\psi|_{2.5}^2+|\partial_t^3\psi|_{1.5}^2\\&+\sum_{k=3}^4\|\lambda\partial_t^k(v,\check{q})\|_{4-k}^2+|\sqrt{\sigma}\lambda\partial_t^k\psi|_{5-k}^2+|\lambda\partial_t^4\psi|_{0.5}^2\end{aligned}\tag{7.2}$$

$$E_5(t):=\sum_{k=0}^5\left\|\lambda^2\partial_t^k(v,\lambda^{(k-4)_+}\check{q})\right\|_{5-k}^2+\left|\sqrt{\sigma}\lambda^2\partial_t^k\psi\right|_{6-k}^2+\left|\lambda^2\partial_t^k\psi\right|_{5-k}^2\tag{7.3}$$

We now introduce the following div-curl inequality

Lemma 7.1 (Hodge-type elliptic estimates)**.** For any sufficiently smooth vector field X and $s\geq 1$, one has

$$\|X\|_s^2\lesssim C(|\psi|_{s+\frac{1}{2}},|\overline{\nabla}\psi|_{W^{1,\infty}})\left(\|X\|_0^2+\|\nabla^\varphi\cdot X\|_{s-1}^2+\|\nabla^\varphi\times X\|_{s-1}^2+|X\cdot N|_{s-\frac{1}{2}}^2\right),\tag{7.4}$$

where the constant $C(|\psi|_{s+\frac{1}{2}},|\overline{\nabla}\psi|_{W^{1,\infty}})>0$ depends linearly on $|\psi|_{s+\frac{1}{2}}^2$.

Applying this inequality to $X = \partial_t^2 v$ and $s = 2$, we obtain that

$$\|\partial_t^2 v\|_2^2 \lesssim C(|\psi|_{2.5}, |\overline{\nabla}\psi|_{W^{1,\infty}}) \left(\|\partial_t^2 v\|_0^2 + \|\nabla^\varphi \cdot \partial_t^2 v\|_1^2 + \|\nabla^\varphi \times \partial_t^2 v\|_1^2 + |\partial_t^2 v \cdot N|_{s-\frac{1}{2}}^2 \right). \tag{7.5}$$

The divergence and vorticity are controlled in the same way as in Section 4.3. As for the boundary term, we have

$$\partial_t^2 v \cdot N = \partial_t^3 \psi + \overline{v}_j \overline{\partial}_j \partial_t^2 \psi,$$

so we shall turn to control $|\partial_t^3\psi|_{1.5}$ and $|\partial_t^2\psi|_{2.5}$ without any weights of λ, σ.

7.1 Time-differentiated evolution equation of the free surface

We derive the evolution equation of the free surface by further differentiating the kinematic boundary condition in time variable.

7.1.1 Time-differentiated kinematic boundary condition

Let $\overline{D_t} := D_t^\varphi|_\Sigma = \partial_t + \overline{v} \cdot \overline{\nabla}$. The kinematic boundary condition then implies

$$\overline{D_t}\psi = v_3, \quad \text{ on } \Sigma. \tag{7.6}$$

Taking one more $\overline{D_t}$ to (7.6), we infer from the momentum equation that

$$\rho \overline{D_t}^2 \psi = -\partial_3 \check{q} - (\rho - 1) g, \quad \text{ on } \Sigma. \tag{7.7}$$

Since $[\partial_t, \overline{D_t}] f = \partial_t \overline{v}_j \overline{\partial}_j f$, we obtain

$$[\partial_t^2, \overline{D_t}] f = \partial_t^2 \overline{v}_j \overline{\partial}_j f + 2\partial_t \overline{v}_j \, \partial_t \overline{\partial}_j f.$$

From this and $[\overline{\partial}_j, \overline{D_t}] = \overline{\partial}_i \overline{v}_j \, \overline{\partial}_j f$, we see that

$$\begin{aligned}
[\partial_t^2, \overline{D_t}^2] f = & \overline{D_t} \left(\partial_t^2 \overline{v}_j \overline{\partial}_j f + 2\partial_t \overline{v}_j \partial_t \overline{\partial}_j f \right) + \partial_t^2 \overline{v}_j \, \overline{\partial}_j \overline{D_t} f + 2\partial_t \overline{v}_j \, \overline{\partial}_j \partial_t \overline{D_t} f \\
= & \partial_t^2 \overline{D_t} \overline{v}_j \, \overline{\partial}_j f - 2\partial_t^2 \overline{v}_j \, \overline{\partial}_j \overline{v}_k \, \overline{\partial}_k f - 2\partial_t \overline{v}_j \, \partial_t \overline{\partial}_j \overline{v}_k \, \overline{\partial}_k f \\
& + 2\partial_t^2 \overline{v}_j \, \overline{\partial}_j \overline{D_t} f + 2\partial_t \overline{D_t} \overline{v}_j \, \overline{\partial}_j \partial_t f - 4\partial_t \overline{v}_j \, \overline{\partial}_j \overline{v}_k \, \overline{\partial}_k \partial_t f \\
& + 4\partial_t \overline{v}_j \, \overline{\partial}_j \partial_t \overline{D_t} f - 2\partial_t \overline{v}_j \, \overline{\partial}_j (\partial_t \overline{v}_k \, \overline{\partial}_k f).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\partial_t^2 \overline{D_t}^2 \psi = & \overline{D_t}^2 \partial_t^2 \psi + \partial_t^2 \overline{D_t} \overline{v}_j \, \overline{\partial}_j \psi - 2\partial_t^2 \overline{v}_j \, \overline{\partial}_j \overline{v}_k \, \overline{\partial}_k \psi - 2\partial_t \overline{v}_j \, \partial_t \overline{\partial}_j \overline{v}_k \, \overline{\partial}_k \psi \\
& + 2\partial_t^2 \overline{v}_j \, \overline{\partial}_j \overline{D_t} \psi + 2\partial_t \overline{D_t} \overline{v}_j \, \overline{\partial}_j \partial_t \psi - 4\partial_t \overline{v}_j \, \overline{\partial}_j \overline{v}_k \, \overline{\partial}_k \partial_t \psi \\
& + 4\partial_t \overline{v}_j \, \overline{\partial}_j \partial_t \overline{D_t} \psi - 2\partial_t \overline{v}_j \, \overline{\partial}_j (\partial_t \overline{v}_k \, \overline{\partial}_k \psi).
\end{aligned} \tag{7.8}$$

Combining this with (7.7) yields

$$\overline{D_t}^2 \partial_t^2 \psi = -\frac{1}{\rho} \partial_t^2 \partial_3 \check{q} - \partial_t^2 \overline{D_t} \overline{v} \cdot \overline{\nabla} \psi + \mathbf{R}_\psi + \mathbf{R}_\rho, \quad \text{ on } \Sigma, \tag{7.9}$$

where

$$\begin{aligned}
-\mathbf{R}_\psi = & -2\partial_t^2 \overline{v}_j \, \overline{\partial}_j \overline{v}_k \, \overline{\partial}_k \psi - 2\partial_t \overline{v}_j \, \partial_t \overline{\partial}_j \overline{v}_k \, \overline{\partial}_k \psi + 2\partial_t^2 \overline{v}_j \, \overline{\partial}_j \overline{D_t} \psi + 2\partial_t \overline{D_t} \overline{v}_j \, \overline{\partial}_j \partial_t \psi \\
& - 4\partial_t \overline{v}_j \, \overline{\partial}_j \overline{v}_k \, \overline{\partial}_k \partial_t \psi + 4\partial_t \overline{v}_j \, \overline{\partial}_j \partial_t \overline{D_t} \psi - 2\partial_t \overline{v}_j \, \overline{\partial}_j (\partial_t \overline{v}_k \, \overline{\partial}_k \psi)
\end{aligned} \tag{7.10}$$

and

$$\mathbf{R}_\rho = -\partial_t^2 \left(\frac{(\rho - 1) g}{\rho} \right) - \left[\partial_t^2, \frac{1}{\rho} \right] \partial_3 \check{q}. \tag{7.11}$$

For $i = 1, 2$, since $\rho\overline{D_t}v_i = -\overline{\partial}_i^{\varphi}\check{q}$ with $\overline{\partial}_i^{\varphi} := \partial_i^{\varphi}|_{\Sigma} = \partial_i - \partial_i\psi\partial_3$, we have

$$-\partial_t^2\overline{D_t}\overline{v}\cdot\overline{\nabla}\psi = \frac{1}{\rho}\partial_t^2\overline{\partial}^{\varphi}\check{q}\cdot\overline{\nabla}\psi + \underbrace{\left[\partial_t^2, \frac{1}{\rho}\right]\overline{\partial}^{\varphi}\check{q}\cdot\overline{\nabla}\psi}_{=\mathbf{R}_{\psi,\rho}} \quad \text{on } \Sigma. \tag{7.12}$$

Also, since $\overline{\partial}_3^{\varphi} := \partial_3^{\varphi}|_{\Sigma} = \partial_3$, then

$$-\frac{1}{\rho}\partial_t^2\partial_3\check{q} + \frac{1}{\rho}\partial_t^2\overline{\partial}^{\varphi}\check{q}\cdot\overline{\nabla}\psi = -\frac{1}{\rho}N\cdot\partial_t^2\overline{\partial}^{\varphi}\check{q} \quad \text{on } \Sigma.$$

This leads to the following evolution equation of the moving interface:

$$\overline{D_t}^2\partial_t^2\psi = -\frac{1}{\rho}N\cdot\partial_t^2\overline{\partial}^{\varphi}\check{q} + \mathbf{R}_{\psi} + \mathbf{R}_{\rho} + \mathbf{R}_{\psi,\rho}, \quad \text{on } \Sigma, \tag{7.13}$$

where $\mathbf{R}_{\psi}, \mathbf{R}_{\rho}$, and $\mathbf{R}_{\psi,\rho}$ are given respectively in (7.10), (7.11) and (7.12).

7.1.2 The reformulation in Alinhac good unknowns

In the next, we introduce Q to be the Alinhac's good unknown of $\check{q}$ associated with ∂_t^2:

$$Q := \partial_t^2\check{q} - \partial_t^2\varphi\partial_3^{\varphi}\check{q}, \quad \text{in } \Omega. \tag{7.14}$$

For $j = 1, 2, 3$, similar to (4.36), we have

$$\partial_t^2\nabla_j^{\varphi}\check{q} = \nabla_j^{\varphi}Q + C_j(\check{q}), \quad \text{in } \Omega. \tag{7.15}$$

Here, for a generic function f, we define

$$C_i(f) = \partial_i^{\varphi}\partial_3^{\varphi}f\partial_t^2\varphi + C_i'(f), \quad i = 1, 2, \text{ and } \quad C_3(\check{q}) = (\partial_3^{\varphi})^2 f\partial_t^2\varphi + C_3'(f), \tag{7.16}$$

where

$$C_i'(f) = -\left[\partial_t^2, \frac{\partial_i\varphi}{\partial_3\varphi}, \partial_3 f\right] - \partial_3 f\left[\partial_t^2, \partial_i\varphi, \frac{1}{\partial_3\varphi}\right] - \partial_i\varphi\partial_3 f\partial_t\left(\frac{1}{(\partial_3\varphi)^2}\right)\partial_t\partial_3\varphi,$$

and

$$C_3'(f) = \left[\partial_t^2, \frac{1}{\partial_3\varphi}, \partial_3 f\right] + \partial_3 f\partial_t\left(\frac{1}{(\partial_3\varphi)^2}\right)\partial_t\partial_3\varphi.$$

Note that $\partial_3\varphi|_{\Sigma} = 1$, (7.15) then yields

$$\partial_t^2\overline{\partial}_j^{\varphi}\check{q} = \overline{\partial}_j^{\varphi}Q + \mathbf{C}_j(\check{q}), \quad \text{on } \Sigma, \tag{7.17}$$

where $\mathbf{C}_i(\check{q}) = \overline{\partial}_i^{\varphi}\partial_3\check{q}\partial_t^2\psi - [\partial_t^2, \partial_i\psi, \partial_3\check{q}]$ when $i = 1, 2$, and $\mathbf{C}_3(\check{q}) = \partial_3^2\check{q}\partial_t^2\psi$. Therefore, the equation (7.13) turns into

$$\overline{D_t}^2\partial_t^2\psi = -\frac{1}{\rho}N\cdot\nabla^{\varphi}Q - \frac{1}{\rho}N\cdot\mathbf{C}(\check{q}) + \mathbf{R}_{\psi} + \mathbf{R}_{\rho} + \mathbf{R}_{\psi,\rho}, \quad \text{on } \Sigma. \tag{7.18}$$

Parallel to Q, we define $\mathcal{V}$ to be the Alinhac's good unknown of v associated with ∂_t^2:

$$\mathcal{V} := \partial_t^2 v - \partial_t^2\varphi\partial_3^{\varphi}v, \quad \text{in } \Omega. \tag{7.19}$$

Then, similar to (4.43)–(4.44), $(\mathcal{V}, Q)$ verifies

$$\begin{aligned} &\rho D_t^{\varphi}\mathcal{V} + \nabla^{\varphi}Q = G^1, && \text{in } \Omega,\\ &\lambda^2 D_t^{\varphi}Q + \nabla^{\varphi}\cdot\mathcal{V} = G^2 - C_i(v^i), && \text{in } \Omega, \end{aligned} \tag{7.20}$$

where we write $\mathcal{F}'(q) = \lambda^2$ for simplicity of notations (which is reasonable when discussing the incompressible limit according to the discussion in Section 1.3)

$$
\begin{aligned}
G_i^1 &= -[\partial_t^2, \rho] D_t^\varphi v_i - \rho D(v_i) - C_i(\check{q}) - (\partial_t^2 \rho) g \delta_{i3}, \quad i = 1,2,3 \\
G^2 &= -\lambda^2 D(\check{q}) + \lambda^2 g \partial_t^2 v_3.
\end{aligned}
$$

Here, for a generic function f, we define

$$
D(f) = (D_t^\varphi \partial_3^\varphi f)(\partial_t^2 \varphi) + D'(f),
$$

with

$$
\begin{aligned}
D'(f) = [\partial_t^2, \overline{v}] \cdot \overline{\partial} f &+ \left[\partial_t^2, \frac{1}{\partial_3 \varphi}(v \cdot \mathbf{N} - \partial_t \varphi), \partial_3 f\right] + \left[\partial_t^2, v \cdot \mathbf{N} - \partial_t \varphi, \frac{1}{\partial_3 \varphi}\right] \partial_3 f \\
&+ \frac{1}{\partial_3 \varphi}[\partial_t^2, v] \cdot \mathbf{N} \partial_3 f - (v \cdot \mathbf{N} - \partial_t \varphi) \partial_3 f \partial_t \left(\frac{1}{(\partial_3 \varphi)^2}\right) \partial_t \partial_3 \varphi.
\end{aligned}
$$

In the next, we commute the divergence operator $\nabla^\varphi \cdot$ to the first equation of (7.20) to obtain:

$$
\rho \lambda^2 (D_t^\varphi)^2 Q - \Delta^\varphi Q = \rho \partial_i^\varphi v^k \partial_k^\varphi \mathcal{V}^i + \nabla^\varphi \rho \cdot D_t^\varphi \mathcal{V} + \rho D_t^\varphi \left(G^2 - C_i(v^i)\right) - \nabla^\varphi \cdot G^1. \tag{7.21}
$$

7.1.3 Decomposition of the pressure: Dirichlet-to-Neumann operator

Since

$$
Q = \partial_t^2 \check{q} - \partial_t^2 \psi \partial_3 \check{q} = \sigma \partial_t^2 \mathcal{H} - \partial_3 q \partial_t^2 \psi, \quad \text{on } \Sigma,
$$

we define $Q = Q_h + Q_w$, where Q_h solves the elliptic equation

$$
\begin{aligned}
-\Delta^\varphi Q_h &= 0, && \text{in } \Omega, \\
Q_h &= \sigma \partial_t^2 \mathcal{H} - \partial_3 q \partial_t^2 \psi, && \text{on } \Sigma, \\
\partial_3 Q_h &= 0, && \text{on } \Sigma_b,
\end{aligned} \tag{7.22}
$$

and Q_w satisfies

$$
\begin{aligned}
-\Delta^\varphi Q_w = -\rho \lambda^2 (D_t^\varphi)^2 Q + \rho \partial_i^\varphi v^k \partial_k^\varphi \mathcal{V}^i + \nabla^\varphi \rho \cdot D_t^\varphi \mathcal{V} + \rho D_t^\varphi \left(G^2 - C_i(v^i)\right) - \nabla^\varphi \cdot G^1, &\quad \text{in } \Omega, \\
Q_w = 0, &\quad \text{on } \Sigma, \\
\partial_3 Q_w = \partial_3 Q = -\partial_t^2 \rho g, &\quad \text{on } \Sigma_b,
\end{aligned} \tag{7.23}
$$

where $\partial_3 Q|_\Sigma$ is computed by restricting the third component of the first equation in (7.20) on Σ_b.

With this decomposition, we can further reduce the evolution equation of the free surface (7.18) by introducing the Dirichlet-to-Neumann (DtN) operator.

Definition 7.1 (Dirichlet-to-Neumann (DtN) operator)**.** For a function $f : \Sigma \to \mathbb{R}$, we define the Dirichlet-to-Neumann (DtN) operator associated with (Ω, ψ) by

$$
\mathfrak{N}_\psi f := N \cdot \nabla^\varphi (\mathcal{E}_\psi f), \tag{7.24}
$$

where $\mathcal{E}_\psi f$ is defined to be the harmonic extension of f into Ω, namely

$$
-\Delta^\varphi (\mathcal{E}_\psi f) = 0 \quad \text{in } \Omega, \quad \mathcal{E}_\psi f = f \quad \text{on } \Sigma, \quad \partial_3 (\mathcal{E}_\psi f) = 0 \quad \text{on } \Sigma_b. \tag{7.25}
$$

With this definition, we can rewrite

$$
\begin{aligned}
N \cdot \nabla^\varphi Q = N \cdot \nabla^\varphi Q_h + N \cdot \nabla^\varphi Q_w &= \mathfrak{N}_\psi (\sigma \partial_t^2 \mathcal{H} - \partial_3 q \partial_t^2 \psi) + N \cdot \nabla^\varphi Q_w \\
&= \sigma \mathfrak{N}_\psi (\partial_t^2 \mathcal{H}) - \mathfrak{N}_\psi (\partial_3 q \partial_t^2 \psi) + N \cdot \nabla^\varphi Q_w,
\end{aligned}
$$

and thus the evolution equation (7.18) becomes

$$
\rho \overline{D_t}^2 \partial_t^2 \psi + \sigma \mathfrak{N}_\psi (\partial_t^2 \mathcal{H}) - \mathfrak{N}_\psi (\partial_3 q \partial_t^2 \psi) = -N \cdot \nabla^\varphi Q_w - N \cdot \mathbf{C}(\check{q}) + \rho (\mathbf{R}_\psi + \mathbf{R}_\rho + \mathbf{R}_{\psi,\rho}) \quad \text{on } \Sigma \tag{7.26}
$$

7.2 Preliminaries on paradifferential calculus

In the equation (7.26), the term involving DtN operators are fully nonlinear, so we shall find out their concrete forms in order for an explicit energy estimate. In the remaining part of this paper, we will introduce several preliminary lemmas about paradifferential calculus that have been proven in Alazard-Burq-Zuily [2]. Following the notations in Métivier [52], we first introduce the basic definition of a paradifferential operator. Note that the dimension d in this section is actually the Hausdorff dimension of the free surface.

Definition 7.2 (Symbols)**.** Given $r \geq 0,\ m \in \mathbb{R}$, we denote $\Gamma_r^m(\mathbb{R}^d)$ to be the space of locally bounded functions $a(x',\xi)$ on $\mathbb{R}^d \times (\mathbb{R}^d\backslash\{0\})$, which are C^∞ with respect to $\xi(\xi \neq \mathbf{0})$, such that for any $\alpha \in \mathbb{N}^d, \xi \neq \mathbf{0}$, the function $x' \mapsto \partial_\xi^\alpha a(x',\xi)$ belongs to $W^{r,\infty}(\mathbb{R}^d)$ and there exists a constant C_α such that

$$\left|\partial_\xi^\alpha a(\cdot,\xi)\right|_{W^{r,\infty}(\mathbb{R}^d)} \leq C_\alpha(1+|\xi|)^{m-|\alpha|},\quad \forall|\xi| \geq 1/2.$$

Definition 7.3 (Paradifferential operator)**.** Given a symbol a, we shall define the **paradifferential operator** T_a by

$$\widehat{T_a u}(\xi) := (2\pi)^{-d}\int_{\mathbb{R}^d}\tilde{\chi}(\xi-\eta,\eta)\hat{a}(\xi-\eta,\eta)\phi(\eta)\hat{u}(\eta)\,\mathrm{d}\eta \tag{7.27}$$

where $\hat{a}(\theta,\xi) = \int_{\mathbb{R}^d}\exp(-ix'\cdot\theta)a(x',\xi)\,\mathrm{d}x'$ is the Fourier transform of a in variable $x' \in \mathbb{R}^d$. Here $\tilde{\chi}$ and ϕ are two given cut-off functions such that

$$\phi(\eta) = 0 \ \text{ for } |\eta| \leq 1,\quad \phi(\eta) = 1 \ \text{ for } |\eta| \geq 2,$$

and $\tilde{\chi}(\theta,\eta)$ is homogeneous of degree 0 and satisfies that for $0 < \varepsilon_1 < \varepsilon_2 \ll 1$, $\tilde{\chi}(\theta,\eta) = 1$ if $|\theta| \leq \varepsilon_1|\eta|$ and $\tilde{\chi}(\theta,\eta) = 0$ if $|\theta| \geq \varepsilon_2|\eta|$. We also introduce the semi-norm

$$M_r^a(a) := \sup_{|\alpha|\leq\frac{d}{2}+1+r}\ \sup_{|\xi|\geq 1/2}\left|(1+|\xi|)^{|\alpha|-m}\partial_\xi^\alpha a(\cdot,\xi)\right|_{W^{r,\infty}(\mathbb{R}^d)}. \tag{7.28}$$

For $m \in \mathbb{R}$, we say T is of order m if for all $s \in \mathbb{R}$, T is bounded from H^s to H^{s-m}.

Proposition 7.2. Let $m \in \mathbb{R}$. If $a \in \Gamma_0^m(\mathbb{R}^d)$, then T_a is of order m. Moreover, for any $s \in \mathbb{R}$, there exists a constant K such that $\|T_a\|_{H^s\to H^{s-m}} \leq KM_0^m(a)$.

Proposition 7.3 (Composition, [2, Theorem 3.7])**.** Let $m \in \mathbb{R}$ and $r > 0$. If $a \in \Gamma_r^m(\mathbb{R}^d),\ b \in \Gamma_r^{m'}(\mathbb{R}^d)$, then $T_aT_b - T_{a\#b}$ is of order $m + m' - r$ where

$$a\#b := \sum_{|\alpha|<r}\frac{1}{i^{|\alpha|}\alpha!}\partial_\xi^\alpha a\partial_{x'}^\alpha b$$

and $\partial_{x'} = (\overline{\partial}_{x_1},\overline{\partial}_{x_2})$. Moreover, for all $s \in \mathbb{R}$, there exists a constant K such that

$$\|T_aT_b - T_{a\#b}\|_{H^s\to H^{s-m-m'+r}} \leq KM_r^m(a)M_r^{m'}(b). \tag{7.29}$$

Proposition 7.4 (Adjoint, [2, Theorem 3.10])**.** Let $m \in \mathbb{R}$, $r > 0$ and $a \in \Gamma_r^m(\mathbb{R}^d)$. We denote by $(T_a)^*$ the adjoint operator of T_a. Then $(T_a)^* - T_{a^*}$ is of order $m - r$ where

$$a^* := \sum_{|\alpha|<r}\frac{1}{i^\alpha\alpha!}\partial_\xi^\alpha\partial_{x'}^\alpha\bar{a}.$$

Moreover, for any $s \in \mathbb{R}$, there exists a constant K such that $\|(T_a)^* - T_{a^*}\|_{H^s\to H^{s-m+r}} \leq KM_r^m(a)$.

Here and thereafter in this section, $\psi \in C([0,T];H^{s+\frac{1}{2}}(\mathbb{R}^d))$ is a given function with $s > 2 + \frac{d}{2}$. The symbolic calculus is not defined for C^∞ symbols, so we need to introduce the following symbols.

Definition 7.4. Given $m \in \mathbb{R}$, we denote Σ^m to be the class of symbols a of the form $a = a^{(m)} + a^{(m-1)}$ with

$$a^{(m)}(t,x',\xi) = F(\partial_{x'}\psi(t,x'),\xi),\quad a^{(m-1)}(t,x',\xi) = \sum_{|\alpha|_2}G_\alpha(\partial_{x'}\psi(t,x'),\xi)\partial_{x'}^\alpha\psi(t,x')$$

such that

i. T_a maps real-valued functions to real-valued functions;

ii. F is a C^∞ real-valued functions of $(\zeta, \xi) \in \mathbb{R}^d \times (\mathbb{R}^d \backslash \{0\})$, homogeneous of degree m in ξ, such that there exists a continuous function $K = K(\zeta) > 0$ such that $F(\zeta, \xi) \geq K(\zeta)|\xi|^m$ for all $(\zeta, \xi) \in \mathbb{R}^d \times (\mathbb{R}^d \backslash \{0\})$;

iii. G_α is a C^∞ complex-valued function of $(\zeta, \xi) \in \mathbb{R}^d \times (\mathbb{R}^d \backslash \{0\})$, homogeneous of degree $m-1$ in ξ.

Definition 7.5 (Equivalence of paradifferential operators)**.** Given $m \in \mathbb{R}$ and consider two families of operators of order m: $\{A(t) : t \in [0, T]\}$ and $\{B(t) : t \in [0, T]\}$, we say $A \sim B$ if $A - B$ has order $m - 1.5$ and satisfies the estimate: for all $r \in \mathbb{R}$ there exists a continuous function $C(\cdot)$ such that

$$\forall t \in [0, T], \quad \|A(t) - B(t)\|_{H^r \to H^{r-(m-1.5)}} \leq C(|\psi(t)|_{s+\frac{1}{2}}).$$

From now on, we use the notation $|\cdot|_{s_1 \to s_2}$ to represent the operator norm $\|\cdot\|_{H^{s_1} \to H^{s_2}}$, and use the notation $|\cdot|_s$ to represent $\|\cdot\|_{H^s(\mathbb{R}^d)}$. We have the following theorem for the composition

Proposition 7.5 ([2, Prop. 4.3])**.** Let $m, m' \in \mathbb{R}$. Then

1. If $a \in \Sigma^m,\ b \in \Sigma^{m'}$, then $T_a T_b \sim T_{a\#b}$ where $a\#b$ is given by

$$a\#b = a^{(m)} b^{(m')} + a^{(m-1)} b^{(m')} + a^{(m)} b^{(m'-1)} + \frac{1}{i} \partial_\xi a^{(m)} \cdot \partial_{x'} b^{(m')}.$$

2. If $a \in \Sigma^m$, then $(T_a)^* \sim T_b$ where $b \in \Sigma^m$ is given by

$$b = a^{(m)} + \overline{a^{(m-1)}} + \frac{1}{i}(\partial_{x'} \cdot \partial_\xi) a^{(m)}.$$

We denote $\Re z$ and $\Im z$ to be the real part and the imaginary part of a complex number z, respectively. As a corollary, we have

Corollary 7.6 ([2, Prop. 4.3(2)])**.** If $a \in \Sigma^m$ satisfies $\Im a^{(m-1)} = -\frac{1}{2}(\partial_\xi \cdot \partial_{x'}) a^{(m)}$, then $(T_a)^* \sim T_a$.

The next proposition is significant for estimates in Sobolev norms via paradifferential calculus.

Proposition 7.7 ([2, Prop. 4.4 and 4.6])**.** Let $m \in \mathbb{R},\ r \in \mathbb{R}$. Then for all symbol $a \in \Sigma^m$ and $t \in [0, T]$, the following estimate holds.

$$|T_{a(t)} u|_{r-m} \leq C(|\psi(t)|_{s-1}) |u|_r, \tag{7.30}$$

$$|u|_{r+m} \leq C(|\psi(t)|_{s-1}) \left(|T_{a(t)} u|_r + |u|_0\right). \tag{7.31}$$

7.3 Paralinearization of evolution equation of the free surface

Now we can start to paralinearize the term involving $\mathfrak{N}_\psi$ and $\mathcal{H}$ in (7.26).

Lemma 7.8 (Paralinearization of the DtN operator, [5, Sect. 4.4])**.** For $f, \psi \in H^{s+\frac{1}{2}}(\mathbb{R}^d)$, we have

$$\mathfrak{N}_\psi f = T_\Lambda f + R^\psi_\Lambda(f), \tag{7.32}$$

with the symbols $\Lambda = \Lambda^{(1)} + \Lambda^{(0)}$ give by

$$\Lambda^{(1)} = \sqrt{(1 + |\overline{\nabla}_{x'} \psi|^2)|\xi|^2 - (\overline{\nabla}_{x'} \psi \cdot \xi)^2}, \tag{7.33}$$

$$\Lambda^{(0)} = \frac{1 + |\overline{\nabla}_{x'} \psi|^2|}{2\Lambda^{(1)}} \left(\overline{\nabla}_{x'} \cdot (\alpha^{(1)} \overline{\nabla}_{x'} \psi) + i \partial_\xi \Lambda^{(1)} \cdot \partial_{x'} \alpha^{(1)}\right), \tag{7.34}$$

and $\alpha^{(1)} := (\Lambda^{(1)} + i \overline{\nabla}_{x'} \psi \cdot \xi)/(1 + |\overline{\nabla}_{x'} \psi|^2)$. The remainder terms satisfy the following estimates

$$|R^\psi_\Lambda(f)|_r \leq C(|\psi|_{s+\frac{1}{2}}) |f|_r \quad \forall \frac{1}{2} \leq r \leq s - \frac{1}{2},\ s > 2 + \frac{d}{2}. \tag{7.35}$$

Next, we paralinearize the mean curvature term. Let $\mathcal{H}(\psi) = -\overline{\nabla} \cdot \left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}}\right)$. We have

Lemma 7.9 (Paralinearization of the mean curvature, [2, Lemma 3.25])**.** There holds $\mathcal{H}(\psi) = T_{\mathfrak{H}}\psi + R_{\mathfrak{H}}$ where $\mathfrak{H} = \mathfrak{H}^{(2)} + \mathfrak{H}^{(1)}$ is defined by

$$\mathfrak{H}^{(2)} = \frac{1}{\sqrt{1+|\overline{\nabla}_{x'}\psi|^2}}\left(|\xi|^2 - \frac{(\overline{\nabla}_{x'}\psi\cdot\xi)^2}{1+|\overline{\nabla}_{x'}\psi|^2}\right), \tag{7.36}$$

$$\mathfrak{H}^{(1)} = -\frac{i}{2}(\partial_{x'}\cdot\partial_\xi)\mathfrak{H}^{(2)}, \tag{7.37}$$

and the remainder term $R_{\mathfrak{H}}$ satisfies

$$|R_{\mathfrak{H}}|_{2s-3} \le C(|\psi|_{s+\frac12}). \tag{7.38}$$

Now, we can treat the nonlinear terms on the left side of (7.26). The term involving surface tension is treated as follows

$$\begin{aligned}\sigma\mathfrak{N}_\psi(\partial_t^2\mathcal{H}(\psi)) &= \sigma\mathfrak{N}_\psi(\partial_t^2 T_{\mathfrak{H}}\psi) + \sigma\mathfrak{N}_\psi(\partial_t^2 R_{\mathfrak{H}})\\ &= \sigma T_\Lambda T_{\mathfrak{H}}(\partial_t^2\psi) + \sigma\mathfrak{N}_\psi([\partial_t^2, T_{\mathfrak{H}}]\psi + \partial_t^2 R_{\mathfrak{H}}) + \sigma R_\Lambda^\psi(T_{\mathfrak{H}}\partial_t^2(\psi))\\ &=: \sigma T_\Lambda T_{\mathfrak{H}}(\partial_t^2\psi) + \mathbf{R}_\psi^{ST}\end{aligned} \tag{7.39}$$

The term involving the Rayleigh-Taylor sign is treated as follows

$$\begin{aligned}\mathfrak{N}_\psi(\partial_3 q\partial_t^2\psi) &= (\partial_3 q)\mathfrak{N}_\psi^{\frac12}\mathfrak{N}_\psi^{\frac12}(\partial_t^2\psi) + [\mathfrak{N}_\psi, \partial_3 q]\partial_t^2\psi.\\ &=: (\partial_3 q)\mathfrak{N}_\psi^{\frac12}\mathfrak{N}_\psi^{\frac12}(\partial_t^2\psi) + \mathbf{R}_\psi^{RT}\end{aligned} \tag{7.40}$$

Now, the evolution equation (7.26) becomes

$$\begin{aligned}\rho\overline{D_t}^2\partial_t^2\psi + \sigma T_\Lambda T_{\mathfrak{H}}(\partial_t^2\psi) + (-\partial_3 q)\mathfrak{N}_\psi^{\frac12}\mathfrak{N}_\psi^{\frac12}(\partial_t^2\psi) &= -N\cdot\nabla^\varphi Q_w + \mathbf{R}_\psi^\sigma + \mathbf{R}_\psi^{RT}\\ &\quad - N\cdot\mathbf{C}(\check{q}) + \rho(\mathbf{R}_\psi + \mathbf{R}_\rho + \mathbf{R}_{\psi,\rho}) \quad \text{on } \Sigma.\end{aligned} \tag{7.41}$$

7.4 Uniform estimates for the free surface

In order for an explicit energy estimate via (7.41), we shall symmetrize the 3-rd order paradifferential operator $T_\Lambda T_{\mathfrak{H}}$. That is, find suitable symbols $\mathfrak{m}\in\Sigma^{1.5}$ and $\mathfrak{n}\in\Sigma^0$ such that $T_{\mathfrak{n}}T_\Lambda T_{\mathfrak{H}} \sim T_{\mathfrak{m}}T_{\mathfrak{m}}T_{\mathfrak{n}}$ and $T_{\mathfrak{m}} \sim (T_{\mathfrak{m}})^*$.

Proposition 7.10 (Symmetrization, [2, Prop. 4.8])**.** Let $\mathfrak{n}\in\Sigma^0$ and $\mathfrak{m}\in\Sigma^{1.5}$ be defined by

$$\mathfrak{n} := \frac{1}{\sqrt[4]{1+|\overline{\nabla}\psi|^2}} = |N|^{-\frac12}, \tag{7.42}$$

$$\mathfrak{m} := \underbrace{\sqrt{\mathfrak{H}^{(2)}\Lambda^{(1)}}}_{=:\mathfrak{m}^{(1.5)}} + \underbrace{\frac{1}{2i}(\partial_\xi\cdot\partial_{x'})\sqrt{\mathfrak{H}^{(2)}\Lambda^{(1)}}}_{=:\mathfrak{m}^{(0.5)}}. \tag{7.43}$$

Then $T_{\mathfrak{n}}T_\lambda T_{\mathfrak{H}} \sim T_{\mathfrak{m}}T_{\mathfrak{m}}T_{\mathfrak{n}}$ and $T_{\mathfrak{m}} \sim (T_{\mathfrak{m}})^*$ are both fulfilled.

Recall that we need the uniform bounds for $|\partial_t^3\psi|_{1.5}^2$, so we shall take 1.5-th order derivative in (7.41). Since the symbol $\mathfrak{m}$ also belongs to $\Sigma^{1.5}$, we alternatively consider the $T_{\mathfrak{m}}$-differentiated evolution equation thanks to the symmetrization result. We introduce the following energy functional

$$\mathfrak{M}(t) := \frac12\int_\Sigma \rho\left|T_{\mathfrak{m}}T_{\mathfrak{n}}\overline{D_t}\partial_t^2\psi\right|^2 + \sigma\left|T_{\mathfrak{m}}T_{\mathfrak{m}}T_{\mathfrak{n}}\partial_t^2\psi\right|^2 + \frac{c_0}{2}\left|\mathfrak{N}_\psi^{\frac12}T_{\mathfrak{m}}T_{\mathfrak{n}}\partial_t^2\psi\right|^2 \mathrm{d}x'. \tag{7.44}$$

In view of Proposition 7.7 and Lemma D.2, we have the comparison between $\mathfrak{M}(t)$ and standard Sobolev norms

$$\mathfrak{M}(t) \lesssim \left|\overline{D_t}\partial_t^2\psi\right|_{1.5}^2 + \sigma|\partial_t^2\psi|_3^2 + \frac{c_0}{4}|\partial_t^2\psi|_2^2; \tag{7.45}$$

$$\left|\overline{D_t}\partial_t^2\psi\right|_{1.5}^2 \lesssim \left|T_{\mathfrak{m}}T_{\mathfrak{n}}\overline{D_t}\partial_t^2\psi\right|_0^2 + \left|\overline{D_t}\partial_t^2\psi\right|_0^2, \tag{7.46}$$

$$\sigma|\partial_t^2\psi|_3^2 \lesssim \sigma\left|T_{\mathfrak{m}}T_{\mathfrak{m}}T_{\mathfrak{n}}\partial_t^2\psi\right|^2 + \sigma|\partial_t^2\psi|_0^2, \quad |\partial_t^2\psi|_2^2 \lesssim \left|\mathfrak{N}_\psi^{\frac12}T_{\mathfrak{m}}T_{\mathfrak{n}}\partial_t^2\psi\right|_0^2 + |\partial_t^2\psi|_0^2. \tag{7.47}$$

For those $L^2(\Sigma)$ norms, we invoke the kinematic boundary condition, trace lemma and Young's inequality to get

$$|\partial_t^2\psi|_0^2 \lesssim \|v_t\|_1^2|\overline{\nabla}\psi|_{L^\infty}^2 + \|v\|_1^2|\overline{\nabla}\partial_t\psi|_{L^\infty}^2$$

and

$$|\partial_t^3\psi|_0^2 \lesssim \varepsilon\|\partial_t^2 v\|_2^2 + \|\partial_t^2 v\|_0^2|\overline{\nabla}\psi|_{L^\infty}^4 + \|v_t\|_1^2|\overline{\nabla}\partial_t\psi|_{L^\infty}^2 + \|v\|_2^2|\overline{\nabla}\partial_t^2\psi|_0^2.$$

The ε-term contributes to $\varepsilon\mathfrak{E}_4(t)$. The term $\|\partial_t^2 v\|_0^2$ can be controlled via ∂_t^2-estimates of (1.18) in which there is no loss of λ-weight in the corresponding commutators of Alinhac good unknowns, as we take a full-time derivative ∂_t^2. The other terms contain at most one time derivatives and thus can be controlled directly. Thus, we have

$$\mathfrak{M}(t) \lesssim \mathfrak{E}_4(t) \text{ and } \left|\partial_t^3\psi\right|_{1.5}^2 + \sigma|\partial_t^2\psi|_3^2 + |\partial_t^2\psi|_2^2 \lesssim \mathfrak{M}(t) + \varepsilon\mathfrak{E}_4(t) + \text{controllable terms}. \tag{7.48}$$

So, it suffices to control $\mathfrak{M}(t)$ in order to establish the bound for $\left|\partial_t^3\psi\right|_{1.5}^2 + \sigma|\partial_t^2\psi|_3^2 + |\partial_t^2\psi|_2^2$ in $\mathfrak{E}_4(t)$.

Now we start to control $\mathfrak{M}(t)$. Taking the partial derivative in the first term, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Sigma \rho\left|T_{\mathfrak{m}}T_{\mathfrak{n}}\overline{D_t}\partial_t^2\psi\right|^2\,\mathrm{d}x' \\
=&\int_\Sigma T_{\mathfrak{m}}T_{\mathfrak{n}}(\rho\overline{D_t}^2\partial_t^2\psi)(T_{\mathfrak{m}}T_{\mathfrak{n}}\overline{D_t}\partial_t^2\psi)\,\mathrm{d}x' + \int_\Sigma[\rho-1, T_{\mathfrak{m}}T_{\mathfrak{n}}]\overline{D_t}^2\partial_t^2\psi\,(T_{\mathfrak{m}}T_{\mathfrak{n}}\overline{D_t}\partial_t^2\psi)\,\mathrm{d}x' \\
&+\frac{1}{2}\int_\Sigma(\partial_t\rho + \overline{\nabla}\cdot(\rho\overline{v}))\left|T_{\mathfrak{m}}T_{\mathfrak{n}}\overline{D_t}\partial_t^2\psi\right|^2\,\mathrm{d}x' + \int_\Sigma\rho[\overline{D_t}, T_{\mathfrak{m}}T_{\mathfrak{n}}](\overline{D_t}\partial_t^2\psi)\,(T_{\mathfrak{m}}T_{\mathfrak{n}}\overline{D_t}\partial_t^2\psi)\,\mathrm{d}x' \\
=:&\int_\Sigma T_{\mathfrak{m}}T_{\mathfrak{n}}(\rho\overline{D_t}^2\partial_t^2\psi)(T_{\mathfrak{m}}T_{\mathfrak{n}}\overline{D_t}\partial_t^2\psi)\,\mathrm{d}x' + R_1^M
\end{aligned} \tag{7.49}$$

Next, plugging the paralinearized equation (7.41) into the above equality, we obtain

$$\begin{aligned}
&\int_\Sigma T_{\mathfrak{m}}T_{\mathfrak{n}}(\rho\overline{D_t}^2\partial_t^2\psi)(T_{\mathfrak{m}}\overline{D_t}\partial_t^2\psi)\,\mathrm{d}x' \\
=&-\sigma\int_\Sigma T_{\mathfrak{m}}T_{\mathfrak{n}}T_\Lambda T_{\mathfrak{H}}(\partial_t^2\psi)(T_{\mathfrak{m}}T_{\mathfrak{n}}\overline{D_t}\partial_t^2\psi)\,\mathrm{d}x' - \int_\Sigma(-\partial_3 q)T_{\mathfrak{m}}T_{\mathfrak{n}}\mathfrak{N}_\psi^{\frac12}\mathfrak{N}_\psi^{\frac12}(\partial_t^2\psi)(T_{\mathfrak{m}}T_{\mathfrak{n}}\overline{D_t}\partial_t^2\psi)\,\mathrm{d}x' \\
&+\int_\Omega[T_{\mathfrak{m}}T_{\mathfrak{n}}, \partial_3 q]\mathfrak{N}_\psi(\partial_t^2\psi)\,(T_{\mathfrak{m}}T_{\mathfrak{n}}\overline{D_t}\partial_t^2\psi)\,\mathrm{d}x' \\
\overset{L}{=}&-\sigma\int_\Sigma T_{\mathfrak{m}}T_{\mathfrak{m}}T_{\mathfrak{n}}(\partial_t^2\psi)\,T_{\mathfrak{m}}(T_{\mathfrak{m}}T_{\mathfrak{n}}\overline{D_t}\partial_t^2\psi)\,\mathrm{d}x' - \int_\Sigma(-\partial_3 q)\mathfrak{N}_\psi^{\frac12}(T_{\mathfrak{m}}T_{\mathfrak{n}}\partial_t^2\psi)\,\mathfrak{N}_\psi^{\frac12}(T_{\mathfrak{m}}T_{\mathfrak{n}}\overline{D_t}\partial_t^2\psi)\,\mathrm{d}x' \\
&-\int_\Sigma T_{\mathfrak{m}}T_{\mathfrak{n}}(N\cdot\nabla^\varphi Q_w)\,(T_{\mathfrak{m}}T_{\mathfrak{n}}\overline{D_t}\partial_t^2\psi)\,\mathrm{d}x' + \int_\Omega[T_{\mathfrak{m}}T_{\mathfrak{n}}, \partial_3 q]\mathfrak{N}_\psi(\partial_t^2\psi)\,(T_{\mathfrak{m}}T_{\mathfrak{n}}\overline{D_t}\partial_t^2\psi)\,\mathrm{d}x' \\
&+\int_\Omega[T_{\mathfrak{m}}T_{\mathfrak{n}}, \partial_3 q]\mathfrak{N}_\psi(\partial_t^2\psi)\,(T_{\mathfrak{m}}T_{\mathfrak{n}}\overline{D_t}\partial_t^2\psi)\,\mathrm{d}x' - \int_\Omega(-\partial_3 q)[T_{\mathfrak{m}}T_{\mathfrak{n}}, \mathfrak{N}_\psi]\partial_t^2\psi\,(T_{\mathfrak{m}}T_{\mathfrak{n}}\overline{D_t}\partial_t^2\psi)\,\mathrm{d}x' \\
&-\int_\Sigma\mathfrak{N}_\psi^{\frac12}(T_{\mathfrak{m}}T_{\mathfrak{n}}\partial_t^2\psi)\,[\mathfrak{N}_\psi^{\frac12}, \partial_3 q](T_{\mathfrak{m}}T_{\mathfrak{n}}\overline{D_t}\partial_t^2\psi)\,\mathrm{d}x' \\
&+\int_\Sigma(\mathbf{R}_\psi^{ST} + \mathbf{R}_\psi^{RT} - N\cdot\mathbf{C}(\check{q}) + \rho(\mathbf{R}_\psi + \mathbf{R}_\rho + \mathbf{R}_{\psi,\rho}))\,(T_{\mathfrak{m}}T_{\mathfrak{n}}\overline{D_t}\partial_t^2\psi)\,\mathrm{d}x' \\
=:&\ M^{ST} + M^{RT} + M^W + R_2^M + R_3^M + R_4^M + R_5^M + R_6^M.
\end{aligned} \tag{7.50}$$

Here we use $T_{\mathfrak{m}}T_{\mathfrak{m}}T_{\mathfrak{n}} \sim T_{\mathfrak{n}}T_\Lambda T_{\mathfrak{H}}$ and $T_{\mathfrak{m}} \sim T_{\mathfrak{m}}^*$ to derive M^{ST} and omit the low-order error terms in this equivalence. We also use the self-adjointness of DtN operator in $L^2(\Sigma)$ to derive M^{RT}. Note that we may not use T_Λ to replace $\mathfrak{N}_\psi$ in the term involving the Rayleigh-Taylor sign, as we do not have $T_\Lambda \sim T_\Lambda^*$. The major terms are M^{ST}, M^{RT} and M^W, among which the first two terms contribute to the boundary regularity with or without σ-weight, while the term M^W contributes to a fifth-order term that motivates us to involve $E_5(t)$ in the energy functional $\mathfrak{E}(t)$. The control of $R_2^M \sim R_6^M$ and other commutators generated by M^{ST}, M^{RT} and M^W will be postponed at the end of this section.

The term M^{ST} contributes to $\sqrt{\sigma}$-weighted boundary regularity. We have

$$
\begin{aligned}
M^{ST} =& -\sigma\int_\Sigma T_{\mathfrak{m}}T_{\mathfrak{m}}T_{\mathfrak{n}}(\partial_t^2\psi)\,T_{\mathfrak{m}}(T_{\mathfrak{m}}T_{\mathfrak{n}}\overline{D_t}\partial_t^2\psi)\,\mathrm{d}x' \\
=& -\frac{\sigma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left|T_{\mathfrak{m}}T_{\mathfrak{m}}T_{\mathfrak{n}}\partial_t^2\psi\right|^2\,\mathrm{d}x' - \sigma\int_\Sigma T_{\mathfrak{m}}T_{\mathfrak{m}}T_{\mathfrak{n}}(\partial_t^2\psi)\,[T_{\mathfrak{m}}T_{\mathfrak{m}}T_{\mathfrak{n}},\overline{D_t}]\partial_t^2\psi\,\mathrm{d}x' - \frac{\sigma}{2}(\overline{\nabla}\cdot\overline{v})\left|T_{\mathfrak{m}}T_{\mathfrak{m}}T_{\mathfrak{n}}\partial_t^2\psi\right|^2\,\mathrm{d}x' \\
=:& -\frac{\sigma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left|T_{\mathfrak{m}}T_{\mathfrak{m}}T_{\mathfrak{n}}\partial_t^2\psi\right|^2\,\mathrm{d}x' + R_7^M.
\end{aligned}
\tag{7.51}
$$

The term M^{RT} contributes to non-weighted boundary regularity with the help of Rayleigh-Taylor sign condition $-\partial_3 q \geq c_0/2 > 0$. We have

$$
\begin{aligned}
M^{RT} =& -\int_\Sigma(-\partial_3 q)\mathfrak{R}_\psi^{\frac12}(T_{\mathfrak{m}}T_{\mathfrak{n}}\partial_t^2\psi)\,\mathfrak{R}_\psi^{\frac12}(T_{\mathfrak{m}}T_{\mathfrak{n}}\overline{D_t}\partial_t^2\psi)\,\mathrm{d}x' \\
=& -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(-\partial_3 q)\left|\mathfrak{R}_\psi^{\frac12}(T_{\mathfrak{m}}T_{\mathfrak{n}}\partial_t^2\psi)\right|^2\,\mathrm{d}x' \\
&+\int_\Sigma \partial_3 q\mathfrak{R}_\psi^{\frac12}(T_{\mathfrak{m}}T_{\mathfrak{n}}\partial_t^2\psi)\,[\mathfrak{R}_\psi^{\frac12}T_{\mathfrak{m}}T_{\mathfrak{n}},\overline{D_t}]\partial_t^2\psi\,\mathrm{d}x' - \frac12\int_\Omega \partial_t\partial_3 q + \overline{\nabla}\cdot(\partial_3 q\overline{v})\left|\mathfrak{R}_\psi^{\frac12}(T_{\mathfrak{m}}T_{\mathfrak{n}}\partial_t^2\psi)\right|^2\,\mathrm{d}x' \\
=:& -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(-\partial_3 q)\left|\mathfrak{R}_\psi^{\frac12}(T_{\mathfrak{m}}T_{\mathfrak{n}}\partial_t^2\psi)\right|^2\,\mathrm{d}x' + R_8^M.
\end{aligned}
\tag{7.52}
$$

Currently, we have arrived at the following energy inequality

$$
\mathfrak{M}(t) \leq \mathfrak{M}(0) + \int_0^t M^W(\tau)\,\mathrm{d}\tau + \sum_{j=1}^8\int_0^t R_j^M(\tau)\,\mathrm{d}\tau. \tag{7.53}
$$

7.5 Weighted fifth-order energy

7.5.1 Necessity of fifth-order energy

Recall that $M^W = -\int_\Sigma T_{\mathfrak{m}}T_{\mathfrak{n}}(N\cdot\nabla^\varphi Q_w)\,(T_{\mathfrak{m}}T_{\mathfrak{n}}\overline{D_t}\partial_t^2\psi)\,\mathrm{d}x'$ and $T_{\mathfrak{m}}T_{\mathfrak{n}}$ is a 1.5-th order paradifferential operator, so it remains to control $|N\cdot\nabla^\varphi Q_w|_{1.5}$ in order for the control of M^W. Using trace theorem, we have $|N\cdot\nabla^\varphi Q_w|_{1.5} \leq |\overline{\nabla}\psi|_{1.5}\|\nabla^\varphi Q_w\|_2$. Then, we use the following div-curl inequality

$$
\|\nabla^\varphi Q_w\|_2^2 \leq C(|\overline{\nabla}\psi|_{W^{1,\infty}}, |\psi|_{2.5})\left(\|\nabla^\varphi Q_w\|_0^2 + \|\Delta^\varphi Q_w\|_1^2 + \|\nabla^\varphi\times\nabla^\varphi Q_w\|_1^2 + |N\times\nabla^\varphi Q_w|_{1.5}^2 + |\partial_3 Q_w|_{H^{1.5}(\Sigma_b)}^2\right), \tag{7.54}
$$

where the third and the fourth terms are all zero because $\nabla^\varphi\times\nabla^\varphi f = \mathbf{0}$ and Q_w has zero boundary value on Σ. The fifth term is easy to control, we have

$$
|\partial_3 Q_w|_{H^{1.5}(\Sigma_b)}^2 = |\partial_t^2\rho g|_{H^{1.5}(\Sigma_b)}^2 \lesssim \lambda^2\|\partial_t^2 q\|_2^2. \tag{7.55}
$$

The first term is of lower-order and we omit the treatment. For the second term, invoking (7.23), we have

$$
\|\Delta^\varphi Q_w\|_1 \lesssim C\left(\sum_{k=2}^4 |\lambda^{(k-2)_+}\partial_t^k\psi|_{4-k}\right)\left(\sum_{k=2}^4 \|\lambda^2\partial_t^k\check{q}\|_{5-k} + \|\partial_t^2 v\|_2\|\partial v\|_2 + \lambda^2\|\partial q\|_2\|\partial_t^3 v\|_1 + P(\mathfrak{E}_4(t))\right), \tag{7.56}
$$

where the first term requires the control of $E_5(t)$. It should be noted that the λ^2 in the third term is generated from $\nabla^\varphi\rho$ such that the term $D_t^\varphi\mathcal{V}$ can be controlled without loss of λ-weight. The last two term on the right side of (7.23) can be directly controlled by $P(\mathfrak{E}_4(t))$, as the number of derivatives does not exceed 4 and the number of time derivatives does not exceed 2. Therefore, the energy inequality (7.57) becomes

$$
\mathfrak{M}(t) \leq \mathfrak{M}(0) + \int_0^t P(\mathfrak{E}_4(\tau))E_5(\tau)\,\mathrm{d}\tau + \sum_{j=1}^8\int_0^t R_j^M(\tau)\,\mathrm{d}\tau, \tag{7.57}
$$

and it remains to control

$$
E_5(t) := \sum_{k=0}^5 \left\|\lambda^2\partial_t^k(v,\lambda^{(k-4)_+}\check{q})\right\|_{5-k}^2 + \left|\sqrt{\sigma}\lambda^2\partial_t^k\psi\right|_{6-k}^2 + \left|\lambda^2\partial_t^k\psi\right|_{5-k}^2
$$

uniformly in λ,σ.

7.5.2 Control of $E_5(t)$ and the remaining terms in $\mathfrak{E}_4(t)$

Notice that $E_5(t)$ has exactly the same structure as the energy $E(t)$ used to prove the local existence in Theorem 1.1 if we remove the weight λ^2: all quantities except the top-order time derivative of $\check{q}$ share the same weight of Mach number. This indicates us to use the div-curl inequality in Lemma 4.2 and follow the same strategies as in Section 4 instead of using the one in Lemma 7.1 to establish the following energy inequality

$$E_5(t) \leq P(\mathfrak{E}(0)) + P(\mathfrak{E}(t)) \int_0^t P(\mathfrak{E}(\tau))\, \mathrm{d}\tau. \tag{7.58}$$

Also, notice that $\partial_t^3 v, \partial_t^3 \check{q}$ and $\partial_t^4 v, \partial_t^4 \check{q}$ in $\mathfrak{E}_4(t)$ also share the same weight of Mach number, so we can still control them by following the same strategies as in Section 4. What's different is that the high-order time derivatives in both $\mathfrak{E}_4(t)$ and $E_5(t)$ may need more weights in order for the uniform boundedness. Thus, it remains to carefully check if there is any loss of λ-weight in the control of commutators $\mathfrak{C}'(\check{q}), \mathfrak{C}_i'(v_i)$ and $\mathfrak{D}'(\check{q}), \mathfrak{D}'(v_i)$ in the tangential estimates. Let us recall the concrete forms of these commutators when $\mathcal{T}^\alpha$ has the form $\overline{\partial}^k \partial_t^l$.

$$\mathfrak{C}_i'(f) = \left[\mathcal{T}^\alpha, \frac{\mathbf{N}_i}{\partial_3\varphi}, \partial_3 f\right] + \partial_3 f \left[\mathcal{T}^\alpha, \mathbf{N}_i, \frac{1}{\partial_3\varphi}\right] + \mathbf{N}_i \partial_3 f \left[\mathcal{T}^{\alpha-\gamma}, \frac{1}{(\partial_3\varphi)^2}\right] \mathcal{T}^\gamma \partial_3\varphi \tag{7.59}$$

with $|\gamma| = 1$, and

$$\begin{aligned}\mathfrak{D}'(f) = {} & [\mathcal{T}^\alpha, \overline{v}] \cdot \overline{\partial} f + \left[\mathcal{T}^\alpha, \frac{1}{\partial_3\varphi}(v\cdot\mathbf{N} - \partial_t\varphi), \partial_3 f\right] + \left[\mathcal{T}^\alpha, v\cdot\mathbf{N} - \partial_t\varphi, \frac{1}{\partial_3\varphi}\right] \partial_3 f + \frac{1}{\partial_3\varphi}[\mathcal{T}^\alpha, v]\cdot \mathbf{N}\partial_3 f \\ & - (v\cdot\mathbf{N} - \partial_t\varphi)\partial_3 f \left[\mathcal{T}^{\alpha-\gamma}, \frac{1}{(\partial_3\varphi)^2}\right] \mathcal{T}^\gamma \partial_3\varphi. \end{aligned} \tag{7.60}$$

Here, we only check the most difficult cases and omit the other easier ones: $\mathfrak{C}(\check{q})$ and $\mathfrak{D}(\check{q})$ in $\lambda\partial_t^3\overline{\partial}$-estimates, $\lambda\partial_t^4$-estimates (for $\mathfrak{E}_4(t)$), $\lambda^2\partial_t^4\overline{\partial}$-estimates and $\lambda^2\partial_t^5$-estimates (for $E_5(t)$). Note that there is no need to check the same commutators for v_i because the power of λ weight that $\partial_t^k v_i$ needs never exceeds that for $\partial_t^k \check{q}$.

$\lambda\partial_t^3\overline{\partial}$-estimates for $\mathfrak{E}_4(t)$. We shall control $\|\lambda\mathfrak{C}_i(\check{q})\|_0$ uniformly in λ when $\mathcal{T}^\alpha = \partial_t^3\overline{\partial}$ and $f = \check{q}$. The worst case is that all time derivatives fall on $\check{q}$ and such terms have the following forms

$$\lambda\overline{\partial}(\mathbf{N}_i/\partial_3\varphi)\partial_t^3\partial_3\check{q}, \quad \lambda\overline{\partial}\left(\frac{1}{\partial_3\varphi}(v\cdot\mathbf{N} - \partial_t\varphi)\right)\partial_t^3\partial_3\check{q}$$

whose L^2 norms are bounded by $P(|\overline{\nabla}\psi, \partial_t\psi|_{W^{1,\infty}}, \|\overline{\partial}v\|_{L^\infty})\|\lambda\partial_t^3\check{q}\|_1$.

$\lambda\partial_t^4$-estimates for $\mathfrak{E}_4(t)$. We shall control $\|\lambda\mathfrak{C}_i(\check{q})\|_0$ uniformly in λ when $\mathcal{T}^\alpha = \partial_t^4$ and $f = \check{q}$. The worst case is that three out of the four time derivatives fall on $\check{q}$ and such terms have the following forms

$$\lambda\partial_t(\mathbf{N}_i/\partial_3\varphi)\partial_t^3\partial_3\check{q}, \quad \lambda\partial_t\left(\frac{1}{\partial_3\varphi}(v\cdot\mathbf{N} - \partial_t\varphi)\right)\partial_t^3\partial_3\check{q}$$

whose L^2 norms are bounded by $P(|\overline{\nabla}\psi, \partial_t\psi|_{W^{1,\infty}}, \|\overline{\partial}v\|_{L^\infty}, |\partial_t^2\psi|_{L^\infty})\|\lambda\partial_t^3\check{q}\|_1$.

$\lambda^2\partial_t^4\overline{\partial}$-estimates for $E_5(t)$. We shall control $\|\lambda^2\mathfrak{C}_i(\check{q})\|_0$ uniformly in λ when $\mathcal{T}^\alpha = \partial_t^4\overline{\partial}$ and $f = \check{q}$. Although every quantity in $E_5(t)$ needs λ^2-weight, the terms involving $\check{q}$ becomes a lower order term and contains at most one time derivative if there are 5 derivatives falling on $\mathbf{N}$ or $v\cdot\mathbf{N}$. Since $\partial_t\nabla\check{q}$ is uniformly bounded in $L^\infty(\Omega)$, there is no need to out extra effort on such terms. The worst case is still that all time derivatives fall on $\check{q}$ and such terms have the following forms

$$\lambda^2\overline{\partial}(\mathbf{N}_i/\partial_3\varphi)\partial_t^4\partial_3\check{q}, \quad \lambda^2\overline{\partial}\left(\frac{1}{\partial_3\varphi}(v\cdot\mathbf{N} - \partial_t\varphi)\right)\partial_t^4\partial_3\check{q}$$

whose L^2 norms are bounded by $P(|\overline{\nabla}\psi, \partial_t\psi|_{W^{1,\infty}}, \|\overline{\partial}v\|_{L^\infty})\|\lambda^2\partial_t^4\check{q}\|_1$. As for the intermediate terms, we check the case that ∂_t^3 falls on $\partial_3\check{q}$ and $\overline{\partial}\partial_t$ falls on $(v\cdot\mathbf{N} - \partial_t\varphi)$ because neither of these two terms can be uniformly bounded in L^∞. We have that

$$\begin{aligned} & \left\|\lambda^2\partial_t\overline{\partial}((\partial_3\varphi)^{-1}(v\cdot\mathbf{N} - \partial_t\varphi))\partial_t^3\partial_3\check{q}\right\|_0 \lesssim \|\lambda^2\partial_t^3\partial_3\check{q}\|_{L^6}\|\partial_t\overline{\partial}((\partial_3\varphi)^{-1}(v\cdot\mathbf{N} - \partial_t\varphi))\|_{L^3} \\ \lesssim {} & \|\lambda^2\partial_t^3\check{q}\|_2(|\psi_t|_{2.5} + |\psi_{tt}|_{1.5})\|v_t\|_3 P(|\overline{\nabla}\psi, \partial_t\psi|_{W^{1,\infty}}) \leq \sqrt{E_5(t)}P(\mathfrak{E}_4(t)). \end{aligned}$$

$\lambda^2\partial_t^5$-estimates for $E_5(t)$. We shall control $\|\lambda^2\mathfrak{C}_i(\check{q})\|_0$ uniformly in λ when $\mathcal{T}^\alpha = \partial_t^5$ and $f = \check{q}$. Again, the worst case is still that all time derivatives fall on $\check{q}$ and such terms have the following forms

$$\lambda^2\partial_t(\mathbf{N}_i/\partial_3\varphi)\partial_t^4\partial_3\check{q}, \quad \lambda^2\partial_t\left(\frac{1}{\partial_3\varphi}(v\cdot\mathbf{N}-\partial_t\varphi)\right)\partial_t^4\partial_3\check{q}$$

whose L^2 norms are bounded by $P(|\overline{\nabla}\psi, \partial_t\psi|_{W^{1,\infty}}, \|\overline{\partial}v\|_{L^\infty}, |\partial_t^2\psi|_{L^\infty})\|\lambda^2\partial_t^4\check{q}\|_1$. As for the intermediate terms, we check the case that ∂_t^3 falls on $\partial_3\check{q}$ and ∂_t^2 falls on $(v\cdot\mathbf{N}-\partial_t\varphi)$. We have that

$$\left\|\lambda^2\partial_t^2((\partial_3\varphi)^{-1}(v\cdot\mathbf{N}-\partial_t\varphi))\partial_t^3\partial_3\check{q}\right\|_0 \lesssim \|\lambda\partial_t^3\check{q}\|_0(|\psi_t|_{2.5}+|\psi_{tt}|_{2.5}+|\lambda\psi_{ttt}|_{1.5})\|v_t\|_3 P(|\overline{\nabla}\psi, \partial_t\psi|_{W^{1,\infty}}) \le P(\mathfrak{E}_4(t)).$$

The omitted terms can be controlled in a similar or easier manner. Thus, we conclude that inequality (7.58) holds true.

7.5.3 Uniform estimates for $\mathfrak{E}(t)$ and the incompressible limit

So far, we already obtain the following energy inequalities.

1. The terms involving less than 2 time derivatives in $\mathfrak{E}_4(t)$:

$$\sum_{k=0}^{1}\|\partial_t^k v\|_{4-k}^2 + \sigma|\partial_t^k\psi|_{5-k}^2 + |\psi|_4^2 + |\partial_t\psi|_{3.5}^2 + \|\check{q}\|_4^2 \lesssim P(\mathfrak{E}_4(0)) + \int_0^t P(\mathfrak{E}_4(\tau))\,\mathrm{d}\tau, \tag{7.61}$$

$$\|\partial_t\check{q}\|_3^2 \lesssim \|\partial_t^2 v\|_2^2 + P(\mathfrak{E}_4(0)) + \int_0^t P(\mathfrak{E}_4(\tau))\,\mathrm{d}\tau. \tag{7.62}$$

   These two inequalities are proved in the same way as in Section 4.
2. The terms involving 3 and 4 time derivatives in $\mathfrak{E}_4(t)$:

$$\sum_{k=3}^{4}\|\lambda\partial_t^k v, \lambda^{1+(k-3)_+}\partial_t^k\check{q}\|_{4-k}^2 + \sigma|\lambda\partial_t^k\psi|_{5-k}^2 + |\lambda\partial_t^3\psi|_{1.5}^2 + |\lambda\partial_t^4\psi|_{0.5}^2 \lesssim P(\mathfrak{E}_4(0)) + \int_0^t P(\mathfrak{E}_4(\tau))\,\mathrm{d}\tau, \tag{7.63}$$

   which is obtained by following the same strategy as in Section 4 and the analysis of commutator in Section 7.5.2.
3. Control of $E_5(t)$:

$$E_5(t) \le P(\mathfrak{E}(0)) + P(\mathfrak{E}(t))\int_0^t P(\mathfrak{E}(\tau))\,\mathrm{d}\tau. \tag{7.64}$$

4. Control of $\|v_{tt}\|_2^2$ in $\mathfrak{E}_4(t)$ via paradifferential calculus:

$$\|v_{tt}\|_2^2 \lesssim \varepsilon\mathfrak{E}_4(t) + \sum_{j=1}^{8}\int_0^t R_j^M(\tau)\,\mathrm{d}\tau + P(\mathfrak{E}_4(0)) + \int_0^t P(\mathfrak{E}_4(\tau))\,\mathrm{d}\tau, \tag{7.65}$$

$$\|\lambda\partial_t^2\check{q}\|_2^2 \lesssim \|\lambda\partial_t^3 v\|_1^2 + P(\mathfrak{E}_4(0)) + \int_0^t P(\mathfrak{E}_4(\tau))\,\mathrm{d}\tau. \tag{7.66}$$

Summing up the above estimates, we can prove the Grönwall-type inequality for the energy functional $\mathfrak{E}(t)$

$$\mathfrak{E}(t) \le P(\mathfrak{E}(0)) + P(\mathfrak{E}(t))\int_0^t P(\mathfrak{E}(\tau))\,\mathrm{d}\tau \quad \underline{\text{uniformly in } \lambda, \sigma}, \tag{7.67}$$

provided that we have the bounds for the remainders $|R_j^M(t)|_{L^2} \le P(\mathfrak{E}(t))$ for $1 \le j \le 8$. Since the first-order time derivatives in $\mathfrak{E}_4(t)$ still remain uniformly bounded, we can obtain the same convergence result as in Section 6.2, and we no longer repeat the statement here.

7.6 Control of commutators involving paradifferential operators

At the end of this paper, it remains to prove that $|R_j^M(t)|_{L^2} \leq P(\mathfrak{E}(t))$ for $1 \leq j \leq 8$. It should be noted that there are many time derivatives involved in these remainders, so the commutator estimates shown in [2] may not be directly applied. In particular, we only show the control of most difficult ones:

- Two commutators in R_1^M: $[T_{\mathfrak{m}}T_{\mathfrak{n}}, \rho-1]\overline{D_t}^2\partial_t^2\psi$ and $[T_{\mathfrak{m}}T_{\mathfrak{n}}, \overline{D_t}]\overline{D_t}\partial_t^2\psi$.
- The commutator in R_5^M: $[\mathfrak{N}_\psi^{\frac{1}{2}}, \partial_3 q]T_{\mathfrak{m}}T_{\mathfrak{n}}\partial_t^2\psi$.
- The concrete forms of $\mathbf{R}_\psi^{ST}$ and $\mathbf{R}_\psi^{RT}$.

The control of the other terms in these remainders will be omitted. In fact $R_2^M, R_3^M, R_7^M, R_8^M$ and can be controlled in the same way as R_1^M. The control of R_4^M is easier than that of R_3^M as the function contains less time derivatives. The other terms in R_6^M can be controlled in $L^2(\Sigma)$ uniformly in λ, σ by directly counting the number of derivatives.

We start with the first one. Writing $f = \overline{D_t}^2\partial_t^2\psi$ and $a = \rho - 1$ for convenience, we have

$$[T_{\mathfrak{m}}T_{\mathfrak{n}}, a]f = T_{\mathfrak{m}}([T_{\mathfrak{n}}, a]f) + [T_{\mathfrak{m}}, a](T_{\mathfrak{n}}f),$$

where the two terms share similar structures and we only show the control of the first one. Using Bony's paraproduct decomposition in Appendix D.1, we rewrite this commutator as

$$\begin{aligned}[T_{\mathfrak{n}}, a]f &= T_{\mathfrak{n}}T_a f + T_{\mathfrak{n}}T_f a + T_{\mathfrak{n}}(R(a,f)) - T_a T_{\mathfrak{n}} f - T_{T_{\mathfrak{n}}f}a - R(a, T_{\mathfrak{n}}f)\\ &= [T_{\mathfrak{n}}, T_a]f + T_{\mathfrak{n}}T_f a - T_{T_{\mathfrak{n}}f}a + T_{\mathfrak{n}}(R(a,f)) - R(a, T_{\mathfrak{n}}f)\end{aligned} \tag{7.68}$$

Here we must $a := \rho - 1$ instead of ρ because $\rho \gtrsim 1$ does not belong to L^2. This also avoids the loss of λ-weight in $f = \overline{D_t}^2\partial_t^2\psi$, as $\rho - 1 = O(\lambda^2)$. The last two terms on the right side of (7.68) are controlled by using Lemma D.1

$$|T_{\mathfrak{n}}(R(a,f))|_{1.5} \lesssim |R(a,f)|_{1.5} \lesssim |\rho-1|_{2.5}|f|_0 \lesssim |\lambda^2\overline{D_t}^2\partial_t^2\psi|_0, \tag{7.69}$$

$$|R(a, T_{\mathfrak{n}}f)|_{1.5} \lesssim |a|_{1.5}|T_{\mathfrak{n}}f|_{0.5} \lesssim |\lambda^2\overline{D_t}^2\partial_t^2\psi|_{0.5}. \tag{7.70}$$

Next, we control the commutator $[T_{\mathfrak{n}}, T_a]f$. Since $\mathfrak{n}, a$ are both function depending on $x' \in \mathbb{R}^2$, not a symbol depending on both x' and the frequency variable $\xi \in \mathbb{R}^2$, we have $a\#\mathfrak{n} = \mathfrak{n}\#a = a\mathfrak{n}$ and thus

$$\|T_aT_{\mathfrak{n}} - T_{\mathfrak{n}}T_a\|_{0.5\to1.5} \leq \|T_aT_{\mathfrak{n}} - T_{\mathfrak{n}a}\|_{0.5\to1.5} + \|T_{\mathfrak{n}}T_a - T_{\mathfrak{n}a}\|_{0.5\to1.5} \lesssim M_1^0(\mathfrak{n})M_1^0(a) \lesssim C(|\psi|_{C^2})|a|_{W^{1,\infty}},$$

which leads to

$$|[T_{\mathfrak{n}}, T_a]f|_{1.5} \lesssim C(|\psi|_{C^2})|a|_{W^{1,\infty}}|f|_{0.5} \lesssim C(|\psi|_{C^2})|\lambda^2\overline{D_t}^2\partial_t^2\psi|_{0.5}. \tag{7.71}$$

The other two terms in (7.68) are controlled in the same way and we only show the control of $T_{\mathfrak{n}}T_f a$. Since $\mathfrak{n} \in \Sigma^0$, it suffices to control $|T_f a|_{1.5}$. Using Plancherel's identity and the definition (7.27) of paradifferential operators, we have

$$|T_f a|_{1.5} = |\langle\xi\rangle^{1.5}\widehat{T_f a}(\xi)|_{L^2_\xi(\mathbb{R}^2)} = (2\pi)^{-2}\left|\int_{\mathbb{R}^2}\langle\xi\rangle^{1.5}\tilde{\chi}(\xi-\eta,\eta)\hat{f}(\xi-\eta)\phi(\eta)\hat{a}(\eta)\,\mathrm{d}\eta\right|_{L^2_\xi}. \tag{7.72}$$

By definition of $\tilde{\chi}$ and ϕ (see Appendix D.1), we know that the integrand is nonzero only if $|\eta| > 1$ and $|\xi - \eta| < \varepsilon_2|\eta|$ for some $0 < \varepsilon_2 \ll 1$, which means $\langle\xi\rangle$ and $\langle\eta\rangle$ are comparable: $(1-\varepsilon_2)|\eta| \leq |\xi| \leq (1+\varepsilon_2)|\eta|$. Then, using this, $|\tilde{\chi}| \leq 1, |\phi| \leq 1$ and Minkowski's inequality for integrals, we have

$$\begin{aligned}|T_f a|_{1.5} &\lesssim \left|\int_{\mathbb{R}^2}\tilde{\chi}(\xi-\eta,\eta)\hat{f}(\xi-\eta)\langle\eta\rangle^{1.5}\phi(\eta)\hat{a}(\eta)\,\mathrm{d}\eta\right|_{L^2_\xi}\\ (\forall 0<\delta<1) \qquad &\lesssim |\hat{f}|_{L^2(\mathbb{R}^2)}\left|\langle\eta\rangle^{-1-\delta}\langle\eta\rangle^{2.5+\delta}\hat{a}(\eta)\right|_{L^1_\eta}\\ &\lesssim |\hat{f}|_{L^2(\mathbb{R}^2)}|\langle\eta\rangle^{-1-\delta}|_{L^2_\eta(\mathbb{R}^2)}|\langle\eta\rangle^{2.5+\delta}\hat{a}(\eta)|_{L^2_\eta(\mathbb{R}^2)} \lesssim |f|_0|a|_3 \leq |\lambda^2\overline{D_t}^2\partial_t^2\psi|_0|q|_3.\end{aligned} \tag{7.73}$$

Next, we analyze the commutator $[T_{\mathfrak{m}}T_{\mathfrak{n}}, \overline{D_t}]f$ for $f = \overline{D_t}\partial_t^2\psi$. Since $\overline{D_t} = \partial_t + \overline{v}\cdot\overline{\nabla}$ and ∂_t is a time derivative, we only show the details for the control of $[T_{\mathfrak{m}}T_{\mathfrak{n}}, \partial_t]f$. Expanding this commutator, we have

$$[T_{\mathfrak{m}}T_{\mathfrak{n}}, \partial_t]f = T_{\mathfrak{m}}([T_{\mathfrak{n}}, \partial_t]f) + [T_{\mathfrak{m}}, \partial_t]T_{\mathfrak{n}}f.$$

Again, these two terms have similar structures, so we only focus on the first one, that is, the control of $|[T_{\mathfrak{n}}, \partial_t]f|_{1.5}$. We have that $[T_{\mathfrak{n}}, \partial_t]f = -T_{\partial_t \mathfrak{n}} f$, so using Lemma D.1, we have

$$|T_{\partial_t \mathfrak{n}} f|_{1.5} \lesssim |\partial_t \mathfrak{n}|_{L^\infty} |f|_{1.5} \lesssim C(|\overline{\nabla}\psi|_{L^\infty}) |\overline{\nabla}\psi_t|_{L^\infty} |\overline{D_t}\partial_t^2 \psi|_{1.5} \leq P(\mathfrak{E}_4(t)). \tag{7.74}$$

Next, we analyze the commutator $[\mathfrak{N}_\psi^{\frac{1}{2}}, a]f$ with $a := \partial_3 q$ and $f := T_{\mathfrak{m}} T_{\mathfrak{n}} \partial_t^2 \psi$. Using Lemma D.5, we have

$$\forall s > 3, \quad |[\mathfrak{N}_\psi^{\frac{1}{2}}, a]f|_0 \lesssim C(|\psi|_s)|a|_{1.5}|f|_{0.5} \lesssim C(|\psi|_s)\|q\|_3 |\partial_t^2 \psi|_2 \leq P(\mathfrak{E}_4(t)). \tag{7.75}$$

Also, the term $\mathbf{R}_\psi^{RT} := [\mathfrak{N}_\psi, \partial_3 q]\partial_t^2 \psi$ is controlled in the same way

$$\forall s > 3, \quad |[\mathfrak{N}_\psi, \partial_3 q]\partial_t^2 \psi|_0 \lesssim C(|\psi|_s)|\partial_3 q|_{1.5}|\partial_t^2 \psi|_1 \leq P(\mathfrak{E}_4(t)). \tag{7.76}$$

Finally, we need to establish the $L^2(\Sigma)$ of $\mathbf{R}_\psi^{ST} := \sigma \mathfrak{N}_\psi([\partial_t^2, T_{\mathfrak{H}}]\psi + \partial_t^2 R_{\mathfrak{H}}) + \sigma R_\Lambda^\psi (T_{\mathfrak{H}} \partial_t^2(\psi))$. The difficulty is that this term simultaneously contains the commutators between a paradifferential operator and ∂_t^2, the time derivatives of $R_{\mathfrak{H}}$ which is not explicitly calculated in previous works about incompressible fluids [3, 2, 63], and the control of remainders for the DtN operator. Among the three terms in $\mathbf{R}_\psi^{ST}$, the last one is directly controlled by using Lemma D.3 and Proposition 7.7

$$|\sigma R_\Lambda^\psi (T_{\mathfrak{H}} \partial_t^2(\psi))|_0 \lesssim \sigma |T_{\mathfrak{H}} \partial_t^2(\psi)|_0 \lesssim C(|\psi|_{W^{1,\infty}})|\psi_{tt}|_2 \leq P(\mathfrak{E}_4(t)). \tag{7.77}$$

Next, we control the first term in $\mathbf{R}_\psi^{ST}$. In view of Lemma D.2, it remains to control $|\sigma[\partial_t^2, T_{\mathfrak{H}}]\psi|_1$. Expanding the commutators, we have

$$[\partial_t^2, T_{\mathfrak{H}}]\psi = T_{\partial_t^2 \mathfrak{H}} \psi + 2T_{\partial_t \mathfrak{H}} \partial_t \psi.$$

We only analyze the first one as the symbol contains second-order time derivative and $\partial_t^2 \mathfrak{H} \notin C^2(\Sigma)$ and the second one is directly controlled with the help of Proposition 7.7. Again, using the definition (7.27), Plancherel's identity and Minkowski's inequality, we have

$$\begin{aligned} |T_{\partial_t^2 \mathfrak{H}} \psi|_1 &= \left| \langle \xi \rangle \int_{\mathbb{R}^2} \chi(\xi - \eta) \widehat{\partial_t^2 \mathfrak{H}}(\xi - \eta) \phi(\eta) \hat{\psi}(\eta) \, \mathrm{d}\eta \right|_{L_\xi^2(\mathbb{R}^2)} \\ (\forall 0 < \delta < 1) \qquad &\lesssim |\widehat{\partial_t^2 \mathfrak{H}}|_{L^2} |\langle \eta \rangle^{-1-\delta}|_{L_\eta^2(\mathbb{R}^2)} |\langle \eta \rangle^{2+\delta} \hat{\psi}(\eta)|_{L_\eta^2(\mathbb{R}^2)} \\ &\lesssim C(|\overline{\nabla}\psi, \overline{\nabla}\psi_t|_{L^\infty}) |\overline{\nabla}\psi_{tt}|_0 |\psi|_{2+\delta} \lesssim P(\mathfrak{E}_4(t)). \end{aligned} \tag{7.78}$$

The last step is to control $\mathfrak{N}_\psi(\partial_t^2 R_{\mathfrak{H}})$ and it suffices to control $|\partial_t^2 R_{\mathfrak{H}}|_1$. This step is actually a refinement of [2, Lemma 3.25]. Recall that the mean curvature is given by $\mathcal{H} = -\overline{\nabla} \cdot F(\overline{\nabla}\psi)$ with $F(x) := \frac{x}{\sqrt{1+x^2}}$ and $F(0) = 0$. We expand F into 2nd-order term to get

$$F(a) = 0 + F'(a)a + \frac{F''(\zeta)}{2}a^2 = T_{F'(a)}a + T_a(F'(a)) + R(a, F'(a)) + \frac{F''(\zeta)}{2}a^2.$$

Let $a = \overline{\nabla}\eta$ and then $T_{F'(a)}a$ is exactly the term $T_{\mathfrak{H}}\psi$ defined in Lemma 7.9 and

$$R_{\mathfrak{H}} = T_{\overline{\nabla}\psi} \left(\frac{Id}{\sqrt{1 + |\overline{\nabla}\psi|^2}} - \frac{\overline{\nabla}\psi \otimes \overline{\nabla}\psi}{(\sqrt{1 + |\overline{\nabla}\psi|^2})^3} \right) + R(\overline{\nabla}\psi, F'(\overline{\nabla}\psi)) + (\overline{\nabla}\psi)^\top \cdot \mathbf{M}(\overline{\nabla}\psi) \cdot (\overline{\nabla}\psi)$$

where $\mathbf{M}(\overline{\nabla}\psi)$ is a 2×2 matrix depending on $\overline{\nabla}\psi$. Thus, the leading-order part in the last two terms of $\partial_t^2 R_{\mathfrak{H}}$ must have the form $(\overline{\nabla}\psi_{tt} + \overline{\nabla}\psi_t \cdot \overline{\nabla}\psi_t)C'(\overline{\nabla}\psi)$ for some continuous function $C'(\cdot)$, while the first term is controlled by either using Lemma D.1 or following the same way as in the control of $[\partial_t^2, T_{\mathfrak{H}}]\psi$. We conclude the result as follows

$$\sigma |\partial_t^2 R_{\mathfrak{H}}|_0 \lesssim C(|\overline{\nabla}\psi, \overline{\nabla}\psi_t|_{L^\infty})(|\sigma \overline{\nabla}\psi_{tt}|_{L^2} + |\sqrt{\sigma}\overline{\nabla}\psi_t|_{L^\infty} |\sqrt{\sigma}\overline{\nabla}\psi_t|_0) \leq P(\mathfrak{E}_4(t)). \tag{7.79}$$

Now, we have finished all estimates and the proof of improved incompressible limit ends here.

Acknowledgment. We would like to thank Tao Wang and Kai Zhou for reading and commenting on the first draft of this paper.

Ethic Declarations.

Conflict of interest. The authors declare that they have no conflict of interest.

Data avaliability. Data sharing is not applicable as no datasets were generated or analyzed in this article.

A The Reynold transport theorems

Below, the formulas involving $\widetilde{\varphi}, \widetilde{\psi}$ are used for the nonlinear κ-problem (3.11) and the formulas involving $\mathring{\widetilde{\varphi}}, \mathring{\widetilde{\psi}}$ are used for the linearized κ-problem (5.6).

Lemma A.1. Let f, g be smooth functions defined on $[0,T]\times\Omega$. Then:

$$\frac{d}{dt}\int_\Omega fg\partial_3\widetilde{\varphi}\,dx = \int_\Omega (\partial_t^{\widetilde{\varphi}} f)g\partial_3\widetilde{\varphi}\,dx + \int_\Omega f(\partial_t^{\widetilde{\varphi}} g)\partial_3\widetilde{\varphi}\,dx + \int_{x_3=0} fg\partial_t\psi\,dx' + \int_\Omega fg\partial_3\partial_t(\widetilde{\varphi}-\varphi)\,dx, \tag{A.1}$$

$$\frac{d}{dt}\int_\Omega fg\partial_3\mathring{\widetilde{\varphi}}\,dx = \int_\Omega (\partial_t^{\mathring{\widetilde{\varphi}}} f)g\partial_3\mathring{\widetilde{\varphi}}\,dx + \int_\Omega f(\partial_t^{\mathring{\widetilde{\varphi}}} g)\partial_3\mathring{\widetilde{\varphi}}\,dx + \int_{x_3=0} fg\partial_t\mathring{\psi}\,dx' + \int_\Omega fg\partial_3\partial_t(\mathring{\widetilde{\varphi}}-\mathring{\varphi})\,dx. \tag{A.2}$$

Proof. In view of (3.12),

$$\begin{aligned}\text{LHS of (A.1)} &= \int_\Omega (\partial_t f)g\partial_3\widetilde{\varphi}\,\mathrm{d}x + \int_\Omega f(\partial_t g)\partial_3\widetilde{\varphi}\,\mathrm{d}x + \int_\Omega fg\partial_3\partial_t\widetilde{\varphi}\,\mathrm{d}x\\ &= \int_\Omega fg\partial_3\partial_t\widetilde{\varphi}\,\mathrm{d}x + \int_\Omega (\partial_t^{\widetilde{\varphi}} f)g\partial_3\widetilde{\varphi}\,\mathrm{d}x + \int_\Omega f(\partial_t^{\widetilde{\varphi}} g)\partial_3\widetilde{\varphi}\,\mathrm{d}x + \overbrace{\int_\Omega \partial_t\varphi(\partial_3 f)g\,\mathrm{d}x}^{i} + \overbrace{\int_\Omega \partial_t\varphi(\partial_3 g)f\,\mathrm{d}x}^{ii}.\end{aligned}$$

Integrating ∂_3 in ii by parts, we have

$$ii = \int_{x_3=0} fg\partial_t\psi\,dx' - \int_{x_3=-b} fg\underbrace{\partial_t\varphi}_{=\partial_t(-b)=0}\,\mathrm{d}x' - \int_\Omega fg\partial_3\partial_t\varphi\,\mathrm{d}x - i.$$

This concludes the proof of (A.1). Moreover, in light of (5.8),

$$\begin{aligned}\text{LHS of (A.2)} &= \int_\Omega (\partial_t f)g\partial_3\mathring{\widetilde{\varphi}}\,\mathrm{d}x + \int_\Omega f(\partial_t g)\partial_3\mathring{\widetilde{\varphi}}\,\mathrm{d}x + \int_\Omega fg\partial_3\partial_t\mathring{\widetilde{\varphi}}\,\mathrm{d}x\\ &= \int_\Omega fg\partial_3\partial_t\mathring{\widetilde{\varphi}}\,\mathrm{d}x + \int_\Omega (\partial_t^{\mathring{\widetilde{\varphi}}} f)g\partial_3\mathring{\widetilde{\varphi}}\,\mathrm{d}x + \int_\Omega f(\partial_t^{\mathring{\widetilde{\varphi}}} g)\partial_3\mathring{\widetilde{\varphi}}\,\mathrm{d}x + \overbrace{\int_\Omega \partial_t\mathring{\varphi}(\partial_3 f)g\,\mathrm{d}x}^{\mathring{i}} + \overbrace{\int_\Omega \partial_t\mathring{\varphi}(\partial_3 g)f\,\mathrm{d}x}^{\mathring{ii}}.\end{aligned}$$

Integrating ∂_3 in $\mathring{ii}$ by parts, we have

$$\mathring{ii} = \int_{x_3=0} fg\partial_t\mathring{\psi}\,dx' - \int_\Omega fg\partial_3\partial_t\mathring{\varphi}\,\mathrm{d}x - \mathring{i},$$

and thus (A.2) follows. □

Lemma A.2 (**Integration by parts for covariant derivatives**)**.** Let f, g be defined as in Lemma A.1. Then:

$$\int_\Omega (\partial_i^{\widetilde{\varphi}} f)g\partial_3\widetilde{\varphi}\,\mathrm{d}x = -\int_\Omega f(\partial_i^{\widetilde{\varphi}} g)\partial_3\widetilde{\varphi}\,\mathrm{d}x + \int_{x_3=0} fg\widetilde{N}_i\,dx', \tag{A.3}$$

$$\int_\Omega (\partial_i^{\mathring{\widetilde{\varphi}}} f)g\partial_3\mathring{\widetilde{\varphi}}\,\mathrm{d}x = -\int_\Omega f(\partial_i^{\mathring{\widetilde{\varphi}}} g)\partial_3\mathring{\widetilde{\varphi}}\,\mathrm{d}x + \int_{x_3=0} fg\mathring{\widetilde{N}}_i\,dx'. \tag{A.4}$$

Proof. (A.3) follows from the fact that $\partial_i^{\widetilde{\varphi}}$ is the covariant spatial derivative and $\partial_3\widetilde{\psi}\,\mathrm{d}x$ is the associated volume element. (A.4) follows from a parallel argument. □

Let $D_t^{\widetilde{\varphi}}$ be the smoothed material derivative defined in (3.15). Then the following theorem holds.

Theorem A.3 (**Reynold transport theorem for nonlinear κ-problem**). Let f be a smooth function defined on $[0,T]\times\Omega$. Then:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|f|^2\partial_3\widetilde{\varphi}\,\mathrm{d}x = \int_\Omega \rho(D_t^{\widetilde{\varphi}}f)f\partial_3\widetilde{\varphi}\,\mathrm{d}x + \frac{1}{2}\int_\Omega \rho|f|^2\partial_3\partial_t(\widetilde{\varphi}-\varphi)\,\mathrm{d}x. \tag{A.5}$$

Proof. First, we express

$$\int_\Omega \rho(D_t^{\widetilde{\varphi}}f)f\partial_3\widetilde{\varphi}\,\mathrm{d}x = \int_\Omega \rho(\partial_t^{\widetilde{\varphi}}f)f\partial_3\widetilde{\varphi}\,\mathrm{d}x + \int_\Omega \rho(v\cdot\nabla^{\widetilde{\varphi}}f)f\partial_3\widetilde{\varphi}\,\mathrm{d}x.$$

Invoking (A.1), we have

$$\int_\Omega \rho(\partial_t^{\widetilde{\varphi}}f)f\partial_3\widetilde{\varphi}\,\mathrm{d}x = \partial_t\int_\Omega \rho|f|^2\partial_3\widetilde{\varphi}\,\mathrm{d}x - \int_\Omega \partial_t^{\widetilde{\varphi}}(\rho f)f\partial_3\widetilde{\varphi}\,\mathrm{d}x - \int_{x_3=0}\rho|f|^2\partial_t\psi\,dx' - \int_\Omega \rho|f|^2\partial_3\partial_t(\widetilde{\varphi}-\varphi)\,\mathrm{d}x,$$

and this indicates that

$$\int_\Omega \rho(\partial_t^{\widetilde{\varphi}}f)f\partial_3\widetilde{\varphi}\,\mathrm{d}x = \frac{1}{2}\frac{d}{dt}\int_\Omega \rho|f|^2\partial_3\widetilde{\varphi}\,\mathrm{d}x \overbrace{-\frac{1}{2}\int_\Omega(\partial_t^{\widetilde{\varphi}}\rho)|f|^2\partial_3\widetilde{\varphi}\,\mathrm{d}x}^{A}\overbrace{-\frac{1}{2}\int_{x_3=0}\rho|f|^2\partial_t\psi\,dx'}^{C} - \frac{1}{2}\int_\Omega \rho|f|^2\partial_3\partial_t(\widetilde{\varphi}-\varphi)\,\mathrm{d}x. \tag{A.6}$$

Furthermore, invoking (A.3), we have

$$\begin{aligned}
&\int_\Omega \rho(v\cdot\nabla^{\widetilde{\varphi}}f)f\partial_3\widetilde{\varphi}\,\mathrm{d}x = \int_\Omega \nabla^{\widetilde{\varphi}}\cdot(\rho v f)f\partial_3\widetilde{\varphi}\,\mathrm{d}x - \int_\Omega \nabla^{\widetilde{\varphi}}\cdot(\rho v)|f|^2\partial_3\widetilde{\varphi}\,\mathrm{d}x\\
=& -\int_\Omega \rho f(v\cdot\nabla^{\widetilde{\varphi}}f)\partial_3\widetilde{\varphi}\,\mathrm{d}x + \int_{x_3=0}\rho|f|^2 v\cdot\widetilde{N}\,dx' - \int_\Omega \nabla^{\widetilde{\varphi}}\cdot(\rho v)|f|^2\partial_3\widetilde{\varphi}\,\mathrm{d}x,
\end{aligned}$$

and thus

$$\int_\Omega \rho(v\cdot\nabla^{\widetilde{\varphi}}f)f\partial_3\widetilde{\varphi}\,\mathrm{d}x = \overbrace{\frac{1}{2}\int_{x_3=0}\rho|f|^2 v\cdot\widetilde{N}\,dx'}^{D}\overbrace{-\frac{1}{2}\int_\Omega \nabla^{\widetilde{\varphi}}\cdot(\rho v)|f|^2\partial_3\widetilde{\varphi}\,\mathrm{d}x}^{B}. \tag{A.7}$$

We have $A+B=C+D=0$ thanks to the second and fifth equations of (3.11), respectively. Hence, (A.5) follows after adding (A.6) and (A.7) up. □

Theorem A.3 leads to the following two corollaries. The first one records the integration by parts formula for $D_t^{\widetilde{\varphi}}$.

Corollary A.4 (**Reynold transport theorem for nonlinear κ-problem**). It holds that

$$\frac{d}{dt}\int_\Omega fg\partial_3\widetilde{\varphi}\,\mathrm{d}x = \int_\Omega (D_t^{\widetilde{\varphi}}f)g\partial_3\widetilde{\varphi}\,\mathrm{d}x + \int_\Omega f(D_t^{\widetilde{\varphi}}g)\partial_3\widetilde{\varphi}\,\mathrm{d}x + \int_\Omega(\nabla^{\widetilde{\varphi}}\cdot v)fg\partial_3\widetilde{\varphi}\,\mathrm{d}x + \int_\Omega fg\partial_3\partial_t(\widetilde{\varphi}-\varphi)\,\mathrm{d}x. \tag{A.8}$$

Proof. Given (A.1), we have

$$\int_\Omega (\partial_t^{\widetilde{\varphi}}f)g\partial_3\widetilde{\varphi}\,\mathrm{d}x = \frac{d}{dt}\int_\Omega fg\partial_3\widetilde{\varphi}\,\mathrm{d}x - \int_\Omega f(\partial_t^{\widetilde{\varphi}}g)\partial_3\widetilde{\varphi}\,\mathrm{d}x - \int_{x_3=0} fg\partial_t\psi\,dx' - \int_\Omega fg\partial_3\partial_t(\widetilde{\varphi}-\varphi)\,\mathrm{d}x,$$

Also, (A.3) yields

$$\begin{aligned}
&\int_\Omega (v\cdot\nabla^{\widetilde{\varphi}}f)g\partial_3\widetilde{\varphi}\,\mathrm{d}x = \int_\Omega \nabla^{\widetilde{\varphi}}\cdot(vf)g\partial_3\widetilde{\varphi}\,\mathrm{d}x - \int_\Omega(\nabla^{\widetilde{\varphi}}\cdot v)fg\partial_3\widetilde{\varphi}\,\mathrm{d}x\\
=& -\int_\Omega f(v\cdot\nabla^{\widetilde{\varphi}}g)\partial_3\widetilde{\varphi}\,\mathrm{d}x + \int_{x_3=0} fg(v\cdot\widetilde{N})\,dx' - \int_\Omega(\nabla^{\widetilde{\varphi}}\cdot v)fg\partial_3\widetilde{\varphi}\,\mathrm{d}x.
\end{aligned}$$

Then we obtain (A.8) by adding these up. □

The second corollary concerns the transport theorem as well as the integration by parts formula for the linearized material derivative $D_t^{\mathring{\tilde{\varphi}}}$, defined in (5.7).

Corollary A.5 (**Reynold transport theorem for linearized κ-problem**). Let $D_t^{\mathring{\tilde{\varphi}}} := \partial_t + (\bar{\mathring{v}} \cdot \overline{\nabla}) + \frac{1}{\partial_3 \mathring{\tilde{\varphi}}}(\mathring{v} \cdot \dot{\tilde{\mathbf{N}}} - \partial_t \mathring{\varphi})\partial_3$ be the linearized material derivative defined in (5.7). Then:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \mathring{\rho}|f|^2 \partial_3\mathring{\tilde{\varphi}}\,\mathrm{d}x = \int_\Omega \mathring{\rho}(D_t^{\mathring{\tilde{\varphi}}} f) f \partial_3 \mathring{\tilde{\varphi}}\,\mathrm{d}x + \frac{1}{2}\int_\Omega \left(D_t^{\mathring{\tilde{\varphi}}}\mathring{\rho} + \mathring{\rho}\nabla^{\mathring{\tilde{\varphi}}}\cdot\mathring{v}\right)|f|^2\partial_3\mathring{\tilde{\varphi}}\,dx$$
$$+\frac{1}{2}\int_\Omega \mathring{\rho}|f|^2\left(\partial_3\partial_t(\dot{\tilde{\varphi}} - \mathring{\varphi}) + \partial_3(\partial_t + \bar{\mathring{v}}\cdot\overline{\nabla})(\mathring{\tilde{\varphi}} - \dot{\tilde{\varphi}})\right)\mathrm{d}x. \tag{A.9}$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |f|^2 \partial_3\mathring{\tilde{\varphi}}\,\mathrm{d}x = \int_\Omega (D_t^{\mathring{\tilde{\varphi}}} f) f \partial_3 \mathring{\tilde{\varphi}}\,\mathrm{d}x + \frac{1}{2}\int_\Omega \nabla^{\mathring{\tilde{\varphi}}}\cdot\mathring{v}|f|^2\partial_3\mathring{\tilde{\varphi}}\,\mathrm{d}x$$
$$+\frac{1}{2}\int_\Omega |f|^2\left(\partial_3\partial_t(\dot{\tilde{\varphi}} - \mathring{\varphi}) + \partial_3(\partial_t + \bar{\mathring{v}}\cdot\overline{\nabla})(\mathring{\tilde{\varphi}} - \dot{\tilde{\varphi}})\right)\mathrm{d}x. \tag{A.10}$$

Proof. It suffices to show (A.9) only since the proof of (A.10) follows by setting $\mathring{\rho} = 1$. We write the first term on the RHS of (A.9) as

$$\int_\Omega \mathring{\rho}(D_t^{\mathring{\tilde{\varphi}}} f) f\partial_3\mathring{\tilde{\varphi}}\,\mathrm{d}x = \int_\Omega \mathring{\rho}(\partial_t f) f \partial_3\mathring{\tilde{\varphi}}\,\mathrm{d}x + \int_\Omega \mathring{\rho}\left((\bar{\mathring{v}}\cdot\overline{\nabla})f\right) f\partial_3\mathring{\tilde{\varphi}}\,\mathrm{d}x + \int_\Omega \mathring{\rho}\left((\mathring{v}\cdot\dot{\tilde{\mathbf{N}}} - \partial_t\mathring{\varphi})\partial_3 f\right) f\,\mathrm{d}x, \tag{A.11}$$

and then integrate ∂_t, $\overline{\nabla}$ and ∂_3 by parts respectively in these terms to get:

$$\begin{aligned}\int_\Omega \mathring{\rho}(D_t^{\mathring{\tilde{\varphi}}} f) f\partial_3\mathring{\tilde{\varphi}}\,\mathrm{d}x =& \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega \mathring{\rho}|f|^2\partial_3\mathring{\tilde{\varphi}}\,\mathrm{d}x - \frac{1}{2}\int_\Omega\left(\partial_t\mathring{\rho} + \bar{\mathring{v}}\cdot\overline{\nabla}\mathring{\rho} + \frac{1}{\partial_3\mathring{\tilde{\varphi}}}(\mathring{v}\cdot\dot{\tilde{\mathbf{N}}} - \partial_t\mathring{\varphi})\partial_3\mathring{\rho}\right)|f|^2\partial_3\mathring{\tilde{\varphi}}\,\mathrm{d}x\\
&-\frac{1}{2}\int_\Omega \mathring{\rho}(\overline{\nabla}\cdot\bar{\mathring{v}})|f|^2\partial_3\mathring{\tilde{\varphi}}\,\mathrm{d}x - \frac{1}{2}\int_\Omega \mathring{\rho}|f|^2(\partial_t + \bar{\mathring{v}}\cdot\overline{\nabla})\partial_3\mathring{\tilde{\varphi}}\,\mathrm{d}x\\
&-\frac{1}{2}\int_\Omega \mathring{\rho}\partial_3(-(\bar{\mathring{v}}\cdot\overline{\nabla})\dot{\tilde{\varphi}} + \mathring{v}_3 - \partial_t\mathring{\varphi})|f|^2\,\mathrm{d}x,\end{aligned} \tag{A.12}$$

where we used $\mathring{v}\cdot\dot{\tilde{\mathbf{N}}} = -(\bar{\mathring{v}}\cdot\overline{\nabla})\dot{\tilde{\varphi}} + \mathring{v}_3$ in the last line. We find that the second integral in the first line is $\int_\Omega D_t^{\mathring{\tilde{\varphi}}}\mathring{\rho}|f|^2\partial_3\mathring{\tilde{\varphi}}\,\mathrm{d}x$. Also, the term in the last line can be written as

$$\begin{aligned}&-\frac{1}{2}\int_\Omega \mathring{\rho}\partial_3(-(\bar{\mathring{v}}\cdot\overline{\nabla})\dot{\tilde{\varphi}} + \mathring{v}_3 - \partial_t\mathring{\varphi})|f|^2\,\mathrm{d}x\\
=&-\frac{1}{2}\int_\Omega \mathring{\rho}|f|^2\left(\frac{1}{\partial_3\mathring{\tilde{\varphi}}}\partial_3 v_3 - \frac{\overline{\partial}_1\mathring{\tilde{\varphi}}}{\partial_3\mathring{\tilde{\varphi}}}\partial_3\mathring{v}_1 - \frac{\overline{\partial}_2\mathring{\tilde{\varphi}}}{\partial_3\mathring{\tilde{\varphi}}}\partial_3\mathring{v}_2\right)\partial_3\mathring{\tilde{\varphi}}\,\mathrm{d}x\\
&+\frac{1}{2}\int_\Omega \mathring{\rho}|f|^2\partial_3\bar{\mathring{v}}\cdot\overline{\nabla}(\dot{\tilde{\varphi}} - \mathring{\tilde{\varphi}})\,\mathrm{d}x + \frac{1}{2}\int_\Omega \mathring{\rho}|f|^2(\partial_t\partial_3\mathring{\varphi} + (\bar{\mathring{v}}\cdot\overline{\nabla})\partial_3\dot{\tilde{\varphi}})\,\mathrm{d}x.\end{aligned} \tag{A.13}$$

The first term on the RHS together with the third term in (A.12) contributes to

$$\frac{1}{2}\int_\Omega \mathring{\rho}(\nabla^{\mathring{\tilde{\varphi}}}\cdot\mathring{v})|f|^2\partial_3\mathring{\tilde{\varphi}}\,\mathrm{d}x$$

in (A.9). Meanwhile, the terms in the last line of (A.13) together with the fourth term in (A.12) give the terms in (A.9) with mismatches. □

B Construction of initial data for the original system

This section aims to construct the initial data for Theorem 1.2 and Theorem 1.3 satisfying the compatibility conditions

$$(D_t^{\varphi})^j q|_{\{t=0\}\times\Sigma} = (D_t^{\varphi})^j(\sigma\mathcal{H})|_{\{t=0\}\times\Sigma}, \quad \partial_t^j v^3|_{\{t=0\}\times\Sigma_b} = 0, \quad j = 0,1,2,3.$$

Since $D_t^{\varphi}|_{\Sigma} = \partial_t + \overline{v}\cdot\overline{\partial}$ and $\mathcal{H} = -\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}}\right)$, we rewrite the compatibility conditions in terms of $\check{q}$ as

$$(\partial_t + \overline{v}\cdot\overline{\partial})^j \check{q}|_{\{t=0\}\times\Sigma} = (\partial_t + \overline{v}\cdot\overline{\partial})^j \left(-\sigma\overline{\nabla}\cdot\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}} + g\psi\right)\Bigg|_{\{t=0\}\times\Sigma}, \quad j = 0,1,2,3. \tag{B.1}$$

Here, we use the modified pressure $\check{q}$ since we want $\partial\check{q}_0 \in L^2(\Omega)$ for the sake of convenience. Such compatibility conditions are required to show that $E(t)$ (defined as (1.33)), and $E^{\lambda,\sigma}(t)$ (defined as (1.39)) are bounded at $t = 0$ by adapting the arguments in [18, Section 4.3].

B.1 Formal construction

We shall adapt the method developed in [18] to construct smooth data $(\psi_0, v_0, \check{q}_0)$ that satisfies (B.1). We first describe the method formally which serves as a good guideline. The key difference, however, is that in [18] we constructed the initial data in Lagrangian coordinates, where (B.1) has a different formulation.

By identifying $\mathcal{F}_\lambda'(q) = \lambda^2$ without loss of generality, and since $\partial_1\varphi|_\Sigma = \partial_1\psi$, $\partial_2\varphi|_\Sigma = \partial_2\psi$, $\partial_3\varphi|_\Sigma = 1$, the momentum and continuity equations reduce respectively to

$$\rho(\partial_t + \overline{v}\cdot\overline{\partial})v + \nabla^{\varphi}\check{q} = -g(\rho-1)e_3, \ \text{ on } \Sigma \tag{B.2}$$

$$\lambda^2(\partial_t + \overline{v}\cdot\overline{\partial})\check{q} + \operatorname{div} v = \partial_1\psi\partial_3 v^1 + \partial_2\psi\partial_3 v^2 + \lambda^2 g v^3, \ \text{ on } \Sigma, \tag{B.3}$$

where $\nabla^{\varphi} q = (\partial_1 q - \partial_1\psi\partial_3 q, \partial_2 q - \partial_2\psi\partial_3 q, \partial_3 q)^\top$ and $\operatorname{div} v = \partial\cdot v$. By ignoring the terms contributed by the denominator, we have $\mathcal{H} \sim -\overline{\Delta}\psi$. Invoking the kinematic boundary condition $\partial_t\psi = v\cdot N$, we have

$$(\partial_t + \overline{v}\cdot\overline{\partial})\psi = v^3, \quad \text{on } \Sigma,$$

we obtain from the zeroth compatibility condition $\check{q} \sim -\sigma\overline{\Delta}\psi$ that

$$(\partial_t + \overline{v}\cdot\overline{\partial})\check{q} \sim -\sigma\overline{\Delta}v^3, \quad \text{on } \Sigma, \tag{B.4}$$

which is the first compatibility condition. Since the continuity equation (B.3) implies $\lambda^2(\partial_t + \overline{v}\cdot\overline{\partial})\check{q} \sim -\operatorname{div} v$, we can deduce from (B.4) that:

$$\operatorname{div} v \sim \sigma\lambda^2\overline{\Delta}v^3, \quad \text{on } \Sigma. \tag{B.5}$$

Furthermore, the momentum equation (B.2) implies $(\partial_t + \overline{v}\cdot\overline{\partial})v^3 \sim -\partial_3\check{q}$, and thus the second compatibility condition on $\check{q}$ becomes:

$$(\partial_t + \overline{v}\cdot\overline{\partial})^2\check{q} \sim -\sigma(\partial_t + \overline{v}\cdot\overline{\partial})\overline{\Delta}v^3 \sim \sigma\overline{\Delta}\partial_3\check{q}, \quad \text{on } \Sigma. \tag{B.6}$$

Taking $\partial_t + \overline{v}\cdot\overline{\partial}$ to the continuity equation to obtain $\lambda^2(\partial_t + \overline{v}\cdot\overline{\partial})^2\check{q} \sim -\operatorname{div}(\partial_t + \overline{v}\cdot\overline{\partial})v \sim \Delta\check{q}$, and this gives

$$\partial_3^2\check{q} \sim \sigma\lambda^2\overline{\Delta}\partial_3\check{q} - \overline{\Delta}\check{q}, \quad \text{on } \Sigma. \tag{B.7}$$

Finally, we derive from the third compatibility condition on $\check{q}$ that

$$(\partial_t + \overline{v}\cdot\overline{\partial})^3\check{q} \sim \sigma\overline{\Delta}\partial_3(\partial_t + \overline{v}\cdot\overline{\partial})\check{q} \sim \sigma\lambda^{-2}\overline{\Delta}\partial_3\operatorname{div} v, \quad \text{on } \Sigma, \tag{B.8}$$

together with the relation $\lambda^2(\partial_t + \overline{v}\cdot\overline{\partial})^3\check{q} \sim \Delta(\partial_t + \overline{v}\cdot\overline{\partial})\check{q} \sim \lambda^{-2}\Delta\operatorname{div} v$ obtained by taking $(\partial_t + \overline{v}\cdot\overline{\partial})^2$ to the continuity equation that

$$\Delta\operatorname{div} v \sim \sigma\lambda^2\overline{\Delta}\partial_3\operatorname{div} v, \quad \text{on } \Sigma. \tag{B.9}$$

In other words,

$$\partial_3^3 v \sim \sigma\lambda^2\overline{\Delta}\partial_3\operatorname{div} v - \Delta\partial_1 v - \Delta\partial_2 v - \overline{\Delta}\partial_3 v, \quad \text{on } \Sigma. \tag{B.10}$$

Therefore, the first order compatibility condition on $\check{q}$ yields an "identity in terms of v" (B.5), the second order compatibility condition on $\check{q}$ yields an "identity in terms of q" (B.7), and lastly, the third order compatibility condition on $\check{q}$ yields an "identity in terms of v" again (B.10).

We construct our data by the following iterative procedure. To begin with, let $(\xi_0, \mathbf{w}_0, p_0)$ be the generic smooth *localized* incompressible data that verifies the zeroth order compatibility condition $\check{p}_0 = -\sigma\overline{\nabla}\cdot\frac{\overline{\nabla}\xi_0}{\sqrt{1+|\overline{\nabla}\xi_0|^2}} + g\xi_0$ on Σ. In the first step, we fixed a smooth function ψ_0 which represents the moving interface, and constructed the data satisfying the first compatibility condition. Given (B.5), we shall need to construct the appropriate velocity vector field denoted by $\mathbf{u}_0 = (\mathbf{u}_0^1, \mathbf{u}_0^2, \mathbf{u}_0^3)$. We achieve this by setting $\mathbf{u}_0^1 = \mathbf{w}_0^1$, $\mathbf{u}_0^2 = \mathbf{w}_0^2$, and construct $\mathbf{u}_0^3$ by solving a poly-harmonic equation of order 2:

$$\begin{cases} \Delta^2\mathbf{u}_0^3 = \Delta^2\mathbf{w}_0^3, & \text{in } \Omega, \\ \mathbf{u}_0^3 = \mathbf{w}_0^3, \quad \partial_3\mathbf{u}_0^3 \sim -\partial_1\mathbf{w}_0^1 - \partial_2\mathbf{w}_0^2 + \sigma\lambda^2\overline{\Delta}\mathbf{w}_0^3, & \text{on } \Sigma, \\ \mathbf{u}_0^3 = \mathbf{w}_0^3, \quad \partial_3\mathbf{u}_0^3 = \partial_3\mathbf{w}_0^3 & \text{on } \Sigma_b. \end{cases} \tag{B.11}$$

In particular, the boundary condition $\partial_3\mathbf{u}_0^3 \sim -\partial_1\mathbf{w}_0^1 - \partial_2\mathbf{w}_0^2 + \sigma\lambda^2\overline{\Delta}\mathbf{w}_0^3$ is derived from (B.5).

In the second step, we construct the data verifying the second compatibility condition. We shall construct $\check{q}_0$ here because of (B.7). This is achieved by solving a poly-harmonic equation of order 3:

$$\begin{cases} \Delta^3\check{q}_0 = \Delta^3\check{p}_0, & \text{in } \Omega, \\ \check{q}_0 = \check{p}_0, \quad \partial_3\check{q}_0 = \partial_3\check{p}_0, & \text{on } \Sigma, \\ \partial_3^2\check{q}_0 \sim \sigma\lambda^2\overline{\Delta}\partial_3\check{p}_0 - \overline{\Delta}\check{p}_0, & \text{on } \Sigma, \\ \partial_3^j\check{q}_0 = 0 \ \ (0 \le j \le 2), & \text{on } \Sigma_b. \end{cases} \tag{B.12}$$

It can be seen that the boundary condition $\partial_3^2\check{q}_0 \sim \sigma\lambda^2\overline{\Delta}\partial_3\check{p}_0$ is a consequence of (B.7).

In the third (and final) step, we construct the data verifying the compatibility conditions up to order 3 with a fixed smooth function representing the moving interface still denoted by ψ_0. Since q_0 has been constructed, we need only to construct $v_0 = (v_0^1, v_0^2, v_0^3)$ by setting $\mathbf{w}_0^1 = v_0^1, \mathbf{w}_0^2 = v_0^2$, and solving the following order 4 poly-harmonic equation for v_0^3:

$$\begin{cases} \Delta^4 v_0^3 = \Delta^4\mathbf{u}_0^3, & \text{in } \Omega, \\ v_0^3 = \mathbf{u}_0^3, \quad \partial_3 v_0^3 \sim -\partial_1\mathbf{u}_0^1 - \partial_2\mathbf{u}_0^2 + \sigma\lambda^2\overline{\Delta}\mathbf{u}_0^3 & \text{on } \Sigma, \\ \partial_3^2 v_0^3 \sim -\partial_3\partial_1\mathbf{u}_0^1 - \partial_3\partial_2\mathbf{u}_0^2 + \sigma\lambda^2\overline{\Delta}\partial_3\mathbf{u}_0^3, & \text{on } \Sigma, \\ \partial_3^3 v_0^3 = -\Delta\partial_1\mathbf{u}_0^1 - \Delta\partial_2\mathbf{u}_0^2 + \sigma\lambda^2\overline{\Delta}\partial_3 \operatorname{div}\mathbf{u}_0 - \overline{\Delta}\partial_3\mathbf{u}_0^3, & \text{on } \Sigma, \\ \partial_3^j v_0^3 = \partial_3^j\mathbf{u}_0^3 \ \ (0 \le j \le 3) & \text{on } \Sigma_b. \end{cases} \tag{B.13}$$

The second and third boundary conditions arise from (B.5), whereas the fourth boundary condition is derived from (B.10).

B.2 The full construction procedure

We shall repeat the method introduced in Subsection B.1 with detailed boundary conditions generated by the compatibility conditions. We will use $\mathcal{P}, \mathcal{Q}$ to denote generic non-negative continuous functions. Apart from this, we will set

$$0 \le k' \le 1, \quad 0 \le k \le 2, \quad 0 \le l \le 3,$$

throughout.

By invoking the commutator

$$[\overline{\partial}^s, \partial_t + \overline{v}\cdot\overline{\partial}] = [\overline{\partial}^s, \overline{v}]\cdot\overline{\partial}, \tag{B.14}$$

and since it holds on Σ that

$$(\partial_t + \overline{v}\cdot\overline{\partial})\psi = v^3, \quad \check{q} = -\sigma\left(\frac{\overline{\Delta}\psi}{|N|} - \frac{\overline{\partial}\psi\cdot\overline{\partial}\overline{\nabla}\psi}{|N|^3}\right) + g\psi, \quad |N| = \sqrt{1+|\overline{\nabla}\psi|^2},$$

the first compatibility condition on $\check{q}$ reads:

$$(\partial_t + \overline{v}\cdot\overline{\partial})\check{q} = \sigma\mathcal{P}(\frac{1}{|N|}, \overline{\partial}^k\psi, \overline{\partial}^k\overline{v}, \overline{\partial}^k v^3), \quad \text{on } \Sigma. \tag{B.15}$$

In addition, the continuity equation (B.3) gives

$$\lambda^2(\partial_t + \overline{v} \cdot \overline{\partial})\check{q} = -\mathrm{div}\, v + \overline{\partial}\psi \cdot \partial_3 \overline{v} + \lambda^2 g v^3, \quad \text{on } \Sigma. \tag{B.16}$$

Hence, we combine (B.15) and (B.16) to get

$$\mathrm{div}\, v = \sigma\lambda^2 \mathcal{P}(|N|^{-1}, \overline{\partial}^k\psi, \overline{\partial}^k\overline{v}, \overline{\partial}^k v^3, \partial_3\overline{v}), \quad \text{on } \Sigma. \tag{B.17}$$

and the equation used to determine $\mathbf{u}_0^3$ is

$$\begin{cases} \Delta^2\mathbf{u}_0^3 = \Delta^2\mathbf{w}_0^3, & \text{in } \Omega, \\ \mathbf{u}_0^3 = \mathbf{w}_0^3, & \text{on } \Sigma\cup\Sigma_b, \\ \partial_3\mathbf{u}_0^3 = -\partial_1\mathbf{w}_0^1 - \partial_2\mathbf{w}_0^2 + \sigma\lambda^2\mathcal{P}(|N_0|^{-1}, \overline{\partial}^k\psi_0, \overline{\partial}^k\overline{\mathbf{w}}_0, \overline{\partial}^k\mathbf{w}_0^3, \partial_3\overline{\mathbf{w}}_0), & \text{on } \Sigma, \\ \partial_3\mathbf{u}_0^3 = \partial_3\mathbf{w}_0^3 & \text{on } \Sigma_b. \end{cases} \tag{B.18}$$

whose rough version is given by (B.11). Let $s_0 \geq 8$. The poly-harmonic estimate yields

$$\|\mathbf{u}_0^3 - \mathbf{w}_0^3\|_{s_0} \lesssim \underbrace{\|\Delta^2(\mathbf{u}_0^3 - \mathbf{w}_0^3)\|_{s_0-4}}_{=0} + \underbrace{|\mathbf{u}_0^3 - \mathbf{w}_0^3|_{s_0-0.5}}_{=0} + |\partial_3(\mathbf{u}_0^3 - \mathbf{w}_0^3)|_{s_0-1.5} \leq \lambda^2 C(|\psi_0|_s, \|\mathbf{w}_0\|_s), \tag{B.19}$$

for some $s > s_0$, and hence $\|\mathbf{u}_0^3 - \mathbf{w}_0^3\|_{s_0} \to 0$ as $\lambda \to 0$.

We construct $\check{q}_0$ using the second-order compatibility condition in the next stage. Owing to (B.2), the identities

$$\rho(\partial_t + \overline{v}\cdot\overline{\partial})\overline{v} + \overline{\partial}\check{q} = \overline{\partial}\psi\partial_3\check{q}, \quad \text{and } \rho(\partial_t + \overline{v}\cdot\overline{\partial})v_3 + \partial_3\check{q} = -g(\rho-1), \tag{B.20}$$

hold on Σ, and we view $\rho = \rho(\check{q})$ here and throughout. Taking $\partial_t + \overline{v}\cdot\overline{\partial}$ to (B.15) and invoking (B.14), we have

$$(\partial_t + \overline{v}\cdot\overline{\partial})^2\check{q} = \sigma\mathcal{P}(\rho^{-1}, |N|^{-1}, \overline{\partial}^l\psi, \overline{\partial}^k\overline{v}, \overline{\partial}^k v^3, \overline{\partial}^l\check{q}, \overline{\partial}^k\partial_3\check{q}), \quad \text{on } \Sigma. \tag{B.21}$$

Moreover, by taking $\partial_t + \overline{v}\cdot\overline{\partial}$ to the continuity equation (B.3), we get

$$\lambda^2(\partial_t + \overline{v}\cdot\overline{\partial})^2\check{q} = -\mathrm{div}\,(\partial_t + \overline{v}\cdot\overline{\partial})v + [\mathrm{div}\,, (\partial_t + \overline{v}\cdot\overline{\partial})]v + (\partial_t + \overline{v}\cdot\overline{\partial})(\overline{\partial}\psi\cdot\partial_3\overline{v} + \lambda^2 g v^3), \tag{B.22}$$

where $[\mathrm{div}\,, (\partial_t + \overline{v}\cdot\overline{\partial})]v = \partial_i\overline{v}\cdot\overline{\partial}v^i$,

$$-\mathrm{div}\,(\partial_t + \overline{v}\cdot\overline{\partial})\overline{v} = \partial^\tau(\rho^{-1}\partial_\tau\check{q}) - \partial^\tau(\rho^{-1}\partial_\tau\psi\partial_3\check{q}) + \underbrace{\partial_3(\rho^{-1}\partial_3\check{q})}_{=\rho^{-1}\partial_3^2\check{q} + \partial_3\rho^{-1}\partial_3\check{q}} + g\partial_3\Big(\rho^{-1}(\rho-1)\Big), \quad \tau = 1,2, \tag{B.23}$$

and

$$\begin{aligned} (\partial_t + \overline{v}\cdot\overline{\partial})(\overline{\partial}\psi\cdot\partial_3\overline{v} + \lambda^2 g v^3) = {} & \overline{\partial}v^3\cdot\partial_3\overline{v} + \overline{\partial}\psi\cdot\partial_3(-\rho^{-1}\overline{\partial}\check{q} + \rho^{-1}\overline{\partial}\psi\partial_3\check{q}) \\ & -\overline{\partial}\overline{v}\cdot\overline{\partial}\psi\cdot\partial_3\overline{v} - \overline{\partial}\psi\cdot\partial_3\overline{v}\cdot\partial_3\overline{v} + \lambda^2 g(-\rho^{-1}\partial_3\check{q} - g\rho^{-1}(\rho-1)). \end{aligned} \tag{B.24}$$

Since the third term on the RHS of (B.24) contributes to $\rho^{-1}|\overline{\partial}\psi|^2\partial_3^2\check{q}$, it holds that

$$\lambda^2(\partial_t + \overline{v}\cdot\overline{\partial})^2\check{q} = \rho^{-1}(1 + |\overline{\partial}\psi|^2)\partial_3^2\check{q} + \mathcal{Q}(\rho^{-1}, |N|^{-1}, \overline{\partial}^k\psi, \overline{\partial}^{k'}\partial_3 v, \overline{\partial}^{k'}\partial_3\check{q}), \quad \text{on } \Sigma. \tag{B.25}$$

Therefore, we combine (B.21) and (B.25) to get

$$\rho^{-1}(1 + |\overline{\partial}\psi|^2)\partial_3^2 q = \sigma\lambda^2\mathcal{P}(\rho^{-1}, |N|^{-1}, \overline{\partial}^l\psi, \overline{\partial}^k\overline{v}, \overline{\partial}^k v^3, \overline{\partial}^l q, \overline{\partial}^k\partial_3 q) + \mathcal{Q}(\rho^{-1}, |N|^{-1}, \overline{\partial}^k\psi, \overline{\partial}^{k'}\partial_3 v, \overline{\partial}^{k'}\partial_3 q), \quad \text{on } \Sigma, \tag{B.26}$$

and we set $\check{q}_0$ by solving

$$\begin{cases} \Delta^3\check{q}_0 = \Delta^3\check{p}_0, & \text{in } \Omega, \\ \check{q}_0 = \check{p}_0, \quad \partial_3\check{q}_0 = \partial_3\check{p}_0, & \text{on } \Sigma, \\ \partial_3^2\check{q}_0 = \rho_0(1 + |\overline{\partial}\psi_0|^2)^{-1}\Big(\sigma\lambda^2\mathcal{P}(\rho_0^{-1}, |N_0|^{-1}, \overline{\partial}^l\psi_0, \overline{\partial}^k\overline{\mathbf{u}}_0, \overline{\partial}^k\mathbf{u}_0^3, \overline{\partial}^l\check{p}_0, \overline{\partial}^k\partial_3\check{p}_0) & \\ +\mathcal{Q}(\rho_0^{-1}, |N_0|^{-1}, \overline{\partial}^k\psi_0, \overline{\partial}^{k'}\partial_3\mathbf{u}_0, \overline{\partial}^{k'}\partial_3\check{p}_0)\Big), & \text{on } \Sigma, \\ \partial_3^j\check{q}_0 = 0 \ \ (0 \leq j \leq 2) & \text{on } \Sigma_b. \end{cases} \tag{B.27}$$

whose rough version is (B.12). Also, the poly-harmonic estimate implies

$$\|\check{q}_0\|_{s_0} \lesssim \|\Delta^3 \check{p}_0\|_{s_0-6} + |\check{p}_0|_{s_0-0.5} + |\partial_3 \check{p}_0|_{s_0-1.5} + |\partial_3^2 \check{q}_0|_{s-2.5} \leq \lambda^2 C_1(|\psi_0|_s, \|\mathbf{u}_0\|_s, \|\check{p}_0\|_s) + C_2(|\psi_0|_s, \|\mathbf{u}_0\|_s, \|\check{p}_0\|_s), \tag{B.28}$$

for some $s > s_0$.

Finally, we construct v_0^3 using the third-order compatibility condition in the last stage. We obtain

$$(\partial_t + \overline{v} \cdot \overline{\partial})^3 \check{q} = \sigma \mathcal{P}(\rho^{-1}, |N|^{-1}, \overline{\partial}^l \psi, \overline{\partial}^l \overline{v}, \overline{\partial}^l v^3, \partial^l \check{q}) \left(\lambda^{-2} \overline{\partial}^4 \psi + \lambda^{-2} \overline{\partial}^4 v + \lambda^{-2} \overline{\partial}^l \partial_3 v + \lambda^{-2} \overline{\partial}^k \partial_3^2 v\right), \quad \text{on } \Sigma, \tag{B.29}$$

by taking $(\partial_t + \overline{v} \cdot \overline{\partial})$ to (B.21). Further, taking $(\partial_t + \overline{v} \cdot \overline{\partial})$ to (B.25) to get

$$\lambda^2 (\partial_t + \overline{v} \cdot \overline{\partial})^3 \check{q} = -\lambda^{-2} \rho^{-1} (1 + |\overline{\partial} \psi|^2) \partial_3^2 \operatorname{div} v + Q(\rho^{-1}, |N|^{-1}, \overline{\partial}^l \psi, \overline{\partial}^l v, \overline{\partial}^k \partial_3 v, \overline{\partial}^{k'} \partial_3^2 v, \overline{\partial}^l \check{q}), \quad \text{on } \Sigma. \tag{B.30}$$

Therefore, we combine (B.29) and (B.30) to obtain

$$\begin{aligned} \rho^{-1}(1 + |\overline{\partial}\psi|^2) \partial_3^2 \operatorname{div} v = \sigma \lambda^2 \mathcal{P}(\rho^{-1}, |N|^{-1}, \overline{\partial}^l \psi, \overline{\partial}^l \overline{v}, \overline{\partial}^l v^3, \partial^l \check{q}) \left(\overline{\partial}^4 \psi + \overline{\partial}^4 v + \overline{\partial}^l \partial_3 v + \overline{\partial}^k \partial_3^2 v\right) \\ + \lambda^2 Q(\rho^{-1}, |N|^{-1}, \overline{\partial}^l \psi, \overline{\partial}^l \psi, \overline{\partial}^l v, \overline{\partial}^k \partial_3 v, \overline{\partial}^{k'} \partial_3^2 v, \overline{\partial}^l \check{q}), \quad \text{on } \Sigma, \end{aligned} \tag{B.31}$$

and we set v_0^3 by solving

$$\begin{cases} \Delta^4 v_0^3 = \Delta^4 \mathbf{u}_0^3, & \text{in } \Omega, \\ v_0^3 = \mathbf{u}_0^3, & \text{on } \Sigma, \\ \partial_3 v_0^3 = -\partial_1 \mathbf{u}_0^1 - \partial_2 \mathbf{u}_0^2 + \sigma \lambda^2 \mathcal{P}(|N_0|^{-1}, \overline{\partial}^k \psi_0, \overline{\partial}^k \mathbf{u}_0, \overline{\partial}^k \mathbf{u}_0^3, \partial_3 \overline{\mathbf{u}}_0), \quad 0 \leq k \leq 2, & \text{on } \Sigma, \\ \partial_3^2 v_0^3 = -\partial_1 \partial_3 \mathbf{u}_0^1 - \partial_2 \partial_3 \mathbf{u}_0^2 + \sigma \lambda^2 \partial_3 \mathcal{P}(|N_0|^{-1}, \overline{\partial}^k \psi_0, \overline{\partial}^k \mathbf{u}_0, \overline{\partial}^k \mathbf{u}_0^3, \partial_3 \overline{\mathbf{u}}_0), & \text{on } \Sigma, \\ \partial_3^3 v_0^3 = \rho_0 (1 + |\overline{\partial} \psi_0|^2)^{-1} \Big(\sigma \lambda^2 \mathcal{P}(\rho_0^{-1}, |N_0|^{-1}, \overline{\partial}^l \psi_0, \overline{\partial}^l \overline{\mathbf{u}}_0, \overline{\partial}^l \mathbf{u}_0^3, \partial^l \check{q}_0) \left(\overline{\partial}^4 \psi_0 + \overline{\partial}^4 \mathbf{u}_0 + \overline{\partial}^l \partial_3 \mathbf{u}_0 + \overline{\partial}^k \partial_3^2 \mathbf{u}_0\right) & \\ + \lambda^2 Q(\rho_0^{-1}, |N_0|^{-1}, \overline{\partial}^l \psi_0, \overline{\partial}^l \psi, \overline{\partial}^l \mathbf{u}_0, \overline{\partial}^k \partial_3 \mathbf{u}_0, \overline{\partial}^{k'} \partial_3^2 \mathbf{u}_0, \overline{\partial}^l \check{q}_0) \Big) - \rho_0^{-1} (1 + |\overline{\partial} \psi_0|^2) \partial_3^2 (\partial_1 \mathbf{u}_0^1 + \partial_2 \mathbf{u}_0^2), & \text{on } \Sigma, \\ \partial_3^j v_0^3 = \partial_3^j \mathbf{u}_0^3 \ \ (0 \leq j \leq 3) & \text{on } \Sigma_b. \end{cases} \tag{B.32}$$

whose rough version is (B.13). By the poly-harmonic estimate, we have

$$\|v_0^3 - \mathbf{u}_0^3\|_{s_0} \lesssim \|\Delta^4 (v_0^3 - \mathbf{u}_0^3)\|_{s_0-8} + |v_0^3 - \mathbf{u}_0^3|_{s_0-0.5} + |\partial^3 (v_0^3 - \mathbf{u}_0^3)|_{s_0-1.5} + |\partial_3^2 (v_0^3 - \mathbf{u}_0^3)|_{s_0-2.5} + |\partial^3 (v_0^3 - \mathbf{u}_0^3)|_{s_0-3.5}. \tag{B.33}$$

The first two terms on the RHS are 0. Invoking (B.19), (B.28), we have, for some s, s' satisfying $s > s' > s_0$, that

$$|v_0^3 - \mathbf{u}_0^3|_{s_0-0.5} + |\partial^3 (v_0^3 - \mathbf{u}_0^3)|_{s_0-1.5} \leq \lambda^2 C(|\psi_0|_{s'}, \|\mathbf{u}_0\|_{s'}) \leq \lambda^2 C(|\psi_0|_s, \|\mathbf{w}_0\|_s),$$

and

$$|\partial_3^2 (v_0^3 - \mathbf{u}_0^3)|_{s_0-2.5} \leq \lambda^2 C(|\psi_0|_{s'}, \|\mathbf{u}_0\|_{s'}, \|\check{q}_0\|_{s'}) \leq \lambda^2 C(|\psi_0|_s, \|\mathbf{w}_0\|_s, \|\check{p}_0\|_s).$$

Thus,

$$\|v_0^3 - \mathbf{u}_0^3\|_{s_0} \leq \lambda^2 C(|\psi_0|_s, \|\mathbf{w}_0\|_s, \|\check{p}_0\|_s). \tag{B.34}$$

In particular, since we have set $\mathbf{w}_0^\tau = \mathbf{u}_0^\tau = v_0^\tau$, $\tau = 1, 2$, we deduce from (B.19) and (B.34) that

$$\|v_0 - \mathbf{w}_0\|_{s_0} \leq \|v_0^3 - \mathbf{u}_0^3\|_{s_0} + \|\mathbf{u}_0^3 - \mathbf{w}_0^3\|_{s_0} = O(\lambda^2). \tag{B.35}$$

In addition, we deduce from $\nabla^\varphi \cdot \mathbf{w}_0 = 0$ and (B.35) that

$$\|\nabla^\varphi \cdot v_0\|_{C^1} = O(\lambda^2). \tag{B.36}$$

Apart from these, it can be seen from (B.27) and (B.32) that $\|v_0\|_{s_0}$ and $\|\check{q}_0\|_{s_0}$ are uniform in both σ and λ. This allows us to take the zero surface tension and incompressible limits at the same time.

C Construction of initial data for the nonlinear κ-approximate system

The construction of smooth initial data for the κ-problem (3.11) is parallel to what has been done in the previous section and thus we shall only sketch the details. We will set

$$0 \le k' \le 1, \quad 0 \le k \le 2, \quad 0 \le l \le 3, \quad 0 \le m \le 4, \quad 0 \le n \le 5$$

in the sequel.

Let (ψ_0, v_0, q_0) be the smooth initial data constructed in the previous section. Our goal is to construct $(\psi_{\kappa,0}, v_{\kappa,0}, q_{\kappa,0})$ that satisfies the κ-compatibility conditions up to the third order:

$$(\partial_t + \overline{v} \cdot \overline{\partial})^j q|_{\{t=0\}\times\Sigma} = \sigma(\partial_t + \overline{v} \cdot \overline{\partial})^j \mathcal{H}|_{\{t=0\}\times\Sigma} + \kappa^2 (\partial_t + \overline{v} \cdot \overline{\partial})^j \Big((1-\overline{\Delta})(-\partial_1 \widetilde{\psi} v^1 - \partial_2 \widetilde{\psi} v^2 + v^3)\Big)|_{\{t=0\}\times\Omega}, \qquad j = 0,1,2,3, \tag{C.1}$$

$$\partial_t^j v^3|_{\{t=0\}\times\Sigma_b} = 0, \qquad j = 0,1,2,3. \tag{C.2}$$

Setting $\psi_{\kappa,0} = \psi_0$, we need only to compute the last term on the RHS to formulate the poly-harmonic equations for $q_{\kappa,0}$ and $v_{\kappa,0}$. Since

$$[(1-\overline{\Delta}), \partial_t + \overline{v} \cdot \overline{\partial}] = -[\overline{\Delta}, \overline{v}] \cdot \overline{\partial},$$

we have, when $j = 1$:

$$(\partial_t + \overline{v} \cdot \overline{\partial})\Big((1-\overline{\Delta})(-\overline{\partial}\widetilde{\psi} \cdot \overline{v} + v^3)\Big) = \mathcal{R}(\overline{\partial}^l \psi, \overline{\partial}^l v, \overline{\partial}^l \widetilde{\psi}, \overline{\partial}^l \widetilde{v}^3, \overline{\partial}^l \check{q}, \overline{\partial}^k \partial_3 \check{q}), \text{ on } \Sigma. \tag{C.3}$$

This implies that the equation used to determine $\mathbf{u}_{\kappa,0}^3$ is

$$\begin{cases} \Delta^2 \mathbf{u}_{\kappa,0}^3 = \Delta^2 v_0^3, & \text{in } \Omega, \\ \mathbf{u}_{\kappa,0}^3 = v_0^3, & \text{on } \Sigma, \\ \partial_3 \mathbf{u}_{\kappa,0}^3 = -\partial_1 v_0^1 - \partial_2 v_0^2 + \sigma\lambda^2 \mathcal{P}(|N_0|^{-1}, \overline{\partial}^k \psi_0, \overline{\partial}^k v_0, \overline{\partial}^k v_0^3, \partial_3 \overline{v}_0) & \\ +\kappa^2 \lambda^2 \mathcal{R}(\overline{\partial}^l \psi_0, \overline{\partial}^l v_0, \overline{\partial}^l \widetilde{\psi}_0, \overline{\partial}^l \widetilde{v}_0^3, \overline{\partial}^l \check{q}_0, \overline{\partial}^k \partial_3 \check{q}_0), & \text{on } \Sigma, \\ \partial_3^j v_{\kappa,0}^3 = \partial_3^j \mathbf{u}_0^3 \ \ (0 \le j \le 1) & \text{on } \Sigma_b. \end{cases} \tag{C.4}$$

which is parallel to (B.18).

Then, when $j = 2$, we have

$$\begin{aligned} &(\partial_t + \overline{v} \cdot \overline{\partial})^2\Big((1-\overline{\Delta})(-\overline{\partial}\widetilde{\psi} \cdot \overline{v} + v^3)\Big) = (\partial_t + \overline{v} \cdot \overline{\partial})\mathcal{R}(\overline{\partial}^l \psi, \overline{\partial}^l v, \overline{\partial}^l \widetilde{\psi}, \overline{\partial}^l \widetilde{v}^3, \overline{\partial}^l \check{q}, \overline{\partial}^k \partial_3 \check{q}) \\ &= \mathcal{R}(\overline{\partial}^l \psi, \overline{\partial}^l \widetilde{\psi}, \overline{\partial}^l v^3, \overline{\partial}^l \widetilde{v}^3, \overline{\partial}^m \check{q}, \overline{\partial}^l \partial_3 \check{q}, \lambda^{-2}\overline{\partial}^4 v, \lambda^{-2}\overline{\partial}^l \partial_3 v, \lambda^{-2}\overline{\partial}^k \partial_3^2 v, \lambda^{-2}\overline{\partial}^4 \psi), \quad \text{on } \Sigma, \end{aligned} \tag{C.5}$$

where the power of λ^{-1} does not exceed 2. Thus, we determine $q_{\kappa,0}$ by solving

$$\begin{cases} \Delta^3 \check{q}_{\kappa,0} = \Delta^3 \check{q}_0, & \text{in } \Omega, \\ \check{q}_{\kappa,0} = \check{q}_0, \quad \partial_3 \check{q}_{\kappa,0} = \partial_3 \check{q}_0, & \text{on } \Sigma, \\ \partial_3^2 \check{q}_{\kappa,0} = \rho_0 (1 + |\overline{\partial}\psi_0|^2)^{-1} \Big(\sigma\lambda^2 \mathcal{P}(\rho_0^{-1}, |N_0|^{-1}, \overline{\partial}^l \psi_0, \overline{\partial}^k \overline{\mathbf{u}}_{\kappa,0}, \overline{\partial}^k \mathbf{u}_{\kappa,0}^3, \overline{\partial}^l \check{q}_0, \overline{\partial}^k \partial_3 \check{q}_0) & \\ +\mathcal{Q}(\rho_0^{-1}, |N_0|^{-1}, \overline{\partial}^k \psi_0, \overline{\partial}^{k'} \partial_3 \mathbf{u}_{\kappa,0}, \overline{\partial}^{k'} \partial_3 \check{q}_0) & \\ +\kappa^2 \lambda^2 \mathcal{R}(\overline{\partial}^l \psi_0, \overline{\partial}^l \widetilde{\psi}_0, \overline{\partial}^l \mathbf{u}_{\kappa,0}^3, \overline{\partial}^l \widetilde{\mathbf{u}}_{\kappa,0}^3, \overline{\partial}^m \check{q}_0, \overline{\partial}^l \partial_3 \check{q}_0, \overline{\partial}^4 \mathbf{u}_{\kappa,0}, \overline{\partial}^l \partial_3 \mathbf{u}_{\kappa,0}, \overline{\partial}^k \partial_3^2 \mathbf{u}_{\kappa,0}, \overline{\partial}^4 \psi_0)\Big), & \text{on } \Sigma, \\ \partial_3^j \check{q}_0^3 = 0 \ \ (0 \le j \le 2) & \text{on } \Sigma_b. \end{cases} \tag{C.6}$$

Finally, when $j = 3$, we have

$$(\partial_t + \overline{v} \cdot \overline{\partial})^3\Big((1-\overline{\Delta})(-\overline{\partial}\widetilde{\psi} \cdot \overline{v} + v^3)\Big) = \mathcal{R}(\overline{\partial}^m \psi, \overline{\partial}^m \widetilde{\psi}, \overline{\partial}^m v^3, \overline{\partial}^m \widetilde{v}^3, \overline{\partial}^n \check{q}, \overline{\partial}^m \partial_3 \check{q}, \lambda^{-2}\overline{\partial}^5 v, \lambda^{-2}\overline{\partial}^m \partial_3 v, \lambda^{-2}\overline{\partial}^l \partial_3^2 v, \lambda^{-2}\overline{\partial}^5 \psi), \quad \text{on } \Sigma, \tag{C.7}$$

where the power of λ^{-1} does not exceed 4. Therefore, we construct $v^3_{\kappa,0}$ by solving

$$\begin{cases} \Delta^4 v^3_{\kappa,0} = \Delta^4 \mathbf{u}^3_{\kappa,0}, & \text{in } \Omega, \\ v^3_{\kappa,0} = \mathbf{u}^3_{\kappa,0}, & \text{on } \Sigma, \\ \partial_3 v^3_{\kappa,0} = -\partial_1 \mathbf{u}^1_{\kappa,0} - \partial_2 \mathbf{u}^2_{\kappa,0} + \sigma\lambda^2 \mathcal{P}(|N_0|^{-1}, \overline{\partial}^k \psi_0, \overline{\partial}^k \mathbf{u}_{\kappa,0}, \overline{\partial}^k \mathbf{u}^3_{\kappa,0}, \partial_3 \overline{\mathbf{u}}_{\kappa,0}) & \\ +\kappa^2\lambda^2 \mathcal{R}(\overline{\partial}^l \psi_0, \overline{\partial}^l \mathbf{u}_{\kappa,0}, \overline{\partial}^l \widetilde{\psi}_0, \overline{\partial}^l \widetilde{\mathbf{u}}^3_{\kappa,0}, \overline{\partial}^l \check{q}_{\kappa,0}, \overline{\partial}^k \partial_3 \check{q}_{\kappa,0}), & \text{on } \Sigma, \\ \partial_3^2 v^3_{\kappa,0} = -\partial_1\partial_3 \mathbf{u}^1_{\kappa,0} - \partial_2\partial_3 \mathbf{u}^2_{\kappa,0} + \sigma\lambda^2 \partial_3 \mathcal{P}(|N_0|^{-1}, \overline{\partial}^k \psi_0, \overline{\partial}^k \mathbf{u}_{\kappa,0}, \overline{\partial}^k \mathbf{u}^3_{\kappa,0}, \partial_3 \overline{\mathbf{u}}_{\kappa,0}) & \\ +\kappa^2\lambda^2 \partial_3 \mathcal{R}(\overline{\partial}^l \psi_0, \overline{\partial}^l \mathbf{u}_{\kappa,0}, \overline{\partial}^l \widetilde{\psi}_0, \overline{\partial}^l \widetilde{\mathbf{u}}^3_{\kappa,0}, \overline{\partial}^l \check{q}_{\kappa,0}, \overline{\partial}^k \partial_3 \check{q}_{\kappa,0}), & \text{on } \Sigma, \\ \partial_3^3 v^3_{\kappa,0} = \rho_0 (1 + |\overline{\partial}\psi_0|^2)^{-1} \Big(\sigma\lambda^2 \mathcal{P}(\rho_0^{-1}, |N_0|^{-1}, \overline{\partial}^l \psi_0, \overline{\partial}^l \overline{\mathbf{u}}_{\kappa,0}, \overline{\partial}^l \mathbf{u}^3_{\kappa,0}, \partial^l \check{q}_0) \left(\overline{\partial}^4 \psi_0 + \overline{\partial}^4 \mathbf{u}_{\kappa,0} + \overline{\partial}^3 \partial_3 \mathbf{u}_{\kappa,0} + \overline{\partial}^2 \partial_3^2 \mathbf{u}_{\kappa,0} \right) & \\ +\lambda^2 \mathcal{Q}(\rho_0^{-1}, |N_0|^{-1}, \overline{\partial}^l \psi_0, \overline{\partial}^l \mathbf{u}_0, \overline{\partial}^k \partial_3 \mathbf{u}_0, \overline{\partial}^{k'} \partial_3^2 \mathbf{u}_0, \overline{\partial}^l \check{q}_{\kappa,0}) & \\ +\mathcal{R}(\overline{\partial}^m \psi_0, \overline{\partial}^m \widetilde{\psi}_0, \overline{\partial}^m \mathbf{u}^3_{\kappa,0}, \overline{\partial}^m \widetilde{\mathbf{u}}^3_{\kappa,0}, \overline{\partial}^n \check{q}_{\kappa,0}, \overline{\partial}^m \partial_3 \check{q}_{\kappa,0}, \overline{\partial}^5 \mathbf{u}_{\kappa,0}, \overline{\partial}^m \partial_3 \mathbf{u}_{\kappa,0}, \overline{\partial}^l \partial_3^2 \mathbf{u}_{\kappa,0}, \overline{\partial}^5 \psi_0) \Big) & \\ -\rho_0^{-1} (1 + |\overline{\partial}\psi_0|^2) \partial_3^2 (\partial_1 \mathbf{u}^1_{\kappa,0} + \partial_2 \mathbf{u}^2_{\kappa,0}), & \text{on } \Sigma, \\ \partial_3^j v^3_{\kappa,0} = \partial_3^j \mathbf{u}^3_0 \ \ (0 \le j \le 3) & \text{on } \Sigma_b. \end{cases} \tag{C.8}$$

Let $\lambda > 0$ be fixed. Invoking the poly-harmonic estimate subsequently to (C.4), (C.6), and (C.8), we obtain that $\|v_{\kappa,0}\|_{s_0}$ and $\|\check{q}_{\kappa,0}\|_{s_0}$ are bounded for some $s_0 \ge 8$. Thus, the energy $E^\kappa(t)$ (defined as (4.1)) is bounded at $t = 0$. In addition,

$$\|v_{\kappa,0} - v_0\|_{s_0}, \text{ and } \|\check{q}_{\kappa,0} - \check{q}\|_{s_0} \to 0, \quad \text{as } \kappa \to 0.$$

D Paraproducts and the Dirichlet-to-Neumann operator

D.1 Bony's paraproduct decomposition

We already introduce the paradifferential operator in Section 7.2. Here we present the relations between paradifferential operators and paraproducts. The cutoff function $\tilde{\chi}(\xi, \eta)$ in the definition of $T_a u$ is

$$\tilde{\chi}(\xi, \eta) = \sum_{k=0}^{\infty} \Theta_{k-3}(\xi) \vartheta(\eta),$$

where $\Theta(\xi) = 1$ when $|\xi| \le 1$ and $\Theta(\xi) = 0$ when $|\xi| \ge 2$ and

$$\Theta_k(\xi) := \Theta(\frac{\xi}{2}), \ \ k \in \mathbb{Z}, \quad \vartheta_0 = \Theta, \quad \vartheta_k := \Theta_k - \Theta_{k-1}, \ \ k \ge 1.$$

Based on this, we can introduce the Littlewood-Paley projections $\mathbf{P}_k$ and $\mathbf{P}_{\le k}$ as follows

$$\widehat{\mathbf{P}_k u}(\xi) := \vartheta_k(\xi) \hat{u}(\xi), \ \ \forall k \ge 0, \quad \mathbf{P}_k u := 0 \ \ \forall k < 0, \quad \mathbf{P}_{\le k} u := \sum_{l \le k} \mathbf{P}_l u.$$

When the symbol $a(x, \xi)$ (in the paradifferential operator T_a) does not depend on ξ, we can take $\psi(\eta) \equiv 1$ and then we have

$$T_a u = \sum_k \mathbf{P}_{\le k-3} a (\mathbf{P}_k u)$$

which is the usual Bony's paraproduct. In general, the well-known Bony's paraproduct decomposition is

$$au = T_a u + T_u a + R(u, a), \quad R(u, a) = \sum_{|k-l| \le 2} (\mathbf{P}_k a)(\mathbf{P}_l u).$$

We have the following estimates for the remainder $R(u, a)$

Lemma D.1 ([2, Section 2.3])**.** For $s \in \mathbb{R}, \ r < d/2, \ \delta > 0$, we have

$$|T_a u|_{H^s} \lesssim \min\{|a|_{L^\infty} |u|_{H^s}, |a|_{H^r} |u|_{H^{s+\frac{d}{2}-r}}, |a|_{H^{\frac{d}{2}}} |u|_{H^{s+\delta}}\}$$

and for any $s > 0, s_1, s_2 \in \mathbb{R}$ satisfying $s_1 + s_2 = s + \frac{d}{2}$, we have

$$|R(u, a)|_{H^s} \lesssim |a|_{H^{s_1}} |u|_{H^{s_2}}.$$

D.2 Basic properties of the Dirichlet-to-Neumann operator

Let the space dimension $d = 3$ for simplicity. Given a function $f : \Sigma = \mathbb{T}^2 \to \mathbb{R}$, we define the Dirichlet-to-Neumann (DtN) operator (with respect to ψ and region $\Omega^\pm$) by

$$\mathfrak{N}_\psi f := \mp N \cdot \nabla^\varphi(\mathcal{E}^\pm_\psi f)|_\Sigma, \quad -\Delta^\varphi(\mathcal{E}^\pm_\psi f) = 0 \text{ in } \Omega^\pm, \;\; \mathcal{E}^\pm_\psi f|_\Sigma = f, \;\; \partial_3(\mathcal{E}^\pm_\psi f)|_{\Sigma^\pm} = 0.$$

Here the Laplacian operator is defined by $\Delta^\varphi := \nabla^\varphi \cdot \nabla^\varphi = \partial_i(\boldsymbol{E}^{ij}\partial_j)$ with

$$\boldsymbol{E} = \frac{1}{\partial_3\varphi}\begin{bmatrix} \partial_3\varphi & 0 & -\overline{\partial}_1\varphi \\ 0 & \partial_3\varphi & -\overline{\partial}_2\varphi \\ -\overline{\partial}_1\varphi & -\overline{\partial}_2\varphi & \frac{1+|\overline{\nabla}\varphi|^2}{\partial_3\varphi} \end{bmatrix} = \frac{1}{\partial_3\varphi}\boldsymbol{P}\boldsymbol{P}^\top, \;\; \boldsymbol{P} := \begin{bmatrix} \partial_3\varphi & 0 & 0 \\ 0 & \partial_3\varphi & 0 \\ -\overline{\partial}_1\varphi & -\overline{\partial}_2\varphi & 1 \end{bmatrix},$$

and $\varphi(t,x) := x_3 + \chi(x_3)\psi(t,x')$ is defined as the extension of ψ into $\Omega^\pm$. The choice of $\chi(x_3)$ is slightly different from [2, 3, 5], but it does not introduce any substantial difference because the expression of Δ^φ is still written to be $\Delta^\varphi := \nabla^\varphi \cdot \nabla^\varphi = \partial_i(\boldsymbol{E}^{ij}\partial_j)$ and we have $\Delta^\varphi\varphi = 0$ in $\Omega^\pm$. The DtN operators satisfy the following estimates and we refer to [63, Appendix A.4] for the proof.

Lemma D.2 (Sobolev estimates for DtN operators)**.** For $s > 2 + \frac{d}{2}$, $-\frac{1}{2} \le r \le s-1$ and $\psi \in H^s(\mathbb{R}^d)$, we have

$$|\mathfrak{N}_\psi f|_r \le C(|\psi|_s)|f|_{r+1}.$$

Lemma D.3 (Remainder estimates for DtN operators)**.** For $s > 2 + \frac{d}{2}$ and $\psi \in H^s(\mathbb{R}^d)$, we have

$$\mathfrak{N}_\psi f = T_\Lambda f + R^\psi_\Lambda(f)$$

with Λ defined in Proposition 7.8. The remainder $R^\psi_\Lambda(f)$ satisfies

$$|R^\psi_\Lambda(f)|_r \le C(|\psi|_{s+\frac{1}{2}})|f|_r.$$

Lemma D.4 (Sobolev estimates for the inverse of the DtN operator)**.** For $s > 2 + \frac{d}{2}$, $-\frac{1}{2} \le r \le s-1$ and $\psi \in H^s(\mathbb{R}^d)$, we have

$$|(\mathfrak{N}_\psi)^{-1} f|_{r+1} \le C(|\psi|_s)|f|_r.$$

Lemma D.5 (Commutator estimate for the DtN operator and its square root)**.** For $s > 2 + \frac{d+1}{2}$ and $\psi \in H^s(\mathbb{R}^d)$, we have

$$|[\mathfrak{N}_\psi, a]f|_{r-1} \le C(|\psi|_s)|a|_{r+1}|f|_r \quad \forall 0 < r \le s - \frac{1}{2},$$

and

$$|[(\mathfrak{N}_\psi)^{\frac{1}{2}}, a]f|_{r-\frac{1}{2}} \le C(|\psi|_s)|a|_{r+1}|f|_r \quad \forall -\frac{1}{2} < r \le s-1.$$

References

[1] T. Alazard. Incompressible limit of the nonisentropic Euler equations with the solid wall boundary conditions. *Adv. Differ. Equ.*, 10(1):19–44, 2005.

[2] T. Alazard, N. Burq, C. Zuily. On the water-wave equations with surface tension. *Duke Math. J.*, 158(3):413–499, 2011.

[3] T. Alazard, N. Burq, C. Zuily. On the Cauchy problem for gravity water waves. *Invent. Math.*, 198(1):71–163, 2014.

[4] T. Alazard, J.-M. Delort. Global solutions and asymptotic behavior for two-dimensional gravity water waves. *Ann. Sci. Éc. Norm. Supér.(4)*, 48(5):1149–1238, 2015.

[5] T. Alazard, G. Métivier, Paralinearization of the Dirichlet to Neumann operator, and regularity of three-dimensional water waves. *Commun. Partial Differ. Equ.*, 34(12), 1632-1704, 2009.

[6] D. Ambrose, N. Masmoudi. The zero surface tension limit of two-dimensional water waves. *Commun. Pure Appl. Math.*, 58(10), 1287-1315, 2005.

[7] S. Alinhac. Existence d'ondes de rarefaction pour des systems quasi-lineaires hyperboliques multidimensionnels. *Commun. Partial differ. Equ.*, 14(2):173–230, 1989.

[8] K. Asano. On the incompressible limit of the compressible Euler equation. *Japan J. Appl. Math.*, 4(3):455–488, 1987.

[9] L. Bieri, S. Miao, S. Shahshahani, S. Wu. On the motion of a self-gravitating incompressible fluid with free boundary. *Commun. Math. Phys.*, 355(1):161–243, 2017.

[10] C.-H. A. Cheng, S. Shkoller. Solvability and regularity for an elliptic system prescribing the curl, divergence, and a partial trace of a vector field on Sobolev-class domains. *J. Math. Fluid Mech.*, 19(3):375–422, 2017.

[11] D. Christodoulou, H. Lindblad. On the motion of the free surface of a liquid. *Commun. Pure. Appl. Math.*, 53(12):1536–1602, 2000.

[12] D. Coutand, J. Hole, S. Shkoller. Well-posedness of the free-boundary compressible 3-D Euler equations with surface tension and the zero surface tension limit. *SIAM J. Math. Anal.*, 45(6):3690–3767, 2013.

[13] D. Coutand, H. Lindblad, S. Shkoller. A priori estimates for the free-boundary 3D compressible Euler equations in a physical vacuum. *Commun. Math. Phys.*, 296(2):559–587, 2010.

[14] D. Coutand, S. Shkoller. Well-posedness of the free-surface incompressible Euler equations with or without surface tension. *J. Amer. Math. Soc.*, 20(3):829–930, 2007.

[15] D. Coutand, S. Shkoller. Well-posedness in smooth function spaces for the moving boundary three-dimensional compressible Euler equations in the physical vacuum. *Arch. Rational Mech. Anal.*, 206(2):515–616, 2012.

[16] Y. Deng, A. D. Ionescu, B. Pausader, F. Pusateri. Global solutions of the gravity-capillary water-wave system in three dimensions. *Acta Math.*, 219(2):213–402, 2017.

[17] M. M. Disconzi, D. G. Ebin. Motion of slightly compressible fluids in a bounded domain. II. *Commun. Contemp. Math.*, 19(04):1650054, 2017.

[18] M. M. Disconzi, C. Luo. On the incompressible limit for the compressible free-boundary Euler equations with surface tension in the case of a liquid. *Arch. Rational Mech. Anal.*, 237(2), 829-897, 2020.

[19] D. G. Ebin. Motion of slightly compressible fluids in a bounded domain. I. *Commun. Pure. Appl. Math.*, 35(4):451–485, 1982.

[20] P. Germain, N. Masmoudi, J. Shatah. Global solutions for the gravity water waves equation in dimension 3. *Ann. Math.*, 175(2), 691–754, 2012.

[21] P. Germain, N. Masmoudi, J. Shatah. Global existence for capillary water waves. *Commun. Pure. Appl. Math.*, 68(4):625–687, 2015.

[22] D. Ginsberg, H. Lindblad, C. Luo. Local well-posedness for the motion of a compressible, self-gravitating liquid with free surface boundary. *Arch. Rational Mech. Anal.*, 236(2):603–733, 2020.

[23] B. Harrop-Griffiths, M.Ifrim, D.Tataru. Finite depth gravity water waves in holomorphic coordinates. *Ann. PDE*, 3(1):4, 2017.

[24] J. K. Hunter, M. Ifrim, D. Tataru. Two-dimensional water waves in holomorphic coordinates. *Commun. Math. Phys.*, 346(2):483–552, 2016.

[25] M. Ifrim, D. Tataru. Two-dimensional water waves in holomorphic coordinates II: global solutions. *Bulletin de la Société mathématique de France*, 144(2):366–394, 2016.

[26] M. Ifrim, D. Tataru. The lifespan of small data solutions in two-dimensional capillary water waves. *Arch. Rational Mech. Anal.*, 225(3):1279–1346, 2017.

[27] M. Ifrim, D. Tataru. Two-dimensional gravity water waves with constant vorticity: I. cubic lifespan. *Anal. & PDE*, 12(4):903–967, 2018.

[28] M. Ifrim, D.Tataru. The compressible Euler equations in a physical vacuum: a comprehensive Eulerian approach. *arXiv preprint arXiv:2007.05668*, 2020.

[29] T. Iguchi. The incompressible limit and the initial layer of the compressible Euler equation in $\mathbb{R}^n_+$. *Math. Methods Appl. Sci.*, 20(11):945–958, 1997.

[30] T. Iguchi. Well-posedness of the initial value problem for capillary-gravity waves. *Funkcial. Ekvac.*, 44(2):219–242, 2001.

[31] A. Ionescu, F. Pusateri. Global solutions for the gravity water waves system in 2D. *Invent. Math.*, 199(3):653–804, 2015.

[32] H. Isozaki. Singular limits for the compressible Euler equations in an exterior domain *J. Reine Angew. Math.*, 381:1-36, 1987.

[33] J. Jang, N.Masmoudi. Well-posedness for compressible Euler equations with physical vacuum singularity. *Commun. Pure. Appl. Math.*, 62(10):1327–1385, 2009.

[34] J. Jang, N. Masmoudi. Well-posedness of compressible Euler equations in a physical vacuum. *Commun. Pure. Appl. Math.*, 68(1):61–111, 2015.

[35] S. Klainerman, A. Majda. Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids. *Commun. Pure. Appl. Math.*, 34(4):481–524, 1981.

[36] S. Klainerman, A. Majda. Compressible and incompressible fluids. *Commun. Pure. Appl. Math.*, 35(5):629–651, 1982.

[37] I. Kukavica, A. Tuffaha, V. Vicol. On the local existence and uniqueness for the 3D Euler equation with a free interface. *Appl. Math. & Optim.*, 76(3):535–563, 2017.

[38] J. Jang, I. Tice, Y. Wang. The Compressible Viscous Surface-Internal Wave Problem: Stability and Vanishing Surface Tension Limit. *Commun. Math. Phys.*, 343(3): 1039-1113, 2016.

[39] D. Lannes. Well-posedness of the water-waves equations. *J. Amer. Math. Soc.*, 18(3):605–654, 2005.

[40] P. D. Lax, R. S. Phillips. Local boundary conditions for dissipative symmetric linear differential operators. *Commun. Pure. Appl. Math.*, 13(3):427–455, 1960.

[41] H. Lindblad. Well-posedness for the linearized motion of an incompressible liquid with free surface boundary. *Commun. Pure. Appl. Math.*, 56(02):153–197, 2002.

[42] H. Lindblad. Well-posedness for the linearized motion of a compressible liquid with free surface boundary. *Commun. Math. Phys.*, 236(2):281–310, 2003.

[43] H. Lindblad. Well-posedness for the motion of a compressible liquid with free surface boundary. *Commun. Math. Phys.*, 260(2):319–392, 2005.

[44] H. Lindblad. Well-posedness for the motion of an incompressible liquid with free surface boundary. *Ann. Math.*, 162(1), 109–194, 2005.

[45] H. Lindblad, C. Luo. A priori estimates for the compressible Euler equations for a liquid with free surface boundary and the incompressible limit. *Commun. Pure. Appl. Math.*, 2018.

[46] H. Lindblad, K.H. Nordgren. A priori estimates for the motion of a self-gravitating incompressible liquid with free surface boundary. *J. Hyperbolic Differ. Equ.*, 6(02):407–432, 2009.

[47] C. Luo. On the motion of a compressible gravity water wave with vorticity. *Ann. PDE*, 4(2):1–71, 2018.

[48] C. Luo, J. Zhang. Local well-posedness for the motion of a compressible gravity water wave with vorticity. *J. Differ. Equ.*, 332:333–403, 2022.

[49] T. Luo, Z. Xin, H. Zeng. Well-posedness for the motion of physical vacuum of the three-dimensional compressible Euler equations with or without self-gravitation. *Arch. Rational Mech. Anal.*, 213(3):763–831, 2014.

[50] N. Masmoud, F. Roussét. Uniform regularity and vanishing viscosity limit for the free surface Navier-Sstokes equations. *Arch. Rational Mech. Anal*, 223(1):301–417, 2017.

[51] G. Métivier. *Small viscosity and boundary layer methods: Theory, stability analysis, and applications.* Springer Science & Business Media, 2004.

[52] G. Métivier *Para-differential calculus and applications to the Cauchy problem for nonlinear systems.* Edizioni della Normale, Pisa, (5) (2008).

[53] G. Métivier, S. Schochet. The incompressible limit of the non-isentropic Euler equations. *Arch. Rational Mech. Anal.*, 158(1):61–90, 2001.

[54] M. Ming, Z. Zhang. Well-posedness of the water-wave problem with surface tension. *J. Math. Pures Appl.*, 92(5), 429-455, 2009.

[55] V. Nalimov. The Cauchy-Poisson problem. *Dinamika Splošn. Sredy,(Vyp. 18 Dinamika Zidkost. so Svobod. Granicami)*, 254:104–210, 1974.

[56] J. Rauch. Symmetric Positive Systems with Boundary Characteristic of Constant Multiplicity *Trans. Amer. Math. Soc.*, 291(1), 167-187, 1985.

[57] S. Schochet. The compressible Euler equations in a bounded domain: Existence of solutions and the incompressible limit. *Commun. Math. Phys.*, 104(1):49–75, 1986.

[58] J. Shatah, C. Zeng. Geometry and a priori estimates for free boundary problems of the Euler's equation. *Commun. Pure. Appl. Math.*, 61(5):698–744, 2008.

[59] J. Shatah, C. Zeng. A priori estimates for fluid interface problems. *Commun. Pure. Appl. Math.*, 61(6):848–876, 2008.

[60] J. Shatah, C. Zeng. Local well-posedness for fluid interface problems. *Arch. Rational Mech. Anal.*, 199(2):653–705, 2011.

[61] B. Stevens Short-time structural stability of compressible vortex sheets with surface tension. *Arch. Rational Mech. Anal.*, 222(2), 603-730, 2016.

[62] Q. Su. Long time behavior of 2D water waves with point vortices. *Commun. Math. Phys.*, 380(3):1173–1266, 2020.

[63] Y. Sun, W. Wang, Z. Zhang. Nonlinear Stability of the Current-Vortex Sheet to the Incompressible MHD Equations. *Commun. Pure Appl. Math.*, 71(2), 356-403, 2018.

[64] Y. Trakhinin. Local existence for the free boundary problem for nonrelativistic and relativistic compressible Euler equations with a vacuum boundary condition. *Commun. Pure. Appl. Math.*, 62(11):1551–1594, 2009.

[65] Y. Trakhinin, T. Wang. Well-posedness of free boundary problem in non-relativistic and relativistic ideal compressible magnetohydrodynamics. *Arch. Rational Mech. Anal.*, 239(2):1131–1176, 2021.

[66] Y. Trakhinin, T. Wang. Well-posedness for the free-boundary ideal compressible magnetohydrodynamic equations with surface tension. *Math. Ann.*, 383(1):761–808, 2022.

[67] S. Ukai. The incompressible limit and the initial layer of the compressible Euler equation. *J. Math. Kyoto Univ.*, 26(2):323–331, 1986.

[68] C. Wang, Z. Zhang, W. Zhao, Y. Zheng. Local well-posedness and break-down criterion of the incompressible Euler equations with free boundary, *Memoirs Amer. Math. Soc.*, vol. 270, 2021.

[69] X. Wang. Global infinite energy solutions for the 2D gravity water waves system. *Commun. Pure. Appl. Math.*, 71(1):90–162, 2018.

[70] Y. Wang, Z. Xin. Vanishing viscosity and surface tension limits of incompressible viscous surface waves. *SIAM J. Math. Anal.*, 53(1):574–648, 2021.

[71] S. Wu. Well-posedness in Sobolev spaces of the full water wave problem in 2-D. *Invent. Math.*, 130(1):39–72, 1997.

[72] S. Wu. Well-posedness in Sobolev spaces of the full water wave problem in 3-D. *J. Amer. Math. Soc.*, 12(2):445–495, 1999.

[73] S. Wu. Almost global wellposedness of the 2-D full water wave problem. *Invent. Math.*, 177(1):45, 2009.

[74] S. Wu. Global wellposedness of the 3-D full water wave problem. *Invent. Math.*, 184(1):125–220, 2011.

[75] H. Yosihara. Gravity waves on the free surface of an incompressible perfect fluid of finite depth. *Publ. Res. Inst. Math. Sci.*, 18(1):49–96, 1982.

[76] J. Zhang. Local well-posedness and incompressible limit of the free-boundary problem in compressible elastodynamics. *Arch. Rational Mech. Anal.*, 244(3):599–697, 2022.

[77] J. Zhang. On the incompressible limit of current-vortex sheets with or without surface tension. arXiv:2405.00421, preprint, 2024.

[78] P. Zhang, Z. Zhang. On the free boundary problem of three-dimensional incompressible Euler equations. *Commun. Pure. Appl. Math.*, 61(7):877–940, 2008.

[79] F. Zheng. Long-term regularity of 3D gravity water waves. *Commun. Pure Appl. Math.*, 75(5):1074–1180, 2022.