Existence, Nonlinear Stability and Incompressible Limit of Current-Vortex Sheets with or without Surface Tension in Compressible Ideal MHD

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Abstract

Current-vortex sheet is one of the characteristic discontinuities in ideal compressible magnetohydrodynamics (MHD). The motion of current-vortex sheets is described by a free-interface problem of two-phase MHD flows with magnetic fields tangential to the interface. First, we prove local well-posedness of current-vortex sheets with surface tension by developing a robust framework that does not rely on Nash-Moser iteration nor tangential smoothing. Second, the energy estimates are uniform in Mach number and are also uniform in surface-tension coefficient under suitable stability conditions. Thus, we present a comprehensive study within one attempt, including well-posedness, nonlinear structural stability and incompressible limit of current-vortex sheets with or without surface tension.

Our result demonstrates that either suitable magnetic fields or surface tension could suppress the analogue of Kelvin-Helmholtz instability for compressible vortex sheets. The key observation is a hidden structure of Lorentz force in the vorticity analysis which motivates us to establish the uniform estimates in some anisotropic Sobolev spaces with suitable weights of Mach number determined by the number of tangential derivatives. Moreover, for isentropic two-phase flows whose density functions converge to the same constant when taking the incompressible limit, we can drop the boundedness assumption (with respect to Mach number) on high-order time derivatives by paralinearizing the evolution equation of the free interface. To our knowledge, this is the first result that rigorously justifies the incompressible limit of compressible vortex sheets.

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1 Introduction

The equations of compressible ideal magnetohydrodynamics (MHD) in $\mathbb{R}^d$ ($d = 2, 3$) can be written in the following form

$$\begin{cases} \varrho D_t u = B \cdot \nabla B - \nabla Q, \quad Q := P + \frac{1}{2}|B|^2, \\ D_t \varrho + \varrho \nabla \cdot u = 0, \\ D_t B = B \cdot \nabla u - B \nabla \cdot u, \\ \nabla \cdot B = 0, \\ D_t \mathfrak{s} = 0. \end{cases} \tag{1.1}$$

Here $\nabla := (\partial_{x_1}, \cdots, \partial_{x_d})$ is the standard spatial derivative. $D_t := \partial_t + u \cdot \nabla$ is the material derivative. The fluid velocity, the magnetic field, the fluid density, the fluid pressure and the entropy are denoted by $u =$

$(u_1, \cdots, u_d)$, $B = (B_1, \cdots, B_d)$, ϱ, P and $\mathfrak{s}$ respectively. The quantity $Q := P + \frac{1}{2}|B|^2$ is the total pressure. Note that the fourth equation in (1.1) is just an initial constraint instead of an independent equation. The last equation of (1.1) is derived from the equation of total energy and Gibbs relation and we refer to [31, Ch. 4.3] for more details. To close system (1.1), we need to introduce the equation of state

$$P = P(\varrho, \mathfrak{s}) \text{ satisfying } \frac{\partial P}{\partial \varrho} > 0. \tag{1.2}$$

A typical choice in this paper would be the polytropic gas parametrized by $\lambda > 0$ [69]:

$$P_\lambda(\varrho, \mathfrak{s}) = \lambda^2(\varrho^\gamma \exp(\mathfrak{s}/C_V) - 1), \quad \gamma > 1, \ C_V > 0. \tag{1.3}$$

We also need to assume $\varrho \geq \bar{\rho_0} > 0$ for some constant $\bar{\rho_0} > 0$, which together with $\frac{\partial P}{\partial \varrho} > 0$ guarantees the hyperbolicity of system (1.1).

1.1 Mathematical formulation of current-vortex sheets

Let $H > 10$ be a given real number, $x = (x_1, \cdots, x_d)$ and $x' := (x_1, \cdots, x_{d-1})$ and the space dimension $d = 2, 3$. We define the regions $\Omega^+(t) := \{x \in \mathbb{T}^{d-1} \times \mathbb{R} : \psi(t, x') < x_d < H\}$, $\Omega^-(t) := \{x \in \mathbb{T}^{d-1} \times \mathbb{R} : -H < x_d < \psi(t, x')\}$ and the moving interface $\Sigma(t) := \{x \in \mathbb{T}^{d-1} \times \mathbb{R} : x_d = \psi(t, x')\}$ between $\Omega^+(t)$ and $\Omega^-(t)$. We assume $U^\pm = (u^\pm, B^\pm, P^\pm, \mathfrak{s}^\pm)^\top$ to be a smooth solution to (1.1) in $\Omega^\pm(t)$ respectively. We say $\Sigma(t)$ is a *current-vortex sheet* (or an *MHD tangential discontinuity*) if the following conditions are satisfied:

$$[\![Q]\!] = \sigma \mathcal{H}, \quad B^\pm \cdot N = 0, \quad \partial_t \psi = u^\pm \cdot N \quad \text{on } \Sigma(t), \tag{1.4}$$

where $N := (-\partial_1 \psi, \cdots, -\partial_{d-1}\psi, 1)^\top$ is the normal vector to $\Sigma(t)$ (pointing towards $\Omega^+(t)$), $\sigma \geq 0$ is the constant coefficient of surface tension and the quantity $\mathcal{H} := \overline{\nabla} \cdot \left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}} \right)$ is twice the mean curvature of $\Sigma(t)$ with $\overline{\nabla} = (\partial_1, \cdots, \partial_{d-1})$. The jump of a function f on $\Sigma(t)$ is denoted by $[\![f]\!] := f^+|_{\Sigma(t)} - f^-|_{\Sigma(t)}$ with $f^\pm := f|_{\Omega^\pm(t)}$. The first condition shows that the jump of total pressure is balanced by surface tension. The second condition shows that both plasmas are perfect conductors. The third condition shows that there is no mass flow across the interface and thus the two plasmas are physically contact and mutually impermeable. These conditions on $\Sigma(t)$ are given by the Rankine-Hugoniot conditions for ideal compressible MHD when the magnetic fields are tangential to the interface, and we refer to Trakhinin-Wang [83, Appendix A] for detailed derivation. Besides, we impose the slip boundary conditions on the rigid boundaries $\Sigma^\pm := \mathbb{T}^{d-1} \times \{\pm H\}$

$$u_d^\pm = B_d^\pm = 0 \quad \text{on } \Sigma^\pm. \tag{1.5}$$

Remark 1.1 (Initial constraints for the magnetic field)**.** The conditions $\nabla \cdot B^\pm = 0$ in $\Omega^\pm(t)$, $B^\pm \cdot N|_{\Sigma(t)} = 0$ and $B_d^\pm = 0$ on $\Sigma^\pm$ are constraints for initial data so that system (1.1) with jump conditions (1.4) is not over-determined. One can use the continuity equation, the evolution equation of B and the kinematic boundary condition to show that $D_t^\pm(\frac{1}{\rho^\pm}\nabla \cdot B^\pm) = 0$ in $\Omega^\pm(t)$ and $D_t^\pm(\frac{B^\pm}{\rho^\pm} \cdot N) = 0$ on $\Sigma(t)$ and $\Sigma^\pm$ with $D_t^\pm := \partial_t + u^\pm \cdot \nabla$. Thus, the initial constraints can propagate within the lifespan of solutions if initially hold.

To make the initial-boundary-valued problem (1.1)-(1.5) solvable, we have to require the initial data to satisfy certain compatibility conditions. Let $(u_0^\pm, B_0^\pm, \varrho_0^\pm, \mathfrak{s}_0^\pm, \psi_0) := (u^\pm, B^\pm, \varrho^\pm, \mathfrak{s}^\pm, \psi)|_{t=0}$ be the initial data of system (1.1)-(1.4). We say the initial data satisfies the compatibility condition up to m-th order ($m \in \mathbb{N}$) if

$$\begin{aligned} (D_t^\pm)^j [\![Q]\!]|_{t=0} &= \sigma (D_t^\pm)^j \mathcal{H}|_{t=0} \quad \text{on } \Sigma(0), \quad 0 \leq j \leq m, \\ (D_t^\pm)^j \partial_t \psi|_{t=0} &= (D_t^\pm)^j (u^\pm \cdot N)|_{t=0} \quad \text{on } \Sigma(0), \quad 0 \leq j \leq m, \\ \partial_t^j u_d^\pm &= 0 \quad \text{on } \Sigma^\pm, \quad 0 \leq j \leq m. \end{aligned} \tag{1.6}$$

With these compatibility conditions, one can show that the magnetic fields also satisfy (cf. [80, Section 4.1])

$$(D_t^\pm)^j (B^\pm \cdot N)|_{t=0} = 0 \quad \text{on } \Sigma(0) \text{ and } \Sigma^\pm, \quad 0 \leq j \leq m.$$

We also note that the fulfillment of the first condition implicitly requires the fulfillment of the second one.

For $T>0$, we denote $\Omega_T^\pm := \bigcup_{0\leq t\leq T}\{t\}\times\Omega^\pm(t)$ and $\Sigma_T := \bigcup_{0\leq t\leq T}\{t\}\times\Sigma(t)$. We consider the Cauchy problem of (1.1): Given the initial data $(u_0^\pm, B_0^\pm, \varrho_0^\pm, \mathfrak{s}_0^\pm, \psi_0)$ satisfying the compatibility conditions (1.6) up to certain order, the vortex-sheet condition $|[\![u_0\cdot\tau]\!]|_\Sigma > 0$ for any vector τ tangential to $\Sigma(0)$, the constraints $\nabla\cdot B_0^\pm = 0$ in $\Omega^\pm(0)$, $(B_0^\pm\cdot N)|_{\Sigma(0)} = 0$ and $B_{0d}^\pm|_{\Sigma^\pm} = 0$, we want to study the well-posedness and the incompressible limit of the following system for both the case $\sigma>0$ and the case $\sigma=0$, and also the zero-surface-tension limit under suitable stability conditions on Σ_T which will be specified later.

$$
\begin{cases}
\varrho^\pm(\partial_t + u^\pm\cdot\nabla)u^\pm - B^\pm\cdot\nabla B^\pm + \nabla Q^\pm = 0, \quad Q^\pm := P^\pm + \frac{1}{2}|B^\pm|^2 & \text{in } \Omega_T^\pm,\\
(\partial_t + u^\pm\cdot\nabla)\varrho^\pm + \varrho^\pm\nabla\cdot u^\pm = 0 & \text{in } \Omega_T^\pm,\\
(\partial_t + u^\pm\cdot\nabla)B^\pm = B^\pm\cdot\nabla u^\pm - B^\pm\nabla\cdot u^\pm & \text{in } \Omega_T^\pm,\\
\nabla\cdot B^\pm = 0 & \text{in } \Omega_T^\pm,\\
(\partial_t + u^\pm\cdot\nabla)\mathfrak{s}^\pm = 0 & \text{in } \Omega_T^\pm,\\
P^\pm = P^\pm(\varrho^\pm,\mathfrak{s}^\pm), \quad \frac{\partial P^\pm}{\partial\varrho^\pm} > 0, \quad \varrho^\pm \geq \bar{\rho}_0 > 0 & \text{in } \overline{\Omega_T^\pm},\\
[\![Q]\!] = \sigma\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}}\right) & \text{on } \Sigma_T,\\
B^\pm\cdot N = 0 & \text{on } \Sigma_T,\\
\partial_t\psi = u^\pm\cdot N & \text{on } \Sigma_T,\\
u_d^\pm = B_d^\pm = 0 & \text{on } [0,T]\times\Sigma^\pm,\\
(u^\pm, B^\pm, \varrho^\pm, \mathfrak{s}^\pm)|_{t=0} = (u_0^\pm, B_0^\pm, \varrho_0^\pm, \mathfrak{s}_0^\pm) \quad \text{in } \Omega^\pm(0), \qquad \psi|_{t=0} = \psi_0 & \text{on } \Sigma(0).
\end{cases}
\tag{1.7}
$$

System (1.7), as a hyperbolic conservation law, admits a conserved L^2 energy

$$
E_0(t) := \sum_\pm \frac{1}{2}\int_{\Omega^\pm(t)} \varrho^\pm|u^\pm|^2 + |B^\pm|^2 + 2\mathfrak{P}(\varrho^\pm,\mathfrak{s}^\pm) + \varrho^\pm|\mathfrak{s}^\pm|^2\,\mathrm{d}x + \sigma\ \mathrm{Area}(\Sigma(t))
$$

where $\mathfrak{P}(\varrho^\pm,\mathfrak{s}^\pm) = \int_{\bar{\rho}_0}^{\varrho^\pm}\frac{P^\pm(z,\mathfrak{s}^\pm)}{z^2}\,\mathrm{d}z$. See Section 3.1 for proof.

1.2 Reformulation in flattened domains

1.2.1 Flattening the fluid domains

We shall convert (1.7) into a PDE system defined in fixed domains $\Omega^\pm := \mathbb{T}^{d-1}\times\{0 < \pm x_d < H\}$. One way to achieve this is to use the Lagrangian coordinates, but it would bring lots of unnecessary technical difficulties when analyzing the surface tension. Here, we consider a family of diffeomorphisms $\Phi(t,\cdot):\Omega^\pm\to\Omega^\pm(t)$ characterized by the moving interface. In particular, let

$$
\Phi(t,x',x_d) = (x', \varphi(t,x_d)), \tag{1.8}
$$

where

$$
\varphi(t,x) = x_d + \chi(x_d)\psi(t,x') \tag{1.9}
$$

and $\chi\in C_c^\infty([-H,H])$ is a smooth cut-off function satisfying the following bounds:

$$
\|\chi'\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\|\psi_0\|_\infty + 20}, \quad \sum_{j=1}^{8}\|\chi^{(j)}\|_{L^\infty(\mathbb{R})} \leq C, \quad \chi = 1 \ \text{ on } (-1,1) \tag{1.10}
$$

for some generic constant $C>0$. We assume $|\psi_0|_{L^\infty(\mathbb{T}^2)} \leq 1$. One can prove that there exists some $T_0>0$ such that $\sup_{[0,T_0]}|\psi(t,\cdot)|_{L^\infty(\mathbb{T}^2)} < 10 < H$, the free interface is still a graph within the time interval $[0,T_0]$ and

$$
\partial_d\varphi(t,x',x_d) = 1 + \chi'(x_d)\psi(t,x') = 1 - \frac{1}{20}\times 10 \geq \frac{1}{2}, \quad t\in[0,T_0],
$$

which ensures that $\Phi(t)$ is a diffeomorphism in $[0,T_0]$.

Based on this, we introduce the new variables

$$\begin{aligned} v^{\pm}(t,x) &= u^{\pm}(t,\Phi(t,x)), \quad b^{\pm}(t,x) = B^{\pm}(t,\Phi(t,x)), \quad \rho^{\pm}(t,x) = \varrho^{\pm}(t,\Phi(t,x)), \\ S^{\pm}(t,x) &= \mathfrak{s}^{\pm}(t,\Phi(t,x)), \quad q^{\pm}(t,x) = Q^{\pm}(t,\Phi(t,x)), \quad p^{\pm}(t,x) = P(t,\Phi(t,x)) \end{aligned} \tag{1.11}$$

that represent the velocity fields, the magnetic fields, the densities, the entropy functions, the total pressure functions and the fluid pressure functions defined in the fixed domains $\Omega^{\pm}$ respectively. Also, we introduce the differential operators

$$\nabla^{\varphi} = (\partial_1^{\varphi}, \cdots, \partial_d^{\varphi}), \quad \partial_a^{\varphi} = \partial_a - \frac{\partial_a \varphi}{\partial_d \varphi}\partial_d, \ \ a = t, 1, \cdots, d-1; \ \ \partial_d^{\varphi} = \frac{1}{\partial_d \varphi}\partial_d. \tag{1.12}$$

Moreover, setting the tangential gradient operator and the tangential derivatives as

$$\overline{\nabla} := (\partial_1, \cdots, \partial_{d-1}), \ \ \overline{\partial}_i := \partial_i, \ \ i = 1, \cdots, d-1,$$

then the boundary conditions (1.4) on the free interface $\Sigma(t)$ are turned into

$$[\![q]\!] = \sigma \mathcal{H}(\psi) := \sigma \overline{\nabla} \cdot \left(\frac{\overline{\nabla}\psi}{\sqrt{1 + |\overline{\nabla}\psi|^2}} \right) \quad \text{on } [0,T] \times \Sigma, \tag{1.13}$$

$$\partial_t \psi = v^{\pm} \cdot N, \quad N = (-\overline{\partial}_1 \psi, -\overline{\partial}_2 \psi, 1)^{\top} \quad \text{on } [0,T] \times \Sigma, \tag{1.14}$$

$$b^{\pm} \cdot N = 0 \quad \text{on } [0,T] \times \Sigma, \tag{1.15}$$

where $\Sigma = \mathbb{T}^{d-1} \times \{x_d = 0\}$.

Let $D_t^{\varphi\pm} := \partial_t^{\varphi} + v^{\pm} \cdot \nabla^{\varphi}$. Then system (1.7) is converted into

$$\begin{cases} \rho^{\pm} D_t^{\varphi\pm} v^{\pm} - (b^{\pm} \cdot \nabla^{\varphi}) b^{\pm} + \nabla^{\varphi} q^{\pm} = 0, \ \ q^{\pm} = p^{\pm} + \frac{1}{2}|b^{\pm}|^2 & \text{in } [0,T] \times \Omega^{\pm}, \\ D_t^{\varphi\pm} \rho^{\pm} + \rho^{\pm} \nabla^{\varphi} \cdot v^{\pm} = 0 & \text{in } [0,T] \times \Omega^{\pm}, \\ p^{\pm} = p^{\pm}(\rho^{\pm}, S^{\pm}), \ \ \frac{\partial p^{\pm}}{\partial \rho^{\pm}} > 0, \ \rho^{\pm} \geq \bar{\rho_0} > 0 & \text{in } [0,T] \times \Omega^{\pm}, \\ D_t^{\varphi\pm} b^{\pm} - (b^{\pm} \cdot \nabla^{\varphi}) v^{\pm} + b^{\pm} \nabla^{\varphi} \cdot v^{\pm} = 0 & \text{in } [0,T] \times \Omega^{\pm}, \\ \nabla^{\varphi} \cdot b^{\pm} = 0 & \text{in } [0,T] \times \Omega^{\pm}, \\ D_t^{\varphi\pm} S^{\pm} = 0 & \text{in } [0,T] \times \Omega^{\pm}, \\ [\![q]\!] = \sigma \overline{\nabla} \cdot \left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}} \right) & \text{on } [0,T] \times \Sigma, \\ \partial_t \psi = v^{\pm} \cdot N & \text{on } [0,T] \times \Sigma, \\ b^{\pm} \cdot N = 0 & \text{on } [0,T] \times \Sigma, \\ v_d^{\pm} = b_d^{\pm} = 0 & \text{on } [0,T] \times \Sigma^{\pm}, \\ (v^{\pm}, b^{\pm}, \rho^{\pm}, S^{\pm}, \psi)|_{t=0} = (v_0^{\pm}, b_0^{\pm}, \rho_0^{\pm}, S_0^{\pm}, \psi_0). \end{cases} \tag{1.16}$$

Invoking (1.12), we can alternatively write the material derivative D_t^{φ} as

$$D_t^{\varphi\pm} = \partial_t + \bar{v}^{\pm} \cdot \overline{\nabla} + \frac{1}{\partial_d \varphi}(v^{\pm} \cdot \mathbf{N} - \partial_t \varphi)\partial_d, \tag{1.17}$$

where $\bar{v}^{\pm} := (v_1^{\pm}, \cdots, v_{d-1}^{\pm})^{\top}$ is the horizontal components of the fluid velocity, $\bar{v}^{\pm} \cdot \overline{\nabla} := \sum_{j=1}^{d-1} v_j^{\pm} \partial_j$, and $\mathbf{N} := (-\partial_1 \varphi, \cdots, -\partial_{d-1}\varphi, 1)^{\top}$ is the extension of the normal vector N into $\Omega^{\pm}$. This formulation will be helpful for us to define the linearized material derivative when using Picard iteration to construct the solution.

1.2.2 Parametrization of the equation of state

We assume the fluids in Ω^+ and Ω^- satisfy the same equation of state of polytropic gases. Specifically, we parametrize the equation of state to be $\rho = \rho(p/\lambda^2, S)$ where $\lambda > 0$ is proportional to the sound speed $c_s := \sqrt{\partial_p \rho}$. For a polytropic gas, the equation of state is parametrized [69] in terms of $\lambda > 0$:

$$p_{\lambda}(\rho, S) = \lambda^2 \left(\rho^{\gamma} \exp(S/C_V) - 1 \right), \quad \gamma > 1, \ \ C_V > 0. \tag{1.18}$$

When viewing the density as a function of the pressure and the entropy, this indicates

$$\rho_\lambda(p/\lambda^2, S) = \left(\left(1+\frac{p}{\lambda^2}\right)e^{-\frac{S}{C_V}}\right)^{\frac{1}{\gamma}}, \quad \text{and } \log\left(\rho_\lambda(p/\lambda^2, S)\right) = \gamma^{-1}\log\left(\left(1+\frac{p}{\lambda^2}\right)e^{-\frac{S}{C_V}}\right). \tag{1.19}$$

Let $\mathcal{F}^\pm(p^\pm, S^\pm) := \log\rho^\pm(p^\pm, S^\pm)$. Since $\frac{\partial p^\pm}{\partial\rho^\pm} > 0$ and $\rho^\pm > 0$ imply $\frac{\partial\mathcal{F}^\pm}{\partial p^\pm} = \frac{1}{\rho^\pm}\frac{\partial\rho^\pm}{\partial p^\pm} > 0$, then the second equation in (1.16) is equivalent to

$$\frac{\partial\mathcal{F}^\pm}{\partial p^\pm}(p^\pm, S^\pm)D_t^{\varphi\pm}p^\pm + \nabla^\varphi\cdot v^\pm = 0. \tag{1.20}$$

Also, we assume there exists a constant $C > 0$, such that the following inequality holds for $0 \le k \le 8$:

$$|\partial_p^k\mathcal{F}(p,S)| \le C, \quad |\partial_p^k\mathcal{F}(p,S)| \le C|\partial_p\mathcal{F}(p,S)|^k \le C\partial_p\mathcal{F}(p,S). \tag{1.21}$$

Hence, we can view $\mathcal{F} = \log\rho$ as a parametrized family $\{\mathcal{F}_\varepsilon(p,S)\}$ as well, where $\varepsilon = \frac{1}{\lambda}$. Indeed, we have

$$\frac{\partial\mathcal{F}_\varepsilon}{\partial p} = \gamma^{-1}\log\left((1+\varepsilon^2 p)e^{-\frac{S}{C_V}}\right). \tag{1.22}$$

Since we work on the case when the entropy and velocity are both bounded (later we will assume $u, S \in H^4(\Omega)$), there exists $A > 0$ such that

$$\frac{\partial\mathcal{F}_\varepsilon}{\partial p}(p,S) = \frac{1}{\rho}\frac{\partial\rho}{\partial p}(p,S) \le A\varepsilon^2. \tag{1.23}$$

We slightly abuse the terminology and call λ the sound speed and call ε the Mach number. When $\lambda \gg 1$ ($\varepsilon \ll 1$), the constant A in (1.23) can be greater than 1 such that

$$A^{-1}\varepsilon^2 \le \frac{\partial\mathcal{F}_\varepsilon}{\partial p}(p,S) \le A\varepsilon^2. \tag{1.24}$$

We sometimes write $\mathcal{F}_p^\pm := \frac{\partial\mathcal{F}_\varepsilon^\pm}{\partial p}(p^\pm, S^\pm) = \varepsilon^2$ for simplicity when discussing the incompressible limit.

1.2.3 Stability conditions for the zero-surface-tension limit

Finally, we need to add some extra stability conditions on the free interface when surface tension is neglected, that is, when $\sigma = 0$. We introduce the quantities

$$a^\pm := \sqrt{\rho^\pm\left(1+\left(\frac{c_A^\pm}{c_s^\pm}\right)^2\right)}$$

where $c_A^\pm := |b^\pm|/\sqrt{\rho^\pm}$ represents the Alfvén speed (the speed of magneto-sonic waves), $c_s^\pm := \sqrt{\partial p^\pm/\partial\rho^\pm}$ represents the sound speed. The stability conditions are

$$d = 3: \ 0 < a^\pm\left|\bar{b}^\mp \times [\![\bar{v}]\!]\right| < |\bar{b}^+ \times \bar{b}^-| \quad \text{on } [0,T]\times\Sigma, \tag{1.25}$$

$$d = 2: \ \left(\frac{|b_1^+|}{a^+} + \frac{|b_1^-|}{a^-}\right) > |\,[\![v_1]\!]\,| > 0 \quad \text{on } [0,T]\times\Sigma, \tag{1.26}$$

where we view the horizontal magnetic field $\bar{b} = (b_1, b_2, 0)^\top$ and the horizontal velocity $\bar{v} = (v_1, v_2, 0)^\top$ as vectors lying on $\mathbb{T}^2 \times \{x_3 = 0\} \subset \mathbb{R}^3$ to define the exterior product. The "> 0" part in (1.25) and (1.26) is necessary because we are considering the vortex sheets which automatically require the tangential discontinuity of velocity is nonzero. Thus, the stability conditions require that the strength of the magnetic fields cannot be too weak. Moreover, the condition for 3D case implies that b^+ and b^- are not collinear on Σ and the condition for 2D case requires certain quantitative relation between the strength of magnetic fields and the jump of tangential velocities. Note that the stability conditions are just initial constraints that can propagate within a short time interval instead of imposed boundary conditions. We will explain in later sections why such stability conditions are needed.

1.3 History and background

1.3.1 An overview of previous results

There have been a lot of studies about free-boundary problems in ideal MHD, of which the original models in physics are mainly three types: plasma-vacuum interface model, current-vortex sheets and MHD contact discontinuities. The plasma-vacuum problem is related to plasma confinement problems [31, Chap. 4] in laboratory plasma physics, which describes the motion of *one isolated* perfectly conducting fluid in an electro-magentic field confined in a vacuum region (in which there is another vacuum magnetic field satisfying the pre-Maxwell system). When the vacuum magnetic fields are neglected, the plasma-vacuum model is reduced the free-boundary problem of one-phase MHD flows and we refer to [39, 53, 36, 35, 34, 41] for local well-posedness (LWP) theory in incompressible ideal MHD. It should be noted that, when the surface tension is neglected, the Rayleigh-Taylor sign condition $-\nabla_N Q|_{\Sigma(t)} \geq c_0 > 0$ should be added as an initial constraint for LWP which is the analogue of Euler equations [26, 38] and we refer to Hao-Luo [40] for the proof. For the full plasma-vacuum model without surface tension in incompressible ideal MHD, we refer to [32, 33, 76, 51]. As for the compressible case, in a series of works [72, 81, 82, 84], Secchi, Trakhinin and Wang used Nash-Moser iteration to construct the solution due to the derivative loss in the linearized problems. Very recently, Lindblad and the author [49] proved the LWP and a continuation criterion for the one-phase free-boundary problem in compressible ideal MHD without surface tension, which gave the first result about the energy estimates without loss of regularity.

In view of fluid mechanics, a vortex sheet is an interface between two "impermeable" fluids across which there is a tangential discontinuity in fluid velocity. However, the study of one-phase free-boundary problem does not tell us how to analyze the free interface between two fluids (e.g., shock fronts, contact discontinuities), which in fact is quite different from the study of free-surface one-phase flow. For incompressible inviscid fluids without surface tension, vortex sheets tend to be violently unstable, which exhibit the so-called Kelvin-Helmholtz instability. There have been numerous mathematical studies in this direction, especially for 2D irrotational flows, and we refer to [27, 88, 89] and references therein. On the other hand, surface tension is expected to "suppress" the Kelvin-Helmholtz instability. Ambrose-Masmoudi [6] rigorously justified this for irrotational flows and Cheng-Coutand-Shkoller [19], Shatah-Zeng [73] proved this for incompressible Euler equations with nonzero vorticity.

When the compressibility is taken into account, we shall consider not only the motion of the interface of discontinuities but also its interaction with the wave propagation in the interior. Let $\mathbf{j} = \varrho(u \cdot N - \partial_t \psi)$ be the mass transfer flux. In view of hyperbolic conservation laws, strong discontinuities can be classified into shock waves ($\mathbf{j} \neq 0, [\![\varrho]\!] \neq 0$) and characteristic discontinuities. According to Lax [45], characteristic discontinuities are called contact discontinuities, which are physically contact ($\mathbf{j} = 0$). For compressible Euler equations, contact discontinuities are further classified to be compressible vortex sheets ($[\![u_\tau]\!] \neq \mathbf{0}$) and entropy waves ($[\![u]\!] = \vec{0},\ [\![\varrho]\!], [\![\mathfrak{s}]\!] \neq 0$). The existence and the structural stability of multi-dimensional shocks for compressible Euler equations was proved by Majda [56, 57] (see also Blokhin [10]) provided that the uniform Kreiss-Lopatinskiĭ condition [43] is satisfied. Indeed, shock fronts under the uniform Kreiss-Lopatinskiĭ condition are non-characteristic discontinuities, while compressible vortex sheets are characteristic discontinuities and the uniform Kreiss-Lopatinskiĭ condition is never satisfied. Thus, there is a potential loss of normal derivatives for compressible vortex sheets, which makes the proof of existence and structural stability more difficult. For 3D Euler equations, compressible vortex sheets are always violently unstable [29, 63, 78] which exhibit an analogue of Kelvin-Helmholtz instability; whereas for 2D Euler equations, Coulombel-Secchi [22, 23] proved the existence of "supersonic" vortex sheets when the Mach number for the rectilinear background solution $(\pm\underline{v}, \underline{\rho})$ exceeds $\sqrt{2}$ and the violent instability when the Mach number is lower than $\sqrt{2}$ by adapting Majda's frequency analysis [56] to the linearized problem and Nash-Moser iteration. See also Chen-Secchi-Wang [13] for the study of supersonic relativistic vortex sheets in (1+2)-dimensional Minkowski space-time. Similarly as the incompressible case, surface tension again prevents such violent instability and we refer to Stevens [74] for the proof of structural stability.

1.3.2 Strong discontinuities in ideal compressible MHD

Apart from surface tension, suitable magnetic fields or elasticity also have stabilization effect on the vortex sheets. For example, one can see a series of work [14, 15, 16, 17] by Chen, Hu, Wang, et al., about the compressible vortex sheets in elastodynamics. As for MHD, after excluding MHD shocks ($\mathbf{j} \neq 0$, $[\![\varrho]\!] \neq 0$) which are non-characteristic discontinuities, there are three different types of characteristic discontinuities: current-vortex sheets ($\mathbf{j} = 0$, $B^\pm \cdot N|_{\Sigma(t)} = 0$), MHD contact discontinuities ($\mathbf{j} = 0$, $B^\pm \cdot N|_{\Sigma(t)} \neq 0$) and Alfvén (rotational) discontinuities ($\mathbf{j} \neq 0$, $[\![\varrho]\!] = 0$). Current-vortex sheets and MHD contact discontinuities are physically contact, while Alfvén discontinuities are not. The Rankine-Hugoniot conditions for current-vortex sheets and MHD contact discontinuities (cf. [31, Chap. 4.5] and [83, Appendix A]) are

- (Current-vortex sheets/Tangential discontinuities) $[\![Q]\!] = \sigma\mathcal{H}$, $B^\pm \cdot N = 0$, $\partial_t\psi = u^\pm \cdot N$ on $\Sigma(t)$.
- (MHD contact discontinuities) $[\![P]\!] = \sigma\mathcal{H}$, $[\![u]\!] = [\![B]\!] = \vec{0}$, $B^\pm \cdot N \neq 0$, $\partial_t\psi = u^\pm \cdot N$ on $\Sigma(t)$.

MHD contact discontinuities usually arise from astrophysical plasmas [31], where the magnetic fields typically originate in a rotating object, such as a star or a dynamo operating inside, and intersect the surface of discontinuity. An example is the photosphere of the sun. In contrast, current-vortex sheets require the magnetic fields to be tangential to the interface. An example in laboratory plasma physics is that the discontinuities confine a high-density plasma by a lower-density one, which is isolated thermally from an outer rigid wall. In particular, when the plasma is liquid metal, the effect of surface tension cannot be neglected [64]. In astrophysics, a generally accepted model for compressible current-vortex sheets is the heliopause [8] (in some sense, the "boundary" of the solar system[1]) that separates the interstellar plasma compressed at the bow shock (outside the solar system) from the solar wind plasma compressed at the termination shock (inside the solar system). Besides, the night-side magnetopause of the earth is also considered to be current-vortex sheets. These facts demonstrate the existence of current-vortex sheets, so the corresponding mathematical modelling becomes very important.

For MHD contact discontinuities, the transversality of magnetic fields could enhance the regularity of the free interface and avoid the possible normal derivative loss in the interior. We refer to Morando-Trakhinin-Trebeschi [66] for the 2D case under Rayleigh-Taylor sign condition $N\cdot\nabla[\![Q]\!]|_{\Sigma(t)} \geq c_0 > 0$, Trakhinin-Wang [83] for the case with nonzero surface tension, and Wang-Xin [87] for both 2D and 3D cases without surface tension or Rayleigh-Taylor sign condition. In other words, Wang-Xin [87] showed that transversal magnetic fields across the interface could suppress the Rayleigh-Taylor instability.

As for current-vortex sheets, Kelvin-Helmholtz instability can also be suppressed, but, unlike the transversal magnetic fields in MHD contact discontinuities, the tangential magnetic fields must satisfy certain constraints. For 3D incompressible ideal MHD, Syrovatskiĭ [77] introduced a stability condition by using normal mode analysis:

$$\varrho^+|B^+ \times [\![u]\!]|^2 + \varrho^-|B^- \times [\![u]\!]|^2 < (\varrho^+ + \varrho^-)|B^+ \times B^-|^2, \tag{1.27}$$

which corresponds to the transition to violent instability, that is, ill-posedness of the linearized problem. Coulombel-Morando-Secchi-Trebeschi [21] proved the a priori estimate for the nonlinear problem under a more restrictive condition

$$\max\left\{\left|\frac{B^+}{\sqrt{\varrho^+}} \times [\![u]\!]\right|, \left|\frac{B^-}{\sqrt{\varrho^-}} \times [\![u]\!]\right|\right\} < \left|\frac{B^+}{\sqrt{\varrho^+}} \times \frac{B^-}{\sqrt{\varrho^-}}\right|. \tag{1.28}$$

Sun-Wang-Zhang [75] proved local well-posedness of the nonlinear problem under the original Syrovatskiĭ condition (1.27) by adapting the framework of Shatah-Zeng [73]. Very recently, Liu-Xin [50] gave a comprehensive study for both $\sigma > 0$ and $\sigma = 0$ cases (see also Li-Li [47]).

For compressible current-vortex sheets without surface tension, Trakhinin [79] showed that the uniform Kreiss-Lopatinskiĭ condition [43] for the linearized problem is never satisfied, so only the neutral stability can be expected for the linearized problem. However, the specific range for the neutral stability cannot be explicitly calculated [79, Section 4.2]. Thus, it is still unknown if there is any *necessary and sufficient condition* for the linear (neutral) stability. To avoid testing the Kreiss-Lopatinskiĭ condition, Trakhinin [79] used the method of "Friedrichs secondary symmetrizer" to raise a sufficient condition for the problem linearized around a background planar current-vortex sheet $(\hat{v}^\pm, \hat{b}^\pm, \hat{\rho}^\pm, \hat{S}^\pm)$ in flattened domains $\Omega^\pm$, which reads

$$|[\![\hat{v}]\!]| < |\sin(\alpha^+ - \alpha^-)| \min\left\{\frac{\gamma^+}{|\sin\alpha^-|}, \frac{\gamma^-}{|\sin\alpha^+|}\right\}, \tag{1.29}$$

[1]On August 25, 2012, Voyager 1 flew beyond the heliopause and entered interstellar space. At the time, it was at a distance about 122 A.U. (around 18 billion kilometers) from the sun. On November 5, 2018, Voyager 2 also traversed the heliopause.

where $\gamma^\pm := \frac{c_A^\pm}{\sqrt{1+(c_A^\pm/c_s^\pm)^2}}$, $c_A^\pm := |\hat{b}^\pm|/\sqrt{\hat{\rho}^\pm}$ represents the Alfvén speed, $c_s^\pm := \sqrt{\partial\hat{p}^\pm/\partial\hat{\rho}^\pm}$ represents the sound speed, and $\alpha^\pm$ represents the oriented angle between $[\![\hat{v}]\!]$ and $\hat{b}^\pm$. Indeed, (1.29) is equivalently to

$$\max\left\{|\hat{b}^- \times [\![\hat{v}]\!]|\sqrt{\hat{\rho}^+\left(1+(c_A^+/c_s^+)^2\right)}, |\hat{b}^+ \times [\![\hat{v}]\!]|\sqrt{\hat{\rho}^-\left(1+(c_A^-/c_s^-)^2\right)}\right\} < |\hat{b}^+ \times \hat{b}^-|, \tag{1.30}$$

which is exactly the same as (1.25). If we formally take the incompressible limit $\hat{\rho}^\pm \to 1$ and $c_s^\pm \to +\infty$, then the above inequality exactly converges to (1.28) used in [21], and it is easy to see (1.28) implies (1.27).

Under (1.25), Chen-Wang [12] and Trakhinin [80] proved the well-posedness for the 3D problem without surface tension by using Nash-Moser iteration. Using similar techniques as [80], Morando-Secchi-Trebeschi-Yuan [65] proved the well-posedness for the 2D problem without surface tension under the stability condition (1.26), which covers part of the "subsonic zone" for the neutral stability obtained by Wang-Yu [86]. That is to say, non-collinear magnetic fields in 3D and sufficiently strong magnetic fields in 2D can also suppress the analogue of Kelvin-Helmholtz instability for compressible vortex sheets.

The abovementioned results only give the local existence of free-boundary ideal compressible MHD, but many behaviors of the solutions are still unclear. For example, Ohno-Shirota [67] showed that the linearized problem in a fixed domain with magnetic fields tangential to the boundary is ill-posed in standard Sobolev spaces $H^l (l \geq 2)$, but the corresponding incompressible problem is well-posed in standard Sobolev spaces [36, 75, 76, 50, 51]. The anisotropic Sobolev spaces defined in Section 1.4.1, first introduced by Chen [18], have been adopted in previous works about ideal compressible MHD [90, 70, 71, 80, 12, 72, 81, 82]. The author [91] studied compressible inviscid-resistive MHD in standard Sobolev spaces, but the vanishing resistivity limit is still unknown. In other words, there is no explanation for the mismatch of the function spaces for local existence yet. Besides, it is also unclear about the comparison between the stabilization mechanism brought by surface tension and the one brought by certain magnetic fields when the plasma is compressible. These questions should be answered by rigorously justifying the incompressible limit and the zero-surface-tension limit. In particular, the existing literature about the incompressible limit of free-boundary problems in inviscid fluids is only avaliable for the one-phase problems [48, 52, 25, 91, 92, 55, 37]. The incompressible limits of free-boundary MHD and vortex sheet in inviscid fluids remain completely open.

1.3.3 Our goal in this paper

In this paper, we give a comprehensive study for the local-in-time solution to current-vortex sheets in ideal MHD and particularly give affirmative answers to the abovementioned questions. Specifically, we aim the prove the following results:

- Well-posedness of current-vortex sheets with surface tension in both 2D and 3D, which corresponds to the local well-posedness and energy estimates (without loss of regularity) of system (1.16).
- Incompressible and zero-surface tension limits of current-vortex sheets. The incompressible limit results from the estimates that are uniform in Mach number (which will be achieved together with local existence) when the initial data is "well-prepared". Taking the zero surface tension limit requires the estimates to be uniform in $\sigma > 0$ under the stability conditions (1.25) or (1.26) in 3D and 2D respectively. It should be noted that these two limit processes are independent of each other, that is, our energy estimates are uniform in both Mach number and σ under (1.25) or (1.26).

To our knowledge, this is the first result about the incompressible limit of compressible vortex sheets and free-boundary MHD and also the first result about compressible current-vortex sheets with surface tension. The incompressible limit also ties our result to the suppression effect on Kelvin-Helmholtz instability brought by either surface tension or suitable magnetic fields.

1.4 Main results

1.4.1 Anisotropic Sobolev spaces

Following the notations in [18, 85], we first define the anisotropic Sobolev space $H_*^m(\Omega^\pm)$ for $m \in \mathbb{N}$ and $\Omega^\pm = \mathbb{T}^{d-1} \times \{0 < \pm x_d < H\}$. Let $\omega = \omega(x_d) = (H^2 - x_d^2)x_d^2$ be a smooth function on $[-H, H]$. The choice of $\omega(x_d)$ is not unique, as we just need $\omega(x_d)$ vanishes on $\Sigma\cup\Sigma^\pm$ and is comparable to the distance function near the interface and the boundaries. Then we define $H_*^m(\Omega^\pm)$ for $m\in\mathbb{N}^*$ as follows

$$H_*^m(\Omega^\pm):=\left\{f\in L^2(\Omega^\pm)\middle|(\omega\partial_d)^{\alpha_{d+1}}\partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d}f\in L^2(\Omega^\pm),\ \ \forall\alpha\text{ with }\sum_{j=1}^{d-1}\alpha_j+2\alpha_d+\alpha_{d+1}\le m\right\},$$

equipped with the norm

$$\|f\|^2_{H_*^m(\Omega^\pm)}:=\sum_{\sum_{j=1}^{d-1}\alpha_j+2\alpha_d+\alpha_{d+1}\le m}\|(\omega\partial_d)^{\alpha_{d+1}}\partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d}f\|^2_{L^2(\Omega)}. \tag{1.31}$$

For any multi-index $\alpha:=(\alpha_0,\alpha_1,\cdots,\alpha_d,\alpha_{d+1})\in\mathbb{N}^{d+2}$, we define

$$\partial_*^\alpha:=\partial_t^{\alpha_0}(\omega\partial_d)^{\alpha_{d+1}}\partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d},\ \ \langle\alpha\rangle:=\sum_{j=0}^{d-1}\alpha_j+2\alpha_d+\alpha_{d+1},$$

and define the **space-time anisotropic Sobolev norm** $\|\cdot\|_{m,*,\pm}$ to be

$$\|f\|^2_{m,*,\pm}:=\sum_{\langle\alpha\rangle\le m}\|\partial_*^\alpha f\|^2_{L^2(\Omega^\pm)}=\sum_{\alpha_0\le m}\|\partial_t^{\alpha_0}f\|^2_{H_*^{m-\alpha_0}(\Omega^\pm)}. \tag{1.32}$$

We also write the interior Sobolev norm to be $\|f\|_{s,\pm}:=\|f(t,\cdot)\|_{H^s(\Omega^\pm)}$ for any function $f(t,x)$ on $[0,T]\times\Omega^\pm$ and denote the boundary Sobolev norm to be $|f|_s:=|f(t,\cdot)|_{H^s(\Sigma)}$ for any function $f(t,x')$ on $[0,T]\times\Sigma$.

From now on, we assume the dimension $d=3$, that is, $\Omega^\pm=\mathbb{T}^2\times\{0<\pm x_3<H\}$, $\Sigma^\pm=\mathbb{T}^2\times\{x_3=\pm H\}$ and $\Sigma=\mathbb{T}^2\times\{x_3=0\}$. We will see the 2D case follows in the same manner as the 3D case up to slight modifications in the vorticity analysis and the analysis of stability condition when $\sigma=0$.

1.4.2 Main result 1: Well-posedness and uniform estimates in Mach number

Invoking (1.20) and writing $\mathcal{F}_p^\pm:=\frac{\partial\mathcal{F}^\pm}{\partial p^\pm}$, system (1.16) is equivalent to

$$\begin{cases}\rho^\pm D_t^{\varphi\pm}v^\pm-(b^\pm\cdot\nabla^\varphi)b^\pm+\nabla^\varphi q^\pm=0,\ \ q^\pm=p^\pm+\frac12|b^\pm|^2 & \text{in }[0,T]\times\Omega^\pm,\\ \mathcal{F}_p^\pm D_t^{\varphi\pm}p^\pm+\nabla^\varphi\cdot v^\pm=0 & \text{in }[0,T]\times\Omega^\pm,\\ p^\pm=p^\pm(\rho^\pm,S^\pm),\ \ \mathcal{F}^\pm=\log\rho^\pm,\ \ \mathcal{F}_p^\pm>0,\ \rho^\pm\ge\bar{\rho_0}>0 & \text{in }[0,T]\times\Omega^\pm,\\ D_t^{\varphi\pm}b^\pm-(b^\pm\cdot\nabla^\varphi)v^\pm+b^\pm\nabla^\varphi\cdot v^\pm=0 & \text{in }[0,T]\times\Omega^\pm,\\ \nabla^\varphi\cdot b^\pm=0 & \text{in }[0,T]\times\Omega^\pm,\\ D_t^{\varphi\pm}S^\pm=0 & \text{in }[0,T]\times\Omega^\pm,\\ [\![q]\!]=\sigma\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}}\right) & \text{on }[0,T]\times\Sigma,\\ \partial_t\psi=v^\pm\cdot N & \text{on }[0,T]\times\Sigma,\\ b^\pm\cdot N=0 & \text{on }[0,T]\times\Sigma,\\ v_d^\pm=b_d^\pm=0 & \text{on }[0,T]\times\Sigma^\pm,\\ (v^\pm,b^\pm,\rho^\pm,S^\pm,\psi)|_{t=0}=(v_0^\pm,b_0^\pm,\rho_0^\pm,S_0^\pm,\psi_0).\end{cases} \tag{1.33}$$

Since the material derivatives are tangential to the boundary, that is, $D_t^{\varphi\pm}=\overline{D_t^\pm}:=\partial_t+\bar{v}^\pm\cdot\overline{\nabla}$ on Σ and $\Sigma^\pm$, the compatibility conditions (1.6) for initial data up to m-th order ($m\in\mathbb{N}$) are now written as:

$$\begin{aligned}\left[\!\left[\partial_t^j q\right]\!\right]|_{t=0}=\sigma\partial_t^j\mathcal{H}|_{t=0}\ \ &\text{on }\Sigma,\ \ 0\le j\le m,\\ \partial_t^{j+1}\psi|_{t=0}=\partial_t^j(v^\pm\cdot N)|_{t=0}\ \ &\text{on }\Sigma,\ \ 0\le j\le m,\\ \partial_t^j v_d^\pm|_{t=0}=0\ \ &\text{on }\Sigma^\pm,\ \ 0\le j\le m.\end{aligned} \tag{1.34}$$

Under (1.34), one can prove that $\partial_t^j(b^\pm\cdot N)|_{t=0}=0$ is also satisfied on Σ and $\Sigma^\pm$ for $0\le j\le m$ and we refer to Trakhinin [80, Section 4] for details.

The first result shows the local well-posedness and the energy estimates of (1.33) for each fixed $\sigma>0$.

Theorem 1.1 (**Well-posedness and uniform estimates for fixed** $\sigma > 0$)**.** Fix the constant $\sigma > 0$. Let $\mathbf{U}_0^\pm := (v_0^\pm, b_0^\pm, \rho_0^\pm, S_0^\pm)^\top \in H_*^8(\Omega^\pm)$ and $\psi_0 \in H^{9.5}(\Sigma)$ be the initial data of (1.33) satisfying

- the compatibility conditions (1.34) up to 7-th order;
- the constraints $\nabla^{\varphi_0} \cdot b_0^\pm = 0$ in $\Omega^\pm$, $b^\pm \cdot N|_{\{t=0\}\times(\Sigma\cup\Sigma^\pm)} = 0$;
- $|[\![\bar{v}_0]\!]| > 0$ on Σ, $|\psi_0|_{L^\infty(\Sigma)} \le 1$, and $E(0) \le M$ for some constant $M > 0$.

Then there exists $T_\sigma > 0$ depending only on M and σ, such that (1.33) admits a unique solution $(v^\pm(t), b^\pm(t), \rho^\pm(t), S^\pm(t), \psi(t))$ verifies the energy estimate

$$\sup_{t\in[0,T]} E(t) \le C(\sigma^{-1})P(E(0)) \tag{1.35}$$

and $\sup_{t\in[0,T_\sigma]} |\psi(t)| < 10 < H$, where $P(\cdots)$ is a generic polynomial in its arguments. The energy $E(t)$ is defined to be

$$\begin{aligned} E(t) &:= E_4(t) + E_5(t) + E_6(t) + E_7(t) + E_8(t), \\ E_{4+l}(t) &:= \sum_{\pm}\sum_{\langle\alpha\rangle=2l}\sum_{k=0}^{4-l} \left\| \left(\varepsilon^{2l}\mathcal{T}^\alpha \partial_t^k \left(v^\pm, b^\pm, S^\pm, (\mathcal{F}_p^\pm)^{\frac{(k+\alpha_0-l-3)_+}{2}} p^\pm \right)\right) \right\|_{4-k-l,\pm}^2 \\ &\quad + \sum_{k=0}^{4+l} \left| \sqrt{\sigma}\varepsilon^{2l}\partial_t^k\psi \right|_{5+l-k}^2 \qquad 0 \le l \le 4, \end{aligned} \tag{1.36}$$

where $k_+ := \max\{k, 0\}$ for $k \in \mathbb{R}$ and we denote $\mathcal{T}^\alpha := (\omega(x_3)\partial_3)^{\alpha_4}\partial_t^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}$ to be a high-order tangential derivative for the multi-index $\alpha = (\alpha_0, \alpha_1, \alpha_2, 0, \alpha_4)$ with length (for the anisotropic Sobolev spaces) $\langle\alpha\rangle = \alpha_0 + \alpha_1 + \alpha_2 + 2\times 0 + \alpha_4$. The quantity ε is the parameter defined in (1.22). Moreover, the $H^{9.5}(\Sigma)$-regularity of ψ can be recovered in the sense that

$$\sum_{l=0}^{4}\sum_{k=0}^{3+l} \left|\sigma\varepsilon^{2l}\partial_t^k\psi\right|_{5.5+l-k}^2 \le P(E(t)), \quad \forall t \in [0, T_\sigma]. \tag{1.37}$$

Remark 1.2 (Correction of $E_4(t)$)**.** The norm $\|p^\pm\|_{4,\pm}^2$ in $E_4(t)$ defined by (1.36) should be replaced by $\|(\mathcal{F}_p^\pm)^{\frac{1}{2}} p^\pm\|_{0,\pm}^2 + \|\nabla p^\pm\|_{3,\pm}^2$ because we do not have L^2 estimates of $p^\pm$ without $\mathcal{F}_p^\pm$-weight. We still write $\|p^\pm\|_{4,\pm}^2$ as above for simplicity of notations.

Remark 1.3 (Weights of Mach number of $p^\pm$)**.** In (1.36), the weight of Mach number of p is slightly different from (v, b, S), but such difference only occurs when $\mathcal{T}^\alpha$ are full time derivatives and $k = 4 - l$. In fact, due to $k \le 4 - l$ and $\alpha_0 \le \langle\alpha\rangle = 2l$, we know $(k + \alpha_0 - l - 3)_+$ is always equal to zero unless $\alpha_0 = 2l$ and $k = 4 - l$ simultanously hold.

Remark 1.4 (Relations with anisotropic Sobolev space)**.** The energy functional $E(t)$ above is considered as a variant of $\|\cdot\|_{8,*,\pm}$ norm at time $t > 0$. For different multi-index α, we set suitable weights of Mach number according to the number of tangential derivatives that appear in ∂_*^α, such that the energy estimates for the modified norms are uniform in ε.

Remark 1.5 (Nonlinear structural stability)**.** System (1.33) is studied in a bounded domain $\mathbb{T}^2 \times (-H, H)$. Indeed, our proof also applies to the case of an unbounded domain, such as $\mathbb{T}^2 \times \mathbb{R}_\pm$, $\mathbb{R}^2 \times \mathbb{R}_\pm$, for non-localised initial data $\mathbf{U}_0^\pm$ satisfying $(\mathbf{U}_0^\pm - \underline{\mathbf{U}}^\pm, \psi_0) \in H_*^8(\Omega) \times H^{9.5}(\Sigma)$ where $\underline{\mathbf{U}}^\pm$ represents a given piecewise-smooth background solution of planar current-vortex sheet $(\underline{v}_1^\pm, \underline{v}_2^\pm, 0, \underline{b}_1^\pm, \underline{b}_2^\pm, 0, \underline{p}^\pm, \underline{S}^\pm)^\top$ in $\Omega^\pm$. The result corresponding to this initial data exactly justifies *the existence and the local-in-time nonlinear structural stability* of the piecewise-smooth planar current-vortex sheet $\underline{\mathbf{U}}^\pm$.

1.4.3 Main result 2: Incompressible and zero-surface-tension limits

Next we are concerned with the incompressible limit and the zero-surface-tension limit. For any fixed $\sigma > 0$, the energy estimates obtained in Theorem 1.1 are already uniform in ε. Also, $\|\partial_t(v, b, S)\|_3 + |\psi_t|_{4.5}$ is uniformly bounded in ε. Thus, using compactness argument, we can prove the incompressible limit for current-vortex sheets with surface tension. Specifically, the motion of incompressible current-vortex sheets with surface tension are characterised by the equations of $(\xi^\sigma, w^{\pm,\sigma}, h^{\pm,\sigma})$ with incompressible initial data $(\xi_0^\sigma, w_0^{\pm,\sigma}, h_0^{\pm,\sigma})$ and a transport equation of $\mathfrak{S}^{\pm,\sigma}$:

$$
\begin{cases}
\mathfrak{R}^{\pm,\sigma}(\partial_t + w^{\pm,\sigma}\cdot\nabla^{\Xi^\sigma})w^{\pm,\sigma} - (h^{\pm,\sigma}\cdot\nabla^{\Xi^\sigma})h^{\pm,\sigma} + \nabla^{\Xi^\sigma}\Pi^{\pm,\sigma} = 0 & \text{in } [0,T]\times\Omega,\\
\nabla^{\Xi^\sigma}\cdot w^{\pm,\sigma} = 0 & \text{in } [0,T]\times\Omega,\\
(\partial_t + w^{\pm,\sigma}\cdot\nabla^{\Xi^\sigma})h^{\pm,\sigma} = (h^{\pm,\sigma}\cdot\nabla^{\Xi^\sigma})w^{\pm,\sigma} & \text{in } [0,T]\times\Omega,\\
\nabla^{\Xi^\sigma}\cdot h^{\pm,\sigma} = 0 & \text{in } [0,T]\times\Omega,\\
(\partial_t + w^{\pm,\sigma}\cdot\nabla^{\Xi^\sigma})\mathfrak{S}^{\pm,\sigma} = 0 & \text{in } [0,T]\times\Omega,\\
[\![\Pi^\sigma]\!] = \sigma\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\xi^\sigma}{\sqrt{1+|\overline{\nabla}\xi^\sigma|^2}}\right) & \text{on } [0,T]\times\Sigma,\\
\partial_t\xi^\sigma = w^{\pm,\sigma}\cdot N^\sigma & \text{on } [0,T]\times\Sigma,\\
h^{\pm,\sigma}\cdot N^\sigma = 0 & \text{on } [0,T]\times\Sigma,\\
w_3^\pm = h_3^\pm = 0 & \text{on } [0,T]\times\Sigma^\pm,\\
(w^{\pm,\sigma}, h^{\pm,\sigma}, \mathfrak{S}^{\pm,\sigma}, \xi^\sigma)|_{t=0} = (w_0^{\pm,\sigma}, h_0^{\pm,\sigma}, \mathfrak{S}_0^{\pm,\sigma}, \xi_0^\sigma),
\end{cases}
\tag{1.38}
$$

where $\Xi^\sigma(t,x) = x_3 + \chi(x_3)\xi^\sigma(t,x')$ is the extension of ξ^σ in Ω and $N^\sigma := (-\overline{\partial}_1\xi^\sigma, -\overline{\partial}_2\xi^\sigma, 1)^\top$. The quantity $\Pi^\pm := \bar{\Pi}^\pm + \frac{1}{2}|h^\pm|^2$ represent the total pressure functions for the incompressible equations with $\bar{\Pi}^\pm$ the fluid pressure functions. The quantity $\mathfrak{R}^\pm$ satisfies the evolution equation $(\partial_t + w^{\pm,\sigma}\cdot\nabla^{\Xi^\sigma})\mathfrak{R}^{\pm,\sigma} = 0$ with initial data $\mathfrak{R}_0^{\pm,\sigma} := \rho^{\pm,\sigma}(0, \mathfrak{S}_0^{\pm,\sigma})$.

Denoting $(\psi^{\varepsilon,\sigma}, v^{\pm,\varepsilon,\sigma}, b^{\pm,\varepsilon,\sigma}, \rho^{\pm,\varepsilon,\sigma}, S^{\pm,\varepsilon,\sigma})$ to be the solution of (1.33) indexed by σ and ε, we prove that $(\psi^{\varepsilon,\sigma}, v^{\pm,\varepsilon,\sigma}, b^{\pm,\varepsilon,\sigma}, \rho^{\pm,\varepsilon,\sigma}, S^{\pm,\varepsilon,\sigma})$ converges to $(\xi^\sigma, w^{\pm,\sigma}, h^{\pm,\sigma}, \mathfrak{R}^{\pm,\sigma}, \mathfrak{S}^{\pm,\sigma})$ as $\varepsilon\to 0$ provided the convergence of initial datum.

Theorem 1.2 (**Incompressible limit for fixed** $\sigma > 0$)**.** Fix $\sigma > 0$. Let $(\psi_0^{\varepsilon,\sigma}, v_0^{\pm,\varepsilon,\sigma}, b_0^{\pm,\varepsilon,\sigma}, \rho_0^{\pm,\varepsilon,\sigma}, S_0^{\pm,\varepsilon,\sigma})$ be the initial data of (1.33) for each fixed $(\varepsilon,\sigma)\in\mathbb{R}^+\times\mathbb{R}^+$, satisfying

a. The sequence of initial data $(\psi_0^{\varepsilon,\sigma}, v_0^{\pm,\varepsilon,\sigma}, b_0^{\pm,\varepsilon,\sigma}, \rho_0^{\pm,\varepsilon,\sigma}, S_0^{\pm,\varepsilon,\sigma}) \in H^{9.5}(\Sigma)\times H_*^8(\Omega^\pm)\times H_*^8(\Omega^\pm)\times H_*^8(\Omega^\pm)\times H^8(\Omega^\pm)$ satisfies the compatibility conditions (1.34) up to 7-th order, and $|\psi_0^{\varepsilon,\sigma}|_{L^\infty}\le 1$.
b. $(\psi_0^{\varepsilon,\sigma}, v_0^{\pm,\varepsilon,\sigma}, b_0^{\pm,\varepsilon,\sigma}, S_0^{\pm,\varepsilon,\sigma}) \to (\xi_0^\sigma, w_0^{\pm,\sigma}, h_0^{\pm,\sigma}, \mathfrak{S}_0^{\pm,\sigma})$ in $H^{5.5}(\Sigma)\times H^4(\Omega^\pm)\times H^4(\Omega^\pm)\times H^4(\Omega^\pm)$ as $\varepsilon,\sigma\to 0$.
c. The incompressible initial data satisfies $|[\![\bar{w}_0^\sigma]\!]| > 0$ on Σ, the constraints $\nabla^{\xi_0^\sigma}\cdot h_0^{\pm,\sigma} = 0$ in $\Omega^\pm$, $h^{\pm,\sigma}\cdot N^\sigma|_{\{t=0\}\times\Sigma} = 0$.

Then it holds that

$$
(\psi^{\varepsilon,\sigma}, v^{\pm,\varepsilon,\sigma}, b^{\pm,\varepsilon,\sigma}, S^{\pm,\varepsilon,\sigma}) \to (\xi^\sigma, w^{\pm,\sigma}, h^{\pm,\sigma}, \mathfrak{S}^{\pm,\sigma}),
\tag{1.39}
$$

weakly-* in $L^\infty([0,T_\sigma]; H^{5.5}(\Sigma)\times(H^4(\Omega^\pm))^3)$ and strongly in $C([0,T_\sigma]; H^{5.5-\delta}_{\text{loc}}(\Sigma)\times(H^{4-\delta}_{\text{loc}}(\Omega^\pm))^3)$ after possibly passing to a subsequence, where T_σ is the time obtained in Theorem 1.1.

Remark 1.6 (The "compatibility conditions" for the incompressible problem)**.** For the incompressible problem, there is no need to require the so-called "compatibility conditions" for the initial data, for example [19]. The convergence of compressible data automatically implies the fulfillment of time-differentiated kinematic boundary conditions and the time-differentiated slip conditions at $t = 0$. The time-differentiated jump conditions can also be easily fulfilled by adjusting the boundary values of Π, as the pressure function Π is NOT uniquely determined by the other variables for the incompressible problem.

When taking the limit $\sigma\to 0$, we shall impose suitable stability conditions on Σ to ensure the well-posedness of "$\sigma = 0$"-problem. Assume there exists a constant $\delta_0\in(0,\frac{1}{8})$ such that

$$
\delta_0 \le a^\pm\left|\bar{b}^\mp\times[\![\bar{v}]\!]\right| \le (1-\delta_0)|\bar{b}^+\times\bar{b}^-| \quad \text{on } [0,T]\times\Sigma,
\tag{1.40}
$$

where

$$
a^\pm := \sqrt{\rho^\pm\left(1+\left(\frac{c_A^\pm}{c_s^\pm}\right)^2\right)}
\tag{1.41}
$$

and $c_A^\pm := |b^\pm|/\sqrt{\rho^\pm}$ represents the Alfvén speed (the speed of magnetosonic wave), $c_s^\pm := \sqrt{\partial p^\pm/\partial \rho^\pm}$ represents the sound speed. It should be noted that (1.40) is not an imposed boundary condition for the "$\sigma = 0$"-problem. Instead, it is just a constraint for initial data which can propagate within a short time. In other words, we only need to assume

$$2\delta_0 \leq (a^\pm|_{t=0}) \left|\bar{b}_0^\mp \times [\![\bar{v}_0]\!]\right| \leq (1-2\delta_0)|\bar{b}_0^+ \times \bar{b}_0^-| \quad \text{on } \Sigma. \tag{1.42}$$

Under the stability condition, we can establish the uniform-in-(ε,σ) energy estimates.

Theorem 1.3 (**Uniform-in-(ε,σ) estimates**)**.** Under the hypothesis of Theorem 1.1, if the stability condition (1.40) holds, then there exists a time $T > 0$ only depending on M, such that

$$\sup_{0\leq t\leq T} \widetilde{E}(t) \leq P(\widetilde{E}(0)), \tag{1.43}$$

where $\widetilde{E}(t)$ is defined by

$$\widetilde{E}(t) := \sum_{l=0}^{4} \widetilde{E}_{4+l}(t), \quad \widetilde{E}_{4+l}(t) = E_{4+l}(t) + \sum_{k=0}^{4+l} \left|\varepsilon^{2l}\partial_t^k \psi\right|^2_{4.5+l-k}. \tag{1.44}$$

Denote $(\psi^{\varepsilon,\sigma}, v^{\pm,\varepsilon,\sigma}, b^{\pm,\varepsilon,\sigma}, \rho^{\pm,\varepsilon,\sigma}, S^{\pm,\varepsilon,\sigma})$ to be the solution of (1.33) indexed by σ and ε. Under the stability conditions, we prove that $(\psi^{\varepsilon,\sigma}, v^{\pm,\varepsilon,\sigma}, b^{\pm,\varepsilon,\sigma}, \rho^{\pm,\varepsilon,\sigma}, S^{\pm,\varepsilon,\sigma})$ converges to $(\xi^0, w^{\pm,0}, h^{\pm,0}, \mathfrak{R}^{\pm,0}, \mathfrak{S}^{\pm,0})$ as $\varepsilon,\sigma \to 0$ provided the convergence of initial datum. Here $(\xi^0, w^{\pm,0}, h^{\pm,0}, \mathfrak{R}^{\pm,0}, \mathfrak{S}^{\pm,0})$ represents the solution to incompressible current-vortex sheets system (1.38) with initial data $(\xi_0^0, w_0^{\pm,0}, h_0^{\pm,0}, \mathfrak{S}_0^{\pm,0})$ when $\sigma = 0$.

Corollary 1.4 (**Incompressible and zero-surface-tension limits**)**.** Let $(\psi_0^{\varepsilon,\sigma}, v_0^{\pm,\varepsilon,\sigma}, b_0^{\pm,\varepsilon,\sigma}, \rho_0^{\pm,\varepsilon,\sigma}, S_0^{\pm,\varepsilon,\sigma})$ be the initial data of (1.33) for each fixed $(\varepsilon,\sigma) \in \mathbb{R}^+ \times \mathbb{R}^+$, satisfying

a. The sequence of initial data $(\psi_0^{\varepsilon,\sigma}, v_0^{\pm,\varepsilon,\sigma}, b_0^{\pm,\varepsilon,\sigma}, S_0^{\pm,\varepsilon,\sigma}) \in H^{9.5}(\Sigma)\times H_*^8(\Omega^\pm)\times H_*^8(\Omega^\pm)\times H_*^8(\Omega^\pm)$ satisfies the hypothesis of Theorem 1.1.
b. $(\psi_0^{\varepsilon,\sigma}, v_0^{\pm,\varepsilon,\sigma}, b_0^{\pm,\varepsilon,\sigma}, S_0^{\pm,\varepsilon,\sigma}) \to (\xi_0^0, w_0^{\pm,0}, h_0^{\pm,0}, \mathfrak{S}_0^{\pm,0})$ in $H^{5.5}(\Sigma)\times H^4(\Omega^\pm)\times H^4(\Omega^\pm)\times H^4(\Omega^\pm)$ as $\varepsilon,\sigma \to 0$.
c. The incompressible initial data satisfies $|[\![\bar{w}_0^0]\!]| > 0$ on Σ, the constraints $\nabla^{\xi_0}\cdot h_0^{0,\pm} = 0$ in $\Omega^\pm$, $h^{0,\pm}\cdot N^0|_{\{t=0\}\times(\Sigma\cup\Sigma^\pm)} = 0$, the stability condition

$$2\delta_0 \leq \sqrt{\mathfrak{R}_0^{\pm,0}} \left|\bar{h}_0^{\mp,0} \times [\![\bar{w}_0^0]\!]\right| \leq (1-2\delta_0)|\bar{h}_0^{+,0} \times \bar{h}_0^{-,0}| \quad \text{on } \Sigma, \tag{1.45}$$

where $\delta_0 > 0$ is the same constant as in (1.40).

Then it holds that

$$(\psi^{\varepsilon,\sigma}, v^{\pm,\varepsilon,\sigma}, b^{\pm,\varepsilon,\sigma}, S^{\pm,\varepsilon,\sigma}) \to (\xi^0, w^{\pm,0}, h^{\pm,0}, \mathfrak{S}^{\pm,0}), \tag{1.46}$$

weakly-* in $L^\infty([0,T]; H^{4.5}(\Sigma)\times(H^4(\Omega^\pm))^3)$ and strongly in $C([0,T]; H^{4.5-\delta}_{\text{loc}}(\Sigma)\times(H^{4-\delta}_{\text{loc}}(\Omega^\pm))^3)$ after possibly passing to a subsequence. Here $T > 0$ is the time obtained in Theorem 1.3.

Remark 1.7 (Stability conditions in 2D)**.** When taking the zero-surface-tension limit, the stability condition for compressible current-vortex sheets in 2D is

$$\left(\frac{|b_1^+|}{a^+} + \frac{|b_1^-|}{a^-}\right) \geq (1+\delta_0)|\,[\![v_1]\!]\,| > 0 \quad \text{on } [0,T]\times\Sigma, \tag{1.47}$$

which is again propagated by the initial constraint

$$\left.\left(\frac{|b_1^+|}{a^+} + \frac{|b_1^-|}{a^-}\right)\right|_{t=0} \geq (1+2\delta_0)|\,[\![v_{01}]\!]\,| > 0 \quad \text{on } \Sigma,. \tag{1.48}$$

The corresponding stability condition for the incompressible data is

$$\left(\frac{|h_{01}^{+,0}|}{\sqrt{\mathfrak{R}_0^{+,0}}} + \frac{|h_{01}^{-,0}|}{\sqrt{\mathfrak{R}_0^{-,0}}}\right) \geq (1+2\delta_0)\left|[\![w_{01}^0]\!]\right| > 0. \tag{1.49}$$

1.4.4 Main result 3: Dropping redundant assumptions on initial data for the incompressible limit

The uniform-in-ε estimates obtained in Theorem 1.1 and Theorem 1.3 require $\nabla^\varphi \cdot v_0 = O(\varepsilon^2)$ and $\partial_t^k v|_{t=0} = O(1)$ for $k \leq 4$. Such assumption is much stronger than the widely used definition of "well-prepared" initial data (cf. [58, Chap. 2.4] and [62, 69]), that is, $\nabla^\varphi \cdot v_0 = O(\varepsilon)$ and $\partial_t v|_{t=0} = O(1)$. When $[\![\rho]\!] = O(\varepsilon)$ on the interface Σ, we can still prove the incompressible limit under the assumption $\nabla^\varphi \cdot v_0 = O(\varepsilon)$, $\partial_t v|_{t=0} = O(1)$ without any boundedness assumptions on higher-order time derivatives. However, the energy functional should also be modified. We define

$$\mathfrak{E}(t) := \mathfrak{E}_4(t) + E_5(t) + E_6(t) + E_7(t) + E_8(t) \tag{1.50}$$
$$\widetilde{\mathfrak{E}}(t) := \widetilde{\mathfrak{E}}_4(t) + \widetilde{E}_5(t) + \widetilde{E}_6(t) + \widetilde{E}_7(t) + \widetilde{E}_8(t) \tag{1.51}$$

where $\mathfrak{E}_4(t)$ is defined as the following

$$\begin{aligned}\mathfrak{E}_4(t) = \sum_\pm \left\|(v^\pm, b^\pm, p^\pm)\right\|_{4,\pm}^2 + \left\|\partial_t(v^\pm, b^\pm, \varepsilon p^\pm)\right\|_{3,\pm}^2 + \sum_{k=2}^4 \left\|\varepsilon\partial_t^k \left(v^\pm, b^\pm, (\mathcal{F}_p^\pm)^{\frac{(k-3)_+}{2}} p^\pm\right)\right\|_{4-k,\pm}^2 \\ + \left|\sqrt{\sigma}\psi\right|_5^2 + \left|\sqrt{\sigma}\partial_t\psi\right|_4^2 + \sum_{k=2}^4 \left|\sqrt{\sigma}\varepsilon\partial_t^k\psi\right|_{5-k}^2,\end{aligned} \tag{1.52}$$

and

$$\widetilde{\mathfrak{E}}_4(t) = \mathfrak{E}_4(t) + |\psi|_{4.5}^2 + |\partial_t\psi|_{3.5}^2 + \left|\partial_t^2\psi\right|_{2.5}^2 + \left|\varepsilon\partial_t^3\psi\right|_{1.5}^2 + \left|\varepsilon\partial_t^4\psi\right|_{0.5}^2. \tag{1.53}$$

Theorem 1.5 (**Improved uniform estimates**)**.** Assume the fluids in $\Omega^\pm$ are isentropic and the initial density functions satisfy $|[\![\rho_0]\!]|_{1.5} \leq C_0\varepsilon$ on Σ for some $C_0 > 0$. Under the hypothesis of Theorem 1.1, the assumption $\mathfrak{E}(0) \leq M'$ for some constant $M' > 0$, there exists $T'_\sigma > 0$ depending only on M' and σ^{-1} such that the solution $(v^\pm(t), b^\pm(t), \rho^\pm(t), \psi(t))$ to system (1.33) verifies the uniform-in-ε energy estimate

$$\sup_{t\in[0,T'_\sigma]} \mathfrak{E}(t) \leq C(\sigma^{-1})P(\mathfrak{E}(0)). \tag{1.54}$$

Furthermore, under the stability condition (1.40) and $\widetilde{\mathfrak{E}}(0) \leq M'$, there exists $T' > 0$ depending only on M' such that such that the solution $(v^\pm(t), b^\pm(t), \rho^\pm(t), \psi(t))$ to system (1.33) verifies the uniform-in-(ε, σ) energy estimate

$$\sup_{t\in[0,T']} \widetilde{\mathfrak{E}}(t) \leq P(\widetilde{\mathfrak{E}}(0)). \tag{1.55}$$

Remark 1.8. Since $\partial_t v|_{t=0} = O(1)$ still remains bounded, the above uniform estimates directly give the same stronge convergence results as in Theorem 1.2 and Corollary 1.4 with the help of the Aubin-Lions compactness lemma. We do not repeat the statement of convergence theorems here. The result is also true for 2D case under the stability condition (1.47).

Remark 1.9 (The smallness assumption on the density jump)**.** The assumption $|[\![\rho_0]\!]|_{1.5} \leq C_0\varepsilon$ on Σ implies that $|[\![\rho(t)]\!]|_{1.5} \leq C_1\varepsilon$ on $[0, T'] \times \Sigma$ for some $C_1 > 0$. To achieve the assumption, one may have to assume the fluids are isentropic and that is why the entropy S is deleted in $\mathfrak{E}$ and $\widetilde{\mathfrak{E}}$. Indeed, taking the incompressible limit yields $[\![\mathfrak{R}]\!] = 0$ on Σ for the incompressible density functions $\mathfrak{R}^\pm$. If the fluids are non-isentropic, then $\mathfrak{R}^\pm$ are not constants and only satisfy $(\partial_t + \bar{w}^\pm \cdot \overline{\nabla})\mathfrak{R}^\pm = 0$ on Σ. Since the vortex sheet problems require $[\![\bar{w}]\!] \neq 0$ on Σ, it is not possible to have $\mathfrak{R}^+(t) = \mathfrak{R}^-(t)$ on Σ even if it holds at $t = 0$.

List of Notations: In the rest of this paper, we sometimes write $\mathcal{T}^k$ to represent a tangential derivative $\mathcal{T}^\alpha$ in $\Omega^\pm$ with order $\langle\alpha\rangle = k$ when we do not need to specify what the derivative $\mathcal{T}^\alpha$ contains. We also list all the notations used in this manuscript.

- $\Omega^\pm := \mathbb{T}^{d-1} \times \{0 < \pm x_d < H\}$, $\Sigma := \mathbb{T}^{d-1} \times \{x_d = 0\}$ and $\Sigma^\pm := \mathbb{T}^{d-1} \times \{x_d = \pm H\}$, $d = 2, 3$.
- $\|\cdot\|_{s,\pm}$: We denote $\|f\|_{s,\pm} := \|f(t,\cdot)\|_{H^s(\Omega^\pm)}$ for any function $f(t, x)$ on $[0, T] \times \Omega^\pm$.
- $|\cdot|_s$: We denote $|f|_s := |f(t,\cdot)|_{H^s(\Sigma)}$ for any function $f(t, x')$ on $[0, T] \times \Sigma$.

- $\|\cdot\|_{m,*}$: For any function $f(t,x)$ on $[0,T]\times\Omega$, $\|f\|_{m,*,\pm}^2 := \sum_{\langle\alpha\rangle\le m} \|\partial_*^\alpha f(t,\cdot)\|_{0,\pm}^2$ denotes the m-th order space-time anisotropic Sobolev norm of f.
- $P(\cdots)$: A generic polynomial with positive coefficients in its arguments;
- $[T,f]g := T(fg) - fT(g)$, and $[T,f,g] := T(fg) - T(f)g - fT(g)$, where T denotes a differential operator and f,g are arbitrary functions.
- $\overline{\partial}$: $\overline{\partial} = \partial_1,\cdots,\partial_{d-1}$ denotes the spatial tangential derivative.
- $A \stackrel{L}{=} B$: A is equal to B plus some lower-order terms that are easily controlled.

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2 Strategy of the proof

Before going to the detailed proof, we would like to briefly introduce the strategies to tackle this complicated problem. We will decompose the problem into the following parts:

1. Uniform-in-ε estimates for one-phase compressible ideal MHD in a fixed domain with boundary.
2. Generalization to the free-boundary setting by using Alinhac good unknowns.
3. Analysis of three crucial terms that contributes to the boundary regularity, shows a cancellation structure to reach the incompressible limit and exhibits the crucial difficulty caused by the tangential velocity jump in vortex sheets respectively.
4. The stability conditions ensure the estimates to be uniform in σ.
5. Design an appropriate approximate system to prove the local well-posedness without using Nash-Moser or tangential smoothing.

Moreover, we will make comparison between the compressible problem and the incompressible problem, bewteen the Lagrangian coordinates and the "flattened coordinates", among the vortex sheet problem, the one-phase problem and the MHD contact discontinuity.

2.1 Uniform estimates for one-phase MHD flows in a fixed domain

First, let us temporarily forget about the free-boundary setting and recall how to derive uniform estimates in Mach number for the one-phase problem of (1.1) in the fixed domain $\Omega = \mathbb{T}^2\times(-H,H)$ with the slip conditions $u_3 = B_3 = 0$ on $\partial\Omega$ in the preparatory work [85] by Wang and the author.

2.1.1 Div-Curl analysis: a hidden structure of Lorentz force

The entropy is easy to control thanks to $D_t\mathfrak{s} = 0$, so it suffices to analyze the relations between (u,B) and $Q := P + \frac{1}{2}|B|^2$. Using div-curl decomposition, we shall prove the H^3-estimates for the divergence part and the curl part in order to control $\|u,B\|_4$. The divergence part is reduced to the tangential derivatives $\|\mathcal{F}_p D_t P\|_3$. To control the curl part, we take $\nabla\times$ in the momentum equation and invoke the evolution equation of B to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho|\partial^3(\nabla\times u)|^2 + |\partial^3(\nabla\times B)|^2\,\mathrm{d}x = -\int_\Omega \partial^3\nabla\times(B(\nabla\cdot u))\cdot\partial^3(\nabla\times B)\,\mathrm{d}x + \text{ controllable terms}, \tag{2.1}$$

where we find that there is a normal derivative loss in the term $\partial^3\nabla\times(B(\nabla\cdot u))$. Indeed, invoking $\nabla\cdot u = -\mathcal{F}_p D_t P$, commuting ∇ with D_t and inserting the momentum equation $-\nabla P = \varrho D_t u + B\times(\nabla\times B)$, we find a hidden structure of the Lorentz force $B\times(\nabla\times B)$ that eliminates the normal derivative in the curl operator:

$$\mathcal{F}_p B\times(\partial^3\nabla D_t P) = -\mathcal{F}_p\varrho B\times(\partial^3 D_t^2 u) - \mathcal{F}_p B\times(B\times\partial^3 D_t(\nabla\times B)) + \text{ lower order terms},$$

in which the second term contributes to an energy term $-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathcal{F}_p|B\times(\partial^3\nabla\times B)|^2\,\mathrm{d}x$ plus controllable remainder terms. Thus, the vorticity analysis for compressible ideal MHD motivates us to **trade one normal derivative (in curl) for two tangential derivatives together with square weights of Mach number, namely**

$\varepsilon^2 D_t^2$. Furthermore, it can be seen that the anisotropic Sobolev spaces defined in Section 1.4.1 should be the appropriate function spaces to study compressible ideal MHD with magnetic fields tangential to the boundary. This structure was first observed by the author and Wang in the recent preparatory work [85] and gives a definitive explanation on the "mismatch" of fucntion spaces for the well-posedness of incompressible MHD (H^m) and compressible MHD (H_*^{2m}): the "anisotropic part", namely the part containing more than m derivatives, must have weight ε^2 or higher power which converges to zero when taking the incompressible limit. The 2D case is also similarly treated and we refer to Section 6.3 for details.

2.1.2 Reduction of pressure: motivation to design the energy functional

We still need to reduce the normal derivative falling on Q (or $P = Q - \frac{1}{2}|B|^2$). To do this, we just need to use $-\nabla Q = \varrho D_t u - (B \cdot \nabla)B$ and the fact that D_t and $B \cdot \nabla$ are both tangential. Repeatedly, all normal derivatives are reduced to tangential derivatives, and the tangential estimates are expected to be parallel to the proof of L^2 energy conservation.

A remaining task is to determine the weights of Mach number assigned on u, B, P when we invoke the momentum equation to reduce ∇P. One thing we already know from the momentum equation is that $\nabla(P + \frac{1}{2}|B|^2) \sim (B \cdot \nabla)B - D_t u$, which suggests that $\partial_t^k \nabla P$ should share the same weights of Mach number as $\partial_t^{k+1} u$. Apart from this, we recall that the L^2 energy conservation shows that $u, B, \sqrt{\mathcal{F}_p} P, \mathfrak{s} \in L^2(\Omega)$, which suggest that $\partial_t^k(u, B, \mathfrak{s})$ should share the same weights of Mach number as $\varepsilon \partial_t^k P$ when doing tangential estimates.

Thus, we can conclude our reduction scheme as follows

a. Using div-curl analysis to reduce any normal derivatives on u, B. In this process, we have $(\nabla \cdot u, \nabla \cdot B) \to \varepsilon^2 \mathcal{T} P$ and $(\nabla \times u, \nabla \times B) \to \varepsilon^2 \mathcal{T}^2 u$, where $\mathcal{T}$ can be any one of the tangential derivatives $\partial_t, \overline{\partial}_1, \overline{\partial}_2, \omega(x_3)\partial_3$.
b. Using the momentum equation to reduce ∇P to $\mathcal{T}(u, B)$ and $\nabla(\frac{1}{2}|B|^2)$ (this term should be further reduced via div-curl analysis).
c. Tangential estimates: When estimating $E_{4+l}(t)$ (defined in (1.36)), $\mathcal{T}^\gamma(u, B)$ is controlled together with $\sqrt{\mathcal{F}_p}\mathcal{T}^\gamma P$ in the estimates of full tangential derivatives, i.e., when $\langle \gamma \rangle = 4 + l$.

Based on the above three properties, we design the energy functional $E(t)$ in (1.36) and we expect to establish uniform-in-ε estimates for this energy functional.

2.2 Analysis in the free-interface setting: Alinhac good unknowns

For the current-vortex sheets problem, one has to take into account of the free-interface motion. The regularity of free interface is unknown a priori, σ-dependent and determined by the solutions. We only focus on the uniform a priori estimates of (1.33) in the following 3 subsections and postpone the solvability of the current-vortex sheets problem to Section 2.4.

2.2.1 The choice of div-curl inequality: different from fixed-domain problems

Compared with the fixed-domain problem, we may not apply the same div-curl inequality which is not appropriate for us to derive the uniform estimates for either $E(t)$ or $\widetilde{E}(t)$, because the kinematic boundary condition $v \cdot N = \psi_t$ introduces one more time derivative. One may alternatively use the following one:

$$\forall s \geq 1, \ \|X\|_s^2 \lesssim C(|\psi|_s, |\overline{\nabla}\psi|_{W^{1,\infty}}) \left(\|X\|_0^2 + \|\nabla^\varphi \cdot X\|_{s-1}^2 + \|\nabla^\varphi \times X\|_{s-1}^2 + \|\overline{\partial}^s X\|_s^2\right), \tag{2.2}$$

Remark 2.1. One may notice that the boundary energy terms in $E(t)$ also depends on σ, which fails when taking vanishing surface tension limit. Indeed, for $0 \leq k \leq 3+l, \ 0 \leq l \leq 4$, one can prove $\varepsilon^{2l}\partial_t^k \psi \in H^{4.5+l-k}(\Sigma)$ without σ-weight under the stability conditions (1.40) or (1.47). See Section 2.3.5 for explanations.

2.2.2 Tangential estimates: Alinhac good unknowns

The div-curl analysis converts all normal derivatives falling on v, b to tangential derivatives. According to the reduction scheme, we need to control $\|\varepsilon^{2l}\mathcal{T}^\alpha \mathcal{T}^\beta \partial_t^k (v, b, \sqrt{\mathcal{F}_p} p, S)\|_0^2$ where $\mathcal{T}^\alpha = (\omega(x_3)\partial_3)^{\alpha_4}\partial_t^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}$ and

α, β, k, l satisfy the following relations

$$\langle \alpha \rangle = 2l,\ \langle \beta \rangle = 4 - l - k,\ 0 \leq k \leq 4 - l,\ 0 \leq l \leq 4 \text{ and } \beta_0 = 0. \tag{2.3}$$

In fact, the $\varepsilon^{2l}\mathcal{T}^\alpha$-part comes from the vorticity analysis for E_{4+l} and the $\mathcal{T}^\beta \partial_t^k$-part comes from the interior tangential derivatives in div-curl inequality (2.2).

When commuting $\mathcal{T}^\gamma$ with ∇^φ, the commutator $[\mathcal{T}^\gamma, \partial_i^\varphi] f$ contains the term $(\partial_3\varphi)^{-1}\mathcal{T}^\gamma \partial_i \varphi \partial_3 f$ whose $L^2(\Omega)$-norm is controlled by $|\mathcal{T}^\gamma \overline{\nabla} \psi|_0$. However, the regularity of ψ obtained in $\mathcal{T}^\gamma$-estimate is $|\sqrt{\sigma}\mathcal{T}^\gamma \overline{\nabla}\psi|_0$, which is σ-dependent. Even if we assume the stability conditions hold when taking the zero-surface-tension limit, Remark 2.1 shows that we still have a 1/2-order derivative loss. To overcome this difficulty, we introduce the Alinhac good unknown method which reveals that the "essential" leading order term in $\mathcal{T}^\gamma(\nabla^\varphi f)$ is not simply $\nabla^\varphi(\mathcal{T}^\gamma f)$, but the covariant derivative of the "Alinhac good unknown" $\mathbf{F}$. Namely, the Alinhac good unknown for a function f with respect to $\mathcal{T}^\gamma$ is defined by $\mathbf{F}^\gamma := \mathcal{T}^\gamma f - \mathcal{T}^\gamma \varphi \partial_3^\varphi f$ and satisfies

$$\mathcal{T}^\gamma \nabla_i^\varphi f = \nabla_i^\varphi \mathbf{F}^\gamma + \mathfrak{C}_i^\gamma(f),\ \ \mathcal{T}^\gamma D_t^\varphi f = D_t^\varphi \mathbf{F}^\gamma + \mathfrak{D}^\gamma(f), \tag{2.4}$$

where $\|\mathfrak{C}_i^\gamma(f)\|_0$ and $\|\mathfrak{D}^\gamma(f)\|_0$ can be directly controlled. Therefore, we can reformulate the $\mathcal{T}^\gamma$-differentiated current-vortex sheets system (1.33) in terms of $\mathbf{V}^{\gamma,\pm}, \mathbf{B}^{\gamma,\pm}, \mathbf{P}^{\gamma,\pm}, \mathbf{Q}^{\gamma,\pm}, \mathbf{S}^{\gamma,\pm}$ (the Alinhac good unknowns of $v^\pm, b^\pm, p^\pm, q^\pm, S^\pm$ in $\Omega^\pm$) with boundary conditions

$$[\![\mathbf{Q}^\gamma]\!] = \sigma \mathcal{T}^\gamma \mathcal{H} - [\![\partial_3 q]\!] \mathcal{T}^\gamma \psi \quad \text{on } [0,T]\times\Sigma, \tag{2.5}$$

$$\mathbf{V}^{\gamma,\pm} \cdot N = \partial_t \mathcal{T}^\gamma \psi + \bar{v}^\pm \cdot \overline{\nabla} \mathcal{T}^\gamma \psi - \mathcal{W}^{\gamma,\pm} \ \text{ on } [0,T]\times\Sigma, \tag{2.6}$$

$$b^\pm \cdot N = 0 \ \text{ on } [0,T]\times\Sigma, \tag{2.7}$$

where the boundary term $\mathcal{W}^{\gamma,\pm}$ is

$$\mathcal{W}^{\gamma,\pm} := (\partial_3 v^\pm \cdot N)\mathcal{T}^\gamma \psi + [\mathcal{T}^\gamma, N_i, v_i^\pm] \tag{2.8}$$

which will be an important role when proving the uniform-in-ε estimates.

Because of (2.4), the reformulated system of Alinhac good unknowns shares the same structure as the original MHD system (1.33). We expect to obtain the $L^2(\Omega)$ estimates of these good unknowns in a similar manner as L^2 energy conservation and then it is easy to obtain the $\mathcal{T}^\gamma$-estimates by using the definition of Alinhac good unknowns. This fact was first observed by Alinhac [5] in the study of rarefaction waves and was applied (implicitly) to the study of free-surface fluids by Christodoulou-Lindblad [20]. See also Masmoudi-Roussét [59] for an explicit formulation that has been widely adopted by related works.

2.3 Crucial terms for boundary regularity, vortex sheets and incompressible limit

Dropping the superscript γ for convenience and applying L^2 estimates to the good unknowns, we get the following equality which includes four major terms

$$\sum_\pm \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\Omega^\pm} \rho^\pm |\mathbf{V}^\pm|^2 + |\mathbf{B}^\pm|^2 + \mathcal{F}_p^\pm |\mathbf{P}^\pm|^2 \,\mathrm{d}\mathcal{V}_t = \text{ ST} + \text{ RT} + \text{ VS} + \sum_\pm (Z^\pm + ZB^\pm) + \cdots \tag{2.9}$$

where $\mathrm{d}\mathcal{V}_t := \partial_3 \varphi \,\mathrm{d}x$. These four major terms are

$$\text{ST} := \varepsilon^{4l} \int_\Sigma \mathcal{T}^\gamma(\sigma\mathcal{H}) \partial_t \mathcal{T}^\gamma \psi \,\mathrm{d}x', \quad \text{RT} := -\varepsilon^{4l}\int_\Sigma [\![\partial_3 q]\!] \mathcal{T}^\gamma \psi \mathcal{T}^\gamma \partial_t \psi \,\mathrm{d}x', \tag{2.10}$$

$$\text{VS} := \varepsilon^{4l} \int_\Sigma \mathcal{T}^\gamma q^- ([\![\bar{v}]\!] \cdot \overline{\nabla}) \mathcal{T}^\gamma \psi \,\mathrm{d}x', \tag{2.11}$$

$$ZB^\pm := \mp \varepsilon^{4l} \int_\Sigma \mathbf{Q}^\pm \mathcal{W}^\pm \,\mathrm{d}x', \quad Z^\pm := -\varepsilon^{4l} \int_{\Omega^\pm} \mathbf{Q}^\pm \mathfrak{C}_i(v_i^\pm) \,\mathrm{d}\mathcal{V}_t. \tag{2.12}$$

2.3.1 Surface tension gives boundary regularity

On the interface Σ, the weight function $\omega(x_3) = 0$, so it remains to consider $\mathcal{T}^\gamma = \partial_t^{k+\alpha_0}\overline{\partial}^{4-l-k+(\alpha_1+\alpha_2)} = \partial_t^{k+\alpha_0}\overline{\partial}^{4+l-(k+\alpha_0)}$. For simplicity of notations, we replace $k+\alpha_0$ by k. It is easy to see that

$$\text{ST} := \varepsilon^{4l}\int_\Sigma \partial_t^k\overline{\partial}^{4+l-k}(\sigma\mathcal{H})\partial_t^{k+1}\overline{\partial}^{4+l-k}\psi\,\mathrm{d}x' = -\frac{\sigma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma \frac{|\varepsilon^{2l}\partial_t^k\overline{\partial}^{4+l-k}\overline{\nabla}\psi|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}^3} + \cdots \tag{2.13}$$

gives the $\sqrt{\sigma}\varepsilon^{2l}$-weighted boundary regularity in $E(t)$. The term RT is supposed to give us boundary regularity $|\varepsilon^{2l}\partial_t^k\psi|_{4+l-k}^2$ without σ-weight provided the Rayleigh-Taylor sign condition $[\![\partial_3 q]\!] \geq c_0 > 0$. However, in the presence of surface tension, we cannot impose the Rayleigh-Taylor sign condition. Thus, we have to use the $\sqrt{\sigma}$-weighted boundary energy, contributed by surface tension, to control RT.

2.3.2 A crucial term for vortex sheets

Let us consider the term VS that exhibits an essential difficulty in the study of vortex sheets.

$$\text{VS} := \varepsilon^{4l}\int_\Sigma \partial_t^k\overline{\partial}^{4+l-k}q^-([\![\bar{v}]\!]\cdot\overline{\nabla})\partial_t^k\overline{\partial}^{4+l-k}\psi\,\mathrm{d}x'. \tag{2.14}$$

The difficulty is that we only have a jump condition for $[\![q]\!]$ but no conditions for $q^\pm$ individually. Thus, when $0\leq k\leq 3+l$, we integrate $\overline{\partial}^{1/2}$ by parts and control $q^\pm$ by using Lemma B.4

$$\text{VS} \leq |\varepsilon^{2l}\partial_t^k\overline{\partial}^{3.5+l-k}q^-|_0|\varepsilon^{2l}([\![\bar{v}]\!]\cdot\overline{\nabla})\partial_t^k\overline{\partial}^{4+l-k}\psi|_{1/2} \leq \|\varepsilon^{2l}\partial_t^k\overline{\partial}^{4+l-k}q^-\|_{0,-}^{1/2}\|\varepsilon^{2l}\partial_t^k\overline{\partial}^{3+l-k}\partial_3 q^-\|_{0,-}^{1/2}|\bar{v}|_2|\varepsilon^{2l}\partial_t^k\psi|_{5.5+l-k}.$$

This indicates us to seek for the control of $|\varepsilon^{2l}\partial_t^k\psi|_{5.5+l-k}$ for $0\leq k\leq 3+l$, which is exactly given by the surface tension. Indeed, the jump condition $\mathcal{H}(\psi) = \sigma^{-1}[\![q]\!]$ and the ellipticity of the mean curvature operator indicates that we can control $|\varepsilon^{2l}\partial_t^k\psi|_{5.5+l-k}$ by $|\sigma^{-1}\varepsilon^{2l}\partial_t^k[\![q]\!]|_{3.5+l-k}$ plus lower-order terms. Thus, **surface tension significantly enhances the regularity of the free interface such that VS is directly controlled.**

Remark 2.2 (**Comparison with one-phase problems and MHD contact discontinuities**)**.** The above estimate of VS term is not uniform in σ as the elliptic estimate is completely contributed by surface tension. This corresponds to the fact that one cannot take the vanishing surface tension limit of vortex sheets for Euler equations as they are usually violently unstable (except the 2D supersonic case [22, 23]). In the absence of surface tension, the term VS loses control even if the Rayleigh-Taylor sign condition holds because the Rayleigh-Taylor sign condition only gives the energy of $|\varepsilon^{2l}\partial_t^k\psi|_{4+l-k}$ which is 1.5-order lower than the desired regularity. *For one-phase problems, the term VS does not appear* because everything in Ω^- is assumed to be vanishing, so the Rayleigh-Taylor sign condition is usually enough to guarantee the well-posedness [81, 49]. *For MHD contact discontinuities, the jump condition* $[\![v]\!] = \vec{0}$ *also eliminates the term VS* and the transversality of magnetic fields automatically give the bound for $|\partial_t^k\psi|_{4-k}$ (cf. Wang-Xin [87]). However, the term VS must appear in the vortex sheet problems due to $|[\![\bar{v}]\!]| > 0$ on Σ. Thus, the appearance of the term VS shows an essential difference from one-phase flow problems and MHD contact discontinuities.

Remark 2.3 (Treatment of full time derivatives)**.** It should be noted that when the tangential derivatives are the full time derivatives $\varepsilon^{2l}\partial_t^{4+l}$, the above analysis is no longer valid as we cannot integrate by part $\partial_t^{1/2}$. Instead, one has to replace one ∂_t by $D_t^{\varphi,-}$ and repeatedly use the Gauss-Green formula, the symmetric structure, the continuity equation. In fact, this is the most difficult step in the proof of uniform estimates and we refer to step 2 in Section 3.4.3 for those rather technical computations.

2.3.3 A crucial cancellation structure for incompressible limit

So far, we still have an interior term and a boundary term to control:

$$Z^\pm + ZB^\pm = -\int_{\Omega^\pm}\mathbf{Q}^\pm\mathfrak{C}_i(v_i^\pm)\,\mathrm{d}\mathcal{V}_t \mp \int_\Sigma \mathbf{Q}^\pm\mathcal{W}^\pm\,\mathrm{d}x'. \tag{2.15}$$

The problem is that we only obtain the regularity for $\sqrt{\mathcal{F}_p^\pm}\mathbf{Q}^\pm$ in tangential estimates, but the first term contains $\mathbf{Q}^\pm$ without ε-weights. When $\mathcal{T}^\gamma$ contains at least one spatial derivative ($\gamma_0 < \langle\gamma\rangle$), one can invoke the momentum equation to replace $\mathcal{T}_i q$ ($i = 1,2,4$) by tangential derivatives of v, b. There is no loss of ε-weight in this process, as only the full time derivatives of $q^\pm$ require one more ε-weight. However, there may be a loss of ε-weight in this term when $\mathcal{T}^\gamma$ only contains time derivatives, e.g., in $\varepsilon^{2l}\partial_t^{4+l}$-estimates for $0 \le l \le 4$. To get rid of this, there is a cancellation structure that is observed by comparing the concrete forms of $\mathcal{W}^\pm$ and $\mathfrak{C}_i(v_i^\pm)$ (see (3.9))

$$
\begin{aligned}
ZB^\pm &= \mp\,\varepsilon^{4l}\int_\Sigma \partial_t^{4+l}q^\pm[\partial_t^{4+l}, N_i, v_i^\pm]\,\mathrm{d}x' + \cdots,\\
Z^\pm &= -\,\varepsilon^{4l}\int_{\Omega^\pm} \partial_t^{4+l}q^\pm\,[\partial_t^{4+l}, \mathbf{N}_i, \partial_3 v_i^\pm]\,\mathrm{d}x + \cdots
\end{aligned}
$$

Using Gauss-Green formula and integrating by parts in ∂_t, it is easy to see that the leading-order part is

$$
\begin{aligned}
ZB^\pm + Z^\pm \overset{\partial_t}{=} {}& \varepsilon^{4l}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm} \partial_3\partial_t^{3+l}q^\pm\,\partial_t^{3+l}v^\pm\cdot\partial_t\mathbf{N}\,\mathrm{d}x\\
&-\varepsilon^{4l}\int_{\Omega^\pm}\partial_3\partial_t^{3+l}q^\pm\partial_t(\partial_t^{3+l}v^\pm\cdot\partial_t\mathbf{N})\,\mathrm{d}x - \varepsilon^{4l}\int_{\Omega^\pm}\partial_t^{3+l}q^\pm\partial_t(\partial_t^{3+l}v^\pm\cdot\partial_3\partial_t\mathbf{N})\,\mathrm{d}x + \cdots
\end{aligned}
\tag{2.16}
$$

where the first term can be controlled by using Young's inequality after integrating in time t and the other two terms can be directly controlled uniformlly in ε because the full time derivatives of q no longer appear. There are also several other terms involving the full time derivatives of $q^\pm$, but they can be directly controlled via delicate calculation and we refer to the author's previous work (jointly with C. Luo) [55, Section 4.6.2] for details. Hence, the problematic terms in (2.15) are controlled uniformly in ε.

2.3.4 Notes on calculations in anisotropic Sobolev space

There is also an important technical difficulty in ideal compressible MHD: we use anisotropic Sobolev spaces, and the normal derivative ∂_3 should be considered as a second-order derivative. Such difficulty is mainly presented in the following two ways: the control of commutators $\mathfrak{C}$ and $\mathfrak{D}$ is more subtle in the analysis of $E_8(t)$ (purely tangential regularity), and the standard Sobolev trace lemma is no longer useful. The latter issue can be resolved by using Lemma B.3 and Lemma B.4, and now we focus on the former one.

For $\mathfrak{C}_i(v_i)$ and $\mathfrak{C}(q)$ in $\mathcal{T}^\gamma$-estimates for $\langle\gamma\rangle = 8$, the problematic terms have the form

$$\mathcal{T}^{\gamma'}(\mathbf{N}_i/\partial_3\varphi)\mathcal{T}^{\gamma-\gamma'}\partial_3 f,\quad f = v_i \text{ or } q,\ i = 1,2,3,\quad \langle\gamma'\rangle = 1.$$

Such terms are part of the commutator $[\mathcal{T}^\gamma, \mathbf{N}_i/\partial_3\varphi, \partial_3 f]$. Since ∂_3 is considered as a 2nd-order derivative when analyzing $E_8(t)$, $\mathcal{T}^{\gamma-\gamma'}\partial_3 f$ cannot be directly controlled by $E_8(t)$. However, for $f = q$ or $v\cdot\mathbf{N}$, one can invoke the momentum equation and the continuity equation to convert this ∂_3 to a tangential derivative. The control of the commutator $\mathfrak{D}(f)$ is much easier, as the only problematic term is $\mathcal{T}^{\gamma'}(v\cdot\mathbf{N} - \partial_t\varphi)(\mathcal{T}^{\gamma-\gamma'}\partial_3^\varphi f)$ with $\langle\gamma'\rangle = 1$. Thanks to $\mathcal{T}^{\gamma'}(v\cdot\mathbf{N} - \partial_t\varphi)|_{\Sigma=0}$, we can still view $\mathcal{T}^{\gamma'}(v\cdot\mathbf{N} - \partial_t\varphi)\partial_3^\varphi$ as a tangential derivative. We refer to Section 3.3.2 for detailed reduction procedures.

Remark 2.4 (Comparison with the Lagrangian setting)**.** In the author's previous paper [49] about the one-phase MHD without surface tension under the setting of Lagrangian coordinates, the "modified Alinhac good unknowns" were introduced to avoid the derivative loss in these commutators, that is, lots of modification terms were added to $\mathbf{F}$ such that the corresponding $\mathfrak{C}(f)$ is L^2-controllable. Those modification terms are necessary when using Lagrangian coordinates but are redundant in the setting of this paper when the free interface is a graph. The precise reason is that, in the Lagrangian setting, the boundary regularity we obtain from tangential estimates has the form $|\overline{\partial}^r\eta\cdot N|_0^2$ where η represents the flow map of v, which is not enough to control the top-order derivatives of the co-factor matrix $A := [\partial\eta]^{-1}$ and the Eulerian normal vector $N = \overline{\partial}\eta\times\overline{\partial}\eta$. In contrast, the setting in this paper allows us to *explicitly express the Eulerian normal vector, the surface tension, the boundary energy in terms of* $\overline{\nabla}\psi$, and we can also *explicitly write the normal derivative of the "non-characteristic variables"* $(q, v\cdot\mathbf{N})$ *in terms of tangential derivatives of the other quantities.*

2.3.5 Zero-surface-tension limit under the stability conditions

Now, it remains to discuss the zero-surface-tension limit under the stability condition (1.40) or (1.47) and we take the 3D case for an example. Roughly speaking, the condition (1.40), namely

$$|\bar{b}^+ \times \bar{b}^-| \geq \delta_0 \text{ and } a^\pm \left|\bar{b}^\mp \times [\![\bar{v}]\!]\right| \leq (1-\delta_0)|\bar{b}^+ \times \bar{b}^-| \text{ on } [0,T]\times\Sigma, \ a^\pm := \sqrt{\rho^\pm\left(1+\left(c_A^\pm/c_s^\pm\right)^2\right)},$$

brings the following two benefits that Euler equations do not enjoy:

a. Enhance the regularity of the free surface to $H^{s+\frac{1}{2}}(\Sigma)$ in $\overline{\partial}^s$-estimates (possibly with suitable ε-weights). This gives 1/2-order higher regularity of ψ than the Rayleigh-Taylor sign condition does.
b. **Completely eliminates the problematic term VS** for vortex sheets problem.

The enhanced regularity in (a) is easy to prove, as the "non-collinearity" allows us to resolve $\overline{\nabla}\psi$ in terms of $b^\pm$ without any derivative. The benefit (b) is rather important. When taking the vanishing surface tension limit, the enhanced regularity obtained in (a) is still not enough to control the term VS. Thus, we would like to completely eliminate the contribution of $[\![\bar{v}]\!]$ in the term VS by inserting a suitable term involving the magnetic fields, that is, we want to insert a term $\mu^\pm\bar{b}^\pm$ into VS to get

$$\mathrm{VS}' := \int_\Sigma \mathcal{T}^\gamma q^- \left(\left[\!\!\left[\bar{v}-\mu\bar{b}\right]\!\!\right]\cdot\overline{\nabla}\right)\mathcal{T}^\gamma\psi\,\mathrm{d}x' \tag{2.17}$$

and find suitable functions $\mu^\pm$ such that $\left[\!\!\left[\bar{v}-\mu\bar{b}\right]\!\!\right] = \mathbf{0}$ on Σ. Under the stability condition (1.40), the functions $\mu^\pm$ uniquely exist: $\mu^\pm = (\bar{b}^+\times\bar{b}^-)_3^{-1}(\bar{b}^\mp\times[\![\bar{v}]\!])_3$. To construct the term VS′ from MHD equations (1.33), we shall replace the variable $v^\pm$ in the momentum equation by $v^\pm - \mu^\pm b^\pm$. However, this operation makes the MHD system not symmetric and consequently the energy estimates cannot be closed. To overcome this difficulty, we introduce the "Friedrichs secondary symmetrization" [30], which was first applied to compressible idea MHD by Trakhinin [79], to re-symmetrize the MHD system.

The final step is to determine the range for $\mu^\pm$ such that the energy estimates for the secondary-symmetrized MHD system can be closed. Using the Alinhac good unknowns, the energy for $\mathbf{V},\mathbf{B},\mathbf{P}$ becomes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}\rho^\pm|\mathbf{V}^\pm|^2+|\mathbf{B}^\pm|^2+\mathcal{F}_p^\pm|\mathbf{P}^\pm|^2-2\mu^\pm\rho^\pm\mathbf{V}^\pm\cdot\mathbf{B}^\pm-2\mu^\pm\rho^\pm\mathcal{F}_p^\pm\mathbf{P}^\pm(b^\pm\cdot\mathbf{V}^\pm)\,\mathrm{d}\mathcal{V}_t, \tag{2.18}$$

where $\mathcal{F}_p^\pm = 1/(\rho^\pm(c_s^\pm)^2)$. Thus, we must guarantee the above quadratic form of $(\mathbf{V},\mathbf{B},\mathbf{P})$ to be positive-definite, which is equivalent to guarantee the hyperbolicity. This requires $\mu^\pm$ to satisfy $(\mu^\pm)^2\rho^\pm(1+(c_A^\pm/c_s^\pm)^2) < 1$, which gives the range of $\mu^\pm$ that exactly coincides with the stability condition (1.40).

Remark 2.5 (Stabilization effects on 2D subsonic vortex sheets)**.** In the 2D case, the non-collinearity property no longer holds because the interface is 1D. The functions $\mu^\pm$ still exist but are not unique. For a rectilinear piecewise-smooth background solution $(\pm\underline{v},0,\pm\underline{b},0,\underline{p}^\pm,\underline{S}^\pm)^\top$, condition (1.47) implies that $|\underline{v}|^2 < c_s^2\frac{c_A^2}{c_A^2+c_s^2} < c_s^2$, that is, the background solution must be a subsonic flow, whereas the linear stability only holds for supersonic flow, that is, $|v|/c_s > \sqrt{2}$, for 2D vortex sheets of compressible Euler equations. Thus, sufficiently strong magnetic fields have stabilization effects on 2D subsonic vortex sheets. However, this range is only a subset of the subsonic zone for the linear neutral stability obtained in Wang-Yu [86]. It still remains open to justify the nonlinear stability *in the whole domain* for the linear stability obtained in Wang-Yu [86].

2.4 A robust method to solve the compressible vortex sheets problem

As pointed out in a series of the author's previous works [54, 91, 92, 35, 55], the local existence for inviscid fluids is not a direct consequence of the a priori estimates without loss of regularity. There is a loss of one tangential spatial derivative in ψ arising from the analogues of ST and RT terms when doing the Picard iteration. Besides, due to the presence of surface tension and compressibility, one has to control the full time derivatives of v,b,p,S which only belong to $L^2(\Omega^\pm)$ and their boundary regularity is unknown due to the failure of trace lemma. The delicate cancellation structures for the original nonlinear problem (1.33) no longer exist for the linearized problem. Therefore, we shall enhance the regularity of ψ in both tangential spatial variables x' and the time variable t. Our method is to introduce a nonlinear approximate problem (3.1), indexed by $\kappa > 0$, for (1.33) by adding two regularization terms to the jump condition. Namely, the "regularized" jump condition for q is

$$[\![q]\!] = \sigma\mathcal{H} - \kappa(1-\overline{\Delta})^2\psi - \kappa(1-\overline{\Delta})\partial_t\psi, \tag{2.19}$$

where $\overline{\Delta} := \overline{\partial}_1^2 + \overline{\partial}_2^2$ is the tangential Laplacian operator on Σ.

These two regularization terms help us to get $\sqrt{\kappa}$-weighted enhanced regularity for both ψ and ψ_t which is enough for us to compensate the loss of derivatives in the Picard iteration process. So, we can solve the nonlinear approximate problem for each fixed $\kappa > 0$. As for the uniform-in-κ estimates for the nonlinear approximate problem, the appearance of these two regularization terms will not introduce any uncontrollable terms with the help of some delicate technical modifications. In particular, the term VS remains the same as (2.14), and the elliptic estimate for $[\![q]\!]$ in Section 2.3.2 is still vaild and uniform in κ for the approximate problem. Hence, the local existence of (1.33) is proven after passing the limit $\kappa \to 0$.

There are mainly two methods to prove the existence in previous related works

1. **Nash-Moser iteration**. Although there may be some derivative loss for the linearized problem, the order of regularity loss is a fixed number, so one can use Nash-Moser iteration to prove the local existence of smooth solution or solution in Sobolev spaces with a loss of regularity from initial data to solution (cf. [23, 80, 72, 13, 81, 82, 84]).
2. **Tangential smoothing**. This method has been widely used in the study of free-surface inviscid fluids in Lagrangian coordinate [19, 36, 54, 91, 92, 35]. In the paper [55], Luo and the author first introduced the tangential smoothing scheme for Euler equations in the "flattened coordinate". However, the constraint $b\cdot N|_\Sigma = 0$ no longer propagates from the initial data after doing tangential smoothing on N.

Remark 2.6. We choose the "flattened coordinate" because of the reasons mentioned in Remark 2.4. It should be noted that the design of the linearized problem and the Picard iteration process in the "flattened coordinate" is much more difficult than in the Lagrangian coordinate because one has to "define" the free surface in each step of the iteration, whereas the free surface is not explicitly computed and the flow map η is completely determined by the velocity in Lagrangian coordinates.

Compared with these two methods that have been widely used in previous works, none of the above difficulties appear in our new approximation scheme. The estimates obtained in our paper have no loss of regularity and are uniform in Mach number, and are also uniform in the surface tension coefficient under suitable stability conditions. Hence, **we believe that the approximation scheme is a robust method to prove the local existence (and the incompressible limit) for a large class of free-boundary problems in inviscid fluids, especially the vortex sheets problem with surface tension.** Furthermore, taking zero-surface-tension limit seems to be an alternative way, other than Nash-Moser iteration, to prove the local existence of compressible vortex sheets problems under certain stability conditions.

2.5 Dropping redundant assumptions on the prepared initial data

As stated in Section 1.4.4, when ε is suitably small, the incompressible limit can be established under the assumption $\nabla^\varphi\cdot v_0 = O(\varepsilon)$, $\partial_t v|_{t=0} = O(1)$ without any redundant restrictions on higher-order time derivatives. This is not difficult under the fixed-domain setting by adding $\varepsilon^{(k-1)_+}$ to ∂_t^k-differentiated variables (cf. [85]), but for the free-boundary problems, we need to use the energy defined in (1.50)-(1.53) and new essential difficulties caused by the free-interface motion will appear. Let us consider $\overline{\partial}^3\partial_t$-estimate (without ε-weight) that arises in $\mathfrak{E}_4(t)$. Following (2.9), we analyze $\mp\int_{\Omega^\pm}\mathbf{Q}^\pm\mathfrak{C}_i(v_i^\pm)\,\mathrm{d}\mathcal{V}_t$ and $\mp\int_{\Omega^\pm}\mathbf{V}_i^\pm\mathfrak{C}_i(q^\pm)\,\mathrm{d}\mathcal{V}_t$, in which $\mathfrak{C}_i(f)$ contains the term $(\partial_3\varphi)^{-1}(\overline{\partial}^2\partial_t\partial_3 f)(\overline{\partial}\partial_i\varphi)$. Thus, we have to control integrals in the following form

$$\int_{\Omega^\pm}(\overline{\partial}^2\partial\partial_t v^\pm)(\overline{\partial}\mathbf{N})(\overline{\partial}^2\partial\partial_t q^\pm)\,\mathrm{d}x,$$

in which the simultaneous appearance of $\overline{\partial}^2\partial\partial_t v$ and $\overline{\partial}^2\partial\partial_t q$ causes a loss of ε-weight. The appearance of such loss of ε-weight is actually necessary when $\mathcal{T}^\gamma v$ ($\gamma_0 < \langle\gamma\rangle$) is assigned with a different ε-weight from that of $\mathcal{T}^\gamma p$, because the normal vector $\mathbf{N}$ may not necessarily absorb a time derivative when $\mathcal{T}^\gamma$ contains both $\overline{\partial}$ and ∂_t. Also, **this difficulty is completely caused by the free-interface motion** because the commutators $\mathfrak{C}(f)$ do not appear in the study of fixed-domain problem.

2.5.1 Improved estimates for double limits: paralinearization of the free-interface motion

According to the setting of $\mathfrak{E}_4$ and $\widetilde{\mathfrak{E}}_4$ in (1.52)-(1.53), we only need to re-consider the estimates of $\|v_t\|_3$ and $\|b_t\|_3$. To avoid interior tangential estimates, we apply the div-curl inequality (B.2) to v_t, b_t and reduce the control of $\|v_t\|_3$ and $\|b_t\|_3$ to their normal traces $|v_t \cdot N|_{2.5}$ and $|b_t \cdot N|_{2.5}$. In view of the boundary conditions, we must seek for other ways to control $|\partial_t^k \psi|_{4.5-k}$ for $0 \leq k \leq 2$ and they must be σ-indepedent when taking the double limits $\varepsilon, \sigma \to 0$. Consider the time-differentiated kinematic boundary condition, which together with the momentum equation gives

$$\rho^\pm \psi_{tt} = -N \cdot \nabla^\varphi q^\pm + (\bar{b}_i^\pm \bar{b}_j^\pm - \rho^\pm \bar{v}_i^\pm \bar{v}_j^\pm)\overline{\partial}_i \overline{\partial}_j \psi + \cdots$$

Motivated by Shatah-Zeng [73], we try to separate the boundary values of $q^\pm$ from the interior contribution of $q^\pm$. Specifically, $q^\pm$ satisfies a two-phase wave equation (we write $\mathcal{F}_p^\pm = \varepsilon^2$ for convenience)

$$\varepsilon^2 (D_t^{\varphi\pm})^2 q^\pm - \Delta^\varphi q^\pm = \varepsilon^2 (D_t^{\varphi\pm})^2 (\frac{1}{2}|b^\pm|^2) + (\partial_i^\varphi v_j^\pm)(\partial_j^\varphi v_i^\pm) - (\partial_i^\varphi b_j^\pm)(\partial_j^\varphi b_i^\pm), \quad \partial_3 q^\pm|_{\Sigma^\pm} = 0, \quad [\![q]\!]|_\Sigma = \sigma \mathcal{H}(\psi).$$

and we introduce the decomposition $q^\pm = q_\psi^\pm + q_w^\pm$ with

$$-\Delta^\varphi q_\psi^\pm = 0 \text{ in } \Omega^\pm, \ \ q_\psi^\pm = q^\pm \text{ on } \Sigma, \ \ \partial_3 q_\psi^\pm = 0 \text{ on } \Sigma^\pm,$$

$$-\Delta^\varphi q_w^\pm = -\varepsilon^2 (D_t^{\varphi\pm})^2 (q^\pm - \frac{1}{2}|b^\pm|^2) + (\partial_i^\varphi v_j^\pm)(\partial_j^\varphi v_i^\pm) - (\partial_i^\varphi b_j^\pm)(\partial_j^\varphi b_i^\pm), \ \ q_w^\pm = 0 \text{ on } \Sigma, \ \ \partial_3 q_w^\pm = 0 \text{ on } \Sigma^\pm.$$

Under this setting, we can write $-N \cdot \nabla^\varphi q^\pm = \pm \mathfrak{N}_\psi^\pm (q^\pm|_\Sigma) - N \cdot \nabla^\varphi q_w^\pm$ where $\mathfrak{N}_\psi^\pm$ represents the Dirichlet-to-Neumann (DtN) operators with respect to $\Omega^\pm$ and ψ (defined in Section 7.2). The traces of $q^\pm$ on Σ can be resolved by inverting the DtN operators, which then gives us the following evolution equation

$$\begin{aligned}(\rho^+ + \rho^-)\partial_t^2 \psi = &\frac{\sigma}{2}\left(\mathfrak{N}_\psi^+ + \mathfrak{N}_\psi^-\right)(\mathcal{H}(\psi)) + \left(\bar{b}_i^+ \bar{b}_j^+ - \rho^+ \bar{v}_i^+ \bar{v}_j^+ + \bar{b}_i^- \bar{b}_j^- - \rho^- \bar{v}_i^- \bar{v}_j^-\right)\overline{\partial}_i \overline{\partial}_j \psi \\ &- N \cdot \nabla^\varphi q_w^+ - N \cdot \nabla^\varphi q_w^- + \cdots \end{aligned} \tag{2.20}$$

Using the paralinearization in Alazard-Burq-Zuily [2, 3] and Alazard-Métivier [4], the principal symbol of the major term $\left(\mathfrak{N}_\psi^+ + \mathfrak{N}_\psi^-\right)(\mathcal{H}(\psi))$ is negative and of the third order. Besides, the stability condition (1.40) ensures the ellipticity of the second term on the right side. The contribution of q_w in this equation is completely reduced to the source term of the wave equation of q_w thanks to $q_w|_\Sigma = 0$. Thus, we can simultaneously obtain the estimates of $|\psi|_{4.5}$, $|\sqrt{\sigma}\psi|_5$ and $|\psi_t|_{3.5}$ by taking a suitable 3.5-th order paradifferential operator in (2.20) and we refer to Section 7.3 for details.

For the control of $|\psi_{tt}|_{2.5}$, it suffices to take ∂_t in (2.20) and take a suitable 2.5-th order paradifferential operator. However, the omitted source term in (2.20), after taking ∂_t, contains $(\mathfrak{N}_\psi^+ - \mathfrak{N}_\psi^-)(\mathfrak{N}_\psi^+ + \mathfrak{N}_\psi^-)^{-1}([\![\rho]\!]\,\psi_{ttt})$ whose $H^{2.5}(\Sigma)$ norm is bounded by $|[\![\rho]\!]|_{1.5}|\partial_t^3\psi|_{1.5}$. In general, we have a loss of ε-weight, as the energy $\mathfrak{E}(t)$ only gives $\partial_t^3 \psi = O(\varepsilon^{-1})$. So, if we additionally require $|[\![\rho]\!]|_{1.5} = O(\varepsilon)$, which is mathematically reasonable according to Remark 1.9, this extra ε-weight could compensate the loss of ε-weight arising in this term. Hence, under the extra assumption $|[\![\rho]\!]|_{1.5} = O(\varepsilon)$, we can control the accelaration of the free interface uniformly in ε without assuming the boundedness of high-order time derivatives of v.

2.5.2 Comparison with the Syrovatskiĭ condition for incompressible MHD

The stability condition used in this paper is

$$a^\pm \left|\bar{b}^\mp \times [\![\bar{v}]\!]\right| < |\bar{b}^+ \times \bar{b}^-| \ \text{ on } \Sigma, \text{ where } a^\pm := \sqrt{\rho^\pm (1 + (c_A^\pm / c_s^\pm)^2)}.$$

Taking the formal incompressible limit $\rho^\pm \to 1$, we get $|\bar{h}^\pm \times [\![\bar{w}]\!]| < |\bar{h}^+ \times \bar{h}^-|$ on Σ. The original Syrovatskiĭ stability condition (cf. Syrovatskiĭ [77] or Landau-Lifshitz-Pitaevskiĭ [44, §71]) for $\rho^\pm = 1$ is

$$|\bar{h}^+ \times [\![\bar{w}]\!]|^2 + |\bar{h}^- \times [\![\bar{w}]\!]|^2 < 2|\bar{h}^+ \times \bar{h}^-|^2 \quad \text{on } \Sigma,$$

which is less restrictive than the above formal incompressible limit of (1.40). We recall that, in the analysis of equation (2.20), the compressibility introduces an extra term $\varepsilon^2 (D_t^\varphi)^2 p$ in the source term of q_w, so we have to prove the interior L^2 estimates of full time derivatives. Since we use the Friedrichs secondary symmetrization to prove the double limits and Ω^+, Ω^- are disconnected domains, it is reasonable to have restrictions for solutions in Ω^+ and Ω^- respectively. Besides, the extra term $\varepsilon^2(D_t^{\varphi})^2 p$ presents a loss of derivative in standard Sobolev spaces and again indicates that one should trade a normal derivative for 2 tangential derivatives together with the square weight of Mach number.

The difference between the two stability conditions is related to the singular nature of incompressible limit: The pressure function for compressible fluids is a variable of a symmetric hyperbolic system satisfying a wave equation (so the interior estimates seem to be necessary) and is uniquely determined together with the fluid density, while the pressure function for incompressible fluids is *not uniquely determined* by the other variables without a Dirichlet-type boundary condition (because the equation of state no longer holds).

Remark 2.7. One can further see such difference from the derivation of these stability conditions. The orginal Syrovatskiĭ condition is obtained via the normal mode analysis *only for the displacement of the interface* which can be explicitly calculated [44, §71]. However, *for the compressible case, one has to take into account of all variables in the interior together with the interface motion*. It is also impossible to explicitly derive a sufficient and necessary condition for the (linear) neutral stability [28, 79]. In spite of this, Fejer [28] pointed out that the condition for neutral stability is more restrictive than the incompressible counterpart for some special cases even if the compressiblility has very slightly effect on the fluid motion. Similar situation also occurs in the case of 2D. That is, the formal incompressible limit of (1.47) is more restrictive than the stability condition (cf. [7]) for 2D incompressible counterpart.

2.5.3 Comparison with one-phase problems

Finally, we briefly discuss the differences between one-phase problems and vortex sheet problems, which are mainly reflected in the study of incompressible limit. Without loss of generality, we assume everything in Ω^- is vanishing for one-phase problems and thus the term VS is vanishing. When $\sigma = 0$, the Rayleigh-Taylor sign condition $\partial_3 q^+ \geq c_0 > 0$ is necessary for the local existence, so the analysis of ST, RT and $Z + ZB$ is parallel to the study of vortex sheet problems. Our framework is then applicable to one-phase problems. A slight difference is that, we may have to start adding ε-weight to $\partial_t^2 q$ instead of $\partial_t q$ in order to control the evolution of the Rayleigh-Taylor sign. We also note that, very recently, Gu-Wang [37] proved the incompressible limit for free-surface Euler equations with heat conduction under the Rayleigh-Taylor sign condition, in which L_t^2-type bound for $\partial_t q$ can be established because the heat conduction contributes to a parabolic part in the system.

When considering the incompressible limit without boundedness assumptions on higher-order time derivatives, the way to add ε-weights is quite different. The reason is that the bad term $(\mathfrak{N}_\psi^+ - \mathfrak{N}_\psi^-)(\mathfrak{N}_\psi^+ + \mathfrak{N}_\psi^-)^{-1}([\![\rho]\!]\, \partial_t^2 \psi)$ on the right side of (2.20) already contains two time derivative. Without the assumption $[\![\rho]\!] = O(\varepsilon)$ on Σ, which actually requires the fluids to be isentropic and the density functions converge to the same constant, there exhibits a loss of ε-weight in the control of ψ_{tt} in general. That is to say, under the assumption $\nabla^{\varphi_0} \cdot v_0 = O(\varepsilon)$, the accelaration of the free interface may be still not uniformly bounded in ε for compressible vortex sheets.

The above problematic term is produced when we invert the DtN operators to resolve the traces $q^\pm|_\Sigma$. For incompressible (current-)vortex sheets, $\rho^\pm$ are constants and $\int_\Sigma \partial_t \psi \, \mathrm{d}x' = \int_\Sigma \partial_t^2 \psi \, \mathrm{d}x' = 0$, so one can directly apply $(\mathfrak{N}_\psi^\pm)^{-1}$ to $(1/\rho^\pm)\mathfrak{N}_\psi^\pm(q^\pm|_\Sigma)$ as in [50, 47] and thus $[\![\rho]\!]\, \partial_t^2 \psi$ no longer appears. For the one-phase problem, the trace of q^+ is already equal to the surface tension, so we do not need to take the inverse of the DtN operators. Such loss of ε-weight never appears in the fixed-domain problems [1, 85], one-phase problems [92, 55] or the incompressible (current-)vortex sheets [75, 50, 47].

3 Uniform estimates of the nonlinear approximate system

Now we introduce the approximate system of (1.33) indexed by $\kappa > 0$.

$$\begin{cases} \rho^\pm D_t^{\varphi\pm} v^\pm - (b^\pm\cdot\nabla^\varphi) b^\pm + \nabla^\varphi q^\pm = 0,\ \ q^\pm = p^\pm + \frac12|b^\pm|^2 & \text{in } [0,T]\times\Omega^\pm,\\ \mathcal{F}_p D_t^{\varphi\pm} p^\pm + \nabla^\varphi\cdot v^\pm = 0 & \text{in } [0,T]\times\Omega^\pm,\\ p^\pm = p^\pm(\rho^\pm,S^\pm),\ \ \mathcal{F}^\pm = \log\rho^\pm,\ \ \mathcal{F}_p^\pm>0,\ \rho^\pm\geq\bar\rho_0>0 & \text{in } [0,T]\times\Omega^\pm,\\ D_t^{\varphi\pm} b^\pm - (b^\pm\cdot\nabla^\varphi)v^\pm + b^\pm\nabla^\varphi\cdot v^\pm = 0 & \text{in } [0,T]\times\Omega^\pm,\\ \nabla^\varphi\cdot b^\pm = 0 & \text{in } [0,T]\times\Omega^\pm,\\ D_t^{\varphi\pm} S^\pm = 0 & \text{in } [0,T]\times\Omega^\pm,\\ [\![q]\!] = \sigma\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}}\right) - \kappa(1-\overline{\Delta})^2\psi - \kappa(1-\overline{\Delta})\partial_t\psi & \text{on } [0,T]\times\Sigma,\\ \partial_t\psi = v^\pm\cdot N & \text{on } [0,T]\times\Sigma,\\ b^\pm\cdot N = 0 & \text{on } [0,T]\times\Sigma,\\ v_3^\pm = b_3^\pm = 0 & \text{on } [0,T]\times\Sigma^\pm,\\ (v^\pm,b^\pm,\rho^\pm,S^\pm,\psi)|_{t=0} = (v_0^{\kappa,\pm},b_0^{\kappa,\pm},\rho_0^{\kappa,\pm},S_0^{\kappa,\pm},\psi_0^\kappa). \end{cases} \tag{3.1}$$

Note that this system is not over-determined: the continuity equation, the evolution equation of $b^\pm$ and the kinematic boundary condition stay unchanged, so one can still prove $\nabla^\varphi\cdot b^\pm = 0$, $b^\pm\cdot N|_\Sigma = 0$ and $b_3^\pm|_{\Sigma^\pm} = 0$ propagates from the initial data.

The energy functional associated with system (3.1) is defined by

$$\begin{aligned} E^\kappa(t) &:= E_4^\kappa(t) + E_5^\kappa(t) + E_6^\kappa(t) + E_7^\kappa(t) + E_8^\kappa(t)\\ E_{4+l}^\kappa(t) &:= \sum_\pm\sum_{\langle\alpha\rangle=2l}\sum_{k=0}^{4-l}\left\|\left(\varepsilon^{2l}\mathcal{T}^\alpha\partial_t^k\left(v^\pm,b^\pm,S^\pm,(\mathcal{F}_p^\pm)^{\frac{(k+\alpha_0-l-3)_+}{2}}p^\pm\right)\right)\right\|_{4-k-l,\pm}^2\\ &\quad + \sum_{k=0}^{4+l}\left|\sqrt{\sigma}\varepsilon^{2l}\partial_t^k\psi\right|_{5+k-l}^2 + \left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^k\psi\right|_{6+k-l}^2 + \int_0^t\left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^{k+1}\psi(\tau)\right|_{5+k-l}^2\,\mathrm{d}\tau, \end{aligned} \tag{3.2}$$

where $0\leq l\leq 4$ and we denote $\mathcal{T}^\alpha := (\omega(x_3)\partial_3)^{\alpha_4}\partial_t^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}$ to be a tangential derivative for the multi-index $\alpha = (\alpha_0,\alpha_1,\alpha_2,0,\alpha_4)$ with length $\langle\alpha\rangle = \alpha_0+\alpha_1+\alpha_2+2\times0+\alpha_4$. The quantity $(k+\alpha_0-l-3)_+ = 1$ only when $\alpha_0 = 2l$ and $k = 4-l$ and it is equal to 0 otherwise.

We aim to establish the a priori estimates of system (3.1) that is uniform in $\kappa > 0$, which allows us taking the limit $\kappa\to 0_+$ to construct the local-in-time solution to the original system (1.33) for fixed $\sigma > 0$. Spefically, we want to prove the following proposition

Proposition 3.1. There exists some $T_\sigma > 0$ independent of κ,ε such that

$$\sup_{0\leq t\leq T_\sigma} E^\kappa(t) \leq C(\sigma^{-1})P(E^\kappa(0)). \tag{3.3}$$

Remark 3.1. The initial data of the approximate system (3.1) is not the same as the initial data of the original system (1.33) because of the different compatibility conditions. The compatibility conditions (up to 7-th order) for system (3.1) are

$$\begin{aligned} \left[\!\left[\partial_t^j q\right]\!\right]\Big|_{t=0} = \partial_t^j\left(\sigma\mathcal{H} - \kappa(1-\overline{\Delta})^2\psi - \kappa(1-\overline{\Delta})\partial_t\psi\right)\Big|_{t=0} &\quad \text{on } \Sigma,\ \ 0\leq j\leq 7,\\ \partial_t^{j+1}\psi|_{t=0} = \partial_t^j(v^\pm\cdot N)|_{t=0} &\quad \text{on } \Sigma,\ \ 0\leq j\leq 7,\\ \partial_t^j v_3^\pm|_{t=0} = 0 &\quad \text{on } \Sigma^\pm,\ \ 0\leq j\leq 7. \end{aligned} \tag{3.4}$$

In Appendix D, we construct the initial data of (3.1) satisfying the compatibility conditions (3.4) that is uniformly bounded in κ and converges to a given initial data of (1.33) satisfying the compatibility conditions (1.34) up to 7-th order.

3.1 L^2 energy conservation

Proposition 3.2. The approximate system (3.1) admits the following conserved quantity: Let

$$
\begin{aligned}
E_0^\kappa(t) := &\sum_\pm \frac{1}{2}\int_{\Omega^\pm} \rho^\pm|v^\pm|^2 + |b^\pm|^2 + 2\mathfrak{P}(\rho^\pm, S^\pm) + \rho^\pm|S^\pm|^2 \,\mathrm{d}\mathcal{V}_t \\
&+ \frac{1}{2}\int_\Sigma \sigma\sqrt{1+|\overline{\nabla}\psi|^2} + \kappa|(1-\overline{\Delta})\psi|^2\,\mathrm{d}x' + \int_0^t\int_\Sigma \kappa|\langle\overline{\partial}\rangle\psi_t|^2\,\mathrm{d}x'\,\mathrm{d}\tau.
\end{aligned}
\tag{3.5}
$$

Then $\frac{\mathrm{d}}{\mathrm{d}t}E_0^\kappa(t) = 0$ with in the lifespan of the solution to (3.1). Here $\langle\overline{\partial}\rangle := \sqrt{1-\overline{\Delta}}$, that is, $\widehat{\langle\overline{\partial}\rangle f}(\xi) = \sqrt{1+|\xi|^2}\hat{f}(\xi)$ in $\mathbb{T}^2$ and $\mathrm{d}\mathcal{V}_t := \partial_3\varphi\,\mathrm{d}x$.

Proof. The proof of L^2 estimate is straightforward. Taking $L^2(\Omega^\pm)$-inner product of v and the first equation in (3.1) and using Reynolds transport formula (A.3), we get

$$
\begin{aligned}
&\sum_\pm \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\Omega^\pm}\rho^\pm|v^\pm|^2\,\mathrm{d}\mathcal{V}_t = \sum_\pm\int_{\Omega^\pm}(\rho^\pm D_t^{\varphi\pm}v^\pm)\cdot v^\pm\,\mathrm{d}\mathcal{V}_t \\
=&\int_\Sigma [\![q]\!]\,\partial_t\psi\,\mathrm{d}x' + \sum_\pm\int_{\Omega^\pm}p^\pm(\nabla^\varphi\cdot v^\pm)\,\mathrm{d}\mathcal{V}_t - \int_{\Omega^\pm}(b^\pm\cdot\nabla^\varphi)v^\pm\cdot b^\pm\,\mathrm{d}\mathcal{V}_t + \int_{\Omega^\pm}\frac{1}{2}|b^\pm|^2(\nabla^\varphi\cdot v^\pm)\,\mathrm{d}\mathcal{V}_t,
\end{aligned}
\tag{3.6}
$$

where the integral on $\Sigma^\pm$ vanishes thanks to the slip conditions. Let $\mathfrak{P}(\rho^\pm, S^\pm) = \int_{\bar\rho_0}^{\rho^\pm}\frac{p^\pm(z,S^\pm)}{z^2}\,\mathrm{d}z$. Then the first integral above together with $D_t^{\varphi\pm}S^\pm = 0$ gives

$$
\int_{\Omega^\pm}p^\pm(\nabla^\varphi\cdot v^\pm)\,\mathrm{d}\mathcal{V}_t = -\int_{\Omega^\pm}\frac{p^\pm}{(\rho^\pm)^2}D_t^{\varphi\pm}\rho^\pm\,\mathrm{d}\mathcal{V}_t = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}\rho^\pm\mathfrak{P}(\rho^\pm)\,\mathrm{d}\mathcal{V}_t.
$$

The boundary term gives $\sqrt{\sigma}$-weighted and $\sqrt{\kappa}$-weighted regularity of ψ and ψ_t. One has

$$
\int_\Sigma [\![q]\!]\,\partial_t\psi\,\mathrm{d}x' = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Sigma\sigma\sqrt{1+|\overline{\nabla}\psi|^2} + \kappa|(1-\overline{\Delta})\psi|^2\,\mathrm{d}x' - \int_\Sigma\kappa|\langle\overline{\partial}\rangle\psi_t|^2\,\mathrm{d}x'.
$$

Then we insert the evolution equation of $b^\pm$ in the third term in (3.6) to get the energy of $b^\pm$.

$$
\begin{aligned}
-\int_{\Omega^\pm}(b^\pm\cdot\nabla^\varphi)v^\pm\cdot b^\pm\,\mathrm{d}\mathcal{V}_t &= -\int_{\Omega^\pm}D_t^{\varphi\pm}b^\pm\cdot b^\pm\,\mathrm{d}\mathcal{V}_t - \int_{\Omega^\pm}|b^\pm|^2(\nabla^\varphi\cdot v^\pm)\,\mathrm{d}\mathcal{V}_t \\
&= -\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\Omega^\pm}|b^\pm|^2\,\mathrm{d}\mathcal{V}_t + \frac{1}{2}\int_{\Omega^\pm}|b^\pm|^2(\nabla^\varphi\cdot v^\pm)\,\mathrm{d}\mathcal{V}_t - \int_{\Omega^\pm}|b^\pm|^2(\nabla^\varphi\cdot v^\pm)\,\mathrm{d}\mathcal{V}_t,
\end{aligned}
$$

where the last two terms exactly cancels with the last term in (3.6). Finally, $D_t^{\varphi\pm}S^\pm = 0$ and the Reynolds transport theorem shows that $\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\Omega^\pm}\rho^\pm|S^\pm|^2\,\mathrm{d}\mathcal{V}_t = 0$. Therefore, we conclude that system (3.1) admits the following conserved quantity

$$
\begin{aligned}
E_0^\kappa(t) := &\sum_\pm \frac{1}{2}\int_{\Omega^\pm} \rho^\pm|v^\pm|^2 + |b^\pm|^2 + 2\mathfrak{P}(\rho^\pm, S^\pm) + \rho^\pm|S^\pm|^2 \,\mathrm{d}\mathcal{V}_t \\
&+ \frac{1}{2}\int_\Sigma \sigma\sqrt{1+|\overline{\nabla}\psi|^2} + \kappa|(1-\overline{\Delta})\psi|^2\,\mathrm{d}x' + \int_0^t\int_\Sigma \kappa|\langle\overline{\partial}\rangle\psi_t|^2\,\mathrm{d}x'\,\mathrm{d}\tau,
\end{aligned}
\tag{3.7}
$$

which can also be inherited to the original current-vortex sheet system (1.33) after taking $\kappa\to 0_+$. □

3.2 Reformulations in Alinhac good unknowns

Let $\mathcal{T}^\gamma := (\omega(x_3)\partial_3)^{\gamma_4}\partial_t^{\gamma_0}\partial_1^{\gamma_1}\partial_2^{\gamma_2}$ be a tangential derivative with $\langle\gamma\rangle = \gamma_0+\gamma_1+\gamma_2+\gamma_4$. We define the Alinhac good unknown of a given function f with respect to $\mathcal{T}^\gamma$ by $\mathbf{F}^\gamma := \mathcal{T}^\gamma f - \mathcal{T}^\gamma\varphi\partial_3^\varphi f$. The good unknown $\mathbf{F}$ satisfies

$$
\mathcal{T}^\gamma\nabla_i^\varphi f = \nabla_i^\varphi\mathbf{F}^\gamma + \mathfrak{C}_i^\gamma(f),\quad \mathcal{T}^\gamma D_t^\varphi f = D_t^\varphi\mathbf{F}^\gamma + \mathfrak{D}^\gamma(f),
\tag{3.8}
$$

where the commutators $\mathfrak{C}_i^\gamma(f)$ and $\mathfrak{D}^\gamma(f)$ are defined by

$$\mathfrak{C}_i^\gamma(f) = (\partial_3^\varphi\partial_i^\varphi f)\mathcal{T}^\gamma\varphi + \left[\mathcal{T}^\gamma, \frac{\mathbf{N}_i}{\partial_3\varphi}, \partial_3 f\right] + \partial_3 f\left[\mathcal{T}^\gamma, \mathbf{N}_i, \frac{1}{\partial_3\varphi}\right] + \mathbf{N}_i\partial_3 f\left[\mathcal{T}^{\gamma-\gamma'}, \frac{1}{(\partial_3\varphi)^2}\right]\mathcal{T}^{\gamma'}\partial_3\varphi$$
$$+ \frac{\mathbf{N}_i}{\partial_3\varphi}[\mathcal{T}^\gamma, \partial_3]f - \frac{\mathbf{N}_i}{(\partial_3\varphi)^2}\partial_3 f[\mathcal{T}^\gamma, \partial_3]\varphi, \quad i = 1,2,3, \tag{3.9}$$

and

$$\mathfrak{D}^\gamma(f) = (D_t^\varphi\partial_3^\varphi f)\mathcal{T}^\gamma\varphi + [\mathcal{T}^\gamma, \bar{v}]\cdot\overline{\partial}f + \left[\mathcal{T}^\gamma, \frac{1}{\partial_3\varphi}(v\cdot\mathbf{N} - \partial_t\varphi), \partial_3 f\right] + \left[\mathcal{T}^\gamma, v\cdot\mathbf{N} - \partial_t\varphi, \frac{1}{\partial_3\varphi}\right]\partial_3 f$$
$$+ \frac{1}{\partial_3\varphi}[\mathcal{T}^\gamma, v]\cdot\mathbf{N}\partial_3 f - (v\cdot\mathbf{N} - \partial_t\varphi)\partial_3 f\left[\mathcal{T}^{\gamma-\gamma'}, \frac{1}{(\partial_3\varphi)^2}\right]\mathcal{T}^{\gamma'}\partial_3\varphi$$
$$+ \frac{1}{\partial_3\varphi}(v\cdot\mathbf{N} - \partial_t\varphi)[\mathcal{T}^\gamma, \partial_3]f + (v\cdot\mathbf{N} - \partial_t\varphi)\frac{\partial_3 f}{(\partial_3\varphi)^2}[\mathcal{T}^\gamma, \partial_3]\varphi \tag{3.10}$$

with $\langle\gamma'\rangle = 1$. Here $\mathbf{N} := (-\overline{\partial}_1\varphi, -\overline{\partial}_2\varphi, 1)^\top$ is the extension of normal vector N in $\Omega^\pm$. The third term on the right side of (3.9) is zero when $i = 3$ because $\mathbf{N}_3 = 1$ is a constant.

Therefore, we can reformulate the $\mathcal{T}^\gamma$-differentiated current-vortex sheets system (3.1) in terms of $\mathbf{V}^{\gamma,\pm}$, $\mathbf{B}^{\gamma,\pm}$, $\mathbf{P}^{\gamma,\pm}$, $\mathbf{S}^{\gamma,\pm}$ (the Alinhac good unknowns of $v^\pm, b^\pm, p^\pm, S^\pm$ in $\Omega^\pm$) as follows

$$\rho^\pm D_t^{\varphi\pm}\mathbf{V}^{\gamma,\pm} - (b^\pm\cdot\nabla^\varphi)\mathbf{B}^{\gamma,\pm} + \nabla^\varphi\mathbf{Q}^{\gamma,\pm} = \mathcal{R}_v^{\gamma,\pm} - \mathfrak{C}^\gamma(q^\pm) \quad \text{in } [0,T]\times\Omega^\pm, \tag{3.11}$$
$$\mathcal{F}_p D_t^{\varphi\pm}\mathbf{P}^{\gamma,\pm} + \nabla^\varphi\cdot\mathbf{V}^{\gamma,\pm} = \mathcal{R}_p^{\gamma,\pm} - \mathfrak{C}_i^\gamma(v_i^\pm) \quad \text{in } [0,T]\times\Omega^\pm, \tag{3.12}$$
$$D_t^{\varphi\pm}\mathbf{B}^{\gamma,\pm} - (b^\pm\cdot\nabla^\varphi)\mathbf{V}^{\gamma,\pm} + b^\pm(\nabla^\varphi\cdot\mathbf{V}^{\gamma,\pm}) = \mathcal{R}_b^{\gamma,\pm} - b^\pm\mathfrak{C}_i^\gamma(v_i^\pm) \quad \text{in } [0,T]\times\Omega^\pm, \tag{3.13}$$
$$\nabla^\varphi\cdot b^\pm = 0 \quad \text{in } [0,T]\times\Omega^\pm, \tag{3.14}$$
$$D_t^{\varphi\pm}\mathbf{S}^{\pm,\alpha} = \mathfrak{D}^\gamma(S^\pm) \quad \text{in } [0,T]\times\Omega^\pm, \tag{3.15}$$

with boundary conditions

$$[\![\mathbf{Q}^\gamma]\!] = \sigma\mathcal{T}^\gamma\mathcal{H} - \kappa\mathcal{T}^\gamma(1-\overline{\Delta})^2\psi - \kappa\mathcal{T}^\gamma(1-\overline{\Delta})\partial_t\psi - [\![\partial_3 q]\!]\mathcal{T}^\gamma\psi \quad \text{on } [0,T]\times\Sigma, \tag{3.16}$$
$$\mathbf{V}^{\gamma,\pm}\cdot N = \partial_t\mathcal{T}^\gamma\psi + \bar{v}^\pm\cdot\overline{\nabla}\mathcal{T}^\gamma\psi - \mathcal{W}^{\gamma,\pm} \quad \text{on } [0,T]\times\Sigma, \tag{3.17}$$
$$b^\pm\cdot N = 0 \quad \text{on } [0,T]\times\Sigma, \tag{3.18}$$
$$b_3^\pm = v_3^\pm = \mathbf{B}_3^\pm = \mathbf{V}_3^\pm = 0 \quad \text{on } [0,T]\times\Sigma^\pm, \tag{3.19}$$

where $\mathcal{R}_v, \mathcal{R}_p, \mathcal{R}_b$ terms consist of the following commutators

$$\mathcal{R}_v^{\gamma,\pm} := [\mathcal{T}^\gamma, b^\pm]\cdot\nabla^\varphi b^\pm - [\mathcal{T}^\gamma, \rho^\pm]D_t^{\varphi\pm}v^\pm - \rho^\pm\mathfrak{D}^\gamma(v^\pm) \tag{3.20}$$
$$\mathcal{R}_p^{\gamma,\pm} := -[\mathcal{T}^\gamma, \mathcal{F}_p^\pm]D_t^{\varphi\pm}p^\pm - \mathcal{F}_p^\pm\mathfrak{D}^\gamma(p^\pm) \tag{3.21}$$
$$\mathcal{R}_b^{\gamma,\pm} := [\mathcal{T}^\gamma, b^\pm]\cdot\nabla^\varphi v^\pm - \mathfrak{D}^\gamma(b^\pm), \tag{3.22}$$

and the boundary term $\mathcal{W}^{\gamma,\pm}$ is

$$\mathcal{W}^{\gamma,\pm} := (\partial_3 v^\pm\cdot N)\mathcal{T}^\gamma\psi + [\mathcal{T}^\gamma, N_i, v_i^\pm], \tag{3.23}$$

Note that $\omega(x_3) = 0$ on $\Sigma\cup\Sigma^\pm$, so all boundary conditions are vanishing when $\gamma_4 > 0$. Thus, $\mathcal{T}^\gamma$ can be written as $\partial_t^{k+\alpha_0}\overline{\partial}^{(4+l)-(k+\alpha_0)}$ on Σ. We can replace $k+\alpha_0$ by k ($0\le k\le 4+l$) in the boundary energy terms.

In the rest of Section 3, we aim to prove the following tangential estimates

Proposition 3.3 (Tangential estimates for the approximate system)**.** For fixed $l \in \{0,1,2,3,4\}$ and any $\delta \in$

(0, 1), the following uniform-in-(κ,ε) energy inequalities hold:

$$\sum_{\pm}\sum_{\langle\alpha\rangle=2l}\sum_{\substack{0\leq k\leq 4-l\\ k+\alpha_0<4+l}}\left\|\left(\varepsilon^{2l}\overline{\partial}^{4-k-l}\mathcal{T}^{\alpha}\partial_t^k(v^\pm,b^\pm,S^\pm,p^\pm)\right)\right\|_{0,\pm}^2$$
$$+\sum_{k=0}^{3+l}\left|\sqrt{\sigma}\varepsilon^{2l}\partial_t^k\psi\right|_{5-k-l}^2+\left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^k\psi\right|_{6+k-l}^2+\int_0^t\left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^{k+1}\psi(\tau)\right|_{5+k-l}^2\,\mathrm{d}\tau$$
$$\lesssim\delta E^\kappa_{4+l}(t)+\sum_{k=0}^{3+l}\left|\varepsilon^{2l}\partial_t^k\psi(0)\right|_{5.5+l-k}^2+P\left(\sigma^{-1},\sum_{j=0}^{l}E^\kappa_{4+j}(0)\right)+P\left(\sum_{j=0}^{l}E^\kappa_{4+j}(t)\right)\int_0^t P\left(\sigma^{-1},\sum_{j=0}^{l}E^\kappa_{4+j}(\tau)\right)\mathrm{d}\tau \tag{3.24}$$

and

$$\sum_{\pm}\sum_{k=0}^{4-l}\left\|\left(\varepsilon^{2l}\partial_t^{4+l}(v^\pm,b^\pm,S^\pm,(\mathcal{F}_p)^{\frac12}p^\pm)\right)\right\|_{4-k-l,\pm}^2+\left|\sqrt{\sigma}\varepsilon^{2l}\partial_t^{4+l}\psi\right|_1^2+\left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^{4+l}\psi\right|_2^2+\int_0^t\left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^{5+l}\psi(\tau)\right|_1^2\,\mathrm{d}\tau$$
$$\lesssim\delta E^\kappa_{4+l}(t)+\left|\varepsilon^{2l}\partial_t^{3+l}\psi(0)\right|_{2.5}^2+P\left(\sigma^{-1},\sum_{j=0}^{l}E^\kappa_{4+j}(0)\right)+P\left(\sum_{j=0}^{l}E^\kappa_{4+j}(t)\right)\int_0^t P\left(\sigma^{-1},\sum_{j=0}^{l}E^\kappa_{4+j}(\tau)\right)\mathrm{d}\tau. \tag{3.25}$$

Here the first inequality represents the case when there are at least one spatial tangential derivatives and the second inequality represents the case of full time derivatives. Moreover, the term $\left|\varepsilon^{2l}\partial_t^k\psi(0)\right|_{5.5+l-k}^2$ on the right side does not appear when $\kappa=0$.

3.3 Tangential estimates: full spatial derivatives

We first study the case when all tangential derivatives are spatial derivatives $\overline{\partial}_1$ and $\overline{\partial}_2$, namely $\gamma_0=\gamma_4=0$ in $\mathcal{T}^\gamma:=(\omega(x_3)\partial_3)^{\gamma_4}\partial_t^{\gamma_0}\partial_1^{\gamma_1}\partial_2^{\gamma_2}$. In view of the definition of $E(t)$ and the div-curl decomposition, we need to prove the L^2 estimates for the $\varepsilon^{2l}\overline{\partial}^{4+l}$-differentiated system $(0\leq l\leq 4)$. We now consider the case $l=0$, that is, the $\overline{\partial}^4$-estimate for the approximate system (3.1) and aim to prove the following estimate

Proposition 3.4. Fix $l\in\{0,1,2,3,4\}$. For the tangential derivative $\mathcal{T}^\gamma=\overline{\partial}^{4+l}$, $(\gamma_0+\gamma_4=0,\ \gamma_1+\gamma_2=4+l)$, the $\varepsilon^{2l}\overline{\partial}^{4+l}$-differentiated approximate system admits the following uniform-in-(κ,ε) estimate: For any $0<\delta<1$

$$\left\|\varepsilon^{2l}\left(\mathbf{V}^{\gamma,\pm},\mathbf{B}^{\gamma,\pm},\mathbf{S}^{\gamma,\pm},\sqrt{\mathcal{F}_p^\pm}\mathbf{P}^{\gamma,\pm}\right)(t)\right\|_0^2+\left|\sqrt{\sigma}\varepsilon^{2l}\overline{\partial}^{4+l}\psi(t)\right|_1^2+\left|\sqrt{\kappa}\varepsilon^{2l}\overline{\partial}^{4+l}\psi(t)\right|_2^2+\int_0^t\left|\sqrt{\kappa}\varepsilon^{2l}\overline{\partial}^{4+l}\partial_t\psi(\tau)\right|_1^2\,\mathrm{d}\tau$$
$$\lesssim\delta E^\kappa_{4+l}(t)+\left|\varepsilon^{2l}\psi_0\right|_{5.5+l}^2+\sum_{j=0}^{l}\int_0^t P(\sigma^{-1},E^\kappa_{4+j}(\tau))\,\mathrm{d}\tau,\quad 0\leq l\leq 4. \tag{3.26}$$

3.3.1 The case $l=0$: $\overline{\partial}^4$-estimates

As stated in Section 2.2, we introduce the Alinhac good unknowns for $\mathcal{T}^\gamma=\overline{\partial}^4$ and drop the script γ for simplicity of notations

$$\mathbf{V}^\pm:=\overline{\partial}^4v^\pm-\overline{\partial}^4\varphi\partial_3^\varphi v^\pm,\ \mathbf{B}^\pm:=\overline{\partial}^4b^\pm-\overline{\partial}^4\varphi\partial_3^\varphi b^\pm,\ \mathbf{P}^\pm:=\overline{\partial}^4p^\pm-\overline{\partial}^4\varphi\partial_3^\varphi p^\pm,\ \mathbf{Q}^\pm:=\overline{\partial}^4q^\pm-\overline{\partial}^4\varphi\partial_3^\varphi q^\pm.$$

Note that we have

$$\mathbf{Q}^\pm=\mathbf{P}^\pm+b\cdot\mathbf{B}^\pm+\underbrace{\sum_{k=1}^{3}c_k\overline{\partial}^kb^\pm\cdot\overline{\partial}^{4-k}b^\pm}_{=:\mathcal{R}_q^{\gamma,\pm}}$$

for some constants $c_k\in\mathbb{N}^*$.

Step 1: Interior energy structure.

We test the equation (3.11) by $\mathbf{V}^\pm$ in $\Omega^\pm$ and integrate by parts to get one boundary term and several interior terms

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}\rho^\pm|\mathbf{V}^\pm|^2\,\mathrm{d}\mathcal{V}_t=\int_{\Omega^\pm}\rho^\pm D_t^{\varphi\pm}\mathbf{V}^\pm\cdot\mathbf{V}^\pm\,\mathrm{d}\mathcal{V}_t\\
=&\int_{\Omega^\pm}(b^\pm\cdot\nabla^\varphi)\mathbf{B}^\pm\cdot\mathbf{V}^\pm\,\mathrm{d}\mathcal{V}_t-\int_{\Omega^\pm}\mathbf{V}^\pm\cdot\nabla^\varphi\mathbf{Q}^\pm\,\mathrm{d}\mathcal{V}_t+\underbrace{\int_{\Omega^\pm}\mathbf{V}^\pm\cdot(\mathcal{R}_v^\pm-\mathfrak{C}(q^\pm))\,\mathrm{d}\mathcal{V}_t}_{=:R_1^\pm}\\
=&-\int_{\Omega^\pm}\mathbf{B}^\pm\cdot(b^\pm\cdot\nabla^\varphi)\mathbf{V}^\pm\,\mathrm{d}\mathcal{V}_t+\int_{\Omega^\pm}b^\pm\cdot\mathbf{B}^\pm(\nabla^\varphi\cdot\mathbf{V}^\pm)\,\mathrm{d}\mathcal{V}_t+\int_{\Omega^\pm}\mathbf{P}^\pm(\nabla^\varphi\cdot\mathbf{V}^\pm)\,\mathrm{d}\mathcal{V}_t\\
&\pm\int_\Sigma\mathbf{Q}^\pm(\mathbf{V}^\pm\cdot N)\,\mathrm{d}x'+R_1^\pm+\int_{\Omega^\pm}\mathcal{R}_q^\pm(\nabla^\varphi\cdot\mathbf{V}^\pm)\,\mathrm{d}\mathcal{V}_t.
\end{aligned}
\tag{3.27}
$$

Invoking the equation (3.13) for the evolution of $\mathbf{B}$ in the first integral above, the energy of $\mathbf{B}^\pm$ is produced.

$$
\begin{aligned}
&-\int_{\Omega^\pm}\mathbf{B}^\pm\cdot(b^\pm\cdot\nabla^\varphi)\mathbf{V}^\pm\,\mathrm{d}\mathcal{V}_t\\
=&-\int_{\Omega^\pm}\mathbf{B}^\pm\cdot D_t^{\varphi\pm}\mathbf{B}^\pm\,\mathrm{d}\mathcal{V}_t-\int_{\Omega^\pm}(\mathbf{B}^\pm\cdot b^\pm)(\nabla^\varphi\cdot\mathbf{V}^\pm)\,\mathrm{d}\mathcal{V}_t+\int_{\Omega^\pm}\mathbf{B}^\pm\cdot\mathcal{R}_b^\pm\,\mathrm{d}\mathcal{V}_t-\int_{\Omega^\pm}(\mathbf{B}^\pm\cdot b^\pm)\mathfrak{C}_i(v_i^\pm)\,\mathrm{d}\mathcal{V}_t\\
=&-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}|\mathbf{B}^\pm|^2\,\mathrm{d}\mathcal{V}_t\underbrace{-\frac{1}{2}\int_{\Omega^\pm}(\nabla^\varphi\cdot v^\pm)|\mathbf{B}^\pm|^2\,\mathrm{d}\mathcal{V}_t+\int_{\Omega^\pm}\mathbf{B}^\pm\cdot\mathcal{R}_b^\pm\,\mathrm{d}\mathcal{V}_t}_{R_2^\pm}\\
&-\int_{\Omega^\pm}(\mathbf{B}^\pm\cdot b^\pm)(\nabla^\varphi\cdot\mathbf{V}^\pm)\,\mathrm{d}\mathcal{V}_t-\int_{\Omega^\pm}(\mathbf{B}^\pm\cdot b^\pm)\mathfrak{C}_i(v_i^\pm)\,\mathrm{d}\mathcal{V}_t,
\end{aligned}
\tag{3.28}
$$

where the first term in the last line is cancelled with the second integral in (3.27), and the analysis of the second term in the last line will be postponed.

The third term in (3.27) produces the energy of $(\mathcal{F}_p^\pm)^{\frac{1}{2}}\mathbf{P}^\pm$ with the help of equation (3.12).

$$
\begin{aligned}
&\int_{\Omega^\pm}\mathbf{P}^\pm(\nabla^\varphi\cdot\mathbf{V}^\pm)\,\mathrm{d}\mathcal{V}_t\\
=&-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}\mathcal{F}_p^\pm(\mathbf{P}^\pm)^2\,\mathrm{d}\mathcal{V}_t\underbrace{-\frac{1}{2}\int_{\Omega^\pm}(D_t^{\varphi\pm}\mathcal{F}_p^\pm+\mathcal{F}_p^\pm\nabla^\varphi\cdot v^\pm)|\mathbf{P}^\pm|^2\,\mathrm{d}\mathcal{V}_t+\int_{\Omega^\pm}\mathbf{P}^\pm\mathcal{R}_p^\pm\,\mathrm{d}\mathcal{V}_t}_{R_3^\pm}-\int_{\Omega^\pm}\mathbf{P}^\pm\mathfrak{C}_i(v_i^\pm)\,\mathrm{d}\mathcal{V}_t.
\end{aligned}
\tag{3.29}
$$

The last term in (3.27) can be controlled by inserting again the continuity equation and integrating $D_t^{\varphi\pm}$ by parts. We have

$$
\begin{aligned}
&\int_{\Omega^\pm}\mathcal{R}_q^\pm(\nabla^\varphi\cdot\mathbf{V}^\pm)\,\mathrm{d}\mathcal{V}_t=-\int_{\Omega^\pm}\mathcal{F}_p^\pm\mathcal{R}_q^\pm D_t^{\varphi\pm}\mathbf{P}^\pm\,\mathrm{d}\mathcal{V}_t+\int_{\Omega^\pm}\mathcal{R}_q^\pm\mathcal{R}_p^\pm\,\mathrm{d}\mathcal{V}_t-\int_{\Omega^\pm}\mathcal{R}_q^\pm\mathfrak{C}_i(v_i^\pm)\,\mathrm{d}\mathcal{V}_t\\
=&-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}\left(\sqrt{\mathcal{F}_p^\pm}\mathcal{R}_q^\pm\right)\left(\sqrt{\mathcal{F}_p^\pm}\mathbf{P}^\pm\right)\mathrm{d}\mathcal{V}_t+\int_{\Omega^\pm}\left(\sqrt{\mathcal{F}_p^\pm}D_t^{\varphi\pm}\mathcal{R}_q^\pm\right)\left(\sqrt{\mathcal{F}_p^\pm}\mathbf{P}^\pm\right)\mathrm{d}\mathcal{V}_t+\int_{\Omega^\pm}\mathcal{R}_q^\pm\mathcal{R}_p^\pm\,\mathrm{d}\mathcal{V}_t\\
&-\int_{\Omega^\pm}\mathcal{R}_q^\pm\mathfrak{C}_i(v_i^\pm)\,\mathrm{d}\mathcal{V}_t,
\end{aligned}
\tag{3.30}
$$

where the first term on the right side is controlled under time integral by

$$
\delta\left\|\sqrt{\mathcal{F}_p^\pm}\mathbf{P}^\pm(t)\right\|_0^2+P(E_4^\kappa(0))+\int_0^t P(E_4^\kappa(\tau))\,\mathrm{d}\tau,\quad\forall 0<\delta\ll 1
$$

and the second term, the third term on the right side can be both controlled by $P(E_4(t))$ via direct computation because $\mathcal{R}_q$ only contains 3-rd order tangential derivative of b.

The entropy is directly bounded by testing the transport equation of $\mathbf{S}^\pm$ with $\mathbf{S}^\pm$ itself

$$
\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\Omega^\pm}\rho^\pm(\mathbf{S}^\pm)^2\,\mathrm{d}\mathcal{V}_t=\int_{\Omega^\pm}\rho^\pm\mathfrak{D}(S^\pm)\,\mathbf{S}^\pm\,\mathrm{d}\mathcal{V}_t\le\|\mathbf{S}^\pm\|_0\|\rho^\pm\|_{L^\infty}\sqrt{E_4^\kappa(t)}.
\tag{3.31}
$$

The remainder terms are controlled by direct computation. For the commutator $\mathfrak{C}, \mathfrak{D}$, we have $\|\mathfrak{C}(f^\pm)\|_{0,\pm} \lesssim C(|\psi|_4)\|f^\pm\|_{0,\pm}$ and $\|\mathfrak{D}(f^\pm)\|_{0,\pm} \lesssim C(|\psi|_4, |\partial_t\psi|_{L^\infty}|)\|f^\pm\|_{0,\pm}$ when $\mathcal{T}^\gamma = \overline{\partial}^4$ by straightforward computation. Note that the initial data is well-prepared in the sense that $\partial_t v|_{t=0} = O(1)$ with respect to Mach number, so there is no loss of ε-weight in $\mathcal{R}_v$ term. We have

$$R_1^\pm + R_2^\pm + R_3^\pm \le P(E_4^\kappa(t)). \tag{3.32}$$

Step 2: The boundary regularity contributed by surface tension.

We denote $Z^\pm := -\int_{\Omega^\pm}(\mathbf{P}^\pm + b^\pm\cdot\mathbf{B}^\pm + \mathcal{R}_q^\pm)\mathfrak{C}_i(v_i^\pm)\,\mathrm{d}\mathcal{V}_t = -\int_{\Omega^\pm}\mathbf{Q}^\pm\mathfrak{C}_i(v_i^\pm)\,\mathrm{d}\mathcal{V}_t$ to be the remaining interior terms presented above which should be controlled together with some boundary terms involving $\mathcal{W}^\pm$. Now we analyze the boundary integral in (3.27). The sum of two boundary integrals can be written as

$$\begin{aligned}
&\int_\Sigma \mathbf{Q}^+(\mathbf{V}^+\cdot N)\,\mathrm{d}x' - \int_\Sigma \mathbf{Q}^-(\mathbf{V}^-\cdot N)\,\mathrm{d}x'\\
=&\int_\Sigma (\overline{\partial}^4 q^+ - \overline{\partial}^4\psi\partial_3 q^+)(\partial_t\overline{\partial}^4\psi + (\bar v^+\cdot\overline{\nabla})\overline{\partial}^4\psi - \mathcal{W}^+)\,\mathrm{d}x'\\
&-\int_\Sigma (\overline{\partial}^4 q^- - \overline{\partial}^4\psi\partial_3 q^-)(\partial_t\overline{\partial}^4\psi + (\bar v^-\cdot\overline{\nabla})\overline{\partial}^4\psi - \mathcal{W}^-)\,\mathrm{d}x'\\
=&\int_\Sigma \overline{\partial}^4\,[\![q]\!]\,\overline{\partial}^4\partial_t\psi\,\mathrm{d}x' + \int_\Sigma \overline{\partial}^4\,[\![q]\!]\,(\bar v^+\cdot\overline{\nabla})\overline{\partial}^4\psi\,\mathrm{d}x' + \int_\Sigma \overline{\partial}^4 q^-([\![\bar v]\!]\cdot\overline{\nabla})\overline{\partial}^4\psi\,\mathrm{d}x'\\
&-\int_\Sigma [\![\partial_3 q]\!]\,\overline{\partial}^4\psi\partial_t\overline{\partial}^4\psi\,\mathrm{d}x' - \int_\Sigma \partial_3 q^+\overline{\partial}^4\psi(\bar v^+\cdot\overline{\nabla})\overline{\partial}^4\psi\,\mathrm{d}x' + \int_\Sigma \partial_3 q^-\overline{\partial}^4\psi(\bar v^-\cdot\overline{\nabla})\overline{\partial}^4\psi\,\mathrm{d}x'\\
&-\int_\Sigma \mathbf{Q}^+\mathcal{W}^+\,\mathrm{d}x' + \int_\Sigma \mathbf{Q}^-\mathcal{W}^-\,\mathrm{d}x'\\
=:&\ \mathrm{ST} + \mathrm{ST}' + \mathrm{VS} + \mathrm{RT} + \mathrm{RT}^+ + \mathrm{RT}^- + ZB^+ + ZB^-.
\end{aligned}\tag{3.33}$$

We will see that the term ST gives the $\sqrt{\sigma}$-weighted boundary regularity (contributed by surface tension) and the $\sqrt{\kappa}$-weighted boundary regularity (contributed by the two regularization terms) which help us control the terms ST′, VS, RT, RT$^\pm$. The terms $ZB^\pm$ will be controlled together with $Z^\pm$ by using Gauss-Green formula. Do note that the slip conditions imply $\mathbf{V}_3^\pm = \mathbf{B}_3^\pm = \overline{\partial}\psi = 0$ on $\Sigma^\pm$, which eliminates all integrals on $\Sigma^\pm$.

Inserting the jump condition $[\![q]\!] = \sigma\mathcal{H} - \kappa(1-\overline{\Delta})^2\psi - \kappa(1-\overline{\Delta})\psi_t$ into the term ST, we get

$$\begin{aligned}
\mathrm{ST} =&\ \sigma\int_\Sigma \overline{\partial}^4\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}}\right)\overline{\partial}^4\partial_t\psi\,\mathrm{d}x' - \int_\Sigma \kappa(1-\overline{\Delta})^2\overline{\partial}^4\psi\overline{\partial}^4\partial_t\psi\,\mathrm{d}x' - \int_\Sigma \kappa(1-\overline{\Delta})\overline{\partial}^4\partial_t\psi\overline{\partial}^4\partial_t\psi\,\mathrm{d}x'\\
=&\ \sigma\int_\Sigma \overline{\partial}^4\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}}\right)\overline{\partial}^4\partial_t\psi\,\mathrm{d}x' - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma \left|\sqrt{\kappa}\langle\overline{\partial}\rangle^2\overline{\partial}^4\psi\right|^2\,\mathrm{d}x' - \int_\Sigma \left|\sqrt{\kappa}\overline{\partial}^4\langle\overline{\partial}\rangle\psi_t\right|^2\,\mathrm{d}x'.
\end{aligned}\tag{3.34}$$

Integrating by parts in the mean curvature term and using

$$\overline{\partial}\left(\frac{1}{|N|}\right) = \frac{\overline{\nabla}\psi\cdot\overline{\nabla}\overline{\partial}\psi}{|N|^3},\quad |N| = \sqrt{1+|\overline{\nabla}\psi|^2},$$

we get

$$\begin{aligned}
&\sigma\int_\Sigma \overline{\partial}^4\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}}\right)\overline{\partial}^4\partial_t\psi\,\mathrm{d}x'\\
=&-\sigma\int_\Sigma \frac{\overline{\partial}^4\overline{\nabla}\psi}{|N|}\cdot\partial_t\overline{\nabla}\overline{\partial}^4\psi\,\mathrm{d}x' + \sigma\int_\Sigma \frac{\overline{\partial}\psi\cdot\overline{\nabla}\overline{\partial}^4\psi}{|N|^3}\overline{\nabla}\psi\cdot\partial_t\overline{\nabla}\overline{\partial}^4\psi\,\mathrm{d}x'\\
&\underbrace{-\sigma\int_\Sigma\left(\left[\overline{\partial}^3, \frac{1}{|N|}\right]\overline{\partial}\overline{\nabla}_i\psi + \left[\overline{\partial}^3, \frac{1}{|N|^3}\right](\overline{\nabla}_k\psi\cdot\overline{\partial}\overline{\nabla}_k\psi\overline{\nabla}_i\psi) - \frac{1}{|N|^3}[\overline{\partial}^3, \overline{\nabla}_i\psi\overline{\nabla}_k\psi]\overline{\partial}\overline{\nabla}_k\psi\right)\cdot\partial_t\overline{\nabla}_i\overline{\partial}^4\psi\,\mathrm{d}x'}_{=:\mathrm{ST}_1^R}
\end{aligned}\tag{3.35}$$

which is further equal to

$$
\begin{aligned}
&-\frac{\sigma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma \frac{|\overline{\partial}^4\overline{\nabla}\psi|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}} - \frac{|\overline{\nabla}\psi\cdot\overline{\partial}^4\overline{\nabla}\psi|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}^3}\,\mathrm{d}x' \\
&+\underbrace{\frac{\sigma}{2}\int_\Sigma \partial_t\left(\frac{1}{|N|}\right)\left|\overline{\partial}^4\overline{\nabla}\psi\right|^2 - \partial_t\left(\frac{1}{|N|^3}\right)\left|\overline{\nabla}\psi\cdot\overline{\nabla}\overline{\partial}^4\psi\right|^2\,\mathrm{d}x'}_{=:\,\mathrm{ST}_2^R} + \mathrm{ST}_1^R
\end{aligned}
\tag{3.36}
$$

The control of $\mathrm{ST}_1^R,\ \mathrm{ST}_2^R$ is straightforward which has been analyzed in the author's previous paper [55, (4.77)-(4.78)], so we only record the result here

$$
\mathrm{ST}_1^R + \mathrm{ST}_2^R \lesssim P(|\overline{\nabla}\psi|_{L^\infty})|\overline{\nabla}\psi|_{W^{1,\infty}}\left|\sqrt{\sigma}\overline{\partial}^4\overline{\nabla}\psi\right|_0\left|\sqrt{\sigma}\partial_t\overline{\partial}^4\psi\right|_0 \le P(E_4^\kappa(t)).
$$

Using Cauchy's inequality

$$
\forall \mathbf{a}\in\mathbb{R}^2,\quad \frac{|\mathbf{a}|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}} - \frac{|\overline{\nabla}\psi\cdot\mathbf{a}|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}^3} \ge \frac{|\mathbf{a}|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}^3},
\tag{3.37}
$$

we obtain the $\sqrt{\sigma}$-weighted boundary regularity

$$
\begin{aligned}
&\int_0^t \mathrm{ST}\,\mathrm{d}\tau + \frac{\sigma}{2}\int_\Sigma \frac{|\overline{\nabla}\overline{\partial}^4\psi|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}^3}\,\mathrm{d}x' + \int_\Sigma\left|\sqrt{\kappa}\langle\overline{\partial}\rangle^2\overline{\partial}^4\psi\right|^2\,\mathrm{d}x' + \int_0^t\int_\Sigma\left|\sqrt{\kappa}\overline{\partial}^4\langle\overline{\partial}\rangle\psi_t\right|^2\,\mathrm{d}x'\,\mathrm{d}\tau \\
\le &\int_0^t \mathrm{ST}_1^R + \mathrm{ST}_2^R\,\mathrm{d}x' \le \int_0^t P(E_4^\kappa(\tau))\,\mathrm{d}\tau.
\end{aligned}
\tag{3.38}
$$

So far, we already obtain the boundary regularity $\sqrt{\sigma}\psi\in H^5(\Sigma)$, $\sqrt{\kappa}\psi\in H^6(\Sigma)$ and $\sqrt{\kappa}\psi_t\in L_t^2H_{x'}^5([0,T]\times\Sigma)$. Using this, we can easily control ST' term in (3.33). Invoking again the boundary condition for $[\![q]\!]$, we get

$$
\mathrm{ST}' = \int_\Sigma \sigma\mathcal{H}(\overline{v}^+\cdot\overline{\nabla})\overline{\partial}^4\psi\,\mathrm{d}x' - \kappa\int_\Sigma(1-\overline{\Delta})^2\overline{\partial}^4\psi\ (\overline{v}^+\cdot\overline{\nabla})\overline{\partial}^4\psi\,\mathrm{d}x' - \kappa\int_\Sigma(1-\overline{\Delta})\overline{\partial}^4\psi_t(\overline{v}^+\cdot\overline{\nabla})\overline{\partial}^4\psi\,\mathrm{d}x'. \tag{3.39}
$$

Integrating by parts $1-\overline{\Delta}$ in the second term and $\langle\overline{\partial}\rangle=\sqrt{1-\overline{\Delta}}$ in the third term above, we can easily use the $\sqrt{\kappa}$-weighted energy to control the last two terms.

$$
\begin{aligned}
&-\kappa\int_\Sigma(1-\overline{\Delta})^2\overline{\partial}^4\psi\ (\overline{v}^+\cdot\overline{\nabla})\overline{\partial}^4\psi\,\mathrm{d}x' \\
=&-\kappa\int_\Sigma\left((1-\overline{\Delta})\overline{\partial}^4\psi\right)(\overline{v}^+\cdot\overline{\nabla})(1-\overline{\Delta})\overline{\partial}^4\psi\,\mathrm{d}x' - \kappa\int_\Sigma\left((1-\overline{\Delta})\overline{\partial}^4\psi\right)\ [1-\overline{\Delta},\overline{v}^+\cdot\overline{\nabla}]\overline{\partial}^4\psi\,\mathrm{d}x',
\end{aligned}
\tag{3.40}
$$

where the first term is controlled by $|\overline{v}^+|_{W^{1,\infty}}|\sqrt{\kappa}(1-\overline{\Delta})\overline{\partial}^4\psi|_0^2$ after integrating $\overline{v}^+\cdot\overline{\nabla}$ by parts and using the symmetry, and the second term is directly controlled by $|\overline{v}^+|_{W^{2,\infty}}|\sqrt{\kappa}(1-\overline{\Delta})\overline{\partial}^4\psi|_0|\sqrt{\kappa}\overline{\partial}^4\psi|_2$. Similarly, we have for any $\delta\in(0,1)$

$$
\begin{aligned}
&-\kappa\int_0^t\int_\Sigma(1-\overline{\Delta})\overline{\partial}^4\psi_t(\overline{v}^+\cdot\overline{\nabla})\overline{\partial}^4\psi\,\mathrm{d}x'\,\mathrm{d}\tau = -\kappa\int_0^t\int_\Sigma\langle\overline{\partial}\rangle\overline{\partial}^4\psi_t\ \langle\overline{\partial}\rangle((\overline{v}^+\cdot\overline{\nabla})\overline{\partial}^4\psi)\,\mathrm{d}x'\,\mathrm{d}\tau \\
\le&\ \delta\left|\sqrt{\kappa}\langle\overline{\partial}\rangle\overline{\partial}^4\psi_t\right|^2_{L_t^2L_{x'}^2} + \frac{1}{4\delta}\int_0^t\int_\Sigma|\overline{v}^+|^2_{W^{1,\infty}}\left|\sqrt{\kappa}\overline{\partial}^4\psi\right|_2^2\,\mathrm{d}x'\,\mathrm{d}\tau \le \delta E_4^\kappa(t) + \int_0^t P(E_4^\kappa(\tau))\,\mathrm{d}\tau.
\end{aligned}
\tag{3.41}
$$

Picking $\delta>0$ to be sufficiently small, the δ-term can be absorbed by $E_4^\kappa(t)$. The first term in ST' is controlled in the same way if we integrating $\overline{\nabla}\cdot$ by parts. Here we only list the result and refer the details to [55, (4.87)-(4.89)]

$$
\int_\Sigma \sigma\mathcal{H}(\overline{v}^+\cdot\overline{\nabla})\overline{\partial}^4\psi\,\mathrm{d}x' \le P(|\overline{\nabla}\psi|_{W^{1,\infty}})|\overline{v}^+|_{W^{1,\infty}}\left|\sqrt{\sigma}\overline{\nabla}\overline{\partial}^4\psi\right|_0^2 \le P(E_4^\kappa(t)). \tag{3.42}
$$

Next we control the terms RT and $\text{RT}^{\pm}$ in (3.33). Note that we do not have the Rayleight-Taylor sign condition $[\![\partial_3 q]\!]|_{\Sigma} \geq c_0 > 0$, so we have to use the $\sqrt{\sigma}$-weighted energy to control these terms, we have

$$\text{RT} \leq |\partial_3 q|_{L^\infty} |\psi|_4 |\psi_t|_4 \leq \sigma^{-1} P(E_4^\kappa(t)). \tag{3.43}$$

Similarly, integrating $\bar{v}^{\pm} \cdot \nabla$ by parts in $\text{RT}^{\pm}$ and using symmetry, the terms $\text{RT}^{\pm}$ can be directly controlled

$$\text{RT}^{\pm} \leq |\bar{v}^{\pm} \partial_3 q|_{W^{1,\infty}} |\psi|_4^2 \leq \sigma^{-1} P(E_4^\kappa(t)). \tag{3.44}$$

Step 3: The crucial term for vortex sheets problem.

Now we study the term VS in (3.33) which appears to be the most problematic term for the vortex sheets problem. Note that we do not have any boundary condition for $q^{\pm}$ individually. Thus, we may alternatively integrate $\overline{\partial}^{1/2}$ by parts and use (B.5) to control VS.

$$\text{VS} = \int_\Sigma \overline{\partial}^4 q^- ([\![\bar{v}]\!] \cdot \overline{\nabla}) \overline{\partial}^4 \psi \, \mathrm{d}x' \lesssim \|\overline{\partial}^4 q^-\|_{0,-}^{\frac{1}{2}} \|\partial_3 \overline{\partial}^3 q^-\|_{0,-}^{\frac{1}{2}} \|\bar{v}^{\pm}\|_{2,\pm} |\overline{\nabla} \overline{\partial}^4 \psi|_{1/2} \leq P(E_4^\kappa(t)) |\psi|_{5.5}, \tag{3.45}$$

where we have used the Kato-Ponce inequality (cf. Lemma B.6) for $s = 1/2,\ p_1 = 2,\ p_2 = \infty,\ q_1 = q_2 = 4$ and Sobolev embedding $H^{1/2}(\mathbb{T}^2) \hookrightarrow L^4(\mathbb{T}^2)$. Now we need to control $|\psi|_{5.5}$ via the jump condition of $[\![q]\!]$. Without the κ-regularization terms, we may use the ellipticity of the mean curvature operator to control $|\psi|_{5.5}$ by $\sigma^{-1} |\,[\![q]\!]\,|_{3.5}$. Now, we can still prove analogous result for the κ-regularized jump condition.

Lemma 3.5 (Elliptic estimate for the free interface)**.** For any $s \geq 0.5$ and $\kappa > 0$, we have the uniform-in-κ estimate

$$|\psi|_{s+1.5} \leq |\psi_0|_{s+1.5} + \sigma^{-1} \left(P(|\overline{\nabla}\psi|_{L^\infty}) |\overline{\nabla}\psi|_{W^{1,\infty}} |\overline{\partial}\psi|_{s-0.5} + |\,[\![q]\!]\,|_{s-0.5} \right).$$

Moreover, when $\kappa = 0$, $|\psi_0|_{s+1.5}$ is not needed

$$|\sigma\psi|_{s+1.5} \leq\ P(|\overline{\nabla}\psi|_{L^\infty}) |\overline{\nabla}\psi|_{W^{1,\infty}} |\sigma\overline{\partial}\psi|_{s-0.5} + |\,[\![q]\!]\,|_{s-0.5}. \tag{3.46}$$

Proof. We take $\langle\overline{\partial}\rangle^{s+0.5}$ in the jump condition to get

$$-\langle\overline{\partial}\rangle^{s+0.5} [\![q]\!] = -\sigma \langle\overline{\partial}\rangle^{s+0.5} \overline{\nabla} \cdot \left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}} \right) + \kappa(1-\overline{\Delta})^2 \langle\overline{\partial}\rangle^{s+0.5}\psi + \kappa(1-\overline{\Delta})\langle\overline{\partial}\rangle^{s+0.5}\psi_t.$$

Testing this equation with $\langle\overline{\partial}\rangle^{s+0.5}\psi$ in $L^2(\Sigma)$, we get

$$-\int_\Sigma \langle\overline{\partial}\rangle^{s+0.5} [\![q]\!]\ \langle\overline{\partial}\rangle^{s+0.5}\psi \, \mathrm{d}x' \leq |\langle\overline{\partial}\rangle^{s-0.5} [\![q]\!]\,|_0 |\langle\overline{\partial}\rangle^{s+1.5}\psi|_0.$$

For the right side, we can mimic the treatment of ST term to obtain the boundary regularity. The two regularization terms can be directly controlled

$$\int_\Sigma \kappa(1-\overline{\Delta})^2 \langle\overline{\partial}\rangle^{s+0.5}\psi\ \langle\overline{\partial}\rangle^{s+0.5}\psi \, \mathrm{d}x' = \int_\Sigma \kappa(1-\overline{\Delta})\langle\overline{\partial}\rangle^{s+0.5}\psi\ (1-\overline{\Delta})\langle\overline{\partial}\rangle^{s+0.5}\psi \, \mathrm{d}x' = \left|\sqrt{\kappa}\psi\right|_{s+2.5}^2,$$

$$\int_\Sigma \kappa(1-\overline{\Delta})\langle\overline{\partial}\rangle^{s+0.5}\psi_t\ \langle\overline{\partial}\rangle^{s+0.5}\psi \, \mathrm{d}x' \overset{\langle\overline{\partial}\rangle}{=} \frac{\mathrm{d}}{\mathrm{d}t} \left|\sqrt{\kappa}\psi\right|_{s+1.5}^2.$$

The term involving surface tension is controlled as follows

$$\begin{aligned}
&-\sigma \int_\Sigma \langle\overline{\partial}\rangle^{s+0.5} \overline{\nabla} \cdot \left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}} \right) \langle\overline{\partial}\rangle^{s+0.5}\psi \, \mathrm{d}x' = \sigma \int_\Sigma \langle\overline{\partial}\rangle^{s+0.5} \left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}} \right) \cdot \langle\overline{\partial}\rangle^{s+0.5} \overline{\nabla}\psi \, \mathrm{d}x' \\
&= \sigma \int_\Sigma \frac{|\langle\overline{\partial}\rangle^{s+0.5}\overline{\nabla}\psi|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}} - \frac{|\overline{\nabla}\psi \cdot \langle\overline{\partial}\rangle^{s+0.5}\overline{\nabla}\psi|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}^3} \, \mathrm{d}x' \\
&\quad + \sigma \int_\Sigma \left(\left[\langle\overline{\partial}\rangle^{s-0.5}, \frac{1}{|N|} \right] \langle\overline{\partial}\rangle \overline{\nabla}_i \psi + \left[\langle\overline{\partial}\rangle^{s-0.5}, \frac{1}{|N|^3} \right] (\overline{\nabla}_k\psi \cdot \langle\overline{\partial}\rangle \overline{\nabla}_k \psi \overline{\nabla}_i \psi) - \frac{1}{|N|^3} [\langle\overline{\partial}\rangle^{s-0.5}, \overline{\nabla}\psi] \langle\overline{\partial}\rangle \overline{\nabla}_i \psi \right) \cdot \overline{\nabla}_i \langle\overline{\partial}\rangle^{s+0.5}\psi \, \mathrm{d}x'
\end{aligned}$$

Using Kato-Ponce commutator estimate (cf. (B.8) in Lemma B.6), the commutators in the last line of the above identity are controlled by $P(|\overline{\nabla}\psi|_{L^\infty})|\overline{\nabla}\psi|_{W^{1,\infty}}|\overline{\partial}\psi|_{s-0.5}$. Using again Cauchy's inequality (3.37), we conclude the elliptic estimate by

$$\sigma|\psi|_{s+1.5}^2+\kappa|\psi|_{s+2.5}^2+\kappa\frac{\mathrm{d}}{\mathrm{d}t}|\psi|_{s+1.5}^2\leq\left(P(|\overline{\nabla}\psi|_{L^\infty})|\overline{\nabla}\psi|_{W^{1,\infty}}|\sigma\overline{\partial}\psi|_{s-0.5}+|\,[\![q]\!]\,|_{s-0.5}\right)|\psi|_{s+1.5}.$$

In particular, Lemma B.7 suggests that we have

$$|\psi|_{s+1.5}\leq|\psi_0|_{s+1.5}+\sigma^{-1}\left(P(|\overline{\nabla}\psi|_{L^\infty})|\overline{\nabla}\psi|_{W^{1,\infty}}|\sigma\overline{\partial}\psi|_{s-0.5}+|\,[\![q]\!]\,|_{s-0.5}\right).$$

Moreover, when $\kappa=0$, $|\psi_0|_{s+1.5}$ no longer appears as we do not need Lemma B.7

$$|\sigma\psi|_{s+1.5}\leq P(|\overline{\nabla}\psi|_{L^\infty})|\overline{\nabla}\psi|_{W^{1,\infty}}|\sigma\overline{\partial}\psi|_{s-0.5}+|\,[\![q]\!]\,|_{s-0.5}. \tag{3.47}$$

□

Now we can easily obtain the control for the problematic term VS by setting $s=4$ in Lemma 3.5

$$\mathrm{VS}\lesssim|\psi_0|_{5.5}+\sigma^{-1}P(E_4^\kappa(t)). \tag{3.48}$$

Step 4: A cancellation structure for the incompressible limit.

It remains to control the term $Z^\pm$ and $ZB^\pm$. In $\overline{\partial}^4$-estimates, each of these terms can be directly controlled. However, in the control of $E_8^\kappa(t)$ and the control of full time derivatives, there will be extra technical difficulties due to the loss of Mach number or the anisotropy of the function spaces. Thus, we would like to present a robust approach to control these terms. We take Z^-+ZB^- as an example and the "+" case is controlled in the same way by reversing the sign when integrating by parts. Recall that $\mathbf{Q}^-=\overline{\partial}^4q^--\overline{\partial}^4\varphi\partial_3^\varphi q^-$, so we have

$$\begin{aligned}ZB^-=&\int_\Sigma\overline{\partial}^4q^-(\partial_3v^-\cdot N)\overline{\partial}^4\psi\,\mathrm{d}x'-\int_\Sigma\overline{\partial}^4\psi\partial_3q^-(\partial_3v^-\cdot N)\overline{\partial}^4\psi\,\mathrm{d}x'\\&+\sum_{k=1}^3\int_\Sigma\binom{4}{k}\mathbf{Q}^-\overline{\partial}^{4-k}v^-\cdot\overline{\partial}^kN\,\mathrm{d}x'.\end{aligned} \tag{3.49}$$

The first two terms in ZB^- can be directly controlled

$$\begin{aligned}&\int_\Sigma\overline{\partial}^4q^-(\partial_3v^-\cdot N)\overline{\partial}^4\psi\,\mathrm{d}x'-\int_\Sigma\overline{\partial}^4\psi\partial_3q^-(\partial_3v^-\cdot N)\overline{\partial}^4\psi\,\mathrm{d}x'\\\leq&\left(|\overline{\partial}^{7/2}q^-|_0|\psi|_{4.5}+|\psi|_4^2\right)|\partial_3v\cdot N|_{1.5}\leq\left(\|q^-\|_{4,-}|\psi|_{4.5}+|\psi|_4^2\right)\|\partial v^-\|_{2,-}|\psi|_{2.5}\leq P(\sigma^{-1},E_4^\kappa(t)).\end{aligned}$$

The last term in ZB^- is controlled together with $Z^-:=-\int_{\Omega^-}\mathbf{Q}^-\mathfrak{C}_i(v_i^-)\,\mathrm{d}\mathcal{V}_t$. Recall that

$$\mathfrak{C}_i(v_i^-)=(\partial_3^\varphi\partial_i^\varphi v_i^-)\overline{\partial}^4\varphi-\left[\overline{\partial}^4,\frac{\partial_i\varphi}{\partial_3\varphi},\partial_3v_i^-\right]-\partial_3v_i^-\left[\overline{\partial}^4,\partial_i\varphi,\frac{1}{\partial_3\varphi}\right]+\partial_i\varphi\partial_3v_i^-\left[\overline{\partial}^3,\frac{1}{(\partial_3\varphi)^2}\right]\overline{\partial}\partial_3\varphi,\ i=1,2$$

and

$$\mathfrak{C}_3(f)=(\partial_3^\varphi)^2v_3^-\overline{\partial}^4\varphi+\left[\overline{\partial}^4,\frac{1}{\partial_3\varphi},\partial_3v_3^-\right]-\partial_3v_3^-\left[\overline{\partial}^3,\frac{1}{(\partial_3\varphi)^2}\right]\overline{\partial}\partial_3\varphi.$$

Note that $\mathbf{N}_i=-\partial_i\varphi$ for $i=1,2$, so we have

$$\begin{aligned}-\left[\overline{\partial}^4,\frac{\partial_i\varphi}{\partial_3\varphi},\partial_3v_i^-\right]=\left[\overline{\partial}^4,\frac{\mathbf{N}_i}{\partial_3\varphi},\partial_3v_i^-\right]=\sum_{k=1}^3\binom{4}{k}\overline{\partial}^k\left(\frac{\mathbf{N}_i}{\partial_3\varphi}\right)\overline{\partial}^{4-k}\partial_3v_i^-\\=\sum_{k=1}^3\binom{4}{k}\overline{\partial}^k\mathbf{N}_i\partial_3^\varphi\overline{\partial}^{4-k}v_i^--\binom{4}{k}\left[\overline{\partial}^k,\frac{1}{\partial_3\varphi}\right]\partial_i\varphi\partial_3\overline{\partial}^{4-k}v_i^-,\end{aligned}$$

where the contribution of the first term above gives us (using Gauss-Green formula)

$$
\begin{aligned}
& ZB^- - \sum_{k=1}^{3} \int_{\Omega^-} \binom{4}{k} \mathbf{Q}^- \overline{\partial}^k \mathbf{N}_i \partial_3^\varphi \overline{\partial}^{4-k} v_i^- \, \mathrm{d}\mathcal{V}_t \\
=& \sum_{k=1}^{3} \binom{4}{k} \left(\int_\Sigma \mathbf{Q}^- \overline{\partial}^{4-k} v^- \cdot \overline{\partial}^k N \, \mathrm{d}x' - \int_{\Omega^-} \mathbf{Q}^- \overline{\partial}^k \mathbf{N}_i \partial_3 \overline{\partial}^{4-k} v_i^- \, \mathrm{d}x \right) \\
=& \sum_{k=1}^{3} \binom{4}{k} \left(\int_{\Omega^-} \partial_3 \mathbf{Q}^- \overline{\partial}^k \mathbf{N}_i \overline{\partial}^{4-k} v_i^- \, \mathrm{d}x + \int_{\Omega^-} \mathbf{Q}^- \overline{\partial}^k \mathbf{N}_i \partial_3 \overline{\partial}^{4-k} v_i^- \, \mathrm{d}x - \int_{\Omega^-} \mathbf{Q}^- \overline{\partial}^k \mathbf{N}_i \partial_3 \overline{\partial}^{4-k} v_i^- \, \mathrm{d}x \right) \\
=& \sum_{k=1}^{3} \binom{4}{k} \int_{\Omega^-} \partial_3 \mathbf{Q}^- \overline{\partial}^k \mathbf{N}_i \overline{\partial}^{4-k} v_i^- \, \mathrm{d}x.
\end{aligned}
\tag{3.50}
$$

Now invoking $\mathbf{Q}^- = \overline{\partial}^4 q^- - \overline{\partial}^4 \varphi \partial_3^\varphi q^-$ and integrating one $\overline{\partial}$ by parts, we find that

$$
\sum_{k=1}^{3} \binom{4}{k} \int_{\Omega^-} \partial_3 \mathbf{Q}^- \overline{\partial}^k \mathbf{N}_i \overline{\partial}^{4-k} v_i^- \, \mathrm{d}x \lesssim (\|\overline{\partial}^3 \partial_3 q^-\|_{0,-} + |\psi|_4 \|\partial_3 q^-\|_{L^\infty(\Omega^-)}) |\psi|_4 \|v_i^-\|_{4,-}. \tag{3.51}
$$

Among other terms in $\mathfrak{C}_i(v_i^-)$, we shall focus on the case when there are 4 derivatives falling on v_i^- and φ, and the control of these terms (lised below) appears to be easier.

$$
\begin{aligned}
& - \int_{\Omega^-} \mathbf{Q}^- \overline{\partial}^4 \varphi \partial_3^\varphi (\nabla^\varphi \cdot v^-) \, \mathrm{d}\mathcal{V}_t \quad \text{from the first term in } \mathfrak{C}_i(v_i^-) \\
& 4 \sum_{i=1}^{3} \int_{\Omega^-} \mathbf{Q}^- \partial_3 \overline{\partial} \varphi \, \partial_3^\varphi \overline{\partial}^3 v^- \cdot \mathbf{N} \, \mathrm{d}x \quad \text{from the second term in } \mathfrak{C}_i(v_i^-) \text{ when } \overline{\partial}^3 \text{ falls on } \partial_3 v_i^-.
\end{aligned}
\tag{3.52}
$$

Note that $\partial_3^\varphi v^- \cdot N = \nabla^\varphi \cdot v^- - \overline{\nabla} \cdot \bar{v}^-$, we have

$$
- \int_{\Omega^-} \mathbf{Q}^- \overline{\partial}^4 \varphi \partial_3^\varphi (\nabla^\varphi \cdot v) \, \mathrm{d}\mathcal{V}_t \lesssim \left\| \sqrt{\mathcal{F}_p^-} \mathbf{Q}^- \right\|_{0,-} \left\| \sqrt{\mathcal{F}_p^-} \partial_3 D_t^{\varphi,-} p^- \right\|_{0,-} |\psi|_4 \,, \tag{3.53}
$$

and

$$
\begin{aligned}
4 \int_{\Omega^-} \mathbf{Q}^- \partial_3 \overline{\partial} \varphi \, \partial_3^\varphi \overline{\partial}^3 v^- \cdot \mathbf{N} \, \mathrm{d}x \overset{L}{=} & \, 4 \int_{\Omega^-} \mathbf{Q}^- \partial_3 \overline{\partial} \varphi \, \overline{\partial}^3 (\nabla^\varphi \cdot v^-) \, \mathrm{d}x - 4 \int_{\Omega^-} \mathbf{Q}^- \partial_3 \overline{\partial} \varphi \, \overline{\partial}^3 (\overline{\nabla} \cdot \bar{v}^-) \, \mathrm{d}x \\
\lesssim & \, |\overline{\partial} \psi|_{L^\infty} \left(\left\| \sqrt{\mathcal{F}_p^-} \mathbf{Q}^- \right\|_{0,-} \left\| \sqrt{\mathcal{F}_p^-} \overline{\partial}^3 D_t^{\varphi,-} p^- \right\|_{0,-} + \|\overline{\nabla} \overline{\partial}^3 \mathbf{Q}^-\|_{0,-} \|\overline{\partial}^4 v^-\|_{0,-} \right).
\end{aligned}
\tag{3.54}
$$

Thus, combining the estimates in the above four steps, we conclude the $\overline{\partial}^4$-estimate by: For the tangential derivative $\mathcal{T}^\gamma = \overline{\partial}^4$ ($\gamma_0 = \gamma_4 = 0,\ \gamma_1 + \gamma_2 = 4$) and for any $0 < \delta < 1$, we have

$$
\begin{aligned}
& \left\| \left(\mathbf{V}^{\gamma,\pm}, \mathbf{B}^{\gamma,\pm}, \mathbf{S}^{\gamma,\pm}, \sqrt{\mathcal{F}_p^\pm} \mathbf{P}^{\gamma,\pm} \right)(t) \right\|_0^2 + \left| \sqrt{\sigma} \varepsilon^{2l} \overline{\partial}^{4+l} \overline{\nabla} \psi(t) \right|_0^2 + \left| \sqrt{\kappa} \varepsilon^{2l} \overline{\partial}^{4+l} \psi(t) \right|_2^2 + \int_0^t \left| \sqrt{\kappa} \varepsilon^{2l} \overline{\partial}^{4+l} \partial_t \psi(\tau) \right|_1^2 \mathrm{d}\tau \\
\lesssim & \, \delta E_{4+l}^\kappa(t) + \left| \varepsilon^{2l} \psi_0 \right|_{5.5+l}^2 + \sum_{j=0}^{l} \int_0^t P(\sigma^{-1}, E_{4+j}^\kappa(\tau)) \, \mathrm{d}\tau, \quad 0 \le l \le 4.
\end{aligned}
\tag{3.55}
$$

Remark 3.2. It should be noted that we only have the L^2 control of $\mathbf{V}, \mathbf{B}, \mathbf{S}$ and $(\mathcal{F}_p)^{\frac{1}{2}} \mathbf{P}$ in the tangential estimates, but the term $\mathbf{Q}$ without $\mathcal{F}_p$-weight does appear in tangential estimates. When $\mathcal{T}^\gamma$ contains at least one spatial derivative, that is, $\gamma_0 < \langle \gamma \rangle$, one can invoke the momentum equation to replace $\mathcal{T} q$ by $D_t^\varphi v$ and $(b \cdot \nabla^\varphi) b$ to avoid the loss of Mach number. This also suggests that we can actually control $\|\mathbf{P}\|_0$ instead of only $\|\mathcal{F}_p^{1/2} \mathbf{P}\|_0$ when there is at least one spatial derivatives. However, when $\mathcal{T}^\gamma$ only consists of time derivatives, we cannot do such substitution any longer. Thus, we have to use the above cancellation structure between ZB and Z to control these two terms together.

3.3.2 The case $l>0$: No loss of regularity or weights of Mach number

Next we consider the tangential estimates for ε-weighted spatial derivatives, namely $\varepsilon^{2l}\overline{\partial}^{4+l}$ for $1\leq l\leq 4$. The proof is parallel to the case $\mathcal{T}^\gamma=\overline{\partial}^4$, but we have to check the following aspects

a. We have to guarantee that there is no loss of $\mathcal{F}_p$-weight in various commutators, especially those involving q.
b. When $l=4$, we only have tangential regularity for 8 derivatives. Due to the anisotropy of the function space H^8_*, we have to put extra efforts to reduce the terms involving the derivative $\overline{\partial}^7\partial_3$.

We only show the detailed modifications for the case $l=4$, that is, the $\varepsilon^8\overline{\partial}^8$-estimate. When $1\leq l\leq 3$, similar modifications can be made in the same way.

Commutators of type $\varepsilon^8[\overline{\partial}^8,f]\mathcal{T}g$ for $\mathcal{T}=\overline{\partial}$ or D_t^φ

This type of commutator includes the following terms

$$-[\mathcal{T}^\gamma,\rho]D_t^\varphi v \text{ in } \mathcal{R}_v,\quad -[\mathcal{T}^\gamma,\mathcal{F}_p]D_t^\varphi p \text{ in } \mathcal{R}_p,\\ [\mathcal{T}^\gamma,\bar v]\cdot\overline{\partial}f \text{ and } \partial_3^\varphi f[\mathcal{T}^\gamma,v]\cdot\mathbf{N} \text{ in } \mathfrak{D}^\gamma(f)$$

It is controlled directly by expanding the commutator. We have

$$\begin{aligned}\varepsilon^8[\overline{\partial}^8,f]\mathcal{T}g=&(\varepsilon^8\overline{\partial}^8f)\mathcal{T}g+8(\varepsilon^6\overline{\partial}^7f)(\varepsilon^2\overline{\partial}\mathcal{T}g)+28(\varepsilon^6\overline{\partial}^6f)(\varepsilon^2\overline{\partial}^2\mathcal{T}g)+56(\varepsilon^6\overline{\partial}^5f)(\varepsilon^2\overline{\partial}^3\mathcal{T}g)\\&+70(\varepsilon^2\overline{\partial}^4f)(\varepsilon^6\overline{\partial}^4\mathcal{T}g)+56(\varepsilon^2\overline{\partial}^3f)(\varepsilon^6\overline{\partial}^5\mathcal{T}g)+28(\varepsilon^2\overline{\partial}^2f)(\varepsilon^6\overline{\partial}^6\mathcal{T}g)+8(\overline{\partial}f)(\varepsilon^8\overline{\partial}^7\mathcal{T}g),\end{aligned}$$

whose $L^2(\Omega)$ norm is controlled by

$$\begin{aligned}&\|\varepsilon^8\overline{\partial}^8f\|_0\|\mathcal{T}g\|_{L^\infty}+8\varepsilon^2\|\varepsilon^6\overline{\partial}^7f\|_0\|\overline{\partial}\mathcal{T}g\|_{L^\infty}+28\|\varepsilon^6\overline{\partial}^6f\|_{L^6}\|\varepsilon^2\overline{\partial}^2\mathcal{T}g\|_{L^3}+56\|\varepsilon^6\overline{\partial}^5f\|_{L^6}\|\varepsilon^2\overline{\partial}^3\mathcal{T}g\|_{L^3}\\&+70\|\varepsilon^2\overline{\partial}^4f\|_{L^3}\|\varepsilon^6\overline{\partial}^4\mathcal{T}g\|_{L^6}+56\|\varepsilon^6\overline{\partial}^3f\|_{L^3}\|\varepsilon^2\overline{\partial}^5\mathcal{T}g\|_{L^6}+28\|\varepsilon^6\overline{\partial}^6g\|_{L^6}\|\varepsilon^2\overline{\partial}^2\mathcal{T}f\|_{L^3}+8\varepsilon^2\|\varepsilon^6\overline{\partial}^7g\|_0\|\overline{\partial}\mathcal{T}f\|_{L^\infty}\\&\lesssim(1+\varepsilon^2)\left(\sqrt{E_8^\kappa(t)E_4^\kappa(t)}+\sqrt{E_7^\kappa(t)E_4^\kappa(t)}+\sqrt{E_7^\kappa(t)E_5^\kappa(t)}\right),\end{aligned}$$

where we use the Sobolev embedding $H^1\hookrightarrow L^6$ and $H^1\hookrightarrow H^{1/2}\hookrightarrow L^3$ in 3D. In 2D case, we can replace (L^6,L^3) by (L^4,L^4) and use Ladyženskaya's inequality $\|f\|_{L^4}^2\leq\|f\|_{L^2}\|\partial f\|_{L^2}\leq\|f\|_1^2$ to obtain the same bound.

Commutator $\varepsilon^8[\overline{\partial}^8,b]\cdot\nabla^\varphi f$ for $f=b,v$

This term appears when we commute $\mathcal{T}^\gamma$ with $(b\cdot\nabla^\varphi)$. Note that we can rewrite the directional derivative to be $(b\cdot\nabla^\varphi)=\bar b\cdot\overline{\nabla}+(\partial_3\varphi)^{-1}(b\cdot\mathbf{N})\partial_3$. When commuting $\overline{\partial}^8$ with $\bar b\cdot\overline{\nabla}$, the estimate is exactly the same as $\varepsilon^8[\overline{\partial}^8,f]\mathcal{T}g$. For the commutator $[\overline{\partial}^8,(\partial_3\varphi)^{-1}(b\cdot\mathbf{N})]\partial_3 f$, we just need to put extra effort on the term $8\overline{\partial}((\partial_3\varphi)^{-1}(b\cdot\mathbf{N}))\overline{\partial}^7\partial_3 f$ because the length of the multi-index exceeds 8 when $|x_3|\lesssim 1$. (Recall that the weight function $\omega(x_3)$ is comparable to $|x_3|$ when $x_3\lesssim 1$ and is comparable to 1 when $|x_3|\gg 1$.) In this case, we notice that $b\cdot\mathbf{N}|_\Sigma=0$, and thus its interior value can be expressed via the fundamental theorem of calculus

$$(\partial_3\varphi)^{-1}(b\cdot\mathbf{N})(x',x_3)=0+\int_0^{x_3}\partial_3\left((\partial_3\varphi)^{-1}(b\cdot\mathbf{N})(x',\xi_3)\right)\mathrm{d}\xi_3,$$

whose $L^\infty(\Omega)$ norm is controlled by $C\omega(x_3)\|\partial_3(b\cdot\mathbf{N})\|_{L^\infty(\Omega)}$.

Commutator $\mathfrak{D}(f)$ for $f=v,\ p,\ b,\ S$

Among all terms in (3.10), we need to further analyze the third term, that is, the commutator $\varepsilon^8\left[\overline{\partial}^8,\frac{1}{\partial_3\varphi}(v\cdot\mathbf{N}-\partial_t\varphi),\partial_3 f\right]$ for $f=v,b,p$. The problem is the same as above, that is, $\overline{\partial}^7$ may fall on $\partial_3 f$ which is not directly controllable. Again, we notice that there is only one $\overline{\partial}$ falling on $\frac{1}{\partial_3\varphi}(v\cdot\mathbf{N}-\partial_t\varphi)$ and $(v\cdot\mathbf{N}-\partial_t\varphi)|_\Sigma=0$, so we can use the same method (as in the control of $\varepsilon^8[\overline{\partial}^8,b]\cdot\nabla^\varphi f$) to control this commutator.

Commutator $\mathfrak{C}(q)$

The problematic term is $-8(\partial_3\varphi)^{-1}(\overline{\partial}\mathbf{N}_i)(\overline{\partial}^7\partial_3 q)$ arising from $[\mathcal{T}^\gamma, \mathbf{N}_i/\partial_3\varphi, \partial_3 q]$. To control this term, we can invoke the third component of the momentum equation to convert $\partial_3 q$ to tangential derivatives of other quantities

$$-\partial_3 q = (\partial_3\varphi)\left(\rho D_t^\varphi v_3 - (b\cdot\nabla^\varphi)b_3\right),$$

where $D_t^\varphi = \partial_t + \bar{v}\cdot\overline{\nabla} + (\partial_3\varphi)^{-1}(v\cdot\mathbf{N} - \partial_t\varphi)\partial_3$ and $(b\cdot\nabla^\varphi) = \bar{b}\cdot\overline{\nabla} + (\partial_3\varphi)^{-1}(b\cdot\mathbf{N})\partial_3$ are both tangential derivatives. Also, there is no loss of weight of Mach number in this term because one can always replace $\overline{\partial}q$ by $D_t^\varphi v$ and $(b\cdot\nabla^\varphi)b$.

Commutator $\mathfrak{C}_i(v_i)$

The problematic term is $-8(\partial_3\varphi)^{-1}(\overline{\partial}\mathbf{N}_i)(\overline{\partial}^7\partial_3 v_i)$ arising from $[\mathcal{T}^\gamma, \mathbf{N}_i/\partial_3\varphi, \partial_3 v_i]$. In fact, this term may not be controlled independently, but its contribution only appears in $-\int_\Omega \mathbf{Q}\mathfrak{C}_i(v_i)\,\mathrm{d}\mathcal{V}_t$ which has been analyzed in step 4 of Section 3.3.1. Specifically, its contribution in the term Z, after combining it with ZB term, is

$$8\varepsilon^{16}\int_\Omega \partial_3(\overline{\partial}^8 q - \overline{\partial}^8\varphi\partial_3^\varphi q)\,\overline{\partial}\mathbf{N}_i\,\overline{\partial}^7 v_i\,\mathrm{d}x,$$

which is controlled by $(\|\varepsilon^8\overline{\partial}^7\partial_3 q\|_0 + |\varepsilon^8\overline{\partial}^8\psi|_0\|\partial q\|_{L^\infty})|\overline{\partial}\psi|_{W^{1,\infty}}\|\varepsilon^8\overline{\partial}^8 v\|_0$ after integrating one $\overline{\partial}$ by parts. Then we convert $\partial_3 q$ to tangential derivatives of other quantities via the momentum equation, which has been presented in the control of $\mathfrak{C}(q)$.

Based on the above analysis, we can follow the same method as in $\overline{\partial}^4$-estimate to prove the following inequality for $\varepsilon^{2l}\overline{\partial}^{4+l}$-estimates ($1\le l\le 4$) for the nonlinear κ-approximate problem (3.1): For any $0<\delta<1$ and fixed $l\in\{1,2,3,4\}$.

$$\begin{aligned}&\left\|\varepsilon^{2l}\left(\mathbf{V}^{\gamma,\pm}, \mathbf{B}^{\gamma,\pm}, \mathbf{S}^{\gamma,\pm}, \sqrt{\mathcal{F}_p^\pm}\mathbf{P}^{\gamma,\pm}\right)(t)\right\|_{0,\pm}^2 + \left|\sqrt{\sigma}\varepsilon^{2l}\overline{\partial}^{4+l}\overline{\nabla}\psi(t)\right|_0^2 + \left|\sqrt{\kappa}\varepsilon^{2l}\overline{\partial}^{4+l}\psi(t)\right|_2^2 + \int_0^t\left|\sqrt{\kappa}\varepsilon^{2l}\overline{\partial}^{4+l}\partial_t\psi(\tau)\right|_1^2\mathrm{d}\tau\\ &\lesssim \delta E_{4+l}^\kappa(t) + \left|\varepsilon^{2l}\psi_0\right|_{5.5+l}^2 + \sum_{j=0}^{l}\int_0^t P(\sigma^{-1}, E_{4+j}^\kappa(\tau))\,\mathrm{d}\tau,\end{aligned} \tag{3.56}$$

where $(\mathbf{V}^{\gamma,\pm}, \mathbf{B}^{\gamma,\pm}, \mathbf{S}^{\gamma,\pm}, \mathbf{P}^{\gamma,\pm})$ represent that Alinhac good unknowns of $(v^\pm, b^\pm, S^\pm, p^\pm)$ with respect to $\overline{\partial}^{4+l}$.

3.4 Tangential estimates: full time derivatives

Now we control the full time derivatives, that is, the $\varepsilon^{2l}\partial_t^{4+l}$ estimates for $0\le l\le 4$. We will take the most difficult case $l=4$ for an example, that is, the $\varepsilon^8\partial_t^8$-estimate. The other cases ($0\le l\le 3$) can be treated in the same way.

3.4.1 Replacing one time derivative by a material derivative

Following the analysis in Section 3.3.1 and Section 3.3.2, we expect to control the following norms

$$\left\|\varepsilon^8\left(\mathbf{V}^\pm, \mathbf{B}^\pm, \sqrt{\mathcal{F}_p^\pm}\mathbf{P}^\pm, \mathbf{S}^\pm\right)\right\|_{0,\pm}^2 + \left|\varepsilon^8\sqrt{\sigma}\partial_t^8\psi\right|_1^2 + \left|\varepsilon^8\sqrt{\kappa}\partial_t^8\psi\right|_2^2 + \left|\varepsilon^8\sqrt{\kappa}\partial_t^9\psi\right|_{L_t^2H_{x'}^1}^2,$$

which further gives the control of $\left\|\varepsilon^8\partial_t^8(v^\pm, b^\pm, \sqrt{\mathcal{F}_p^\pm}p^\pm, S^\pm)\right\|_0^2$. However, there are several extra difficulties that may make our previous method invalid.

a. We cannot substitute ∂q by $\mathcal{T}(v,b)$ because there is no spatial derivative.

b. $\partial_t^{4+l}p$ has weight $\sqrt{\mathcal{F}_p^\pm}\varepsilon^{2l} = O(\varepsilon^{1+2l})$ instead of ε^{2l}. There might be a loss of ε-weight.

c. $\sqrt{\mathcal{F}_p^\pm}\varepsilon^{2l}\partial_t^{4+l}q$ only has $L^2(\Omega)$ regularity, so the trace lemma is no longer valid.

d. We cannot integrate by parts for "half-order time derivative" $\partial_t^{1/2}$. Thus, the control of VS term will be rather different.

To overcome the abovementioned difficulties, especially (c) and (d) in the control of the crucial boundary term VS, we would like to replace the full-time derivative ∂_t^{4+l} by $D_t^{\varphi,-}\partial_t^{3+l}$ where $D_t^{\varphi,-} = \partial_t + \bar{v}^- \cdot \overline{\nabla} + (\partial_3\varphi)^{-1}(v^- \cdot \mathbf{N} - \partial_t\varphi)\partial_3$ and $v^-|_{\Omega^+}$ is defined to be the Sobolev extension of v^- in Ω^+. We aim to prove the following estimates.

Proposition 3.6. Fix $l \in \{0,1,2,3,4\}$. For $\varepsilon^{2l}D_t^{\varphi,-}\partial_t^{3+l}$-differentiated approximate system ($0 \le l \le 4$), we have the following uniform-in-(κ,ε) estimate for any $0<\delta<1$

$$\begin{aligned}
&\left\|\varepsilon^{2l}\left(\mathbf{V}^{*,\gamma,\pm},\mathbf{B}^{*,\gamma,\pm},\mathbf{S}^{*,\gamma,\pm},(\mathcal{F}_p^{\pm})^{1/2}\mathbf{P}^{*,\gamma,\pm}\right)(t)\right\|_0^2 + \left|\sqrt{\sigma}\varepsilon^{2l}D_t^{\varphi,-}\partial_t^{3+l}\overline{\nabla}\psi(t)\right|_0^2 \\
&+\left|\sqrt{\kappa}\varepsilon^{2l}D_t^{\varphi,-}\partial_t^{3+l}\psi(t)\right|_2^2 + \int_0^t \left|\sqrt{\kappa}\varepsilon^{2l}D_t^{\varphi,-}\partial_t^{3+l}\partial_t\psi(\tau)\right|_1^2\,\mathrm{d}\tau \\
&\lesssim \delta E_{4+l}^{\kappa}(t) + \sum_{j=0}^{l} P(E_{4+j}^{\kappa}(0)) + \int_0^t P(\sigma^{-1},E_{4+j}^{\kappa}(\tau))\,\mathrm{d}\tau, \quad 0\le l\le 4,
\end{aligned} \tag{3.57}$$

where $\left(\mathbf{V}^{*,\gamma,\pm},\mathbf{B}^{*,\gamma,\pm},\mathbf{S}^{*,\gamma,\pm},(\mathcal{F}_p^{\pm})^{1/2}\mathbf{P}^{*,\gamma,\pm}\right)$ represent the Alinhac good unknowns of $(v^\pm,b^\pm,S^\pm,p^\pm)$ with respect to $D_t^{\varphi,-}\partial_t^{3+l}$, that is, $\mathbf{F}^{*,\gamma,\pm} = D_t^{\varphi,-}\partial_t^{3+l}f^\pm - (D_t^{\varphi,-}\partial_t^{3+l}\varphi)\partial_3^{\varphi}f^\pm$.

For the case $l=4$, we introduce the Alinhac good unknowns with respect to $D_t^{\varphi,-}\partial_t^7$

$$(\mathbf{V}^{*,\pm},\mathbf{B}^{*,\pm},\mathbf{P}^{*,\pm},\mathbf{Q}^{*,\pm},\mathbf{S}^{*,\pm}) := D_t^{\varphi,-}\partial_t^7(v^\pm,b^\pm,p^\pm,q^\pm,S^\pm) - (D_t^{\varphi,-}\partial_t^7\varphi)\partial_3^{\varphi}(v^\pm,b^\pm,p^\pm,q^\pm,S^\pm).$$

They satisfy

$$D_t^{\varphi,-}\partial_t^7\partial_i^{\varphi}f^\pm = \partial_i^{\varphi}\mathbf{F}^{*,\pm} + \mathfrak{C}_i^*(f^\pm),\quad D_t^{\varphi,-}\partial_t^7 D_t^{\varphi,-}f^\pm = D_t^{\varphi,-}\mathbf{F}^{*,\pm} + \mathfrak{D}_i^*(f^\pm)f,$$

where $\mathfrak{C}^*(f),\mathfrak{D}^*(f)$ are defined in the same way as (3.9)-(3.10) by replacing $\mathcal{T}^\gamma$ with $D_t^{\varphi,-}\partial_t^7$. The boundary conditions of these good unknowns are

$$[\![\mathbf{Q}^*]\!] = \sigma\overline{D_t^-}\partial_t^7\mathcal{H} - \kappa\overline{D_t^-}\partial_t^7(1-\overline{\Delta})^2\psi - \kappa\overline{D_t^-}\partial_t^7(1-\overline{\Delta})\partial_t\psi - [\![\partial_3 q]\!]\,\overline{D_t^-}\partial_t^7\psi \tag{3.58}$$

$$\mathbf{V}^{*,\pm}\cdot N = \partial_t\overline{D_t^-}\partial_t^7\psi + (\bar{v}^\pm\cdot\overline{\nabla})\overline{D_t^-}\partial_t^7\psi - \overline{D_t^\pm}\bar{v}^-\cdot\overline{\nabla}\partial_t^7\psi - \mathcal{W}^{*,\pm}, \tag{3.59}$$

with

$$\mathcal{W}^{*\pm} = (\partial_3 v^\pm\cdot N)\overline{D_t^-}\partial_t^7\psi + [D_t^{\varphi,-}\partial_t^7, N_i, v_i^\pm], \tag{3.60}$$

where we use the fact that $D_t^{\varphi\pm}|_\Sigma = \overline{D_t^\pm} = \partial_t + \bar{v}^\pm\cdot\overline{\nabla}$. Note that $\overline{D_t^-}$ does not directly commute with ∂_t or ∂_i, so there is an extra term $-\overline{D_t^\pm}\bar{v}^-\cdot\overline{\nabla}\partial_t^7\psi$ in the expression of $\mathbf{V}^{\pm,*}\cdot N$.

3.4.2 Analysis of the interior commutators

Since we replaced ∂_t^8 with $D_t^{\varphi,-}\partial_t^7$ and $D_t^{\varphi,-}$ does not directly commute with ∂_3, we need to further analyze the commutators $\mathfrak{C}_i(f)$ for $f=q$ and v_i and $\mathfrak{D}(f)$ for $f=v,b,p,S$. The problematic thing is that ∂_3 may fall on $(\partial_3\varphi)^{-1}(v^-\cdot\mathbf{N}-\partial_t\varphi)$ (in $D_t^{\varphi,-}$) and produce a normal derivative without a weight function that vanishes on Σ, which may introduce a second-order derivative in the setting of anisotropic Sobolev space. This problem does not appear in $\mathfrak{D}(f)$, as we find that such commutator has the form $(\partial_3\varphi)^{-1}(v\cdot\mathbf{N}-\partial_t\varphi)[D_t^{\varphi,-}\partial_t^7,\partial_3]f$ which already includes a weight $(v\cdot\mathbf{N}-\partial_t\varphi)$ that vanishes on Σ. In $\mathfrak{C}_i(f)$, according to (3.9), we need to further analyze the term $\mathbf{N}_i(\partial_3\varphi)^{-1}[D_t^{\varphi,-}\partial_t^7,\partial_3]f$ for $f=q,v_i$. Using $D_t^{\varphi,-} = \partial_t + \bar{v}^-\cdot\overline{\nabla} + (\partial_3\varphi)^{-1}(v^-\cdot\mathbf{N}-\partial_t\varphi)\partial_3$, we have

$$\begin{aligned}
&\mathbf{N}_i(\partial_3\varphi)^{-1}[D_t^{\varphi,-}\partial_t^7,\partial_3]f = \mathbf{N}_i(\partial_3\varphi)^{-1}[D_t^{\varphi,-},\partial_3]\partial_t^7 f \\
&= -\mathbf{N}_i(\partial_3\varphi)^{-1}\partial_3\bar{v}^-\cdot\overline{\nabla}\partial_t^7 f + \mathbf{N}_i\partial_3\left((\partial_3\varphi)^{-1}(v^-\cdot\mathbf{N}-\partial_t\varphi)\right)\partial_3^{\varphi}\partial_t^7 f.
\end{aligned}$$

The first term above can be directly controlled in L^2 because only tangential derivative falls on $\partial_t^7 f$. For the second term, we can invoke the momentum equation and the continuity equation to convert this normal derivative to a tangential derivative.

- When $f = q$, we use $-\partial_3^\varphi q = \rho D_t^\varphi v_3 - (b\cdot\nabla^\varphi) b_3$.
- When $f = v_i$, using $\nabla^\varphi\cdot v = \overline{\nabla}\cdot\bar{v} + \partial_3^\varphi v\cdot\mathbf{N}$, the continuity equation becomes $\partial_3^\varphi v\cdot\mathbf{N} = -\varepsilon^2 D_t^\varphi p - \overline{\nabla}\cdot\bar{v}$. Thus we have $\partial_t^7\partial_3^\varphi v\cdot\mathbf{N} = -\partial_t^7(\varepsilon^2 D_t^\varphi p + \overline{\nabla}\cdot\bar{v}) + [\partial_t^7, \mathbf{N}]\cdot\partial_3 v$ in which both terms can be directly controlled in $\|\cdot\|_{8,*}$ norm.

Also note that there is no extra loss of Mach number even if $\partial_t^8 p$ requires one more ε-weight. In fact, the only term in the commutators $\mathfrak{C}, \mathfrak{D}$ that contains $\partial_t^8 p$ is $\mathcal{R}_p$, but there is an extra weight $\mathcal{F}_p = O(\varepsilon^2)$ multiplying on it. Therefore, we can follow the same strategy presented in Section 3.3.1 and Section 3.3.2 to analyze the interior part. We can prove the following energy identity

$$\begin{aligned}&\sum_\pm \frac{\mathrm{d}}{\mathrm{d}t}\frac{\varepsilon^{16}}{2}\int_{\Omega^\pm}\rho^\pm|\mathbf{V}^{*,\pm}|^2 + |\mathbf{B}^{*,\pm}|^2 + \mathcal{F}_p^\pm(\mathbf{P}^{*,\pm})^2 + \rho^\pm(\mathbf{S}^{*,\pm})^2\,\mathrm{d}\mathcal{V}_t\\ &= \mathrm{ST}^* + \mathrm{ST}^{*\prime} + \mathrm{VS}^* + \mathrm{RT}^* + \sum_\pm \mathrm{RT}^{*,\pm} + ZB^{*,\pm} + Z^{*,\pm} + R_\Sigma^{*,\pm} + R_\Omega^{*,\pm},\end{aligned}\tag{3.61}$$

where

$$\mathrm{ST}^* := \varepsilon^{16}\int_\Sigma \overline{D_t^-}\partial_t^7 [\![q]\!]\ \partial_t\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x',\tag{3.62}$$

$$\mathrm{ST}^{*\prime} := \varepsilon^{16}\int_\Sigma \overline{D_t^-}\partial_t^7 [\![q]\!]\ (\bar{v}^+\cdot\overline{\nabla})\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x',\tag{3.63}$$

$$\mathrm{VS}^* := \varepsilon^{16}\int_\Sigma \overline{D_t^-}\partial_t^7 q^-\ ([\![\bar{v}]\!]\cdot\overline{\nabla})\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x',\tag{3.64}$$

$$\mathrm{RT}^* := -\varepsilon^{16}\int_\Sigma [\![\partial_3 q]\!]\,\overline{D_t^-}\partial_t^7\psi\,\partial_t\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x',\tag{3.65}$$

$$\mathrm{RT}^{*,\pm} := \mp\varepsilon^{16}\int_\Sigma \partial_3 q^\pm\,\overline{D_t^-}\partial_t^7\psi\,(\bar{v}^\pm\cdot\overline{\nabla})\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x',\tag{3.66}$$

$$R_\Sigma^{*,\pm} := \pm\varepsilon^{16}\int_\Sigma \mathbf{Q}^{*,\pm}\overline{D_t^\pm}\bar{v}^-\cdot\overline{\nabla}\partial_t^7\psi\,\mathrm{d}x',\tag{3.67}$$

$$ZB^{*,\pm} := \mp\varepsilon^{16}\int_\Sigma \mathbf{Q}^{*,\pm}\mathcal{W}^{*,\pm}\,\mathrm{d}x',\quad Z^{*,\pm} = -\int_{\Omega^\pm}\varepsilon^{16}\mathbf{Q}^{\pm,*}\mathfrak{C}_i^*(v_i^\pm)\,\mathrm{d}\mathcal{V}_t,\tag{3.68}$$

and $R_\Omega^{*,\pm}$ represents the controllable terms in the interior containing the analogues of $R_1^\pm, R_2^\pm, R_3^\pm$. Specifically, we have

$$\begin{aligned}\varepsilon^{-16}R_\Omega^{*,\pm} =& \int_{\Omega^\pm}\mathbf{V}^{*,\pm}\cdot(\mathcal{R}_v^{*,\pm} - \mathfrak{C}^*(q^\pm))\,\mathrm{d}\mathcal{V}_t + \int_{\Omega^\pm}\mathcal{R}_q^{*,\pm}(\nabla^\varphi\cdot\mathbf{V}^{*,\pm})\,\mathrm{d}\mathcal{V}_t + \int_{\Omega^\pm}\mathbf{B}^{*,\pm}\cdot\mathcal{R}_b^{*,\pm}\,\mathrm{d}\mathcal{V}_t + \int_{\Omega^\pm}\mathbf{P}^{*,\pm}\mathcal{R}_p^{*,\pm}\,\mathrm{d}\mathcal{V}_t\\ &-\frac12\int_{\Omega^\pm}(\nabla^\varphi\cdot v^\pm)|\mathbf{B}^{*,\pm}|^2\,\mathrm{d}\mathcal{V}_t - \frac12\int_{\Omega^\pm}(D_t^{\varphi\pm}\mathcal{F}_p^\pm + \mathcal{F}_p^\pm\nabla^\varphi\cdot v^\pm)|\mathbf{P}^{*,\pm}|^2\,\mathrm{d}\mathcal{V}_t + \int_{\Omega^\pm}\rho^\pm\mathfrak{D}^*(S^\pm)\,\mathbf{S}^{*,\pm}\,\mathrm{d}\mathcal{V}_t\end{aligned}\tag{3.69}$$

where

$$\begin{aligned}\mathcal{R}_v^{*,\pm} &:= [D_t^{\varphi,-}\partial_t^7, b^\pm]\cdot\nabla^\varphi b^\pm - [D_t^{\varphi,-}\partial_t^7, \rho^\pm]D_t^{\varphi\pm}v^\pm - \rho^\pm\mathfrak{D}^*(v^\pm),\\ \mathcal{R}_p^{*,\pm} &:= -[D_t^{\varphi,-}\partial_t^7, \mathcal{F}_p^\pm]D_t^{\varphi\pm}p^\pm - \mathcal{F}_p^\pm\mathfrak{D}^*(p^\pm),\\ \mathcal{R}_b^{*,\pm} &:= [D_t^{\varphi,-}\partial_t^7, b^\pm]\cdot\nabla^\varphi v^\pm - \mathfrak{D}^*(b^\pm),\quad \mathcal{R}_q^{*,\pm} := \mathbf{Q}^{*,\pm} - \mathbf{P}^{*,\pm} - b^\pm\cdot\mathbf{B}^{*,\pm}.\end{aligned}$$

These terms can be directly controlled in the same way as presented in Section 3.3.1, so we omit the details

$$\int_0^t R_\Omega^{*,\pm}\,\mathrm{d}\tau \lesssim P(E^\kappa(0)) + \int_0^t P(E_4^\kappa(t))E_8^\kappa(t)\,\mathrm{d}\tau.\tag{3.70}$$

3.4.3 Analysis of the boundary integrals

Similarly as in Section 3.3.1, we can decompose the control of these terms in the following steps.

Step 1: Boundary regularity of full time derivatives given by surface tension.

Invoking the boundary condition (3.58) for $[\![\mathbf{Q}^*]\!]$, the term ST becomes

$$
\begin{aligned}
\text{ST}^* =& \sigma\varepsilon^{16}\int_\Sigma \overline{D_t^-}\partial_t^7\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}}\right)\partial_t\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x' \\
&-\kappa\varepsilon^{16}\int_\Sigma \overline{D_t^-}\partial_t^7(1-\overline{\Delta})^2\psi\,\partial_t\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x' - \kappa\varepsilon^{16}\int_\Sigma \overline{D_t^-}\partial_t^7(1-\overline{\Delta})\partial_t\psi\,\partial_t\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x' \\
=:& \text{ST}_0^* + \text{ST}_{1,\kappa}^* + \text{ST}_{2,\kappa}^*.
\end{aligned}
\tag{3.71}
$$

Commuting $\overline{\nabla}\cdot$ with $\overline{D_t^-}$, we have

$$
\text{ST}_0^* = \sigma\varepsilon^{16}\int_\Sigma \overline{\nabla}\cdot\overline{D_t^-}\partial_t^7(\overline{\nabla}\psi/|N|)\,\partial_t\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x' + \underbrace{\sigma\varepsilon^{16}\int_\Sigma \overline{\partial}_i\bar{v}_j^-\overline{\partial}_j\partial_t^7(\overline{\partial}_i\psi/|N|)\partial_t\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x'}_{\text{ST}_0^{*,R}}
\tag{3.72}
$$

Integrating $\overline{\nabla}\cdot$ by parts in the mean curvature term, we get an analogous energy term contributed by surface tension as in Section 3.3.1

$$
\begin{aligned}
&\sigma\varepsilon^{16}\int_\Sigma \overline{\nabla}\cdot\overline{D_t^-}\partial_t^7\left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}}\right)\partial_t\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x' \\
=& -\sigma\varepsilon^{16}\int_\Sigma \frac{\overline{D_t^-}\partial_t^7\overline{\nabla}\psi}{|N|}\cdot\partial_t\overline{D_t^-}\partial_t^7\overline{\nabla}\psi\,\mathrm{d}x' + \sigma\varepsilon^{16}\int_\Sigma\frac{\overline{\nabla}\psi\cdot\overline{D_t^-}\partial_t^7\overline{\nabla}\psi}{|N|^3}\overline{\nabla}\psi\cdot\partial_t\overline{D_t^-}\partial_t^7\overline{\nabla}\psi\,\mathrm{d}x' \\
&\underbrace{-\sigma\varepsilon^{16}\int_\Sigma\frac{\overline{D_t^-}\partial_t^7\overline{\partial}_i\psi}{|N|}\cdot\partial_t(\overline{\partial}_i\bar{v}_j^-\overline{\partial}_j\partial_t^7\psi)\,\mathrm{d}x' - \sigma\varepsilon^{16}\int_\Sigma\frac{\overline{\nabla}\psi\cdot\overline{D_t^-}\partial_t^7\overline{\partial}_i\psi}{|N|^3}\overline{\nabla}\psi\,\partial_t(\overline{\partial}_i\bar{v}_j^-\overline{\partial}_j\partial_t^7\psi)\,\mathrm{d}x'}_{=:\,\text{ST}_1^{*,R}} \\
&\underbrace{-\sigma\varepsilon^{16}\int_\Sigma\left(\left[\overline{D_t^-}\partial_t^6,\frac{1}{|N|}\right]\partial_t\overline{\nabla}\psi + \left[\overline{D_t^-}\partial_t^6,\frac{1}{|N|^3}\right]((\overline{\nabla}\psi\cdot\partial_t\overline{\nabla}\psi)\overline{\nabla}\psi) - \frac{1}{|N|^3}[\overline{D_t^-}\partial_t^6,\overline{\nabla}\psi]\partial_t\overline{\nabla}\psi\right)\cdot\partial_t\overline{\nabla}\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x'}_{=:\,\text{ST}_2^{*,R}}
\end{aligned}
\tag{3.73}
$$

where the right side is further equal to

$$
\begin{aligned}
&-\frac{\sigma\varepsilon^{16}}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma\frac{|\overline{D_t^-}\partial_t^7\overline{\nabla}\psi|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}} - \frac{|\overline{\nabla}\psi\cdot\overline{D_t^-}\partial_t^7\overline{\nabla}\psi|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}^{\,3}}\,\mathrm{d}x' \\
&+\underbrace{\frac{\sigma\varepsilon^{16}}{2}\int_\Sigma\partial_t\left(\frac{1}{|N|}\right)\left|\overline{D_t^-}\partial_t^7\overline{\nabla}\psi\right|^2 - \partial_t\left(\frac{1}{|N|^3}\right)\left|\overline{\nabla}\psi\cdot\overline{D_t^-}\partial_t^7\overline{\nabla}\psi\right|^2\,\mathrm{d}x'}_{=:\,\text{ST}_3^{*,R}} + \text{ST}_1^{*,R} + \text{ST}_2^{*,R}.
\end{aligned}
\tag{3.74}
$$

The first line above together with the inequality (3.37) gives the $\sqrt{\sigma}$-weighted boundary regularity as in step 2 in Section 3.3.1. The term $\text{ST}_1^{*,R}$ is generated by commuting $\overline{D_t^-}$ with $\overline{\nabla}$ (the one falling on $\partial_t^7\psi$) and is directly controlled by the energy. The term $\text{ST}_3^{*,R}$ is controlled in the same way as ST_2^R in step 2 of Section 3.3. The term $\text{ST}_2^{*,R}$ is controlled by integrating ∂_t by parts under time integral, which was also analyzed in [55, Section 4.6]. The term $\text{ST}_0^{*,R}$ is controlled by integrating by parts in ∂_t and then in $\overline{\partial}_j$ under time integral (which is similar to $\text{ST}_2^{*,R}$). Thus, we conclude their estimates by

$$
\int_0^t \text{ST}_0^{*,R} + \text{ST}_1^{*,R} + \text{ST}_2^{*,R} + \text{ST}_3^{*,R}\,\mathrm{d}\tau \lesssim P(E^\kappa(0)) + \int_0^t P(E^\kappa(\tau))\,\mathrm{d}\tau.
\tag{3.75}
$$

Next we analyze the terms $\mathrm{ST}^*_{1,\kappa}$, $\mathrm{ST}^*_{2,\kappa}$ involving the κ-regularization terms. Note that we have to commute $\overline{D_t^-}$ with $1-\overline{\Delta}$ or $\langle\overline{\partial}\rangle = \sqrt{1-\overline{\Delta}}$ when deriving the $\sqrt{\kappa}$-weighte terms. Integrating $1-\overline{\Delta}$ by parts in $\mathrm{ST}^*_{1,\kappa}$

$$
\begin{aligned}
\mathrm{ST}^*_{1,\kappa} &= -\kappa\varepsilon^{16}\int_\Sigma \overline{D_t^-}\partial_t^7(1-\overline{\Delta})^2\psi\,\partial_t\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x' \\
=&-\kappa\varepsilon^{16}\int_\Sigma D_t^{\varphi,-}\partial_t^7(1-\overline{\Delta})\psi\,\partial_t\left(\overline{D_t^-}\partial_t^7(1-\overline{\Delta})\psi\right)\mathrm{d}x' \\
&-\kappa\varepsilon^{16}\int_\Sigma [\overline{D_t^-},1-\overline{\Delta}]\left(\partial_t^7(1-\overline{\Delta})\psi\right)\partial_t\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x' - \kappa\varepsilon^{16}\int_\Sigma \overline{D_t^-}\partial_t^7(1-\overline{\Delta})\psi\,\partial_t\left([1-\overline{\Delta},\overline{D_t^-}]\partial_t^7\psi\right)\mathrm{d}x' \\
=:&-\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\left|\sqrt{\kappa}\varepsilon^8\overline{D_t^-}\partial_t^7(1-\overline{\Delta})\psi\right|_0^2 + \mathrm{ST}^{*,R}_{11,\kappa} + \mathrm{ST}^{*,R}_{12,\kappa}
\end{aligned}
\tag{3.76}
$$

On Σ, the material derivative $D_t^{\varphi,-} = \overline{D_t^-} = \partial_t + \bar{v}^-\cdot\overline{\nabla}$, so the commutator is

$$[\overline{D_t^-},1-\overline{\Delta}]f = [\overline{\Delta},\bar{v}^-\cdot\overline{\nabla}]f = \overline{\Delta}\bar{v}^-\cdot\overline{\nabla}f + 2\overline{\partial}_i\bar{v}_j^-\overline{\partial}_j\overline{\partial}_i f.$$

Then $\mathrm{ST}^{*,R}_{11,\kappa}$ is controlled under time integral by integrating $\overline{\partial}_j$ by parts in the second term

$$
\begin{aligned}
\int_0^t \mathrm{ST}^{*,R}_{11,\kappa}\,\mathrm{d}\tau =& -\kappa\varepsilon^{16}\int_0^t\int_\Sigma \overline{\Delta}\bar{v}_j^-\overline{\partial}_j\left(\partial_t^7(1-\overline{\Delta})\psi\right)\partial_t\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x'\,\mathrm{d}\tau \\
&+2\kappa\varepsilon^{16}\int_0^t\int_\Sigma \overline{\partial}_i\bar{v}_j^-\overline{\partial}_i\left(\partial_t^7(1-\overline{\Delta})\psi\right)\overline{\partial}_j\partial_t\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x'\,\mathrm{d}\tau + \text{ lower order terms} \\
&\lesssim \delta\left|\sqrt{\kappa}\varepsilon^8\overline{D_t^-}\partial_t^8\psi\right|_1^2 + \frac{1}{4\delta}\int_0^t\left|\sqrt{\kappa}\varepsilon^8\overline{\partial}\partial_t^7\psi\right|_2^2|\overline{\partial}\bar{v}^-|^2_{W^{1,\infty}}\,\mathrm{d}\tau \le \delta E_8^\kappa(t) + \int_0^t P(E_8^\kappa(\tau),E_4^\kappa(\tau))\,\mathrm{d}\tau.
\end{aligned}
\tag{3.77}
$$

The control of $\mathrm{ST}^{*,R}_{12,\kappa}$ is easier because there is no term containing 9 time derivatives of ψ. It is directly controlled by using the $\sqrt{\kappa}$-weighted boundary energy obtained above.

$$\mathrm{ST}^{*,R}_{12,\kappa} \lesssim \left|\sqrt{\kappa}\varepsilon^8\overline{D_t^-}\partial_t^7(1-\overline{\Delta})\psi\right|_0\left(\left|\sqrt{\kappa}\varepsilon^8\partial_t^8\psi\right|_2 + \left|\sqrt{\kappa}\varepsilon^8\partial_t^7\psi\right|_2\right)\sqrt{E_4^\kappa(t)} \le E_8^\kappa(t)\sqrt{E_4^\kappa(t)}.$$

The control of $\mathrm{ST}^*_{2,\kappa}$ is similar to $\mathrm{ST}^*_{1,\kappa}$. Using $\langle\overline{\partial}\rangle^2 = 1-\overline{\Delta}$, we have

$$
\begin{aligned}
\mathrm{ST}^*_{2,\kappa} =& -\int_\Sigma\left|\sqrt{\kappa}\varepsilon^8 D_t^{\varphi,-}\partial_t^8\langle\overline{\partial}\rangle\psi\right|^2\mathrm{d}x' \\
&+\kappa\varepsilon^{16}\int_\Sigma[\overline{D_t^-},\overline{\partial}_i]\left(\partial_t^8\overline{\partial}_i\psi\right)\partial_t\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x' + \kappa\varepsilon^{16}\int_\Sigma\overline{D_t^-}\partial_t^8\overline{\partial}_i\psi\left([\overline{\partial}_i\partial_t,\overline{D_t^-}]\partial_t^7\psi\right)\mathrm{d}x' \\
=:&-\int_\Sigma\left|\sqrt{\kappa}\varepsilon^8 D_t^{\varphi,-}\partial_t^8\langle\overline{\partial}\rangle\psi\right|^2\mathrm{d}x' + \mathrm{ST}^{*,R}_{21,\kappa} + \mathrm{ST}^{*,R}_{22,\kappa},
\end{aligned}
\tag{3.78}
$$

where we use the concrete form of the commutators

$$[\overline{D_t^-},\overline{\partial}_i] = -\overline{\partial}_i\bar{v}_j^-\overline{\partial}_j f,\quad [\overline{\partial}_i\partial_t,\overline{D_t^-}]f = \partial_t\left(\overline{\partial}_i\bar{v}_j^-\overline{\partial}_j\overline{D_t^-}f\right) + \partial_t\bar{v}_j^-\overline{\partial}_j\overline{\partial}_i f$$

to get estimates similar to $\mathrm{ST}^{*,R}_{11,\kappa}$ and $\mathrm{ST}^{*,R}_{12,\kappa}$

$$\int_0^t \mathrm{ST}^{*,R}_{21,\kappa} + \mathrm{ST}^{*,R}_{22,\kappa}\,\mathrm{d}\tau \lesssim \delta E_8^\kappa(t) + \int_0^t P(E_8^\kappa(\tau),E_4^\kappa(\tau))\,\mathrm{d}\tau.$$

Hence, the control of ST^* in (3.61) is concluded by

$$\int_0^t \text{ST}\,\mathrm{d}\tau + \frac{1}{2}\int_\Sigma \frac{|\sqrt{\sigma}\varepsilon^8\overline{D_t^-}\partial_t^7\overline{\nabla}\psi|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}^3}\,\mathrm{d}x'$$
$$+\int_\Sigma\left|\sqrt{\kappa}\varepsilon^8\overline{D_t^-}\partial_t^7(1-\overline{\Delta})\psi\right|^2\,\mathrm{d}x' + \int_0^t\int_\Sigma\left|\sqrt{\kappa}\varepsilon^8\overline{D_t^-}\partial_t^8\langle\overline{\partial}\rangle\psi\right|^2\,\mathrm{d}x'\,\mathrm{d}\tau \tag{3.79}$$
$$\le \delta E_8^\kappa(t) + P(E^\kappa(0)) + \int_0^t P(E^\kappa(\tau))\,\mathrm{d}\tau.$$

The term $\text{ST}^{*\prime}$ is controlled in the same way as ST^* by replacing $\partial_t\overline{D_t^-}\partial_t^7\psi$ with $(\bar{v}^+\cdot\overline{\nabla})\overline{D_t^-}\partial_t^7\psi$. We no longer get energy terms, but we can integrate $(\bar{v}^+\cdot\overline{\nabla})$ by parts and use symmetry and the above boundary regularity to control them. Invoking the jump condition, we have

$$\begin{aligned}\text{ST}^{*\prime} &= \sigma\varepsilon^{16}\overline{D_t^-}\partial_t^7\mathcal{H}\,(\bar{v}^+\cdot\overline{\nabla})\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x'\\ &\quad-\kappa\varepsilon^{16}\int_\Sigma\overline{D_t^-}\partial_t^7(1-\overline{\Delta})^2\psi\,(\bar{v}^+\cdot\overline{\nabla})\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x' - \kappa\varepsilon^{16}\int_\Sigma\overline{D_t^-}\partial_t^7(1-\overline{\Delta})\partial_t\psi\,(\bar{v}^+\cdot\overline{\nabla})\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x'\\ &=: \text{ST}_0^{*\prime} + \text{ST}_{1,\kappa}^{*}{}' + \text{ST}_{2,\kappa}^{*}{}'.\end{aligned}\tag{3.80}$$

Following the analysis (3.72)-(3.75), the first term is controlled thanks to the boundary regularity and symmetric structure after integrating $(\bar{v}^+\cdot\overline{\nabla})$ by parts.

$$\text{ST}_0^{*\prime} \overset{L}{=} \frac{1}{2}\sigma\varepsilon^{16}\int_\Sigma(\overline{\nabla}\cdot\bar{v}^+)\left(\frac{|\overline{D_t^-}\partial_t^7\psi|^2}{|N|} - \frac{|\overline{\nabla}\psi\cdot\overline{\nabla}\overline{D_t^-}\partial_t^7\psi|^2}{|N|^3}\right)\mathrm{d}x' \le P(|\overline{\nabla}\psi|_{L^\infty})|\bar{v}^+|_{W^{1,\infty}}\left|\sqrt{\sigma}\varepsilon^8\overline{\nabla}\overline{D_t^-}\partial_t^7\psi\right|_0^2. \tag{3.81}$$

Similarly, we can use the symmetric structure to control $\text{ST}_{1,\kappa}^{*}{}' + \text{ST}_{2,\kappa}^{*}{}'$. We only check the commutators arising in the control of $\text{ST}_{1,\kappa}^{*}{}'$ as an example.

$$\begin{aligned}\text{ST}_{1,\kappa}^{*,R\prime} &:= -\kappa\varepsilon^{16}\int_\Sigma[\overline{D_t^-},1-\overline{\Delta}]\left(\partial_t^7(1-\overline{\Delta})\psi\right)(\bar{v}^+\cdot\overline{\nabla})\left(\overline{D_t^-}\partial_t^7\psi\right)\mathrm{d}x'\\ &\quad-\kappa\varepsilon^{16}\int_\Sigma\left(\overline{D_t^-}\partial_t^7(1-\overline{\Delta})\psi\right)(\bar{v}^+\cdot\overline{\nabla})\left([1-\overline{\Delta},\overline{D_t^-}]\partial_t^7\psi\right)\mathrm{d}x'\\ &\quad-\kappa\varepsilon^{16}\int_\Sigma\overline{D_t^-}\left(\partial_t^7(1-\overline{\Delta})\psi\right)[1-\overline{\Delta},(\bar{v}^+\cdot\overline{\nabla})]\left(\overline{D_t^-}\partial_t^7\psi\right)\mathrm{d}x'\\ &=: \text{ST}_{11,\kappa}^{*,R}{}' + \text{ST}_{12,\kappa}^{*,R}{}' + \text{ST}_{13,\kappa}^{*,R}{}'.\end{aligned}\tag{3.82}$$

The control of $\text{ST}_{11,\kappa}^{*,R}{}' + \text{ST}_{12,\kappa}^{*,R}{}'$ is similar to $\text{ST}_{11,\kappa}^{*,R} + \text{ST}_{12,\kappa}^{*,R}$. We have

$$\begin{aligned}\text{ST}_{11,\kappa}^{*,R}{}' &\overset{L}{=} -\kappa\varepsilon^{16}\int_\Sigma\overline{\Delta}\bar{v}_j^-\overline{\partial}_j\left(\partial_t^7(1-\overline{\Delta})\psi\right)(\bar{v}^+\cdot\overline{\nabla})\left(\overline{D_t^-}\partial_t^7\psi\right)\mathrm{d}x'\\ &\quad+2\kappa\varepsilon^{16}\int_\Sigma\overline{\partial}_j\bar{v}_j^-\overline{\partial}_i\left(\partial_t^7(1-\overline{\Delta})\psi\right)\overline{\partial}_j(\bar{v}^+\cdot\overline{\nabla})\left(\overline{D_t^-}\partial_t^7\psi\right)\mathrm{d}x'\\ &\lesssim |\bar{v}^-|_{W^{2,\infty}}|\bar{v}^+|_{L^\infty}\left|\sqrt{\kappa}\varepsilon^8\partial_t^7\psi\right|_3\left(\left|\sqrt{\kappa}\varepsilon^8\partial_t^7\psi\right|_3 + \left|\sqrt{\kappa}\varepsilon^8\partial_t^8\psi\right|_2\right) \lesssim E_4^\kappa(t)E_8^\kappa(t),\end{aligned}\tag{3.83}$$

and

$$\text{ST}_{12,\kappa}^{*,R}{}' \lesssim |\bar{v}^-|_{W^{2,\infty}}|\bar{v}^+|_{L^\infty}\left(\left|\sqrt{\kappa}\varepsilon^8\partial_t^8\psi\right|_2 + \left|\sqrt{\kappa}\varepsilon^8\partial_t^7\psi\right|_3\right)^2 \lesssim E_4^\kappa(t)E_8^\kappa(t). \tag{3.84}$$

The extra term $\text{ST}_{13,\kappa}^{*,R}{}'$ is also directly controlled

$$\begin{aligned}\text{ST}_{13,\kappa}^{*,R}{}' &= \kappa\varepsilon^{16}\int_\Sigma\overline{D_t^-}\left(\partial_t^7(1-\overline{\Delta})\psi\right)\left(\overline{\Delta}\bar{v}_j^+\overline{\partial}_j + 2\overline{\partial}_i\bar{v}_j^+\overline{\partial}_j\overline{\partial}_i\right)\left(\overline{D_t^-}\partial_t^7\psi\right)\mathrm{d}x'\\ &\lesssim |\bar{v}^+|_{W^{2,\infty}}|\bar{v}^-|_{L^\infty}\left(\left|\sqrt{\kappa}\varepsilon^8\partial_t^8\psi\right|_2 + \left|\sqrt{\kappa}\varepsilon^8\partial_t^7\psi\right|_3\right)^2 \lesssim E_4^\kappa(t)E_8^\kappa(t).\end{aligned}\tag{3.85}$$

Thus we have

$$\mathrm{ST}^{*}_{1,\kappa}{}' \lesssim \frac{1}{2}\int_\Sigma (\overline{\nabla}\cdot\bar{v}^+)\left|\sqrt{\kappa}\varepsilon^8\overline{D_t^-}\partial_t^7(1-\overline{\Delta})\psi\right|^2 + E_4^\kappa(t)E_8^\kappa(t). \tag{3.86}$$

Similarly, we have

$$\begin{aligned}\int_0^t \mathrm{ST}^{*}_{2,\kappa}{}'\,\mathrm{d}\tau =&\ \kappa\varepsilon^{16}\int_0^t\int_\Sigma [\overline{D_t^-},\overline{\partial}_i](\partial_t^8\overline{\partial}_i\psi)\,(\bar{v}^+\cdot\overline{\nabla})\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x'\,\mathrm{d}\tau\\ &-\kappa\varepsilon^{16}\int_0^t\int_\Sigma (\overline{D_t^-}\partial_t^8\overline{\partial}_i\psi)\,\overline{\partial}_i\left((\bar{v}^+\cdot\overline{\nabla})\overline{D_t^-}\partial_t^7\psi\right)\mathrm{d}x'\,\mathrm{d}\tau\\ &-\kappa\varepsilon^{16}\int_0^t\int_\Sigma \overline{D_t^-}\partial_t^7\partial_t\psi\,(\bar{v}^+\cdot\overline{\nabla})\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x'\\ \lesssim&\ \delta\left|\sqrt{\kappa}\varepsilon^8\overline{D_t^-}\partial_t^8\langle\overline{\partial}\rangle\psi\right|_0^2 + \int_0^t |\bar{v}^\pm|^2_{W^{1,\infty}}\left(\left|\sqrt{\kappa}\varepsilon^8\partial_t^7\psi\right|_3^2 + \left|\sqrt{\kappa}\varepsilon^8\partial_t^8\psi\right|_2^2\right)\mathrm{d}\tau.\end{aligned} \tag{3.87}$$

Hence, we have the estimate of $\mathrm{ST}^{*}{}'$:

$$\int_0^t \mathrm{ST}^{*}{}'\,\mathrm{d}\tau \lesssim \delta E_8^\kappa(t) + \int_0^t E_8^\kappa(\tau)E_4^\kappa(\tau)\,\mathrm{d}\tau. \tag{3.88}$$

What's more, we can also control the remainder term $R_\Sigma^{*,\pm} := \pm\varepsilon^{16}\int_\Sigma \mathbf{Q}^{*,\pm}\overline{D_t^\pm}\bar{v}^-\cdot\overline{\nabla}\partial_t^7\psi\,\mathrm{d}x'$. Indeed, we use Gauss-Green formula to write it to be an interior intergral.

$$R_\Sigma^{*,\pm} \overset{L}{=} -\varepsilon^{16}\int_{\Omega^\pm}\partial_3\mathbf{Q}^{*,\pm}\,\overline{D_t^\pm}\bar{v}^-\cdot\overline{\nabla}\partial_t^7\varphi\,\mathrm{d}x \tag{3.89}$$

Recall that $\mathbf{Q}^{*,\pm} = D_t^{\varphi,-}\partial_t^7q^\pm - D_t^{\varphi,-}\partial_t^7\psi\partial_3q^\pm$. Note that $[\partial_3^\varphi, D_t^{\varphi,-}]\partial_t^7q = \partial_3^\varphi v_j^-\partial_j^\varphi q = (\partial_3^\varphi\bar{v}^-\cdot\overline{\nabla})\partial_t^7q + (\partial_3^\varphi v^-\cdot\mathbf{N})\partial_t^7\partial_3q$, so one can still convert ∂q to a tangential derivative of v, b. We now integrate by parts $\overline{D_t^-}$ to get

$$\begin{aligned}\int_0^t R_\Sigma^{*,\pm}\,\mathrm{d}\tau \overset{L}{=}&\ \varepsilon^{16}\int_0^t\int_{\Omega^\pm}\partial_3(\partial_t^7q^\pm - \partial_t^7\varphi\partial_3q^\pm)\overline{D_t^\pm}\bar{v}^-\cdot D_t^{\varphi,-}\overline{\nabla}\partial_t^7\varphi\,\mathrm{d}x\,\mathrm{d}\tau\\ &-\varepsilon^{16}\int_0^t\int_{\Omega^\pm}\partial_3(\partial_t^7q^\pm - \partial_t^7\varphi\partial_3q^\pm)\overline{D_t^\pm}\bar{v}^-\cdot\overline{\nabla}\partial_t^7\varphi\,\mathrm{d}x.\end{aligned} \tag{3.90}$$

Using the reduction for ∂_3q again, we can control the above integral by

$$\int_0^t R_\Sigma^{*,\pm}\,\mathrm{d}\tau \lesssim \delta\|\varepsilon^8\partial_3\partial_t^7q^\pm\|^2_{0,\pm} + \int_0^t P(E_4^\kappa(\tau))E_8^\kappa(\tau)\,\mathrm{d}\tau \lesssim \delta E_8^\kappa(t) + \int_0^t P(E_4^\kappa(\tau))E_8^\kappa(\tau)\,\mathrm{d}\tau. \tag{3.91}$$

Step 2: Control of VS term.

Now we start to analyze the most difficult boundary term

$$\mathrm{VS}^* := \varepsilon^{16}\int_\Sigma \overline{D_t^-}\partial_t^7q^-\,([\![\bar{v}]\!]\cdot\overline{\nabla})\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x'. \tag{3.92}$$

Note that there is no spatial derivative $\overline{\partial}$ in VS^*, so we cannot integrate $\overline{\partial}^{1/2}$ by parts as in step 3 in Section 3.3.1. To overcome this difficulty, we try to rewrite the term $\overline{D_t^-}\partial_t^7\psi$ by invoking the kinematic boundary condition

$$\begin{aligned}\overline{D_t^-}\partial_t^7\psi &= \partial_t^8\psi + \bar{v}^-\cdot\overline{\nabla}\partial_t^7\psi = \partial_t^7(v^-\cdot N) - v^-\cdot\partial_t^7N\\ &= \partial_t^7v^-\cdot N + [\partial_t^7, N_i, v_i^-],\end{aligned}$$

and thus

$$\begin{aligned}\mathrm{VS}^* =& \varepsilon^{16}\int_\Sigma \overline{D_t^-}\partial_t^7 q^- \, ([\![\bar v]\!]\cdot\overline{\nabla})\partial_t^7 v^- \cdot N \,\mathrm{d}x' \\ &+ \varepsilon^{16}\int_\Sigma \overline{D_t^-}\partial_t^7 q^- \, \partial_t^7 v^- \cdot ([\![\bar v]\!]\cdot\overline{\nabla})N\,\mathrm{d}x' + \varepsilon^{16}\int_\Sigma \overline{D_t^-}\partial_t^7 q^- \, ([\![\bar v]\!]\cdot\overline{\nabla})[\partial_t^7, N_i, v_i^-]\,\mathrm{d}x' \\ =:& \ \mathrm{VS}_0^* + \ \mathrm{VS}_1^{*,ZB} + \ \mathrm{VS}_2^{*,ZB} \end{aligned} \tag{3.93}$$

Using divergence theorem, we convert VS_0^* to an interior integral in Ω^-

$$\begin{aligned}\mathrm{VS}_0^* =& \varepsilon^{16}\int_{\Omega^-} D_t^{\varphi,-}\partial_t^7 q^- \, \nabla^\varphi\cdot\left(([\![\bar v]\!]\cdot\overline{\nabla})\partial_t^7 v^-\right)\mathrm{d}\mathcal{V}_t + \varepsilon^{16}\int_{\Omega^-}\partial_i^\varphi D_t^{\varphi,-}\partial_t^7 q^- \, ([\![\bar v]\!]\cdot\overline{\nabla})\partial_t^7 v_i^- \,\mathrm{d}\mathcal{V}_t \\ =:& \ \mathrm{VS}_{01}^* + \ \mathrm{VS}_{02}^*, \end{aligned} \tag{3.94}$$

where $[\![\bar v]\!] = \bar v^+ - \bar v^-$ is defined via Sobolev extension in Ω^-. In VS_{01}^*, we want to commute $\nabla^\varphi\cdot$ with $([\![\bar v]\!]\cdot\overline{\nabla})$ in order to get a similar cancellation structure as in $ZB + Z$. The commutator is

$$\begin{aligned} [\partial_i^\varphi, [\![\bar v]\!]\cdot\overline{\nabla}]f =& \overline{\partial}_i([\![\bar v]\!]\cdot\overline{\nabla}f) - \frac{\overline{\partial}_i\varphi}{\partial_3\varphi}\partial_3([\![\bar v]\!]\cdot\overline{\nabla}f) - ([\![\bar v]\!]\cdot\overline{\nabla})\left(\overline{\partial}_i f - \frac{\overline{\partial}_i\varphi}{\partial_3\varphi}\partial_3 f\right) \\ =& (\overline{\partial}_i\,[\![\bar v]\!]_j)(\overline{\partial}_j f) - \frac{\overline{\partial}_i\varphi}{\partial_3\varphi}(\partial_3\,[\![\bar v]\!]_j)(\overline{\partial}_j f) + [\![\bar v]\!]_j\,\overline{\partial}_j\left(\frac{\overline{\partial}_i\varphi}{\partial_3\varphi}\right)\partial_3 f \\ =& \partial_i^\varphi\,[\![\bar v]\!]\cdot\overline{\nabla}f - ([\![\bar v]\!]\cdot\overline{\nabla})\mathbf{N}_i\partial_3^\varphi f + \mathbf{N}_i\frac{([\![\bar v]\!]\cdot\overline{\nabla})\partial_3\varphi}{\partial_3\varphi}\partial_3^\varphi f, \quad i=1,2, \\ [\partial_3^\varphi, [\![\bar v]\!]\cdot\overline{\nabla}]f =& \partial_3^\varphi\,[\![\bar v]\!]\cdot\overline{\nabla}f + \frac{([\![\bar v]\!]\cdot\overline{\nabla})\partial_3\varphi}{\partial_3\varphi}\partial_3^\varphi f. \end{aligned}$$

Commuting $\nabla^\varphi\cdot$ with $([\![\bar v]\!]\cdot\overline{\nabla})$, we get

$$\begin{aligned}\mathrm{VS}_{01}^* =& \varepsilon^{16}\int_{\Omega^-} D_t^{\varphi,-}\partial_t^7 q^- \, \nabla^\varphi\cdot\left(([\![\bar v]\!]\cdot\overline{\nabla})\partial_t^7 v^-\right)\mathrm{d}\mathcal{V}_t \\ =& \varepsilon^{16}\int_{\Omega^-} D_t^{\varphi,-}\partial_t^7 q^- \, ([\![\bar v]\!]\cdot\overline{\nabla})\left(\nabla^\varphi\cdot\partial_t^7 v^-\right)\mathrm{d}\mathcal{V}_t - \varepsilon^{16}\int_{\Omega^-} D_t^{\varphi,-}\partial_t^7 q^- \, \partial_3\partial_t^7 v^-\cdot([\![\bar v]\!]\cdot\overline{\nabla})\mathbf{N}\,\mathrm{d}x \\ &+ \varepsilon^{16}\int_{\Omega^-} D_t^{\varphi,-}\partial_t^7 q^- \, \partial_i^\varphi\,[\![\bar v]\!]\cdot\overline{\nabla}\partial_t^7 v_i^-\,\mathrm{d}\mathcal{V}_t + \varepsilon^{16}\int_{\Omega^-} D_t^{\varphi,-}\partial_t^7 q^- \, ([\![\bar v]\!]\cdot\overline{\nabla})\partial_3\varphi\,\partial_3^\varphi\partial_t^7 v^-\cdot\mathbf{N}\,\mathrm{d}x \\ =:& \ \mathrm{VS}_{011}^* + \ \mathrm{VS}_{011}^{*,Z} + \ \mathrm{VS}_{011}^{*,R} + \ \mathrm{VS}_{012}^{*,R} \end{aligned} \tag{3.95}$$

Next we introduce $\mathbf{F}^\sharp := \partial_t^7 f - \partial_t^7\psi\partial_3^\varphi f$ to be the Alinhac good unknown of f with respect to ∂_t^7 in order to commute ∇^φ with ∂_t^7. Namely, we have

$$\partial_t^7\partial_i^\varphi f = \partial_i^\varphi\mathbf{F}^\sharp + \mathfrak{C}_i^\sharp(f), \quad \partial_t^7 D_t^\varphi f = D_t^\varphi\mathbf{F}^\sharp + \mathfrak{D}^\sharp(f),$$

where $\mathfrak{C}^\sharp, \mathfrak{D}^\sharp$ are defined in the same way as (3.9)-(3.10) with $\mathcal{T}^\gamma = \partial_t^7$. With this formulation, we have

$$\nabla^\varphi\cdot\partial_t^7 v^- = \nabla^\varphi\cdot\mathbf{V}^{\sharp,-} + \partial_i^\varphi(\partial_t^7\varphi\partial_3^\varphi v_i^-) = \partial_t^7(\nabla^\varphi\cdot v^-) - \mathfrak{C}_i^\sharp(v_i^-) + \partial_i^\varphi(\partial_t^7\varphi\partial_3^\varphi v_i^-).$$

Now we insert the good unknowns in VS_{011}^* to get

$$\begin{aligned}\mathrm{VS}_{011}^* =& \varepsilon^{16}\int_{\Omega^-} D_t^{\varphi,-}\partial_t^7 q^- \, ([\![\bar v]\!]\cdot\overline{\nabla})\partial_t^7(\nabla^\varphi\cdot v^-)\,\mathrm{d}\mathcal{V}_t \underbrace{-\varepsilon^{16}\int_{\Omega^-} D_t^{\varphi,-}\partial_t^7 q^- \, ([\![\bar v]\!]\cdot\overline{\nabla})\left(\mathfrak{C}_i^\sharp(v_i^-) - \partial_i^\varphi(\partial_t^7\varphi\partial_3^\varphi v_i^-)\right)\mathrm{d}\mathcal{V}_t}_{\mathrm{VS}_{012}^{*,Z}} \\ =& -\varepsilon^{16}\int_{\Omega^-}\mathcal{F}_p^{\pm} D_t^{\varphi,-}\partial_t^7 p^- \, ([\![\bar v]\!]\cdot\overline{\nabla})\partial_t^7 D_t^{\varphi,-}p^-\,\mathrm{d}\mathcal{V}_t + \varepsilon^{16}\int_{\Omega^-} D_t^{\varphi,-}\partial_t^7(\frac{1}{2}|b^-|^2)\,([\![\bar v]\!]\cdot\overline{\nabla})\partial_t^7(\nabla^\varphi\cdot v^-)\,\mathrm{d}\mathcal{V}_t + \ \mathrm{VS}_{012}^{*,Z} \\ =:& \ \mathrm{VS}_{0111}^* + \ \mathrm{VS}_{0111}^{*,B} + \ \mathrm{VS}_{012}^{*,Z}. \end{aligned} \tag{3.96}$$

By the definition of $\mathbf{P}^{\sharp,-}$

$$D_t^{\varphi,-}\partial_t^7 p^- = D_t^{\varphi,-}\mathbf{P}^{\sharp,-} + D_t^{\varphi,-}(\partial_t^7\varphi\partial_3^\varphi p^-),\ \ \partial_t^7 D_t^{\varphi,-}p^- = D_t^{\varphi,-}\mathbf{P}^{\sharp} + \mathfrak{D}^{\sharp}(p^-).$$

Then we integrate $(\llbracket\bar{v}\rrbracket\cdot\overline{\nabla})$ by parts and use symmetry to find

$$\begin{aligned}\text{VS}^*_{0111} =& -\varepsilon^{16}\int_{\Omega^-}\mathcal{F}_p^{\pm}D_t^{\varphi,-}\mathbf{P}^{\sharp,-}(\llbracket\bar{v}\rrbracket\cdot\overline{\nabla})D_t^{\varphi,-}\mathbf{P}^{\sharp,-}\,\mathrm{d}\mathcal{V}_t \\ &\underbrace{+\varepsilon^{16}\int_{\Omega^-}\mathcal{F}_p^{\pm}D_t^{\varphi,-}\mathbf{P}^{\sharp,-}\left((\overline{\nabla}\cdot\llbracket\bar{v}\rrbracket)D_t^{\varphi,-}(\partial_t^7\varphi\partial_3^\varphi p^-)-(\llbracket\bar{v}\rrbracket\cdot\overline{\nabla})\left(\mathfrak{D}^{\sharp}(p^-)-D_t^{\varphi,-}(\partial_t^7\varphi\partial_3^\varphi p^-)\right)\right)\mathrm{d}\mathcal{V}_t}_{\text{VS}^{*,R}_{0111}} \\ &\underbrace{-\varepsilon^{16}\int_{\Omega^-}\mathcal{F}_p^{\pm}D_t^{\varphi,-}(\partial_t^7\varphi\partial_3^\varphi p^-)(\llbracket\bar{v}\rrbracket\cdot\overline{\nabla})\mathfrak{D}^{\sharp}(p^-)\,\mathrm{d}\mathcal{V}_t}_{\text{VS}^{*,R}_{0112}} \\ \overset{\llbracket\bar{v}\rrbracket\cdot\overline{\nabla}}{=\!=\!=}& -\frac{1}{2}\int_{\Omega^-}(\overline{\nabla}\cdot\llbracket\bar{v}\rrbracket)(\sqrt{\mathcal{F}_p^{\pm}}\varepsilon^8 D_t^{\varphi,-}\mathbf{P}^{\sharp,-})^2\,\mathrm{d}\mathcal{V}_t + \text{VS}^{*,R}_{0111} + \text{VS}^{*,R}_{0112},\end{aligned} \tag{3.97}$$

where the first term on the right side is controlled by $\left\|(\mathcal{F}_p^{\pm})^{\frac{1}{2}}\varepsilon^8\mathbf{P}^{*,-}\right\|_0^2\|\overline{\nabla}\llbracket\bar{v}\rrbracket\|^2_{L^\infty}$. Next we adapt the analysis for $Z^{\pm}+ZB^{\pm}$ term to the control of $\text{VS}_1^{*,ZB}+\text{VS}^{*,Z}_{011}$ and $\text{VS}_2^{*,ZB}+\text{VS}^{*,Z}_{012}$. Using Gauss-Green formula, we have

$$\begin{aligned}\text{VS}_1^{*,ZB}+\text{VS}^{*,Z}_{011} =& \varepsilon^{16}\int_{\Omega^-}(D_t^{\varphi,-}\partial_3^\varphi\partial_t^7 q^-)(\llbracket\bar{v}\rrbracket\cdot\overline{\nabla})\mathbf{N}\cdot\partial_t^7 v^-\,\mathrm{d}\mathcal{V}_t \\ &+\varepsilon^{16}\int_{\Omega^-}[\partial_3^\varphi, D_t^{\varphi,-}](\partial_t^7 q^-)(\llbracket\bar{v}\rrbracket\cdot\overline{\nabla})\mathbf{N}\cdot\partial_t^7 v^- + D_t^{\varphi,-}\partial_t^7 q^-\,\partial_3^\varphi\left((\llbracket\bar{v}\rrbracket\cdot\overline{\nabla})\mathbf{N}\right)\cdot\partial_t^7 v^-\,\mathrm{d}\mathcal{V}_t \\ =:& \varepsilon^{16}\int_{\Omega^-}(D_t^{\varphi,-}\partial_3^\varphi\partial_t^7 q^-)(\llbracket\bar{v}\rrbracket\cdot\overline{\nabla})\mathbf{N}\cdot\partial_t^7 v^-\,\mathrm{d}\mathcal{V}_t + \text{VS}_1^{*,ZR}.\end{aligned} \tag{3.98}$$

The main term is controlled by integrating $D_t^{\varphi,-}$ by parts under time integral and invoking the momentum equation to replace $\partial_3^\varphi q^-$ by tangential derivatives of v^-, b^-:

$$\begin{aligned}&\varepsilon^{16}\int_0^t\int_{\Omega^-}(D_t^{\varphi,-}\partial_3^\varphi\partial_t^7 q^-)(\llbracket\bar{v}\rrbracket\cdot\overline{\nabla})\mathbf{N}\cdot\partial_t^7 v^-\,\mathrm{d}\mathcal{V}_t\,\mathrm{d}\tau \\ \overset{L}{=}& -\varepsilon^{16}\int_0^t\int_{\Omega^-}(\partial_3^\varphi\partial_t^7 q^-)D_t^{\varphi,-}\left((\llbracket\bar{v}\rrbracket\cdot\overline{\nabla})\mathbf{N}\cdot\partial_t^7 v^-\right)\mathrm{d}\mathcal{V}_t\,\mathrm{d}\tau + \varepsilon^{16}\int_{\Omega^-}(\partial_3^\varphi\partial_t^7 q^-)(\llbracket\bar{v}\rrbracket\cdot\overline{\nabla})\mathbf{N}\cdot\partial_t^7 v^-\,\mathrm{d}\mathcal{V}_t\Big|_0^t \\ \lesssim& \delta\|\varepsilon^8\partial_3^\varphi\partial_t^7 q^-\|^2_{0,-} + P(E_4^\kappa(0))E_8^\kappa(0) + \int_0^t\|\varepsilon^8\partial_3^\varphi\partial_t^7 q^-\|_{0,-}\|\varepsilon^8(\overline{\partial}\partial_t^7 v^-, \partial_t^8 v^-)\|_0 P(E_4^\kappa(\tau))\,\mathrm{d}\tau \\ \leq& \delta E_8^\kappa(t) + P(E_4^\kappa(0))E_8^\kappa(0) + \int_0^t P(E_4^\kappa(\tau))E_8^\kappa(\tau)\,\mathrm{d}\tau,\quad \forall\delta\in(0,1).\end{aligned} \tag{3.99}$$

For $\text{VS}_2^{*,ZB}+\text{VS}^{*,Z}_{012}$, we recall that the term $\mathfrak{C}_i^{\sharp}(v_i^-)$ in $\text{VS}^{*,Z}_{012}$ includes a term $[\partial_t^7, \mathbf{N}_i/\partial_3\varphi, v_i^-]$ which also appears in $\text{VS}_2^{*,ZB}$. Thus we can again use the Gauss-Green formula to analyze this term. Let us first compute the commutator in $\text{VS}^{*,Z}_{012}$:

$$\begin{aligned}\mathfrak{C}_i^{\sharp}(v_i^-) - \partial_i^\varphi(\partial_t^7\varphi\partial_3^\varphi v_i^-) =& -\partial_i^\varphi\partial_t^7\varphi\partial_3^\varphi v_i + \left[\partial_t^7, \frac{\mathbf{N}_i}{\partial_3\varphi}, \partial_3 v_i^-\right] - \partial_3 v_i^-\left[\partial_t^7\mathbf{N}_i, \frac{1}{\partial_3\varphi}\right] - \mathbf{N}_i\partial_3 v_i^-\left[\partial_t^6, \frac{1}{(\partial_3\varphi)^2}\right]\partial_t\partial_3\varphi \\ =& -\partial_i^\varphi\partial_t^7\varphi\partial_3^\varphi v_i + \left[\partial_t^7, \mathbf{N}_i, \partial_3 v_i^-\right] + \sum_{k=1}^6\binom{7}{k}[\partial_t^k, (\partial_3\varphi)^{-1}]\mathbf{N}_i\,\partial_t^{7-k}\partial_3 v_i^- \\ &- \partial_3 v_i^-\left[\partial_t^7, \mathbf{N}_i, \frac{1}{\partial_3\varphi}\right] - \mathbf{N}_i\partial_3 v_i^-\left[\partial_t^6, \frac{1}{(\partial_3\varphi)^2}\right]\partial_t\partial_3\varphi \\ =:& \frac{1}{\partial_3\varphi}\left[\partial_t^7, \mathbf{N}_i, \partial_3 v_i^-\right] + \mathfrak{C}_i^{\sharp,R}(v_i^-)\end{aligned}$$

Then

$$\mathrm{VS}_2^{*,ZB}+\ \mathrm{VS}_{012}^{*,Z}=\varepsilon^{16}\int_\Sigma D_t^{\varphi,-}\partial_t^7 q^-\,([\![\bar v]\!]\cdot\overline{\nabla})[\partial_t^7,N_i,v_i^-]\,\mathrm{d}x'-\varepsilon^{16}\int_{\Omega^-}D_t^{\varphi,-}\partial_t^7 q^-\,([\![\bar v]\!]\cdot\overline{\nabla})\left(\frac{1}{\partial_3\varphi}\left[\partial_t^7,\mathbf{N}_i,\partial_3 v_i^-\right]\right)\mathrm{d}\mathcal{V}_t$$
$$\underbrace{-\varepsilon^{16}\int_{\Omega^-}D_t^{\varphi,-}\partial_t^7 q^-\,([\![\bar v]\!]\cdot\overline{\nabla})\mathfrak{C}_i^{\sharp,R}(v_i^-)\,\mathrm{d}\mathcal{V}_t}_{\mathrm{VS}_2^{*,ZR}}$$
$$=\varepsilon^{16}\int_{\Omega^-}\partial_3^\varphi\left(D_t^{\varphi,-}\partial_t^7 q^-\right)([\![\bar v]\!]\cdot\overline{\nabla})[\partial_t^7,\mathbf{N}_i,v_i^-]\,\mathrm{d}x+\ \mathrm{VS}_2^{*,ZR}+\ \text{lower order terms,}\tag{3.100}$$

where the first term on the right side is again controlled by integrating $D_t^{\varphi,-}$ by parts under time integral. We omit the details and just list the result

$$\int_0^t\int_{\Omega^-}\varepsilon^{16}\partial_3^\varphi\left(D_t^{\varphi,-}\partial_t^7 q^-\right)([\![\bar v]\!]\cdot\overline{\nabla})[\partial_t^7,\mathbf{N}_i,v_i^-]\,\mathrm{d}x\,\mathrm{d}\tau\le\delta E_8^\kappa(t)+P(E_4^\kappa(0))E_8^\kappa(0)+\int_0^t P(E_4^\kappa(\tau))E_8^\kappa(\tau)\,\mathrm{d}\tau,\quad\forall\delta\in(0,1).$$

Now the term VS_1^* is controlled except for those remainder terms $\mathrm{VS}_{011}^{*,R}$, $\mathrm{VS}_{012}^{*,R}$, $\mathrm{VS}_{0111}^{*,B}$, $\mathrm{VS}_{0111}^{*,R}$, $\mathrm{VS}_{0112}^{*,R}$, $\mathrm{VS}_1^{*,ZR}$ and $\mathrm{VS}_2^{*,ZR}$. In fact, apart from $\mathrm{VS}_{0111}^{*,B}$, the other remainder terms can be directly controlled by counting the number of derivatives and invoking the reduction for $\partial_3^\varphi\partial_t^7 v^-\cdot\mathbf{N}$ and $\partial_3^\varphi\partial_t^7 q^-$. There is no loss of Mach number in these remainder terms. In fact, when $\partial_t^8 p^-$ appears in the remainder terms, either we have $\varepsilon^{16}\mathcal{F}_p^\pm$-weight to control it directly, or we can integrate by parts $D_t^{\varphi,-}$ and $([\![\bar v]\!]\cdot\overline{\nabla})$ under time integral to move one time derivative to v_i^-. Besides, the control of $\partial_t^7\varphi,\partial_t^8\varphi$ depends on the boundary regularity contributed by surface tension and so depends on σ^{-1}. Therefore, we can conclude the estimates of VS_1^* by

$$\mathrm{VS}_{01}^*+\ \mathrm{VS}_1^{*,ZB}+\ \mathrm{VS}_2^{*,ZB}\le\ \mathrm{VS}_{0111}^{*,B}+\delta E_8^\kappa(t)+P(E_4^\kappa(0))E_8^\kappa(0)+\int_0^t P(\sigma^{-1},E^\kappa(\tau))E_8^\kappa(\tau)\,\mathrm{d}\tau\quad\forall\delta\in(0,1).\tag{3.101}$$

Next we control $\mathrm{VS}_{02}^*=\varepsilon^{16}\int_{\Omega^-}\partial_i^\varphi D_t^{\varphi,-}\partial_t^7 q^-\,([\![\bar v]\!]\cdot\overline{\nabla})\partial_t^7 v_i^-\,\mathrm{d}\mathcal{V}_t$. First, we commute $D_t^{\varphi,-}$ with ∂_i^φ to get

$$\mathrm{VS}_{02}^*=\varepsilon^{16}\int_{\Omega^-}D_t^{\varphi,-}\partial_i^\varphi\partial_t^7 q^-\,([\![\bar v]\!]\cdot\overline{\nabla})\partial_t^7 v_i^-\,\mathrm{d}\mathcal{V}_t+\varepsilon^{16}\int_{\Omega^-}\partial_i^\varphi v_j^-\partial_j^\varphi\partial_t^7 q^-\,([\![\bar v]\!]\cdot\overline{\nabla})\partial_t^7 v_i^-\,\mathrm{d}\mathcal{V}_t$$
$$=:\mathrm{VS}_{021}^*+\ \mathrm{VS}_{021}^{*,R}.\tag{3.102}$$

In the first term, we integrate by parts $D_t^{\varphi,-}$ under time integral and commute $D_t^{\varphi,-}$ with $([\![\bar v]\!]\cdot\overline{\nabla})$ to get

$$\int_0^t\ \mathrm{VS}_{021}^*\,\mathrm{d}\tau\overset{L}{=}-\varepsilon^{16}\int_0^t\int_{\Omega^-}\partial_i^\varphi\partial_t^7 q^-\,([\![\bar v]\!]\cdot\overline{\nabla})D_t^{\varphi,-}\partial_t^7 v_i^-\,\mathrm{d}\mathcal{V}_t\,\mathrm{d}\tau+\varepsilon^{16}\int_{\Omega^-}\partial_i^\varphi\partial_t^7 q^-\,([\![\bar v]\!]\cdot\overline{\nabla})\partial_t^7 v_i^-\,\mathrm{d}\mathcal{V}_t\Big|_0^t$$
$$-\varepsilon^{16}\int_0^t\int_{\Omega^-}\partial_i^\varphi\partial_t^7 q^-\,[D_t^{\varphi,-},([\![\bar v]\!]\cdot\overline{\nabla})]\partial_t^7 v_i^-\,\mathrm{d}\mathcal{V}_t\,\mathrm{d}\tau$$
$$=:\int_0^t\ \mathrm{VS}_{0211}^*\,\mathrm{d}\tau+\ \mathrm{VS}_{022}^{*,R}+\int_0^t\ \mathrm{VS}_{023}^{*,R}\,\mathrm{d}\tau.\tag{3.103}$$

Next we insert the good unknowns $\mathbf{Q}^{\sharp,-}$ and $\mathbf{V}^{\sharp,-}$ and invoke again the momentum equation $\rho^-D_t^{\varphi,-}\mathbf{V}^{\sharp,-}-(b^-\cdot\nabla^\varphi)\mathbf{B}^{\sharp,-}=-\nabla^\varphi\mathbf{Q}^{\sharp,-}+\mathcal{R}_v^{\sharp,-}-\mathfrak{C}^\sharp(q^-)$ to get

$$\mathrm{VS}_{0211}^*=\ -\varepsilon^{16}\int_{\Omega^-}\partial_i^\varphi(\mathbf{Q}^{\sharp,-}-\partial_t^7\varphi\partial_3^\varphi q^-)\,([\![\bar v]\!]\cdot\overline{\nabla})D_t^{\varphi,-}(\mathbf{V}^{\sharp,-}-\partial_t^7\varphi\partial_3^\varphi v_i)\,\mathrm{d}\mathcal{V}_t$$
$$=\varepsilon^{16}\int_{\Omega^-}\rho^-D_t^{\varphi,-}\mathbf{V}^{\sharp,-}\cdot([\![\bar v]\!]\cdot\overline{\nabla})D_t^{\varphi,-}\mathbf{V}^{\sharp,-}\,\mathrm{d}\mathcal{V}_t-\varepsilon^{16}\int_{\Omega^-}(b^-\cdot\nabla^\varphi)\mathbf{B}^{\sharp,-}\cdot([\![\bar v]\!]\cdot\overline{\nabla})D_t^{\varphi,-}\mathbf{V}^{\sharp,-}\,\mathrm{d}\mathcal{V}_t$$
$$+\varepsilon^{16}\int_{\Omega^-}\left(\mathfrak{C}^\sharp(q^-)-\mathcal{R}_v^{\sharp,-}\right)\cdot([\![\bar v]\!]\cdot\overline{\nabla})D_t^{\varphi,-}\mathbf{V}^{\sharp,-}\,\mathrm{d}\mathcal{V}_t+\varepsilon^{16}\int_{\Omega^-}\partial_i^\varphi(\partial_t^7\varphi\partial_3^\varphi q^-)\,([\![\bar v]\!]\cdot\overline{\nabla})D_t^{\varphi,-}\partial_t^7 v_i^-\,\mathrm{d}\mathcal{V}_t$$
$$=:\varepsilon^{16}\int_{\Omega^-}\rho^-D_t^{\varphi,-}\mathbf{V}^{\sharp,-}\,([\![\bar v]\!]\cdot\overline{\nabla})D_t^{\varphi,-}\mathbf{V}^{\sharp,-}\,\mathrm{d}\mathcal{V}_t+\ \mathrm{VS}_{0211}^{*,B}+\ \mathrm{VS}_{0211}^{*,R}+\ \mathrm{VS}_{0212}^{*,R},\tag{3.104}$$

where the first term is again controlled by integrating by parts in $(\llbracket \bar{v} \rrbracket \cdot \overline{\nabla})$ and using symmetry

$$\varepsilon^{16} \int_{\Omega^-} \rho^- D_t^{\varphi,-} \mathbf{V}^{\sharp,-} (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) D_t^{\varphi,-} \mathbf{V}^{\sharp,-} \,\mathrm{d}\mathcal{V}_t = \frac{\varepsilon^{16}}{2} \int_{\Omega^-} \left(\overline{\nabla} \cdot (\rho^- \llbracket \bar{v} \rrbracket) \right) \left| D_t^{\varphi,-} \mathbf{V}^{\sharp,-} \right|^2 \,\mathrm{d}\mathcal{V}_t \leq P(E_4^\kappa(t)) E_8^\kappa(t). \tag{3.105}$$

Next we wish to combine $\mathrm{VS}_{0211}^{*,B}$ with $\mathrm{VS}_{0111}^{*,B} := \varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \partial_t^7 (\frac{1}{2}|b^-|^2) (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) \partial_t^7 (\nabla^\varphi \cdot v^-) \,\mathrm{d}\mathcal{V}_t$ to get a cancellation structure. In $\mathrm{VS}_{0111}^{*,B}$, we invoke the evolution equation $D_t^{\varphi,-} b_j^- = (b^- \cdot \nabla^\varphi) v^- - b^- (\nabla^\varphi \cdot v^-)$ to get

$$\begin{aligned}
\mathrm{VS}_{0111}^{*,B} &\overset{L}{=} \varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \mathbf{B}_j^{\sharp,-} b_j^- (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) \partial_t^7 (\nabla^\varphi \cdot v^-) \,\mathrm{d}\mathcal{V}_t \\
&= \varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \mathbf{B}_j^{\sharp,-} (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) \partial_t^7 \left(b_j^- (\nabla^\varphi \cdot v^-) \right) \mathrm{d}\mathcal{V}_t + \underbrace{\varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \mathbf{B}_j^{\sharp,-} \left[b_j, (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) \partial_t^7 \right] (\nabla^\varphi \cdot v^-) \,\mathrm{d}\mathcal{V}_t}_{\mathrm{VS}_{0111}^{*,BR}} \\
&= -\varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \mathbf{B}_j^{\sharp,-} (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) D_t^{\varphi,-} \mathbf{B}_j^{\sharp,-} \,\mathrm{d}\mathcal{V}_t + \varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \mathbf{B}_j^{\sharp,-} (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) \partial_t^7 ((b^- \cdot \nabla^\varphi) v_j^-) \,\mathrm{d}\mathcal{V}_t \\
&\quad - \varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \mathbf{B}_j^{\sharp,-} (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) \mathfrak{D}^\sharp (b_j^-) \,\mathrm{d}\mathcal{V}_t + \mathrm{VS}_{0111}^{*,BR},
\end{aligned} \tag{3.106}$$

where the first term on the right side is again controlled by integrating by parts in $(\llbracket \bar{v} \rrbracket \cdot \overline{\nabla})$ and using symmetry, and the third term on the right side is controlled directly after inserting the expression of $\mathfrak{D}^\sharp(b)$. We denote

$$\mathrm{VS}_{0112}^{*,B} := \varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \mathbf{B}_j^{\sharp,-} (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) \partial_t^7 ((b^- \cdot \nabla^\varphi) v_j^-) \,\mathrm{d}\mathcal{V}_t$$

to be the second term on the right side above. Inserting the good unknown $\mathbf{V}^{\sharp,-}$, the term $\mathrm{VS}_{0112}^{*,B}$ is equal to

$$\begin{aligned}
&\varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \mathbf{B}_i^{\sharp,-} (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) \left((b^- \cdot \nabla^\varphi) \mathbf{V}_i^{\sharp,-} \right) \mathrm{d}\mathcal{V}_t + \underbrace{\varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \mathbf{B}_i^{\sharp,-} (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) \left([\partial_t^7, b_j^-] \partial_j^\varphi v_i^- + b_j^- \mathfrak{C}_j^\sharp (v_i^-) \right) \mathrm{d}\mathcal{V}_t}_{\mathrm{VS}_{0112}^{*,BR}} \\
&= \varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \mathbf{B}_i^{\sharp,-} (b^- \cdot \nabla^\varphi) \left((\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) \mathbf{V}_i^{\sharp,-} \right) \mathrm{d}\mathcal{V}_t + \varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \mathbf{B}_i^{\sharp,-} \left[(b^- \cdot \nabla^\varphi), (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) \right] \mathbf{V}_i^{\sharp,-} \,\mathrm{d}\mathcal{V}_t + \mathrm{VS}_{0112}^{*,BR} \\
&=: \mathrm{VS}_{0113}^{*,B} + \mathrm{VS}_{0113}^{*,BR} + \mathrm{VS}_{0112}^{*,BR}.
\end{aligned} \tag{3.107}$$

Now we can integrate by parts $(b^- \cdot \nabla^\varphi)$ and then $D_t^{\varphi,-}$ in $\mathrm{VS}_{0113}^{*,B}$ in order to produce the cancellation with $\mathrm{VS}_{0211}^{*,B}$. Under time integral, $\int_0^t \mathrm{VS}_{0113}^{*,B} \,\mathrm{d}\tau$ is equal to

$$\begin{aligned}
&\int_0^t \underbrace{\varepsilon^{16} \int_{\Omega^-} (b^- \cdot \nabla^\varphi) \mathbf{B}_i^{\sharp,-} (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) D_t^{\varphi,-} \mathbf{V}_i^{\sharp,-} \,\mathrm{d}\mathcal{V}_t}_{=-\mathrm{VS}_{0211}^{*,B}} \mathrm{d}\tau + \varepsilon^{16} \int_{\Omega^-} (b^- \cdot \nabla^\varphi) \mathbf{B}_i^{\sharp,-} (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) \mathbf{V}_i^{\sharp,-} \,\mathrm{d}\mathcal{V}_t \Big|_0^t \\
&+ \varepsilon^{16} \int_0^t \int_{\Omega^-} [(b^- \cdot \nabla^\varphi), D_t^{\varphi,-}] \mathbf{B}_i^{\sharp,-} (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) \mathbf{V}_i^{\sharp,-} \,\mathrm{d}\mathcal{V}_t \,\mathrm{d}\tau + \varepsilon^{16} \int_0^t \int_{\Omega^-} (b^- \cdot \nabla^\varphi) \mathbf{B}_i^{\sharp,-} [D_t^{\varphi,-}, (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla})] \mathbf{V}_i^{\sharp,-} \,\mathrm{d}\mathcal{V}_t \,\mathrm{d}\tau \\
&=: -\int_0^t \mathrm{VS}_{0211}^{*,B} \,\mathrm{d}\tau + \mathrm{VS}_{0211}^{*,BR} + \int_0^t \mathrm{VS}_{0212}^{*,BR} + \mathrm{VS}_{0213}^{*,BR} \,\mathrm{d}\tau.
\end{aligned} \tag{3.108}$$

Note that $[D_t^{\varphi,-}, (b^- \cdot \nabla^\varphi)] = -(\nabla^\varphi \cdot v^-)(b^- \cdot \nabla^\varphi) f$ and when we commute $(\llbracket \bar{v} \rrbracket \cdot \overline{\nabla})$ with either $D_t^{\varphi,-}$ or $(b^- \cdot \nabla^\varphi)$,, no normal derivative will be generated because the weight functions in front of ∂_3 (namely, $b^- \cdot \mathbf{N}$ and $(v^- \cdot \mathbf{N} - \partial_t \varphi)$) are still vanishing on the interface Σ after taking $(\llbracket \bar{v} \rrbracket \cdot \overline{\nabla})$. Therefore, the commutators above are all controllable in $\|\cdot\|_{8,*,-}$ norm and no loss of Mach number occurs. The following remainder terms are controlled directly

$$\begin{aligned}
&\mathrm{VS}_{022}^{*,R} + \mathrm{VS}_{0211}^{*,BR} + \int_0^t \mathrm{VS}_{021}^{*,R} + \mathrm{VS}_{023}^{*,R} + \mathrm{VS}_{0111}^{*,BR} + \mathrm{VS}_{0112}^{*,BR} + \mathrm{VS}_{0212}^{*,BR} + \mathrm{VS}_{0213}^{*,BR} \,\mathrm{d}\tau \\
&\leq \delta E_8^\kappa(t) + P(E^\kappa(0)) + \int_0^t P(E^\kappa(\tau)) \,\mathrm{d}\tau.
\end{aligned} \tag{3.109}$$

In the terms $\mathrm{VS}^{*,R}_{0211} + \mathrm{VS}^{*,R}_{0212}$, we can integrate $([\![\bar{v}]\!] \cdot \overline{\nabla})$ by parts to get to get the desired control thanks to the $\sqrt{\sigma}$-weighted boundary regularity of ψ

$$\mathrm{VS}^{*,R}_{0211} + \mathrm{VS}^{*,R}_{0212} \lesssim_{\sigma^{-1}} \left(|\varepsilon^8\partial_t^7\psi|_2 + |\varepsilon^8\partial_t^8\psi|_1\right)\|\varepsilon^8 D_t^{\varphi,-}\partial_t^7 v\|_{0,-}P(E_4^\kappa(t)). \tag{3.110}$$

Thus, the control of VS^*_{02} term is concluded by

$$\int_0^t \mathrm{VS}^*_{02} + \mathrm{VS}^{*,B}_{0111}\,\mathrm{d}\tau \leq \delta E_8^\kappa(t) + P(E^\kappa(0)) + \int_0^t P(\sigma^{-1}, E^\kappa(\tau))\,\mathrm{d}\tau. \tag{3.111}$$

Finally, combining (3.92), (3.93), (3.101) and (3.111), we get the estimate of VS^* term

$$\int_0^t \mathrm{VS}^*\,\mathrm{d}\tau \leq \delta E_8^\kappa(t) + P(E^\kappa(0)) + \int_0^t P(\sigma^{-1}, E^\kappa(\tau))\,\mathrm{d}\tau. \tag{3.112}$$

Step 3: Control of RT term.

In step 3, we control the terms RT^* and $\mathrm{RT}^{*,\pm}$ defined in (3.65)-(3.66), The latter one can be directly controlled by using symmetry

$$\mathrm{RT}^{*,\pm} = \mp\frac{1}{2}\int_\Sigma (\overline{\nabla}\cdot(\partial_3 q^\pm\, v^\pm))\left|\overline{D_t^-}\partial_t^7\psi\right|^2\,\mathrm{d}x' \leq \sigma^{-1}E_4^\kappa(t)E_8^\kappa(t). \tag{3.113}$$

The term $\mathrm{RT}^* = -\varepsilon^{16}\int_\Sigma [\![\partial_3 q]\!]\,\overline{D_t^-}\partial_t^7\psi\,\partial_t\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x'$ cannot be controlled in the same way as in the estimates of spatial derivatives because we do not have $L^2(\Sigma)$-control for $\partial_t\overline{D_t^-}\partial_t^7\psi$ without κ-weight nor can we integrate by parts $\partial_t^{1/2}$. To overcome this difficulty, we need to invoke the kinematic boundary condition to reduce the number of time derivatives. We have

$$\overline{D_t^-}\partial_t^7\psi = \partial_t^7 v^-\cdot N + [\partial_t^7, v^-, N],\quad \partial_t\overline{D_t^-}\partial_t^7\psi = \partial_t^8 v^-\cdot N + 8\partial_t^7 v^-\cdot\partial_t N + \text{ lower order terms.}$$

Plugging it to RT^*, we find

$$\mathrm{RT}^* \overset{L}{=} -\varepsilon^{16}\int_\Sigma [\![\partial_3 q]\!]\,\partial_t^7 v^-\cdot N\,\partial_t^8 v^-\cdot N\,\mathrm{d}x' - 8\varepsilon^{16}\int_\Sigma [\![\partial_3 q]\!]\,\partial_t^7 v^-\cdot N\,\partial_t^7 v^-\cdot\partial_t N\,\mathrm{d}x' =: \mathrm{RT}^*_1 + \mathrm{RT}^*_2. \tag{3.114}$$

The term RT^*_2 can be controlled by using Gauss-Green formula

$$\mathrm{RT}^*_2 \overset{L}{=} -8\varepsilon^{16}\int_{\Omega^-} [\![\partial_3 q]\!]\,(\partial_3^\varphi\partial_t^7 v^-\cdot\mathbf{N})(\partial_t^7 v^-\cdot\partial_t\mathbf{N})\,\mathrm{d}\mathcal{V}_t - 8\varepsilon^{16}\int_{\Omega^-} [\![\partial_3 q]\!]\,(\partial_t^7 v^-\cdot\mathbf{N})(\partial_3\partial_t^7 v^-\cdot\partial_t\mathbf{N})\,\mathrm{d}x, \tag{3.115}$$

where $[\![\partial_3 q]\!]$ is defined via Sobolev extension. The first term above is directly controlled after invoking the reduction $\partial_3^\varphi\partial_t^7 v^-\cdot\mathbf{N} \overset{L}{=} -\partial_t^7(\varepsilon^2 D_t^{\varphi,-}p^- + \overline{\nabla}\cdot\bar{v}^-)$. For the second term, it suffices to integrate ∂_t by parts under time integral

$$\begin{aligned}
&-8\varepsilon^{16}\int_0^t\int_{\Omega^-} [\![\partial_3 q]\!]\,(\partial_t^7 v^\pm\cdot\mathbf{N})(\partial_3\partial_t^7 v^-\cdot\partial_t\mathbf{N})\,\mathrm{d}x\,\mathrm{d}\tau\\
\overset{L}{=}& -8\varepsilon^{16}\int_{\Omega^-} [\![\partial_3 q]\!]\,(\partial_t^7 v^-\cdot\mathbf{N})(\partial_3\partial_t^6 v^-\cdot\partial_t\mathbf{N})\,\mathrm{d}x\Big|_0^t + 8\int_0^t\int_{\Omega^-} [\![\partial_3 q]\!]\,(\partial_t^8 v^-\cdot\mathbf{N})(\partial_3\partial_t^6 v^-\cdot\partial_t\mathbf{N})\,\mathrm{d}x\\
\lesssim& \,\delta\|\varepsilon^8\partial_3\partial_t^6 v^-\|^2_{0,-} + P(E^\kappa(0)) + \int_0^t P(E_4^\kappa(\tau))E_8^\kappa(\tau)\,\mathrm{d}\tau
\end{aligned} \tag{3.116}$$

Using the same trick as above, the term RT^*_1 is directly controlled by repeated invoking $\partial_3^\varphi\partial_t^7 v^-\cdot\mathbf{N} \overset{L}{=}$

$-\partial_t^7(\varepsilon^2 D_t^{\varphi,-} p^- + \overline{\nabla}\cdot\bar{v}^-)$

$$\int_0^t \mathrm{RT}_1^*\,\mathrm{d}\tau \stackrel{L}{=} -\varepsilon^{16}\int_0^t\int_{\Omega^-} [\![\partial_3 q]\!]\left((\partial_3^\varphi\partial_t^7 v^-\cdot\mathbf{N})(\partial_t^8 v^-\cdot\mathbf{N}) + (\partial_t^7 v^-\cdot\mathbf{N})(\partial_3^\varphi\partial_t^8 v^-\cdot\mathbf{N})\right)\mathrm{d}\mathcal{V}_t\,\mathrm{d}\tau$$

$$\stackrel{\partial_t,\,L}{===}\int_0^t\int_{\Omega^-} [\![\partial_3 q]\!]\left((\partial_3^\varphi\partial_t^7 v^-\cdot\mathbf{N})(\partial_t^8 v^-\cdot\mathbf{N}) - \partial_t(\partial_t^7 v^-\cdot\mathbf{N})(\partial_3^\varphi\partial_t^7 v^-\cdot\mathbf{N})\right)\mathrm{d}\mathcal{V}_t\,\mathrm{d}\tau$$

$$-\varepsilon^{16}\int_{\Omega^-}[\![\partial_3 q]\!]\,(\partial_t^7 v^-\cdot\mathbf{N})(\partial_3^\varphi\partial_t^7 v^-\cdot\mathbf{N})\,\mathrm{d}\mathcal{V}_t\Big|_0^t$$

$$\lesssim \delta E_8^\kappa(t) + P(E^\kappa(0)) + \int_0^t P(E_4^\kappa(\tau))E_8^\kappa(\tau)\,\mathrm{d}\tau. \tag{3.117}$$

Hence, we conclude the estimate of RT^* by

$$\int_0^t \mathrm{RT}^*\,\mathrm{d}\tau \lesssim \delta E_8^\kappa(t) + P(E^\kappa(0)) + \int_0^t P(E_4^\kappa(\tau))E_8^\kappa(\tau)\,\mathrm{d}\tau. \tag{3.118}$$

Step 4: The cancellation structure between ZB^* and Z^*.

Now we control the term $ZB^{*,\pm} + Z^{*,\pm}$. Note that we cannot integrate by parts $\overline{\partial}^{1/2}$ due to the lack of spatial derivatives. First, $ZB^{*,\pm}$ can be written as

$$ZB^{*,\pm} = \mp\varepsilon^{16}\int_\Sigma \overline{D_t^-}\partial_t^7 q^\pm(\partial_3 v^\pm\cdot N)\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x' \pm \varepsilon^{16}\int_\Sigma \overline{D_t^-}\partial_t^7\psi\,\partial_3 q^\pm(\partial_3 v^\pm\cdot N)\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x'$$

$$\mp\varepsilon^{16}\int_\Sigma \mathbf{Q}^{*,\pm}\left[D_t^{\varphi,-}\partial_t^7, N_i, v_i^\pm\right]\mathrm{d}x'$$

$$=: ZB_1^{*,R,\pm} + ZB_2^{*,R,\pm} + ZB_0^{*,\pm}. \tag{3.119}$$

The second term on the right side can be directly controlled. We have

$$ZB_2^{*,R,\pm} \le \left|D_t^{\varphi,-}\partial_t^7\psi\right|_0^2\left|\partial_3 q^\pm\,(\partial_3 v^\pm\cdot N)\right|_{L^\infty} \le P(\sigma^{-1}, E_4^\kappa(t))E_8^\kappa(t). \tag{3.120}$$

For the first term, using again $\overline{D_t^-}\partial_t^7\psi = \partial_t^7 v\cdot N$+lower order terms, we can convert it to an interior integral.

$$\int_0^t ZB_1^{*,R,\pm}\,\mathrm{d}\tau \stackrel{L}{=} \varepsilon^{16}\int_0^t\int_{\Omega^\pm}(\partial_3 v^\pm\cdot\mathbf{N})\left(\partial_3^\varphi D_t^{\varphi,-}\partial_t^7 q^\pm\,\partial_t^7 v^\pm\cdot\mathbf{N} + D_t^{\varphi,-}\partial_t^7 q^\pm\,\partial_3^\varphi\partial_t^7 v^\pm\cdot\mathbf{N}\right)\mathrm{d}\mathcal{V}_t$$

$$\stackrel{D_t^{\varphi,-}}{==}\varepsilon^{16}\int_{\Omega^\pm}(\partial_3 v^\pm\cdot\mathbf{N})\partial_3^\varphi\partial_t^7 q^\pm\,\partial_t^7 v^\pm\cdot\mathbf{N}\,\mathrm{d}\mathcal{V}_t\Big|_0^t + \varepsilon^{16}\int_0^t\int_{\Omega^\pm}(\partial_3 v^\pm\cdot\mathbf{N})\left([\partial_3^\varphi, D_t^{\varphi,-}]\partial_t^7 q^\pm\right)\partial_t^7 v^\pm\cdot\mathbf{N}\,\mathrm{d}\mathcal{V}_t\,\mathrm{d}\tau$$

$$+\varepsilon^{16}\int_0^t\int_{\Omega^\pm}(\partial_3 v^\pm\cdot\mathbf{N})\left(\partial_3^\varphi\partial_t^7 q^\pm\,D_t^{\varphi,-}\partial_t^7 v^\pm\cdot\mathbf{N} + D_t^{\varphi,-}\partial_t^7 q^\pm\,\partial_3^\varphi\partial_t^7 v^\pm\cdot\mathbf{N}\right)\mathrm{d}\mathcal{V}_t + \text{l.o.t} \tag{3.121}$$

Now we can invoke the reduction for $\partial_3^\varphi q$ and $\partial_3^\varphi v\cdot\mathbf{N}$ to convert ∂_3^φ to a tangential derivative

$$\partial_3^\varphi\partial_t^7 q \stackrel{L}{=} \partial_t^7(\rho D_t^\varphi v - (b\cdot\nabla^\varphi)b),\quad \partial_3^\varphi\partial_t^7 v\cdot\mathbf{N} \stackrel{L}{=} -\partial_t^7(\mathcal{F}_p D_t^\varphi p + \overline{\nabla}\cdot\bar{v}).$$

Note that the second equation above produces an extra $\mathcal{F}_p = O(\varepsilon^2)$ weight, so there is no loss of Mach number when $D_t^{\varphi,-}\partial_t^7 q$ appears. When $D_t^{\varphi,-}\partial_t^7 q$ is multiplied by $\partial_t^7\overline{\nabla}\cdot\bar{v}$, we can further integrate by parts in ∂_t and then in $\overline{\nabla}\cdot$ to move one time derivative to v. Hence, $ZB_1^{*,R,\pm}$ is controlled in $\|\cdot\|_{8,*}$ norm without loss of ε-weights

$$\int_0^t ZB_1^{*,R,\pm}\,\mathrm{d}\tau \lesssim\ \delta E_8^\kappa(t) + P(E^\kappa(0)) + \int_0^t P(E^\kappa(\tau))\,\mathrm{d}\tau. \tag{3.122}$$

Next we will see again the cancellation structure in $ZB_0^{*,\pm} + Z^{*,\pm}$. From (3.9), we have

$$\sum_{i=1}^3 \mathfrak{C}_i^*(v_i) = \partial_3^\varphi(\nabla^\varphi\cdot v)D_t^{\varphi,-}\partial_t^7\varphi + \left[D_t^{\varphi,-}\partial_t^7, \frac{\mathbf{N}_i}{\partial_3\varphi}, \partial_3 v_i\right] + \partial_3 v_i\left[D_t^{\varphi,-}\partial_t^7, \mathbf{N}_i, (\partial_3\varphi)^{-1}\right]$$

$$+(\partial_3 v\cdot\mathbf{N})\left[D_t^{\varphi,-}\partial_t^6, \frac{1}{(\partial_3\varphi)^2}\right]\partial_t\partial_3\varphi + (\partial_3\varphi)^{-1}\mathbf{N}\cdot[D_t^{\varphi,-}\partial_t^7, \partial_3]v + (\partial_3\varphi)^{-1}(\partial_3^\varphi v\cdot\mathbf{N})[D_t^{\varphi,-}\partial_t^7, \partial_3]\varphi,$$

where we have further analysis on the second term and the fifth term

$$\left[D_t^{\varphi,-}\partial_t^7, \frac{\mathbf{N}_i}{\partial_3\varphi}, \partial_3 v_i\right] \stackrel{L}{=} (\partial_3\varphi)^{-1}\left[D_t^{\varphi,-}\partial_t^7, \mathbf{N}_i, \partial_3 v_i\right] - \partial_3^\varphi\partial_t\varphi\partial_3^\varphi D_t^{\varphi,-}\partial_t^6 v\cdot\mathbf{N}, \tag{3.123}$$

$$(\partial_3\varphi)^{-1}\mathbf{N}\cdot[D_t^{\varphi,-}\partial_t^7, \partial_3]v = \left(\partial_3^\varphi\bar{v}^-\cdot\overline{\nabla}\right)\partial_t^7 v\cdot\mathbf{N} + \partial_3\left((\partial_3\varphi)^{-1}(v^-\cdot\mathbf{N}-\partial_t\varphi)\right)\partial_3^\varphi\partial_t^7 v\cdot\mathbf{N}. \tag{3.124}$$

Thus, we find that, apart from the term $(\partial_3\varphi)^{-1}\left[D_t^{\varphi,-}\partial_t^7, \mathbf{N}_i, \partial_3 v_i\right]$, all the other terms in $\mathfrak{C}_i^*(v_i)$ include either a tangential derivative falling on the leading order term or the term $\partial_3^\varphi v\cdot\mathbf{N}$ (possibly with some derivatives) such that $\mathcal{F}_p D_t^\varphi p$ and $\overline{\nabla}\cdot\bar{v}$ are produced by invoking the continuity equation. Thus, when $\mathbf{Q}^*$ is multiplied with these terms, its contribution in $Z^{*,\pm}$ can be directly controlled without any loss of weights of Mach number.

It now remains to control $ZB_0^{*,\pm} + Z^{*,\pm}$ with $Z_0^{*,\pm} := \varepsilon^{16}\int_{\Omega^\pm}\mathbf{Q}^{*,\pm}(\partial_3\varphi)^{-1}\left[D_t^{\varphi,-}\partial_t^7, \mathbf{N}_i, \partial_3 v_i\right]\mathrm{d}\mathcal{V}_t$. Using $\mathrm{d}\mathcal{V}_t = \partial_3\varphi\,\mathrm{d}x$ and Gauss-Green formula, we have

$$\begin{aligned}
&ZB_0^{*,\pm} + Z_0^{*,\pm} = \mp\varepsilon^{16}\int_\Sigma \mathbf{Q}^{*,\pm}\left[D_t^{\varphi,-}\partial_t^7, N_i, v_i^\pm\right]\mathrm{d}x' + \varepsilon^{16}\int_{\Omega^\pm}\mathbf{Q}^{*,\pm}\left[D_t^{\varphi,-}\partial_t^7, \mathbf{N}_i, \partial_3 v_i^\pm\right]\mathrm{d}x\\
&= \sum_{j=0}^1\sum_{k=1}^6 \varepsilon^{16}\binom{7}{k}\left(\pm\int_\Sigma \mathbf{Q}^{*,\pm}(D_t^{\varphi,-})^j\partial_t^k v_i^\pm\,(D_t^{\varphi,-})^{1-j}\partial_t^{6-k}N_i\,\mathrm{d}x' + \int_{\Omega^\pm}\mathbf{Q}^{*,\pm}(D_t^{\varphi,-})^j\partial_t^k\partial_3 v_i^\pm\,(D_t^{\varphi,-})^{1-j}\partial_t^{6-k}\mathbf{N}_i\,\mathrm{d}x\right)\\
&\stackrel{L}{=} -\sum_{j=0}^1\sum_{k=1}^6 \varepsilon^{16}\binom{7}{k}\int_{\Omega^\pm}\partial_3^\varphi\mathbf{Q}^{*,\pm}(D_t^{\varphi,-})^j\partial_t^k v_i^\pm\,(D_t^{\varphi,-})^{1-j}\partial_t^{6-k}\mathbf{N}_i\,\mathrm{d}\mathcal{V}_t.
\end{aligned} \tag{3.125}$$

Recall that $\mathbf{Q}^* = D_t^{\varphi,-}\partial_t^7 q^- - D_t^{\varphi,-}\partial_t^7\varphi\partial_3^\varphi q$, we can integrate by parts this $D_t^{\varphi,-}$ under time integral and invoke the momentum equation to reduce $\partial_3^\varphi\mathbf{Q}^*$ to $-\rho D_t^\varphi\mathbf{V}^* + (b\cdot\nabla^\varphi)\mathbf{B}^*$+lower order terms. Note that $[\partial_3^\varphi, D_t^{\varphi,-}]\partial_t^7 q = \partial_3^\varphi v_j^-\partial_j^\varphi q = (\partial_3^\varphi\bar{v}^-\cdot\overline{\nabla})\partial_t^7 q + (\partial_3^\varphi v^-\cdot\mathbf{N})\partial_t^7\partial_3 q$, so one can still convert the normal derivative ∂q to a tangential derivative of v, b. Thus, we have

$$\begin{aligned}
\int_0^t ZB_0^{*,\pm} + Z_0^{*,\pm}\,\mathrm{d}\tau &\stackrel{L}{=} \sum_{j=0}^1\sum_{k=1}^6 \varepsilon^{16}\binom{7}{k}\int_0^t\int_{\Omega^\pm}\partial_3^\varphi\mathbf{Q}^{\sharp,\pm}D_t^{\varphi,-}\left((D_t^{\varphi,-})^j\partial_t^k v_i^\pm\cdot,(D_t^{\varphi,-})^{1-j}\partial_t^{6-k}\mathbf{N}\right)\mathrm{d}\mathcal{V}_t\,\mathrm{d}\tau\\
&\quad + \varepsilon^{16}\binom{7}{k}\int_0^t\int_{\Omega^\pm}\partial_3^\varphi\mathbf{Q}^{\sharp,\pm}\left((D_t^{\varphi,-})^j\partial_t^k v_i^\pm\cdot,(D_t^{\varphi,-})^{1-j}\partial_t^{6-k}\mathbf{N}\right)\mathrm{d}\mathcal{V}_t\Big|_0^t\\
&\lesssim \delta\|\partial_3^\varphi\mathbf{Q}^{\sharp,\pm}\|_0^2 + P(E^\kappa(0)) + \int_0^t P(\sigma^{-1}, E^\kappa(\tau))\,\mathrm{d}\tau, \quad \forall\delta\in(0,1).
\end{aligned} \tag{3.126}$$

Combining this with the control of remainder terms and commutators, we can easily obtain that

$$\int_0^t ZB^{*,\pm} + Z^{*,\pm}\,\mathrm{d}\tau \lesssim \delta E_8^\kappa(0) + P(E^\kappa(0)) + \int_0^t P(\sigma^{-1}, E^\kappa(\tau))\,\mathrm{d}\tau, \quad \forall\delta\in(0,1). \tag{3.127}$$

3.5 Tangential estimates: general cases and summary

Let $\mathcal{T}^\gamma = (\omega(x_3)\partial_3)^{\gamma_4}\partial_t^{\gamma_0}\overline{\partial}_1^{\gamma_1}\overline{\partial}_2^{\gamma_2}$ be a tangential derivative with length of the multi-index $\langle\gamma\rangle := \gamma_0+\gamma_1+\gamma_2+2\times 0+\gamma_4$. Section 3.3.1-Section 3.4 are devoted to the control of full spatial derivatives ($\gamma_1+\gamma_2 = \langle\gamma\rangle$) and full time derivatives ($\gamma_0 = \langle\gamma\rangle$). Now we analyze how to handle the general case.

Space-time mixed derivatives: $\gamma_0 > 0$ **and** $\gamma_1+\gamma_2 > 0$

Let us temporarily assume $\gamma_4 = 0$. In this case, the tangential derivatives that we need to consider have the form $\overline{\partial}^{4-l-k}\partial_t^k\mathcal{T}^\alpha$ with $\langle\alpha\rangle = 2l$, $\alpha_4 = 0$ and weights of Mach number ε^{2l}. That is, we need to consider $\varepsilon^{2l}\partial_t^{k+\alpha_0}\overline{\partial}^{4+l-k-\alpha_0}$-estimates. Following the previous paper [55] by Luo and the author, the control of space-time mixed tangential derivatives ($0 < k+\alpha_0 < 4+l$) is the same as the control of purely spatial tangential derivatives. In particular, compared with the one-phase fluid problem [55], we only need one spatial derivative to do integration by parts in order for the control of the extra problematic term

$$\mathrm{VS} := \varepsilon^{4l}\int_\Sigma \partial_t^{k+\alpha_0}\overline{\partial}^{4+l-k-\alpha_0}q^-\,([\![\bar{v}]\!]\cdot\overline{\nabla})\partial_t^{k+\alpha_0}\overline{\partial}^{4+l-k-\alpha_0}\psi\,\mathrm{d}x'$$

in which we need to integrate by parts $\overline{\partial}^{1/2}$ and seek for the control of $\varepsilon^{2l}\left|\partial_t^{k+\alpha_0}\overline{\partial}^{4+l-k-\alpha_0}\right|_{1.5}$. Mimicing the proof of Lemma 3.5, we can show that (replacing $k+\alpha_0$ by k)

Lemma 3.7 (Elliptic estimate for the time derivatives of the free interface)**.** Fix $l\in\{0,1,2,3,4\}$. For $0<k<4+l$, we have the following uniform-in-(ε,κ) inequality, in which the first term on the right side disappears when $\kappa=0$.

$$\left|\varepsilon^{2l}\partial_t^k\psi\right|_{5.5+l-k}\leq\left|\varepsilon^{2l}\partial_t^k\psi(0)\right|_{5.5+l-k}+\sigma^{-1}\left|\varepsilon^{2l}\partial_t^k[\![q]\!]\right|_{3.5+l-k}$$
$$+P\left(\sigma^{-1},|\overline{\nabla}\psi|_{L^\infty},\sum_{j=0}^{l}E_{4+j}^{\kappa}(t)\right)\left(\left|\varepsilon^{2l}\partial_t^k\psi\right|_{4.5+l-k}+\left|\varepsilon^{2l}\partial_t^{k-1}\psi\right|_{5.5+l-k}\right).$$

Weighted normal derivatives: $\gamma_4>0$

In the most general case, $\mathcal{T}^\gamma$ may contain weighted normal derivative $\omega(x_3)\partial_3$, so we have to analyze the commutator involving $[\mathcal{T}^\gamma,\partial_3]$ in $\mathfrak{C}(f)$ and $\mathfrak{D}(f)$ defined in (3.9)-(3.10). The problematic thing is that ∂_3 may fall on $\omega(x_3)$ which converts a "tangential" derivative $\omega(x_3)\partial_3$ (a first-order derivative) to a normal derivative ∂_3 (considered to be second-order under the setting of anisotropic Sobolev spaces). Such terms in $\mathfrak{D}(f)$ are

$$(\partial_3\varphi)^{-1}(v\cdot\mathbf{N}-\partial_t\varphi)[\mathcal{T}^\gamma,\partial_3]f+(v\cdot\mathbf{N}-\partial_t\varphi)\frac{\partial_3 f}{(\partial_3\varphi)^2}[\mathcal{T}^\gamma,\partial_3]\varphi.$$

They can be directly controlled because an extra weight $(v\cdot\mathbf{N}-\partial_t\varphi)$, which vanishes on Σ, is automatically generated to compensate the possible loss of weight function. As for $\mathfrak{C}(f)$, we notice that the terms involving $[\mathcal{T}^\gamma,\partial_3]$ can be written to be

$$\frac{\mathbf{N}_i}{\partial_3\varphi}[\mathcal{T}^\gamma,\partial_3]f-\frac{\mathbf{N}_i}{\partial_3\varphi}\partial_3^\varphi f[\mathcal{T}^\gamma,\partial_3]\varphi,\quad f=q\text{ or }v_i.$$

The second term above is easy to control because $\varphi(t,x)=x_3+\chi(x_3)\psi(t,x')$ implies the C^∞-regularity of φ in x_3 direction. For the first term, it may generate a term $\mathcal{T}^\beta\partial_3 f\mathbf{N}_i$ with $\beta_i=\gamma_i(i=0,1,2)$, $\beta_4=\gamma_4-1$, whose $L^2(\Omega)$ norm may be not directly bounded. Luckily, for $f=q$ or v_i, we can again invoke the momentum equation or the continuity equation to reduce $-\partial_3^\varphi q$ and $\partial_3^\varphi v\cdot\mathbf{N}$ to tangential derivatives $\rho D_t^\varphi v-(b\cdot\nabla^\varphi)b$ and $-\mathcal{F}_p D_t^\varphi p-\overline{\nabla}\cdot\bar{v}$ respectively. Therefore, there is no extra loss of derivative in the commutators $\mathfrak{C}(f)$ and $\mathfrak{D}(f)$ when $\gamma_4>0$.

Summary of tangential estimates

Finally, we need to recover the estimates of $\mathcal{T}^\gamma(v,b,S,\sqrt{\mathcal{F}_p}p)$ from the L^2-estimates of their Alinhac good unknowns. By definition, we have

$$\left\|\mathcal{T}^\gamma f^\pm\right\|_{0,\pm}^2\leq\left\|\mathbf{F}^{\gamma,\pm}\right\|_{0,\pm}^2+|\mathcal{T}^\gamma\psi|_0^2\,\|\partial_3^\varphi f^\pm\|_{L^\infty(\Omega^\pm)}^2,$$

in which $\|\mathbf{F}^{\gamma,\pm}\|_{0,\pm}$ and $|\mathcal{T}^\gamma\psi|_0$ have been controlled by $\delta E^\kappa(t)+\int_0^t P(\sigma^{-1},E^\kappa(\tau))\,\mathrm{d}\tau$. When $\mathcal{T}^\gamma$ contains at least one spatial derivative, we can use $-\mathcal{T}q\sim D_t^\varphi v+(b\cdot\nabla^\varphi)b$ to get the control of $\mathcal{T}q$ instead of $\sqrt{\mathcal{F}_p}\mathcal{T}q$. For the full time derivatives, we use $D_t^{\varphi,-}=\partial_t+(\bar{v}^-\cdot\overline{\nabla})+(\partial_3\varphi)^{-1}(v^-\cdot\mathbf{N}-\partial_t\varphi)\partial_3$ to convert the $\varepsilon^{2l}\partial_t^{4+l}$-estimate to $\varepsilon^{2l}D_t^{\varphi,-}\partial_t^{3+l}$-estimate, $\varepsilon^{2l}\overline{\partial}\partial_t^{3+l}$-estimate and $\varepsilon^{2l}(\omega\partial_3)\partial_t^{3+l}$-estimate, in the second part of which the norm $|\varepsilon^{2l}\partial_t^{3+l}\psi(0)|_{2.5}$ is needed to control the VS term. Also, since $\omega(x_3)=0$ on the interface, $\mathcal{T}^\gamma$ can be expressed as $\overline{\partial}^{4+l-k}\partial_t^k$ for $0\leq k\leq 4+l$, $0\leq l\leq 4$. Hence, we conclude the tangential estimates by the following inequalities

$$\begin{aligned}
&\sum_{\pm}\sum_{\langle\alpha\rangle=2l}\sum_{\substack{0\le k\le 4-l\\ k+\alpha_0<4+l}}\left\|\left(\varepsilon^{2l}\overline{\partial}^{4-k-l}\mathcal{T}^{\alpha}\partial_t^k(v^{\pm},b^{\pm},S^{\pm},p^{\pm})\right)\right\|_{0,\pm}^2\\
&+\sum_{k=0}^{4+l}\left|\sqrt{\sigma}\varepsilon^{2l}\partial_t^k\psi\right|_{5+l-k}^2+\left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^k\psi\right|_{6+l-k}^2+\int_0^t\left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^{k+1}\psi(\tau)\right|_{5+l-k}^2\,\mathrm{d}\tau\\
\lesssim{}&\delta E_{4+l}^{\kappa}(t)+\sum_{k=0}^{3+l}\left|\varepsilon^{2l}\partial_t^k\psi(0)\right|_{5.5+l-k}^2+\sum_{j=0}^{l}P\left(\sigma^{-1},E_{4+j}^{\kappa}(0)\right)\\
&+P\left(\sum_{j=0}^{l}E_{4+j}^{\kappa}(t)\right)\int_0^t P\left(\sigma^{-1},\sum_{j=0}^{l}E_{4+j}^{\kappa}(\tau)\right)\mathrm{d}\tau
\end{aligned}\tag{3.128}$$

and

$$\begin{aligned}
&\sum_{\pm}\sum_{k=0}^{4-l}\left\|\left(\varepsilon^{2l}\partial_t^{4+l}(v^{\pm},b^{\pm},S^{\pm},(\mathcal{F}_p)^{\frac12}p^{\pm})\right)\right\|_{4-k-l,\pm}^2\\
&+\left|\sqrt{\sigma}\varepsilon^{2l}\partial_t^{4+l}\psi\right|_1^2+\left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^{4+l}\psi\right|_2^2+\int_0^t\left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^{5+l}\psi(\tau)\right|_1^2\,\mathrm{d}\tau\\
\lesssim{}&\delta E_{4+l}^{\kappa}(t)+\left|\varepsilon^{2l}\partial_t^{3+l}\psi(0)\right|_{2.5}^2+P\left(\sigma^{-1},\sum_{j=0}^{l}E_{4+j}^{\kappa}(0)\right)+P\left(\sum_{j=0}^{l}E_{4+j}^{\kappa}(t)\right)\int_0^t P\left(\sigma^{-1},\sum_{j=0}^{l}E_{4+j}^{\kappa}(\tau)\right)\mathrm{d}\tau.
\end{aligned}\tag{3.129}$$

These are exactly the desired uniform-in-(κ,ε) energy estimates in Proposition 3.3.

3.6 Div-Curl analysis and reduction of pressure

The tangential derivatives of the variables (v,b,p) are analyzed in Section 3.3-Section 3.5. Here we show the reduction of normal derivatives of pressure and the analysis for the divergence and vorticity. We use the div-curl decomposition (cf. Lemma B.1) such that the normal derivatives of (v,b) are controlled via their divergence and curl parts. For $0\le l\le 3,\ 0\le k\le 3-l,\ \langle\alpha\rangle=2l,\ \alpha_3=0$, we have

$$\begin{aligned}
\left\|\varepsilon^{2l}\partial_t^k\mathcal{T}^{\alpha}(v^{\pm},b^{\pm})\right\|_{4-k-l,\pm}^2\le C\Big(&\left\|\varepsilon^{2l}\partial_t^k\mathcal{T}^{\alpha}(v^{\pm},b^{\pm})\right\|_{0,\pm}^2+\left\|\varepsilon^{2l}\nabla^{\varphi}\cdot\partial_t^k\mathcal{T}^{\alpha}(v^{\pm},b^{\pm})\right\|_{3-k-l,\pm}^2\\
&+\left\|\varepsilon^{2l}\nabla^{\varphi}\times\partial_t^k\mathcal{T}^{\alpha}(v^{\pm},b^{\pm})\right\|_{3-k-l,\pm}^2+\left\|\varepsilon^{2l}\overline{\partial}^{4-k-l}\partial_t^k\mathcal{T}^{\alpha}(v^{\pm},b^{\pm})\right\|_{0,\pm}^2\Big)
\end{aligned}\tag{3.130}$$

with

$$C=C\left(\sum_{j=0}^{l}\sum_{k=0}^{3+j}|\varepsilon^{2j}\partial_t^j\psi|_{4+l-j}^2,|\overline{\nabla}\psi|_{W^{1,\infty}}\right)>0$$

a positive continuious function linear in $|\varepsilon^{2j}\partial_t^j\psi|_{4+l-j}^2$. The conclusion for the div-curl analysis is

Proposition 3.8. Fix $l\in\{0,1,2,3\}$. For any $0\le k\le l-1$, any multi-index α satisfying $\langle\alpha\rangle=2l$ and any constant $\delta\in(0,1)$, we can prove the following estimates for the curl part

$$\begin{aligned}
&\left\|\varepsilon^{2l}\nabla^{\varphi}\times\partial_t^k\mathcal{T}^{\alpha}v^{\pm}\right\|_{3-k-l,\pm}^2+\left\|\varepsilon^{2l}\nabla^{\varphi}\times\partial_t^k\mathcal{T}^{\alpha}b^{\pm}\right\|_{3-k-l,\pm}^2\\
\lesssim{}&\delta E_{4+l}^{\kappa}(t)+P\left(\sigma^{-1},\sum_{j=0}^{l}E_{4+j}^{\kappa}(0)\right)+P(E_4^{\kappa}(t))\int_0^t P\left(\sigma^{-1},\sum_{j=0}^{l}E_{4+j}^{\kappa}(\tau)\right)+E_{l+1}^{\kappa}(\tau)\,\mathrm{d}\tau,
\end{aligned}\tag{3.131}$$

and for the divergence part

$$\begin{aligned}
&\left\|\varepsilon^{2l}\nabla^{\varphi}\cdot\partial_t^k\mathcal{T}^{\alpha}v^{\pm}\right\|_{3-k-l,\pm}^2+\left\|\varepsilon^{2l}\nabla^{\varphi}\cdot\partial_t^k\mathcal{T}^{\alpha}v^{\pm}\right\|_{3-k-l,\pm}^2\\
\lesssim{}&\delta E_{4+l}^{\kappa}(t)+P\left(\sigma^{-1},\sum_{j=0}^{l}E_{4+j}^{\kappa}(0)\right)+P(E_4^{\kappa}(t))\int_0^t P\left(\sigma^{-1},\sum_{j=0}^{l}E_{4+j}^{\kappa}(\tau)\right)\mathrm{d}\tau.
\end{aligned}\tag{3.132}$$

3.6.1 Reduction of pressure and divergence

Let us start with $l = 0$. The spatial derivative of q is controlled by invoking the momentum equation:

$$-\partial_3 q = (\partial_3\varphi)\left(\rho D_t^{\varphi} v_3 - (b\cdot\nabla^{\varphi})b_3\right); \tag{3.133}$$

$$-\overline{\partial}_i q = -(\partial_3\varphi)^{-1}\overline{\partial}_i\varphi\,\partial_3 q + \rho D_t^{\varphi} v_i - (b\cdot\nabla^{\varphi})b_i,\ \ i=1,2. \tag{3.134}$$

Let $\mathcal{T}$ be ∂_t or $\overline{\partial}$ or $\omega(x_3)\partial_3$. Then we have

$$\|\partial_t^k\partial_3 q\|_{3-k} \lesssim \|\partial_t^k(\rho\mathcal{T}v_3)\|_{3-k} + \|\partial_t^k(b\mathcal{T}b_3)\|_{3-k} \tag{3.135}$$

$$\|\partial_t^k\overline{\partial}_i q\|_{3-k} \lesssim \|\partial_t^k(\overline{\partial}_i\varphi\partial_3 q)\|_{3-k} + \|\partial_t^k(\rho\mathcal{T}v_i)\|_{3-k} + \|\partial_t^k(b\mathcal{T}b_i)\|_{3-k}, \tag{3.136}$$

in which the leading order terms are $\|\partial_t^k\mathcal{T}(v,b)\|_{3-k}$ and $|\partial_t^k\psi|_{4-k}$. This shows that we can convert the control of spatial derivative of q to *tangential estimates* of v and b.

Next we turn to the div-curl analysis for v, b. Let us first analyze $E_4(t)$. For $0\le k\le 3$, we have

$$\|\partial_t^k v\|_{4-k}^2 \le C(|\psi|_{4-k}, |\overline{\nabla}\psi|_{W^{1,\infty}})\left(\|\partial_t^k v\|_0^2 + \|\nabla^{\varphi}\cdot\partial_t^k v\|_{3-k}^2 + \|\nabla^{\varphi}\times\partial_t^k v\|_{3-k}^2 + \|\overline{\partial}^{4-k}\partial_t^k v\|_0^2\right), \tag{3.137}$$

$$\|\partial_t^k b\|_{4-k}^2 \le C(|\psi|_{4-k}, |\overline{\nabla}\psi|_{W^{1,\infty}})\left(\|\partial_t^k b\|_0^2 + \|\nabla^{\varphi}\cdot\partial_t^k b\|_{3-k}^2 + \|\nabla^{\varphi}\times\partial_t^k b\|_{3-k}^2 + \|\overline{\partial}^{4-k}\partial_t^k b\|_0^2\right). \tag{3.138}$$

For the divergence, we can directly invoke the continuity equation to convert $\nabla^{\varphi}\cdot v$ to time derivative of p together with square weights of Mach number. When $k=0$, we have

$$\|\nabla^{\varphi}\cdot v\|_3^2 = \|\mathcal{F}_p D_t^{\varphi} p\|_3^2, \tag{3.139}$$

which is further reduced to the tangential derivatives of v and b by using the above reduction of q. Note that the magnetic tension term $\frac12|b|^2$ in the total pressure q does not involve extra normal derivatives thanks to $\mathcal{T}(\frac12|b|^2) = b\cdot\mathcal{T}b$. Taking ∂_t in the continuity equation and omitting lower order terms, we have

$$\nabla^{\varphi}\cdot\partial_t^k v \overset{L}{=} -\mathcal{F}_p\partial_t^k D_t^{\varphi} p + (\partial_3\varphi)^{-1}\overline{\partial}\partial_t^k\varphi\cdot\partial_3 v,$$

which gives

$$\|\nabla^{\varphi}\cdot\partial_t^k v^{\pm}\|_{3-k,\pm}^2 \lesssim C(\|v^{\pm}\|_{W^{1,\infty}(\Omega^{\pm})})\left(\left\|\mathcal{F}_p^{\pm}\partial_t^k\mathcal{T}p^{\pm}\right\|_{3-k,\pm}^2 + \left|\partial_t^k\psi\right|_{4-k}^2\right) + \text{ lower order terms.} \tag{3.140}$$

Again, this can be reduced to tangential derivatives of v, b until there is no spatial derivative falling on p. As for the divergence of magnetic fields, we can invoke the div-free constraint to convert it to lower order terms. Namely, using $\nabla^{\varphi}\cdot b = 0$, we have

$$\nabla^{\varphi}\cdot\partial_t^k b \overset{L}{=} \underbrace{\partial_t^k(\nabla^{\varphi}\cdot b)}_{=0} + (\partial_3\varphi)^{-1}\overline{\partial}\partial_t^k\varphi\cdot\partial_3 b$$

and thus

$$\|\nabla^{\varphi}\cdot\partial_t^k b^{\pm}\|_{3-k,\pm}^2 \lesssim C(\|b^{\pm}\|_{W^{1,\infty}(\Omega^{\pm})})\left|\partial_t^k\psi\right|_{4-k}^2 + \text{ lower order terms.} \tag{3.141}$$

The term $\left|\partial_t^k\psi\right|_{4-k}^2$ has been controlled in tangential estimates of $E_4^{\kappa}(t)$. Combining the result of tangential estimates in Proposition 3.3, the control of divergence of time derivatives is concluded by

$$\begin{aligned}\left\|\nabla^{\varphi}\cdot\partial_t^k(v^{\pm},b^{\pm})\right\|_{3-k,\pm}^2 &\lesssim C(\|v^{\pm}\|_{W^{1,\infty}(\Omega^{\pm})})\left\|\mathcal{F}_p^{\pm}\partial_t^k\mathcal{T}p^{\pm}\right\|_{3-k,\pm}^2 + C(\|v^{\pm},b^{\pm}\|_{W^{1,\infty}(\Omega^{\pm})})\left|\partial_t^k\psi\right|_{4-k}^2\\ &\lesssim C(\|v^{\pm}\|_{W^{1,\infty}(\Omega^{\pm})})\left\|\mathcal{F}_p^{\pm}\partial_t^k\mathcal{T}p^{\pm}\right\|_{3-k,\pm}^2 + \delta E_4^{\kappa}(t) + P(E_4^{\kappa}(0)) + P(E_4^{\kappa}(t))\int_0^t P(E_4^{\kappa}(\tau))\,\mathrm{d}\tau,\end{aligned} \tag{3.142}$$

where the term involving $p^{\pm}$ can be further reduced to $\mathcal{T}(v^{\pm},b^{\pm})$ when $3-k>0$ so that one can further apply the div-curl analysis to it.

3.6.2 Vorticity analysis for E_4

Taking $\nabla^\varphi\times$ in the momentum equation of v and the evolution equation of b, we get the evolution equation for the vorticity $\nabla^\varphi\times v$ and the current density $\nabla^\varphi\times b$

$$\rho D_t^\varphi(\nabla^\varphi\times v)-(b\cdot\nabla^\varphi)(\nabla^\varphi\times b)=-(\nabla^\varphi\rho)\times(D_t^\varphi v)-\rho(\nabla^\varphi v_j)\times(\partial_j^\varphi v)+(\nabla^\varphi b_j)\times(\partial_j^\varphi b), \tag{3.143}$$

$$D_t^\varphi(\nabla^\varphi\times b)-(b\cdot\nabla^\varphi)(\nabla^\varphi\times v)-b\times\nabla^\varphi(\nabla^\varphi\cdot v)=-(\nabla^\varphi\times b)(\nabla^\varphi\cdot v)-(\nabla^\varphi v_j)\times(\partial_j^\varphi b)+(\nabla^\varphi b_j)\times(\partial_j^\varphi v), \tag{3.144}$$

and taking ∂^3 gives

$$\rho D_t^\varphi(\partial^3\nabla^\varphi\times v)-(b\cdot\nabla^\varphi)(\partial^3\nabla^\varphi\times b)=RK_v, \tag{3.145}$$

$$D_t^\varphi(\partial^3\nabla^\varphi\times b)-(b\cdot\nabla^\varphi)(\partial^3\nabla^\varphi\times v)-b\times\partial^3\nabla^\varphi(\nabla^\varphi\cdot v)=RK_b, \tag{3.146}$$

where

$$\begin{aligned}RK_v &:=-[\partial^3,\rho D_t^\varphi](\nabla^\varphi\times v)+[\partial^3,(b\cdot\nabla^\varphi)](\nabla^\varphi\times b)+\partial^3(\text{right side of (3.143)}),\\ RK_b &:=-[\partial^3,D_t^\varphi](\nabla^\varphi\times b)+[\partial^3,(b\cdot\nabla^\varphi)](\nabla^\varphi\times v)+\partial^3(\text{right side of (3.144)}).\end{aligned}$$

Direct computation shows that the highest-order terms in $RK_v,\ RK_b$ only have 4 spatial derivatives and do not contain time derivative of q. Therefore, we can prove the H^3-control of the voriticity and current density by standard energy estimates.

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}\rho^\pm\left|\partial^3(\nabla^\varphi\times v^\pm)\right|^2\,\mathrm{d}\mathcal{V}_t=\int_{\Omega^\pm}\rho^\pm D_t^{\varphi\pm}(\partial^3\nabla^\varphi\times v^\pm)\cdot(\partial^3\nabla^\varphi\times v^\pm)\,\mathrm{d}\mathcal{V}_t\\
=&\int_{\Omega^\pm}(b^\pm\cdot\nabla^\varphi)(\partial^3\nabla^\varphi\times b^\pm)\cdot(\partial^3\nabla^\varphi\times v^\pm)\,\mathrm{d}\mathcal{V}_t+\underbrace{\int_{\Omega^\pm}RK_v^\pm\cdot(\partial^3\nabla^\varphi\times v^\pm)\,\mathrm{d}\mathcal{V}_t}_{=:L_1^\pm}\\
\overset{(b^\pm\cdot\nabla^\varphi)}{=\!=\!=}&-\int_{\Omega^\pm}(\partial^3\nabla^\varphi\times b^\pm)\cdot D_t^{\varphi\pm}(\partial^3\nabla^\varphi\times b^\pm)\,\mathrm{d}\mathcal{V}_t+\underbrace{\int_{\Omega^\pm}(\partial^3\nabla^\varphi\times b^\pm)\cdot\left(b^\pm\times(\partial^3\nabla^\varphi(\nabla^\varphi\cdot v^\pm))\right)\mathrm{d}\mathcal{V}_t}_{=:K_1^\pm}\\
&+\underbrace{\int_{\Omega^\pm}(\partial^3\nabla^\varphi\times b^\pm)\cdot RK_b^\pm\,\mathrm{d}\mathcal{V}_t}_{L_2^\pm}+L_1^\pm,\end{aligned} \tag{3.147}$$

where $L_1^\pm,\ L_2^\pm$ are directly controlled

$$L_1^\pm+L_2^\pm\le P(\|(v^\pm,b^\pm)\|_{4,\pm},|\psi|_4). \tag{3.148}$$

It remains to analyze the term $K_1^\pm$ in which there is a key observation for the energy structure of compressible MHD system. We invoke the continuity equation $\nabla^\varphi\cdot v^\pm=\mathcal{F}_p^\pm D_t^{\varphi\pm}p^\pm$ and commute $D_t^{\varphi\pm}$ with ∇^φ to get

$$\nabla^\varphi(\nabla^\varphi\cdot v^\pm)=-\mathcal{F}_p^\pm D_t^{\varphi\pm}\nabla^\varphi p^\pm+\mathcal{F}_p^\pm(\nabla^\varphi v_j^\pm)(\partial_j^\varphi p^\pm).$$

Next, we rewrite the momentum equation to be $\rho^\pm D_t^{\varphi\pm}v^\pm+b^\pm\times(\nabla^\varphi\times b^\pm)=-\nabla^\varphi p^\pm$ and plug it into the highest-order term $-\mathcal{F}_p^\pm D_t^{\varphi\pm}\nabla^\varphi p^\pm$ to get

$$\begin{aligned}-\mathcal{F}_p^\pm D_t^{\varphi\pm}\nabla^\varphi p^\pm&=\mathcal{F}_p^\pm D_t^{\varphi\pm}(\rho^\pm D_t^{\varphi\pm}v^\pm)+\mathcal{F}_p^\pm D_t^{\varphi\pm}(b^\pm\times(\nabla^\varphi\times b^\pm))\\
&=\mathcal{F}_p^\pm\rho^\pm(D_t^{\varphi\pm})^2v^\pm+\mathcal{F}_p^\pm D_t^{\varphi\pm}(b^\pm\times(\nabla^\varphi\times b^\pm))+\mathcal{F}_p^\pm(D_t^{\varphi\pm}\rho^\pm)(D_t^{\varphi\pm}v^\pm).\end{aligned}$$

Thus, the term $K_1^\pm$ becomes

$$\begin{aligned}K_1^\pm=&\int_{\Omega^\pm}(\partial^3\nabla^\varphi\times b^\pm)\cdot\left(b^\pm\times(\mathcal{F}_p^\pm\rho^\pm\partial^3(D_t^{\varphi\pm})^2v^\pm)\right)\mathrm{d}\mathcal{V}_t\\
&-\int_{\Omega^\pm}\mathcal{F}_p^\pm\left(b^\pm\times(\partial^3\nabla^\varphi\times b^\pm)\right)\cdot D_t^{\varphi\pm}\left(b^\pm\times(\partial^3\nabla^\varphi\times b^\pm)\right)\mathrm{d}\mathcal{V}_t\\
&+\int_{\Omega^\pm}(\partial^3\nabla^\varphi\times b^\pm)\cdot RK_p^\pm\,\mathrm{d}\mathcal{V}_t,\end{aligned} \tag{3.149}$$

where

$$RK_p^\pm := \mathcal{F}_p^\pm \partial^3 \left((\nabla^\varphi v_j^\pm)(\partial_j^\varphi p^\pm) + (D_t^{\varphi\pm}\rho^\pm)(D_t^{\varphi\pm} v^\pm) \right) + [\partial^3, \mathcal{F}_p^\pm \rho^\pm](D_t^{\varphi\pm})^2 v^\pm \\ + \mathcal{F}_p^\pm [\partial^3, D_t^{\varphi\pm}](b^\pm \times (\nabla^\varphi \times b^\pm)) + \mathcal{F}_p^\pm D_t^{\varphi\pm} \left([\partial^3, b^\pm \times](\nabla^\varphi \times b^\pm) \right)$$

consists of ≤ 4 derivatives in each term and its contribution can be directly controlled

$$L_3^\pm := \int_{\Omega^\pm} (\partial^3 \nabla^\varphi \times b^\pm) \cdot RK_p^\pm \, \mathrm{d}\mathcal{V}_t \leq P(\|b^\pm, v^\pm, \mathcal{F}_p^\pm p^\pm\|_{4,\pm}, \|\mathcal{F}_p^\pm D_t^{\varphi\pm}(v^\pm, b^\pm, p^\pm)\|_{3,\pm}) \tag{3.150}$$

Note that the second term on the right side of $K_1^\pm$ is obtained by using the vector identity $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = -\mathbf{c}\cdot(\mathbf{b}\times\mathbf{a})$:

$$(\partial^3 \nabla^\varphi \times b^\pm) \cdot \left(b^\pm \times D_t^{\varphi\pm} \left(b^\pm \times (\partial^3 \nabla^\varphi \times b^\pm) \right) \right) = -D_t^{\varphi\pm} \left(b^\pm \times (\partial^3 \nabla^\varphi \times b^\pm) \right) \cdot \left(b^\pm \times (\partial^3 \nabla^\varphi \times b^\pm) \right).$$

Therefore, we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega^\pm} \rho^\pm \left| \partial^3 (\nabla^\varphi \times v^\pm) \right|^2 \mathrm{d}\mathcal{V}_t \\ =& -\int_{\Omega^\pm} (\partial^3 \nabla^\varphi \times b^\pm) \cdot D_t^{\varphi\pm} (\partial^3 \nabla^\varphi \times b^\pm) \, \mathrm{d}\mathcal{V}_t - \int_{\Omega^\pm} \mathcal{F}_p^\pm \left(b^\pm \times (\partial^3 \nabla^\varphi \times b^\pm) \right) \cdot D_t^{\varphi\pm} \left(b^\pm \times (\partial^3 \nabla^\varphi \times b^\pm) \right) \mathrm{d}\mathcal{V}_t \\ &+ \underbrace{\int_{\Omega^\pm} (\partial^3 \nabla^\varphi \times b^\pm) \cdot \left(b^\pm \times (\mathcal{F}_p \rho^\pm \partial^3 (D_t^{\varphi\pm})^2 v^\pm) \right) \mathrm{d}\mathcal{V}_t}_{K^\pm} + L_1^\pm + L_2^\pm + L_3^\pm \\ =& -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega^\pm} \left| \partial^3 (\nabla^\varphi \times b^\pm) \right|^2 + \mathcal{F}_p^\pm \left| b^\pm \times \partial^3 (\nabla^\varphi \times b^\pm) \right|^2 \mathrm{d}\mathcal{V}_t \\ &+ \frac{1}{2} \int_{\Omega^\pm} (\nabla^\varphi \cdot v^\pm) \left(\left| \partial^3 (\nabla^\varphi \times b^\pm) \right|^2 + \mathcal{F}_p^\pm \left| b^\pm \times \partial^3 (\nabla^\varphi \times b^\pm) \right|^2 \right) \mathrm{d}\mathcal{V}_t + K^\pm + L_1^\pm + L_2^\pm + L_3^\pm, \end{aligned} \tag{3.151}$$

which further gives the control of vorticity and current density simultaneously

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega^\pm} \rho^\pm \left| \partial^3 (\nabla^\varphi \times v^\pm) \right|^2 + \left| \partial^3 (\nabla^\varphi \times b^\pm) \right|^2 + \mathcal{F}_p^\pm \left| b^\pm \times \partial^3 (\nabla^\varphi \times b^\pm) \right|^2 \mathrm{d}\mathcal{V}_t \\ \lesssim & P(E_4^\kappa(t)) + P(|\psi|_4) \|b^\pm\|_{4,\pm} \|b^\pm \rho^\pm\|_{L^\infty(\Omega^\pm)} \left\| \varepsilon^2 (D_t^{\varphi\pm})^2 v^\pm \right\|_3 \leq P(E_4^\kappa(t)) + E_5^\kappa(t). \end{aligned} \tag{3.152}$$

Hence, the vorticity analysis for compressible ideal MHD cannot be closed in standard Sobolev space because of the term $\varepsilon^2 \partial^3 (D_t^{\varphi\pm})^2 v^\pm$ in $K^\pm$. Instead, the appearance of this term indicates us to **trade one normal derivative (in the curl operator) for two tangential derivatives $(D_t^{\varphi\pm})^2$ together with square weights of Mach number ε^2. Besides, the normal derivative part involving $\partial^3 D_t^{\varphi\pm}(\nabla^\varphi \times b^\pm)$ contributes to the energy of current density thanks to the special structure of Lorentz force $-b^\pm \times (\nabla^\varphi \times b^\pm)$. This is exactly the motivation for us to define the energy functional $E(t)$ under the setting the anisotropic Sobolev spaces instead of standard Sobolev spaces.**

Similarly, the curl estimates for the time derivatives (in $E_4(t)$) can be proven in the same way by replacing ∂^3 with $\partial^{3-k}\partial_t^k$ ($1 \leq k \leq 3$). We omit the details and list the conclusion

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega^\pm} \rho^\pm \left| \partial^{3-k}\partial_t^k (\nabla^\varphi \times v^\pm) \right|^2 + \left| \partial^{3-k}\partial_t^k (\nabla^\varphi \times b^\pm) \right|^2 + \mathcal{F}_p^\pm \left| b^\pm \times \partial^{3-k}\partial_t^k (\nabla^\varphi \times b^\pm) \right|^2 \mathrm{d}\mathcal{V}_t \\ \lesssim & P(E_4^\kappa(t)) + \left\| \varepsilon^2 \partial_t^k (D_t^{\varphi\pm})^2 v^\pm \right\|_{3-k,\pm}^2 \leq P(E_4^\kappa(t)) + E_5^\kappa(t). \end{aligned} \tag{3.153}$$

Finally, we need to commute ∂_t^k with $\nabla^\varphi \times$ when $k \geq 1$. We have

$$(\nabla^\varphi \times \partial_t^k v)_i \stackrel{L}{=} \partial_t^k (\nabla^\varphi \times v)_i + \epsilon_{ijl} (\partial_3 \varphi)^{-1} (\overline{\partial}_j \partial_t^k \varphi)(\partial_3 v_l),$$

where ϵ_{ijl} is the sign of permutation $(ijl) \in S_3$. This gives

$$\|\nabla^\varphi \times \partial_t^k v^\pm\|_{3-k,\pm}^2 \lesssim C(\|v^\pm\|_{W^{1,\infty}(\Omega^\pm)}) \left(\left\| \partial_t^k (\nabla^\varphi \times v^\pm) \right\|_{3-k,\pm}^2 + \left| \partial_t^k \psi \right|_{4-k}^2 \right) + \text{ lower order terms,} \tag{3.154}$$

where both leading order terms have been controlled in tangential estimates of $E_4^\kappa(t)$. The same result holds for $b^\pm$. Using the result of tangential estimates of $E_4^\kappa(t)$, we have: for any $k \in \{0,1,2,3\}$ and any $\delta \in (0,1)$

$$\begin{aligned}
&\left\|\nabla^\varphi \times \partial_t^k v^\pm\right\|_{3-k,\pm}^2 + \left\|\nabla^\varphi \times \partial_t^k b^\pm\right\|_{3-k,\pm}^2 \\
\lesssim\ & P(E_4^\kappa(0)) + \int_0^t P(E_4^\kappa(\tau)) + E_5^\kappa(\tau)\,\mathrm{d}\tau + P(\|v^\pm, b^\pm\|_{W^{1,\infty}(\Omega^\pm)}) \left|\partial_t^k \psi\right|_{4-k}^2 \\
\lesssim\ & \delta E_4^\kappa(t) + P(\sigma^{-1}, E_4^\kappa(0)) + P(E_4^\kappa(t)) \int_0^t P(\sigma^{-1}, E_4^\kappa(\tau)) + E_5^\kappa(\tau)\,\mathrm{d}\tau,
\end{aligned} \tag{3.155}$$

3.6.3 Further div-curl analysis for $E_5 \sim E_7$

The vorticity analysis for $E_4(t)$ requires the control of $\left\|\varepsilon^2 \partial_t^k (D_t^{\varphi\pm})^2 v^\pm\right\|_{3-k}^2$ for $0 \le k \le 3$. When $0 \le k \le 2$, there are still normal derivatives in this term. Thus, we shall do further div-curl analysis on $\left\|\varepsilon^2 \partial_t^k (D_t^{\varphi\pm})^2 v^\pm\right\|_{3-k}^2$ for $0 \le k \le 2$. Let $\mathcal{T}^\alpha = \partial_t^{\alpha_0} \overline{\partial}_1^{\alpha_1} \overline{\partial}_2^{\alpha_2} (\omega(x_3)\partial_3)^{\alpha_4}$ with $\langle \alpha \rangle = 2$. The divergence part can be reduced in the same way as in Section 3.6.1. We take $\partial_t^k \mathcal{T}^\alpha$ in the continuity equation and omit the lower order terms to get

$$\nabla^\varphi \cdot \partial_t^k \mathcal{T}^\alpha v \overset{L}{=} -\varepsilon^2 \partial_t^k \mathcal{T}^\alpha D_t^\varphi p + (\partial_3 \varphi)^{-1} \overline{\partial} \partial_t^k \mathcal{T}^\alpha \varphi \cdot \partial_3 v,$$

which gives

$$\|\varepsilon^2 \nabla^\varphi \cdot \partial_t^k \mathcal{T}^\alpha v^\pm\|_{2-k}^2 \lesssim C(\|v\|_{W^{1,\infty}}) \left(\left\|\varepsilon^4 \partial_t^k \mathcal{T}^\alpha \mathcal{T} p^\pm\right\|_{2-k,\pm}^2 + \left|\varepsilon^2 \partial_t^k \mathcal{T}^\alpha \psi\right|_{2-k}^2 \right) + \text{ lower order terms.} \tag{3.156}$$

Remark 3.3. The term generated when commuting $\mathcal{T}^\alpha$ with ∇^φ is actually of lower order. One can check that (see also [85, (3.24)-(3.25)])

$$[(\omega\partial_3)^m, \partial_3] f = \underbrace{(\omega\partial_3)^m \partial_3 f - \partial_3((\omega\partial_3)^m f)}_{\text{both are (m+1)-th order terms}} = \sum_{k \le m-1} c_{m,k} (\omega\partial_3)^k \partial_3 f = \sum_{k \le m-1} d_{m,k} \partial_3 (\omega\partial_3)^k f$$

for some smooth functions $c_{m,k}, d_{m,k}$ depending on m, k and the derivatives (up to order m) of ω, and the right side only contains $\le m$-th order terms.

Similarly, using $\nabla^\varphi \cdot b = 0$, we have

$$\nabla^\varphi \cdot \partial_t^k \mathcal{T}^\alpha b \overset{L}{=} \underbrace{\partial_t^k \mathcal{T}^\alpha (\nabla^\varphi \cdot b)}_{=0} + (\partial_3 \varphi)^{-1} \overline{\partial} \partial_t^k \mathcal{T}^\alpha \varphi \cdot \partial_3 b$$

and thus

$$\|\varepsilon^2 \nabla^\varphi \cdot \partial_t^k b^\pm\|_{2-k,\pm}^2 \lesssim C(\|b^\pm\|_{W^{1,\infty}(\Omega^\pm)}) \left|\varepsilon^2 \partial_t^k \mathcal{T}^\alpha \psi\right|_{2-k}^2 + \text{ lower order terms.} \tag{3.157}$$

The control of divergence part in the analysis of $E_5^\kappa(t)$ is concluded by the following energy inequality. For any $k \in \{0,1,2\}$, any multi-index α with $\langle \alpha \rangle = 2$ and any $\delta \in (0,1)$

$$\begin{aligned}
&\left\|\varepsilon^2 \nabla^\varphi \cdot \partial_t^k \mathcal{T}^\alpha (v^\pm, b^\pm)\right\|_{2-k,\pm}^2 \lesssim C(\|v^\pm\|_{W^{1,\infty}(\Omega^\pm)}) \left\|\varepsilon^4 \partial_t^k \mathcal{T}^\alpha \mathcal{T} p^\pm\right\|_{2-k,\pm}^2 + C(\|v^\pm, b^\pm\|_{W^{1,\infty}(\Omega^\pm)}) \left|\partial_t^k \mathcal{T}^\alpha \psi\right|_{3-k}^2 \\
\lesssim\ & C(\|v^\pm\|_{W^{1,\infty}(\Omega^\pm)}) \left\|\varepsilon^4 \partial_t^k \mathcal{T}^\alpha \mathcal{T} p^\pm\right\|_{2-k,\pm}^2 + \delta E_5^\kappa(t) + P(E_4^\kappa(0), E_5^\kappa(0)) + P(E_4^\kappa(t)) \int_0^t P(E_4^\kappa(\tau), E_5^\kappa(\tau))\,\mathrm{d}\tau,
\end{aligned} \tag{3.158}$$

where the term involving $\mathcal{T} p^\pm$ can be further reduced to $\mathcal{T}(v^\pm, b^\pm)$ when $2 - k > 0$ so that one can further apply the div-curl analysis to it.

As for the curl part, we can still mimic the proof in Section 3.6.2 to get the control of $\left\|\varepsilon^2 \partial_t^k \mathcal{T}^\alpha (\nabla^\varphi \times (v, b))\right\|_{2-k}$ for $0 \le k \le 2$ and $\langle \alpha \rangle = 2$ with $\alpha_3 = 0$

$$\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega^\pm} \rho^\pm \left|\varepsilon^2 \partial^{2-k} \partial_t^k \mathcal{T}^\alpha (\nabla^\varphi \times v^\pm)\right|^2 + \left|\varepsilon^2 \partial^{2-k} \partial_t^k \mathcal{T}^\alpha (\nabla^\varphi \times b^\pm)\right|^2 + \mathcal{F}_p^\pm \left|\varepsilon^2 b^\pm \times \partial^{2-k} \partial_t^k \mathcal{T}^\alpha (\nabla^\varphi \times b^\pm)\right|^2 \mathrm{d}\mathcal{V}_t \\
\lesssim\ & P(E_4^\kappa(t), E_5^\kappa(t)) + \left\|\varepsilon^4 \partial_t^k \mathcal{T}^\alpha (D_t^{\varphi\pm})^2 v^\pm\right\|_{2-k}^2 \le P(E_4^\kappa(t), E_5^\kappa(t)) + E_6^\kappa(t).
\end{aligned} \tag{3.159}$$

Then we commute $\partial^{2-k}\partial_t^k\mathcal{T}^\alpha$ with $\nabla^\varphi\times$ to get: for any $k \in \{0,1,2\}$, any multi-index α with $\langle\alpha\rangle = 2$ and $\alpha_3 = 0$, and any $\delta \in (0,1)$

$$\begin{aligned}
&\left\|\nabla^\varphi \times \partial_t^k\mathcal{T}^\alpha v^\pm\right\|_{2-k,\pm}^2 + \left\|\nabla^\varphi \times \partial_t^k\mathcal{T}^\alpha b^\pm\right\|_{2-k,\pm}^2 \\
\lesssim\ & P(E_4^\kappa(0), E_5^\kappa(0)) + \int_0^t P(E_4^\kappa(\tau), E_5^\kappa(\tau)) + E_6^\kappa(\tau)\,\mathrm{d}\tau + P(\|v^\pm, b^\pm\|_{W^{1,\infty}(\Omega^\pm)}) \left|\varepsilon^2\partial_t^k\mathcal{T}^\alpha\psi\right|_{2-k}^2 \\
\lesssim\ & \delta E_5^\kappa(t) + P(\sigma^{-1}, E_4^\kappa(0), E_5^\kappa(0)) + P(E_4^\kappa(t)) \int_0^t P(\sigma^{-1}, E_4^\kappa(\tau), E_5^\kappa(\tau)) + E_6^\kappa(\tau)\,\mathrm{d}\tau,
\end{aligned} \tag{3.160}$$

where we use the result of tangential estimates to control $\left|\varepsilon^2\partial_t^k\mathcal{T}^\alpha\psi\right|_{2-k}$. When $k \le 1$ in the above energy estimate, we shall continue to apply the div-curl analysis to $\left\|\varepsilon^4\partial_t^k\mathcal{T}^\alpha(D_t^{\varphi\pm})^2 v^\pm\right\|_{2-k}^2$.

For E_6^κ and E_7^κ, we have analogous div-curl inequalities. For $l = 2,3$, we continue to analyze the divergence and the curl according to (3.130). Similarly as above, we have the following estimates for any $k \in \{0,1\}$, any multi-index α with $\langle\alpha\rangle = 4,\ \alpha_3 = 0$ and any $\delta \in (0,1)$

$$\begin{aligned}
&\left\|\varepsilon^4\nabla^\varphi \times \partial_t^k\mathcal{T}^\alpha v^\pm\right\|_{1-k,\pm}^2 + \left\|\varepsilon^4\nabla^\varphi \times \partial_t^k\mathcal{T}^\alpha b^\pm\right\|_{1-k,\pm}^2 \\
\lesssim\ & \delta E_6^\kappa(t) + P\left(\sigma^{-1}, \sum_{l=0}^2 E_{4+l}^\kappa(0)\right) + P(E_4^\kappa(t)) \int_0^t P\left(\sigma^{-1}, \sum_{l=0}^2 E_{4+l}^\kappa(\tau)\right) + E_7^\kappa(\tau)\,\mathrm{d}\tau,
\end{aligned} \tag{3.161}$$

where this E_7^κ term is contributed by $\left\|\varepsilon^6\partial_t^k\mathcal{T}^\alpha(D_t^{\varphi\pm})^2 v^\pm\right\|_{1-k,\pm}$. When $k = 0$ in this term, we again apply the div-curl analysis to it in order to eliminate all normal derivatives falling on v, b. For any multi-index α with $\langle\alpha\rangle = 6$ and $\alpha_3 = 0$ and any $\delta \in (0,1)$, we have

$$\begin{aligned}
&\left\|\varepsilon^6\nabla^\varphi \times \mathcal{T}^\alpha v^\pm\right\|_{0,\pm}^2 + \left\|\varepsilon^6\nabla^\varphi \times \mathcal{T}^\alpha b^\pm\right\|_{0,\pm}^2 \\
\lesssim\ & \delta E_7^\kappa(t) + P\left(\sigma^{-1}, \sum_{l=0}^3 E_{4+l}^\kappa(0)\right) + P(E_4^\kappa(t)) \int_0^t P\left(\sigma^{-1}, \sum_{l=0}^3 E_{4+l}^\kappa(\tau)\right) + E_8^\kappa(\tau)\,\mathrm{d}\tau.
\end{aligned} \tag{3.162}$$

The control of divergence part for $E_6^\kappa(t),\ E_7^\kappa(t)$ also follows the same way as $E_4^\kappa(t), E_5^\kappa(t)$. For any $k \in \{0,1\}$, any multi-index α with $\langle\alpha\rangle = 4,\ \alpha_3 = 0$, we have

$$\begin{aligned}
&\left\|\varepsilon^4\nabla^\varphi \cdot \partial_t^k\mathcal{T}^\alpha(v^\pm, b^\pm)\right\|_{1-k,\pm}^2 \lesssim C(\|v^\pm\|_{W^{1,\infty}(\Omega^\pm)}) \left\|\varepsilon^6\partial_t^k\mathcal{T}^\alpha\mathcal{T} p^\pm\right\|_{1-k,\pm}^2 + C(\|v^\pm, b^\pm\|_{W^{1,\infty}(\Omega^\pm)}) \left|\partial_t^k\mathcal{T}^\alpha\psi\right|_{2-k}^2 \\
\lesssim\ & \delta E_6^\kappa(t) + P\left(\sigma^{-1}, \sum_{l=0}^2 E_{4+l}^\kappa(0)\right) + P(E_4^\kappa(t)) \int_0^t P\left(\sigma^{-1}, \sum_{l=0}^2 E_{4+l}^\kappa(\tau)\right)\mathrm{d}\tau,
\end{aligned} \tag{3.163}$$

where the term $\left\|\varepsilon^6\partial_t^k\mathcal{T}^\alpha\mathcal{T} p^\pm\right\|_{1-k,\pm}^2$ does not appear because it can be converted to full tangential derivatives (part of $E_6^\kappa(t)$) which has been controlled in Section 3.3-Section 3.5. For any multi-index α with $\langle\alpha\rangle = 6,\ \alpha_3 = 0$, we have

$$\begin{aligned}
&\left\|\varepsilon^6\nabla^\varphi \cdot \partial_t^k\mathcal{T}^\alpha(v^\pm, b^\pm)\right\|_{0,\pm}^2 \lesssim C(\|v^\pm\|_{W^{1,\infty}(\Omega^\pm)}) \left\|\varepsilon^8\partial_t^k\mathcal{T}^\alpha\mathcal{T} p^\pm\right\|_{0,\pm}^2 + C(\|v^\pm, b^\pm\|_{W^{1,\infty}(\Omega^\pm)}) \left|\varepsilon^6\mathcal{T}^\alpha\psi\right|_1^2 \\
\lesssim\ & \delta E_7^\kappa(t) + P\left(\sigma^{-1}, \sum_{l=0}^3 E_{4+l}^\kappa(0)\right) + P(E_4^\kappa(t)) \int_0^t P\left(\sigma^{-1}, \sum_{l=0}^3 E_{4+l}^\kappa(\tau)\right)\mathrm{d}\tau,
\end{aligned} \tag{3.164}$$

where the term $\left\|\varepsilon^8\mathcal{T}^\alpha\mathcal{T} p^\pm\right\|_{0,\pm}^2$ does not appear because it has been included in tangential estimates for $E_7^\kappa(t)$.

3.7 Uniform estimates for the nonlinear approximate system

3.7.1 Control of the entropy

It remains to control the full (anisotropic) Sobolev norms of the entropy functions $S^\pm$. This can be easily proven thanks to $D_t^{\varphi\pm}S^\pm = 0$. In the control of $E_{4+l}^\kappa(t)$ for fixed $0 \le l \le 4$, we need to take the derivative

$\partial_*^\alpha := \partial^{4-l-k}\partial_t^k\mathcal{T}^\gamma = \partial_3^{\gamma_3'}(\omega\partial_3)^{\gamma_4}\partial_t^{k+\gamma_0}\overline{\partial}_1^{\gamma_1+\gamma_1'}\overline{\partial}_2^{\gamma_2+\gamma_2'}$ with $\gamma_0+\gamma_1+\gamma_2+\gamma_4 = 2l$, $\gamma_1'+\gamma_2'+\gamma_3' = 4-k-l$ and $0\leq k\leq 4-l$ and also multiply the weight ε^{2l}. Then we can introduce the Alinhac good unknown $\mathbf{S}^\partial$ with respect to this general derivative ∂_*^α by

$$\mathbf{S}^{\partial,\pm} := \partial_*^\alpha S^\pm - \partial_*^\alpha\varphi\partial_3^\varphi S^\pm,$$

which satisfies the evolution equation $D_t^{\varphi\pm}\mathbf{S}^{\partial,\pm} = \mathfrak{D}^\partial(S^\pm)$ in $\Omega^\pm$ where $\mathfrak{D}^\partial(S^\pm)$ is defined by (3.10) after replacing $\mathcal{T}^\gamma$ with ∂_*^α. We will get

$$\begin{aligned}\left\|\varepsilon^{2l}\partial_t^k\mathcal{T}^\gamma S^\pm\right\|_{4-k-l,\pm}^2 &\lesssim \left\|\varepsilon^{2l}\mathbf{S}^{\partial,\pm}\right\|_{0,\pm}^2 + \left|\partial_*^\alpha\psi\right|_0^2\|\partial_3 S^\pm\|_{L^\infty(\Omega^\pm)}^2\\ &\lesssim \delta E_{4+l}^\kappa(t) + P\left(\sigma^{-1},\sum_{j=0}^{l}E_{4+j}^\kappa(0)\right) + E_4^\kappa(t)\int_0^t P\left(\sigma^{-1},\sum_{j=0}^{l}E_{4+j}^\kappa(\tau)\right)\mathrm{d}\tau,\end{aligned}\tag{3.165}$$

where we again invoke the estimate of $\left|\partial_*^\alpha\psi\right|_0$ that has been proven in Section 3.3-Section 3.5.

3.7.2 Uniform-in-κ estimates for the nonlinear approximate system

Summarizing Proposition 3.2 (L^2-energy conservation), Proposition 3.3 (tangential estimates), Proposition 3.8 (div-curl estimates) and (3.165) (entropy estimates), we conclude the estimates of the energy functional $E^\kappa(t)$ for the nonlinear approximate system (3.1) by

$$E^\kappa(t) \lesssim \delta E^\kappa(t) + P(E^\kappa(0)) + P(E^\kappa(t))\int_0^t P(\sigma^{-1},E^\kappa(\tau))\,\mathrm{d}\tau,\quad \forall\delta\in(0,1)\tag{3.166}$$

Thus, choosing δ suitably small such that $\delta E^\kappa(t)$ can be absorbed by the left side and then using Gronwall-type argument, we find that there exists a time $T_\sigma > 0$ that depends on σ and the initial data and is independent of κ and ε, such that

$$\sup_{0\leq t\leq T_\sigma} E^\kappa(t) \leq C(\sigma^{-1})P(E^\kappa(0)),\tag{3.167}$$

which is exactly the conclusion of Proposition 3.1.

4 Well-posedness of the nonlinear approximate system

We already prove the uniform-in-κ estimates for the nonlinear approximate problem (3.1). If we can prove the well-posedness of (3.1) for each fixed $\kappa > 0$, then the uniform estimates allow us to take the limit $\kappa\to 0_+$ and prove the local existence of system (1.33) for the compressible current-vortex sheets with surface tension. Since there is no loss of regularity in the estimate of $E^\kappa(t)$, we would like to use Picard iteration to construct the solution to (3.1) for each fixed κ.

4.1 Construction of the linearized problem

We start with $\psi^{[-1]} = \psi^{[0]} = 0$ and $(v^{[0],\pm}, b^{[0],\pm}, \rho^{[0],\pm}, S^{[0],\pm}) = (\vec{0},\vec{0},\underline{\rho}^\pm, 0)$ for some constants $\underline{\rho}^\pm \geq \bar{\rho}_0$. Then for any $n\geq 0, n\in\mathbb{N}$, given $\{(v^{[k],\pm}, b^{[k],\pm}, \rho^{[k],\pm}, S^{[k],\pm})\}_{k\leq n}$, we define $(v^{[n+1],\pm}, b^{[n+1],\pm}, q^{[n+1],\pm}, S^{[n+1],\pm}, \psi^{[n+1]})$ by the following linear system with variable coefficients only depending on $(v^{[n],\pm}, b^{[n],\pm}, q^{[n],\pm}, S^{[n],\pm}, \psi^{[n]}, \psi^{[n-1]})$

$$\begin{cases}\rho^{[n],\pm}D_t^{\varphi^{[n]},\pm}v^{[n+1],\pm} - (\boldsymbol{b}^{[n],\pm}\cdot\nabla^{\varphi^{[n]},\pm})b^{[n+1],\pm} + \nabla^{\varphi^{[n]},\pm}q^{[n+1],\pm} = 0 & \text{in } [0,T]\times\Omega^\pm,\\ (\mathcal{F}_p^\pm)^{[n]}D_t^{\varphi^{[n]},\pm}q^{[n+1],\pm} - (\mathcal{F}_p^\pm)^{[n]}D_t^{\varphi^{[n]},\pm}b^{[n+1],\pm}\cdot\boldsymbol{b}^{[n],\pm} + \nabla^{\varphi^{[n]},\pm}\cdot v^{[n+1],\pm} = 0 & \text{in } [0,T]\times\Omega^\pm,\\ D_t^{\varphi^{[n]},\pm}b^{[n+1],\pm} - (\boldsymbol{b}^{[n],\pm}\cdot\nabla^{\varphi^{[n]},\pm})v^{[n+1],\pm} + \boldsymbol{b}^{[n],\pm}\nabla^{\varphi^{[n]},\pm}\cdot v^{[n+1],\pm} = 0 & \text{in } [0,T]\times\Omega^\pm,\\ D_t^{\varphi^{[n]},\pm}S^{[n+1],\pm} = 0 & \text{in } [0,T]\times\Omega^\pm,\\ \left[\!\left[q^{[n+1]}\right]\!\right] = \sigma\mathcal{H}(\psi^{[n]}) - \kappa(1-\overline{\Delta})^2\psi^{[n+1]} - \kappa(1-\overline{\Delta})\partial_t\psi^{[n+1]} & \text{on } [0,T]\times\Sigma,\\ \partial_t\psi^{[n+1]} = v^{[n+1],\pm}\cdot N^{[n]} & \text{on } [0,T]\times\Sigma,\\ v_3^{[n+1],\pm} = 0, & \text{on } [0,T]\times\Sigma^\pm,\\ (v^{[n+1],\pm}, b^{[n+1],\pm}, q^{[n+1],\pm}, S^{[n+1],\pm}, \psi^\pm)|_{t=0} = (v_0^{\kappa,\pm}, b_0^{\kappa,\pm}, q_0^{\kappa,\pm}, S_0^{\kappa,\pm}, \psi_0^\kappa), \end{cases}\tag{4.1}$$

where $\boldsymbol{b}_i^{[n],\pm} := b_i^{[n],\pm}$ for $i = 1, 2$ and $\boldsymbol{b}_3^{[n],\pm}$ is defined by

$$\boldsymbol{b}_3^{[n],\pm} := b_3^{[n],\pm} + \mathfrak{R}_T^{\pm}\left(b_1^{[n],\pm}\overline{\partial}_1\psi^{[n]} + b_2^{[n],\pm}\overline{\partial}_2\psi^{[n]} - b_3^{[n],\pm}\right)\Big|_{\Sigma} \tag{4.2}$$

where $\mathfrak{R}_T^{\pm}$ is the lifting operator defined in Lemma B.3. The initial data $(v_0^{\kappa,\pm}, b_0^{\kappa,\pm}, \rho_0^{\kappa,\pm}, S_0^{\kappa,\pm}, \psi_0^{\kappa})$ is the same as (3.1). The basic state $(v^{[n],\pm}, \boldsymbol{b}^{[n],\pm}, \rho^{[n],\pm}, S^{[n],\pm}, \psi^{[n]}, \psi^{[n-1]})$ satisfies

1. (The hyperbolicity assumption) $\rho^{[n],\pm} > 0$ is determined by the equation of state (1.18) where $p^{[n],\pm}$ is defined by $p^{[n],\pm} := q^{[n],\pm} - \frac{1}{2}|b^{[n],\pm}|^2$. Then define $\mathcal{F}^{[n]} = \log\rho^{[n]}$, $\mathcal{F}_p^{[n],\pm} := \frac{\partial\mathcal{F}^{[n],\pm}}{\partial p}(p^{[n+1],\pm}, S^{[n+1],\pm}) > 0$.
2. (Tangential magnetic fields) $\boldsymbol{b}^{[n],\pm}\cdot N^{[n]} = 0$ on Σ, and $\boldsymbol{b}_3^{[n],\pm} = 0$ on $\Sigma^{\pm}$.
3. (Linearized material derivatives and covariant derivatives)

$$D_t^{\varphi^{[n]},\pm} := \partial_t + \bar{v}^{[n]}\cdot\overline{\nabla} + \frac{1}{\partial_3\varphi^{[n]}}(v^{[n]}\cdot\mathbf{N}^{[n-1]} - \partial_t\varphi^{[n]})\partial_3, \tag{4.3}$$

$$\partial_t^{\varphi^{[n]}} := \partial_t - \frac{\partial_t\varphi^{[n]}}{\partial_3\varphi^{[n]}}\partial_3,\ \nabla_a^{\varphi^{[n]}} = \partial_a^{\varphi^{[n]}} := \partial_a - \frac{\partial_a\varphi^{[n]}}{\partial_3\varphi^{[n]}}\partial_3,\ \ a = 1,2, \quad \nabla_a^{\varphi^{[n]}} = \partial_3^{\varphi^{[n]}} := \frac{1}{\partial_3\varphi^{[n]}}\partial_3 \tag{4.4}$$

where $N^{[n]} := (-\partial_1\psi^{[n]}, -\partial_2\psi^{[n]}, 1)^\top$ and $\mathbf{N}^{[n]}$ is the extension of $N^{[n]}$ with $\varphi^{[n]} = x_3 + \chi(x_3)\psi^{[n]}(t, x')$.

After solving the linear problem (4.1), we define $p^{[n+1],\pm} = q^{[n+1],\pm} - \frac{1}{2}|b^{[n+1],\pm}|^2$ and use the equation of state $p^{[n+1]} = p^{[n+1]}(\rho^{[n+1]}, S^{[n+1]})$ to determine the density $\rho^{[n+1]} > 0$. We shall also define the "modified magnteic fields" $\boldsymbol{b}^{[n+1],\pm}$ as follows in order to guarantee $\boldsymbol{b}^{[n+1],\pm}\cdot N^{[n+1]} = 0$ on Σ and $\Sigma^{\pm}$:

$$\boldsymbol{b}_1^{[n+1],\pm} = b_1^{[n+1],\pm}, \quad \boldsymbol{b}_2^{[n+1],\pm} = b_2^{[n+1],\pm},$$
$$\boldsymbol{b}_3^{[n+1],\pm} = b_3^{[n+1],\pm} + \mathfrak{R}_T^{\pm}\left(b_1^{[n+1],\pm}\overline{\partial}_1\psi^{[n+1]} + b_2^{[n+1],\pm}\overline{\partial}_2\psi^{[n+1]} - b_3^{[n+1],\pm}\right)\Big|_{\Sigma}. \tag{4.5}$$

Remark 4.1 (The boundary constraint of magnetic fields)**.** The modified basic state $\mathring{\boldsymbol{b}}$ is necessary here, because the quantity $b^{[n+1]}$ solved from (4.1) may not be tangential on Σ and so integrating $(b\cdot\nabla^{\varphi})$ by parts produces uncontrollable boundary terms. When taking the limit $n\to\infty$, we can show that the limit function $b^{[\infty]}$ also satisfy the boundary constraint $b^{[\infty]}\cdot N^{[\infty]}|_{\Sigma} = 0$ which then indicates that $\boldsymbol{b}_3^{[\infty]} = b_3^{[\infty]}$ in order to recover the nonlinear approximate system (3.1). We refer to Section 4.4.1 for details.

Remark 4.2 (The divergence constraint of magnetic fields)**.** Notice that the divergence-free condition for $b^{\pm}$ no longer propagates from the initial data for the linear problem, but we will show that the contribution of the divergence of part of $b^{\pm}$ is still controllable and does not introduce extra substantial difficulty. After solving the nonlinear problem (3.1) for each fixed $\kappa > 0$, $\nabla^{\varphi}\cdot b^{\pm} = 0$ in (3.1) is automatically recovered from the initial constraint $\nabla^{\varphi}\cdot b_0^{\kappa,\pm} = 0$.

For simplicity of notations, given any $n\in\mathbb{N}$, we denote $(v^{[n+1],\pm}, b^{[n+1],\pm}, q^{[n+1],\pm}, p^{[n+1],\pm}, \rho^{[n+1],\pm}, S^{[n+1],\pm}, \psi^{[n+1]})$, $(v^{[n],\pm}, b^{[n],\pm}, \boldsymbol{b}^{[n],\pm}, q^{[n],\pm}, \rho^{[n],\pm}, p^{[n],\pm}, S^{[n],\pm}, \psi^{[n]})$, $\psi^{[n-1]}$ respectively by $(v^{\pm}, b^{\pm}, q^{\pm}, p^{\pm}, \rho^{\pm}, S^{\pm})$, $(\mathring{v}^{\pm}, \mathring{b}^{\pm}, \mathring{\boldsymbol{b}}^{\pm}, \mathring{q}^{\pm}, \mathring{\rho}^{\pm}, \mathring{p}^{\pm}, \mathring{S}^{\pm}, \mathring{\psi})$ and $\dot{\psi}$. Also, we denote $D_t^{\varphi^{[n]},\pm}$ and $\partial_i^{\varphi^{[n]}}$, $\nabla_i^{\varphi^{[n]}}$ by $D_t^{\mathring{\varphi}\pm}$ and $\partial_i^{\mathring{\varphi}}$, $\nabla^{\mathring{\varphi}}$. Thus, the linear problem above becomes

$$\begin{cases}
\mathring{\rho}D_t^{\mathring{\varphi}\pm}v^{\pm} - (\mathring{\boldsymbol{b}}\cdot\nabla^{\mathring{\varphi}})b^{\pm} + \nabla^{\mathring{\varphi}}q^{\pm} = 0 & \text{in } [0,T]\times\Omega^{\pm},\\
\mathring{\mathcal{F}}_p^{\pm}D_t^{\mathring{\varphi}\pm}q^{\pm} - \mathring{\mathcal{F}}_p^{\pm}D_t^{\mathring{\varphi}\pm}b^{\pm}\cdot\mathring{\boldsymbol{b}}^{\pm} + \nabla^{\mathring{\varphi}}\cdot v^{\pm} = 0 & \text{in } [0,T]\times\Omega^{\pm},\\
D_t^{\mathring{\varphi}\pm}b^{\pm} - (\mathring{\boldsymbol{b}}^{\pm}\cdot\nabla^{\mathring{\varphi}})v^{\pm} + \mathring{\boldsymbol{b}}^{\pm}\nabla^{\mathring{\varphi}}\cdot v^{\pm} = 0 & \text{in } [0,T]\times\Omega^{\pm},\\
D_t^{\mathring{\varphi}\pm}S^{\pm} = 0 & \text{in } [0,T]\times\Omega^{\pm},\\
[\![q]\!] = \sigma\mathcal{H}(\mathring{\psi}) - \kappa(1-\overline{\Delta})^2\psi - \kappa(1-\overline{\Delta})\partial_t\psi & \text{on } [0,T]\times\Sigma,\\
\partial_t\psi = v^{\pm}\cdot\mathring{N} & \text{on } [0,T]\times\Sigma,\\
v_3^{\pm} = 0 & \text{on } [0,T]\times\Sigma^{\pm},\\
(v^{\pm}, b^{\pm}, q^{\pm}, S^{\pm}, \psi)|_{t=0} = (v_0^{\kappa,\pm}, b_0^{\kappa,\pm}, q_0^{\kappa,\pm}, S_0^{\kappa,\pm}, \psi_0^{\kappa}),
\end{cases} \tag{4.6}$$

where $D_t^{\mathring{\varphi}\pm} = \partial_t + \mathring{\bar{v}}\cdot\overline{\nabla} + \frac{1}{\partial_3\mathring{\varphi}}(\mathring{v}\cdot\dot{\mathbf{N}} - \partial_t\mathring{\varphi})\partial_3$ and $\mathcal{H}(\mathring{\psi}) = \overline{\nabla}\cdot(\overline{\nabla}\mathring{\psi}/|\mathring{N}|)$.

4.2 Well-posedness of the linearized approximate problem

Although (4.6) is a first-order linear symmetric hyperbolic system with characteristic boundary conditions, it is still not easy to apply the duality argument in Lax-Phillips [46] to prove the well-posedness of (4.6) in $L^2([0,T]\times\Omega^\pm)$ because the boundary condition for the dual system of (4.6) may not be explicitly calculated due to the appearance of the regularization term $\kappa(1-\overline{\Delta})^2\psi$. Instead, we apply the classical Galerkin's method to construct the weak solution in $L^2([0,T]\times\Omega^\pm)$ to the linearized problem (4.6). Once this is done, the weak solution is actually a strong solution by the argument in [60, Chapter 2.2.3].

We assume the basic state $(\mathring{v},\mathring{b},\mathring{q},\mathring{\rho},\mathring{p},\mathring{S},\mathring{\psi})$ and $\dot{\psi}$ satisfy the following bounds: There exists some $\mathring{K}_0>0$ and a time $T_\kappa>0$ (depending on $\kappa>0$) such that

$$\sup_{0\le t\le T_\kappa}\sum_{l=0}^{4}\Big(\sum_{\pm}\sum_{\langle\alpha\rangle=2l}\sum_{k=0}^{4-l}\left\|\left(\varepsilon^{2l}\mathcal{T}^\alpha\partial_t^k(\mathring{v}^\pm,\mathring{b}^\pm,\mathring{S}^\pm,(\mathring{\mathcal{F}}_p^\pm)^{\frac{(k+\alpha_0-l-3)_+}{2}}\mathring{q}^\pm)\right)\right\|_{4-k-l,\pm}^2 + \sum_{k=0}^{4+l}\left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^k\mathring{\psi}\right|_{6+l-k}^2+\int_0^t\left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^{5+l}\mathring{\psi}\right|_1^2\,\mathrm{d}\tau\Big)\le\mathring{K}_0, \tag{4.7}$$

where $\mathcal{T}^\alpha:=(\omega(x_3)\partial_3)^{\alpha_4}\partial_t^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}$ with the multi-index $\alpha=(\alpha_0,\alpha_1,\alpha_2,0,\alpha_4)$, $\langle\alpha\rangle=\alpha_0+\alpha_1+\alpha_2+2\times 0+\alpha_4$. Moreover, we have

$$\forall 0\le T\le T_\kappa,\quad \int_0^T\left\|\varepsilon^{2l}\mathcal{T}^\alpha\partial_t^k\mathring{\boldsymbol{b}}^\pm\right\|_{4-k-l,\pm}^2\,\mathrm{d}t\le C(\mathring{K}_0).$$

Remark 4.3. The L_t^2-type bound of $\mathring{\boldsymbol{b}}$ is obtained by using the second part of Lemma B.3 and the $\sqrt{\kappa}$-weighted enhanced regularity for the free interface. Indeed, the modification term $\mathfrak{R}_T^\pm\left(\mathring{b}_1^\pm\overline{\partial}_1\mathring{\psi}+\mathring{b}_2^\pm\overline{\partial}_2\mathring{\psi}-\mathring{b}_3^\pm\right)\big|_\Sigma$ has *vanishing initial value* thank to $b_0^{\kappa,\pm}\cdot N_0^\kappa=0$ on Σ. Thus, one can extend this function to $(-\infty,T]\times\Omega^\pm$ and then apply the trace lemma for anisotropic Sobolev spaces (cf. Trakhinin-Wang [81, Lemma 3.4] or Lemma B.3 in this paper) to show that

$$\int_0^T\left\|\mathcal{T}^\alpha\partial_t^k(\mathring{b}^\pm-\mathring{\boldsymbol{b}}^\pm)\right\|_{4-k-l,\pm}^2\,\mathrm{d}t\le\left\|\mathring{b}^\pm-\mathring{\boldsymbol{b}}^\pm\right\|_{8,*,T,\pm}^2\lesssim\left|\mathring{b}_1^\pm\overline{\partial}_1\mathring{\psi}+\mathring{b}_2^\pm\overline{\partial}_2\mathring{\psi}-\mathring{b}_3^\pm\right|_{7,T}^2$$
$$\lesssim\left\|\mathring{b}_1^\pm\overline{\partial}_1\mathring{\varphi}+\mathring{b}_2^\pm\overline{\partial}_2\mathring{\varphi}-\mathring{b}_3^\pm\right\|_{8,*,T,\pm}^2=\int_0^T\left\|\mathring{b}_1^\pm\overline{\partial}_1\mathring{\varphi}+\mathring{b}_2^\pm\overline{\partial}_2\mathring{\varphi}-\mathring{b}_3^\pm\right\|_{8,*,\pm}^2\,\mathrm{d}t\lesssim T\mathring{K}_0,\quad\forall T\in[0,T_\kappa],$$

where $\|\cdot\|_{m,*,T,\pm},|\cdot|_{m,T}$ norms are defined in Appendix B. Notice that this $\sqrt{\kappa}$-weighted enhanced regularity is necessary here, otherwise we lose the control of $|\overline{\partial}\psi(t)|_8$ and a loss of tangential derivative occurs as in lots of previous works [12, 80, 81, 82] and references therein.

4.2.1 Construction of Galerkin sequence

Since our domain $\Omega:=\mathbb{T}^2\times(-H,H)$ is bounded, there exists an orthonormal basis $\{\boldsymbol{e}_j\}_{j=1}^\infty\subset C^\infty(\Omega)$ for $L^2(\Omega)$ which is also an orthogonal basis of $H_0^1(\Omega)$. To construct the Galerkin sequence, we first write the linearized system (4.6) into a symmetric hyperbolic system of $U^\pm:=(q^\pm,v^\pm,b^\pm,S^\pm)^\top\in\mathbb{R}^8$:

$$A_0(\mathring{U}^\pm)\partial_tU^\pm+A_1(\mathring{U}^\pm)\partial_1U^\pm+A_2(\mathring{U}^\pm)\partial_2U^\pm+A_3(\mathring{U}^\pm)\partial_3U^\pm=0\quad\text{in }\Omega^\pm \tag{4.8}$$

where the coefficient matrices are

$$A_0(\mathring{U}):=\begin{bmatrix}\mathring{\mathcal{F}}_p & \vec{0}^\top & -\mathring{\mathcal{F}}_p\mathring{\boldsymbol{b}}^\top & 0\\ \vec{0} & \mathring{\rho}\boldsymbol{I}_3 & \boldsymbol{O}_3 & \vec{0}\\ -\mathring{\mathcal{F}}_p\mathring{\boldsymbol{b}} & \boldsymbol{O}_3 & \boldsymbol{I}_3+\mathring{\mathcal{F}}_p\mathring{\boldsymbol{b}}\otimes\mathring{\boldsymbol{b}} & \vec{0}\\ 0 & \vec{0}^\top & \vec{0}^\top & 1\end{bmatrix},\ A_i(\mathring{U}):=\begin{bmatrix}\mathring{\mathcal{F}}_p\mathring{\bar{v}}_i & \vec{e_i}^\top & -\mathring{\mathcal{F}}_p\mathring{\bar{v}}_i\mathring{\boldsymbol{b}}^\top & 0\\ \vec{e_i} & \mathring{\rho}\mathring{\bar{v}}_i\boldsymbol{I}_3 & -\mathring{\bar{\boldsymbol{b}}}_i\boldsymbol{I}_3 & \vec{0}\\ -\mathring{\mathcal{F}}_p\mathring{\bar{v}}_i\mathring{\boldsymbol{b}} & -\mathring{\bar{\boldsymbol{b}}}_i\boldsymbol{I}_3 & \mathring{\bar{v}}_i\boldsymbol{I}_3+\mathring{\mathcal{F}}_p\mathring{\bar{v}}_i(\mathring{\boldsymbol{b}}\otimes\mathring{\boldsymbol{b}}) & \vec{0}\\ 0 & \vec{0}^\top & \vec{0}^\top & \mathring{\bar{v}}_i\end{bmatrix}\ (i=1,2),$$

$$A_3(\mathring{U}):=\frac{1}{\partial_3\mathring{\varphi}}\begin{bmatrix}\mathring{\mathcal{F}}_p(\mathring{v}\cdot\dot{\mathbf{N}}-\partial_t\mathring{\varphi}) & \dot{\mathbf{N}}^\top & -\mathring{\mathcal{F}}_p(\mathring{v}\cdot\dot{\mathbf{N}}-\partial_t\mathring{\varphi})\mathring{\boldsymbol{b}}^\top & 0\\ \dot{\mathbf{N}} & \mathring{\rho}(\mathring{v}\cdot\dot{\mathbf{N}}-\partial_t\mathring{\varphi})\boldsymbol{I}_3 & -(\mathring{\boldsymbol{b}}\cdot\dot{\mathbf{N}})\boldsymbol{I}_3 & \vec{0}\\ -\mathring{\mathcal{F}}_p(\mathring{v}\cdot\dot{\mathbf{N}}-\partial_t\mathring{\varphi})\mathring{\boldsymbol{b}} & -(\mathring{\boldsymbol{b}}\cdot\dot{\mathbf{N}})\boldsymbol{I}_3 & (\mathring{v}\cdot\dot{\mathbf{N}}-\partial_t\mathring{\varphi})\boldsymbol{I}_3+\mathring{\mathcal{F}}_p(\mathring{v}\cdot\dot{\mathbf{N}}-\partial_t\mathring{\varphi})(\mathring{\boldsymbol{b}}\otimes\mathring{\boldsymbol{b}}) & \vec{0}\\ 0 & \vec{0}^\top & \vec{0}^\top & \mathring{v}\cdot\dot{\mathbf{N}}-\partial_t\mathring{\varphi}\end{bmatrix}.$$

Also notice that the matrix $A_3(\mathring{U})$ is equal to the following matrix on the boundary

$$A_3(\mathring{U})|_{\Sigma,\Sigma^\pm} := \begin{bmatrix} 0 & \mathring{\mathbf{N}}^\top & \mathbf{0}_4^\top \\ \mathring{\mathbf{N}} & \boldsymbol{O}_3 & \\ \mathbf{0}_4 & & \boldsymbol{O}_4 \end{bmatrix},$$

which has constant rank 2 and has one negative eigenvalue and one positive eigenvalue, so the correct number of boundary conditions to solve $U^\pm$ in (4.8) is $1 + 1 = 2$ (the jump conditions for q and $v \cdot \mathring{\mathbf{N}}$), and we need one more equation, namely $\partial_t \psi = v^\pm \cdot \mathring{N}$, to determine the free interface.

Given $2 \leq m \in \mathbb{N}^*$, we introduce the Galerkin sequence $\{U^{m,\pm}(t,x), \psi^m(t,x')\}$ by

$$U_j^{m,\pm}(t,x) := \sum_{l=1}^m U_{lj}^{m,\pm}(t)\boldsymbol{e}_l(x) \quad 1 \leq j \leq 8, \tag{4.9}$$

$$\psi^m(t,x') := \sum_{l=1}^m \psi_l^m(t)\boldsymbol{e}_l(x',0). \tag{4.10}$$

The Galerkin sequence is assumed to satisfy the boundary conditions

$$\partial_t \psi^m = U_4^{m,\pm} - U_2^{m,\pm}\overline{\partial}_1 \mathring{\psi} - U_3^{m,\pm}\overline{\partial}_2 \mathring{\psi} \tag{4.11}$$

$$\left[\!\left[U_1^m \right]\!\right] = \sigma \mathcal{H}(\mathring{\psi}) - \kappa(1-\overline{\Delta})^2 \psi^m - \kappa(1-\overline{\Delta})\partial_t \psi^m \tag{4.12}$$

Now we introduce an ODE system as the "truncated version" of (4.8) in $\text{Span}\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_m\}$ by testing the Galerkin sequence by a vector field $\boldsymbol{\phi} := (\phi_1, \cdots, \phi_8)^\top$ with

$$\phi_i := \sum_{l=1}^m \phi_{il}(t)\boldsymbol{e}_l(x) \in \text{Span}\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_m\}.$$

$$\int_{\Omega^\pm} A_0^{ij}(\mathring{U}^\pm)(\partial_t U_j^{m,\pm})\phi_i \,\mathrm{d}\mathring{\mathcal{V}}_t + \sum_{k=1}^2 A_k^{ij}(\mathring{U}^\pm)(\overline{\partial}_k U_j^{m,\pm})\phi_i \,\mathrm{d}\mathring{\mathcal{V}}_t + \int_{\Omega^\pm} A_3^{ij}(\mathring{U}^\pm)(\partial_3 U_j^{m,\pm})\phi_i \,\mathrm{d}\mathring{\mathcal{V}}_t = 0 \tag{4.13}$$

where $\mathrm{d}\mathring{\mathcal{V}}_t := \partial_3 \mathring{\varphi}\,\mathrm{d}x$ Integrating by parts in $\overline{\partial}_k$ and ∂_3, we get

$$\int_{\Omega^\pm} A_0^{ij}(\mathring{U}^\pm)(\partial_t U_j^{m,\pm})\phi_i - \sum_{k=1}^3 U_j^{m,\pm}\partial_k(A_k^{ij}(\mathring{U}^\pm)\phi_i)\,\mathrm{d}\mathring{\mathcal{V}}_t \mp \int_\Sigma A_3^{ij}(\mathring{U}^\pm)U_j^{m,\pm}\phi_i \,\mathrm{d}x' = 0. \tag{4.14}$$

Plugging the Galerkin sequence into the above identity, we get

$$\int_{\Omega^\pm} A_0^{ij}(\mathring{U}^\pm)\boldsymbol{e}_l(x)\phi_i(U_{lj}^{m,\pm})'(t) - \sum_{k=1}^3 \partial_k(A_k^{ij}(\mathring{U}^\pm)\phi_i)\boldsymbol{e}_l(x)U_{lj}^{m,\pm}(t)\,\mathrm{d}\mathcal{V}_t = \pm \int_\Sigma A_3^{ij}(\mathring{U}^\pm)U_j^{m,\pm}\phi_i \,\mathrm{d}x'. \tag{4.15}$$

Taking sum for the two parts in $\Omega^\pm$, setting $\phi_i(x) = \boldsymbol{e}_i(x)$ and using the jump condition for $[\![q]\!]$ and $v^\pm \cdot \mathring{\mathbf{N}}$, we obtain a first-order linear ODE system for $\{U_{lj}^\pm(t)\}$

$$\begin{aligned}
&\sum_\pm \left(\int_{\Omega^\pm} A_0^{ij}(\mathring{U}^\pm)\boldsymbol{e}_l(x)\boldsymbol{e}_i(x)\,\mathrm{d}\mathring{\mathcal{V}}_t\right)(U_{lj}^{m,\pm})'(t) - \left(\int_{\Omega^\pm} \partial_k(A_k^{ij}(\mathring{U}^\pm)\boldsymbol{e}_i(x))\boldsymbol{e}_l(x)\,\mathrm{d}\mathring{\mathcal{V}}_t\right)U_{lj}^{m,\pm}(t) \\
=& \int_\Sigma [\![q]\!] \underbrace{\left(-\boldsymbol{e}_2(x',0)\overline{\partial}_1\mathring{\psi}(t,x') - \boldsymbol{e}_3(x',0)\overline{\partial}_2\mathring{\psi}(t,x') + \boldsymbol{e}_4(x',0)\right)}_{=:\phi_v\cdot\mathring{N}}\,\mathrm{d}x' \\
=& -\sigma\int_\Sigma (\overline{\nabla}\mathring{\psi}/|\mathring{N}|)\cdot\overline{\nabla}\left(\phi_v\cdot\mathring{N}\right)\mathrm{d}x' - \kappa\int_\Sigma (1-\overline{\Delta})\psi^m\,(1-\overline{\Delta})\left(\phi_v\cdot\mathring{N}\right)\mathrm{d}x' \\
&- \kappa\int_\Sigma \partial_t\psi^m\left(\phi_v\cdot\mathring{N}\right)\mathrm{d}x' - \kappa\int_\Sigma \overline{\nabla}\partial_t\psi^m\cdot\overline{\nabla}\left(\phi_v\cdot\mathring{N}\right)\mathrm{d}x'.
\end{aligned} \tag{4.16}$$

Since the basis $\{\boldsymbol{e}_l\}$ are smooth and the coefficients $(\mathring{U}^{m,\pm}, \mathring{\psi})$ are sufficiently regular, standard ODE theory guarantees the local existence and uniqueness of the above ODE system (4.16) with initial data

$$U_{lj}^{m,\pm}(0) := \int_{\Omega^\pm} U_j^{m,\pm}(0,x)\boldsymbol{e}_k(x)\partial_3\mathring{\varphi}_0\,\mathrm{d}x.$$

4.2.2 Existence of solutions to the linearized problem

The existence of weak solution is guaranteed by uniform-in-m estimates for the Galerkin sequence $\{U^{m,\pm}(t,x),\psi^m(t,x')\}$. Now we let the test function $\phi = U^{m,\pm}$ in $\Omega^\pm$ respectively to obtain the standard L^2-type energy estimates thanks to the symmetric property of the coefficient matrices and the concrete form of $A_3(\mathring{U})$ on the boundary

$$
\begin{aligned}
&\sum_\pm \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\Omega^\pm} (U^{m,\pm})^\top \cdot A_0(\mathring{U}^\pm)U^{m,\pm}\,\mathrm{d}\mathcal{V}_t \\
=&\sum_\pm \frac{1}{2}\int_{\Omega^\pm} (U^{m,\pm})^\top \cdot \partial_t(A_0(\mathring{U}^\pm))U^{m,\pm}\,\mathrm{d}\mathcal{V}_t + \frac{1}{2}\int_{\Omega^\pm} (U^{m,\pm})^\top \cdot \partial_k(A_k(\mathring{U}^\pm))U^{m,\pm}\,\mathrm{d}\mathcal{V}_t \\
&+\int_\Sigma [\![U_1^m]\!] (U_4^{m,\pm} - U_2^{m,\pm}\overline{\partial}_1\mathring{\psi} - U_3^{m,\pm}\overline{\partial}_2\mathring{\psi})\,\mathrm{d}x'
\end{aligned} \tag{4.17}
$$

where the interior term can be controlled directly by $C(\mathring{K}_0)\|U^{m,\pm}\|^2_{0,\pm}$. For the boundary term, using the boundary conditions (4.11)-(4.12), we get the energy bounds under time integral

$$
\begin{aligned}
&\int_0^t\int_\Sigma [\![U_1^m]\!] (U_4^{m,\pm} - U_2^{m,\pm}\overline{\partial}_1\mathring{\psi} - U_3^{m,\pm}\overline{\partial}_2\mathring{\psi})\,\mathrm{d}x'\,\mathrm{d}\tau \\
=&\int_0^t\int_\Sigma \left(\sigma\mathcal{H}(\mathring{\psi}) - \kappa(1-\overline{\Delta})^2\psi^m - \kappa(1-\overline{\Delta})\partial_t\psi^m\right)\partial_t\psi^m\,\mathrm{d}x'\,\mathrm{d}\tau \\
=&-\sigma\int_0^t\int_\Sigma (\overline{\nabla}\mathring{\psi}/|\mathring{\mathbf{N}}|)\cdot\overline{\nabla}\partial_t\psi^m\,\mathrm{d}x'\,\mathrm{d}\tau - \kappa\int_0^t\int_\Sigma (1-\overline{\Delta})\psi^m\,(1-\overline{\Delta})\partial_t\psi_m\,\mathrm{d}x'\,\mathrm{d}\tau \\
&-\kappa\int_0^t\int_\Sigma \partial_t\psi^m\partial_t\psi_m\,\mathrm{d}x'\,\mathrm{d}\tau - \kappa\int_0^t\int_\Sigma \overline{\nabla}\partial_t\psi^m\cdot\overline{\nabla}\partial_t\psi_m\,\mathrm{d}x'\,\mathrm{d}\tau \\
\lesssim&\frac{\sigma}{\sqrt{\kappa}}\int_0^t P(|\overline{\nabla}\mathring{\psi}|_{L^\infty})|\overline{\nabla}\mathring{\psi}|_0^2\,\mathrm{d}\tau + \delta|\sqrt{\kappa}\psi^m|^2_{L_t^2H_{x'}^1} - \frac{1}{2}\left|\sqrt{\kappa}\psi^m\right|_2^2\Big|_0^t - \int_0^t\left|\sqrt{\kappa}\partial_t\psi^m\right|_1^2\,\mathrm{d}\tau.
\end{aligned} \tag{4.18}
$$

We define

$$
\boldsymbol{N}^m(t) := \sum_\pm \left\|\left(\sqrt{\mathring{\mathcal{F}}_p^\pm}U_1^{m,\pm}, U_2^{m,\pm},\cdots,U_8^{m,\pm}\right)\right\|^2_{0,\pm} + \left|\sqrt{\kappa}\psi^m\right|_2^2 + \int_0^t\left|\sqrt{\kappa}\partial_t\psi^m\right|_1^2\,\mathrm{d}\tau. \tag{4.19}
$$

Since $A_0(\mathring{U}^\pm) > 0$, we obtain the uniform-in-m estimate for the Galerkin sequence $\{U^{m,\pm}(t,x),\psi^m(t,x')\}$.

$$
\boldsymbol{N}^m(t) \le \boldsymbol{N}^m(0) + \int_0^t C(\mathring{K}_0,\kappa^{-1})\boldsymbol{N}^m(\tau)\,\mathrm{d}\tau, \tag{4.20}
$$

and thus there exists a time $T_{\boldsymbol{N}} > 0$ (depending on κ and $\boldsymbol{N}^m(0)$, independent of m) such that

$$
\sup_{0\le t\le T_{\boldsymbol{N}}} \boldsymbol{N}^m(t) \le C'(\mathring{K}_0,\kappa^{-1})\boldsymbol{N}^m(0).
$$

Because $L^\infty([0,T_{\boldsymbol{N}}];L^2(\Omega^\pm))$ is not reflexive, we alternatively consider the weak convergence in $L^2([0,T_{\boldsymbol{N}}];L^2(\Omega^\pm))$. By Eberlein-Šmulian theorem and the uniqueness of expansion in Galerkin basis $\{e_l\}_{l=1}^\infty$, there exists a subsequence $\{U^{m_k,\pm}(t,x),\psi^{m_k}(t,x')\}_{k=1}^\infty$ such that

$$
\left(\sqrt{\mathring{\mathcal{F}}_p^\pm}U_1^{m_k,\pm}, U_2^{m_k,\pm},\cdots,U_8^{m_k,\pm}\right) \rightharpoonup \left(\sqrt{\mathring{\mathcal{F}}_p^\pm}q^\pm, v^\pm,\cdots,b^\pm,S^\pm\right) \text{ in } L^2([0,T_{\boldsymbol{N}}];L^2(\Omega^\pm)) \tag{4.21}
$$

$$
\psi^{m_k} \rightharpoonup \psi \text{ in } L^2([0,T_{\boldsymbol{N}}];H^2(\Sigma)),\quad \partial_t\psi^{m_k} \rightharpoonup \partial_t\psi \text{ in } L^2([0,T_{\boldsymbol{N}}];H^1(\Sigma)). \tag{4.22}
$$

This proves the existence of weak solution to (4.8) (and equivalently (4.6)). The uniqueness easily follows from the estimate of $\boldsymbol{N}(t)$ and the linearity of (4.8). The weak solution is actually a strong solution according to the argument in [60, Chapter 2.2.3].

4.3 Higher-order estimates of the linearized approximate problem

To proceed the Picard iteration, we shall prove that the bounds (4.7) for the coefficients $(\mathring{U}, \mathring{\psi}, \dot{\psi})$ can be preserved by the solution to (4.6). Fix $\kappa > 0$, we define the energy functional for (4.6) to be

$$
\begin{aligned}
&\mathring{E}^\kappa(t) := \mathring{E}_4^\kappa(t) + \cdots + \mathring{E}_8^\kappa(t) \\
&\mathring{E}_{4+l}^\kappa(t) := \sum_{\pm} \sum_{\langle\alpha\rangle=2l} \sum_{k=0}^{4-l} \left\| \left(\varepsilon^{2l} \mathcal{T}^\alpha \partial_t^k (v^\pm, b^\pm, S^\pm, (\mathring{\mathcal{F}}_p^\pm)^{\frac{(k+\alpha_0-l-3)_+}{2}} q^\pm) \right) \right\|_{4-k-l,\pm}^2 \\
&\qquad + \sum_{k=0}^{4+l} \left| \sqrt{\kappa} \varepsilon^{2l} \partial_t^k \psi \right|_{6+l-k}^2 + \int_0^t \left| \sqrt{\kappa} \varepsilon^{2l} \partial_t^{5+l} \psi \right|_1^2 \, \mathrm{d}\tau
\end{aligned}
\tag{4.23}
$$

where $\mathcal{T}^\alpha := (\omega(x_3)\partial_3)^{\alpha_4} \partial_t^{\alpha_0} \partial_1^{\alpha_1} \partial_2^{\alpha_2}$ with the multi-index $\alpha = (\alpha_0, \alpha_1, \alpha_2, 0, \alpha_4)$, $\langle\alpha\rangle = \alpha_0 + \alpha_1 + \alpha_2 + 2\times 0 + \alpha_4$. We aim to prove that

Proposition 4.1. There exists some $T_\kappa > 0$ depending on κ and $\mathring{K}_0$, such that

$$\sup_{0\le t\le T_\kappa} \mathring{E}^\kappa(t) \le C(\kappa^{-1}, \mathring{K}_0) \mathring{E}^\kappa(0).$$

A large part of the proof of proposition (4.1) is similar to the analysis in Section 3. Moreover, since $\kappa > 0$ is fixed, we obtain higher boundary regularity for the free interface ψ, which allows us to avoid some technical steps (such as the analysis in Section 3.4). We will skip some details for the part substantially similar to Section 3 and emphasize the different part. Now we start with div-curl analysis.

4.3.1 Div-Curl analysis

We start with $\mathring{E}_4(t)$. Using (B.1) and the boundary conditions for v, b, we get

$$\left\| v^\pm, b^\pm \right\|_{4,\pm}^2 \lesssim C(|\mathring{\psi}|_4, |\overline{\nabla} \mathring{\psi}|_{W^{1,\infty}}) \left(\|(v^\pm, b^\pm)\|_{0,\pm}^2 + \|\nabla^{\mathring{\varphi}} \cdot (v^\pm, b^\pm)\|_{3,\pm}^2 + \|\nabla^{\mathring{\varphi}} \times (v^\pm, b^\pm)\|_{3,\pm}^2 + \|\overline{\partial}^4 (v^\pm, b^\pm)\|_0^2 \right). \tag{4.24}$$

Remark 4.4. Here we cannot use the div-curl inequality (B.2) to estimate the normal traces because the boundary constraint $b \cdot \mathring{N} = 0$ no longer holds for the linearized problem.

The L^2-estimates are already proven in the uniform estimates of Galerkin sequence, so we no longer repeat it. The treatment of $\nabla^{\mathring{\varphi}} \cdot v$ is also the same as in Section 3.3.1, that is, invoking the continuity equation. For $\nabla^{\mathring{\varphi}} \cdot b$, we no longer have the div-free constraint. Instead, we can take $\nabla^{\mathring{\varphi}} \cdot$ in the linearized evolution equation of b to get

$$D_t^{\mathring{\varphi}\pm} (\nabla^{\mathring{\varphi}} \cdot b^\pm) = \partial_i^{\mathring{\varphi}} (\mathring{\boldsymbol{b}}_j^\pm \partial_j^{\mathring{\varphi}} v_i^\pm) - \partial_i^{\mathring{\varphi}} (\mathring{\boldsymbol{b}}_i^\pm \partial_j^{\mathring{\varphi}} v_j^\pm) + [D_t^{\mathring{\varphi}\pm}, \nabla^{\mathring{\varphi}} \cdot] b^\pm \tag{4.25}$$

$$= (\partial_i^{\mathring{\varphi}} \mathring{\boldsymbol{b}}_j^\pm)(\partial_j^{\mathring{\varphi}} v_i^\pm) - (\nabla^{\mathring{\varphi}} \cdot \mathring{\boldsymbol{b}}^\pm)(\nabla^{\mathring{\varphi}} \cdot v^\pm) + [D_t^{\mathring{\varphi}\pm}, \nabla^{\mathring{\varphi}} \cdot] b^\pm. \tag{4.26}$$

Direct calculation shows that $[D_t^{\mathring{\varphi}\pm}, \partial_i^{\mathring{\varphi}}](\cdot) = -(\partial_i^{\mathring{\varphi}} \mathring{v}_j) \partial_j^{\mathring{\varphi}}(\cdot) - (\partial_i^{\mathring{\varphi}} \partial_t (\mathring{\varphi} - \dot{\varphi})) \partial_3^{\mathring{\varphi}}(\cdot)$. On the other hand, the κ-regularization term provides extra regularity for $\varphi_t, \mathring{\varphi}_t, \dot{\varphi}_t$. Thus, standard H^3 estimates give the control of divergence

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left\| \nabla^{\mathring{\varphi}} \cdot b^\pm \right\|_{3,\pm}^2 \lesssim C(\mathring{K}_0, \kappa^{-1}) \left(\|b^\pm\|_{4,\pm} \|v^\pm\|_{4,\pm} \right) \le C(\mathring{K}_0, \kappa^{-1}) \mathring{E}_4^\kappa(t). \tag{4.27}$$

The vorticity part is analyzed in a similar way as in Section 3.6. The evolution equations are

$$
\begin{aligned}
\mathring{\rho} D_t^{\mathring{\varphi}} (\nabla^{\mathring{\varphi}} \times v) - (\mathring{\boldsymbol{b}} \cdot \nabla^{\mathring{\varphi}})(\nabla^{\mathring{\varphi}} \times b) &= (\nabla^{\mathring{\varphi}} \mathring{\rho}) \times (D_t^{\mathring{\varphi}} v) - (\nabla^{\mathring{\varphi}} \mathring{\boldsymbol{b}}_j) \times (\partial_j^{\mathring{\varphi}} b) \\
&\quad - \mathring{\rho} \left((\nabla^{\mathring{\varphi}} \mathring{v}_j) \times (\partial_j^{\mathring{\varphi}} v) + \nabla^{\varphi} (\partial_t \mathring{\varphi} - \partial_t \dot{\varphi}) \times \partial_3^{\mathring{\varphi}} v \right), \\
D_t^{\mathring{\varphi}} (\nabla^{\mathring{\varphi}} \times b) - (\mathring{\boldsymbol{b}} \cdot \nabla^{\mathring{\varphi}})(\nabla^{\mathring{\varphi}} \times v) - \mathring{\boldsymbol{b}} \times \nabla^{\mathring{\varphi}} (\nabla^{\mathring{\varphi}} \cdot v) &= -(\nabla^{\mathring{\varphi}} \times \mathring{\boldsymbol{b}})(\nabla^{\mathring{\varphi}} \cdot v) - (\nabla^{\mathring{\varphi}} \mathring{\boldsymbol{b}}_j) \times (\partial_j^{\mathring{\varphi}} v) \\
&\quad - (\nabla^{\mathring{\varphi}} \mathring{v}_j) \times (\partial_j^{\mathring{\varphi}} b) - \nabla^{\varphi} (\partial_t \mathring{\varphi} - \partial_t \dot{\varphi}) \times \partial_3^{\mathring{\varphi}} b,
\end{aligned}
$$

on the right side of which the highest-order derivative is 1 (except the mismatch term). Thus, we can still follow the analysis in Section 3.6.2 to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\Omega^\pm}\mathring{\rho}^\pm\left|\partial^3\nabla^{\mathring{\varphi}}\times v^\pm\right|^2+\left|\partial^3\nabla^{\mathring{\varphi}}\times b^\pm\right|^2\,\mathrm{d}\mathring{\mathcal{V}}_t\leq P(\mathring{E}_4^\kappa(t),\mathring{K}_0)+\mathring{K}_1^\pm, \tag{4.28}$$

where

$$\mathring{K}_1^\pm:=\int_{\Omega^\pm}(\partial^3\nabla^{\mathring{\varphi}}\times b^\pm)\cdot\left(\mathring{\boldsymbol{b}}^\pm\times(\partial^3\nabla^{\mathring{\varphi}}(\nabla^{\mathring{\varphi}}\cdot v^\pm))\right)\mathrm{d}\mathring{\mathcal{V}}_t. \tag{4.29}$$

Again, we invoke the continuity equation and the momentum equation to get

$$\begin{aligned}\mathring{\boldsymbol{b}}\times(\partial^3\nabla^{\mathring{\varphi}}(\nabla^{\mathring{\varphi}}\cdot v))&\overset{L}{=}-\mathring{\mathcal{F}}_p\mathring{\boldsymbol{b}}\times(\partial^3\nabla^{\mathring{\varphi}}D_t^{\mathring{\varphi}}q)+\mathring{\mathcal{F}}_p\mathring{\boldsymbol{b}}\times((\partial^3\nabla^{\mathring{\varphi}}D_t^{\mathring{\varphi}}b_j)\mathring{\boldsymbol{b}}_j)\\&\overset{L}{=}\mathring{\mathcal{F}}_p\mathring{\rho}\mathring{\boldsymbol{b}}\times(\partial^3(D_t^{\mathring{\varphi}})^2v)-\mathring{\mathcal{F}}_p\mathring{\boldsymbol{b}}\times D_t^{\mathring{\varphi}}(\mathring{b}_j\partial^3\partial_j^{\mathring{\varphi}}b)+\mathring{\mathcal{F}}_p\mathring{\boldsymbol{b}}\times D_t^{\mathring{\varphi}}((\partial^3\nabla^{\mathring{\varphi}}b_j)\mathring{\boldsymbol{b}}_j)\\&\overset{L}{=}\mathring{\mathcal{F}}_p\mathring{\rho}\mathring{\boldsymbol{b}}\times(\partial^3(D_t^{\mathring{\varphi}})^2v)+\mathring{\mathcal{F}}_p\mathring{\boldsymbol{b}}\times D_t^{\mathring{\varphi}}(\mathring{\boldsymbol{b}}\times(\partial^3\nabla^{\mathring{\varphi}}\times b))\end{aligned}$$

where we use the vector identity $(\mathbf{a}\times(\nabla^{\mathring{\varphi}}\times\mathbf{b}))_i=(\partial_i^{\mathring{\varphi}}\mathbf{b}_j)\mathbf{a}_j-\mathbf{a}_j\partial_j^{\mathring{\varphi}}\mathbf{b}_i$, and the omitted terms are directly controlled by $P(\mathring{E}_4^\kappa(t),\mathring{K}_0)$. Thus, we have

$$\begin{aligned}\mathring{K}_1^\pm&\overset{L}{=}\int_{\Omega^\pm}\mathring{\mathcal{F}}_p^\pm\mathring{\rho}(\partial^3\nabla^{\mathring{\varphi}}\times b)\cdot\left(\mathring{\boldsymbol{b}}\times(\partial^3(D_t^{\mathring{\varphi}})^2v)\right)\mathrm{d}\mathring{\mathcal{V}}_t+\int_{\Omega^\pm}\mathring{\mathcal{F}}_p^\pm(\partial^3\nabla^{\mathring{\varphi}}\times b)\cdot\left(\mathring{\boldsymbol{b}}\times D_t^{\mathring{\varphi}}(\mathring{\boldsymbol{b}}\times(\partial^3\nabla^{\mathring{\varphi}}b))\right)\mathrm{d}\mathcal{V}_t\\&=\int_{\Omega^\pm}\mathring{\mathcal{F}}_p^\pm\mathring{\rho}(\partial^3\nabla^{\mathring{\varphi}}\times b)\cdot\left(\mathring{\boldsymbol{b}}\times(\partial^3(D_t^{\mathring{\varphi}})^2v)\right)\mathrm{d}\mathring{\mathcal{V}}_t-\int_{\Omega^\pm}\mathring{\mathcal{F}}_p^\pm D_t^{\mathring{\varphi}}\left(\mathring{\boldsymbol{b}}\times(\partial^3\nabla^{\mathring{\varphi}}\times b)\right)\cdot\left(\mathring{\boldsymbol{b}}\times(\partial^3\nabla^{\mathring{\varphi}}\times b)\right)\mathrm{d}\mathcal{V}_t\\&\lesssim-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}\mathring{\mathcal{F}}_p^\pm\left|\mathring{\boldsymbol{b}}\times(\partial^3\nabla^{\mathring{\varphi}}\times b)\right|_0^2+P(\mathring{K}_0)\mathring{E}_4^\kappa(t)+\mathring{E}_5^\kappa(t).\end{aligned}$$

So, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}\mathring{\rho}^\pm\left|\partial^3\nabla^{\mathring{\varphi}}\times v^\pm\right|^2+\left|\partial^3\nabla^{\mathring{\varphi}}\times b^\pm\right|^2+\mathring{\mathcal{F}}_p^\pm\left|\mathring{\boldsymbol{b}}^\pm\times(\partial^3\nabla^{\mathring{\varphi}}\times b^\pm)\right|_0^2\,\mathrm{d}\mathring{\mathcal{V}}_t\\&\lesssim P(\mathring{K}_0)\mathring{E}_4^\kappa(t)+\mathring{E}_5^\kappa(t).\end{aligned} \tag{4.30}$$

Similarly as in Section 3.6.2 and Section 3.6.3, we can prove the div-curl estimates for time-differentiated system and $\mathcal{T}^\alpha$-differentiated system. For $0\leq l\leq 3, 0\leq k\leq 3-l, \langle\alpha\rangle=2l,\ \alpha_3=0$, we have

$$\begin{aligned}\left\|\varepsilon^{2l}\partial_t^k\mathcal{T}^\alpha(v^\pm,b^\pm)\right\|_{4-l-k,\pm}^2\leq C(|\mathring{\psi}|_3)\Big(&\left\|\varepsilon^{2l}\partial_t^k\mathcal{T}^\alpha(v^\pm,b^\pm)\right\|_{0,\pm}^2+\left\|\varepsilon^{2l}\nabla^{\mathring{\varphi}}\cdot\partial_t^k\mathcal{T}^\alpha(v^\pm,b^\pm)\right\|_{3-k-l,\pm}^2\\&+\left\|\varepsilon^{2l}\nabla^{\mathring{\varphi}}\times\partial_t^k\mathcal{T}^\alpha(v^\pm,b^\pm)\right\|_{3-k-l,\pm}^2+\left\|\varepsilon^{2l}\overline{\partial}^{4-k-l}\partial_t^k\mathcal{T}^\alpha(v^\pm,b^\pm)\right\|_{0,\pm}^2\Big).\end{aligned} \tag{4.31}$$

Then the curl part has the following control

$$\begin{aligned}&\left\|\varepsilon^{2l}\nabla^{\mathring{\varphi}}\times\partial_t^k\mathcal{T}^\alpha v^\pm\right\|_{3-l-k,\pm}^2+\left\|\varepsilon^{2l}\nabla^{\mathring{\varphi}}\times\partial_t^k\mathcal{T}^\alpha b^\pm\right\|_{3-l-k,\pm}^2+\left\|\varepsilon^{2l}\mathring{\mathcal{F}}_p^\pm\mathring{b}\times(\partial_t^k\mathcal{T}^\alpha b^\pm)\right\|_{3-l-k,\pm}^2\\&\lesssim P(\mathring{K}_0)\left(\sum_{j=0}^{l}E_{4+j}^\kappa(0)\right)+P(\mathring{K}_0)\int_0^t\sum_{j=0}^{l}\mathring{E}_{4+j}^\kappa(\tau)+\mathring{E}_{4+l+1}^\kappa(\tau)\,\mathrm{d}\tau.\end{aligned} \tag{4.32}$$

Similarly, the divergence part is controlled by

$$\begin{aligned}&\left\|\varepsilon^{2l}\nabla^{\mathring{\varphi}}\cdot\partial_t^k\mathcal{T}^\alpha v^\pm\right\|_{3-l-k,\pm}^2+\left\|\varepsilon^{2l}\nabla^{\mathring{\varphi}}\cdot\partial_t^k\mathcal{T}^\alpha b^\pm\right\|_{3-l-k,\pm}^2\\&\lesssim\left\|\varepsilon^{2l}\mathring{\mathcal{F}}_p^\pm\partial_t^k\mathcal{T}^\alpha D_t^{\mathring{\varphi}\pm}(q^\pm,b^\pm)\right\|_{3-l-k,\pm}^2+C(\mathring{K}_0)\left(\sum_{j=0}^{l}E_{4+j}^\kappa(0)\right)+P(\mathring{K}_0)\int_0^t\sum_{j=0}^{l}\mathring{E}_{4+j}^\kappa(\tau)\,\mathrm{d}\tau,\end{aligned} \tag{4.33}$$

in which the first term will be controlled via tangential estimates.

For the pressure q, we still use the linearized momentum equation to convert it to tangential derivatives of v and b. This step is exactly the same as Section 3.6.1, so we do not repeat the details here.

4.3.2 Tangential estimates

The tangential estimates are also proved in the same way as Section 3.3-Section 3.5. What's more, the rather technical part in the estimates of full time derivatives can be simplified a lot thanks to the $\sqrt{\kappa}$-weighted extra regularity of the free interface. For $\mathcal{T}^\gamma$-differentiated linearize system (4.6), we introduce the corresponding Alinhac good unknown $\mathring{\mathbf{F}}^\gamma := \mathcal{T}^\gamma f - \mathcal{T}^\gamma \mathring{\varphi}\partial_3^{\mathring{\varphi}} f$ which satisfies

$$\mathcal{T}^\gamma(\partial_i^{\mathring{\varphi}} f) = \partial_i^{\mathring{\varphi}} \mathring{\mathbf{F}}^\gamma + \mathring{\mathfrak{C}}_i^\gamma(f), \quad \mathcal{T}^\gamma(D_t^{\mathring{\varphi}} f) = D_t^{\mathring{\varphi}} \mathring{\mathbf{F}}^\gamma + \mathring{\mathfrak{D}}^\gamma(f),$$

where

$$\begin{aligned}\mathring{\mathfrak{C}}_i^\gamma(f) = {} & (\partial_3^{\mathring{\varphi}}\partial_i^{\mathring{\varphi}} f)\mathcal{T}^\gamma\mathring{\varphi} + \left[\mathcal{T}^\gamma, \frac{\mathring{\mathbf{N}}_i}{\partial_3\mathring{\varphi}}, \partial_3 f\right] + \partial_3 f\left[\mathcal{T}^\gamma, \mathring{\mathbf{N}}_i, \frac{1}{\partial_3\mathring{\varphi}}\right] + \mathring{\mathbf{N}}_i\partial_3 f\left[\mathcal{T}^{\gamma-\gamma'}, \frac{1}{(\partial_3\mathring{\varphi})^2}\right]\mathcal{T}^{\gamma'}\partial_3\mathring{\varphi} \\ & + \frac{\mathring{\mathbf{N}}_i}{\partial_3\mathring{\varphi}}[\mathcal{T}^\gamma, \partial_3] f - \frac{\mathring{\mathbf{N}}_i}{(\partial_3\mathring{\varphi})^2}\partial_3 f[\mathcal{T}^\gamma, \partial_3]\mathring{\varphi}, \end{aligned} \tag{4.34}$$

and

$$\begin{aligned}\mathring{\mathfrak{D}}^\gamma(f) = {} & (D_t^{\mathring{\varphi}}\partial_3^{\mathring{\varphi}} f)\mathcal{T}^\gamma\varphi + [\mathcal{T}^\gamma, \mathring{\bar{v}}]\cdot\overline{\partial} f + \left[\mathcal{T}^\gamma, \frac{1}{\partial_3\mathring{\varphi}}(\mathring{v}\cdot\dot{\mathbf{N}} - \partial_t\mathring{\varphi}), \partial_3 f\right] + \left[\mathcal{T}^\gamma, (\mathring{v}\cdot\dot{\mathbf{N}} - \partial_t\mathring{\varphi}), \frac{1}{\partial_3\mathring{\varphi}}\right]\partial_3 f \\ & + \frac{1}{\partial_3\varphi}[\mathcal{T}^\gamma, \mathring{v}]\cdot\dot{\mathbf{N}}\partial_3 f - (\mathring{v}\cdot\dot{\mathbf{N}} - \partial_t\mathring{\varphi})\partial_3 f\left[\mathcal{T}^{\gamma-\gamma'}, \frac{1}{(\partial_3\mathring{\varphi})^2}\right]\mathcal{T}^{\gamma'}\partial_3\mathring{\varphi} \\ & + \frac{1}{\partial_3\mathring{\varphi}}(\mathring{v}\cdot\dot{\mathbf{N}} - \partial_t\mathring{\varphi})[\mathcal{T}^\gamma, \partial_3] f + (\mathring{v}\cdot\dot{\mathbf{N}} - \partial_t\mathring{\varphi})\frac{\partial_3 f}{(\partial_3\mathring{\varphi})^2}[\mathcal{T}^\gamma, \partial_3]\mathring{\varphi} + \mathcal{T}^\gamma\partial_t(\varphi - \mathring{\varphi})\partial_3^{\mathring{\varphi}} f \end{aligned} \tag{4.35}$$

with $\langle\gamma'\rangle = 1$. Since $\mathring{\mathbf{N}}_3 = 1$, the third term in $\mathring{\mathfrak{C}}_i^\gamma(f)$ does not appear when $i = 3$. Under this setting, the $\mathcal{T}^\gamma$-differentiated linearized system is reformulated as follows

$$\mathring{\rho}^\pm D_t^{\mathring{\varphi}\pm}\mathring{\mathbf{V}}^{\gamma,\pm} - (\mathring{\boldsymbol{b}}^\pm\cdot\nabla^{\mathring{\varphi}})\mathring{\mathbf{B}}^{\gamma,\pm} + \nabla^{\mathring{\varphi}}\mathring{\mathbf{Q}}^{\gamma,\pm} = \mathring{\mathcal{R}}_v^{\gamma,\pm} - \mathring{\mathfrak{C}}^\gamma(q^\pm) \quad \text{in } [0,T]\times\Omega^\pm, \tag{4.36}$$

$$\mathring{\mathcal{F}}_p^\pm D_t^{\mathring{\varphi}\pm}\mathring{\mathbf{Q}}^{\gamma,\pm} - \mathring{\mathcal{F}}_p^\pm D_t^{\mathring{\varphi}\pm}\mathbf{B}^{\gamma,\pm}\cdot\mathring{\boldsymbol{b}}^\pm + \nabla^{\mathring{\varphi}}\cdot\mathring{\mathbf{V}}^{\gamma,\pm} = \mathring{\mathcal{R}}_p^{\gamma,\pm} - \mathring{\mathfrak{C}}_i^\gamma(v_i^\pm) \quad \text{in } [0,T]\times\Omega^\pm, \tag{4.37}$$

$$D_t^{\mathring{\varphi}\pm}\mathring{\mathbf{B}}^{\gamma,\pm} - (\mathring{\boldsymbol{b}}^\pm\cdot\nabla^{\mathring{\varphi}})\mathring{\mathbf{V}}^{\gamma,\pm} + \mathring{b}^\pm(\nabla^{\mathring{\varphi}}\cdot\mathring{\mathbf{V}}^{\gamma,\pm}) = \mathring{\mathcal{R}}_b^{\gamma,\pm} - \mathring{b}^\pm\mathring{\mathfrak{C}}_i^\gamma(v_i^\pm) \quad \text{in } [0,T]\times\Omega^\pm, \tag{4.38}$$

$$D_t^{\mathring{\varphi}\pm}\mathring{\mathbf{S}}^{\pm,\alpha} = \mathring{\mathfrak{D}}^\gamma(S^\pm) \quad \text{in } [0,T]\times\Omega^\pm, \tag{4.39}$$

with boundary conditions

$$\left[\!\left[\mathring{\mathbf{Q}}^\gamma\right]\!\right] = \sigma\mathcal{T}^\gamma\mathcal{H}(\mathring{\psi}) - \kappa\mathcal{T}^\gamma(1-\overline{\Delta})^2\psi - \kappa\mathcal{T}^\gamma(1-\overline{\Delta})\partial_t\psi - [\![\partial_3 q]\!]\mathcal{T}^\gamma\mathring{\psi} \quad \text{on } [0,T]\times\Sigma, \tag{4.40}$$

$$\mathring{\mathbf{V}}^{\gamma,\pm}\cdot\mathring{N} = \partial_t\mathcal{T}^\gamma\psi + \bar{v}^\pm\cdot\overline{\nabla}\mathcal{T}^\gamma\mathring{\psi} - \mathcal{W}^{\gamma,\pm} \quad \text{on } [0,T]\times\Sigma, \tag{4.41}$$

where $\mathring{\mathcal{R}}_v, \mathring{\mathcal{R}}_p, \mathring{\mathcal{R}}_b$ terms consist of the following commutators

$$\mathring{\mathcal{R}}_v^{\gamma,\pm} := [\mathcal{T}^\gamma, \mathring{\boldsymbol{b}}^\pm]\cdot\nabla^{\mathring{\varphi}} b^\pm - [\mathcal{T}^\gamma, \mathring{\rho}^\pm] D_t^{\mathring{\varphi}\pm} v^\pm - \mathring{\rho}^\pm\mathring{\mathfrak{D}}^\gamma(v^\pm) \tag{4.42}$$

$$\begin{aligned}\mathring{\mathcal{R}}_p^{\gamma,\pm} := {} & -[\mathcal{T}^\gamma, \mathring{\mathcal{F}}_p^\pm] D_t^{\mathring{\varphi}\pm} q^\pm - \mathring{\mathcal{F}}_p^\pm\mathring{\mathfrak{D}}^\gamma(q^\pm) \\ & + [\mathcal{T}^\gamma, \mathring{\mathcal{F}}_p^\pm] D_t^{\mathring{\varphi}\pm} b^\pm\cdot\mathring{\boldsymbol{b}}^\pm + \mathring{\mathcal{F}}_p^\pm\mathring{\mathfrak{D}}^\gamma(b^\pm)\cdot\mathring{\boldsymbol{b}}^\pm + [\mathcal{T}^\gamma, \mathring{\mathcal{F}}_p^\pm\mathring{\boldsymbol{b}}^\pm]\cdot D_t^{\mathring{\varphi}\pm} b^\pm \end{aligned} \tag{4.43}$$

$$\mathring{\mathcal{R}}_b^{\gamma,\pm} := [\mathcal{T}^\gamma, \mathring{\boldsymbol{b}}^\pm]\cdot\nabla^{\mathring{\varphi}} v^\pm - \mathring{\mathfrak{D}}^\gamma(b^\pm), \tag{4.44}$$

and the boundary term $\mathcal{W}^{\gamma,\pm}$ is

$$\mathring{\mathcal{W}}^{\gamma,\pm} := (\partial_3 v^\pm\cdot\mathring{N})\mathcal{T}^\gamma\mathring{\psi} + [\mathcal{T}^\gamma, \mathring{N}_i, v_i^\pm]. \tag{4.45}$$

Given $0 \le l \le 4$, we shall consider the tangential estimates for $\overline{\partial}^{4-k-l}\partial_t^k\mathcal{T}^\alpha$ for $0 \le k \le 4-l$ and $\langle\alpha\rangle = 2l$, $\alpha_3 = 0$. Following the analysis in Section 3.3-Section 3.5, using the linearized Reynolds transport theorem (Lemma A.7), dropping γ for simplicity of notations, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\Omega^\pm}\mathring{\rho}|\varepsilon^{2l}\mathring{\mathbf{V}}^\pm|^2\,\mathrm{d}\mathring{\mathcal{V}}_t =& \int_{\Omega^\pm}\varepsilon^{4l}(\mathring{b}^\pm\cdot\nabla^{\mathring{\varphi}})\mathring{\mathbf{B}}^\pm\cdot\mathring{\mathbf{V}}^\pm\,\mathrm{d}\mathring{\mathcal{V}}_t - \int_{\Omega^\pm}\varepsilon^{4l}\mathring{\mathbf{V}}^\pm\cdot\nabla^{\mathring{\varphi}}\mathring{\mathbf{Q}}^\pm\,\mathrm{d}\mathring{\mathcal{V}}_t\\ &-\int_{\Omega^\pm}\varepsilon^{4l}(\mathring{\mathcal{R}}_v^\pm - \mathring{\mathfrak{C}}(q^\pm))\cdot\mathring{\mathbf{V}}^\pm\,\mathrm{d}\mathring{\mathcal{V}}_t\\ &+\frac{1}{2}\int_{\Omega^\pm}\varepsilon^{4l}\left(D_t^{\mathring{\varphi}\pm}\mathring{\rho}+\mathring{\rho}\nabla^{\mathring{\varphi}}\cdot\mathring{v}^\pm+\mathring{\rho}\partial_3^{\mathring{\varphi}}(\mathring{\bar{v}}\cdot\overline{\nabla})(\mathring{\varphi}-\dot{\varphi})\right)|\mathring{\mathbf{V}}^\pm|^2\,\mathrm{d}\mathring{\mathcal{V}}_t,\end{aligned}\tag{4.46}$$

where the last two terms can be directly controlled by $C(\mathring{K}_0)\mathring{E}^\kappa(t)$. In the rest of this section, we will use the notation $\overset{L}{=}$ to skip some of these remainder terms. We then analyze the first line. Integrating $(\mathring{b}^\pm\cdot\nabla^{\mathring{\varphi}})$ and $\nabla^{\mathring{\varphi}}$ by parts and using $\mathring{b}\cdot\mathring{N}|_\Sigma = 0$, we get

$$\begin{aligned}&\int_{\Omega^\pm}\varepsilon^{4l}(\mathring{b}^\pm\cdot\nabla^{\mathring{\varphi}})\mathring{\mathbf{B}}^\pm\cdot\mathring{\mathbf{V}}^\pm\,\mathrm{d}\mathring{\mathcal{V}}_t = -\int_{\Omega^\pm}\varepsilon^{4l}\mathring{\mathbf{B}}^\pm\cdot(\mathring{b}^\pm\cdot\nabla^{\mathring{\varphi}})\mathring{\mathbf{V}}^\pm\,\mathrm{d}\mathring{\mathcal{V}}_t - \int_{\Omega^\pm}\varepsilon^{4l}(\nabla^{\mathring{\varphi}}\cdot\mathring{b}^\pm)\mathring{\mathbf{B}}^\pm\cdot\mathring{\mathbf{V}}^\pm\,\mathrm{d}\mathring{\mathcal{V}}_t\\ =&-\int_{\Omega^\pm}\varepsilon^{4l}\mathring{\mathbf{B}}^\pm\cdot D_t^{\mathring{\varphi}\pm}\mathring{\mathbf{B}}^\pm\,\mathrm{d}\mathring{\mathcal{V}}_t - \int_{\Omega^\pm}\varepsilon^{4l}(\mathring{\mathbf{B}}^\pm\cdot\mathring{b}^\pm)(\nabla^{\mathring{\varphi}}\cdot\mathring{\mathbf{V}}^\pm)\,\mathrm{d}\mathring{\mathcal{V}}_t - \int_{\Omega^\pm}\varepsilon^{4l}(\nabla^{\mathring{\varphi}}\cdot\mathring{b}^\pm)\mathring{\mathbf{B}}^\pm\cdot\mathring{\mathbf{V}}^\pm\,\mathrm{d}\mathring{\mathcal{V}}_t\\ \overset{L}{=}&-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}|\varepsilon^{2l}\mathring{\mathbf{B}}^\pm|^2\,\mathrm{d}\mathring{\mathcal{V}}_t - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}\mathring{\mathcal{F}}_p^\pm(\varepsilon^{2l}\mathring{\mathbf{B}}^\pm\cdot\mathring{b}^\pm)^2\,\mathrm{d}\mathcal{V}_t + \int_{\Omega^\pm}\varepsilon^{4l}\mathring{\mathcal{F}}_p^\pm(\mathring{\mathbf{B}}^\pm\cdot\mathring{b}^\pm)D_t^{\mathring{\varphi}\pm}\mathring{\mathbf{Q}}^\pm\,\mathrm{d}\mathring{\mathcal{V}}_t\end{aligned}\tag{4.47}$$

and

$$\begin{aligned}&-\int_{\Omega^\pm}\varepsilon^{4l}\mathring{\mathbf{V}}^\pm\cdot\nabla^{\mathring{\varphi}}\mathring{\mathbf{Q}}^\pm\,\mathrm{d}\mathring{\mathcal{V}}_t = \underbrace{\pm\int_\Sigma\varepsilon^{4l}(\mathring{\mathbf{V}}^\pm\cdot\mathring{N})\mathring{\mathbf{Q}}^\pm\,\mathrm{d}x'}_{=:\mathring{I}^\pm} + \int_{\Omega^\pm}\varepsilon^{4l}\mathring{\mathbf{Q}}^\pm(\nabla^{\mathring{\varphi}}\cdot\mathring{\mathbf{V}}^\pm)\,\mathrm{d}\mathring{\mathcal{V}}_t\\ \overset{L}{=}&\,\mathring{I}^\pm - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}\mathring{\mathcal{F}}_p^\pm(\varepsilon^{2l}\mathring{\mathbf{Q}}^\pm)^2\,\mathrm{d}\mathring{\mathcal{V}}_t + \int_{\Omega^\pm}\mathring{\mathcal{F}}_p^\pm\varepsilon^{4l}\mathring{\mathbf{Q}}^\pm D_t^{\mathring{\varphi}\pm}(\mathring{\mathbf{B}}^\pm\cdot\mathring{b}^\pm)\,\mathrm{d}\mathring{\mathcal{V}}_t \underbrace{-\int_{\Omega^\pm}\varepsilon^{4l}\mathring{\mathbf{Q}}^\pm\mathring{\mathfrak{C}}_i(v_i^\pm)\,\mathrm{d}\mathring{\mathcal{V}}_t}_{=:\mathring{Z}^\pm}.\end{aligned}\tag{4.48}$$

Notice that

$$\int_{\Omega^\pm}\varepsilon^{4l}\mathring{\mathcal{F}}_p^\pm(\mathring{\mathbf{B}}^\pm\cdot\mathring{b}^\pm)D_t^{\mathring{\varphi}\pm}\mathring{\mathbf{Q}}^\pm\,\mathrm{d}\mathring{\mathcal{V}}_t + \int_{\Omega^\pm}\varepsilon^{4l}\mathring{\mathcal{F}}_p^\pm\mathring{\mathbf{Q}}^\pm D_t^{\mathring{\varphi}\pm}(\mathring{\mathbf{B}}^\pm\cdot\mathring{b}^\pm)\,\mathrm{d}\mathring{\mathcal{V}}_t \overset{L}{=} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}\varepsilon^{4l}\mathring{\mathcal{F}}_p^\pm\mathring{\mathbf{Q}}^\pm(\mathring{\mathbf{B}}^\pm\cdot\mathring{b}^\pm)\,\mathrm{d}\mathring{\mathcal{V}}_t,\tag{4.49}$$

we find that

$$\begin{aligned}&\int_{\Omega^\pm}\varepsilon^{4l}(\mathring{b}^\pm\cdot\nabla^{\mathring{\varphi}})\mathring{\mathbf{B}}^\pm\cdot\mathring{\mathbf{V}}^\pm\,\mathrm{d}\mathring{\mathcal{V}}_t - \int_{\Omega^\pm}\varepsilon^{4l}\mathring{\mathbf{V}}^\pm\cdot\nabla^{\mathring{\varphi}}\mathring{\mathbf{Q}}^\pm\,\mathrm{d}\mathring{\mathcal{V}}_t\\ \overset{L}{=}&\,\mathring{I}^\pm + \mathring{Z}^\pm - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}|\varepsilon^{2l}\mathring{\mathbf{B}}^\pm|^2\,\mathrm{d}\mathring{\mathcal{V}}_t - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}\mathring{\mathcal{F}}_p^\pm\left|\varepsilon^{2l}\left(\mathring{\mathbf{Q}}^\pm - \mathring{\mathbf{B}}^\pm\cdot\mathring{b}^\pm\right)\right|^2\,\mathrm{d}\mathring{\mathcal{V}}_t.\end{aligned}\tag{4.50}$$

Thus, we already get the energy terms for $\mathring{\mathbf{V}}$, $\mathring{\mathbf{B}}$ and $\mathring{\mathbf{Q}}$, and it remains to analyze the boundary term $\mathring{I}^\pm$. Again, following the analysis in Section 3.3-Section 3.5, we have

$$\mathring{I}^+ + \mathring{I}^- = \mathring{\mathrm{S}}\mathrm{T} + \mathring{\mathrm{S}}\mathrm{T}' + \mathring{\mathrm{V}}\mathrm{S} + \mathring{\mathrm{R}}\mathrm{T} + \mathring{\mathrm{R}}\mathrm{T}^+ + \mathring{\mathrm{R}}\mathrm{T}^- + \mathring{Z}B^+ + \mathring{Z}B^-\tag{4.51}$$

where

$$\mathring{\mathrm{S}}\mathrm{T} := \varepsilon^{4l}\int_\Sigma\mathcal{T}^\gamma\,[\![q]\!]\,\partial_t\mathcal{T}^\gamma\psi\,\mathrm{d}x',\tag{4.52}$$

$$\mathring{\mathrm{S}}\mathrm{T}' := \varepsilon^{4l}\int_\Sigma\mathcal{T}^\gamma\,[\![q]\!]\,(\bar{v}^+\cdot\overline{\nabla})\mathcal{T}^\gamma\mathring{\psi}\,\mathrm{d}x',\tag{4.53}$$

$$\mathring{\mathrm{V}}\mathrm{S} := \varepsilon^{4l}\int_\Sigma\mathcal{T}^\gamma q^-\,([\![\bar{v}]\!]\cdot\overline{\nabla})\mathcal{T}^\gamma\mathring{\psi}\,\mathrm{d}x',\tag{4.54}$$

$$\mathring{\mathrm{R}}\mathrm{T} := -\varepsilon^{4l}\int_\Sigma[\![\partial_3 q]\!]\,\mathcal{T}^\gamma\mathring{\psi}\,\partial_t\mathcal{T}^\gamma\psi\,\mathrm{d}x',\tag{4.55}$$

$$\mathring{\mathrm{R}}\mathrm{T}^\pm := \mp\varepsilon^{4l}\int_\Sigma\partial_3 q^\pm\,\mathcal{T}^\gamma\mathring{\psi}\,(\bar{v}^\pm\cdot\overline{\nabla})\mathcal{T}^\gamma\mathring{\psi}\,\mathrm{d}x',\tag{4.56}$$

$$\mathring{Z}B^\pm := \mp\varepsilon^{4l}\int_\Sigma\mathring{\mathbf{Q}}^\pm\mathring{\mathcal{W}}^\pm\,\mathrm{d}x',\quad \mathring{Z}^\pm = -\int_{\Omega^\pm}\varepsilon^{4l}\mathring{\mathbf{Q}}^\pm\mathring{\mathfrak{C}}_i(v_i^\pm)\,\mathrm{d}\mathring{\mathcal{V}}_t.\tag{4.57}$$

4.3.3 Analysis of the boundary integrals

Since the weight function $\omega(x_3)$ vanishes on Σ, we can alternatively write $\mathcal{T}^\alpha = \partial_t^{\alpha_0}\overline{\partial}^{2l-\alpha_0}$ and $\mathcal{T}^\gamma = \partial_t^{k+\alpha_0}\overline{\partial}^{4+l-(k+\alpha_0)}$. Replacing $k+\alpha_0$ by k, it suffices to analyze the case $\mathcal{T}^\gamma = \partial_t^k\overline{\partial}^{4+l-k}$ for $0 \le k \le 4+l,\ 0 \le l \le 4$. First, there is no need to analyze $\mathring{\text{RT}}$ and $\mathring{\text{RT}}^\pm$ because they can be directly controlled by using the energy bounds (4.7) for the basic state. For the term $\mathring{\text{ST}}$, the boundary regularity is given by the κ-regularization terms instead of the surface tension because we do not need a uniform-in-κ estimate for the linearized problem. Using the jump conditions for $[\![q]\!]$ and integrating by parts, we have

$$\begin{aligned}\int_0^t \mathring{\text{ST}}\,\mathrm{d}\tau =& \sigma\int_0^t\int_\Sigma \varepsilon^{4l}\partial_t^k\overline{\partial}^{4+l-k}\left(\frac{\overline{\nabla}\mathring{\psi}}{\sqrt{1+|\overline{\nabla}\mathring{\psi}|^2}}\right)\cdot\overline{\nabla}\partial_t^{k+1}\overline{\partial}^{4+l-k}\psi\,\mathrm{d}x'\,\mathrm{d}\tau\\
&-\kappa\int_0^t\int_\Sigma \varepsilon^{4l}\partial_t^k\overline{\partial}^{4+l-k}(1-\overline{\Delta})\psi\,\partial_t^{k+1}\overline{\partial}^{4+l-k}(1-\overline{\Delta})\psi\,\mathrm{d}x'\,\mathrm{d}\tau-\kappa\int_0^t\int_\Sigma\left|\varepsilon^{2l}\partial_t^{k+1}\overline{\partial}^{4+l-k}\langle\overline{\partial}\rangle\psi\right|^2\mathrm{d}x'\,\mathrm{d}\tau\\
\lesssim& -\left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^k\overline{\partial}^{4+l-k}\psi\right|_2^2\Big|_0^t-\left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^{k+1}\overline{\partial}^{4+l-k}\psi\right|^2_{L_t^2H^1_{x'}}+\delta\left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^{k+1}\overline{\partial}^{4+l-k}\psi\right|^2_{L_t^2H^1_{x'}}+\frac{\sigma}{\kappa}\int_0^t C(\mathring{K}_0)\,\mathrm{d}\tau.\end{aligned}\tag{4.58}$$

For the term $\mathring{\text{ST}}'$, we have

$$\begin{aligned}\int_0^t \mathring{\text{ST}}'\,\mathrm{d}\tau =& \sigma\int_0^t\int_\Sigma \varepsilon^{4l}\partial_t^k\overline{\partial}^{4+l-k}\left(\frac{\overline{\nabla}\mathring{\psi}}{\sqrt{1+|\overline{\nabla}\mathring{\psi}|^2}}\right)\cdot\overline{\nabla}\left((\bar{v}^+\cdot\overline{\nabla})\partial_t^k\overline{\partial}^{4+l-k}\mathring{\psi}\right)\mathrm{d}x'\,\mathrm{d}\tau\\
&-\kappa\int_0^t\int_\Sigma \varepsilon^{4l}\partial_t^k\overline{\partial}^{4+l-k}(1-\overline{\Delta})\psi\,(1-\overline{\Delta})\left((\bar{v}^+\cdot\overline{\nabla})\partial_t^k\overline{\partial}^{4+l-k}\mathring{\psi}\right)\mathrm{d}x'\,\mathrm{d}\tau\\
&-\kappa\int_0^t\int_\Sigma \varepsilon^{4l}\partial_t^{k+1}\overline{\partial}^{4+l-k}\langle\overline{\partial}\rangle\psi\,\langle\overline{\partial}\rangle\left((\bar{v}^+\cdot\overline{\nabla})\partial_t^k\overline{\partial}^{4+l-k}\mathring{\psi}\right)\mathrm{d}x'\,\mathrm{d}\tau\\
\lesssim& \sigma C(\mathring{K}_0,\kappa^{-1})t+C(\mathring{K}_0)\int_0^t\left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^k\overline{\partial}^{4+l-k}\psi\right|_2\|v^+\|_{4,\pm}\,\mathrm{d}\tau\\
&+\delta\left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^{k+1}\overline{\partial}^{4+l-k}\psi\right|^2_{L_t^2H^1_{x'}}+C(\mathring{K}_0)\int_0^t\|v^+\|^2_{4,\pm}\,\mathrm{d}\tau\\
\lesssim& \sigma C(\mathring{K}_0,\kappa^{-1})t+C(\mathring{K}_0)\int_0^t \mathring{E}^\kappa_{4+l}(\tau)+\mathring{E}^\kappa_4(\tau)\,\mathrm{d}\tau.\end{aligned}\tag{4.59}$$

The term $\mathring{\text{VS}}$ can also be directly controlled even if $\mathcal{T}^\gamma$ only contains time derivatives. When $k<4+l$, we can use the κ-weighted energy to control it after integrating $\overline{\partial}^{\frac12}$ by parts and using Lemma B.4

$$\begin{aligned}\mathring{\text{VS}} =& \int_\Sigma \varepsilon^{4l}\partial_t^k\overline{\partial}^{3.5+l-k}q^-\,\overline{\partial}^{\frac12}\left(([\![\bar{v}]\!]\cdot\overline{\nabla})\partial_t^k\overline{\partial}^{4+l-k}\mathring{\psi}\right)\mathrm{d}x'\\
\lesssim& \|\varepsilon^{2l}\partial_t^k\overline{\partial}^{4+l-k}q^-\|_{0,-}^{\frac12}\|\varepsilon^{2l}\partial_t^k\overline{\partial}^{3+l-k}\partial_3q^-\|_{0,-}^{\frac12}|\bar{v}^\pm|_{W^{\frac12,\infty}}\left|\varepsilon^{2l}\partial_t^k\mathring{\psi}\right|_{5.5+l-k}\\
\lesssim& (\mathring{E}^\kappa_4(t)+\mathring{E}^\kappa_{4+l}(t))C(\mathring{K}_0).\end{aligned}\tag{4.60}$$

When $k=4+l$, we can first integrate ∂_t by parts and then integrate $\overline{\partial}^{\frac12}$ by parts

$$\begin{aligned}\int_0^t \mathring{\text{VS}}\,\mathrm{d}\tau \overset{L}{=}& \int_0^t\int_\Sigma \varepsilon^{4l}\overline{\partial}^{\frac12}([\![\bar{v}]\!]\,\partial_t^{3+l}q^-)\,\overline{\partial}^{\frac12}\partial_t^{5+l}\mathring{\psi}\,\mathrm{d}x'\,\mathrm{d}\tau+\int_\Sigma \varepsilon^{4l}\partial_t^{3+l}q^-\,([\![\bar{v}]\!]\cdot\overline{\nabla})\partial_t^{4+l}\mathring{\psi}\,\mathrm{d}x'\Big|_0^t\\
\lesssim& \int_0^t \|\varepsilon^{2l}\overline{\partial}\partial_t^{3+l}q^-\|_{0,-}^{\frac12}\|\varepsilon^{2l}\partial_t^{3+l}\partial_3q^-\|_{0,-}^{\frac12}|\bar{v}^\pm|_{W^{\frac12,\infty}}\left|\varepsilon^{2l}\partial_t^{5+l}\mathring{\psi}\right|_{0.5}\,\mathrm{d}\tau\\
&+\delta\|\varepsilon^{2l}\partial_t^{3+l}q^-\|^2_{1,-}+|\bar{v}^\pm|^2_{L^\infty}\left|\varepsilon^{2l}\partial_t^4\mathring{\psi}\right|_1^2+C(\mathring{K}_0)\mathring{E}^\kappa(0)\\
\lesssim& \delta\mathring{E}^\kappa_{4+l}(t)+C(\mathring{K}_0,\kappa^{-1})\left(\mathring{E}^\kappa(0)+\int_0^t\mathring{E}^\kappa(\tau)\,\mathrm{d}\tau\right).\end{aligned}\tag{4.61}$$

For $\mathring{Z}B+\mathring{Z}$, the cancellation structure obtained in Section 3.3.1 and Section 3.4 still holds. Following step 4 in Section 3.4, we have

$$\begin{aligned}\mathring{Z}B^{\pm}+\mathring{Z}^{\pm} = &\mp\int_{\Sigma}\varepsilon^{4l}(\partial_t^k\overline{\partial}^{4+l-k}q^{\pm}-\partial_t^k\overline{\partial}^{4+l-k}\mathring{\psi}\partial_3 q^{\pm})(\partial_3 v^{\pm}\cdot\mathring{N})\,\partial_t^k\overline{\partial}^{4+l-k}\mathring{\psi}\,\mathrm{d}x'\\ &\mp\int_{\Sigma}\varepsilon^{4l}\mathring{\mathbf{Q}}^{\pm}\left[\partial_t^k\overline{\partial}^{4+l-k},\mathring{N}_i,v_i^{\pm}\right]\mathrm{d}x'-\int_{\Omega^{\pm}}\varepsilon^{4l}\mathring{\mathbf{Q}}^{\pm}\mathring{\mathfrak{C}}_i(v_i^{\pm})\,\mathrm{d}\mathring{\mathcal{V}}_t,\end{aligned}\tag{4.62}$$

where the first line is controlled in the same way as $\mathring{\mathrm{V}}\mathrm{S}$. Mimicing the proof in step 4 in Section 3.4, we have

$$\begin{aligned}&\mp\int_{\Sigma}\varepsilon^{4l}\mathring{\mathbf{Q}}^{\pm}\left[\partial_t^k\overline{\partial}^{4+l-k},\mathring{N}_i,v_i^{\pm}\right]\mathrm{d}x'-\int_{\Omega^{\pm}}\varepsilon^{4l}\mathring{\mathbf{Q}}^{\pm}\mathring{\mathfrak{C}}_i(v_i^{\pm})\,\mathrm{d}\mathring{\mathcal{V}}_t\\ \stackrel{L}{=}&\int_{\Omega^{\pm}}\varepsilon^{4l}\partial_3^{\mathring{\varphi}}\mathring{\mathbf{Q}}^{\pm}\left[\partial_t^k\overline{\partial}^{4+l-k},\mathring{\mathbf{N}}_i,v_i^{\pm}\right]\mathrm{d}\mathring{\mathcal{V}}_t,\end{aligned}\tag{4.63}$$

whose time integral can be directly controlled by

$$\delta\mathring{E}^{\kappa}_{4+l}(t)+C(\mathring{K}_0,\kappa^{-1})\left(\mathring{E}^{\kappa}(0)+\int_0^t\mathring{E}^{\kappa}(\tau)\,\mathrm{d}\tau\right)$$

after integrating by parts one tangential derivative in $\partial_t^k\overline{\partial}^{4+l-k}$.

4.3.4 Uniform-in-n estimates for the linearized approximate system

Summarizing the estimates obtained in Section 4.3.1-Section 4.3.3, we prove that for any $\delta\in(0,1)$,

$$\mathring{E}^{\kappa}(t)\lesssim\delta\mathring{E}^{\kappa}(t)+C(\mathring{K}_0,\kappa^{-1})\left(\mathring{E}^{\kappa}(0)+\int_0^t\mathring{E}^{\kappa}(\tau)\,\mathrm{d}\tau\right).$$

Choosing $\delta>0$ suitably small such that the δ-term can be absorbed by the left side and using Grönwall's inequality, we find that there exists a time $T_\kappa>0$ (independent of ε and n), such that

$$\sup_{0\le t\le T_\kappa}\mathring{E}^{\kappa}(t)\le C'(\mathring{K}_0,\kappa^{-1})\mathring{E}^{\kappa}(0)$$

for some positive function C' continuous in its arguments. Following the argument in remark 4.3, it is straightforward to show that

$$\sum_{\pm}\sum_{l=0}^{4}\sum_{\langle\alpha\rangle=2l}\sum_{k=0}^{4-l}\int_0^t\left\|\varepsilon^{2l}\mathcal{T}^{\alpha}\partial_t^k\boldsymbol{b}^{\pm}\right\|^2_{4-k-l,\pm}\mathrm{d}\tau<P(\mathring{E}^{\kappa}(t))\quad\forall t\in[0,T_\kappa].$$

4.4 Picard iteration: well-posedness of the nonlinear approximate problem

We already establish the local existence of the linear system (4.1) for each n and the uniform-in-n estimates for the solution to (4.1). It suffices to prove $\{(v^{[n],\pm},b^{[n],\pm},\boldsymbol{b}^{[n],\pm},q^{[n],\pm},\psi^{[n]})\}$ has a strongly convergent subsequence (in certain anisotropic Sobolev norms).

4.4.1 The way to recover the nonlinear approximate system

Let us first see how to construct the solution to the nonlinear system (3.1) for fixed $\kappa>0$ if the strong convergence result has been proven. We assume that the expected limit is denoted by $(v^{[\infty],\pm},b^{[\infty],\pm},\boldsymbol{b}^{[\infty],\pm},q^{[\infty],\pm},\psi^{[\infty]})$.

Then the limit functions satisfy the following system

$$
\begin{cases}
\rho^{[\infty],\pm}D_t^{\varphi^{[\infty]},\pm}v^{[\infty],\pm}-(\boldsymbol{b}^{[\infty],\pm}\cdot\nabla^{\varphi^{[\infty]}})b^{[\infty],\pm}+\nabla^{\varphi^{[\infty]}}q^{[\infty],\pm}=0 & \text{in } [0,T]\times\Omega^\pm,\\
(\mathcal{F}_p^\pm)^{[\infty]}D_t^{\varphi^{[\infty]},\pm}q^{[\infty],\pm}-(\mathcal{F}_p^\pm)^{[\infty]}D_t^{\varphi^{[\infty]},\pm}b^{[\infty],\pm}\cdot\boldsymbol{b}^{[\infty],\pm}+\nabla^{\varphi^{[\infty]}}\cdot v^{[\infty],\pm}=0 & \text{in } [0,T]\times\Omega^\pm,\\
D_t^{\varphi^{[\infty]},\pm}b^{[\infty],\pm}-(\boldsymbol{b}^{[\infty],\pm}\cdot\nabla^{\varphi^{[\infty]}})v^{[\infty],\pm}+\boldsymbol{b}^{[\infty],\pm}\nabla^{\varphi^{[\infty]}}\cdot v^{[\infty],\pm}=0 & \text{in } [0,T]\times\Omega^\pm,\\
D_t^{\varphi^{[\infty]},\pm}S^{[\infty],\pm}=0 & \text{in } [0,T]\times\Omega^\pm,\\
\left[\!\left[q^{[\infty]}\right]\!\right]=\sigma\mathcal{H}(\psi^{[\infty]})-\kappa(1-\overline{\Delta})^2\psi^{[\infty]}-\kappa(1-\overline{\Delta})\partial_t\psi^{[\infty]} & \text{on } [0,T]\times\Sigma,\\
\partial_t\psi^{[\infty]}=v^{[\infty],\pm}\cdot N^{[\infty]} & \text{on } [0,T]\times\Sigma,\\
v_3^{[\infty],\pm}=0 & \text{on } [0,T]\times\Sigma^\pm,\\
(v^{[\infty],\pm},b^{[\infty],\pm},q^{[\infty],\pm},S^{[\infty],\pm},\psi^{[\infty]})|_{t=0}=(v_0^{\kappa,\pm},b_0^{\kappa,\pm},q_0^{\kappa,\pm},S_0^{\kappa,\pm},\psi_0^\kappa),
\end{cases}
\tag{4.64}
$$

where $\rho^{[\infty]}$ is defined via the equation of state $\rho=\rho(p,S)$ and $p^{[\infty]}:=q^{[\infty]}-\frac{1}{2}|b^{[\infty]}|^2$. Also we have

$$
D_t^{\varphi^{[\infty]},\pm}=\partial_t+\bar{v}^{[\infty],\pm}\cdot\overline{\nabla}+\frac{1}{\partial_3\varphi^{[\infty]}}(v^{[\infty],\pm}\cdot\mathbf{N}^{[\infty]}-\partial_t\varphi^{[\infty]})\partial_3,
$$

$$
\boldsymbol{b}^{[\infty],\pm}\cdot\nabla^{\varphi^{[\infty]}}=\bar{\boldsymbol{b}}^{[\infty],\pm}\cdot\overline{\nabla}+\frac{1}{\partial_3\varphi^{[\infty]}}(\boldsymbol{b}^{[\infty],\pm}\cdot N^{[\infty]})\partial_3.
$$

We must prove that $b_3^{[\infty],\pm}=\boldsymbol{b}_3^{[\infty],\pm}$ in $\Omega^\pm$. According to the definition of $\boldsymbol{b}^{[n]}$, the limit function satisfies

$$
\boldsymbol{b}_3^{[\infty],\pm}=b_3^{[\infty],\pm}+\mathfrak{R}_T^\pm\left(b_1^{[\infty],\pm}\overline{\partial}_1\psi^{[\infty]}+b_2^{[\infty],\pm}\overline{\partial}_2\psi^{[\infty]}-b_3^{[\infty],\pm}\right)\Big|_\Sigma\Rightarrow\boldsymbol{b}_3^{[\infty],\pm}\cdot N^{[\infty]}|_\Sigma=0.
$$

Since Lemma B.3 implies that $\mathfrak{R}_T^\pm(0)=0$, then the remaining step is to show $b^{[\infty],\pm}\cdot N^{[\infty]}|_\Sigma=0$ holds with in the lifespan of the solution to (4.64) provided $b^{[\infty],\pm}\cdot N^{[\infty]}|_{t=0}=0$ on Σ. On Σ, we compute that

$$
\begin{aligned}
&D_t^{\varphi^{[\infty]},\pm}(b^{[\infty],\pm}\cdot N^{[\infty]})=D_t^{\varphi^{[\infty]},\pm}b^{[\infty],\pm}\cdot N^{[\infty]}+b^{[\infty],\pm}\cdot D_t^{\varphi^{[\infty]},\pm}N^{[\infty]}\\
=&\underbrace{(\bar{\boldsymbol{b}}^{[\infty],\pm}\cdot\overline{\nabla})}_{=\bar{b}^{[\infty],\pm}\cdot\overline{\nabla}}v^{[\infty],\pm}\cdot N^{[\infty]}+\underbrace{(\boldsymbol{b}^{[\infty],\pm}\cdot N^{[\infty]})}_{=0\text{ on }\Sigma}\partial_3v^{[\infty],\pm}\cdot N^{[\infty]}+\underbrace{(\boldsymbol{b}^{[\infty],\pm}\cdot N^{[\infty]})}_{=0\text{ on }\Sigma}(\nabla^{\varphi^{[\infty]}}\cdot v^{[\infty],\pm})\\
&-\bar{b}_i^{[\infty],\pm}\overline{\partial}_i\partial_t\psi^{[\infty]}-\bar{b}_i^{[\infty],\pm}\bar{v}_j^{[\infty],\pm}\overline{\partial}_j\overline{\partial}_i\psi^{[\infty]}\\
=&(\bar{b}^{[\infty],\pm}\cdot\overline{\nabla})\underbrace{\left(v_3^{[\infty],\pm}-\bar{v}_j^{[\infty],\pm}\overline{\partial}_j\psi^{[\infty]}\right)}_{=\partial_t\psi^{[\infty]}}+\bar{b}_i^{[\infty],\pm}\bar{v}_j^{[\infty],\pm}\overline{\partial}_j\overline{\partial}_i\psi^{[\infty]}-\bar{b}_i^{[\infty],\pm}\overline{\partial}_i\partial_t\psi^{[\infty]}-\bar{b}_i^{[\infty],\pm}\bar{v}_j^{[\infty],\pm}\overline{\partial}_j\overline{\partial}_i\psi^{[\infty]}=0.
\end{aligned}
$$

Thus standard L^2 energy estimate shows that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma\left|b^{[\infty],\pm}\cdot N^{[\infty]}\right|^2\,\mathrm{d}x'=\int_\Sigma(\overline{\nabla}\cdot\bar{v}^{[\infty],\pm})\left|b^{[\infty],\pm}\cdot N^{[\infty]}\right|^2\,\mathrm{d}x'\le|\overline{\partial}v^{[\infty],\pm}|_{L^\infty}\left|b^{[\infty],\pm}\cdot N^{[\infty]}\right|_0^2.
\tag{4.65}
$$

Since $b^{[\infty],\pm}\cdot N^{[\infty]}|_{t=0}=0$ on Σ, we conclude that $b^{[\infty],\pm}\cdot N^{[\infty]}=0$ always holds on Σ by using Grönwall's inequality. Plugging it back to the expression of $\boldsymbol{b}_3^{[\infty],\pm}$, we find $\boldsymbol{b}_3^{[\infty],\pm}=b_3^{[\infty],\pm}$ in $\Omega^\pm$ as desired. Then we can replace $\boldsymbol{b}$ by b in the limit system (4.64)

$$
\begin{cases}
\rho^{[\infty],\pm}D_t^{\varphi^{[\infty]},\pm}v^{[\infty],\pm}-(b^{[\infty],\pm}\cdot\nabla^{\varphi^{[\infty]}})b^{[\infty],\pm}+\nabla^{\varphi^{[\infty]}}q^{[\infty],\pm}=0 & \text{in } [0,T]\times\Omega^\pm,\\
(\mathcal{F}_p^\pm)^{[\infty]}D_t^{\varphi^{[\infty]},\pm}q^{[\infty],\pm}-(\mathcal{F}_p^\pm)^{[\infty]}D_t^{\varphi^{[\infty]},\pm}b^{[\infty],\pm}\cdot b^{[\infty],\pm}+\nabla^{\varphi^{[\infty]}}\cdot v^{[\infty],\pm}=0 & \text{in } [0,T]\times\Omega^\pm,\\
D_t^{\varphi^{[\infty]},\pm}b^{[\infty],\pm}-(b^{[\infty],\pm}\cdot\nabla^{\varphi^{[\infty]}})v^{[\infty],\pm}+b^{[\infty],\pm}\nabla^{\varphi^{[\infty]}}\cdot v^{[\infty],\pm}=0 & \text{in } [0,T]\times\Omega^\pm,\\
D_t^{\varphi^{[\infty]},\pm}S^{[\infty],\pm}=0 & \text{in } [0,T]\times\Omega^\pm,\\
\left[\!\left[q^{[\infty]}\right]\!\right]=\sigma\mathcal{H}(\psi^{[\infty]})-\kappa(1-\overline{\Delta})^2\psi^{[\infty]}-\kappa(1-\overline{\Delta})\partial_t\psi^{[\infty]} & \text{on } [0,T]\times\Sigma,\\
\partial_t\psi^{[\infty]}=v^{[\infty],\pm}\cdot N^{[\infty]},\quad b^{[\infty],\pm}\cdot N^{[\infty]}=0 & \text{on } [0,T]\times\Sigma,\\
v_3^{[\infty],\pm}=b_3^{[\infty],\pm}=0 & \text{on } [0,T]\times\Sigma^\pm,\\
(v^{[\infty],\pm},b^{[\infty],\pm},q^{[\infty],\pm},S^{[\infty],\pm},\psi^{[\infty]})|_{t=0}=(v_0^{\kappa,\pm},b_0^{\kappa,\pm},q_0^{\kappa,\pm},S_0^{\kappa,\pm},\psi_0^\kappa),
\end{cases}
\tag{4.66}
$$

in which $(v^{[\infty],\pm},b^{[\infty],\pm},q^{[\infty],\pm},\psi^{[\infty]})$ exactly gives the solution to the nonlinear approximate system (3.1). Finally, the divergence constraint $\nabla^{\varphi^{[\infty]}}\cdot b^{[\infty],\pm}=0$ in $\Omega^\pm$ automatically holds thanks to the second equation, the fourth equation in (4.66) and $\nabla^{\varphi_0^\kappa}\cdot b_0^{\kappa,\pm}=0$ in $\Omega^\pm$.

4.4.2 Proof of strong convergence

For a function sequence $\{f^{[n],\pm}\}$, we define $[f]^{[n],\pm} := f^{[n+1],\pm} - f^{[n],\pm}$. Then we can write the linear system of $\{([v]^{[n],\pm}, [b]^{[n],\pm}, [q]^{[n],\pm}, [\psi]^{[n]})\}$ as follows

$$\begin{cases} \rho^{[n],\pm} D_t^{\varphi^{[n]}} [v]^{[n],\pm} - (\boldsymbol{b}^{[n],\pm} \cdot \nabla^{\varphi^{[n]}})[b]^{[n],\pm} + \nabla^{\varphi^{[n]}}[q]^{[n],\pm} + \nabla^{[\varphi]^{[n-1]}} q^{[n],\pm} = -\mathring{f}_v^{[n],\pm} & \text{in } [0,T]\times\Omega^\pm \\ \mathcal{F}_p^{[n],\pm} D_t^{\varphi^{[n]}}[q]^{[n],\pm} - \mathcal{F}_p^{[n],\pm} D_t^{\varphi^{[n]}}[b]^{[n],\pm}\cdot \boldsymbol{b}^{[n],\pm} + \nabla^{\varphi^{[n]}}\cdot [v]^{[n],\pm} + \nabla^{[\varphi]^{[n-1]}}\cdot v^{[n],\pm} = -\mathring{f}_p^{[n],\pm} & \text{in } [0,T]\times\Omega^\pm \\ D_t^{\varphi^{[n]}}[b]^{[n],\pm} - (\boldsymbol{b}^{[n],\pm}\cdot\nabla^{\varphi^{[n]}})[v]^{[n],\pm} + \boldsymbol{b}^{[n],\pm}(\nabla^{\varphi^{[n]}}\cdot[v]^{[n],\pm} + \nabla^{[\varphi]^{[n-1]}}\cdot v^{[n],\pm}) = -\mathring{f}_b^{[n],\pm} & \text{in } [0,T]\times\Omega^\pm \\ D_t^{\varphi^{[n]}}[S]^{[n],\pm} = -\mathring{f}_S^{[n],\pm} & \text{in } [0,T]\times\Omega^\pm \\ [\![[q]^{[n],\pm}]\!] = \sigma(\mathcal{H}(\psi^{[n]}) - \mathcal{H}(\psi^{[n-1]})) - \kappa(1-\overline{\Delta})^2[\psi]^{[n]} - \kappa(1-\overline{\Delta})\partial_t[\psi]^{[n]} & \text{on } [0,T]\times\Sigma \\ \partial_t[\psi]^{[n]} = [v]^{[n],\pm}\cdot N^{[n]} + v^{[n],\pm}\cdot[N]^{[n-1]} & \text{on } [0,T]\times\Sigma \\ v_3^{[n],\pm} = v_3^{[n-1],\pm} = \boldsymbol{b}_3^{[n],\pm} = \boldsymbol{b}_3^{[n-1],\pm} = 0 & \text{on } [0,T]\times\Sigma^\pm \\ ([v]^{[n]}, [b]^{[n]}, [q]^{[n]}, [\psi]^{[n]})|_{t=0} = (\vec{0},\vec{0},0,0), \end{cases} \tag{4.67}$$

where the source terms are defined by

$$\begin{aligned} \mathring{f}_v^{[n],\pm} :=\ & [\rho]^{[n-1],\pm}\partial_t v^{[n],\pm} + [\rho\bar{v}]^{[n-1],\pm}\cdot\overline{\nabla}v^{[n],\pm} + [\rho V_{\mathbf{N}}]^{[n-1],\pm}\partial_3 v^{[n],\pm} \\ & - [\boldsymbol{b}]^{[n-1],\pm}\cdot\overline{\nabla}b^{[n],\pm} - [\boldsymbol{B}_{\mathbf{N}}]^{[n-1],\pm}\partial_3 b^{[n],\pm}, \end{aligned} \tag{4.68}$$

$$\begin{aligned} \mathring{f}_q^{[n],\pm} :=\ & [\mathcal{F}_p]^{[n-1],\pm}\partial_t q^{[n],\pm} + [\mathcal{F}_p\bar{v}]^{[n-1],\pm}\cdot\overline{\nabla}q^{[n],\pm} + [\mathcal{F}_p V_{\mathbf{N}}]^{[n-1],\pm}\partial_3 q^{[n],\pm} \\ & - ([\mathcal{F}_p]^{[n-1],\pm}\partial_t b^{[n],\pm} + [\mathcal{F}_p\bar{v}]^{[n-1],\pm}\cdot\overline{\nabla}b^{[n],\pm} + [\mathcal{F}_p V_{\mathbf{N}}]^{[n-1],\pm}\partial_3 b^{[n],\pm})\cdot\boldsymbol{b}^{[n],\pm} \\ & - (\mathcal{F}_p^\pm)^{[n-1]} D_t^{\varphi^{[n-1]}} b^{[n],\pm}\cdot[\boldsymbol{b}]^{[n-1],\pm}, \end{aligned} \tag{4.69}$$

$$\begin{aligned} \mathring{f}_p^{[n],\pm} :=\ & [\bar{v}]^{[n-1],\pm}\cdot\overline{\nabla}b^{[n],\pm} + [V_{\mathbf{N}}]^{[n-1],\pm}\partial_3 b^{[n],\pm} - [\boldsymbol{b}]^{[n-1],\pm}\cdot\overline{\nabla}v^{[n],\pm} - [\boldsymbol{B}_{\mathbf{N}}]^{[n-1],\pm}\partial_3 v^{[n],\pm} \\ & + [\boldsymbol{b}]^{[n-1],\pm}(\nabla^{\varphi^{[n-1]}}\cdot v^{[n],\pm}), \end{aligned} \tag{4.70}$$

$$\mathring{f}_S^{[n],\pm} := [\bar{v}]^{[n-1],\pm}\cdot\overline{\nabla}S^{[n],\pm} + [V_{\mathbf{N}}]^{[n-1],\pm}\partial_3 S^{[n],\pm}, \tag{4.71}$$

with

$$V_{\mathbf{N}}^{[n]} := \frac{1}{\partial_3\varphi^{[n]}}(v^{[n]}\cdot\mathbf{N}^{[n-1]} - \partial_t\varphi^{[n]}), \quad \boldsymbol{B}_{\mathbf{N}}^{[n]} := \frac{1}{\partial_3\varphi^{[n]}}(\boldsymbol{b}^{[n]}\cdot\mathbf{N}^{[n]}), \quad \nabla^{[\varphi]^{[n-1]}} f^{[n]} := -[\mathbf{N}/\partial_3\varphi]^{[n-1]}\partial_3 f^{[n]}$$

For $1 \le n \in \mathbb{N}^*$, we define the energy for the linear system (4.67) as follows

$$\begin{aligned} [\mathring{E}^\kappa]^{[n]}(t) :=\ & [\mathring{E}^\kappa]_3^{[n]}(t) + \cdots + [\mathring{E}^\kappa]_6^{[n]}(t), \\ [\mathring{E}^\kappa]_{3+l}^{[n]}(t) :=\ & \sum_\pm \sum_{k=0}^{3} \sum_{\langle\alpha\rangle=2l} \left\| \varepsilon^{2l}\partial_t^k \mathcal{T}^\alpha([v]^{[n],\pm}, [b]^{[n],\pm}, [q]^{[n],\pm}, [S]^{[n],\pm}) \right\|_{3-k-l}^2 \\ & + \sum_{k=0}^{3+l} \left| \sqrt{\kappa}\varepsilon^{2l}\partial_t^k[\psi]^{[n]} \right|_{5+l-k}^2 + \int_0^t \left| \sqrt{\kappa}\varepsilon^{2l}\partial_t^{4+l}[\psi]^{[n]} \right|_1^2 \,\mathrm{d}\tau, \quad 0 \le l \le 3, \end{aligned} \tag{4.72}$$

where $\mathcal{T}^\alpha := (\omega(x_3)\partial_3)^{\alpha_4}\partial_t^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}$ with the multi-index $\alpha = (\alpha_0, \alpha_1, \alpha_2, 0, \alpha_4)$, $\langle\alpha\rangle = \alpha_0 + \alpha_1 + \alpha_2 + 2\times 0 + \alpha_4$. It should be noted that the initial value of $[\mathring{E}^\kappa]^{[n]}$. Thus, we shall prove the following proposition in order for the strong convergence.

Proposition 4.2. There exists a time $T'_\kappa > 0$ depending on κ and $\mathring{K}_0$, such that

$$\forall 2 \le n \in \mathbb{N}^*, \quad \sup_{0\le t\le T'_\kappa} [\mathring{E}^\kappa]^{[n]}(t) \le \frac{1}{4}\left(\sup_{0\le t\le T'_\kappa} [\mathring{E}^\kappa]^{[n-1]}(t) + \sup_{0\le t\le T'_\kappa} [\mathring{E}^\kappa]^{[n-2]}(t) \right). \tag{4.73}$$

The proof of proposition is substantially similar to the estimates of $\mathring{E}(t)$ in Section 4.3, so we will not go into every detail but only write the sketch of the proof.

Step 1: Div-Curl analysis and reduction of pressure

The reduction of pressure follows in the same way as in Section 3.6.1. Invoking the momentum equation, we have

$$-(\partial_3\varphi^{[n]})^{-1}\partial_3[q]^{[n],\pm} = \rho^{[n],\pm}D_t^{\varphi^{[n]}}[v]^{[n],\pm} - (\boldsymbol{b}^{[n],\pm}\cdot\nabla^{\varphi^{[n]}})[b]^{[n],\pm} + \mathring{f}_v^{[n],\pm} + (\partial_3\varphi^{[n]})^{-1}\partial_3 q^{[n],\pm}.$$

Then using $\partial_i^{\mathring{\varphi}} = \overline{\partial}_i - \overline{\partial}_i\mathring{\varphi}\partial_3^{\mathring{\varphi}}$, we can convert ∂q to a spatial derivative of v and b plus the given term $\partial_3 q^{[n],\pm}$.

For the div-curl analysis, using (B.1), we have for $0\le l\le 2,\ 0\le k\le 2-l$

$$\begin{aligned}&\left\|\varepsilon^{2l}\partial_t^k\mathcal{T}^\alpha([v]^{[n],\pm},[b]^{[n],\pm})\right\|_{3-k-l,\pm}^2\\ \le\ & C(\mathring{K}_0)\Big(\|\varepsilon^{2l}\partial_t^k\mathcal{T}^\alpha([v]^{[n],\pm},[b]^{[n],\pm})\|_{0,\pm}^2 + \left\|\varepsilon^{2l}\nabla^{\varphi^{[n]}}\times\partial_t^k\mathcal{T}^\alpha([v]^{[n],\pm},[b]^{[n],\pm})\right\|_{2-k-l,\pm}^2\\ &+\left\|\varepsilon^{2l}\nabla^{\varphi^{[n]}}\cdot\partial_t^k\mathcal{T}^\alpha([v]^{[n],\pm},[b]^{[n],\pm})\right\|_{2-k-l,\pm}^2 + \left\|\overline{\partial}^{3-k-l}\partial_t^k\mathcal{T}^\alpha([v]^{[n],\pm},[b]^{[n],\pm})\right\|_{0,\pm}^2\Big).\end{aligned}\tag{4.74}$$

The L^2 estimate is straightforward, so we skip the proof. For the curl part, we again analyze the evolution equations of vorticity and current

$$\begin{aligned}&\rho^{[n]}D_t^{\varphi^{[n]}}(\nabla^{\varphi^{[n]}}\times[v]^{[n]}) - (\boldsymbol{b}^{[n]}\cdot\nabla^{\varphi^{[n]}})(\nabla^{\varphi^{[n]}}\times[b]^{[n]})\\ =&-\nabla^{\varphi^{[n]}}\times\mathring{f}_v^{[n]} - \nabla^{\varphi^{[n]}}\rho^{[n]}\times D_t^{\varphi^{[n]}}[v]^{[n]} + (\nabla^{\varphi^{[n]}}\boldsymbol{b}_j^{[n]})\times(\partial_j^{\varphi^{[n]}}[b]^{[n]}) + \rho^{[n]}[D_t^{\varphi^{[n]}},\nabla^{\varphi^{[n]}}\times][v]^{[n]},\end{aligned}\tag{4.75}$$

$$\begin{aligned}&D_t^{\varphi^{[n]}}(\nabla^{\varphi^{[n]}}\times[b]^{[n]}) - (\boldsymbol{b}^{[n]}\cdot\nabla^{\varphi^{[n]}})(\nabla^{\varphi^{[n]}}\times[v]^{[n]}) - \boldsymbol{b}^{[n]}\times(\nabla^{\varphi^{[n]}}(\nabla^{\varphi^{[n]}}\cdot[v]^{[n]}))\\ =&-\nabla^{\varphi^{[n]}}\times\mathring{f}_b^{[n]} + [D_t^{\varphi^{[n]}},\nabla^{\varphi^{[n]}}\times][b]^{[n]} + (\nabla^{\varphi^{[n]}}\boldsymbol{b}_j^{[n]})\times(\partial_j^{\varphi^{[n]}}[v]^{[n]})\\ &-\nabla^{\varphi^{[n]}}\times(\boldsymbol{b}^{[n]}\nabla^{[\varphi]^{[n-1]}}\cdot v^{[n],\pm}) - (\nabla^{\varphi^{[n]}}\times\boldsymbol{b}^{[n]})(\nabla^{\varphi^{[n]}}\cdot[v]^{[n]}).\end{aligned}\tag{4.76}$$

Mimicing the proof in Section 4.3.1 and using the vanishing initial value of system (4.67), we can prove

$$\begin{aligned}&\left\|\varepsilon^{2l}\nabla^{\varphi^{[n]}}\times\partial_t^k\mathcal{T}^\alpha[v]^{[n],\pm}\right\|_{2-k-l,\pm}^2 + \left\|\varepsilon^{2l}\nabla^{\varphi^{[n]}}\times\partial_t^k\mathcal{T}^\alpha[b]^{[n],\pm}\right\|_{2-k-l,\pm}^2 + \left\|\varepsilon^{2l}\sqrt{(\mathcal{F}_p^\pm)^{[n]}}\boldsymbol{b}^{[n],\pm}\times(\partial_t^k\mathcal{T}^\alpha[b]^\pm)\right\|_{2-l-k,\pm}^2\\ \lesssim\ & C(\mathring{K}_0)\int_0^t\sum_{j=0}^l[E]_{3+j}^{[n]}(\tau) + [E]_{3+l+1}^{[n]}(\tau)\,\mathrm{d}\tau.\end{aligned}\tag{4.77}$$

Similarly, the divergence of $[v]^{[n]}$ can be converted to tangential derivatives of $[q]^{[n]}$ and $[b]^{[n]}$ by invoking the continuity equation, and the evolution equation of $\nabla^{\mathring{\varphi}}\cdot[b]^{[n]}$ is

$$\begin{aligned}D_t^{\varphi^{[n]}}(\nabla^{\varphi^{[n]}}\cdot[b]^{[n]}) =\ & (\partial_i^{\varphi^{[n]}}\boldsymbol{b}_j^{[n]})(\partial_j^{\varphi^{[n]}}[v]_i^{[n]}) - (\nabla^{\varphi^{[n]}}\cdot\boldsymbol{b}^{[n]})(\nabla^{\varphi^{[n]}}\cdot[v]^{[n]}) + [D_t^{\varphi^{[n]}},\nabla^{\mathring{\varphi}}\cdot][b]^{[n]}\\ &-\nabla^{\varphi^{[n]}}\cdot(\mathring{f}_b^{[n]} + \boldsymbol{b}^{[n]}\nabla^{[\varphi]^{[n-1]}}\cdot v^{[n],\pm}),\end{aligned}\tag{4.78}$$

so the divergence part is controlled by

$$\begin{aligned}&\left\|\varepsilon^{2l}\nabla^{\varphi^{[n]}}\cdot\partial_t^k\mathcal{T}^\alpha[v]^{[n],\pm}\right\|_{2-k-l,\pm}^2 + \left\|\varepsilon^{2l}\nabla^{\varphi^{[n]}}\cdot\partial_t^k\mathcal{T}^\alpha[b]^{[n],\pm}\right\|_{2-k-l,\pm}^2\\ \lesssim\ & \left\|\varepsilon^{2l}(\mathcal{F}_p^\pm)^{[n]}\partial_t^k\mathcal{T}^\alpha D_t^{\varphi^{[n]}}([q]^\pm,[b]^\pm)\right\|_{2-l-k,\pm}^2 + C(\mathring{K}_0)\int_0^t\sum_{j=0}^l[E]_{3+j}^{[n]}(\tau)\,\mathrm{d}\tau,\end{aligned}\tag{4.79}$$

in which the first term will be reduced to tangential estimates.

Step 2: Tangential estimates

It remains to prove the tangential estimates for $\mathcal{T}^\gamma$-differentiated system where $\mathcal{T}^\gamma = \overline{\partial}^{3-l-k}\partial_t^k\mathcal{T}^\alpha$ satisfies $\alpha_3 = 0, \langle\alpha\rangle = 2l,\ 0\le k\le 3-l,\ 0\le l\le 3$. We shall introduce the Alinhac good unknowns ($[\mathbf{V}],[\mathbf{B}],[\mathbf{Q}]$) as below instead of directly taking tangential derivatives in (4.67).

$$[\mathbf{F}]^{[n]} := \mathring{\mathbf{F}}^{[n+1]} - \mathring{\mathbf{F}}^{[n]} = \mathcal{T}^\gamma[f]^{[n]} - \mathcal{T}^\gamma\varphi^{[n]}\partial_3^{\varphi^{[n]}}[f]^{[n]} - \mathcal{T}^\gamma\varphi^{[n]}\partial_3^{[\varphi]^{[n-1]}}q^{[n]} - \mathcal{T}^\gamma[\varphi]^{[n-1]}\partial_3^{\varphi^{[n-1]}}q^{[n]}$$

and it satisfies

$$\mathcal{T}^\gamma(\partial_i^{\varphi^{[n]}}[f]^{[n]}+\partial_i^{[\varphi]^{[n-1]}}f^{[n]})=\partial_i^{\varphi^{[n]}}[\mathbf{F}]^{[n]}+[\mathfrak{C}]_i^{[n]}(f),\quad \mathcal{T}^\gamma(D_t^{\varphi^{[n]}}[f]^{[n]}+D_t^{[\varphi]^{[n-1]}}f^{[n]})=D_t^{\varphi^{[n]}}[\mathbf{F}]^{[n]}+[\mathfrak{D}]^{[n]}(f)$$

with

$$[\mathfrak{C}]_i^{[n]}(f)=\mathfrak{C}_i^{[n]}(f^{[n+1]})-\mathfrak{C}_i^{[n-1]}(f^{[n]})+\text{lower-order controllable terms}$$
$$[\mathfrak{D}]^{[n]}(f)=\mathfrak{D}^{[n]}(f^{[n+1]})-\mathfrak{D}^{[n-1]}(f^{[n]})+\text{lower-order controllable terms}$$

where $\mathfrak{C}_i^{[n]}(f^{[m]}),\mathfrak{D}^{[n]}(f^{[m]})$ are defined by setting $\mathring{\varphi}=\varphi^{[n]},\ \dot{\varphi}=\varphi^{[n-1]},\ f^{[n+1]}=f,\ f^{[n]}=\mathring{f}$ in (4.34)-(4.35). This can be seen by substracting the corresponding identities of $\mathring{\mathbf{F}}$ with superscript $[n-1]$ from the ones with superscript $[n]$. The evolution equations of the good unknowns are (with $\pm$ dropped)

$$\rho^{[n]}D_t^{\varphi^{[n]}}[\mathbf{V}]^{[n]}-(\boldsymbol{b}^{[n]}\cdot\nabla^{\varphi^{[n]}})[\mathbf{B}]^{[n]}+\nabla^{\varphi^{[n]}}[\mathbf{Q}]^{[n]}=-\mathfrak{C}^{[n]}(q^{[n+1]})+\mathfrak{C}^{[n-1]}(q^{[n]})+[\mathcal{R}]_v\quad\text{in }\Omega\tag{4.80}$$

$$\mathcal{F}_p^{[n]}D_t^{\varphi^{[n]}}[\mathbf{Q}]^{[n]}-\mathcal{F}_p^{[n]}D_t^{\varphi^{[n]}}[\mathbf{B}]^{[n]}\cdot\boldsymbol{b}^{[n]}+\nabla^{\varphi^{[n]}}\cdot[\mathbf{V}]^{[n]}=-\mathfrak{C}_i^{[n]}(v_i^{[n+1]})+\mathfrak{C}_i^{[n-1]}(v_i^{[n]})+[\mathcal{R}]_q\quad\text{in }\Omega\tag{4.81}$$

$$D_t^{\varphi^{[n]}}[\mathbf{B}]^{[n]}-(\boldsymbol{b}^{[n]}\cdot\nabla^{\varphi^{[n]}})[\mathbf{V}]^{[n]}+\boldsymbol{b}^{[n]}(\nabla^{\varphi^{[n]}}\cdot[\mathbf{V}]^{[n]})=-\boldsymbol{b}^{[n]}\left(\mathfrak{C}_i^{[n]}(v_i^{[n+1]})-\mathfrak{C}_i^{[n-1]}(v_i^{[n]})\right)+[\mathcal{R}]_b\quad\text{in }\Omega\tag{4.82}$$

where $[\mathcal{R}]$ terms are controllable in $L^2(\Omega)$ by

$$\|[\mathcal{R}]\|_0^2\le C(\mathring{K}_0)([\mathring{E}^\kappa]^{[n]}(t)+[\mathring{E}^\kappa]^{[n-1]}(t)+[\mathring{E}^\kappa]^{[n-2]}(t)).$$

The boundary conditions of these good unknowns on the interface Σ are

$$\begin{aligned}[\mathbf{Q}]^{[n]}:=&\,\sigma\mathcal{T}^\gamma\left(\mathcal{H}(\psi^{[n]})-\mathcal{H}(\psi^{[n-1]})\right)-\kappa(1-\overline{\Delta})^2\mathcal{T}^\gamma[\psi]^{[n]}-\kappa(1-\overline{\Delta})\partial_t\mathcal{T}^\gamma[\psi]^{[n]}\\&-\mathcal{T}^\gamma\psi^{[n]}\left[\!\left[\partial_3[q]^{[n]}\right]\!\right]-\mathcal{T}^\gamma[\psi]^{[n-1]}\left[\!\left[\partial_3 q^{[n]}\right]\!\right]\end{aligned}\tag{4.83}$$

$$[\mathbf{V}]^{[n]}\cdot N^{[n]}:=\mathcal{T}^\gamma\partial_t[\psi]^{[n]}+[\bar{v}]^{[n]}\cdot\overline{\nabla}\mathcal{T}^\gamma\psi^{[n]}+(\bar{v}^{[n]}\cdot\overline{\nabla})\mathcal{T}^\gamma[\psi]^{[n-1]}+\mathcal{T}^\gamma\bar{v}^{[n]}\cdot\overline{\nabla}[\psi]^{[n-1]}-[\mathcal{W}]^{[n]}\tag{4.84}$$

$$[\mathcal{W}]^{[n]}:=(\partial_3[v]^{[n]}\cdot N^{[n]})\mathcal{T}^\gamma\psi^{[n]}+(\partial_3 v^{[n]}\cdot N^{[n]})\mathcal{T}^\gamma[\psi]^{[n-1]}+\left[\mathcal{T}^\gamma,N_i^{[n]},v_i^{[n+1]}\right]-\left[\mathcal{T}^\gamma,N_i^{[n-1]},v_i^{[n]}\right]\tag{4.85}$$

Given $0\le l\le 3$, following Section 4.3.2, we can similarly prove that

$$\begin{aligned}&\sum_\pm\frac{\mathrm{d}}{\mathrm{d}t}\frac12\int_{\Omega^\pm}\varepsilon^{2l}\rho^{[n]}|[\mathbf{V}]^{[n],\pm}|^2+|[\mathbf{B}]^{[n],\pm}|^2+(\mathcal{F}_p^\pm)^{[n],\pm}([\mathbf{Q}]^{[n],\pm}-[\mathbf{B}]^{[n],\pm}\cdot\boldsymbol{b}^{[n],\pm})^2\,\mathrm{d}\mathcal{V}_t^{[n]}\\&=[\text{ ST}]^{[n]}+[\text{ ST}']^{[n]}+[\text{ VS}]^{[n]}+[\text{ RT}]^{[n]}+\sum_\pm[\text{ RT}]^{[n],\pm}+([ZB]^{[n],\pm}+[Z]^{[n],\pm})\\&\quad+C(\mathring{K}_0)([\mathring{E}^\kappa]^{[n]}(t)+[\mathring{E}^\kappa]^{[n-1]}(t)+[\mathring{E}^\kappa]^{[n-2]}(t))\end{aligned}\tag{4.86}$$

where the term $[\mathring{E}^\kappa]^{[n-1]}+[\mathring{E}^\kappa]^{[n-2]}$ is produced from the estimates of $[\varphi]^{[n-1]},[\varphi]^{[n-2]}$. The above terms on the right side are defined by

$$[\text{ ST}]^{[n]}:=\varepsilon^{4l}\int_\Sigma\mathcal{T}^\gamma\left[\!\left[[q]^{[n]}\right]\!\right]\partial_t\mathcal{T}^\gamma[\psi]^{[n]}\,\mathrm{d}x',\tag{4.87}$$

$$[\text{ ST}']^{[n]}:=\varepsilon^{4l}\int_\Sigma\mathcal{T}^\gamma\left[\!\left[[q]^{[n]}\right]\!\right]([\bar{v}^+]^{[n]}\cdot\overline{\nabla})\mathcal{T}^\gamma\psi^{[n]}\,\mathrm{d}x'+\varepsilon^{4l}\int_\Sigma\mathcal{T}^\gamma\left[\!\left[[q]^{[n]}\right]\!\right](\bar{v}^{[n],+}\cdot\overline{\nabla})\mathcal{T}^\gamma[\psi]^{[n-1]}\,\mathrm{d}x',\tag{4.88}$$

$$[\text{ VS}]^{[n]}:=\varepsilon^{4l}\int_\Sigma\mathcal{T}^\gamma[q]^{[n],-}([\![\bar{v}]\!]^{[n]}\cdot\overline{\nabla})\mathcal{T}^\gamma\psi^{[n]}\,\mathrm{d}x'+\varepsilon^{4l}\int_\Sigma\mathcal{T}^\gamma[q]^{[n],-}\left(\left[\!\left[\bar{v}^{[n]}\right]\!\right]\cdot\overline{\nabla}\right)\mathcal{T}^\gamma[\psi]^{[n-1]}\,\mathrm{d}x',\tag{4.89}$$

$$[\text{ RT}]^{[n]}:=-\varepsilon^{4l}\int_\Sigma\left(\left[\!\left[\partial_3[q]^{[n]}\right]\!\right]\mathcal{T}^\gamma\psi^{[n]}+\mathcal{T}^\gamma[\psi]^{[n-1]}\left[\!\left[\partial_3 q^{[n]}\right]\!\right]\right)\partial_t\mathcal{T}^\gamma\psi\,\mathrm{d}x',\tag{4.90}$$

$$[\text{ RT}]^{[n],\pm}:=\mp\varepsilon^{4l}\int_\Sigma\left(\partial_3[q]^{[n],\pm}\mathcal{T}^\gamma\psi^{[n]}+\mathcal{T}^\gamma[\psi]^{[n-1]}\partial_3 q^{[n],\pm}\right)\left((\bar{v}^\pm\cdot\overline{\nabla})\mathcal{T}^\gamma\mathring{\psi}+(\bar{v}^{[n],\pm}\cdot\overline{\nabla})\mathcal{T}^\gamma[\psi]^{[n-1]}\right)\mathrm{d}x',\tag{4.91}$$

$$[ZB]^{[n],\pm}:=\mp\varepsilon^{4l}\int_\Sigma[\mathbf{Q}]^{[n],\pm}[\mathcal{W}]^{[n],\pm}\,\mathrm{d}x',\quad [Z]^{[n],\pm}=-\int_{\Omega^\pm}\varepsilon^{4l}[\mathbf{Q}]^{[n],\pm}(\mathfrak{C}_i^{[n]}(v_i^{[n+1],\pm})-\mathfrak{C}_i^{[n-1]}(v_i^{[n],\pm}))\,\mathrm{d}\mathcal{V}_t^{[n]}.\tag{4.92}$$

Step 3: Boundary regularity of $[\psi]$

The analysis of the boundary integrals is still similar to Section 4.3.3. Since $\omega(x_3)=0$ on Σ, we can rewrite $\partial_t^k\mathcal{T}^\alpha$ to be $\partial_t^k\overline{\partial}^{3+l-k}$ for $0\le k\le 3+l$, $0\le l\le 3$. Then the term $[\text{ ST}]^{[n]}$ gives the regularity of $[\psi]^{[n]}$ after inserting the jump condition for $[q]^{[n]}$

$$\int_0^t [\text{ ST}]^{[n]}\,\mathrm{d}\tau \lesssim -\left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^k[\psi]^{[n]}\right|_{5+l-k}^2\Big|_0^t - \int_0^t \left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^{k+1}[\psi]^{[n]}\right|_{4+l-k}^2 + \frac{\sigma}{\kappa}C(\mathring{K}_0)\int_0^t [\mathring{E}^\kappa]^{[n]}(\tau)+[\mathring{E}^\kappa]^{[n-1]}(\tau)\,\mathrm{d}\tau. \tag{4.93}$$

The term $[\text{ ST}']^{[n]}$ can be controlled by inserting the jump condition for $[q]^{[n]}$ and then integrating by parts $\overline{\nabla}\cdot$, $1-\overline{\Delta}$, $\sqrt{1-\overline{\Delta}}$ in the three terms in $[q]^{[n]}$ respectively. This is essentially the same as shown in Section 4.3.3, so we only list the result

$$\int_0^t [\text{ ST}]^{[n]}\,\mathrm{d}\tau \lesssim \sigma C(\mathring{K}_0,\kappa^{-1})t + C(\mathring{K}_0)\int_0^t [\mathring{E}^\kappa]^{[n]}(\tau)+[\mathring{E}^\kappa]^{[n-1]}(\tau)\,\mathrm{d}\tau \tag{4.94}$$

The terms $[\text{ RT}]^{[n]}$, $[\text{ RT}]^{[n],\pm}$ are also controlled directly with the help of κ-weighted enhanced regularity. The term $[\text{ VS}]^{[n]}$ is also controlled directly by integrating by parts for one tangential derivative in $\partial_t^k\overline{\partial}^{3+l-k}$ as in Section 4.3.3. Finally, for the term $([ZB]^{[n],\pm}+[Z]^{[n],\pm})$, we still have the previously-used cancellation structure

$$\begin{aligned}[ZB]^{[n],\pm}+[Z]^{[n],\pm} \overset{L}{=} &\mp\int_\Sigma \varepsilon^{4l}[\mathbf{Q}]^{[n],\pm}\left[\partial_t^k\overline{\partial}^{3+l-k}, N_i^{[n]}, v_i^{[n],\pm}\right]\mathrm{d}x' - \int_{\Omega^\pm}\varepsilon^{4l}[\mathbf{Q}]^\pm\mathfrak{C}_i^{[n]}(v_i^{[n],\pm})\,\mathrm{d}\mathcal{V}_t^{[n]}\\ &\pm\int_\Sigma \varepsilon^{4l}[\mathbf{Q}]^{[n],\pm}\left[\partial_t^k\overline{\partial}^{3+l-k}, N_i^{[n-1]}, v_i^{[n+1],\pm}\right]\mathrm{d}x' + \int_{\Omega^\pm}\varepsilon^{4l}[\mathbf{Q}]^{[n],\pm}\mathfrak{C}_i^{[n-1]}(v_i^{[n],\pm})\,\mathrm{d}\mathcal{V}_t^{[n]}\end{aligned} \tag{4.95}$$

where the omitted terms are controlled in the same way as $[\text{ VS}]^{[n]}$. Mimicing the proof in step 4 in Section 3.4, we have

$$\begin{aligned}&\mp\int_\Sigma \varepsilon^{4l}[\mathbf{Q}]^{[n],\pm}\left[\partial_t^k\overline{\partial}^{3+l-k}, N_i^{[n]}, v_i^{[n],\pm}\right]\mathrm{d}x' - \int_{\Omega^\pm}\varepsilon^{4l}[\mathbf{Q}]^\pm\mathfrak{C}_i^{[n]}(v_i^{[n],\pm})\,\mathrm{d}\mathcal{V}_t^{[n]}\\ \overset{L}{=}&\int_{\Omega^\pm}\varepsilon^{4l}\partial_3^{\varphi^{[n]}}[\mathbf{Q}]^{[n],\pm}\left[\partial_t^k\overline{\partial}^{3+l-k}, \mathbf{N}_i^{[n]}, v_i^{[n+1],\pm}\right]\mathrm{d}\mathring{\mathcal{V}}_t,\end{aligned} \tag{4.96}$$

whose time integral can be directly controlled by

$$\delta[\mathring{E}^\kappa]^{[n]}(t) + C(\mathring{K}_0,\kappa^{-1})\int_0^t [\mathring{E}^\kappa]^{[n]}(\tau)+[\mathring{E}^\kappa]^{[n-1]}(\tau)\,\mathrm{d}\tau$$

after integrating by parts for one tangential derivative in $\partial_t^k\overline{\partial}^{3+l-k}$. Similar estimate applies to the second line of $[ZB]^{[n],\pm}+[Z]^{[n],\pm}$:

$$\begin{aligned}&\int_0^t\left(\pm\int_\Sigma \varepsilon^{4l}[\mathbf{Q}]^{[n],\pm}\left[\partial_t^k\overline{\partial}^{3+l-k}, N_i^{[n-1]}, v_i^{[n+1],\pm}\right]\mathrm{d}x' + \int_{\Omega^\pm}\varepsilon^{4l}[\mathbf{Q}]^{[n],\pm}\mathfrak{C}_i^{[n-1]}(v_i^{[n],\pm})\,\mathrm{d}\mathcal{V}_t^{[n]}\right)\mathrm{d}\tau\\ \lesssim\,&\delta[\mathring{E}^\kappa]^{[n]}(t) + C(\mathring{K}_0,\kappa^{-1})\int_0^t [\mathring{E}^\kappa]^{[n]}(\tau)+[\mathring{E}^\kappa]^{[n-1]}(\tau)+[\mathring{E}^\kappa]^{[n-2]}(\tau)\,\mathrm{d}\tau\end{aligned}$$

Step 4: Convergence

Summarizing the above estimates and using $[\mathring{E}^\kappa]^{[n]}(0)=0$, we obtain the energy inequality

$$[\mathring{E}^\kappa]^{[n]}(t) \lesssim \delta[\mathring{E}^\kappa]^{[n]}(t) + C(\mathring{K}_0,\kappa^{-1})\int_0^t [\mathring{E}^\kappa]^{[n]}(\tau)+[\mathring{E}^\kappa]^{[n-1]}(\tau)+[\mathring{E}^\kappa]^{[n-2]}(\tau)\,\mathrm{d}\tau.$$

Choosing $0 < \delta \ll 1$ suitably small, the δ-term can be absorbed by the left side. Thus, there exists a time $T'_\kappa > 0$ depending on $\kappa, \mathring{K}_0$ and independent of n, such that

$$\sup_{0\le t\le T'_\kappa} [\mathring{E}^\kappa]^{[n]}(t) \le \frac{1}{4}\left(\sup_{0\le t\le T'_\kappa} [\mathring{E}^\kappa]^{[n-1]}(t) + \sup_{0\le t\le T'_\kappa} [\mathring{E}^\kappa]^{[n-2]}(t)\right), \tag{4.97}$$

and thus we know by induction that

$$\sup_{0\le t\le T'_\kappa} [\mathring{E}^\kappa]^{[n]}(t) \le C(\mathring{K}_0, \kappa^{-1})/2^{n-1} \to 0 \text{ as } n \to +\infty. \tag{4.98}$$

Hence, for any fixed $\kappa > 0$, the sequence of approximate solutions $\{(v^{[n],\pm}, b^{[n],\pm}, \boldsymbol{b}^{[n],\pm}, q^{[n],\pm}, \psi^{[n]})\}_{n\in\mathbb{N}^*}$ has a strongly convergent subsequence. We write the limit function to be $\{(v^{[\infty],\pm}, b^{[\infty],\pm}, \boldsymbol{b}^{[\infty],\pm} q^{[\infty],\pm}, \psi^{[\infty]})\}_{n\in\mathbb{N}^*}$ to keep consistent with the notations in Section 4.4.1.

4.4.3 Well-posedness of the nonlinear approximate problem for each fixed κ

According to the argument about the limiting process in Section 4.4.1, we know the limit of $b^{[n],\pm}$ coincides with the limit of $\boldsymbol{b}^{[n],\pm}$. Thus, the limit functions $\{(v^{[\infty],\pm}, b^{[\infty],\pm}, q^{[\infty],\pm}, \psi^{[\infty]})\}_{n\in\mathbb{N}^*}$ introduced in Section 4.4.1 exactly give the solution to the nonlinear κ-problem (3.1) in the time interval $[0, T'_\kappa]$ for each fixed $\kappa > 0$. The uniqueness follows from a parallel argument.

5 Well-posedness of current-vortex sheets with surface tension

We are ready to prove the local well-posedness of the original system (1.33) for 3D compressible current-vortex sheets with fixed surface tension coefficient $\sigma > 0$. Recall that we introduce the nonlinear approximate system (3.1) indexed by $\kappa > 0$. In Section 4, we use Galerkin approximation and Picard iteration to prove the well-posedness of (3.1) for each fixed $\kappa > 0$. The lifespan for (3.1) may rely on $\kappa > 0$. Then we prove the uniform-in-κ estimates for (3.1) *without loss of regularity* so that we can extend the solution of (3.1) to a κ-independent lifespan $[0, T]$. In Appendix D, we construct the initial data of (3.1) that converges to the given initial data of (1.33) as $\kappa \to 0$. Thus, by taking $\kappa \to 0$, we obtain the local existence of the original system (1.33) and the energy estimates for $E(t)$ defined in (1.36) without loss of regularity.

It remains to prove the uniqueness. Namely, we assume $(v^{[1],\pm}, b^{[1],\pm}, q^{[1],\pm}, \psi^{[1]})$ and $(v^{[2],\pm}, b^{[2],\pm}, q^{[2],\pm}, \psi^{[2]})$ are two solutions to (1.33) *with the same initial data*. Define $[f] := f^{[1]} - f^{[2]}$, and we need to prove $([v]^\pm, [b]^\pm, [q]^\pm, [\psi])$ are identically zero. In fact, the argument for uniqueness is quite similar to the analysis in Section 4.4.2. The only difference is that the boundary regularity is now given by the surface tension instead of the κ-regularization terms. This has been studied in the previous paper [55, Section 6] by Luo and the author, so we refer to [55, Section 6] and omit the details here.

6 Incompressible and zero-surface-tension limits

This section is devoted to the justification of incompressible limit and zero-surface-tension limit under certain stability conditions, that is the limiting behavior of the local-in-time solution of (1.33) as $\varepsilon \to 0$ and $\sigma \to 0$. Given $\sigma \ge 0$, we introduce the equations of $(\xi^\sigma, w^{\pm,\sigma}, h^{\pm,\sigma})$ incompressible current-vortex sheets together with a transport equation of entropy $\mathfrak{S}^\sigma$

$$
\begin{cases}
\mathfrak{R}^{\pm,\sigma}(\partial_t + w^{\pm,\sigma}\cdot\nabla^{\Xi^\sigma})w^{\pm,\sigma} - (h^{\pm,\sigma}\cdot\nabla^{\Xi^\sigma})h^{\pm,\sigma} + \nabla^{\Xi^\sigma}\Pi^{\pm,\sigma} = 0 & \text{in } [0,T]\times\Omega^\pm,\\
\nabla^{\Xi^\sigma}\cdot w^{\pm,\sigma} = 0 & \text{in } [0,T]\times\Omega^\pm,\\
(\partial_t + w^{\pm,\sigma}\cdot\nabla^{\Xi^\sigma})h^{\pm,\sigma} = (h^{\pm,\sigma}\cdot\nabla^{\Xi^\sigma})w^{\pm,\sigma} & \text{in } [0,T]\times\Omega^\pm,\\
\nabla^{\Xi^\sigma}\cdot h^{\pm,\sigma} = 0 & \text{in } [0,T]\times\Omega^\pm,\\
(\partial_t + w^{\pm,\sigma}\cdot\nabla^{\Xi^\sigma})\mathfrak{S}^{\pm,\sigma} = 0 & \text{in } [0,T]\times\Omega^\pm,\\
[\![\Pi^\sigma]\!] = \sigma\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\xi^\sigma}{\sqrt{1+|\overline{\nabla}\xi^\sigma|^2}}\right) & \text{on } [0,T]\times\Sigma,\\
\partial_t\xi^\sigma = w^{\pm,\sigma}\cdot N^\sigma & \text{on } [0,T]\times\Sigma,\\
h^{\pm,\sigma}\cdot N^\sigma = 0 & \text{on } [0,T]\times\Sigma,\\
(w^{\pm,\sigma}, h^{\pm,\sigma}, \mathfrak{S}^{\pm,\sigma}, \xi^\sigma)|_{t=0} = (w_0^{\pm,\sigma}, h_0^{\pm,\sigma}, \mathfrak{S}_0^{\pm,\sigma}, \xi_0^\sigma),
\end{cases}
\tag{6.1}
$$

where $\Xi^\sigma(t,x) = x_3 + \chi(x_3)\xi^\sigma(t,x')$ to be the extension of ξ^σ in Ω and $N^\sigma := (-\overline{\partial}_1\xi^\sigma, -\overline{\partial}_2\xi^\sigma, 1)^\top$. The quantity $\Pi^\pm := \bar{\Pi}^\pm + \frac{1}{2}|h^\pm|^2$ represent the total pressure for the incompressible equations with $\bar{\Pi}^\pm$ the fluid pressure functions. The quantity $\mathfrak{R}^\pm$ satisfies the evolution equation $(\partial_t + w^\pm\cdot\nabla^\varphi)\mathfrak{R}^\pm = 0$ with initial data $\mathfrak{R}_0^\pm := \rho^\pm(0, \mathfrak{S}_0^\pm)$.

Denote $(\psi^{\varepsilon,\sigma}, v^{\pm,\varepsilon,\sigma}, b^{\pm,\varepsilon,\sigma}, \rho^{\pm,\varepsilon,\sigma}, S^{\pm,\varepsilon,\sigma})$ to be the solution of (1.33) (indexed by σ and ε) with initial data $(\psi_0^{\varepsilon,\sigma}, v_0^{\pm,\varepsilon,\sigma}, b_0^{\pm,\varepsilon,\sigma}, \rho_0^{\pm,\varepsilon,\sigma}, S_0^{\pm,\varepsilon,\sigma})$. We want to show the convergence from the solutions to (1.33) to the solution to (6.1) as $\varepsilon\to 0$ provided the convergence of initial datum. Furthermore, we want to consider the limit process as both ε and $\sigma\to 0$ under certain stability conditions in order for a comprehensive study about the local-in-time solutions of current-vortex sheets.

6.1 Incompressible limit for fixed $\sigma > 0$

We now consider the incompressible limit problem for fixed surface tension coefficient $\sigma > 0$. We assume

1. (Surface tension is not neglected) $\sigma > 0$.
2. (Constraints for compressible initial data) The sequence of initial datum $(\psi_0^{\varepsilon,\sigma}, v_0^{\pm,\varepsilon,\sigma}, b_0^{\pm,\varepsilon,\sigma}, \rho_0^{\pm,\varepsilon,\sigma}, S_0^{\pm,\varepsilon,\sigma}) \in H^{9.5}(\Sigma)\times(H_*^8(\Omega^\pm))^4$ of (1.33) satisfy the constraints $\nabla^\varphi\cdot b_0^{\pm,\varepsilon,\sigma} = 0$ in $\Omega^\pm$, $b^{\pm,\varepsilon,\sigma}\cdot N^\sigma|_{t=0} = 0$ on $\Sigma\cup\Sigma^\pm$, the compatibility conditions (1.34) up to 7-th order, $|\psi_0^{\varepsilon,\sigma}| \le 1$ and $|[\![\bar{v}_0]\!]| > 0$.
3. (Convergence of initial data) $(\psi_0^{\varepsilon,\sigma}, v_0^{\pm,\varepsilon,\sigma}, b_0^{\pm,\varepsilon,\sigma}, \rho_0^{\pm,\varepsilon,\sigma}, S_0^{\pm,\varepsilon,\sigma}) \to (\xi_0^\sigma, w_0^{\pm,\sigma}, h_0^{\pm,\sigma}, \mathfrak{R}_0^{\pm,\sigma}, \mathfrak{S}_0^{\pm,\sigma})$ in $H^{5.5}(\Sigma)\times(H^4(\Omega^\pm))^4$.
4. (Constraints for incompressible initial data) The incompressible data $(\xi_0^\sigma, w_0^{\pm,\sigma}, h_0^{\pm,\sigma}, \mathfrak{R}_0^{\pm,\sigma}, \mathfrak{S}_0^{\pm,\sigma}) \in H^5(\Sigma)\times(H^4(\Omega^\pm))^4$ satisfies the constraints $\nabla^{\xi_0^\sigma}\cdot h_0^\pm = 0$ in $\Omega^\pm$, $h^{\pm,\sigma}\cdot N^\sigma|_{t=0} = 0$ on $\Sigma\cup\Sigma^\pm$, $|\xi_0^{\varepsilon,\sigma}| \le 2$ and $[\![\bar{w}_0]\!] > 0$.

Under these assumptions, we can prove that there exists a time $T_\sigma > 0$ that depends on σ and initial data and is independent of Mach number ε, such that the corresponding solutions to (1.33) converge to the solution to (6.1) as the Mach number $\varepsilon\to 0$

$$
\begin{aligned}
&(\psi^{\varepsilon,\sigma}, v^{\pm,\varepsilon,\sigma}, b^{\pm,\varepsilon,\sigma}, \rho^{\pm,\varepsilon,\sigma}, S^{\pm,\varepsilon,\sigma}) \to (\xi^\sigma, w^{\pm,\sigma}, h^{\pm,\sigma}, \mathfrak{R}^{\pm,\sigma}, \mathfrak{S}^{\pm,\sigma})\\
&\quad\text{strongly in } C([0,T_\sigma]; H^{5.5-\delta}_{\text{loc}}(\Sigma)\times(H^{4-\delta}_{\text{loc}}(\Omega^\pm))^4), \text{ and weakly-* in } L^\infty([0,T_\sigma]; H^{5.5}(\Sigma)\times(H^4(\Omega^\pm))^4).
\end{aligned}
$$

In fact, according to estimates obtained in Theorem 1.1, we already have the uniform-in-ε boundedness for $\psi^{\varepsilon,\sigma}$, $v^{\pm,\varepsilon,\sigma}$, $b^{\pm,\varepsilon,\sigma}$, $S^{\pm,\varepsilon,\sigma}$ as well as their first-order time derivatives. Thus, using Aubin-Lions compactness lemma, the above convergence is a straightforward result of uniform-in-ε estimates. Theorem 1.2 is proven.

6.2 Double limits in 3D: non-collinearity condition

We want to further study the limit process as both $\sigma, \varepsilon\to 0$. The difficulty in taking the zero-surface-tension limit is that we need to seek for the control of ψ (and its time derivatives) without σ-weight to avoid the dependence on $1/\sigma$. Let us first recall what quantities in the estimates of $E(t)$ depend on $1/\sigma$.

a. All the commutators $\mathfrak{C}(f)$, $\mathfrak{D}(f)$ and the modification terms $\mathcal{T}^\gamma\varphi\partial_3^\varphi f$ in the Alinhac good unknowns;

b. The commutators between $\nabla^{\varphi}\cdot$ (or $\nabla^{\varphi}\times$) and $\partial_t^k\mathcal{T}^\alpha$ in the div-curl analysis (Note that this commutator appears without time integral!);
c. The boundary terms RT, $\text{RT}^\pm$;
d. The cancellation structure $ZB^\pm + Z^\pm$;
e. The most difficult boundary integral VS.

Recall that, when controlling $E_{4+l}(t)$ $(0 \le l \le 4)$, we analyze the ε^{2l}-weighted $\overline{\partial}^{4-k-l}\partial_t^k\mathcal{T}^\alpha$-differentiated tangential estimates for $0 \le k \le 4-l,\ \langle\alpha\rangle = 2l,\ \alpha_3 = 0$ and $\partial^{3-k-l}\partial_t^k\mathcal{T}^\alpha$-differentiated divergence equations and vorticity equations for $0 \le k \le 3-l,\ \langle\alpha\rangle = 2l,\ \alpha_3 = 0$. To get rid of the dependence on $1/\sigma$, the above quantities (a)-(e) should be controlled in the following way

- Quantities (a), (d) can be controlled if we have the estimates of $|\varepsilon^{2l}\partial_t^k\mathcal{T}^\alpha\psi|_{4-k-l}$ (under time integral) for $0 \le k \le 4-l,\ \langle\alpha\rangle = 2l,\ \alpha_3 = 0$;
- Quantity (c) can be controlled if we have the estimates of $|\varepsilon^{2l}\partial_t^k\mathcal{T}^\alpha\psi|_{4.5-k-l}$ (under time integral) for $0 \le k \le 4-l,\ k+l \ge 1,\ \langle\alpha\rangle = 2l,\ \alpha_3 = 0$;
- Quantity (b) can be controlled if we have the estimates of $|\varepsilon^{2l}\partial_t^k\mathcal{T}^\alpha\psi|_{4-k-l}$ (NOT under time integral) for $0 \le k \le 4-l,\ \langle\alpha\rangle = 2l,\ \alpha_3 = 0$;
- Quantity (e) must be completely eliminated.

It is easy to see that merely invoking the kinematic boundary condition and using trace lemma does not solve any issue apart from the first bullet above. In order to seek for σ-independent estimates for ψ and its time derivatives, we require the stability condition (1.40) when the space dimension is 3, that is, for some $\delta_0 \in (0, \frac{1}{8})$,

$$0 < \delta_0 \le a^\pm \left|\bar{b}^\mp \times [\![\bar{v}]\!]\right| \le (1-\delta_0)\left|\bar{b}^+ \times \bar{b}^-\right| \quad \text{on } [0,T]\times\Sigma, \tag{6.2}$$

where we view $\bar{b}^\pm = (b_1^\pm, b_2^\pm, 0)^\top, [\![\bar{v}]\!] = ([\![v_1]\!], [\![v_2]\!], 0)^\top$ as vectors lying on the plane $\mathbb{T}^2\times\{x_3 = 0\} \subset \mathbb{R}^3$ to define the exterior product. The quantity $a^\pm$ is defined by

$$a^\pm := \sqrt{\rho^\pm\left(1 + \left(\frac{c_A^\pm}{c_s^\pm}\right)^2\right)}$$

and $c_A^\pm := |b^\pm|/\sqrt{\rho^\pm}$ represents the Alfvén speed, $c_s^\pm := \sqrt{\partial p^\pm/\partial\rho^\pm}$ represents the sound speed. This condition implies the following two important features:

1. Magnetics fields are not collinear on Σ, which allows us to gain 1/2-order regularity of the free interface.
2. Quantitative relations between $b^\pm$ and $[\![\bar{v}]\!]$ on Σ are given, which allows us to completely eliminate the problematic term VS.

We define the following energy functional $\widetilde{E}(t)$ for the compressible current-vortex sheet system (1.33) in order to prove uniform-in-(ε,σ) estimates.

$$\widetilde{E}(t) := \sum_{l=0}^{4}\widetilde{E}_{4+l}(t),\quad \widetilde{E}_{4+l}(t) := E_{4+l}(t) + \sum_{k=0}^{4+l}\left|\varepsilon^{2l}\partial_t^k\psi\right|^2_{4.5+l-k}, \tag{6.3}$$

where term added to $E_{4+l}(t)$ is exactly the enhanced regularity for the free interface contributed by the non-collinearity stability condition (6.2).

6.2.1 Enhanced regularity of the interface: non-collinearity of magnetic fields

Recall that the magnetic fields satisfy the constraint $b^\pm\cdot N = 0$ on Σ, that is, $b_3^\pm = b_1^\pm\overline{\partial}_1\psi + b_2^\pm\overline{\partial}_2\psi$. So, we can solve $\overline{\partial}\psi$ in terms of $b^\pm$ without any derivatives thanks to $\bar{b}^+\times\bar{b}^- \ne \mathbf{0}$. However, due to the anisotropy of the function spaces, we have to take derivatives on the constraint before we use trace lemma. We have the following estimates for the interface function ψ.

Lemma 6.1. For $s \ge 3$, one has

$$|\overline{\partial}\psi|^2_{s-\frac{1}{2}} \le P(\|b^\pm\|_{s-1,*,\pm}, \|b^\pm\|_{3,\pm}, |\overline{\partial}\psi|_{W^{1,\infty}})\left(\|\partial_3\langle\overline{\partial}\rangle^{s-2}b^\pm\|_{0,\pm}\|\langle\overline{\partial}\rangle^s b^\pm\|_{0,\pm} + |\overline{\partial}\psi|^2_{s-1} + \|\langle\overline{\partial}\rangle^s b^\pm\|^2_{0,\pm}\right) \tag{6.4}$$

Proof. Taking $\overline{\partial}^{s-\frac{1}{2}}$ in the constraint $b^\pm\cdot N=0$ for $s\geq 3$, we get

$$\begin{cases} b_1^+\overline{\partial}_1\langle\overline{\partial}\rangle^{s-\frac{1}{2}}\psi+b_2^+\overline{\partial}_2\langle\overline{\partial}\rangle^{s-\frac{1}{2}}\psi=f_b^+\\ b_1^-\overline{\partial}_1\langle\overline{\partial}\rangle^{s-\frac{1}{2}}\psi+b_2^-\overline{\partial}_2\langle\overline{\partial}\rangle^{s-\frac{1}{2}}\psi=f_b^- \end{cases} \xrightarrow{b^+\nparallel b^-} \begin{cases} \overline{\partial}_1\langle\overline{\partial}\rangle^{s-\frac{1}{2}}\psi=\dfrac{-b_2^+f_b^-+b_2^-f_b^+}{b_1^+b_2^--b_1^-b_2^+}\\ \\ \overline{\partial}_2\langle\overline{\partial}\rangle^{s-\frac{1}{2}}\psi=\dfrac{b_1^+f_b^--b_1^-f_b^+}{b_1^+b_2^--b_1^-b_2^+} \end{cases}$$

with $f_b^\pm:=\langle\overline{\partial}\rangle^{\frac{1}{2}}(\langle\overline{\partial}\rangle^{s-1}b^\pm\cdot N)+\langle\overline{\partial}\rangle^{\frac{1}{2}}([\langle\overline{\partial}\rangle^{s-1},b^\pm,N])+[\langle\overline{\partial}\rangle^{\frac{1}{2}},b^\pm\cdot]\langle\overline{\partial}\rangle^{s-1}N$. The $L^2(\Sigma)$ norms of the last two terms in $f_b^\pm$ can be directly controlled via Kato-Ponce type inequality (Lemma B.6)

$$\left|[\langle\overline{\partial}\rangle^{s-1},b^\pm,N]\right|_{\frac{1}{2}}\lesssim|b^\pm|_{s-\frac{3}{2}}|\overline{\partial}\psi|_{W^{1,\infty}}+|b^\pm|_{W^{1,\infty}}|\overline{\partial}\psi|_{s-\frac{3}{2}}\lesssim\|\partial_3\langle\overline{\partial}\rangle^{s-2}b^\pm\|_{0,\pm}^{\frac{1}{2}}\|\langle\overline{\partial}\rangle^{s-1}b^\pm\|_{0,\pm}^{\frac{1}{2}}|\overline{\partial}\psi|_{W^{1,\infty}}+\|b^\pm\|_{3,\pm}|\overline{\partial}\psi|_{s-\frac{3}{2}}$$

$$\left|[\langle\overline{\partial}\rangle^{\frac{1}{2}},b^\pm\cdot]\langle\overline{\partial}\rangle^{s-1}N\right|_0\lesssim|\langle\overline{\partial}\rangle^{\frac{1}{2}}b^\pm|_{L^\infty}|\overline{\partial}\psi|_{s-1}\lesssim\|b^\pm\|_{2.5,\pm}|\overline{\partial}\psi|_{s-1}$$

Then the regularity of the free interface is given by $b^\pm$

$$|\overline{\partial}\psi|_{s-\frac{1}{2}}\leq P(|\bar{b}^\pm|_{L^\infty})|f_b^\pm|_0\leq P(\|b^\pm\|_{3,\pm},|\overline{\partial}\psi|_{W^{1,\infty}})\left(\|b^\pm\|_{H_*^s(\Omega^\pm)}^{\frac{1}{2}}\|b^\pm\|_{H_*^{s-1}(\Omega^\pm)}^{\frac{1}{2}}+|\overline{\partial}\psi|_{s-1}+\left|\langle\overline{\partial}\rangle^{\frac{1}{2}}\left(\overline{\partial}^{s-1}b^\pm\cdot N\right)\right|_0\right).$$

To control the boundary norm of $\langle\overline{\partial}\rangle^{s-\frac{1}{2}}b^\pm\cdot N$, we again convert it to an interior integral and use the divergence constraint $0=\nabla^\varphi\cdot b^\pm=\overline{\nabla}\cdot\bar{b}^\pm+\partial_3^\varphi b^\pm\cdot\mathbf{N}$ in $\Omega^\pm$.

$$\begin{aligned}\left|\langle\overline{\partial}\rangle^{\frac{1}{2}}\left(\langle\overline{\partial}\rangle^{s-1}b^\pm\cdot N\right)\right|_0^2=&\mp2\int_{\Omega^\pm}\langle\overline{\partial}\rangle^{\frac{1}{2}}\partial_3\left(\langle\overline{\partial}\rangle^{s-1}b^\pm\cdot N\right)\langle\overline{\partial}\rangle^{\frac{1}{2}}\left(\langle\overline{\partial}\rangle^{s-1}b^\pm\cdot N\right)\mathrm{d}x\\ \overset{\langle\overline{\partial}\rangle^{\frac{1}{2}}}{=\!=}&\mp2\int_{\Omega^\pm}\partial_3\left(\langle\overline{\partial}\rangle^{s-1}b^\pm\cdot\mathbf{N}\right)\langle\overline{\partial}\rangle\left(\langle\overline{\partial}\rangle^{s-\frac{1}{2}}b^\pm\cdot\mathbf{N}\right)\mathrm{d}x\\ =&\mp2\int_{\Omega^\pm}\langle\overline{\partial}\rangle^{s-1}(\partial_3b^\pm\cdot\mathbf{N})\langle\overline{\partial}\rangle\left(\langle\overline{\partial}\rangle^{s-\frac{1}{2}}b^\pm\cdot\mathbf{N}\right)\mathrm{d}x\\ &\mp2\int_{\Omega^\pm}\left(\langle\overline{\partial}\rangle^{s-1}b^\pm\cdot\partial_3\mathbf{N}-\left[\langle\overline{\partial}\rangle^{s-1},\mathbf{N}\cdot\right]\partial_3b^\pm\right)\langle\overline{\partial}\rangle\left(\langle\overline{\partial}\rangle^{s-\frac{1}{2}}b^\pm\cdot\mathbf{N}\right)\mathrm{d}x\\ =&\pm2\int_{\Omega^\pm}\langle\overline{\partial}\rangle^{s-1}\left(\partial_3\varphi(\overline{\nabla}\cdot\bar{b}^\pm)\right)\langle\overline{\partial}\rangle\left(\langle\overline{\partial}\rangle^{s-\frac{1}{2}}b^\pm\cdot\mathbf{N}\right)\mathrm{d}x\\ &\mp2\int_{\Omega^\pm}\left(\langle\overline{\partial}\rangle^{s-1}b^\pm\cdot\partial_3\mathbf{N}-\left[\langle\overline{\partial}\rangle^{s-1},\mathbf{N}\cdot\right]\partial_3b^\pm\right)\langle\overline{\partial}\rangle\left(\langle\overline{\partial}\rangle^{s-\frac{1}{2}}b^\pm\cdot\mathbf{N}\right)\mathrm{d}x\\ \lesssim&P(|\overline{\partial}\psi|_{W^{1,\infty}})\left(\|\langle\overline{\partial}\rangle^sb^\pm\|_{0,\pm}+\|\langle\overline{\partial}\rangle^{s-1}b^\pm\|_{0,\pm}+\|\langle\overline{\partial}\rangle^{s-2}\partial_3b^\pm\|_{0,\pm}\right)\|\langle\overline{\partial}\rangle^sb^\pm\|_{0,\pm}.\end{aligned}$$

□

Lemma 6.1 shows that the $H^{s+\frac{1}{2}}(\Sigma)$ norm of the free interface ψ can be converted to lower-order terms and $H_*^s(\Omega^\pm)$ norms of $b^\pm$. Thus, the "non-collinearity" of b^+ and b^- on Σ brings the gain of $1/2$-order regularity for the interface. Given $l\in\{0,1,2,3,4\}$, the definition of $E_{4+l}(t)$ suggests that $\varepsilon^{2l}\langle\overline{\partial}\rangle^{2l}b^\pm\in H^{4-l}(\Omega^\pm)$, thus letting $s=4+l$ in Lemma 6.1, we can get

$$\left|\varepsilon^{2l}\overline{\partial}\psi\right|_{3.5+l}^2\leq P(\|b^\pm\|_{3,\pm},|\overline{\partial}\psi|_{W^{1,\infty}})\left(\|\varepsilon^{2l}b^\pm\|_{H_*^{4+l}(\Omega^\pm)}\|\varepsilon^{2l}b^\pm\|_{H_*^{3+l}(\Omega^\pm)}+|\varepsilon^{2l}\overline{\partial}\psi|_{3+l}^2+\|\varepsilon^{2l}b^\pm\|_{H_*^{4+l}(\Omega^\pm)}^2\right).$$

Similarly, we can show the enhanced regularity for the time derivatives of ψ after replacing $\langle\overline{\partial}\rangle^{s-\frac{1}{2}}$ by $\langle\overline{\partial}\rangle^{s-k-\frac{1}{2}}\partial_t^k$

Lemma 6.2. For $3\leq s\in\mathbb{N}^*$ and $1\leq k\leq s-1,\ k\in\mathbb{N}^*$, one has

$$\begin{aligned}|\overline{\partial}\partial_t^k\psi|_{s-k-\frac{1}{2}}^2\leq P\left(\|b^\pm\|_{s-1,*,\pm},|\overline{\nabla}\psi|_{W^{1,\infty}}\right)\Bigg(&\sum_{j=1}^{k-1}|\partial_t^j\overline{\partial}\psi|_{s-j-\frac{3}{2}}^2+|\overline{\partial}^{s-k}\partial_t^k\psi|_0^2\\ &+\|\langle\overline{\partial}\rangle^{s-k}\partial_t^kb^\pm\|_{0,\pm}\|\partial_3\langle\overline{\partial}\rangle^{s-k-2}\partial_t^kb^\pm\|_{0,\pm}+\|\langle\overline{\partial}\rangle^{s-k}\partial_t^kb^\pm\|_{0,\pm}^2\Bigg),\end{aligned}\tag{6.5}$$

where the number of time derivatives in $\|b^\pm\|_{s,*,\pm}$ appearing on the right side does not exceed k. The term $\partial_3\langle\overline{\partial}\rangle^{s-k-2}\partial_t^k b^\pm$ does not appear when $s-k=1$.

In the study of $E_{4+l}(t)$ for $0\le l\le 4$, we have $\varepsilon^{2l}\partial_t^k\mathcal{T}^\alpha b\in L^2(\Omega^\pm)$ for $0\le k\le l$, $\langle\alpha\rangle=2l$, $\alpha_3=0$. It should also be noted that there is no loss of Mach number in the estimates of $\partial_t^k\psi$ because the number of time derivatives appearing on the right side of (6.5) does not exceed that on the left side. Thus, the above estimates directly help us to control the quantities mentioned in (a), (c), (d).

Apart from the term VS in (e), we still need to control the commutators mentioned in (d), namely $\left\|\varepsilon^{2l}[\partial_t^k\mathcal{T}^\alpha,\nabla^\varphi\cdot]f\right\|_{3-k-l}^2$ and $\left\|\varepsilon^{2l}[\partial_t^k\mathcal{T}^\alpha,\nabla^\varphi\times]f\right\|_{3-k-l}^2$ for $f=v,b$ and $0\le l\le 4$, $k+l\ge 1$, $\langle\alpha\rangle=2l$, $\alpha_3=0$, in which the highest order terms have the form $\varepsilon^{2l}(\partial_3\varphi)^{-1}(\overline{\partial}\partial_t^k\mathcal{T}^\alpha\varphi)(\partial_3 f)$ whose estimate requires the bound for $|\varepsilon^{2l}\overline{\partial}\partial_t^k\mathcal{T}^\alpha\psi|_{3-k-l}^2$. We can also assume $\alpha_4=0$ because φ has C^∞-regularity in x_3-direction. Such terms appear without time integral, so we have control them by $P(\widetilde{E}(0))+P(\widetilde{E}(t))\int_0^t P(\widetilde{E}(\tau))\,\mathrm{d}\tau$. Letting $s=3.5+l$ in Lemma 6.1 for $0\le l\le 4$, using interpolation and Young's inequality, we get

$$
\begin{aligned}
|\overline{\partial}\psi|_{3+l}^2 &\le P(\|b^\pm\|_{2+l,*,\pm},\|b^\pm\|_{3,\pm},|\overline{\partial}\psi|_{W^{1,\infty}})\left(\|\partial_3\langle\overline{\partial}\rangle^{1.5+l}b^\pm\|_{0,\pm}\|\langle\overline{\partial}\rangle^{3.5+l}b^\pm\|_{0,\pm}+|\overline{\partial}\psi|_{2.5+l}^2+\|\langle\overline{\partial}\rangle^{3.5+l}b^\pm\|_{0,\pm}^2\right)\\
&\lesssim P(\|b^\pm\|_{2+l,*,\pm},\|b^\pm\|_{3,\pm},|\overline{\partial}\psi|_{W^{1,\infty}})\\
&\quad\left(\|\partial_3\langle\overline{\partial}\rangle^{1+l}b^\pm\|_{0,\pm}^{\frac12}\|\partial_3\langle\overline{\partial}\rangle^{2+l}b^\pm\|_{0,\pm}^{\frac12}\|\langle\overline{\partial}\rangle^{3+l}b^\pm\|_{0,\pm}^{\frac12}\|\langle\overline{\partial}\rangle^{4+l}b^\pm\|_{0,\pm}^{\frac12}+|\overline{\partial}\psi|_{2.5+l}^2+\|\langle\overline{\partial}\rangle^{3+l}b^\pm\|_{0,\pm}\|\langle\overline{\partial}\rangle^{4+l}b^\pm\|_{0,\pm}\right)\\
&\lesssim\delta\left(\|\partial_3\langle\overline{\partial}\rangle^{2+l}b^\pm\|_{0,\pm}^2+\|\langle\overline{\partial}\rangle^{4+l}b^\pm\|_{0,\pm}^2\right)\\
&\quad+P(\|b^\pm\|_{2+l,*,\pm},\|b^\pm\|_{3,\pm},|\overline{\partial}\psi|_{W^{1,\infty}},\delta^{-1})\left(\|\langle\overline{\partial}\rangle^{3+l}b^\pm\|_{0,\pm}^2+\|\langle\overline{\partial}\rangle^{3+l}b^\pm\|_{0,\pm}+|\overline{\partial}\psi|_{2.5+l}^2\right)\\
&\lesssim\delta\|b^\pm\|_{4+l,*,\pm}^2+P(\|b_0^\pm\|_{3+l,*,\pm},|\overline{\partial}\psi_0|_{2.5+l})+\int_0^t P\left(\|\partial_t b^\pm(\tau)\|_{3+l,*,\pm},|\partial_t\overline{\partial}\psi(\tau)|_{2.5+l}\right)\mathrm{d}\tau.
\end{aligned}
$$

Similarly, we can show that for $0\le l\le 4$, $k+l\ge 1$, $\langle\alpha\rangle=\alpha_0+\alpha_1+\alpha_2=2l$ $(\alpha_3=\alpha_4=0)$

$$
\begin{aligned}
\left|\overline{\partial}\partial_t^k\mathcal{T}^\alpha\psi\right|_{3-k-l}^2=\left|\overline{\partial}^{1+2l-\alpha_0}\partial_t^{k+\alpha_0}\psi\right|_{3-k-l}^2&\lesssim\delta\|b^\pm\|_{4+l,*,\pm}^2+\sum_{j=0}^k P\left(\|b^\pm\|_{3+l,*,\pm},|\overline{\partial}\partial_t^{j+\alpha_0}\psi|_{2.5+l-j-\alpha_0}\right)\Big|_{t=0}\\
&\quad+\int_0^t P\left(\|\partial_t b^\pm(\tau)\|_{3+l,*,\pm},|\overline{\partial}\partial_t^{j+\alpha_0+1}\psi(\tau)|_{2.5+l-j-\alpha_0}\right)\mathrm{d}\tau.
\end{aligned}
$$

Since there is no loss of weights of Mach number when applying Lemma 6.1-Lemma 6.2 to the estimates of compressible current-vortex sheet system (1.33), we can conclude the enhanced regularity, which is uniform in (ε,σ), of the free interface by the following proposition.

Proposition 6.3. For $l\in\{0,1,2,3,4\}$ and $k\le 4+l$, $k\in\mathbb{N}$, we have

1. When $0\le k\le 3+l$:

$$
\left|\varepsilon^{2l}\overline{\partial}\partial_t^k\psi(t)\right|_{3.5+l-k}^2\le P\left(\sum_{j=0}^{(l-1)_+}\widetilde{E}_{4+j}(0)\right)+P\left(\sum_{j=0}^{(l-1)_+}\widetilde{E}_{4+j}(t)\right)\left(\int_0^t P\left(\sum_{j=0}^{l}\widetilde{E}_{4+j}(\tau)\right)\mathrm{d}\tau+\left\|b^\pm(t)\right\|_{4+l,*,\pm}^2\right).\tag{6.6}
$$

2. When $k=4+l$:

$$
\begin{aligned}
\left|\varepsilon^{2l}\partial_t^{4+l}\psi(t)\right|_{0.5}^2&\le P(|\overline{\nabla}\psi|_{W^{1,\infty}})\left(\left\|\varepsilon^{2l}\langle\overline{\partial}\rangle\partial_t^{3+l}v^\pm\right\|_{0,\pm}^2+\left\|\varepsilon^{2l+2}\langle\overline{\partial}\rangle\partial_t^{3+l}D_t^{\varphi\pm}p^\pm\right\|_{0,\pm}^2\right)+|\overline{v}|_{L^\infty}\left|\varepsilon^{2l}\overline{\partial}\partial_t^{3+l}\psi(t)\right|_{0.5}\\
&\quad+P\left(\sum_{j=0}^{(l-1)_+}\widetilde{E}_{4+j}(0)\right)+\int_0^t P\left(\sum_{j=0}^{l}\widetilde{E}_{4+j}(\tau)\right)\mathrm{d}\tau.
\end{aligned}\tag{6.7}
$$

Proof. When $k\le 3+l$, the inequality (6.6) is a direct consequence of Lemma 6.1 and Lemma 6.2. Indeed, we just need to write the ψ term to be $P(\widetilde{E}(0))+P(\widetilde{E}(t))\int_0^t P(\widetilde{E}(\tau))\,\mathrm{d}\tau$. This can be done by applying again

6.1 and Lemma 6.2 to the ψ term appearing on the right side of (6.4) and (6.5) by replacing $s = 4 + l - k$ with $s = 3.5 + l - k$. When $k = 4 + l$, we just differentiate the kinematic boundary condition $\partial_t \psi = v^\pm \cdot N$ to get

$$\partial_t^{4+l}\psi = \partial_t^{3+l} v^\pm \cdot N + \bar{v}^\pm \cdot \overline{\nabla}\partial_t^{3+l}\psi + [\partial_t^{3+l}, v^\pm \cdot, N],$$

where the second term contributes to the second term on the right side of (6.7) and the last term contributes to the second line of (6.7). For $\partial_t^{3+l} v^\pm \cdot N$, we again use Gauss-Green formula to covert its boundary norm to an interior integral and use $\partial_3^\varphi v^\pm \cdot \mathbf{N} = \nabla^\varphi \cdot v^\pm - \overline{\nabla} \cdot \bar{v}^\pm = -\varepsilon^2 D_t^{\varphi\pm} p^\pm - \overline{\nabla} \cdot \bar{v}^\pm$ to replace the normal derivative by tangential derivative. □

6.2.2 Elimination of VS term: Friedrichs secondary symmetrization

With the new energy functional (6.3), quantities (a)-(d) mentioned at the beginning of Section 6.2 are all controlled by $P(\widetilde{E}(0)) + P(\widetilde{E}(t)) \int_0^t P(\widetilde{E}(\tau))\,\mathrm{d}\tau$ thanks to Proposition 6.3. The terms that appear without time integral on the right side of (6.6) and (6.7) can also be converted to the form $P(\widetilde{E}(0)) + P(\widetilde{E}(t)) \int_0^t P(\widetilde{E}(\tau))\,\mathrm{d}\tau$ via div-curl analysis or tangential estimates. In other words, we have reached the following energy inequality for $\widetilde{E}(t)$

$$\widetilde{E}(t) \lesssim \delta E(t) + P(\widetilde{E}(0)) + P(\widetilde{E}(t)) \int_0^t P(\widetilde{E}(\tau))\,\mathrm{d}\tau + \text{ VS}, \tag{6.8}$$

so it remains to control or eliminate the term VS arising from the estimates of $E(t)$, such that we can close the energy estimates for $\widetilde{E}(t)$ and also get rid of the dependence on $1/\sigma$.

Motivation for Friedrichs secondary symmetrization

The regularity for the free interface needed in the control of VS is higher than the one we obtain in Proposition 6.3. So, we alternatively try to completely eliminate the term VS by utilizing the jump of tangential magnetic field. Recall that the term VS is generated due to the discontinuity in tangential velocity

$$\text{VS} = \int_\Sigma \mathcal{T}^\gamma q^- \,(\llbracket \bar{v} \rrbracket \cdot \overline{\nabla})\mathcal{T}^\gamma \psi \,\mathrm{d}x',$$

in which we may try to insert a term $\llbracket \mu \bar{b} \rrbracket$ into $\llbracket \bar{v} \rrbracket$ such that $\llbracket \bar{v} - \bar{\mu}\bar{b} \rrbracket = \mathbf{0}$ on Σ for some function $\bar{\mu}^\pm$. Such functions $\bar{\mu}^\pm$ do exist and are unique thanks to the non-collinearity $\bar{b}^+ \nparallel \bar{b}^-$ on Σ:

$$\begin{cases} \llbracket v_1 \rrbracket = \bar{\mu}^+ b_1^+ - \bar{\mu}^- b_1^- \\ \llbracket v_2 \rrbracket = \bar{\mu}^+ b_2^+ - \bar{\mu}^- b_2^- \end{cases} \xRightarrow{b^+ \nparallel b^-} \bar{\mu}^\pm = \frac{b_1^\mp \llbracket v_2 \rrbracket - b_2^\mp \llbracket v_1 \rrbracket}{b_1^+ b_2^- - b_1^- b_2^+} = \frac{(\bar{b}^\mp \times \llbracket \bar{v} \rrbracket)_3}{(\bar{b}^+ \times \bar{b}^-)_3}. \tag{6.9}$$

Next, a natural question is how to produce such $\llbracket \bar{\mu}\bar{b} \rrbracket$-terms in the tangential estimates. Recall that the discontinuity term $(\llbracket \bar{v} \rrbracket \cdot \overline{\nabla})\mathcal{T}^\gamma \psi$ is produced by taking substraction bewteen the equations of $\mathbf{V}^{\gamma,\pm} \cdot N$ which originates from $\frac{1}{2}\int_{\Omega^\pm} \rho^\pm |\mathbf{V}^{\gamma,\pm}|^2 \,\mathrm{d}\mathcal{V}_t$. That is, we need to replace the variable $v^\pm$ by $v^\pm - \bar{\mu}^\pm b^\pm$ in the momentum equation in order to create the elimination $\llbracket \bar{v} - \bar{\mu}\bar{b} \rrbracket = \mathbf{0}$. However, such replacement in the momentum equation will make the compressible ideal MHD system (1.33) no longer symmetric, which will further lead to the failure of L^2 energy conservation. Hence, we must re-symmetrize the hyperbolic system after replacing v by $v - \mu b$ for suitable function μ.

The technique we use is the so-called Friedrichs secondary symmetrization [30]. For compressible ideal MHD system, the symmetrizer was explicitly calculated in Trakhinin [79]. Let $\mu(t,x) = \bar{\mu}(t,x')\eta(x_3)$ where $\eta(x_3) \in C_c^\infty(\mathbb{R})$ is a smooth, non-negative, even function satisfying $\eta(0) = 1$ and $\eta(x_3) = 0$ when $|x_3| > \delta_1$ for some sufficiently small constant $\delta_1 > 0$. Inserting this η is to localise the function $\mu^\pm$ near the interface Σ. The new system takes the form

$$\begin{cases} \rho D_t^\varphi v - (b \cdot \nabla^\varphi) b + \nabla^\varphi (p + \frac{1}{2}|b|^2) - \mu\rho \left(D_t^\varphi b - (b \cdot \nabla^\varphi) v + b(\nabla^\varphi \cdot v) \right) = 0, \\ \mathcal{F}_p D_t^\varphi p + \nabla^\varphi \cdot v + \mu \mathcal{F}_p \left(\rho D_t^\varphi v \cdot b + b \cdot \nabla^\varphi p \right) = 0 \\ D_t^\varphi b - (b \cdot \nabla^\varphi) v + b(\nabla^\varphi \cdot v) - \mu \left(\rho D_t^\varphi v - (b \cdot \nabla^\varphi) b + \nabla^\varphi (p + \frac{1}{2}|b|^2) \right) = 0, \end{cases} \tag{6.10}$$

where these equations are obtained by

$$\begin{aligned}\text{new momentum equation} &= \text{momentum equation} - \mu\rho(\text{evolution equation of } b),\\ \text{new continuity equation} &= \text{continuity equation} + \mu\mathcal{F}_p(\text{momentum equation})\cdot b,\\ \text{new evolution equation of } b &= \text{momentum equation} - \mu(\text{momentum equation}).\end{aligned}$$

Note that in the second equation we use the fact that $\left(\nabla^\varphi(1/2|b|^2)-(b\cdot\nabla^\varphi)b\right)\cdot b = (b\times(\nabla^\varphi\times b))\cdot b = 0$. Now we need to re-consider the tangential estimates in order to avoid the appearance of the term VS. Given a tangential derivative $\mathcal{T}^\gamma$ ($\gamma_3=0$), the $\mathcal{T}^\gamma$-differentiated current-vortex sheet system is reformulated in the corresponding Alinhac good unknowns $(\mathbf{V}^\pm,\mathbf{B}^\pm,\mathbf{P}^\pm,\mathbf{Q}^\pm,\mathbf{S}^\pm)$ as follows

$$\rho^\pm D_t^{\varphi\pm}\mathbf{V}^\pm-(b^\pm\cdot\nabla^\varphi)\mathbf{B}^\pm+\nabla^\varphi\mathbf{Q}^\pm-\mu^\pm\rho^\pm(D_t^{\varphi\pm}\mathbf{B}^\pm-(b\cdot\nabla^\varphi)\mathbf{V}^\pm+b^\pm(\nabla^\varphi\cdot\mathbf{V}^\pm))=\mathcal{R}_v^{\pm,\mu}-\mathfrak{C}(q^\pm)+\mu^\pm\rho^\pm b^\pm\mathfrak{C}_i(v_i^\pm), \tag{6.11}$$

$$\mathcal{F}_p^\pm D_t^{\varphi\pm}\mathbf{P}^\pm+\nabla^\varphi\cdot\mathbf{V}^\pm+\mu^\pm\mathcal{F}_p^\pm(\rho^\pm D_t^{\varphi\pm}\mathbf{V}^\pm\cdot b^\pm+b^\pm\cdot\nabla^\varphi\mathbf{P}^\pm)=\mathcal{R}_p^{\pm,\mu}-\mathfrak{C}_i(v_i^\pm)-\mu^\pm\mathcal{F}_p^\pm b^\pm\cdot\mathfrak{C}(q^\pm), \tag{6.12}$$

$$D_t^{\varphi\pm}\mathbf{B}^\pm-(b^\pm\cdot\nabla^\varphi)\mathbf{V}^\pm+b^\pm(\nabla^\varphi\cdot\mathbf{V}^\pm)-\mu(\rho^\pm D_t^{\varphi\pm}\mathbf{V}^\pm-(b^\pm\cdot\nabla^\varphi)\mathbf{B}^\pm+\nabla^\varphi\mathbf{Q}^\pm)=\mathcal{R}_b^{\pm,\mu}-b^\pm\mathfrak{C}_i(v_i^\pm)+\mu^\pm\mathfrak{C}(q^\pm) \tag{6.13}$$

$$D_t^{\varphi\pm}\mathbf{S}^\pm=\mathfrak{D}(S^\pm), \tag{6.14}$$

where $\mathcal{R}_v^\mu,\mathcal{R}_p^\mu,\mathcal{R}_b^\mu$ terms consist of the following commutators

$$\mathcal{R}_v^{\pm,\mu}:=\mathcal{R}_v^\pm-\mu^\pm\rho^\pm\mathcal{R}_b^\pm+[\mathcal{T}^\gamma,\mu^\pm\rho^\pm]\left(D_t^{\varphi\pm}b^\pm-(b^\pm\cdot\nabla^\varphi)v^\pm+b^\pm(\nabla^\varphi\cdot v^\pm)\right) \tag{6.15}$$

$$\mathcal{R}_p^{\pm,\mu}:=\mathcal{R}_p^\pm+\mu^\pm\mathcal{F}_p^\pm b^\pm\cdot\mathcal{R}_v^\pm-[\mathcal{T}^\gamma,\mu^\pm\mathcal{F}_p^\pm]\left(\rho^\pm D_t^{\varphi\pm}v^\pm\cdot b^\pm-(b^\pm\cdot\nabla^\varphi)p^\pm\right) \tag{6.16}$$

$$\mathcal{R}_b^{\pm,\mu}:=\mathcal{R}_b^\pm-\mu^\pm\rho^\pm\mathcal{R}_v^\pm+[\mathcal{T}^\gamma,\mu^\pm]\left(D_t^{\varphi\pm}v^\pm-(b^\pm\cdot\nabla^\varphi)b^\pm+\nabla^\varphi q^\pm\right), \tag{6.17}$$

with $\mathcal{R}_v^\pm,\mathcal{R}_b^\pm,\mathcal{R}_b^\pm$ defined in (3.11)-(3.13). The boundary conditions on the interface Σ are

$$[\![\mathbf{Q}]\!]=\sigma\mathcal{T}^\gamma\mathcal{H}(\psi)-[\![\partial_3 q]\!]\mathcal{T}^\gamma\psi\quad\text{on }[0,T]\times\Sigma, \tag{6.18}$$

$$\mathbf{V}^\pm\cdot N=\partial_t\mathcal{T}^\gamma\psi+\bar{v}^\pm\cdot\overline{\nabla}\mathcal{T}^\gamma\psi-\mathcal{W}_v^\pm\quad\text{on }[0,T]\times\Sigma, \tag{6.19}$$

$$b^\pm\cdot N=0\Rightarrow\mathbf{B}^\pm\cdot N=\bar{b}^\pm\cdot\overline{\nabla}\mathcal{T}^\gamma\psi-\mathcal{W}_b^\pm\quad\text{on }[0,T]\times\Sigma, \tag{6.20}$$

and the boundary term $\mathcal{W}^{\gamma,\pm}$ is

$$\mathcal{W}_f^\pm:=(\partial_3 f^\pm\cdot N)\mathcal{T}^\gamma\psi+[\mathcal{T}^\gamma,N_i,f_i^\pm],\quad f=v,b. \tag{6.21}$$

Note that $\omega(x_3)|_\Sigma=0$, so all boundary conditions are vanishing when $\gamma_4>0$.

Analysis in the interior

Recall that the term VS originates from the tangential estimates. After doing Friedrichs secondary symmtrisation, we shall consider the tangential estimates for the following functional

$$\mathbf{G}^{\pm,\mu}(t):=\frac{1}{2}\int_{\Omega^\pm}\rho^\pm|\mathbf{V}^\pm|^2+|\mathbf{B}^\pm|^2+\mathcal{F}_p^\pm(\mathbf{P}^\pm)^2-2\mu^\pm\rho^\pm\mathbf{V}^\pm\cdot\mathbf{B}^\pm+2\mu^\pm\rho^\pm\mathcal{F}_p^\pm\mathbf{P}^\pm(b^\pm\cdot\mathbf{V}^\pm)\,\mathrm{d}\mathcal{V}_t \tag{6.22}$$

instead of the one used in Section 3

$$\mathbf{G}^\pm(t):=\frac{1}{2}\int_{\Omega^\pm}\rho^\pm|\mathbf{V}^\pm|^2+|\mathbf{B}^\pm|^2+\mathcal{F}_p^\pm(\mathbf{P}^\pm)^2\,\mathrm{d}\mathcal{V}_t.$$

Using Reynolds's transport theorem, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{G}^{\pm,\mu}(t)&=\int_{\Omega^\pm}\rho^\pm D_t^{\varphi\pm}\mathbf{V}^\pm\cdot(\mathbf{V}^\pm-\mu^\pm\mathbf{B}^\pm+\mu^\pm\mathcal{F}_p^\pm\mathbf{P}^\pm b^\pm)\,\mathrm{d}\mathcal{V}_t+\int_{\Omega^\pm}D_t^{\varphi\pm}\mathbf{B}^\pm\cdot(\mathbf{B}^\pm-\mu^\pm\rho^\pm\mathbf{V}^\pm)\,\mathrm{d}\mathcal{V}_t\\&\quad+\int_{\Omega^\pm}\mathcal{F}_p^\pm D_t^{\varphi\pm}\mathbf{P}^\pm(\mathbf{P}^\pm+\mu^\pm\rho^\pm b^\pm\cdot\mathbf{V}^\pm)\,\mathrm{d}\mathcal{V}_t+R_{1,\mu}^\pm\\&=:G_1^{\pm,\mu}+G_2^{\pm,\mu}+G_3^{\pm,\mu}+R_1^{\pm,\mu}\end{aligned} \tag{6.23}$$

where

$$R_1^{\pm,\mu} := \frac{1}{2}\int_{\Omega^\pm} (\nabla^\varphi\cdot v^\pm)\Big(|\mathbf{B}^\pm|^2 + (\sqrt{\mathcal{F}_p^\pm}\mathbf{P}^\pm)^2\Big)\,\mathrm{d}\mathcal{V}_t + \int_{\Omega^\pm} \mu^\pm\mathcal{F}_p^\pm\mathbf{P}^\pm(D_t^{\varphi\pm}(\rho^\pm b^\pm)\cdot\mathbf{V}^\pm)\,\mathrm{d}\mathcal{V}_t$$
$$-\int_{\Omega^\pm} D_t^{\varphi\pm}\mu\Big(\rho^\pm\mathbf{V}^\pm\cdot\mathbf{B}^\pm + \rho^\pm\mathcal{F}_p^\pm\mathbf{P}^\pm(b^\pm\cdot\mathbf{V}^\pm)\Big)\,\mathrm{d}\mathcal{V}_t. \tag{6.24}$$

Invoking the evolution equations of good unknowns (3.11)-(3.13) (NOT (6.11)-(6.13)!), we get

$$G_1^{\pm,\mu} = \int_{\Omega^\pm} (b^\pm\cdot\nabla^\varphi)\mathbf{B}^\pm\cdot(\mathbf{V}^\pm - \mu^\pm\mathbf{B}^\pm)\,\mathrm{d}\mathcal{V}_t - \int_{\Omega^\pm}\nabla^\varphi\mathbf{Q}^\pm\cdot(\mathbf{V}^\pm-\mu^\pm\mathbf{B}^\pm)\,\mathrm{d}\mathcal{V}_t$$
$$+\int_{\Omega^\pm}\mathcal{F}_p^\pm\mu^\pm(\rho^\pm D_t^{\varphi\pm}\mathbf{V}^\pm\cdot b^\pm)\mathbf{P}^\pm\,\mathrm{d}\mathcal{V}_t - \int_{\Omega^\pm}\mathfrak{C}(q^\pm)\cdot(\mathbf{V}^\pm-\mu^\pm\mathbf{B}^\pm)\,\mathrm{d}\mathcal{V}_t + \int_{\Omega^\pm}\mathcal{R}_v^\pm\cdot(\mathbf{V}^\pm-\mu^\pm\mathbf{B}^\pm)\,\mathrm{d}\mathcal{V}_t$$
$$=: G_{11}^{\pm,\mu} + G_{12}^{\pm,\mu} + G_{13}^{\pm,\mu} + R_2^{\pm,\mu} + R_3^{\pm,\mu}. \tag{6.25}$$

In $G_{11}^{\pm,\mu}$ and $G_{12}^{\pm,\mu}$, we integrate by parts to get

$$G_{11}^{\pm,\mu} = -\int_{\Omega^\pm}\mathbf{B}^\pm\cdot(b^\pm\cdot\nabla^\varphi)\mathbf{V}^\pm\,\mathrm{d}\mathcal{V}_t - \frac{1}{2}\int_{\Omega^\pm}\nabla^\varphi\cdot(\mu^\pm b^\pm)|\mathbf{B}^\pm|^2\,\mathrm{d}\mathcal{V}_t$$
$$=: G_{111}^{\pm,\mu} + R_4^{\pm,\mu} \tag{6.26}$$

and use $\nabla^\varphi\cdot\mathbf{B}^\pm = -\mathfrak{C}_i(b_i)$ to get

$$G_{12}^{\pm,\mu} = \pm\int_\Sigma \mathbf{Q}^\pm(\mathbf{V}^\pm-\mu^\pm\mathbf{B}^\pm)\cdot N\,\mathrm{d}x' + \int_{\Omega^\pm}\mathbf{Q}^\pm(\nabla^\varphi\cdot\mathbf{V}^\pm)\,\mathrm{d}\mathcal{V}_t + \int_{\Omega^\pm}\mu^\pm\mathbf{Q}^\pm\,\mathfrak{C}_i(b_i^\pm)\,\mathrm{d}\mathcal{V}_t$$
$$=: G_0^{\pm,\mu} + G_{121}^{\pm,\mu} + G_{122}^{\pm,\mu}. \tag{6.27}$$

In $G_{13}^{\pm,\mu}$, we notice that

$$(-(b^\pm\cdot\nabla^\varphi)\mathbf{B}^\pm + \nabla^\varphi\mathbf{B}_j^\pm b_j^\pm)\cdot b^\pm = -b_j^\pm(\partial_j^\varphi\mathbf{B}_i^\pm)b_i^\pm + (\partial_i^\varphi\mathbf{B}_j^\pm)b_j^\pm b_i^\pm = 0,$$

so it becomes the following controllable quantities by using symmetry

$$G_{13}^{\pm,\mu} = -\int_{\Omega^\pm}\mathcal{F}_p^\pm\mu^\pm\rho^\pm((b^\pm\cdot\nabla^\varphi)\mathbf{P}^\pm)\,\mathbf{P}^\pm\,\mathrm{d}\mathcal{V}_t + \int_{\Omega^\pm}\mathcal{F}_p^\pm\mu^\pm(\mathcal{R}_v^\pm - \mathfrak{C}(q^\pm))\cdot b^\pm\mathbf{P}^\pm\,\mathrm{d}\mathcal{V}_t$$
$$= -\frac{1}{2}\int_{\Omega^\pm}\nabla^\varphi\cdot(\mathcal{F}_p^\pm\mu^\pm\rho^\pm b^\pm)(\mathbf{P}^\pm)^2\,\mathrm{d}\mathcal{V}_t + \int_{\Omega^\pm}\mathcal{F}_p^\pm\mu^\pm(\mathcal{R}_v^\pm-\mathfrak{C}(q^\pm))\cdot b^\pm\mathbf{P}^\pm\,\mathrm{d}\mathcal{V}_t =: R_5^{\pm,\mu} + R_6^{\pm,\mu}. \tag{6.28}$$

Note that the terms $G_{111}^{\pm,\mu}$ and $G_{122}^{\pm,\mu}$ already appear in the previous analysis for (1.33) (cf. Section 3), so we no longer need to put extra effort on it. Next we analyze $G_2^{\pm,\mu}\sim G_3^{\pm,\mu}$. Invoking (3.12) and (3.13), we get

$$G_2^{\pm,\mu} = \underbrace{\int_{\Omega^\pm}((b^\pm\cdot\nabla^\varphi)\mathbf{V}^\pm)\cdot\mathbf{B}^\pm\,\mathrm{d}\mathcal{V}_t}_{=-G_{111}^{\pm,\mu}} - \int_{\Omega^\pm} b^\pm(\nabla^\varphi\cdot\mathbf{V}^\pm)\cdot\mathbf{B}^\pm\,\mathrm{d}\mathcal{V}_t - \int_{\Omega^\pm}\mathbf{B}^\pm\cdot b^\pm\mathfrak{C}_i(v_i)\,\mathrm{d}\mathcal{V}_t$$
$$+\int_{\Omega^\pm}\mu^\pm\rho^\pm\mathbf{V}^\pm\cdot b^\pm(\nabla^\varphi\cdot\mathbf{V}^\pm)\,\mathrm{d}\mathcal{V}_t + \int_{\Omega^\pm}\mu^\pm\rho^\pm\mathbf{V}^\pm\cdot b^\pm\mathfrak{C}_i(v_i^\pm)\,\mathrm{d}\mathcal{V}_t - \int_{\Omega^\pm}\mu^\pm\rho^\pm(b^\pm\cdot\nabla^\varphi)\mathbf{V}^\pm\cdot\mathbf{V}^\pm\,\mathrm{d}\mathcal{V}_t$$
$$=: -G_{111}^{\pm,\mu} + G_{21}^{\pm,\mu} + G_{22}^{\pm,\mu} + G_{23}^{\pm,\mu} + R_7^{\pm,\mu} + R_8^{\pm,\mu}, \tag{6.29}$$

and

$$G_3^{\pm,\mu} = -\int_{\Omega^\pm}(\nabla^\varphi\cdot\mathbf{V}^\pm)\mathbf{P}^\pm\,\mathrm{d}\mathcal{V}_t - \int_{\Omega^\pm}\mathbf{P}^\pm\mathfrak{C}_i(v_i^\pm)\,\mathrm{d}\mathcal{V}_t\underbrace{-\int_{\Omega^\pm}\rho^\pm\mu^\pm b^\pm(\nabla^\varphi\cdot\mathbf{V}^\pm)\cdot\mathbf{V}^\pm\,\mathrm{d}\mathcal{V}_t}_{=-G_{23}^{\pm,\mu}}$$
$$+\int_{\Omega^\pm}\mathbf{P}^\pm\mathcal{R}_p^\pm + \mu^\pm(\mathcal{R}_p^\pm - \mathfrak{C}_i(v_i^\pm))(\rho^\pm b^\pm\cdot\mathbf{V}^\pm)\,\mathrm{d}\mathcal{V}_t$$
$$=: G_{31}^{\pm,\mu} + G_{32}^{\pm,\mu} - G_{23}^{\pm,\mu} + R_9^{\pm,\mu}. \tag{6.30}$$

Now we can see a lot of cancellation structures among these interior integrals. First, using

$$\mathbf{Q}^\pm = \mathbf{P}^\pm + b^\pm \cdot \mathbf{B}^\pm + \mathcal{R}_q^\pm, \quad \mathcal{R}_q^\pm = \sum_{1\le\langle\gamma'\rangle\le\langle\gamma\rangle-1} \mathcal{T}^\gamma b_j^\pm \cdot \mathcal{T}^{\gamma-\gamma'} b_j^\pm,$$

we have

$$G_{121}^{\pm,\mu} + G_{21}^{\pm,\mu} + G_{31}^{\pm,\mu} = \int_{\Omega^\pm} (\nabla^\varphi \cdot \mathbf{V}^\pm)\mathcal{R}_q^\pm \,\mathrm{d}\mathcal{V}_t =: R_{10}^{\pm,\mu}, \tag{6.31}$$

$$\begin{aligned} G_{122}^{\pm,\mu} + G_{22}^{\pm,\mu} + G_{32}^{\pm,\mu} &= -\int_{\Omega^\pm} \mathbf{Q}^\pm(\mathfrak{C}_i(v_i) - \mu\mathfrak{C}_i(b_i))\,\mathrm{d}\mathcal{V}_t + \int_{\Omega^\pm} \mathcal{R}_q^\pm(\mathfrak{C}_i(v_i) - \mu\mathfrak{C}_i(b_i))\,\mathrm{d}\mathcal{V}_t \\ &=: Z^{\pm,\mu} + R_{11}^{\pm,\mu}. \end{aligned} \tag{6.32}$$

The terms $R_j^{\pm,\mu}$ $(1 \le j \le 11)$ can be directly controlled using the same method in Section 3, so we omit the details.

$$\int_0^t \sum_{j=1}^{11} R_j^{\pm,\mu} \le \delta P(\widetilde{E}(t)) + P(\widetilde{E}(0)) + \int_0^t P(\widetilde{E}(\tau))\,\mathrm{d}\tau, \quad \forall \delta \in (0,1). \tag{6.33}$$

Thus, it again remains to analyze the boundary integral $G_0^{\pm,\mu}$ and the commutator term $Z^{\pm,\mu}$.

Elimination of the term VS

The boundary integral $G_0^{\pm,\mu}$ can be decomposed as follows

$$\begin{aligned} G_0^{+,\mu} + G_0^{-,\mu} &= \int_\Sigma \mathbf{Q}^+(\mathbf{V}^+ - \mu^+\mathbf{B}^+) \cdot N \,\mathrm{d}x' - \int_\Sigma \mathbf{Q}^-(\mathbf{V}^- - \mu^-\mathbf{B}^-) \cdot N \,\mathrm{d}x' \\ &= \mathrm{ST}^\mu + \mathrm{ST}^{\mu\prime} + \mathrm{VS}^\mu + \mathrm{RT}^\mu + \mathrm{RT}^{\pm,\mu} + ZB^{\pm,\mu} \end{aligned} \tag{6.34}$$

where

$$\mathrm{ST}^\mu := \int_\Sigma \mathcal{T}^\gamma [\![q]\!]\, \partial_t \mathcal{T}^\gamma \psi \,\mathrm{d}x', \tag{6.35}$$

$$\mathrm{ST}^{\mu\prime} := \int_\Sigma \mathcal{T}^\gamma [\![q]\!] \left((\bar{v}^+ - \bar{\mu}^+\bar{b}^+) \cdot \overline{\nabla}\right) \mathcal{T}^\gamma \psi \,\mathrm{d}x', \tag{6.36}$$

$$\mathrm{VS}^\mu := \int_\Sigma \mathcal{T}^\gamma q^- \left([\![\bar{v} - \bar{\mu}\bar{b}]\!] \cdot \overline{\nabla}\right) \mathcal{T}^\gamma \psi \,\mathrm{d}x', \tag{6.37}$$

$$\mathrm{RT}^\mu := -\int_\Sigma [\![\partial_3 q]\!]\, \mathcal{T}^\gamma \psi\, \partial_t \mathcal{T}^\gamma \psi \,\mathrm{d}x', \tag{6.38}$$

$$\mathrm{RT}^{\pm,\mu} := \mp \int_\Sigma \partial_3 q^\pm\, \mathcal{T}^\gamma \psi \left((\bar{v}^\pm - \bar{\mu}^\pm\bar{b}^\pm) \cdot \overline{\nabla}\right) \mathcal{T}^\gamma \psi \,\mathrm{d}x', \tag{6.39}$$

$$ZB^{\pm,\mu} := \mp \int_\Sigma \mathbf{Q}^\pm(\mathcal{W}_v^\pm - \mu^\pm \mathcal{W}_b^\pm)\,\mathrm{d}x' \tag{6.40}$$

Among the terms (6.35)-(6.40), ST^μ, $\mathrm{ST}^{\mu\prime}$, RT^μ, $\mathrm{RT}^{\pm,\mu}$ can be analyzed in the same way as in Section 3.3-Section 3.5, so we omit the details. Also, the control of these terms do not depend on $1/\sigma$ thanks to the enhanced regularity of ψ obtained in Section 6.2.1. With the unique choice of $\mu^\pm$ in (6.9), the quantity $\bar{v} - \mu\bar{b}$ has NO jump across the interface Σ and so $\mathrm{VS}^\mu = 0$. Finally, there is a cancellation structure in $ZB^{\pm,\mu} + Z^{\pm,\mu}$ which is similar to the one observed in Section 3.3-Section 3.5. It suffices to replace $v^\pm$ (in Section 3.3-Section 3.5) by $v^\pm - \mu^\pm b^\pm$ and use $\nabla^\varphi \cdot b^\pm = 0$ in $\Omega^\pm$ in order for the same result.

The stability condition ensures the hyperbolicity

So far, we can prove the following estimates for $\mathbf{G}^{\pm,\mu}(t)$ in tangential estimates:

$$\sum_{\pm}\frac{1}{2}\int_{\Omega^\pm}\rho^\pm|\mathbf{V}^\pm|^2+|\mathbf{B}^\pm|^2+\mathcal{F}_p^\pm(\mathbf{P}^\pm)^2-2\mu^\pm\rho^\pm\mathbf{V}^\pm\cdot\mathbf{B}^\pm+2\mu^\pm\rho^\pm\mathcal{F}_p^\pm\mathbf{P}^\pm(b^\pm\cdot\mathbf{V}^\pm)\,\mathrm{d}\mathcal{V}_t+\frac{\sigma}{2}\int_\Sigma\frac{|\mathcal{T}^\gamma\overline{\nabla}\psi|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}^3}\,\mathrm{d}x'$$
$$\lesssim\delta\widetilde{E}(t)+P(\widetilde{E}(0))+P(\widetilde{E}(t))\int_0^t P(\widetilde{E}(\tau))\,\mathrm{d}\tau. \tag{6.41}$$

To replace the tangential estimates in Section 3.3-Section 3.5, we must guarantee the *positive-definiteness* of

$$\begin{aligned}&|\sqrt{\rho^\pm}\mathbf{V}^\pm|^2+|\mathbf{B}^\pm|^2+\mathcal{F}_p^\pm(\mathbf{P}^\pm)^2-2\mu^\pm\rho^\pm\mathbf{V}^\pm\cdot\mathbf{B}^\pm+2\mu^\pm\rho^\pm\mathcal{F}_p^\pm\mathbf{P}^\pm(b^\pm\cdot\mathbf{V}^\pm)\\ \geq\,&|\sqrt{\rho^\pm}\mathbf{V}^\pm|^2+|\mathbf{B}^\pm|^2+\mathcal{F}_p^\pm(\mathbf{P}^\pm)^2-2\left|\mu^\pm\sqrt{\rho^\pm}\right|\left|\sqrt{\rho^\pm}\mathbf{V}^\pm\right|\left|\mathbf{B}^\pm\right|-2\left|\mu^\pm\sqrt{\rho^\pm}\right|\frac{c_A^\pm}{c_s^\pm}\left|\sqrt{\mathcal{F}_p^\pm}\mathbf{P}^\pm\right|\left|\sqrt{\rho^\pm}\mathbf{V}^\pm\right|\end{aligned}$$

as a quadratic form of $(\sqrt{\rho^\pm}\mathbf{V}^\pm,\mathbf{B}^\pm,\sqrt{\mathcal{F}_p^\pm}\mathbf{P}^\pm)$ in $\Omega^\pm$, respectively. Here we use the fact that $\mathcal{F}_p=1/(\rho c_s^2)$ and $c_A:=|b|/\sqrt{\rho}$. This is equivalent to show that the following matrix only has strictly positive eigenvalues

$$\begin{pmatrix}1 & -|\mu^\pm|\sqrt{\rho^\pm} & -|\mu^\pm|\sqrt{\rho^\pm}\frac{c_A^\pm}{c_s^\pm}\\ -|\mu^\pm|\sqrt{\rho^\pm} & 1 & 0\\ -|\mu^\pm|\sqrt{\rho^\pm}\frac{c_A^\pm}{c_s^\pm} & 0 & 1\end{pmatrix},$$

which is further converted to the following inequalities

$$(\mu^\pm)^2\rho^\pm\left(1+\left(\frac{c_A^\pm}{c_s^\pm}\right)^2\right)<1\text{ in }\Omega^\pm\Rightarrow|\bar{b}^+\times\bar{b}^-|>|\bar{b}^\mp\times[\![\bar{v}]\!]|\sqrt{\rho^\pm\left(1+\left(\frac{c_A^\pm}{c_s^\pm}\right)^2\right)}\text{ on }\Sigma.$$

Thus, we find that the stability condition (1.40), that is, for some $\delta_0\in(0,\frac{1}{8})$ there holds

$$(1-\delta_0)|\bar{b}^+\times\bar{b}^-|\geq|\bar{b}^\mp\times[\![\bar{v}]\!]|\sqrt{\rho^\pm\left(1+\left(\frac{c_A^\pm}{c_s^\pm}\right)^2\right)}>0\quad\text{on }\Sigma,$$

exactly ensures the positive-definiteness of $\mathbf{G}^{\pm,\mu}(t)$. Plugging this into the energy inequality (6.42), we find that there exists some constant $\delta_0\in(0,\frac{1}{8})$, such that

$$\begin{aligned}&\sum_{\pm}\frac{\delta_0}{2}\int_{\Omega^\pm}\rho^\pm|\mathbf{V}^\pm|^2+|\mathbf{B}^\pm|^2+\mathcal{F}_p^\pm(\mathbf{P}^\pm)^2\,\mathrm{d}\mathcal{V}_t+\frac{\sigma}{2}\int_\Sigma\frac{|\mathcal{T}^\gamma\overline{\nabla}\psi|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}^3}\,\mathrm{d}x'\\ \lesssim\,&\delta\widetilde{E}(t)+P(\widetilde{E}(0))+P(\widetilde{E}(t))\int_0^t P(\widetilde{E}(\tau))\,\mathrm{d}\tau,\quad\forall\delta\in(0,\delta_0/100).\end{aligned} \tag{6.42}$$

6.2.3 Incompressible and zero-surface-tension-limits under the stability condition

Combining the tangential estimates (6.42), the enhanced regularity for ψ obtained in Section 6.2.1 and the div-curl analysis in Section 3.6, we conclude the uniform-in-(ε,σ) estimates of $\widetilde{E}(t)$ by

$$\forall\delta\in(0,\frac{\delta_0}{100}),\quad\widetilde{E}(t)\lesssim\delta\widetilde{E}(t)+P(\widetilde{E}(0))+P(\widetilde{E}(t))\int_0^t P(\widetilde{E}(\tau))\,\mathrm{d}\tau. \tag{6.43}$$

Using Gronwall-type argument, we know there exists some $T>0$ independent of (ε,σ) such that

$$\sup_{0\leq t\leq T}\widetilde{E}(t)\leq P(\widetilde{E}(0)). \tag{6.44}$$

With the uniform-in-(ε,σ) estimates, we can pass the limit $\varepsilon,\sigma\to 0_+$ to the incompressible current-vortex sheets problem without surface tension under the non-collinearity condition. Given $\varepsilon,\sigma>0$, let $(\psi^{\varepsilon,\sigma}, v^{\pm,\varepsilon,\sigma}, b^{\pm,\varepsilon,\sigma}, \rho^{\pm,\varepsilon,\sigma}, S^{\pm,\varepsilon,\sigma})$ be the solution to (1.33) with initial data $(\psi_0^{\varepsilon,\sigma}, v_0^{\pm,\varepsilon,\sigma}, b_0^{\pm,\varepsilon,\sigma}, \rho_0^{\pm,\varepsilon,\sigma}, S_0^{\pm,\varepsilon,\sigma})$ and let $(\xi^0, w^{\pm,0}, h^{\pm,0}, \mathfrak{S}^{\pm,0})$ be the solution to (6.1) with $\sigma=0$ with initial data $(\xi_0^0, w_0^{\pm,0}, h_0^{\pm,0}, \mathfrak{S}_0^{\pm,0})$. We assume

a. $(\psi_0^{\varepsilon,\sigma}, v_0^{\pm,\varepsilon,\sigma}, b_0^{\pm,\varepsilon,\sigma}, S_0^{\pm,\varepsilon,\sigma})\in H^{9.5}(\Sigma)\times H_*^8(\Omega^\pm)\times H_*^8(\Omega^\pm)\times H_*^8(\Omega^\pm)$ satisfies the compatibility conditions (1.34) up to 7-th order, the stability condition (1.42) and $|\psi_0^{\varepsilon,\sigma}|_\infty\leq 1$.
b. $(\psi_0^{\varepsilon,\sigma}, v_0^{\pm,\varepsilon,\sigma}, b_0^{\pm,\varepsilon,\sigma}, S_0^{\pm,\varepsilon,\sigma})\to(\xi_0^0, w_0^{\pm,0}, h_0^{\pm,0}, \mathfrak{S}_0^{\pm,0})$ in $H^{4.5}(\Sigma)\times H^4(\Omega^\pm)\times H^4(\Omega^\pm)\times H^4(\Omega^\pm)$ as $\varepsilon,\sigma\to 0$.
c. The incompressible initial data satisfies the constraints $\nabla^{\xi_0}\cdot h_0^\pm=0$ in $\Omega^\pm$, $h^\pm\cdot N^0|_{\{t=0\}\times\Sigma}=0$, the stability condition

$$2\delta_0\leq\sqrt{\mathfrak{R}_0^\pm}\left|\bar h_0^\mp\times[\bar w_0]\right|\leq(1-2\delta_0)|\bar h_0^+\times\bar h_0^-|\quad\text{on }\Sigma,\tag{6.45}$$

where $\delta_0>0$ is the same constant as in (1.42).

Then, by the Aubin-Lions compactness lemma, it holds that

$$(\psi^{\varepsilon,\sigma}, v^{\pm,\varepsilon,\sigma}, b^{\pm,\varepsilon,\sigma}, S^{\pm,\varepsilon,\sigma})\to(\xi^0, w^{\pm,0}, h^{\pm,0}, \mathfrak{S}^{\pm,0}),\tag{6.46}$$

weakly-* in $L^\infty([0,T];H^{4.5}(\Sigma)\times(H^4(\Omega^\pm))^3)$ and strongly in $C([0,T];H^{4.5-\delta}_{\text{loc}}(\Sigma)\times(H^{4-\delta}_{\text{loc}}(\Omega^\pm))^3)$ after possibly passing to a subsequence. Theorem 1.3 is proven.

6.3 Double limits in 2D: a subsonic zone

When the space dimension $d=2$, the substantial part of the proof for well-posedness, uniform estimates and limit process remains unchanged. In fact, we shall only re-consider the following aspects

- The curl operator now becomes $\nabla^{\varphi,\perp}\cdot := (-\partial_2^\varphi,\partial_1^\varphi)\cdot$, so we need to check the special structure given by Lorentz force in the vorticity analysis .
- The interface is now a 1D curve instead of a 2D surface, thus it is impossible to have "non-parallel" magnetic fields $b^\pm$ on Σ. The functions $\mu^\pm$ are no longer uniquely determined by $b_1^\pm$.

6.3.1 Modifications in vorticity analysis

In the case of 2D, the equations of vorticity $\nabla^{\varphi,\perp}\cdot v$ and current density $\nabla^{\varphi,\perp}\cdot b$ are

$$\rho D_t^\varphi(\nabla^{\varphi,\perp}\cdot v)-(b\cdot\nabla^\varphi)(\nabla^{\varphi,\perp}\cdot b)=-(\nabla^{\varphi,\perp}\rho)\cdot(D_t^\varphi v)-\rho(\nabla^{\varphi,\perp}v_j)\cdot(\nabla_j^\varphi v)+(\nabla^{\varphi,\perp}b_j)\cdot(\nabla_j^\varphi b),\tag{6.47}$$

$$D_t^\varphi(\nabla^{\varphi,\perp}\cdot b)-(b\cdot\nabla^\varphi)(\nabla^{\varphi,\perp}\cdot v)-b\cdot\nabla^{\varphi,\perp}(\nabla^\varphi\cdot v)=-(\nabla^{\varphi,\perp}\cdot b)(\nabla^\varphi\cdot v)-(\nabla^{\varphi,\perp}v_j)\cdot(\nabla_j^\varphi b)+(\nabla^{\varphi,\perp}b_j)\cdot(\nabla_j^\varphi v),\tag{6.48}$$

which has the same structure as (3.143)-(3.144). Thus, we expect to adopt the strategy in Section 3.6 to prove the div-curl estimates. The only slight difference is the structure of Lorentz force. Let us take the ∂^3-estimate of $\nabla^{\varphi,\perp}\cdot(v,b)$ for an example. In this case, the problematic term (in the analogue of $K_1^\pm$ in (3.147)) becomes

$$K_1^{\pm\prime}=\int_{\Omega^\pm}(\partial^3\nabla^{\varphi,\perp}\cdot b^\pm)\left(b^\pm\cdot\nabla^{\varphi,\perp}(\partial^3\nabla^\varphi\cdot v^\pm)\right)\mathrm{d}\mathcal{V}_t.$$

Again, we invoke the continuity equation, commute ∇^φ with $D_t^{\varphi\pm}$ to get

$$b^\pm\cdot\nabla^{\varphi,\perp}(\partial^3\nabla^\varphi\cdot v^\pm)\overset{L}{=}\varepsilon^2(b_1^\pm\partial^3\partial_2^\varphi D_t^{\varphi\pm}p^\pm-b_2^\pm\partial^3\partial_1^\varphi D_t^{\varphi\pm}p^\pm)\overset{L}{=}\varepsilon^2(b_1^\pm\partial^3 D_t^{\varphi\pm}(\partial_2^\varphi p^\pm)-b_2^\pm\partial^3 D_t^{\varphi\pm}(\partial_1^\varphi p^\pm)).$$

Then we plug the momentum equation

$$-\partial_1^\varphi p=\rho D_t^\varphi v_1-b_1\partial_1^\varphi b_1-b_2\partial_2^\varphi b_1+b_1\partial_1^\varphi b_1+b_2\partial_1^\varphi b_2=\rho D_t^\varphi v_1+b_2(\nabla^{\varphi,\perp}\cdot b)$$
$$-\partial_2^\varphi p=\rho D_t^\varphi v_2-b_1\partial_1^\varphi b_2-b_2\partial_2^\varphi b_2+b_1\partial_2^\varphi b_1+b_2\partial_2^\varphi b_2=\rho D_t^\varphi v_2-b_1(\nabla^{\varphi,\perp}\cdot b)$$

to get

$$b^{\pm}\cdot\nabla^{\varphi,\perp}(\partial^3\nabla^{\varphi}\cdot v^{\pm})\overset{L}{=}\mathcal{F}_p^{\pm}\rho^{\pm}(b^{\pm,\perp}\cdot\partial^3(D_t^{\varphi\pm})^2v^{\pm})-\mathcal{F}_p^{\pm}\left((b_1^{\pm})^2+(b_2^{\pm})^2\right)\partial^3D_t^{\varphi\pm}(\nabla^{\varphi,\perp}\cdot b),\ \ b^{\perp}:=(-b_2,b_1).$$

Thus, the term $K_1^{\pm\prime}$ can be controlled in a similar manner as in Section 3.6

$$\begin{aligned}K_1^{\pm\prime}\overset{L}{=}&\int_{\Omega^{\pm}}(\partial^3\nabla^{\varphi,\perp}\cdot b^{\pm})\left(\mathcal{F}_p^{\pm}\rho^{\pm}b^{\pm,\perp}\cdot\partial^3(D_t^{\varphi\pm})^2v^{\pm}\right)\mathrm{d}\mathcal{V}_t\\&-\int_{\Omega^{\pm}}\mathcal{F}_p^{\pm}|b^{\pm}|^2(\partial^3\nabla^{\varphi,\perp}\cdot b^{\pm})(D_t^{\varphi\pm}\partial^3\nabla^{\varphi,\perp}\cdot b^{\pm})\,\mathrm{d}\mathcal{V}_t\\=&\int_{\Omega^{\pm}}(\partial^3\nabla^{\varphi,\perp}\cdot b^{\pm})\left(\mathcal{F}_p^{\pm}\rho^{\pm}b^{\pm,\perp}\cdot\partial^3(D_t^{\varphi\pm})^2v^{\pm}\right)\mathrm{d}\mathcal{V}_t-\frac{1}{2}\int_{\Omega^{\pm}}(\nabla^{\varphi}\cdot v^{\pm})\mathcal{F}_p^{\pm}\left|b^{\pm}\right|^2\left|\partial^3\nabla^{\varphi,\perp}\cdot b^{\pm}\right|^2\mathrm{d}\mathcal{V}_t\\&-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^{\pm}}\mathcal{F}_p^{\pm}\left|b^{\pm}\right|^2\left|\partial^3\nabla^{\varphi,\perp}\cdot b^{\pm}\right|^2\mathrm{d}\mathcal{V}_t.\end{aligned}$$

Hence, the curl estimate (3.152) should be modified to be

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^{\pm}}\rho^{\pm}\left|\partial^3(\nabla^{\varphi,\perp}\cdot v^{\pm})\right|^2+(1+\mathcal{F}_p^{\pm}|b^{\pm}|^2)\left|\partial^3(\nabla^{\varphi,\perp}\cdot b^{\pm})\right|^2\mathrm{d}\mathcal{V}_t\lesssim P(\widetilde{E}_4(t))+\widetilde{E}_5(t).\tag{6.49}$$

6.3.2 Different choice of $\mu^{\pm}$ and stability condition

Since the non-collinearity of $b^{\pm}$ no longer holds, we need to re-consider the choice of $\bar{\mu}^{\pm}$ used in Fredriches secondary symmetrization. We prove the following lemma, which is analogous to Morando-Trebeschi-Secchi-Yuan [65, Lemma 5.1] for the constant rectilinear background solution of the linearized problem.

Lemma 6.4. There exist functions $\bar{\mu}^{\pm}(t,x_1)$ satisfying

$$[\![v_1-\bar{\mu}b_1]\!]=0\ \text{ and }\ |\bar{\mu}^{\pm}|<1/a^{\pm}\quad\text{on }[0,T]\times\Sigma,\quad a^{\pm}:=\sqrt{\rho^{\pm}\left(1+(c_A^{\pm}/c_s^{\pm})^2\right)}$$

if and only if the following inequality holds

$$|\,[\![v_1]\!]\,|<\frac{|b_1^+|}{a^+}+\frac{|b_1^-|}{a^-}\quad\text{on }[0,T]\times\Sigma.\tag{6.50}$$

Under (6.50), the functions $\bar{\mu}^{\pm}$ are chosen to be

$$\bar{\mu}^{\pm}=\pm\frac{\mathrm{sgn}(b_1^{\pm})\,a^{\mp}\,[\![v_1]\!]}{a^-|b_1^+|+a^+|b_1^-|}.\tag{6.51}$$

Proof. First, we shall exclude the possibility for $b_1^{\pm}=0$ at some point $(x_1,0)\in\Sigma$ because of $[\![v_1]\!]\neq 0$ everywhere on Σ.

Case 1: <u>One of the two magnetic fields is vanishing on Σ</u>, e.g., we assume $|b_1^+|>0=|b_1^-|$ on Σ, then $[\![v_1-\bar{\mu}b_1]\!]=0$ directly gives us $\bar{\mu}^+=[\![v_1]\!]/b_1^+$ and $\bar{\mu}^-$ can be any function satisfying the hyperbolicity constraint $|\bar{\mu}^-|<1/a^-$. For simplicity, we may choose $\bar{\mu}^-=0$. Solving the constraint $|\bar{\mu}^+|<1/a^+$ gives us $|\,[\![v_1]\!]\,|<|b_1^+|/a^+$ as a special case of (6.50). Similarly, when $|b_1^-|>0=|b_1^+|$ on Σ, we can choose $\bar{\mu}^-=-[\![v_1]\!]/b_1^-$ and $\bar{\mu}^+$ can be any function satisfying the hyperbolicity constraint $|\bar{\mu}^+|<1/a^+$ and we may choose $\bar{\mu}^+=0$ for simplicity.

Remark 6.1. One can verify that if $b_0=0$ on Σ, then b must be identically zero on Σ. In fact, restricting the equation of b onto Σ and doing L^2 estimate shows that $\frac{\mathrm{d}}{\mathrm{d}t}|b^{\pm}|_0^2\leq C|\partial v|_{L^\infty}|b^{\pm}|_0^2$. Using Gronwall's inequality and $b_0|_{\Sigma}=0$ yields the result.

Case 2: <u>$b_1^{\pm}$ are not identically zero on Σ</u>. In this case, we may assume $|b_1^{\pm}|>0$ on Σ as well. In fact, if b_1^- vanishes at some point $(x_1,0)\in\Sigma$, then we can follow the choice of $\bar{\mu}^{\pm}$ as in case 1 to determine the function $\bar{\mu}^{\pm}$ at this point. Note that the functions $\bar{\mu}^+=[\![v_1]\!]/b_1^+,\ \bar{\mu}^-=0$ still satisfies (6.51), so they do not break the continuity and differentiability of (6.51) at the points where one of $b_1^\pm$ vanishes. Thus, making the assumption $|b_1^\pm|>0$ on Σ is reasonable.

The "if" part is easy to prove. Indeed, when the stability condition (6.50) holds on Σ, we can set $\bar\mu^\pm$ as in (6.51). The direct computation shows that such $\bar\mu^\pm$ satisfy $[\![v_1-\bar\mu b_1]\!]=0$ on Σ and $|\bar\mu^\pm|<1/a^\pm$. Let us prove the "only if" part. When $|b_1^\pm|>0$ on Σ, we can write

$$\bar\mu^+=\frac{[\![v_1]\!]+\bar\mu^-b_1^-}{b_1^+}.$$

Using $|\bar\mu^+|<1/a^+$, we can solve the inequality by

$$-\frac{1}{a^+}-\frac{[\![v_1]\!]}{b_1^+}<\frac{b_1^-}{b_1^+}\bar\mu^-<\frac{1}{a^+}-\frac{[\![v_1]\!]}{b_1^+},$$

where $\bar\mu^-$ should also satisfy $|\bar\mu^-|<1/a^-$. Assume $b_1^\pm<0$ for simplicity (that is, the horizontal directions of $b^\pm$ on Σ are the same). Combining these two requirements, we find that the following inequality is necessary

$$|\,[\![v_1]\!]\,|<-\frac{b_1^+}{a^+}-\frac{b_1^-}{a^-}=\frac{|b_1^+|}{a^+}+\frac{|b_1^-|}{a^-}. \tag{6.52}$$

Similar calculation for the case $b_1^\pm>0$ and the case $b_1^+b_1^-<0$ also leads to the same inequality as above. □

7 Improved incompressible limit for well-prepared initial data

In this final section, we aim to drop the redundant assumptions on the well-prepared initial data, namely $\partial_t^k v|_{t=0}=O(1)$ for $2\le k\le 4$, when taking the incompressible limit. Compared with the energy $E(t)$ that we use to prove the local existence, there is a new difficulty in the control of the "weaker" energy $\mathfrak{E}(t)$: There exhibits a loss of weight of Mach number in $\overline{\partial}{}^3\partial_t$-tangential estimates when analyzing $\mathfrak{E}_4(t)$. In particular, we have to control the following quantity in the cancellation structure in $Z^\pm+ZB^\pm$,

$$\int_\Omega(\overline{\partial}{}^2\partial_3\partial_t v_i)(\overline{\partial}\mathbf{N}_i)(\overline{\partial}{}^2\partial_3\partial_t q)\,\mathrm{d}x,$$

in which $\partial_t q$ has to be uniformly bounded with respect to Mach number. However, now we only have $\nabla^\varphi\cdot v=O(\varepsilon)$ and $\partial_t q=O(1/\varepsilon)$, which leads to a loss of ε-weight. Besides, similar difficulty also appears in the control of $-\int_{\Omega^\pm}\mathbf{V}^\pm\cdot\mathfrak{C}(q^\pm)\,\mathrm{d}\mathcal{V}_t$. Indeed, such loss of ε-weight necessarily happens in $\overline{\partial}{}^3\partial_t$-tangential estimates because of the following two reasons

1. $\overline{\partial}{}^3\partial_t q$ needs one more ε-weight than $\overline{\partial}{}^3\partial_t v$;
2. The (extension of) normal vector $\mathbf{N}$, which arises from the commutator $[\overline{\partial}{}^3\partial_t,\mathbf{N}_i/\partial_3\varphi,\partial_3 f]$ in $\mathfrak{C}_i(f)$, may NOT absorb a time derivative.

As we can see, *this type of difficulty never appears in the fixed-domain setting because the commutator terms $\mathfrak{C}(f)$ are contributed by the free-interface motion.* To get rid of the loss of Mach number, we have to find a new way to control v_t, b_t and also avoid the appearance of $|\sqrt{\sigma}\overline{\nabla}\partial_t\psi|_3$ without ε-weight.

7.1 The weaker energy for the improved incompressible limit

As in (1.50)-(1.51), we consider the new energy functional

$$\mathfrak{E}(t):=\mathfrak{E}_4(t)+E_5(t)+E_6(t)+E_7(t)+E_8(t)$$
$$\widetilde{\mathfrak{E}}(t):=\widetilde{\mathfrak{E}}_4(t)+\widetilde{E}_5(t)+\widetilde{E}_6(t)+\widetilde{E}_7(t)+\widetilde{E}_8(t)$$

where

$$\mathfrak{E}_4(t)=\sum_\pm\left\|(v^\pm,b^\pm,S^\pm,p^\pm)\right\|_{4,\pm}^2+\left|\sqrt{\sigma}\psi\right|_5^2+\left\|\partial_t(v^\pm,b^\pm,S^\pm,(\mathcal{F}_p^\pm)^{\frac12}p^\pm)\right\|_{3,\pm}^2+\left|\sqrt{\sigma}\partial_t\psi\right|_4^2$$
$$+\sum_{k=2}^4\left\|\varepsilon\partial_t^k(v^\pm,b^\pm,S^\pm,(\mathcal{F}_p^\pm)^{\frac{(k-3)_+}{2}}p^\pm)\right\|_{4-k,\pm}^2+\left|\sqrt{\sigma}\varepsilon\partial_t^k\psi\right|_{5-k}^2 \tag{7.1}$$

and

$$\widetilde{\mathfrak{E}}_4(t) = \mathfrak{E}_4(t) + |\psi|_{4.5}^2 + |\partial_t\psi|_{3.5}^2 + \left|\partial_t^2\psi\right|_{2.5}^2 + \left|\varepsilon\partial_t^3\psi\right|_{1.5}^2 + \left|\varepsilon\partial_t^4\psi\right|_{0.5}^2. \tag{7.2}$$

We aim to prove uniform-in-ε estimates for $\mathfrak{E}(t)$ for each $\sigma > 0$ and prove uniform-in-(ε, σ) estimates for $\widetilde{\mathfrak{E}}(t)$ under the stability condition (1.40) (replaced with (1.47) in the 2D case). Let us analyze the estimates for different k in $\mathfrak{E}_4(t)$ and $\widetilde{\mathfrak{E}}_4(t)$.

The case $k = 0$

When $k = 0$, the reduction of v, b is the same as in Section 3. That is, we use the div-curl analysis to convert normal derivatives to tangential derivatives

$$\|(v^\pm, b^\pm)\|_{4,\pm}^2 \le C(|\psi|_4, |\overline{\nabla}\psi|_{W^{1,\infty}})\left(\|v^\pm, b^\pm\|_{0,\pm}^2 + \left\|\nabla^\varphi\cdot v^\pm, \nabla^\varphi \times (v^\pm, b^\pm)\right\|_{3,\pm}^2 + \left\|\overline{\partial}^4(v^\pm, b^\pm)\right\|_{0,\pm}^2\right), \tag{7.3}$$

$$\left\|\nabla^\varphi\cdot v^\pm\right\|_{3,\pm}^2 \lesssim \left\|\mathcal{F}_p^\pm D_t^{\varphi\pm} p^\pm\right\|_{3,\pm}^2, \quad \left\|\nabla^\varphi \times (v^\pm, b^\pm)\right\|_{3,\pm}^2 \lesssim \delta\mathfrak{E}_4(t) + \int_0^t P(\mathfrak{E}_4(\tau)) + \mathfrak{E}_5(\tau)\,\mathrm{d}\tau, \tag{7.4}$$

$$\|\nabla q\|_{3,\pm}^2 \lesssim \left\|\rho D_t^{\varphi\pm} v^\pm\right\|_{3,\pm}^2 + \left\|(b^\pm\cdot\nabla^\varphi) b^\pm\right\|_{3,\pm}^2. \tag{7.5}$$

The $\overline{\partial}^4$-control is proved in almost the same way as in Section 3.3.1 which gives the control of $\left|\sqrt{\sigma}\overline{\nabla}\psi\right|_4^2$. The only difference is the treatment of RT defined in (3.33) because we need to avoid using $\sqrt{\sigma}$-weight enenrgy when taking the limit $\sigma \to 0$. Using Kato-Ponce type porduct estimate (B.7) in Lemma B.6, we have

$$\mathrm{RT} = \int_\Sigma [\![\partial_3 q]\!]\, \overline{\partial}^4\psi\overline{\partial}^4\psi_t \le \left|[\![\partial_3 q]\!]\,\overline{\partial}^4\psi\right|_{\frac12} |\psi_t|_{3.5} \le \|q^\pm\|_{3,\pm}|\psi|_{4.5}|\psi_t|_{3.5}. \tag{7.6}$$

So, we need to find (ε,σ)-independent control of $|\psi|_{4.5}^2$ and $|\partial_t\psi|_{3.5}^2$.

The case $k = 1$

When $k = 1$, we cannot use the above div-curl inequality because we must avoid $\overline{\partial}^3\partial_t$-estimate. Instead, we use the div-curl inequality (B.2) to get

$$\begin{aligned}\|(\partial_t v^\pm, \partial_t b^\pm)\|_{3,\pm}^2 \le C(|\psi|_{3.5}, |\overline{\nabla}\psi|_{W^{1,\infty}})\Big(&\|\partial_t v^\pm, \partial_t b^\pm\|_{0,\pm}^2 + \left\|\nabla^\varphi\cdot(\partial_t v^\pm, \partial_t b^\pm), \nabla^\varphi\times(\partial_t v^\pm, \partial_t b^\pm)\right\|_{2,\pm}^2 \\ &+ \left|\partial_t v^\pm\cdot N, \partial_t b^\pm \cdot N\right|_{2.5}^2\Big).\end{aligned} \tag{7.7}$$

The divergence part and the curl part are controlled in the same way as Section 3.6, so we do not repeat the analysis here. The boundary normal trace for b_t is easy to control. Using $b^\pm\cdot N = 0$, we have $b_t\cdot N = \bar{b}\cdot\overline{\nabla}\psi_t$ and thus

$$\left|\partial_t b^\pm\cdot N\right|_{2.5}^2 = \left|\bar{b}^\pm\cdot\overline{\nabla}\psi_t\right|_{2.5}^2 \lesssim \|b^\pm\|_{3,\pm}^2|\psi_t|_{3.5}^2. \tag{7.8}$$

For the normal trace $|\partial_t v^\pm\cdot N|_{2.5}^2$, we invoke the kinematic boundary condition $\partial_t\psi = v^\pm\cdot N$ to get

$$\left|\partial_t v^\pm\cdot N\right|_{2.5}^2 \le \left|\partial_t^2\psi\right|_{2.5}^2 + \left|\bar{v}^\pm\cdot\overline{\nabla}\psi_t\right|_{2.5}^2 \lesssim \left|\partial_t^2\psi\right|_{2.5}^2 + \|v^\pm\|_{3,\pm}^2|\psi_t|_{3.5}^2. \tag{7.9}$$

Since we avoid $\overline{\partial}^3\partial_t$-tangential estimates, we must seek for another way to find ε-independent estimates for $\left|\partial_t^2\psi\right|_{2.5}^2$ and $\left|\sqrt{\sigma}\partial_t\psi\right|_4^2$. Also, under the stability condition (1.40), we need to find (ε,σ)-independent control of $|\psi|_{4.5}^2$, $|\partial_t\psi|_{3.5}^2$, $\left|\partial_t^2\psi\right|_{2.5}^2$ and $\left|\sqrt{\sigma}\partial_t\psi\right|_4^2$.

The case $2 \le k \le 4$

When $k = 2, 3, 4$, the reduction stays the same as in Section 3. The reason is that $\partial_t^k q$ share the same weight of Mach number as $\partial_t^k v$ which helps us avoid the loss of ε-weight in $\mathfrak{E}(q)$.

7.2 The evolution equation of the free interface and its paralinearization

To prove the uniform-in-ε estimates for $\mathfrak{E}(t)$ and the uniform-in-(ε,σ) estimates for $\widetilde{\mathfrak{E}}(t)$ under the stability condition (1.40), it remains to prove the ε-indenpedent control of $|\psi|_{4.5}$, $|\psi_t|_{3.5}$, $|\psi_{tt}|_{2.5}$ and $|\sqrt{\sigma}\psi_t|_4$ by $P(\mathfrak{E}(0)) + P(\mathfrak{E}(t))\int_0^t P(\mathfrak{E}(\tau))\,\mathrm{d}\tau$ and (ε,σ)-indenpedent control of them by $P(\widetilde{\mathfrak{E}}(0)) + P(\widetilde{\mathfrak{E}}(t))\int_0^t P(\widetilde{\mathfrak{E}}(\tau))\,\mathrm{d}\tau$. Since we already avoid $\overline{\partial}{}^3\partial_t$-tangential estimates, we shall further analyze the evolution equation of the free interface.

7.2.1 Derivation of the equation

We take ∂_t in the kinematic boundary condition to get $\partial_t^2\psi = \partial_t v^\pm \cdot N - \bar{v}^\pm \cdot \overline{\nabla}\partial_t\psi$. Plugging the momentum equation of (1.33) into the term $\partial_t v^\pm \cdot N$, we get

$$\partial_t v^\pm \cdot N = -\frac{1}{\rho^\pm} N \cdot \nabla^\varphi q^\pm - (\bar{v}^\pm \cdot \overline{\nabla}) v^\pm \cdot N + \frac{1}{\rho^\pm}(\bar{b}^\pm \cdot \overline{\nabla}) b^\pm \cdot N \quad \text{on } \Sigma.$$

Using $\partial_t\psi = v^\pm \cdot N$ and $b^\pm \cdot N = 0$ on Σ, we have

$$\begin{aligned} -(\bar{v}^\pm \cdot \overline{\nabla})v^\pm \cdot N &= -(\bar{v}^\pm \cdot \overline{\nabla})\partial_t\psi + v^\pm \cdot (\bar{v}^\pm \cdot \overline{\nabla})N = -\bar{v}_j^\pm \overline{\partial}_j\partial_t\psi - \bar{v}_i^\pm\bar{v}_j^\pm\overline{\partial}_i\overline{\partial}_j\psi, \\ (\bar{b}^\pm \cdot \overline{\nabla})b^\pm \cdot N &= \bar{b}_i^\pm\bar{b}_j^\pm\overline{\partial}_i\overline{\partial}_j\psi, \end{aligned}$$

and thus

$$\partial_t^2\psi = -\frac{1}{\rho^\pm}N \cdot \nabla^\varphi q^\pm + \left(\frac{1}{\rho^\pm}\bar{b}_i^\pm\bar{b}_j^\pm - \bar{v}_i^\pm\bar{v}_j^\pm\right)\overline{\partial}_i\overline{\partial}_j\psi - 2(\bar{v}^\pm \cdot \overline{\nabla})\partial_t\psi. \tag{7.10}$$

Next we want to separate the boundary value of $q^\pm$ from its interior contribution in order to create an energy term involving the surface tension. First, taking $\nabla^\varphi\cdot$ in the momentum equation and invoking the continuity equation in (1.33), we derive a wave-type equation

$$\mathcal{F}_p^\pm (D_t^{\varphi\pm})^2 p^\pm - \Delta^\varphi q^\pm = (\partial_i^\varphi v_j^\pm)(\partial_j^\varphi v_i^\pm) - (\partial_i^\varphi b_j^\pm)(\partial_j^\varphi b_i^\pm),$$

which can be written as a wave equation of $q^\pm$ thanks to $q^\pm = p^\pm + \frac{1}{2}|b^\pm|^2$

$$\mathcal{F}_p^\pm (D_t^{\varphi\pm})^2 q^\pm - \Delta^\varphi q^\pm = \varepsilon^2 (D_t^{\varphi\pm})^2\left(\frac{1}{2}|b^\pm|^2\right) + (\partial_i^\varphi v_j^\pm)(\partial_j^\varphi v_i^\pm) - (\partial_i^\varphi b_j^\pm)(\partial_j^\varphi b_i^\pm) \quad \text{in } [0,T]\times\Omega^\pm, \tag{7.11}$$

with a jump condition $[\![q]\!] = \sigma\mathcal{H}(\psi)$ on Σ and a Neumann-type boundary condition $\partial_3 q^\pm = 0$ on $\Sigma^\pm$ (got by restricting the momentum equations on $\Sigma^\pm$), where we omit the terms in which $D_t^{\varphi\pm}$ falls on $\mathcal{F}_p^\pm$.

Definition 7.1. For a function $f : \Sigma \to \mathbb{R}$, we now define the Dirichlet-to-Neumann operator with respect to $(\psi, \Omega^\pm)$ by

$$\mathfrak{N}_\psi^\pm f := \mp N \cdot \nabla^\varphi(\mathcal{E}_\psi^\pm f), \tag{7.12}$$

where $\mathcal{E}_\psi^\pm f$ is defined to be the harmonic extension of f into $\Omega^\pm$, namely

$$-\Delta^\varphi(\mathcal{E}_\psi^\pm f) = 0 \quad \text{in } \Omega^\pm, \quad \mathcal{E}_\psi^\pm f = f \quad \text{on } \Sigma, \quad \partial_3(\mathcal{E}_\psi^\pm f) = 0 \quad \text{on } \Sigma^\pm. \tag{7.13}$$

Thus, we can define a decomposition $q^\pm = q_\psi^\pm + q_w^\pm$ satisfying

$$q_\psi^\pm := \mathcal{E}_\psi^\pm(q^\pm|_\Sigma) \text{ in } \Omega^\pm \tag{7.14}$$

and

$$-\Delta^\varphi q_w^\pm = -\mathcal{F}_p^\pm (D_t^{\varphi\pm})^2 q^\pm + \mathcal{F}_p^\pm (D_t^{\varphi\pm})^2\left(\frac{1}{2}|b^\pm|^2\right) + (\partial_i^\varphi v_j^\pm)(\partial_j^\varphi v_i^\pm) - (\partial_i^\varphi b_j^\pm)(\partial_j^\varphi b_i^\pm) \text{ in } \Omega^\pm \tag{7.15}$$

with boundary conditions $q_w^\pm = 0$ on Σ and $N\cdot\nabla^\varphi q_w^\pm = \partial_3 q_w^\pm = 0$ on $\Sigma^\pm$. The second boundary conditions holds thanks to the slip condition for v_3, b_3 on Σ. Thus, the evolution equation of ψ can be written as

$$\rho^\pm\partial_t^2\psi = \pm\,\mathfrak{N}_\psi^\pm(q^\pm|_\Sigma) - N\cdot\nabla^\varphi q_w^\pm + \left(\bar b_i^\pm \bar b_j^\pm - \rho^\pm \bar v_i^\pm \bar v_j^\pm\right)\overline{\partial}_i\overline{\partial}_j\psi - 2(\rho^\pm \bar v^\pm\cdot\overline{\nabla})\partial_t\psi. \tag{7.16}$$

We now want to resolve $q^\pm|_\Sigma$ in terms of $\rho^\pm$ and $F_\psi^\pm$ by inverting the Dirichlet-to-Neumann operators $\mathfrak{N}_\psi^\pm$. However, we no longer have $\int_\Sigma \rho\partial_t\psi\,\mathrm{d}x' = \int_\Sigma \rho\partial_t^2\psi\,\mathrm{d}x' = 0$ due to the compressibility of fluids. Thus, we have to eliminate the zero-frequency part in $\rho^\pm\partial_t^2\psi$ before inverting the Dirichlet-to-Neumann operators. For a function $f:\Sigma=\mathbb{T}^2\to\mathbb{R}$, we define the Littlewood-Paley projection

$$\mathcal{P}_{\neq 0} f := f - (f)_\Sigma,\ \ (f)_\Sigma := \int_{\mathbb{T}^2} f\,\mathrm{d}x'.$$

Under this setting, we have

$$\rho^\pm\partial_t^2\psi = \mathcal{P}_{\neq 0}(\rho^\pm\partial_t^2\psi) + (\rho^\pm\partial_t^2\psi)_\Sigma \tag{7.17}$$

and we insert it back to the evolution equation to get

$$\begin{aligned}\mathcal{P}_{\neq 0}(\rho^\pm\partial_t^2\psi) =&\ \pm\,\mathfrak{N}_\psi^\pm(q^\pm|_\Sigma) - (\rho^\pm\partial_t^2\psi)_\Sigma - N\cdot\nabla^\varphi q_w^\pm + \left(\bar b_i^\pm \bar b_j^\pm - \rho^\pm \bar v_i^\pm \bar v_j^\pm\right)\overline{\partial}_i\overline{\partial}_j\psi - 2(\rho^\pm \bar v^\pm\cdot\overline{\nabla})\partial_t\psi \qquad (7.18)\\ =&:\ \pm\mathfrak{N}_\psi^\pm(q^\pm|_\Sigma) + F_\psi^\pm. \qquad (7.19)\end{aligned}$$

Note that the zero-frequency modes of both $\mathcal{P}_{\neq 0}(\rho^\pm\partial_t^2\psi)$ and $\pm\mathfrak{N}_\psi^\pm(q^\pm|_\Sigma)$ are vanishing on the interface Σ, so we deduce that $\int_\Sigma F_\psi^\pm = 0$ and then $(\mathfrak{N}_\psi^\pm)^{-1}(F_\psi^\pm)$ is well-defined. Now we can resolve the traces $q^\pm|_\Sigma$ from the evolution equations of ψ. We have

$$\begin{aligned}&\mathfrak{N}_\psi^+(q^+|_\Sigma) + \mathfrak{N}_\psi^-(q^-|_\Sigma) = \mathcal{P}_{\neq 0}([\![\rho]\!]\,\partial_t^2\psi) - F_\psi^+ + F_\psi^-\\ \Rightarrow&\ \mp\,\mathfrak{N}_\psi^\mp([\![q]\!]\,|_\Sigma) + \left(\mathfrak{N}_\psi^+ + \mathfrak{N}_\psi^-\right)(q^\pm|_\Sigma) = \mathcal{P}_{\neq 0}([\![\rho]\!]\,\partial_t^2\psi) - F_\psi^+ + F_\psi^-\\ \Rightarrow&\ q^\pm|_\Sigma = \widetilde{\mathfrak{N}}^{-1}\left(\pm\mathfrak{N}_\psi^\mp(\sigma\mathcal{H}(\psi)) + \mathcal{P}_{\neq 0}([\![\rho]\!]\,\partial_t^2\psi) - \left[\!\left[F_\psi\right]\!\right]\right), \qquad (7.20)\end{aligned}$$

where $\widetilde{\mathfrak{N}} := \mathfrak{N}_\psi^+ + \mathfrak{N}_\psi^-$ (and equivalently we have $\mathfrak{N}_\psi^\pm = \frac{1}{2}(\widetilde{\mathfrak{N}} \pm (\mathfrak{N}_\psi^+ - \mathfrak{N}_\psi^-))$ represents the mixed Dirichlet-to-Neumann operator and $\left[\!\left[F_\psi\right]\!\right] := F_\psi^+ - F_\psi^-$.

Plugging (7.20) back into the evolution equation of the free interface, we get

$$\begin{aligned}\mathcal{P}_{\neq 0}(\rho^+\partial_t^2\psi) &= \mathfrak{N}_\psi^+(q^+|_\Sigma) + F_\psi^+ = \mathfrak{N}_\psi^+\widetilde{\mathfrak{N}}^{-1}\left(\mathfrak{N}_\psi^-(\sigma\mathcal{H}(\psi)) + \mathcal{P}_{\neq 0}([\![\rho]\!]\,\partial_t^2\psi) - F_\psi^+ + F_\psi^-\right) + F_\psi^+\\ &= \sigma\mathfrak{N}_\psi^+\widetilde{\mathfrak{N}}^{-1}\mathfrak{N}_\psi^-(\mathcal{H}(\psi)) + \mathfrak{N}_\psi^+\widetilde{\mathfrak{N}}^{-1}F_\psi^- - \mathfrak{N}_\psi^+\widetilde{\mathfrak{N}}^{-1}F_\psi^+ + F_\psi^+ + \mathfrak{N}_\psi^+\widetilde{\mathfrak{N}}^{-1}\left(\mathcal{P}_{\neq 0}([\![\rho]\!]\,\partial_t^2\psi)\right)\\ &= \sigma\mathfrak{N}_\psi^+\widetilde{\mathfrak{N}}^{-1}\mathfrak{N}_\psi^-(\mathcal{H}(\psi)) + \mathfrak{N}_\psi^-\widetilde{\mathfrak{N}}^{-1}F_\psi^+ + \mathfrak{N}_\psi^+\widetilde{\mathfrak{N}}^{-1}F_\psi^- + \mathfrak{N}_\psi^+\widetilde{\mathfrak{N}}^{-1}\left(\mathcal{P}_{\neq 0}([\![\rho]\!]\,\partial_t^2\psi)\right). \qquad (7.21)\end{aligned}$$

Similarly, we have

$$\mathcal{P}_{\neq 0}(\rho^-\partial_t^2\psi) = \sigma\mathfrak{N}_\psi^-\widetilde{\mathfrak{N}}^{-1}\mathfrak{N}_\psi^+(\mathcal{H}(\psi)) + \mathfrak{N}_\psi^-\widetilde{\mathfrak{N}}^{-1}F_\psi^+ + \mathfrak{N}_\psi^+\widetilde{\mathfrak{N}}^{-1}F_\psi^- - \mathfrak{N}_\psi^-\widetilde{\mathfrak{N}}^{-1}\left(\mathcal{P}_{\neq 0}([\![\rho]\!]\,\partial_t^2\psi)\right). \tag{7.22}$$

Now, using the expressions of $\mathfrak{N}_\psi^\pm$ in terms of $\widetilde{\mathfrak{N}}$ and $\mathfrak{N}_\psi^+ - \mathfrak{N}_\psi^-$, we have

$$\mathfrak{N}_\psi^+\widetilde{\mathfrak{N}}^{-1}\mathfrak{N}_\psi^- f = \frac{1}{2}\mathfrak{N}_\psi^+\widetilde{\mathfrak{N}}^{-1}(\widetilde{\mathfrak{N}} f + (\mathfrak{N}_\psi^+ - \mathfrak{N}_\psi^-)f) = \frac{1}{2}\mathfrak{N}_\psi^+ f + \frac{1}{2}\mathfrak{N}_\psi^+\widetilde{\mathfrak{N}}^{-1}(\mathfrak{N}_\psi^+ - \mathfrak{N}_\psi^-)f, \tag{7.23}$$

$$\mathfrak{N}_\psi^-\widetilde{\mathfrak{N}}^{-1}\mathfrak{N}_\psi^+ f = \frac{1}{2}\mathfrak{N}_\psi^-\widetilde{\mathfrak{N}}^{-1}(\widetilde{\mathfrak{N}} f - (\mathfrak{N}_\psi^+ - \mathfrak{N}_\psi^-)f) = \frac{1}{2}\mathfrak{N}_\psi^- f - \frac{1}{2}\mathfrak{N}_\psi^-\widetilde{\mathfrak{N}}^{-1}(\mathfrak{N}_\psi^+ - \mathfrak{N}_\psi^-)f, \tag{7.24}$$

and also for $g^\pm:\Sigma\to\mathbb{R}$ with $\int_\Sigma g^\pm\,\mathrm{d}x' = 0$, we have

$$\begin{aligned}\mathfrak{N}_\psi^-\widetilde{\mathfrak{N}}^{-1}g^+ + \mathfrak{N}_\psi^+\widetilde{\mathfrak{N}}^{-1}g^- =& \frac{1}{2}(\widetilde{\mathfrak{N}} - (\mathfrak{N}_\psi^+ - \mathfrak{N}_\psi^-))\widetilde{\mathfrak{N}}^{-1}g^+ + \frac{1}{2}(\widetilde{\mathfrak{N}} + (\mathfrak{N}_\psi^+ - \mathfrak{N}_\psi^-))\widetilde{\mathfrak{N}}^{-1}g^-\\ =& \frac{g^+ + g^-}{2} - \frac{1}{2}(\mathfrak{N}_\psi^+ - \mathfrak{N}_\psi^-)\widetilde{\mathfrak{N}}^{-1}\,[\![g]\!]\,. \qquad (7.25)\end{aligned}$$

Let $f = \mathcal{H}(\psi)$ and $g^\pm = F^\pm_\psi$ in (7.23)-(7.25). We find that (7.21) + (7.22) can be written as

$$\begin{aligned}\mathcal{P}_{\neq 0}(\rho^+\partial_t^2\psi) + \mathcal{P}_{\neq 0}(\rho^-\partial_t^2\psi) = &\frac{\sigma}{2}\left(\mathfrak{N}^+_\psi + \mathfrak{N}^-_\psi\right)(\mathcal{H}(\psi)) + F^+_\psi + F^-_\psi \\ &+ \frac{\sigma}{2}(\mathfrak{N}^+_\psi - \mathfrak{N}^-_\psi)\widetilde{\mathfrak{N}}^{-1}(\mathfrak{N}^+_\psi - \mathfrak{N}^-_\psi)(\mathcal{H}(\psi)) \\ &- (\mathfrak{N}^+_\psi - \mathfrak{N}^-_\psi)\widetilde{\mathfrak{N}}^{-1}(F^+_\psi - F^-_\psi) + (\mathfrak{N}^+_\psi - \mathfrak{N}^-_\psi)\widetilde{\mathfrak{N}}^{-1}\left(\mathcal{P}_{\neq 0}([\![\rho]\!]\,\partial_t^2\psi)\right). \end{aligned} \tag{7.26}$$

Recall that $F^\pm_\psi = \mathfrak{F}^\pm_\psi - (\rho^\pm\partial_t^2\psi)_\Sigma$ and $\rho^\pm\partial_t^2\psi = \mathcal{P}_{\neq 0}(\rho^\pm\partial_t^2\psi) + (\rho^\pm\partial_t^2\psi)_\Sigma$ where

$$\mathfrak{F}^\pm_\psi := -N\cdot\nabla^\varphi q^\pm_w + \left(\bar{b}^\pm_i\bar{b}^\pm_j - \rho^\pm\bar{v}^\pm_i\bar{v}^\pm_j\right)\overline{\partial}_i\overline{\partial}_j\psi - 2(\rho^\pm\bar{v}^\pm\cdot\overline{\nabla})\partial_t\psi.$$

Thus, the evolution equation of the free interface becomes

$$\begin{aligned}(\rho^+ + \rho^-)\partial_t^2\psi = &\frac{\sigma}{2}\left(\mathfrak{N}^+_\psi + \mathfrak{N}^-_\psi\right)(\mathcal{H}(\psi)) + \left(\bar{b}^+_i\bar{b}^+_j - \rho^+\bar{v}^+_i\bar{v}^+_j + \bar{b}^-_i\bar{b}^-_j - \rho^-\bar{v}^-_i\bar{v}^-_j\right)\overline{\partial}_i\overline{\partial}_j\psi - 2(\rho^+\bar{v}^+_i + \rho^-\bar{v}^-_i)\overline{\partial}_i\partial_t\psi \\ &- N\cdot\nabla^\varphi q^+_w - N\cdot\nabla^\varphi q^-_w \\ &+ \frac{\sigma}{2}(\mathfrak{N}^+_\psi - \mathfrak{N}^-_\psi)\widetilde{\mathfrak{N}}^{-1}(\mathfrak{N}^+_\psi - \mathfrak{N}^-_\psi)(\mathcal{H}(\psi)) - (\mathfrak{N}^+_\psi - \mathfrak{N}^-_\psi)\widetilde{\mathfrak{N}}^{-1}\left([\![\mathfrak{F}_\psi - \rho\partial_t^2\psi]\!]\right), \end{aligned} \tag{7.27}$$

where the first line is expected to give the $\sqrt{\sigma}$-weighted regularity (contributed by surface tension) and the non-weighted regularity (provided that stability condition (1.40)) for the free interface, the second line will be converted to the interior estimate of the right side of (7.15), and the last line consists of remainder terms that can be directly controlled by using paradifferential calculus.

7.2.2 Preliminaries on paradifferential calculus

In the equation (7.27), the term $\left(\mathfrak{N}^+_\psi + \mathfrak{N}^-_\psi\right)(\mathcal{H}(\psi))$ is a fully nonlinear term. Although it is well-known that the Dirichlet-to-Neumann operator is a first-order elliptic operator and the mean-curvature operator is a second-order elliptic operator, it is still necessary for us to find out their concrete forms and "symmetrize" the paradifferential formulations in order for an explicit energy estimate. In the remaining part of this paper, we will introduce several preliminary lemmas about paradifferential calculus that have been proven in Alazard-Burq-Zuily [2]. Following the notations in Métivier [61], we first introduce the basic definition of a paradifferential operator. Note that the dimension d below is not the same as the one in Section 1.

Definition 7.2 (Symbols)**.** Given $r \geq 0$, $m \in \mathbb{R}$, we denote $\Gamma^m_r(\mathbb{T}^d)$ to be the space of locally bounded functions $a(x',\xi)$ on $\mathbb{T}^d\times(\mathbb{R}^d\backslash\{0\})$, which are C^∞ with respect to $\xi(\xi\neq\mathbf{0})$, such that for any $\alpha\in\mathbb{N}^d, \xi\neq\mathbf{0}$, the function $x'\mapsto\partial^\alpha_\xi a(x',\xi)$ belongs to $W^{r,\infty}(\mathbb{T}^d)$ and there exists a constant C_α such that

$$\left|\partial^\alpha_\xi a(\cdot,\xi)\right|_{W^{r,\infty}(\mathbb{T}^d)} \leq C_\alpha(1+|\xi|)^{m-|\alpha|},\quad \forall|\xi|\geq 1/2.$$

Definition 7.3 (Paradifferential operator)**.** Given a symbol a, we shall define the **paradifferential operator** T_a by

$$\widehat{T_a u}(\xi) := (2\pi)^{-d}\int_{\mathbb{R}^d}\tilde{\chi}(\xi-\eta,\eta)\hat{a}(\xi-\eta,\eta)\phi(\eta)\hat{u}(\eta)\,\mathrm{d}\eta \tag{7.28}$$

where $\hat{a}(\theta,\xi) = \int_{\mathbb{T}^d}\exp(-ix'\cdot\theta)a(x',\xi)\,\mathrm{d}x'$ is the Fourier transform of a in variable x'. Here $\tilde{\chi}$ and ϕ are two given cut-off functions such that

$$\phi(\eta) = 0\ \text{ for } |\eta|\leq 1,\quad \phi(\eta) = 1\ \text{ for } |\eta|\geq 2,$$

and $\tilde{\chi}(\theta,\eta)$ is homogeneous of degree 0 and satisfies that for $0<\varepsilon_1<\varepsilon_2\ll 1$, $\tilde{\chi}(\theta,\eta) = 1$ if $|\theta|\leq\varepsilon_1|\eta|$ and $\tilde{\chi}(\theta,\eta) = 0$ if $|\theta|\geq\varepsilon_2|\eta|$. We also introduce the semi-norm

$$M^a_r(a) := \sup_{|\alpha|\leq\frac{d}{2}+1+r}\ \sup_{|\xi|\geq 1/2}\left|(1+|\xi|)^{|\alpha|-m}\partial^\alpha_\xi a(\cdot,\xi)\right|_{W^{r,\infty}(\mathbb{T}^d)}. \tag{7.29}$$

For $m \in \mathbb{R}$, we say T is of order m if for all $s \in \mathbb{R}$, T is bounded from H^s to H^{s-m}.

Proposition 7.1. Let $m \in \mathbb{R}$. If $a \in \Gamma_0^m(\mathbb{T}^d)$, then T_a is of order m. Moreover, for any $s \in \mathbb{R}$, there exists a constant K such that $\|T_a\|_{H^s \to H^{s-m}} \le K M_0^m(a)$.

Proposition 7.2 (Composition, [2, Theorem 3.7])**.** Let $m \in \mathbb{R}$ and $r > 0$. If $a \in \Gamma_r^m(\mathbb{T}^d)$, $b \in \Gamma_r^{m'}(\mathbb{T}^d)$, then $T_a T_b - T_{a\#b}$ is of order $m + m' - r$ where

$$a\#b := \sum_{|\alpha|<r} \frac{1}{i^{|\alpha|}\alpha!} \partial_\xi^\alpha a \partial_{x'}^\alpha b.$$

Moreover, for all $s \in \mathbb{R}$, there exists a constant K such that

$$\|T_a T_b - T_{a\#b}\|_{H^s \to H^{s-m-m'+r}} \le K M_r^m(a) M_r^{m'}(b). \tag{7.30}$$

Proposition 7.3 (Adjoint, [2, Theorem 3.10])**.** Let $m \in \mathbb{R}$, $r > 0$ and $a \in \Gamma_r^m(\mathbb{T}^d)$. We denote by $(T_a)^*$ the adjoint operator of T_a. Then $(T_a)^* - T_{a^*}$ is of order $m - r$ where

$$a^* := \sum_{|\alpha|<r} \frac{1}{i^\alpha \alpha!} \partial_\xi^\alpha \partial_{x'}^\alpha \bar{a}.$$

Moreover, for any $s \in \mathbb{R}$, there exists a constant K such that $\|(T_a)^* - T_{a^*}\|_{H^s \to H^{s-m+r}} \le K M_r^m(a)$.

The symbolic calculus adopted in this paper is not of C^∞-regularity. We shall introduce the following class of symbols. Here and thereafter in this section, $\psi \in C([0,T]; H^{s+\frac{1}{2}}(\mathbb{T}^d))$ is a given function with $s > 2 + \frac{d}{2}$.

Definition 7.4. Given $m \in \mathbb{R}$, we denote Σ^m to be the class of symbols a of the form $a = a^{(m)} + a^{(m-1)}$ with

$$a^{(m)}(t, x', \xi) = F(\overline{\nabla}_{x'}\psi(t, x'), \xi), \quad a^{(m-1)}(t, x', \xi) = \sum_{|\alpha|_2} G_\alpha(\overline{\nabla}_{x'}\psi(t, x'), \xi) \partial_{x'}^\alpha \psi(t, x')$$

such that

i. T_a maps real-valued functions to real-valued functions;
ii. F is a C^∞ real-valued functions of $(\zeta, \xi) \in \mathbb{R}^d \times (\mathbb{R}^d \backslash \{0\})$, homogeneous of degree m in ξ, such that there exists a continuous function $K = K(\zeta) > 0$ such that $F(\zeta, \xi) \ge K(\zeta)|\xi|^m$ for all $(\zeta, \xi) \in \mathbb{R}^d \times (\mathbb{R}^d \backslash \{0\})$;
iii. G_α is a C^∞ complex-valued function of $(\zeta, \xi) \in \mathbb{R}^d \times (\mathbb{R}^d \backslash \{0\})$, homogeneous of degree $m - 1$ in ξ.

Definition 7.5 ("Equivalence" of operators)**.** Given $m \in \mathbb{R}$ and consider two families of operators of order m: $\{A(t) : t \in [0,T]\}$ and $\{B(t) : t \in [0,T]\}$, We say $A \sim B$ if $A - B$ is of order $m - 1.5$ and satisfies the estimate: for all $r \in \mathbb{R}$ there exists a continuous function $C(\cdot)$ such that

$$\forall t \in [0,T], \quad \|A(t) - B(t)\|_{H^r \to H^{r-(m-1.5)}} \le C(|\psi(t)|_{s+\frac{1}{2}}).$$

From now on, we use the notation $|\cdot|_{s_1 \to s_2}$ to represent the operator norm $\|\cdot\|_{H^{s_1} \to H^{s_2}}$, and use the notation $|\cdot|_s$ to represent $\|\cdot\|_{H^s(\mathbb{T}^d)}$, as we only apply paradifferential calculus on the free interface Σ. With this definition, we have

Proposition 7.4 ([2, Prop. 4.3])**.** Let $m, m' \in \mathbb{R}$. Then

1. If $a \in \Sigma^m$, $b \in \Sigma^{m'}$, then $T_a T_b \sim T_{a\#b}$ where $a\#b$ is given by

$$a\#b = a^{(m)} b^{(m')} + a^{(m-1)} b^{(m')} + a^{(m)} b^{(m'-1)} + \frac{1}{i} \partial_\xi a^{(m)} \cdot \partial_{x'} b^{(m')}.$$

2. If $a \in \Sigma^m$, then $(T_a)^* \sim T_b$ where $b \in \Sigma^m$ is given by

$$b = a^{(m)} + \overline{a^{(m-1)}} + \frac{1}{i}(\partial_{x'} \cdot \partial_\xi) a^{(m)}.$$

As a corollary, we have

Corollary 7.5 ([2, Prop. 4.3(2)]). If $a \in \Sigma^m$ satisfies $\mathbf{Im} a^{(m-1)} = -0.5(\partial_\xi \cdot \partial_{x'})a^{(m)}$, then $(T_a)^* \sim T_a$.

The next proposition is significant for the estimate of Sobolev norms via paradifferential calculus.

Proposition 7.6 ([2, Prop. 4.4 and 4.6]). Let $m \in \mathbb{R},\ r \in \mathbb{R}$. Then for all symbol $a \in \Sigma^m$ and $t \in [0, T]$, the following estimate holds.

$$|T_{a(t)}u|_{r-m} \leq C(|\psi(t)|_{s-1})|u|_r, \tag{7.31}$$

$$|u|_{r+m} \leq C(|\psi(t)|_{s-1})\left(|T_{a(t)}u|_r + |u|_0\right). \tag{7.32}$$

7.2.3 Paralinearization of the nonlinear terms

Now we can start to paralinearize the term $\left(\mathfrak{N}^+_\psi + \mathfrak{N}^-_\psi\right)(\mathcal{H}(\psi))$ in (7.27).

Lemma 7.7 (Paralinearization of the Dirichlet-to-Neumann operator, [4, Section 4.4]). For $f, \psi \in H^{s+\frac{1}{2}}(\mathbb{T}^d)$, we have

$$\mathfrak{N}^\pm_\psi f = T_{\Lambda^\pm}\psi + R^\pm_{\Lambda,1}(\psi, f) + R^\pm_{\Lambda,2}(\psi, f), \tag{7.33}$$

with the symbols $\lambda^\pm = \lambda^{(1),\pm} + \lambda^{(0),\pm}$ give by

$$\Lambda^{(1),\pm} = \sqrt{(1 + |\overline{\nabla}\psi|^2)|\xi|^2 - (\overline{\nabla}\psi \cdot \xi)^2}, \tag{7.34}$$

$$\Lambda^{(0),-} = -\overline{\Lambda^{(0),+}} = \frac{1 + |\overline{\nabla}\psi|^2|}{2\Lambda^{(1),-}}\left(\overline{\nabla} \cdot (\alpha^{(1)}\overline{\nabla}\psi) + i\partial_\xi\Lambda^{(1),-} \cdot \overline{\nabla}\alpha^{(1)}\right), \tag{7.35}$$

and $\alpha^{(1)} := (\Lambda^{(1),-} + i\overline{\nabla}\psi \cdot \xi)/(1 + |\overline{\nabla}\psi|^2)$. The remainder terms satisfy the following estimates

$$|R^\pm_{\Lambda,1}(\psi, f)|_{s-\frac{1}{2}} \leq C(|\psi|_{C^2}, |f|_3)|f|_{s+\frac{1}{2}}, \quad |R^\pm_{\Lambda,2}(\psi, f)|_{s-\frac{1}{2}} \leq C(|\psi|_{s-\frac{1}{2}})|\overline{\partial} f|_{s-2}. \tag{7.36}$$

Lemma 7.8 (Paralinearization of the mean curvature operator, [2, Lemma 3.25]). There holds $\mathcal{H}(\psi) = -T_{\mathfrak{H}}f + R_{\mathfrak{H}}$ where $\mathfrak{H} = \mathfrak{H}^{(2)} + \mathfrak{H}^{(1)}$ is defined by

$$\mathfrak{H}^{(2)} = \frac{1}{\sqrt{1 + |\overline{\nabla}\psi|^2}}\left(|\xi|^2 - \frac{(\overline{\nabla}\psi \cdot \xi)^2}{1 + |\overline{\nabla}\psi|^2}\right), \tag{7.37}$$

$$\mathfrak{H}^{(1)} = -\frac{i}{2}(\overline{\nabla}_{x'} \cdot \partial_\xi)\mathfrak{H}^{(2)}, \tag{7.38}$$

and the remainder term $R_{\mathfrak{H}}$ satisfies

$$|R_{\mathfrak{H}}|_{2s-3} \leq C(|\psi|_{s+\frac{1}{2}}). \tag{7.39}$$

With the paralinearization of operators $\mathfrak{N}^\pm_\psi$ and $\mathcal{H}(\psi)$, the term $\left(\mathfrak{N}^+_\psi + \mathfrak{N}^-_\psi\right)(\mathcal{H}(\psi))$ in (7.27) becomes

$$\sigma\left(\mathfrak{N}^+_\psi + \mathfrak{N}^-_\psi\right)(\mathcal{H}(\psi)) = -\sigma T_\Lambda T_{\mathfrak{H}}\psi + \sigma\mathcal{R}^\sigma_\psi, \tag{7.40}$$

where $-\overline{\Lambda^{(0),+}} = \Lambda^{(0),-}$ shows that $\mathbf{Re}(\Lambda^{(0),+}) + \mathbf{Re}(\Lambda^{(0),-}) = 0$, $\mathbf{Im}(\Lambda^{(0),+}) = \mathbf{Im}(\Lambda^{(0),+})$ and thus

$$\Lambda := \underbrace{(\Lambda^{(1),+} + \Lambda^{(1),-})}_{=:\Lambda^{(1)}} + \underbrace{(\Lambda^{(0),+} + \Lambda^{(0),-})}_{=:\Lambda^{(0)}} = 2\Lambda^{(1),-} + 2i\mathbf{Im}(\Lambda^{(0),-}) \tag{7.41}$$

$$\mathcal{R}^\sigma_\psi := \sum_\pm T_{\Lambda^\pm}R_{\mathfrak{H}} + R^\pm_{\Lambda,1}(\psi, \mathcal{H}(\psi)) + R^\pm_{\Lambda,2}(\psi, \mathcal{H}(\psi)) \text{ and } |\mathcal{R}^\sigma_\psi|_{s-\frac{1}{2}} \leq C(|\psi|_{s+\frac{1}{2}})|\psi|_{s+1}. \tag{7.42}$$

In order for an explicit energy estimate for ψ and ψ_t, we shall symmetrize the 3-rd order paradifferential operator $T_\Lambda T_{\mathfrak{H}}$. That is, find suitable symbols $\mathfrak{m} \in \Sigma^{1.5}$ and $\mathfrak{n} \in \Sigma^0$ such that $T_{\mathfrak{n}}T_\Lambda T_{\mathfrak{H}} \sim T_{\mathfrak{m}}T_{\mathfrak{m}}T_{\mathfrak{n}}$ and $T_{\mathfrak{m}} \sim (T_{\mathfrak{m}})^*$.

Proposition 7.9 (Symmetrisation of the composition)**.** Let $\mathfrak{n} \in \Sigma^0$ and $\mathfrak{m} \in \Sigma^{1.5}$ be defined by

$$\mathfrak{n} := \frac{1}{\sqrt[3]{2}\sqrt[4]{1+|\overline{\nabla}\psi|^2}} = 2^{-\frac{1}{3}}|N|^{-\frac{1}{2}}, \tag{7.43}$$

$$\mathfrak{m} := \underbrace{\sqrt{\mathfrak{H}^{(2)}\Lambda^{(1)}}}_{=:\mathfrak{m}^{(1.5)}} + \underbrace{\frac{1}{2i}(\partial_\xi \cdot \partial_{x'})\sqrt{\mathfrak{H}^{(2)}\Lambda^{(1)}}}_{=:\mathfrak{m}^{(0.5)}}. \tag{7.44}$$

Then $T_\mathfrak{n} T_\lambda T_\mathfrak{H} \sim T_\mathfrak{m} T_\mathfrak{m} T_\mathfrak{n}$ and $T_\mathfrak{m} \sim (T_\mathfrak{m})^*$ are both fulfilled.

Proof. Given the symbol Λ and $\mathfrak{H}$, we shall find suitable symbols $\mathfrak{n} \in \Sigma^0, \mathfrak{m} \in \Sigma^{1.5}$ such that $\mathfrak{n}(x', \xi)$ is independent of ξ and $\mathfrak{n}\#(\Lambda\#\mathfrak{H}) = (\mathfrak{m}\#\mathfrak{m})\#\mathfrak{n}$, i.e.,

$$\begin{aligned}&\mathfrak{n}^{(0)}(\Lambda\#\mathfrak{H}) + \mathfrak{n}^{(-1)}\Lambda^{(1)}\mathfrak{H}^{(2)} + \frac{1}{i}\partial_\xi \mathfrak{n}^{(0)} \cdot \partial_{x'}(\Lambda^{(1)}\mathfrak{H}^{(2)}) \\ &= (\mathfrak{m}\#\mathfrak{m})\mathfrak{n}^{(0)} + (\mathfrak{m}^{(1.5)})^2\mathfrak{n}^{(-1)} + \frac{1}{i}\partial_\xi((\mathfrak{m}^{(1.5)})^2) \cdot \partial_{x'}\mathfrak{n}^{(0)}.\end{aligned}$$

Recall that

$$\begin{aligned}(\Lambda\#\mathfrak{H}) &= \Lambda^{(1)}\mathfrak{H}^{(2)} + \Lambda^{(0)}\mathfrak{H}^{(2)} + \Lambda^{(1)}\mathfrak{H}^{(1)} + \frac{1}{i}\partial_\xi \mathfrak{H}^{(2)} \cdot \partial_{x'}\Lambda^{(1)}, \\ (\mathfrak{m}\#\mathfrak{m}) &= (\mathfrak{m}^{(1.5)})^2 + 2(\mathfrak{m}^{(1.5)})(\mathfrak{m}^{(0.5)}) + \frac{1}{i}\partial_\xi(\mathfrak{m}^{(1.5)}) \cdot \partial_{x'}(\mathfrak{m}^{(1.5)}).\end{aligned}$$

We choose the principal symbol $\mathfrak{m}^{(1.5)} := \sqrt{\Lambda^{(1)}\mathfrak{H}^{(2)}}$ in order for cancelling the leading-order symbols. Since we require $(T_\mathfrak{m})^* \sim T_{\mathfrak{m}^*}$, we must have $\mathbf{Im}(\mathfrak{m}^{(0.5)}) = -0.5(\partial_{x'} \cdot \partial_\xi)\mathfrak{m}^{(1.5)}$ (cf. [2, Prop. 4.3]). With this choice for $\mathfrak{m}$, it remains to solve the symbolic equation

$$\mathfrak{n}^{(0)}(\Lambda\#\mathfrak{H} - \mathfrak{m}\#\mathfrak{m}) = \frac{1}{i}\partial_\xi((\Lambda^{(1)}\mathfrak{H}^{(2)}) \cdot \partial_{x'}\mathfrak{n}^{(0)} - \frac{1}{i}\partial_{x'}(\Lambda^{(1)}\mathfrak{H}^{(2)}) \cdot \partial_\xi \mathfrak{n}^{(0)}, \tag{7.45}$$

with

$$\Lambda\#\mathfrak{H} - \mathfrak{m}\#\mathfrak{m} = \Lambda^{(0)}\mathfrak{H}^{(2)} + \Lambda^{(1)}\mathfrak{H}^{(1)} - 2(\mathfrak{m}^{(1.5)})(\mathfrak{m}^{(0.5)}) + \frac{1}{i}\partial_\xi \mathfrak{H}^{(2)} \cdot \partial_{x'}\Lambda^{(1)} - \frac{1}{i}\partial_\xi(\mathfrak{m}^{(1.5)}) \cdot \partial_{x'}(\mathfrak{m}^{(1.5)}).$$

The sub-principle $\mathfrak{n}^{(-1)}$ does not appear, so we can choose $\mathfrak{n}^{(-1)} = 0$. Since the principal symbols of Λ and $\mathfrak{H}$ are real-valued, we now just need to solve

$$\mathbf{Re}(\Lambda\#\mathfrak{H} - \mathfrak{m}\#\mathfrak{m}) = 0, \quad \mathfrak{n}^{(0)}\mathbf{Im}(\Lambda\#\mathfrak{H} - \mathfrak{m}\#\mathfrak{m}) = -\partial_\xi((\Lambda^{(1)}\mathfrak{H}^{(2)}) \cdot \partial_{x'}\mathfrak{n}^{(0)} + \partial_{x'}(\Lambda^{(1)}\mathfrak{H}^{(2)}) \cdot \partial_\xi \mathfrak{n}^{(0)}.$$

The condition for the real part is fulfilled if we have

$$\underbrace{\mathbf{Re}(\Lambda^{(0)})}_{=0}\mathfrak{H}^{(2)} = 2\mathfrak{m}^{(1.5)}\mathbf{Re}(\mathfrak{m}^{(0.5)}) \Rightarrow \mathbf{Re}(\mathfrak{m}^{(0.5)}) = 0.$$

For the imaginary part, inserting the symbols $\mathfrak{H}^{(1)}$, $\mathbf{Im}(\Lambda^{(0)})$, $\mathfrak{m}^{(1.5)}$ and $\mathbf{Im}(\mathfrak{m}^{(0.5)})$, we get

$$\mathbf{Im}(\Lambda\#\mathfrak{H} - \mathfrak{m}\#\mathfrak{m}) = \frac{1}{2}\partial_\xi \mathfrak{H}^{(2)} \cdot \partial_{x'}\Lambda^{(1)} - \frac{1}{2}\partial_{x'}\mathfrak{H}^{(2)} \cdot \partial_\xi \Lambda^{(1)},$$

and thus we need to solve

$$\mathfrak{n}^{(0)}\left(\frac{1}{2}\partial_\xi \mathfrak{H}^{(2)} \cdot \partial_{x'}\Lambda^{(1)} - \frac{1}{2}\partial_{x'}\mathfrak{H}^{(2)} \cdot \partial_\xi \Lambda^{(1)}\right) = -\partial_\xi((\Lambda^{(1)}\mathfrak{H}^{(2)}) \cdot \partial_{x'}\mathfrak{n}^{(0)} + \partial_{x'}(\Lambda^{(1)}\mathfrak{H}^{(2)}) \cdot \partial_\xi \mathfrak{n}^{(0)}. \tag{7.46}$$

Notice that $\mathfrak{H}^{(2)} = (c\Lambda^{(1)})^2$ with $c = \frac{1}{2}(1+|\overline{\nabla}\psi|^2)^{-\frac{3}{4}}$. Plugging it to the above equation, after a long and tedious calculation, we get the following relation

$$\mathfrak{n}^{(0)}\left(c^2\partial_{x'}\Lambda^{(1)}(\Lambda^{(1)}) - (\partial_{x'}c)c(\Lambda^{(1)})^2 - c^2\partial_{x'}(\Lambda^{(1)})\Lambda^{(1)}\right) = -3c^2(\Lambda^{(1)})^2\partial_{x'}\mathfrak{n}^{(0)},$$

that is,

$$\frac{\partial_{x'}\mathfrak{n}^{(0)}}{\mathfrak{n}^{(0)}}=\frac{-(\partial_{x'}c)c(\Lambda^{(1)})^2}{-3c^2(\Lambda^{(1)})^2}=\frac{1}{3}\frac{\partial_{x'}c}{c}\Rightarrow\mathfrak{n}^{(0)}=c^{\frac{1}{3}}=2^{-\frac{1}{3}}(1+|\overline{\nabla}\psi|^2)^{-\frac{1}{4}}.$$

□

We expect to take $(s-\frac{1}{2})$-th order derivatives in (7.27). In view of the paradifferential formulation, we shall alternatively take $T_{\mathfrak{M}}$ with

$$\mathfrak{M}:=(\mathfrak{m}^{(1.5)})^{\frac{2s-1}{3}}=2^{\frac{2s-1}{6}}|\xi|^{s-\frac{1}{2}}\left(1-\left|\frac{N}{|N|}\cdot\frac{\xi}{|\xi|}\right|^2\right)^{\frac{s}{2}-\frac{1}{4}}\in\Sigma^{s-\frac{1}{2}}\tag{7.47}$$

for sake of simplicity. Below, we list several commutator estimates for the paradifferential operators.

Lemma 7.10. For any $r\in\mathbb{R}$, $s>2+\frac{d}{2}$, any functions a and f, the following commutator estimates hold

$$\begin{aligned}|[T_{\mathfrak{M}},T_{\mathfrak{m}}]|_{r+s-1\to r}&\le C\left(|\psi|_{s+0.5}\right),\\ |[T_{\mathfrak{n}},T_a]f|_{s-\frac{1}{2}}&\le C(|\overline{\nabla}\psi|_{W^{1,\infty}})|a|_{W^{1,\infty}}|f|_{s-1.5},\\ |[T_{\mathfrak{M}},a]\,T_{\mathfrak{n}}f|_0+|T_{\mathfrak{M}}\,[T_{\mathfrak{n}},a]\,f|_0&\le C(|\overline{\nabla}\psi|_{W^{1,\infty}})|a|_{s-0.5}|f|_{s-1.5},\\ |[T_{\mathfrak{m}},a]f|_0&\le C(|\overline{\nabla}\psi|_{W^{1,\infty}})|a|_{s-0.5}|f|_{0.5}.\end{aligned}$$

We also need to commute ∂_t with paradifferential operators. These steps will generate paradifferential operators whose symbols are spatial or time derivatives.

Lemma 7.11. For any $r\in\mathbb{R}$, the following estimates hold

$$\begin{aligned}|T_{\partial_t\mathfrak{n}}|_{r\to r}+|T_{\partial_t\mathfrak{M}}|_{r\to r-(s-0.5)}&\le C(|\psi|_{W^{1,\infty}},|\partial_t\psi|_{W^{1,\infty}}),\\ |T_{\partial_t^2\mathfrak{n}}|_{r\to r}+|T_{\partial_t^2\mathfrak{M}}|_{r\to r-(s-0.5)}&\le C\left(\left|\overline{\nabla}\psi,\partial_t\psi,\partial_t^2\psi\right|_{W^{1,\infty}}\right),\\ |T_{\overline{\partial}\mathfrak{n}}|_{r\to r}+|T_{\overline{\partial}\mathfrak{M}}|_{r\to r-(s-0.5)}&\le C(|\overline{\nabla}\psi|_{W^{1,\infty}}),\\ |T_{\partial_t\mathfrak{m}}|_{r\to r-1.5}+|T_{\overline{\partial}\mathfrak{m}}|_{r\to r-1.5}&\le C(|\overline{\nabla}\psi|_{W^{1,\infty}},|\overline{\nabla}\partial_t\psi|_{W^{1,\infty}}).\end{aligned}$$

7.3 Uniform estimates for the free interface

With the symmetrized paralinearization of $(\mathfrak{N}_\psi^++\mathfrak{N}_\psi^-)\mathcal{H}(\psi)$ derived in Section 7.2.2, we can now prove the uniform-in-(ε,σ) estimates of ψ under the stability condition (1.40). The equation (7.27) can be written as

$$\begin{aligned}(\rho^++\rho^-)\partial_t^2\psi=&-\frac{\sigma}{2}T_\Lambda T_{\mathfrak{H}}\psi-(\rho^++\rho^-)(\bar{\mathbf{w}}_i\bar{\mathbf{w}}_j)\overline{\partial}_i\overline{\partial}_j\psi-2(\rho^++\rho^-)\bar{\mathbf{w}}_i\overline{\partial}_i\partial_t\psi\\&+\left(\rho^+(\bar{\mathbf{b}}_i^+\bar{\mathbf{b}}_j^+-\bar{\mathbf{u}}_i\bar{\mathbf{u}}_j)+\rho^-(\bar{\mathbf{b}}_i^-\bar{\mathbf{b}}_j^--\bar{\mathbf{u}}_i\bar{\mathbf{u}}_j)\right)\overline{\partial}_i\overline{\partial}_j\psi\\&-(N\cdot\nabla^\varphi q_w^++N\cdot\nabla^\varphi q_w^-)+\Psi^R,\end{aligned}\tag{7.48}$$

where $T_\Lambda,T_{\mathfrak{H}}$ are the paradifferential operators defined in Proposition 7.7 and Proposition 7.8, the quantities $\mathbf{w},\mathbf{u},\mathbf{b}$ are defined by

$$\mathbf{w}:=\frac{\rho^+v^++\rho^-v^-}{\rho^++\rho^-},\quad \mathbf{u}:=\frac{\sqrt{\rho^+\rho^-}}{\rho^++\rho^-}[\![v]\!],\quad \mathbf{b}^\pm:=\frac{b^\pm}{\sqrt{\rho^\pm}},\tag{7.49}$$

and Ψ^R is defined by

$$\Psi^R:=\frac{\sigma}{2}(\mathfrak{N}_\psi^+-\mathfrak{N}_\psi^-)\widetilde{\mathfrak{N}}^{-1}(\mathfrak{N}_\psi^+-\mathfrak{N}_\psi^-)(\mathcal{H}(\psi))-(\mathfrak{N}_\psi^+-\mathfrak{N}_\psi^-)\widetilde{\mathfrak{N}}^{-1}\left(\left[\!\!\left[\mathfrak{F}_\psi-\rho\partial_t^2\psi\right]\!\!\right]\right)+\frac{\sigma}{2}\mathcal{R}_\psi^\sigma.\tag{7.50}$$

We pick $s=4$ in the paradifferential operator $T_{\mathfrak{M}}$, that is, $\mathfrak{M}=(\mathfrak{m}^{(1.5)})^{\frac{7}{3}}\in\Sigma^{3.5}$ and then consider the energy functionals

$$\mathcal{E}(t):=\frac{1}{2}\int_\Sigma(\rho^++\rho^-)\left|(\partial_t+\bar{\mathbf{w}}\cdot\overline{\nabla})T_{\mathfrak{M}}T_{\mathfrak{n}}\psi\right|^2\,\mathrm{d}x'+\frac{1}{4}\int_\Sigma\left|\sqrt{\sigma}T_{\mathfrak{m}}T_{\mathfrak{M}}T_{\mathfrak{n}}\psi\right|_0^2\,\mathrm{d}x',\tag{7.51}$$

$$\widetilde{\mathcal{E}}(t):=\frac{1}{2}\int_\Sigma\rho^+\left(\left|\bar{\mathbf{b}}^+\cdot\overline{\nabla}T_{\mathfrak{M}}T_{\mathfrak{n}}\psi\right|^2-\left|\bar{\mathbf{u}}\cdot\overline{\nabla}T_{\mathfrak{M}}T_{\mathfrak{n}}\psi\right|^2\right)+\rho^-\left(\left|\bar{\mathbf{b}}^-\cdot\overline{\nabla}T_{\mathfrak{M}}T_{\mathfrak{n}}\psi\right|^2-\left|\bar{\mathbf{u}}\cdot\overline{\nabla}T_{\mathfrak{M}}T_{\mathfrak{n}}\psi\right|^2\right)\mathrm{d}x'.\tag{7.52}$$

Lemma 7.12 (Comparison between $\mathcal{E}, \widetilde{\mathcal{E}}$ and Sobolev norms)**.** For any fixed $\sigma > 0$, we have the following relations between $\mathcal{E}, \widetilde{\mathcal{E}}$ and standard Sobolev norms.

$$\begin{aligned} \left|\sqrt{\sigma}\psi\right|_5^2 &\leq C(|\overline{\nabla}\psi|_{W^{1,\infty}})\left(\mathcal{E}(t) + |\sqrt{\sigma}\psi|_0^2\right), \\ |\psi|_{4.5}^2 &\lesssim |\sqrt{\sigma}\psi|_5^2 + \sigma^{-1}|\psi|_0^2, \\ |\psi_t|_{3.5}^2 &\leq C(|\overline{\nabla}\psi, \overline{\nabla}\psi_t, \bar{v}^\pm, \rho^\pm|_{W^{1,\infty}})\left(\mathcal{E}(t) + |\psi|_{4.5}^2 + |\psi_t|_0^2\right) \end{aligned}$$

where $C(\cdot)$ represents a generic positive continuous function in its arguments. Moreover, when the stability condition (1.40) holds, there exist positive continuous functions C_1, C_1', C_1'' depending on $|\overline{\nabla}\psi, \bar{v}^\pm, \bar{b}^\pm, \rho^\pm|_{W^{1,\infty}}$ and independent of σ, such that

$$C_1\left(|\overline{\nabla}\psi, \bar{v}^\pm, \bar{b}^\pm, \rho^\pm|_{W^{1,\infty}}\right)|\psi|_{4.5}^2 \leq \widetilde{\mathcal{E}}(t) + C_1'|\psi|_0^2, \quad \widetilde{\mathcal{E}}(t) \leq C_1''\left(|\overline{\nabla}\psi, \bar{v}^\pm, \bar{b}^\pm, \rho^\pm|_{W^{1,\infty}}\right)|\psi|_{4.5}^2.$$

Proof. Recall that $T_{\mathfrak{M}}$ and $T_{\mathfrak{m}}$ are paradifferential operators of order 3.5 and 1.5 respectively, thus the first inequality is a direct consequence of Proposition 7.6. The second inequality is a directly consequence of Sobolev interpolation and Young's inequality

$$|\psi|_{4.5}^2 \leq |\sqrt{\sigma}\psi|_5^{1.8}|\sigma^{-\frac{1}{2}}\psi|_0^{0.2} \leq \frac{|\sqrt{\sigma}\psi|_5^2}{10/9} + \frac{|\psi|_0^2}{10\sigma}.$$

To prove the third inequality, we again use Proposition 7.6 to get

$$\begin{aligned} |\psi_t|_{3.5}^2 &\leq C(|\overline{\nabla}\psi|_{W^{1,\infty}})\left(|T_{\mathfrak{M}}T_{\mathfrak{n}}\psi_t|_0^2 + |\psi_t|_0^2\right) \\ &\leq C(|\overline{\nabla}\psi, \bar{v}^\pm, \rho^\pm|_{W^{1,\infty}})\left(\mathcal{E}(t) + |T_{\mathfrak{M}}T_{\partial_t\mathfrak{n}}\psi|_0^2 + |T_{\partial_t\mathfrak{M}}T_{\mathfrak{n}}\psi|_0^2 + |\psi_t|_0^2\right) \\ &\leq C(|\overline{\nabla}\psi, \overline{\nabla}\psi_t, \bar{v}^\pm, \rho^\pm|_{W^{1,\infty}})\left(\mathcal{E}(t) + |\psi|_{3.5}^2 + |\psi_t|_0^2\right). \end{aligned}$$

For the last inequality, the right side is trivial. When the stability condition (1.40) holds, it suffices to prove that $\widetilde{\mathcal{E}}(t)$ is a positive-definite energy, then the left side automatically holds. Multiplying $(\rho^+\rho^-)^{-\frac{1}{2}}$ in (1.40), the stability condition becomes

$$\exists \delta_0 \in (0, \frac{1}{8}), \quad |\bar{\mathbf{b}}^+ \times \bar{\mathbf{b}}^-| \geq (1-\delta_0)^{-1}|\bar{\mathbf{b}}^\pm \times [\![\bar{v}]\!]|\sqrt{1 + (c_A^\pm/c_s^\pm)^2} > (1-\delta_0)^{-1}|\bar{\mathbf{b}}^\pm \times [\![\bar{v}]\!]|. \tag{7.53}$$

Since $[\![v]\!] \cdot N = 0$ and $\mathbf{b}^\pm$ are nonzero and not collinear, we may assume $[\![v]\!] = c_1\mathbf{b}^+ + c_2\mathbf{b}^-$. Plugging this into the stability condition, we get $c_1, c_2 \leq 1 - \delta_0$. Using Cauchy-Schwarz inequality, we derive that

$$\inf_{\substack{\mathbf{z}\in\mathbb{R}^2\\|\mathbf{z}|=1}} (1-\delta_0)^2\left((\bar{\mathbf{b}}^+\cdot\mathbf{z})^2 + 2(\bar{\mathbf{b}}^+\cdot\mathbf{z})(\bar{\mathbf{b}}^-\cdot\mathbf{z}) + (\bar{\mathbf{b}}^-\cdot\mathbf{z})^2\right) - ([\![\bar{v}]\!]\cdot\mathbf{z})^2 \geq 0$$

Invoking $\bar{\mathbf{u}} = \frac{\sqrt{\rho^+\rho^-}}{\rho^++\rho^-}[\![\bar{v}]\!]$ and using the non-collinearity, the above inequality implies that

$$\inf_{\substack{\mathbf{z}\in\mathbb{R}^2\\|\mathbf{z}|=1}} \left(\frac{\rho^+\rho^-}{\rho^++\rho^-}(\bar{\mathbf{b}}^+\cdot\mathbf{z})^2 + 2\frac{\rho^+\rho^-}{\rho^++\rho^-}(\bar{\mathbf{b}}^+\cdot\mathbf{z})(\bar{\mathbf{b}}^-\cdot\mathbf{z}) + \frac{\rho^+\rho^-}{\rho^++\rho^-}(\bar{\mathbf{b}}^-\cdot\mathbf{z})^2 - (\rho^++\rho^-)(\bar{\mathbf{u}}\cdot\mathbf{z})^2\right) > 0.$$

Notice that

$$\begin{aligned} &\rho^+(\bar{\mathbf{b}}^+\cdot\mathbf{z})^2 + \rho^-(\bar{\mathbf{b}}^-\cdot\mathbf{z})^2 - \left(\frac{\rho^+\rho^-}{\rho^++\rho^-}(\bar{\mathbf{b}}^+\cdot\mathbf{z})^2 + 2\frac{\rho^+\rho^-}{\rho^++\rho^-}(\bar{\mathbf{b}}^+\cdot\mathbf{z})(\bar{\mathbf{b}}^-\cdot\mathbf{z}) + \frac{\rho^+\rho^-}{\rho^++\rho^-}(\bar{\mathbf{b}}^-\cdot\mathbf{z})^2\right) \\ =&\frac{1}{\rho^++\rho^-}\left((\rho^+)^2(\bar{\mathbf{b}}^+\cdot\mathbf{z})^2 - 2\rho^+\rho^-(\bar{\mathbf{b}}^+\cdot\mathbf{z})(\bar{\mathbf{b}}^-\cdot\mathbf{z}) + (\rho^-)^2(\bar{\mathbf{b}}^-\cdot\mathbf{z})^2\right) \geq 0. \end{aligned}$$

Thus, it implies that

$$\inf_{\substack{\mathbf{z}\in\mathbb{R}^2\\|\mathbf{z}|=1}} \left(\rho^+(\bar{\mathbf{b}}^+\cdot\mathbf{z})^2 + \rho^-(\bar{\mathbf{b}}^-\cdot\mathbf{z})^2 - (\rho^++\rho^-)(\bar{\mathbf{u}}\cdot\mathbf{z})^2\right) > 0, \tag{7.54}$$

or equivalently, there exists some $\delta_0' > 0$ such that

$$\inf_{\mathbf{z}\in\mathbb{R}^2} \left(\rho^+(\bar{\mathbf{b}}^+ \cdot \mathbf{z})^2 + \rho^-(\bar{\mathbf{b}}^- \cdot \mathbf{z})^2 - (\rho^+ + \rho^-)\,(\bar{\mathbf{u}} \cdot \mathbf{z})^2\right) \geq 2\delta_0'|\mathbf{z}|^2. \tag{7.55}$$

Now let $\mathbf{z} = \overline{\nabla} T_{\mathfrak{M}} T_{\mathfrak{n}}\psi$, the above inequality shows that $\widetilde{\mathcal{E}}(t) \geq \delta_0'|\overline{\nabla} T_{\mathfrak{M}} T_{\mathfrak{n}}\psi|_0^2 \gtrsim |\psi|_{4.5}^2 - |\psi|_0^2$. □

Remark 7.1 (The 2D case)**.** When the space dimension $d = 2$, we no longer have the non-collinearity, but the stability condition (1.47) still guarantees the ellipticity of the corresponding second-order differential operator, i.e.,

$$\exists \delta_0' > 0, \quad \rho^+\left((\mathbf{b}_1^+)^2 - \mathbf{u}_1^2\right) + \rho^-\left((\mathbf{b}_1^-)^2 - \mathbf{u}_1^2\right) \geq \delta_0'.$$

In fact, the stability condition (1.47) implies $|\mathbf{b}_1^+| + |\mathbf{b}_1^-| \geq (1+\delta_0)|\llbracket v_1 \rrbracket|$. Taking square and invoking $\mathbf{u} := \frac{\sqrt{\rho^+\rho^-}}{\rho^+ + \rho^-}\llbracket v \rrbracket$, we get

$$\frac{\rho^+\rho^-}{\rho^+ + \rho^-}\left((\mathbf{b}_1^+)^2 + 2\mathbf{b}_1^+\mathbf{b}_1^- + (\mathbf{b}_1^-)^2\right) \geq (1+\delta_0)(\rho^+ + \rho^-)\mathbf{u}_1^2,$$

in which we find that the left side does not exceed $(\rho^+ + \rho^-)^{-1}\left(\rho^+(\mathbf{b}_1^+)^2 + \rho^-(\mathbf{b}_1^-)^2\right)$ by direct calculation. The desired result immediately follows thanks to $|\mathbf{u}_1| > 0$ (otherwise the interface is not a vortex sheet).

In view of Lemma 7.12, it suffices to prove energy estimates for $\mathcal{E}(t)$ and $\widetilde{\mathcal{E}}(t)$ under the stability condition (1.40). We start with the estimate of $|\psi_t|_{3.5}$.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Sigma (\rho^+ + \rho^-)\left|(\partial_t + \bar{\mathbf{w}}\cdot\overline{\nabla})T_{\mathfrak{M}}T_{\mathfrak{n}}\psi\right|^2\,\mathrm{d}x' \\
=&\int_\Sigma (\rho^+ + \rho^-)(\partial_t^2 T_{\mathfrak{M}}T_{\mathfrak{n}}\psi)\left((\partial_t + \bar{\mathbf{w}}\cdot\overline{\nabla})T_{\mathfrak{M}}T_{\mathfrak{n}}\psi\right)\mathrm{d}x' + \int_\Sigma (\rho^+ + \rho^-)(\bar{\mathbf{w}}\cdot\overline{\nabla}\partial_t T_{\mathfrak{M}}T_{\mathfrak{n}}\psi)\left((\partial_t + \bar{\mathbf{w}}\cdot\overline{\nabla})T_{\mathfrak{M}}T_{\mathfrak{n}}\psi\right)\mathrm{d}x' \\
&+ \int_\Sigma (\rho^+ + \rho^-)(\partial_t\bar{\mathbf{w}}\cdot\overline{\nabla}T_{\mathfrak{M}}T_{\mathfrak{n}}\psi)\left((\partial_t + \bar{\mathbf{w}}\cdot\overline{\nabla})T_{\mathfrak{M}}T_{\mathfrak{n}}\psi\right)\mathrm{d}x' + \frac{1}{2}\int_\Sigma \partial_t(\rho^+ + \rho^-)\left|(\partial_t + \bar{\mathbf{w}}\cdot\overline{\nabla})T_{\mathfrak{M}}T_{\mathfrak{n}}\psi\right|^2\mathrm{d}x' \\
=:&\, I_0 + I_1 + I_1^R + I_2^R.
\end{aligned} \tag{7.56}$$

The remainder terms are easy to control. Using Proposition 7.6, we have

$$I_1^R + I_2^R \leq C\left(\left|\rho^\pm, \partial_t\rho^\pm, \bar{v}^\pm, \partial_t\bar{v}^\pm\right|_{L^\infty}, |\psi|_3\right)\left(|\psi_t|_{3.5}^2 + |\psi|_{4.5}^2\right). \tag{7.57}$$

For the main term I_0, we first commute $(\rho^+ + \rho^-)\partial_t^2$ with $T_{\mathfrak{M}}T_{\mathfrak{n}}$

$$\begin{aligned}
(\rho^+ + \rho^-)\partial_t^2 T_{\mathfrak{M}}T_{\mathfrak{n}}\psi &= (\rho^+ + \rho^-)\left(T_{\mathfrak{M}}T_{\mathfrak{n}}\partial_t^2\psi + (T_{\partial_t^2\mathfrak{M}}T_{\mathfrak{n}} + T_{\mathfrak{M}}T_{\partial_t^2\mathfrak{n}})\psi + 2(T_{\partial_t\mathfrak{M}}T_{\mathfrak{n}} + T_{\mathfrak{M}}T_{\partial_t\mathfrak{n}})\partial_t\psi\right) \\
&= T_{\mathfrak{M}}T_{\mathfrak{n}}\left((\rho^+ + \rho^-)\partial_t^2\psi\right) - \left([T_{\mathfrak{M}}, \rho^+ + \rho^-]T_{\mathfrak{n}}\partial_t^2\psi + T_{\mathfrak{M}}([T_{\mathfrak{n}}, \rho^+ + \rho^-]\partial_t^2\psi)\right) \\
&\quad + (\rho^+ + \rho^-)\left((T_{\partial_t^2\mathfrak{M}}T_{\mathfrak{n}} + T_{\mathfrak{M}}T_{\partial_t^2\mathfrak{n}})\psi + 2(T_{\partial_t\mathfrak{M}}T_{\mathfrak{n}} + T_{\mathfrak{M}}T_{\partial_t\mathfrak{n}})\partial_t\psi\right).
\end{aligned}$$

The commutators can be controlled straightforwardly thanks to Lemma 7.10 and Lemma 7.11:

$$\begin{aligned}
\left|[T_{\mathfrak{M}}, \rho^+ + \rho^-]T_{\mathfrak{n}}\partial_t^2\psi\right|_0 + \left|T_{\mathfrak{M}}([T_{\mathfrak{n}}, \rho^+ + \rho^-]\partial_t^2\psi)\right|_0 &\leq C(|\overline{\nabla}\psi|_{W^{1,\infty}})\left(|\rho^+ + \rho^-|_{3.5}|\partial_t^2\psi|_{2.5}\right), \\
\left|(T_{\partial_t^2\mathfrak{M}}T_{\mathfrak{n}} + T_{\mathfrak{M}}T_{\partial_t^2\mathfrak{n}})\psi\right|_0 + \left|(T_{\partial_t\mathfrak{M}}T_{\mathfrak{n}} + T_{\mathfrak{M}}T_{\partial_t\mathfrak{n}})\partial_t\psi\right|_0 &\leq C(|\psi_{tt}, \psi_t, \overline{\nabla}\psi|_{W^{1,\infty}})\left(|\psi|_{3.5} + |\psi_t|_{3.5}\right).
\end{aligned}$$

In the remaining of this section, we no longer explicitly write the commutators between the paradifferential operators and functions or $\partial_t, \overline{\partial}_i$, as they can be controlled in the same way as above. Instead, we will again use the notation $\stackrel{L}{=}$ to skip these terms and analyze the main terms.

Then we can plug the equation (7.48) into the integral to get

$$\begin{aligned}
I_{00} :=& \int_\Sigma \left(T_{\mathfrak{M}} T_{\mathfrak{n}}((\rho^+ + \rho^-)\partial_t^2\psi)\right)\left((\partial_t + \bar{\mathbf{w}}\cdot\overline{\nabla})T_{\mathfrak{M}} T_{\mathfrak{n}}\psi\right)\,\mathrm{d}x' \\
=& -\frac{\sigma}{2}\int_\Sigma (T_{\mathfrak{M}} T_{\mathfrak{n}} T_\Lambda T_{\mathfrak{H}}\psi)\left((\partial_t + \bar{\mathbf{w}}\cdot\overline{\nabla})T_{\mathfrak{M}} T_{\mathfrak{n}}\psi\right)\,\mathrm{d}x' \\
& - 2\int_\Sigma \left(T_{\mathfrak{M}} T_{\mathfrak{n}}((\rho^+ + \rho^-)\bar{\mathbf{w}}_i\overline{\partial}_i\partial_t\psi)\right)\left((\partial_t + \bar{\mathbf{w}}\cdot\overline{\nabla})T_{\mathfrak{M}} T_{\mathfrak{n}}\psi\right)\,\mathrm{d}x' \\
& - \int_\Sigma \left(T_{\mathfrak{M}} T_{\mathfrak{n}}((\rho^+ + \rho^-)\bar{\mathbf{w}}_i\bar{\mathbf{w}}_j\overline{\partial}_i\overline{\partial}_j\psi)\right)\left((\partial_t + \bar{\mathbf{w}}\cdot\overline{\nabla})T_{\mathfrak{M}} T_{\mathfrak{n}}\psi\right)\,\mathrm{d}x' \\
& + \int_\Sigma \left(T_{\mathfrak{M}} T_{\mathfrak{n}}\left(\rho^+(\bar{\mathbf{b}}_i^+\bar{\mathbf{b}}_j^+ - \bar{\mathbf{u}}_i\bar{\mathbf{u}}_j) + \rho^-(\bar{\mathbf{b}}_i^-\bar{\mathbf{b}}_j^- - \bar{\mathbf{u}}_i\bar{\mathbf{u}}_j)\right)\overline{\partial}_i\overline{\partial}_j\psi\right)\left((\partial_t + \bar{\mathbf{w}}\cdot\overline{\nabla})T_{\mathfrak{M}} T_{\mathfrak{n}}\psi\right)\,\mathrm{d}x' \\
& - \int_\Sigma (T_{\mathfrak{M}} T_{\mathfrak{n}}(N\cdot\nabla^\varphi q_w^+ + N\cdot q_w^-))\left((\partial_t + \bar{\mathbf{w}}\cdot\overline{\nabla})T_{\mathfrak{M}} T_{\mathfrak{n}}\psi\right)\,\mathrm{d}x' \\
& + \int_\Sigma \left(T_{\mathfrak{M}} T_{\mathfrak{n}}\Psi^R\right)\left((\partial_t + \bar{\mathbf{w}}\cdot\overline{\nabla})T_{\mathfrak{M}} T_{\mathfrak{n}}\psi\right)\,\mathrm{d}x' =: I_{000} + I_{001} + I_{002} + I_{003} + I_0^w + I_0^R.
\end{aligned} \tag{7.58}$$

Since Proposition 7.9 indicates that $T_{\mathfrak{n}} T_\Lambda T_{\mathfrak{H}} \sim T_{\mathfrak{m}} T_{\mathfrak{m}} T_{\mathfrak{n}}$ and $(T_{\mathfrak{m}})^* \sim T_{\mathfrak{m}}$, we have

$$\begin{aligned}
& T_{\mathfrak{M}} T_{\mathfrak{n}} T_\Lambda T_{\mathfrak{H}}\psi \overset{L}{=} T_{\mathfrak{M}} T_{\mathfrak{m}} T_{\mathfrak{m}} T_{\mathfrak{n}}\psi \\
=& (T_{\mathfrak{m}})^* T_{\mathfrak{m}} T_{\mathfrak{M}} T_{\mathfrak{n}}\psi + (((T_{\mathfrak{m}})^* - T_{\mathfrak{m}})T_{\mathfrak{M}} + T_{\mathfrak{m}}[T_{\mathfrak{M}}, T_{\mathfrak{m}}] + [T_{\mathfrak{M}}, T_{\mathfrak{m}}]T_{\mathfrak{m}})T_{\mathfrak{n}}\psi \overset{L}{=} (T_{\mathfrak{m}})^* T_{\mathfrak{m}} T_{\mathfrak{M}} T_{\mathfrak{n}}\psi
\end{aligned}$$

and using the duality, we get

$$\begin{aligned}
I_{000} \overset{L}{=}& -\frac{\sigma}{2}\int_\Sigma (T_{\mathfrak{m}} T_{\mathfrak{M}} T_{\mathfrak{n}}\psi)\,T_{\mathfrak{m}}(\partial_t + \bar{\mathbf{w}}\cdot\overline{\nabla})T_{\mathfrak{M}} T_{\mathfrak{n}}\psi\,\mathrm{d}x' \\
=& -\frac{\sigma}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma |T_{\mathfrak{m}} T_{\mathfrak{M}} T_{\mathfrak{n}}\psi|^2\,\mathrm{d}x' \\
& + \frac{\sigma}{2}\int_\Sigma (T_{\mathfrak{m}} T_{\mathfrak{M}} T_{\mathfrak{n}}\psi)\left(T_{\partial_t\mathfrak{m}} + \bar{\mathbf{w}}_i T_{\overline{\partial}_i\mathfrak{m}} + \frac{1}{2}(\overline{\nabla}\cdot\bar{\mathbf{w}}) - [T_{\mathfrak{m}}, \bar{\mathbf{w}}_i]\overline{\partial}_i\right)T_{\mathfrak{M}} T_{\mathfrak{n}}\psi\,\mathrm{d}x',
\end{aligned} \tag{7.59}$$

where the second term can be directly controlled (uniformly in σ) by $\mathcal{E}(t)C\left(|\overline{\nabla}\psi, \psi_t, \bar{v}^\pm, \rho^\pm|_{W^{1,\infty}}\right)$ thanks to Lemma 7.10. Next we analyze $I_1 + I_{001}$ and I_{002}, I_{003}. For a generic function $\mathbf{a} \in H^{3.5}(\Sigma \to \mathbb{R}^2)$ and a generic $\boldsymbol{\rho} \in H^{3.5}(\Sigma \to \mathbb{R}_+)$, we have

$$\begin{aligned}
& \int_\Sigma \left(T_{\mathfrak{M}} T_{\mathfrak{n}}(\boldsymbol{\rho}\mathbf{a}_i\mathbf{a}_j\overline{\partial}_i\overline{\partial}_j\psi)\right)\left((\partial_t + \bar{\mathbf{w}}\cdot\overline{\nabla})T_{\mathfrak{M}} T_{\mathfrak{n}}\psi\right)\,\mathrm{d}x' \\
\overset{L}{=}& \int_\Sigma \boldsymbol{\rho}\mathbf{a}_i\mathbf{a}_j\overline{\partial}_i\overline{\partial}_j T_{\mathfrak{M}} T_{\mathfrak{n}}\psi\,\left((\partial_t + \bar{\mathbf{w}}\cdot\overline{\nabla})T_{\mathfrak{M}} T_{\mathfrak{n}}\psi\right)\,\mathrm{d}x' \overset{L}{=} -\int_\Sigma \boldsymbol{\rho}(\mathbf{a}_i\overline{\partial}_i T_{\mathfrak{M}} T_{\mathfrak{n}}\psi)\,(\mathbf{a}_j\overline{\partial}_j(\partial_t + \bar{\mathbf{w}}\cdot\overline{\nabla})T_{\mathfrak{M}} T_{\mathfrak{n}}\psi)\,\mathrm{d}x' \\
\overset{L}{=}& -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma \boldsymbol{\rho}\left|\mathbf{a}_i\overline{\partial}_i T_{\mathfrak{M}} T_{\mathfrak{n}}\psi\right|^2\,\mathrm{d}x'.
\end{aligned}$$

Setting $\mathbf{a} = \bar{\mathbf{w}}, \bar{\mathbf{u}}, \bar{\mathbf{b}}$ and $\boldsymbol{\rho} = \rho^\pm$ or $\rho^+ + \rho^-$, we immediately get

$$I_{002} \overset{L}{=} +\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma (\rho^+ + \rho^-)\left|(\bar{\mathbf{w}}\cdot\overline{\nabla})T_{\mathfrak{M}} T_{\mathfrak{n}}\psi\right|^2\,\mathrm{d}x, \quad I_{003} \overset{L}{=} -\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathcal{E}}(t).$$

For $I_1 + I_{001}$, we have

$$\begin{aligned}
I_{001} \overset{L}{=}& -2\int_\Sigma (\rho^+ + \rho^-)(\bar{\mathbf{w}}\cdot\overline{\nabla})\partial_t T_{\mathfrak{m}} T_{\mathfrak{n}}\psi\,\left((\partial_t + \bar{\mathbf{w}}\cdot\overline{\nabla})T_{\mathfrak{M}} T_{\mathfrak{n}}\psi\right)\,\mathrm{d}x' \\
\Rightarrow I_1 + I_{001} \overset{L}{=}& -\int_\Sigma (\rho^+ + \rho^-)(\bar{\mathbf{w}}\cdot\overline{\nabla})\partial_t T_{\mathfrak{m}} T_{\mathfrak{n}}\psi\,\left((\partial_t + \bar{\mathbf{w}}\cdot\overline{\nabla})T_{\mathfrak{M}} T_{\mathfrak{n}}\psi\right)\,\mathrm{d}x' \\
\overset{L}{=}& -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma (\rho^+ + \rho^-)\left|(\bar{\mathbf{w}}\cdot\overline{\nabla})T_{\mathfrak{M}} T_{\mathfrak{n}}\psi\right|^2\,\mathrm{d}x,
\end{aligned} \tag{7.60}$$

which cancels with the main term in I_{002}. When $\sigma > 0$ is given and the stability condition (1.40) is not assumed, the quantity $\widetilde{\mathcal{E}}(t)$ is not necessarily positive, but its contribution, namely the term I_{003}, can be controlled by integrating by parts for 1/2-derivative

$$I_{003} \lesssim P(\|v^\pm, \rho^\pm, b^\pm\|_{4,\pm})(|\psi|_5^2 + |\psi|_5|\psi_t|_4) \lesssim \sigma^{-1}P(\mathfrak{E}_4(t)).$$

Now, it remains to control I_0^w and I_0^R. In view of Proposition 7.6, it suffices to control the $H^{3.5}(\Sigma)$ norms of $N \cdot \nabla^\varphi q_w^\pm$ and Ψ^R. For the term $q_w^\pm$, we use trace lemma and div-curl inequality with tangential trace (see (B.3)) to get

$$\begin{aligned}&|N \cdot \nabla^\varphi q_w^\pm|_{3.5}^2 \le |\psi|_{4.5}^2 \|\nabla^\varphi q_w^\pm\|_{4,\pm}^2 \\ \le\ & C(|\psi|_{4.5})\left(\|\nabla^\varphi q_w^\pm\|_{0,\pm}^2 + \|\Delta^\varphi q_w^\pm\|_{3,\pm}^2 + \|\nabla^\varphi \times \nabla^\varphi q_w^\pm\|_{3,\pm}^2 + \left|N \times \nabla^\varphi q_w^\pm\right|_{3.5}^2 + \left|N \cdot \nabla^\varphi q_w^\pm\right|_{H^{3.5}(\Sigma^\pm)}^2\right)\end{aligned} \tag{7.61}$$

where the last three terms are zero because of

$$\nabla^\varphi \times \nabla^\varphi q_w^\pm = \vec{0}, \quad N \cdot \nabla^\varphi q_w^\pm|_{\Sigma^\pm} = 0, \quad q_w^\pm|_\Sigma = 0 \Rightarrow N \times \nabla^\varphi q_w^\pm|_\Sigma = (-\overline{\partial}_2 q_w^\pm, \overline{\partial}_1 q_w^\pm, \overline{\partial}_2\psi\overline{\partial}_1 q_w^\pm - \overline{\partial}_1\psi\overline{\partial}_2 q_w^+)^\top|_\Sigma = \vec{0}.$$

Then invoking the definition of $q_w^\pm$ and using $\mathcal{F}_p^\pm \lesssim \varepsilon^2$, we find

$$\|\Delta^\varphi q_w^\pm\|_{3,\pm}^2 \le C(|\psi|_4, |\psi_t|_3)\left(\left\|\varepsilon^2\mathcal{T}^2(b^\pm, p^\pm)\right\|_{3,\pm}^2 + P\left(\left\|v^\pm, b^\pm\right\|_{4,\pm}\right)\right) \le P(\widetilde{\mathfrak{E}}_4(t))\widetilde{\mathfrak{E}}_5(t). \tag{7.62}$$

Remark 7.2 (Necessity of anisotropic Sobolev spaces)**.** The term bounded by $\widetilde{\mathfrak{E}}_5(t)$is contributed exactly by the extra $\frac{1}{2}|b^\pm|^2$ in the total pressure. From this, we can also see the necessity of the anisotropic Sobolev spaces when studying ideal compressible MHD. For Euler equations, the source terms for the wave equation only contain quadratic first-order terms, and one can use the trick in [48, 92] to close the energy bound by $\widetilde{\mathfrak{E}}_4(t)$. For incompressible MHD, the second-order time derivative term vanishes because $q^\pm$ satisfies an elliptic equation.

The term $|\Psi^R|_{3.5}$ can also be directly controlled. Recall that

$$\Psi^R := \frac{\sigma}{2}(\mathfrak{N}_\psi^+ - \mathfrak{N}_\psi^-)\widetilde{\mathfrak{N}}^{-1}(\mathfrak{N}_\psi^+ - \mathfrak{N}_\psi^-)(\mathcal{H}(\psi)) - (\mathfrak{N}_\psi^+ - \mathfrak{N}_\psi^-)\widetilde{\mathfrak{N}}^{-1}\left(\left[\!\left[\mathfrak{F}_\psi - \rho\partial_t^2\psi\right]\!\right]\right) + \frac{\sigma}{2}\mathcal{R}_\psi^\sigma.$$

Using the Sobolev estimates for the Dirichlet-to-Neumann operators, we have

$$\begin{aligned}&\left|\frac{\sigma}{2}(\mathfrak{N}_\psi^+ - \mathfrak{N}_\psi^-)\widetilde{\mathfrak{N}}^{-1}(\mathfrak{N}_\psi^+ - \mathfrak{N}_\psi^-)(\mathcal{H}(\psi))\right|_{3.5} + \left|(\mathfrak{N}_\psi^+ - \mathfrak{N}_\psi^-)\widetilde{\mathfrak{N}}^{-1}\left(\left[\!\left[\mathfrak{F}_\psi - \rho\partial_t^2\psi\right]\!\right]\right)\right|_{3.5} \\ \lesssim\ & |\sigma\mathfrak{H}(\psi)|_{2.5} + \left|\mathfrak{F}_\psi^\pm - \rho^\pm\partial_t^2\psi\right|_{2.5}^2 \\ \lesssim\ & ((1+\sigma)|\psi|_{4.5} + |\psi_t|_{3.5})\left(\left\|\varepsilon^2\partial_t^2 p^\pm\right\|_{2,\pm} + P(\|v^\pm, b^\pm\|_{3,\pm}, |\psi|_3)\right) \lesssim P(\widetilde{\mathfrak{E}}_4(t)).\end{aligned} \tag{7.63}$$

Setting $s = 4$ in the remainder estimate (7.42), we have $|\sigma\mathcal{R}_\psi^\sigma|_{3.5} \le C(|\psi|_{4.5})|\sigma\psi|_5 \le \sqrt{\sigma}C(\widetilde{\mathcal{E}}(t))\sqrt{\widetilde{\mathcal{E}}(t)} \le \sqrt{\sigma}C(\widetilde{\mathfrak{E}}_4(t))$.

Summarizing the estimate above, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) \lesssim \sigma^{-1}P(\mathfrak{E}_4(t))\mathfrak{E}_5(t), \tag{7.64}$$

and under the stability condition (1.40), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{E}(t) + \widetilde{\mathcal{E}}(t)) \lesssim P(\widetilde{\mathfrak{E}}_4(t))\widetilde{\mathfrak{E}}_5(t). \tag{7.65}$$

Invoking Lemma 7.12, we actually prove the following uniform-in-ε estimate for fixed $\sigma > 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left|\sqrt{\sigma}\psi\right|_5^2 + |\psi_t|_{3.5}^2\right) \lesssim \sigma^{-1}P(\mathfrak{E}_4(t))\mathfrak{E}_5(t), \tag{7.66}$$

and the following uniform-in-(ε,σ) estimate under the stability condition (1.40)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left|\sqrt{\sigma}\psi\right|_5^2+|\psi|_{4.5}^2+|\psi_t|_{3.5}^2\right)\lesssim P(\widetilde{\mathfrak{E}}_4(t))\widetilde{\mathfrak{E}}_5(t). \tag{7.67}$$

It remains to prove the uniform estimates for $|\psi_{tt}|_{2.5}^2$. We just need to take one more ∂_t in the paralinearized evolution equation (7.48). The proof follows in the same way as the above analysis for the $T_{\mathfrak{M}}T_{\mathfrak{n}}$-differentiated version of (7.48). So, we skip the details and only list the differences. The first difference is that we should replace $T_{\mathfrak{M}}$ by $T_{\mathfrak{M}'}$ where $\mathfrak{M}'\in\Sigma^{\frac52}$ is defined by

$$\mathfrak{M}':=(\mathfrak{m}^{(1.5)})^{\frac53}=2^{\frac56}|\xi|^{\frac52}\left(1-\left|\frac{N}{|N|}\cdot\frac{\xi}{|\xi|}\right|^2\right)^{\frac54}\in\Sigma^{\frac52}.$$

The second difference is essential: we notice that the remainder term Ψ^R already contains second-order time derivative $(\mathfrak{N}_\psi^+-\mathfrak{N}_\psi^-)\widetilde{\mathfrak{N}}^{-1}([\![\rho]\!]\,\partial_t^2\psi)$. Taking one more time derivative, we are required to control $(\mathfrak{N}_\psi^+-\mathfrak{N}_\psi^-)\widetilde{\mathfrak{N}}^{-1}([\![\rho]\!]\,\partial_t^3\psi)$. Using Lemma C.3 and Lemma C.4, we have

$$\left|(\mathfrak{N}_\psi^+-\mathfrak{N}_\psi^-)\widetilde{\mathfrak{N}}^{-1}([\![\rho]\!]\,\partial_t^3\psi)\right|_{2.5}\lesssim|\,[\![\rho]\!]\,|_{1.5}|\partial_t^3\psi|_{1.5}.$$

Since we require $|\,[\![\rho]\!]\,|_{1.5}\lesssim\varepsilon$, we can control the right side by $|\varepsilon\partial_t^3\psi|_{1.5}$. Under the stability condition (1.40), this term is already a part of $\widetilde{\mathfrak{E}}_4(t)$. For fixed $\sigma>0$, we again invoke the kinematic boundary condition to get

$$|\varepsilon\partial_t^3\psi|_{1.5}\lesssim\|\varepsilon\partial_t^2v^\pm\|_{2,\pm}|\psi|_{2.5}+\|\varepsilon\partial_tv^\pm\|_{2,\pm}|\psi_t|_{2.5}+\|\varepsilon v^\pm\|_{2,\pm}|\psi_{tt}|_{2.5},$$

where the right side is already controlled by $P(\mathfrak{E}_4(t))$. As for the fifth-order terms arising from $N\cdot\nabla^\varphi q_w^\pm$, they now contribute to $\|\varepsilon^2\partial_t^3q^\pm\|_{2,\pm}$ which is still a part of $E_5(t)$. Thus, we can conclude the following uniform-in-ε estimate for fixed $\sigma>0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left|\sqrt{\sigma}\psi_t\right|_4^2+|\psi_{tt}|_{2.5}^2\right)\lesssim\sigma^{-1}P(\mathfrak{E}_4(t))\mathfrak{E}_5(t), \tag{7.68}$$

and the following uniform-in-(ε,σ) estimate under the stability condition (1.40)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left|\sqrt{\sigma}\psi_t\right|_4^2+|\psi|_{3.5}^2+|\psi_{tt}|_{2.5}^2\right)\lesssim P(\widetilde{\mathfrak{E}}_4(t))\widetilde{\mathfrak{E}}_5(t). \tag{7.69}$$

7.4 Double limits without the boundedness of ≥ 2 time derivatives

7.4.1 Incompressible limit for fixed $\sigma>0$

From the analysis in Section 7.3, we can prove the uniform-in-ε estimates for $\mathfrak{E}_4(t)$. For any $\delta\in(0,1)$

$$\mathfrak{E}_4(t)\lesssim\delta\mathfrak{E}_4(t)+P(\mathfrak{E}_4(0))+P(\mathfrak{E}_4(t))\int_0^tP(\sigma^{-1},\mathfrak{E}_4(\tau))+\mathfrak{E}_5(\tau)\,\mathrm{d}\tau. \tag{7.70}$$

For $1\leq l\leq4$, since we do not change anything $\mathfrak{E}_{4+l}(t)$, we still have

$$l=1,2,3:\ \mathfrak{E}_{4+l}(t)\lesssim\delta\mathfrak{E}_{4+l}(t)+P(\mathfrak{E}_{4+l}(0))+P(\mathfrak{E}_4(t))\int_0^tP\left(\sigma^{-1},\sum_{j=0}^l\mathfrak{E}_{4+j}(\tau)\right)+\mathfrak{E}_{4+l+1}(\tau)\,\mathrm{d}\tau; \tag{7.71}$$

$$l=4:\ \mathfrak{E}_8(t)\lesssim\delta\mathfrak{E}_8(t)+P(\mathfrak{E}_8(0))+P(\mathfrak{E}_4(t))\int_0^tP(\sigma^{-1},\mathfrak{E}_8(\tau))\,\mathrm{d}\tau. \tag{7.72}$$

Therefore, we get the Gronwall-type energy inequality for $\mathfrak{E}(t)$

$$\mathfrak{E}(t)\lesssim\delta\mathfrak{E}(t)+P(\mathfrak{E}(0))+P(\mathfrak{E}(t))\int_0^tP(\sigma^{-1},\mathfrak{E}(\tau))\,\mathrm{d}\tau. \tag{7.73}$$

Choosing $\delta > 0$ suitably small, the term $\delta\mathfrak{E}(t)$ can be absorbed by the left side. Thus, there exists a time $T'_\sigma > 0$ depending on σ^{-1} and the initial data, but independent of ε, such that

$$\sup_{0\le t\le T'_\sigma} \mathfrak{E}(t) \lesssim P(\sigma^{-1}, \mathfrak{E}(0)). \tag{7.74}$$

With the uniform-in-ε estimates for $\mathfrak{E}(t)$, we now take the incompressible limit. Again, since $\|\partial_t(v,b)\|_3$ is uniformly bounded with respect to ε and we still have ε-independent bound $|\psi_t|_{3.5}$, the Aubin-Lions compactness lemma gives the same strong convergence result as in Section 6.1.

7.4.2 Double limits under the stability conditions

With the estimates (7.67) and (7.69) in Section 7.3, we can get

$$|v_t\cdot N|_{2.5}^2 + |b_t\cdot N|_{2.5}^2 \lesssim P(\widetilde{\mathfrak{E}}_4(0)) + P(\widetilde{\mathfrak{E}}_4(t))\int_0^t P(\widetilde{\mathfrak{E}}_4(\tau))\widetilde{\mathfrak{E}}_5(\tau)\,\mathrm{d}\tau.$$

This finishes the control of $\widetilde{\mathfrak{E}}_4(t)$. Since $\widetilde{\mathfrak{E}}_{4+l}(t) = \widetilde{E}_{4+l}(t)$ when $1\le l\le 4$ and the strategies to control them remain unchanged, we can now close the energy estimates for $\widetilde{\mathfrak{E}}(t)$, uniformly in ε and σ, under the stability condition (1.40) ($d=3$) or (1.47) ($d=2$).

$$\widetilde{\mathfrak{E}}(t) \le P(\widetilde{\mathfrak{E}}(0)) + P(\widetilde{\mathfrak{E}}(t))\int_0^t P(\widetilde{\mathfrak{E}}(\tau))\,\mathrm{d}\tau. \tag{7.75}$$

By Grönwall's inequality, there exists $T'>0$ independent of σ and ε such that

$$\sup_{t\in[0,T]} \widetilde{\mathfrak{E}}(t) \le P(\widetilde{\mathfrak{E}}(0)). \tag{7.76}$$

Thus, by Aubin-Lions compactness lemma, we can prove the same convergence result as in Theorem 1.3.

Conflict of interest. The author declares that there is no conflict of interest.

Data avaliability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

A Reynolds transport theorems

We record the Reynolds transport theorems used in this paper. For the proof, we refer to Luo-Zhang [55, Appendix A]

Lemma A.1. Let f, g be smooth functions defined on $[0,T]\times\Omega$. Then:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega fg\partial_3\varphi\,\mathrm{d}x = \int_\Omega (\partial_t^\varphi f)g\partial_3\varphi\,\mathrm{d}x + \int_\Omega f(\widetilde{\varphi}_t g)\partial_3\varphi\,\mathrm{d}x + \int_{x_3=0} fg\partial_t\psi\,\mathrm{d}x', \tag{A.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega fg\partial_3\mathring{\varphi}\,\mathrm{d}x = \int_\Omega (\partial_t^{\mathring{\varphi}} f)g\partial_3\mathring{\varphi}\,\mathrm{d}x + \int_\Omega f(\partial_t^{\mathring{\varphi}} g)\partial_3\mathring{\varphi}\,\mathrm{d}x + \int_{x_3=0} fg\partial_t\mathring{\psi}\,\mathrm{d}x'. \tag{A.2}$$

Lemma A.2 (Integration by parts for covariant derivatives). Let f, g be defined as in Lemma A.1. Then:

$$\int_\Omega (\partial_i^\varphi f)g\partial_3\varphi\,\mathrm{d}x = -\int_\Omega f(\partial_i^\varphi g)\partial_3\varphi\,\mathrm{d}x + \int_{x_3=0} fgN_i\,\mathrm{d}x', \tag{A.3}$$

$$\int_\Omega (\partial_i^{\mathring{\varphi}} f)g\partial_3\mathring{\varphi}\,\mathrm{d}x = -\int_\Omega f(\partial_i^{\mathring{\varphi}} g)\partial_3\mathring{\varphi}\,\mathrm{d}x + \int_{x_3=0} fg\mathring{N}_i\,\mathrm{d}x'. \tag{A.4}$$

The following theorem holds.

Theorem A.3 (**Reynolds transport theorem**)**.** Let f be a smooth function defined on $[0, T] \times \Omega$. Then:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho |f|^2 \partial_3\varphi \,\mathrm{d}x = \int_\Omega \rho (D_t^\varphi f) f \partial_3\varphi \,\mathrm{d}x. \tag{A.5}$$

Theorem A.3 leads to the following two corollaries. The first one records the integration by parts formula for D_t^φ.

Corollary A.4 (**Reynolds transport theorem - a variant**)**.** It holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega fg\partial_3\varphi \,\mathrm{d}x = \int_\Omega (D_t^\varphi f) g\partial_3\varphi \,\mathrm{d}x + \int_\Omega f(D_t^\varphi g)\partial_3\varphi \,\mathrm{d}x + \int_\Omega (\nabla^\varphi \cdot v) fg\partial_3\varphi \,\mathrm{d}x. \tag{A.6}$$

The second corollary concerns the transport theorem as well as the integration by parts formula for the linearized material derivative $D_t^{\mathring{\varphi}}$.

Corollary A.5 (**Reynolds transport theorem for linearized κ-problem**)**.** Let $D_t^{\mathring{\varphi}} := \partial_t + (\overline{\mathring{v}} \cdot \overline{\nabla}) + \frac{1}{\partial_3\mathring{\varphi}}(\mathring{v} \cdot \dot{\mathbf{N}} - \partial_t\mathring{\varphi})\partial_3$ be the linearized material derivative. Then:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathring{\rho}|f|^2 \partial_3\mathring{\varphi} \,\mathrm{d}x = &\int_\Omega \mathring{\rho}(D_t^{\mathring{\varphi}} f) f \partial_3\mathring{\varphi} \,\mathrm{d}x + \frac{1}{2}\int_\Omega \left(D_t^{\mathring{\varphi}}\mathring{\rho} + \mathring{\rho}\nabla^{\mathring{\varphi}} \cdot \mathring{v}\right)|f|^2 \partial_3\mathring{\varphi} \,\mathrm{d}x \\ &+ \frac{1}{2}\int_\Omega \mathring{\rho}|f|^2 \left(\partial_3(\overline{\mathring{v}} \cdot \overline{\nabla})(\mathring{\varphi} - \varphi)\right) \mathrm{d}x.\end{aligned} \tag{A.7}$$

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |f|^2 \partial_3\mathring{\varphi} \,\mathrm{d}x = &\int_\Omega (D_t^{\mathring{\varphi}} f) f \partial_3\mathring{\varphi} \,\mathrm{d}x + \frac{1}{2}\int_\Omega \nabla^{\mathring{\varphi}} \cdot \mathring{v}|f|^2 \partial_3\mathring{\varphi} \,\mathrm{d}x \\ &+ \frac{1}{2}\int_\Omega |f|^2 \left(\partial_3(\overline{\mathring{v}} \cdot \overline{\nabla})(\mathring{\varphi} - \varphi)\right) \mathrm{d}x.\end{aligned} \tag{A.8}$$

B Preliminary lemmas about Sobolev inequalities

Lemma B.1 (Hodge-type elliptic estimates)**.** For any sufficiently smooth vector field X and $s \geq 1$, one has

$$\|X\|_s^2 \leq C(|\psi|_s, |\overline{\nabla}\psi|_{W^{1,\infty}})\left(\|X\|_0^2 + \|\nabla^\varphi \cdot X\|_{s-1}^2 + \|\nabla^\varphi \times X\|_{s-1}^2 + \|\overline{\partial}^\alpha X\|_0^2\right), \tag{B.1}$$

$$\|X\|_s^2 \leq C'(|\psi|_{s+\frac{1}{2}}, |\overline{\nabla}\psi|_{W^{1,\infty}})\left(\|X\|_0^2 + \|\nabla^\varphi \cdot X\|_{s-1}^2 + \|\nabla^\varphi \times X\|_{s-1}^2 + |X \cdot N|_{s-\frac{1}{2}}^2\right), \tag{B.2}$$

$$\|X\|_s^2 \leq C''(|\psi|_{s+\frac{1}{2}}, |\overline{\nabla}\psi|_{W^{1,\infty}})\left(\|X\|_0^2 + \|\nabla^\varphi \cdot X\|_{s-1}^2 + \|\nabla^\varphi \times X\|_{s-1}^2 + |X \times N|_{s-\frac{1}{2}}^2\right), \tag{B.3}$$

for any multi-index α with $|\alpha| = s$. The constant $C(|\psi|_s, |\overline{\nabla}\psi|_{W^{1,\infty}}) > 0$ depends linearly on $|\psi|_s^2$ and the constants $C'(|\psi|_{s+\frac{1}{2}}, |\overline{\nabla}\psi|_{W^{1,\infty}}) > 0$ and $C'(|\psi|_{s+\frac{1}{2}}, |\overline{\nabla}\psi|_{W^{1,\infty}}) > 0$ depend linearly on $|\psi|_{s+\frac{1}{2}}^2$.

Lemma B.2 (Normal trace lemma)**.** For any sufficiently smooth vector field X and $s \geq 0$, one has

$$|X \cdot N|_{s-\frac{1}{2}}^2 \lesssim C'''(|\psi|_{s+\frac{1}{2}}, |\overline{\nabla}\psi|_{W^{1,\infty}})\left(\|\langle\overline{\partial}\rangle^s X\|_0^2 + \|\nabla^\varphi \cdot X\|_{s-1}^2\right) \tag{B.4}$$

where the constant $C'''(|\psi|_{s+\frac{1}{2}}, |\overline{\nabla}\psi|_{W^{1,\infty}}) > 0$ depends linearly on $|\psi|_{s+\frac{1}{2}}^2$.

We list two lemmas for the estimates of traces in the anisotropic Sobolev spaces. Define

$$L_T^2(H_*^m(\Omega^\pm)) = \bigcap_{k=0}^m H^k((-\infty, T]; H_*^{m-k}(\Omega^\pm))$$

with the norm $\|u\|_{m,*,T,\pm} := \int_{-\infty}^T \|u(t)\|_{m,*,\pm}^2 \,\mathrm{d}t$. Similarly, we define

$$L_T^2(H^m(\Sigma)) = \bigcap_{k=0}^m H^k((-\infty, T]; H^{m-k}(\Sigma))$$

with the norm $|u|_{m,T} := \int_{-\infty}^T |u(t)|_m^2 \,\mathrm{d}t$.

Lemma B.3 (Trace lemma for anisotropic Sobolev spaces, [81, Lemma 3.4])**.** Let $m \geq 1,\ m \in \mathbb{N}^*$, then we have the following trace lemma for the anisotropic Sobolev space.

1. If $f \in L^2_T(H^{m+1}_*(\Omega^\pm))$, then its trace $f|_\Sigma$ belongs to $L^2_T(H^m(\Omega^\pm))$ and satisfies
$$|f|_{m,T} \lesssim \|f\|_{m+1,*,T,\pm}.$$
2. There exists a linear continuous operator $\mathfrak{R}^\pm_T : L^2_T(H^m(\Sigma)) \to L^2_T(H^{m+1}_*(\Omega^\pm))$ such that $(\mathfrak{R}^\pm_T g)|_\Sigma = g$ and
$$\|\mathfrak{R}^\pm_T g\|_{m+1,*,T,\pm} \lesssim |g|_{m,T}.$$

Proof. The proof for the above lemma can be found in [68, Theorem 1] when we replace $(-\infty, T)$ by $(-\infty, \infty)$. In our case, we can prove the same result by doing Sobolev extension. Namely, given $f \in L^2_T(H^{m+1}_*(\Omega^+))$, we can extend it to $F(t,x) : R \times \Omega^+ \to \mathbb{R}$ such that
$$\|f\|_{m+1,*,T,+} \lesssim \|F(t,x)\|_{H^{m+1}_*(\mathbb{R}\times\Omega^+)} \lesssim \|f\|_{m+1,*,T,+}.$$
We can apply [68, Theorem 1] to F, and then do the truncation in $(-\infty, T]$
$$|f|_{m,T} \lesssim |F|_{H^m(\mathbb{R}\times\Sigma)} \lesssim \|F(t,x)\|_{H^{m+1}_*(\mathbb{R}\times\Omega^+)} \lesssim \|f\|_{m+1,*,T,+}.$$
□

There is one derivative loss in the above trace lemma, which is 1/2-order more than the trace lemma for standard Sobolev spaces. Indeed, for $\Omega^\pm$ defined in this paper, we have the following estimate that will be applied to control the non-characteristic variables $q, v\cdot\mathbf{N}$ and $b\cdot\mathbf{N}$.

Lemma B.4 (An estimate for traces of non-characteristic variables)**.** Let $\Omega^\pm := \mathbb{T}^{d-1} \times \{0 \leqslant x_d \leqslant \pm H\}$, $\Sigma = \mathbb{T}^{d-1} \times \{x_d = 0\}$ and $\Sigma^\pm = \mathbb{T}^{d-1} \times \{\pm H\}$. Let $\mathcal{T}^\alpha = (\omega(x_d)\partial_d)^{\alpha_{d+1}}\partial_t^{\alpha_0}\overline{\partial}_1^{\alpha_1}\cdots\overline{\partial}_{d-1}^{\alpha_{d-1}}\partial_d^{\alpha_d}$ with $\langle\alpha\rangle := \alpha_0 + \cdots + \alpha_{d-1} + 2\alpha_d + \alpha_{d+1} = m-1,\ m \in \mathbb{N}^*$. Let $q^\pm(t,x) \in H^m_*(\Omega)$ satisfy $\|q^\pm(t)\|_{m,*,\pm} + \|\partial_d q^\pm(t)\|_{m-1,*,\pm} < \infty$ for any $0 \le t \le T$ and let $f^\pm \in H^2_*(\Omega^\pm) \cap H^{\frac32}(\Omega^\pm)$ be a function vanishing on $\Sigma^\pm$. Then we have
$$\int_\Sigma (\langle\overline{\partial}\rangle^{\frac12}\mathcal{T}^\gamma q^\pm)\,(\langle\overline{\partial}\rangle f^\pm)\,\mathrm{d}x' \le (\|\partial_d q^\pm\|_{m-1,*,\pm} + \|q^\pm\|_{m,*,\pm})\|\langle\overline{\partial}\rangle^{\frac12} f^\pm\|_{1,\pm} \tag{B.5}$$
In particular, for $s \ge 1$, we have the following inequality for any $g^\pm \in H^s_*(\Omega^\pm)$ with $g^\pm|_{\Sigma^\pm} = 0$.
$$|g^\pm|^2_{s-1/2} \le \|\langle\overline{\partial}\rangle^s g^\pm\|_{0,\pm}\|\langle\overline{\partial}\rangle^{s-1}\partial_d g^\pm\|_{0,\pm} \le \|g^\pm\|_{s,*,\pm}\|\partial_d g^\pm\|_{s-1,*,\pm}.$$

Proof. This is a direct consequence of Gauss-Green formula. Note that the unit exterior normal vectors for $\Omega^\pm$ are $(0,\cdots,0,\mp1)^\top$ respectively, so we have
$$\begin{aligned}\int_\Sigma (\langle\overline{\partial}\rangle^{\frac12}\mathcal{T}^\gamma q^\pm)\,(\langle\overline{\partial}\rangle f^\pm)\,\mathrm{d}x' &= \mp\int_{\Omega^\pm} (\partial_d\mathcal{T}^\gamma q^\pm)\,(\langle\overline{\partial}\rangle^{\frac32} f^\pm) + (\langle\overline{\partial}\rangle\mathcal{T}^\gamma q^\pm)\,(\langle\overline{\partial}\rangle^{\frac12}\partial_d f^\pm)\,\mathrm{d}x \\ &\le (\|\partial_d q^\pm\|_{m-1,*,\pm} + \|q^\pm\|_{m,*,\pm})\|\langle\overline{\partial}\rangle^{\frac12} f^\pm\|_{1,\pm}\end{aligned} \tag{B.6}$$
In particular, let $q^\pm = g^\pm$ and $f^\pm = \langle\overline{\partial}\rangle^{s-\frac32} g^\pm$ in (B.5) and we get
$$\begin{aligned}|g^\pm|^2_{s-1/2} &= \int_\Sigma (\langle\overline{\partial}\rangle^{s-1/2} g^\pm)(\langle\overline{\partial}\rangle^{s-1/2} g^\pm)\,\mathrm{d}x' = \mp2\int_{\Omega^\pm} (\partial_d\langle\overline{\partial}\rangle^{s-1/2} g^\pm)(\langle\overline{\partial}\rangle^{s-1/2} g^\pm)\,\mathrm{d}x \\ &\overset{\langle\overline{\partial}\rangle^{1/2}}{=} \mp 2\int_{\Omega^\pm} (\partial_d\langle\overline{\partial}\rangle^{s-1} g^\pm)(\langle\overline{\partial}\rangle^s g^\pm)\,\mathrm{d}x.\end{aligned}$$
□

The following lemma concerns the Sobolev embeddings.

Lemma B.5 ([81, Lemma 3.3])**.** We have the following inequalities
$$H^m(\Omega^\pm) \hookrightarrow H^m_*(\Omega^\pm) \hookrightarrow H^{\lfloor m/2\rfloor}(\Omega^\pm),\ \ \forall m \in \mathbb{N}^*$$
$$\|u\|_{L^\infty(\Omega^\pm)} \lesssim \|u\|_{H^3_*(\Omega^\pm)},\ \ \|u\|_{W^{1,\infty}(\Omega^\pm)} \lesssim \|u\|_{H^5_*(\Omega^\pm)},\ \ |u|_{W^{1,\infty}(\Omega^\pm)} \lesssim \|u\|_{H^5_*(\Omega^\pm)}.$$

We also need the following Kato-Ponce type multiplicative Sobolev inequality.

Lemma B.6 ([42])**.** Let $J = (1-\Delta)^{1/2}$, $s \geq 0$. Then the following estimates hold:

$$\|J^s(fg)\|_{L^2} \lesssim \|f\|_{W^{s,p_1}}\|g\|_{L^{p_2}} + \|f\|_{L^{q_1}}\|g\|_{W^{s,q_2}}, \tag{B.7}$$

where $1/2 = 1/p_1 + 1/p_2 = 1/q_1 + 1/q_2$ and $2 \leq p_1, q_2 < \infty$.

$$\|[J^s, f]g\|_{L^p} \lesssim \|\partial f\|_{L^\infty}\|J^{s-1}g\|_{L^p} + \|J^s f\|_{L^p}\|g\|_{L^\infty} \tag{B.8}$$

where $s \geq 0$ and $1 < p < \infty$.

We also need the following transport-type estimate in order to close the uniform estimates for the nonlinear approximate system.

Lemma B.7 ([24, Lemma 1])**.** Let $f(t) \in W^{1,1}(0,T)$ and $g \in L^1(0,T)$ and $\kappa > 0$. Assume that

$$f(t) + \kappa f'(t) \leq g(t) \ \text{ a.e. } t \in (0,T).$$

Then for any $t \in (0,T)$,

$$\sup_{\tau\in[0,t]} f(\tau) \leq f(0) + \operatorname*{ess\,sup}_{\tau\in(0,t)} |g(\tau)|.$$

C Paraproducts and the Dirichlet-to-Neumann operator

C.1 Bony's paraproduct decomposition

We already introduce the paradifferential operator in Section 7.2.2. Here we present the relations between paradifferential operators and paraproducts. The cutoff function $\tilde{\chi}(\xi,\eta)$ in the definition of $T_a u$ is

$$\tilde{\chi}(\xi,\eta) = \sum_{k=0}^{\infty} \Theta_{k-3}(\xi)\vartheta(\eta),$$

where $\Theta(\xi) = 1$ when $|\xi| \leq 1$ and $\Theta(\xi) = 0$ when $|\xi| \geq 2$ and

$$\Theta_k(\xi) := \Theta(\frac{\xi}{2}), \ \ k \in \mathbb{Z}, \quad \vartheta_0 = \Theta, \quad \vartheta_k := \Theta_k - \Theta_{k-1}, \ \ k \geq 1.$$

Based on this, we can introduce the Littlewood-Paley projections $\mathcal{P}_k$ and $\mathcal{P}_{\leq k}$ as follows

$$\widehat{\mathcal{P}_k u}(\xi) := \vartheta_k(\xi)\hat{u}(\xi), \ \ \forall k \geq 0, \quad \mathcal{P}_k u := 0 \ \ \forall k < 0, \quad \mathcal{P}_{\leq k} u := \sum_{l\leq k} \mathcal{P}_l u.$$

When the symbol $a(x,\xi)$ (in the paradifferential operator T_a) does not depend on ξ, we can take $\psi(\eta) \equiv 1$ and then we have

$$T_a u = \sum_k \mathcal{P}_{\leq k-3} a(\mathcal{P}_k u)$$

which is the usual Bony's paraproduct. In general, the well-known Bony's paraproduct decomposition is

$$au = T_a u + T_u a + R(u,a), \quad R(u,a) = \sum_{|k-l|\leq 2} (\mathcal{P}_k a)(\mathcal{P}_l u).$$

We have the following estimates for the remainder $R(u,a)$

Lemma C.1 ([3, Section 2.3])**.** For $s \in \mathbb{R}$, $r < d/2$, $\delta > 0$, we have

$$|T_a u|_{H^s} \lesssim \min\{|a|_{L^\infty}|u|_{H^s}, |a|_{H^r}|u|_{H^{s+\frac{d}{2}-r}}, |a|_{H^{\frac{d}{2}}}|u|_{H^{s+\delta}}\}$$

and for any $s > 0$, $s_1, s_2 \in \mathbb{R}$ satisfying $s_1 + s_2 = s + \frac{d}{2}$, we have

$$|R(u,a)|_{H^s} \lesssim |a|_{H^{s_1}}|u|_{H^{s_2}}.$$

C.2 Basic properties of the Dirichlet-to-Neumann operator

Let the space dimension $d = 3$ for simplicity. Given a function $f : \Sigma = \mathbb{T}^2 \to \mathbb{R}$, we define the Dirichlet-to-Neumann (DtN) operator (with respect to ψ and region $\Omega^\pm$) by

$$\mathfrak{N}_\psi^\pm f := \mp N \cdot \nabla^\varphi (\mathcal{E}_\psi^\pm f)|_\Sigma, \quad -\Delta^\varphi(\mathcal{E}_\psi^\pm f) = 0 \text{ in } \Omega^\pm, \ \ \mathcal{E}_\psi^\pm f|_\Sigma = f, \ \ \partial_3(\mathcal{E}_\psi^\pm f)|_{\Sigma^\pm} = 0.$$

Here the Laplacian operator is defined by $\Delta^\varphi := \nabla^\varphi \cdot \nabla^\varphi = \partial_i(\boldsymbol{E}^{ij}\partial_j)$ with

$$\boldsymbol{E} = \frac{1}{\partial_3\varphi}\begin{bmatrix} \partial_3\varphi & 0 & -\overline{\partial}_1\varphi \\ 0 & \partial_3\varphi & -\overline{\partial}_2\varphi \\ -\overline{\partial}_1\varphi & -\overline{\partial}_2\varphi & \frac{1+|\overline{\nabla}\varphi|^2}{\partial_3\varphi} \end{bmatrix} = \frac{1}{\partial_3\varphi}\boldsymbol{P}\boldsymbol{P}^\top, \ \ \boldsymbol{P} := \begin{bmatrix} \partial_3\varphi & 0 & 0 \\ 0 & \partial_3\varphi & 0 \\ -\overline{\partial}_1\varphi & -\overline{\partial}_2\varphi & 1 \end{bmatrix},$$

and $\varphi(t,x) := x_3 + \chi(x_3)\psi(t,x')$ is defined in (1.9) as the extension of ψ into $\Omega^\pm$. The choice of $\chi(x_3)$ is slightly different from [2, 3, 4], but it does not introduce any substantial difference because the expression of Δ^φ is still written to be $\Delta^\varphi := \nabla^\varphi \cdot \nabla^\varphi = \partial_i(\boldsymbol{E}^{ij}\partial_j)$. The DtN operators satisfy the following estimates and we refer to [75, Appendix A.4] for the proof.

Lemma C.2 (Sobolev estimates for DtN operators)**.** For $s > 2 + \frac{d}{2}$, $-\frac{1}{2} \leq r \leq s-1$ and $\psi \in H^s(\mathbb{T}^d)$, we have

$$|\mathfrak{N}_\psi^\pm f|_r \leq C(|\psi|_s)|f|_{r+1}.$$

Lemma C.3 (Remainder estimates for DtN operators)**.** For $s > 2 + \frac{d}{2}$ and $\psi \in H^s(\mathbb{T}^d)$, we have

$$\mathfrak{N}_\psi^\pm f = T_{\Lambda^{(1),\pm}} f + R_1^\pm(f)$$

with $\Lambda^{(1),+} = \Lambda^{(1),\pm}$ defined in Proposition 7.7 and

$$\forall r \in [\frac{1}{2}, s-1], \quad |R_1^\pm(f)|_r + |(\mathfrak{N}_\psi^+ - \mathfrak{N}_\psi^-)f|_r \leq C(|\psi|_s)|f|_r.$$

Lemma C.4 (Sobolev estimates for the inverse of DtN operators)**.** For $s > 2 + \frac{d}{2}$, $-\frac{1}{2} \leq r \leq s-1$ and $\psi \in H^s(\mathbb{T}^d)$, we have

$$|(\mathfrak{N}_\psi^\pm)^{-1} f|_{r+1} \leq C(|\psi|_s)|f|_r.$$

D Construction of initial data satisfying the compatibility conditions

Given initial data $(v_0^\pm, b_0^\pm, q_0^\pm, S_0^\pm, \psi_0)$ of the original current-vortex sheets problem (1.33) satisfying the compatibility conditions (1.34) up to 7-th order, we need to construct a sequence of initial data $(v_0^{\kappa,\pm}, b_0^{\kappa,\pm}, q_0^{\kappa,\pm}, S_0^{\kappa,\pm}, \psi_0^\kappa)$ to the nonlinear κ-approximate system (3.1) satisfying the compatibility conditions (3.4) up to 7-th order that converge to the given data as $\kappa \to 0_+$.

D.1 Reformulation of the compatibility conditions

Let us first ignore the κ-regularization terms and consider the compatibility conditions (1.34) for the original system. Also, let us omit the fixed boundaries $\Sigma^\pm$, omit the density functions, consider the isentropic case and write $\varepsilon^2 = \mathcal{F}_p^\pm$ for convenience. The heuristic idea is that the odd ($m = 2r+1$) order compatibility condition is rewritten to be

$$-\left[\!\left[\Lambda_{\varepsilon,b_0}^{r+1}(\Delta^{\varphi_0})^r(\nabla^{\varphi_0}\cdot v_0)\right]\!\right] = \cdots \quad \text{on } \Sigma$$

and the even ($m = 2r$) order compatibility condition is rewritten to be

$$\left[\!\left[\Lambda_{\varepsilon,b_0}^{r}(\Delta^{\varphi_0})^r q_0\right]\!\right] = \cdots \quad \text{on } \Sigma$$

with $\Lambda_{\varepsilon,b_0} := \varepsilon^{-2} + |b_0|^2$. Such reformulation is convenient for us to add κ-perturbation terms to construct the desired data for (3.1). More specifically, let us start with the zero-th order compatibility conditions:

$$[\![q_0]\!] = \sigma\mathcal{H}(\psi_0), \quad \psi_t|_{t=0} = v_0^\pm \cdot N_0 = v_{03}^\pm - \bar{v}_0^\pm \cdot \overline{\partial}\psi_0. \tag{D.1}$$

The first-order compatibility conditions are

$$\partial_t [\![q]\!]|_{t=0} = \sigma\partial_t\mathcal{H}(\psi)|_{t=0}, \quad \psi_{tt}|_{t=0} = \partial_t(v^\pm \cdot N)|_{t=0}, \tag{D.2}$$

which are not easy to compute, especially the first one. The left side is equal to

$$\partial_t q^+ - \partial_t q^- = \overline{D_t^+} q^+ - \overline{D_t^-} q^- - (\bar{v}^+ \cdot \overline{\nabla}) [\![q]\!] - ([\![\bar{v}]\!] \cdot \overline{\nabla}) q^-.$$

Using the continuity equation, the evolution equation of b, we get

$$\overline{D_t} q = -\varepsilon^{-2}(\nabla^\varphi \cdot v) + \overline{D_t} b \cdot b = -\underbrace{(\varepsilon^{-2} + |b|^2)}_{=:\Lambda_{\varepsilon,b}}(\nabla^\varphi \cdot v) + (\bar{b} \cdot \overline{\nabla}) v \cdot b \text{ on } \Sigma,$$

and thus the time-differentiated jump condition becomes

$$[\![\Lambda_{\varepsilon,b}(\nabla^{\varphi_0} \cdot v_0)]\!] = \left[\!\!\left[(\bar{b}_0 \cdot \overline{\nabla}) v_0 \cdot b_0\right]\!\!\right] - ([\![\bar{v}_0]\!] \cdot \overline{\nabla}) q_0^- + \overline{D_t^+}(\sigma\mathcal{H}(\psi))|_{t=0} \text{ on } \Sigma.$$

Here and thereafter, we will repeatedly use $\overline{D_t^\pm}\psi = v_3^\pm$ on Σ and omit lots of redundant terms in order for simplicity of notations. For example, we will write $\mathcal{H}(\psi) \sim \overline{\Delta}\psi$, write $(1 - \overline{\Delta})$ to be $-\overline{\Delta}$, and omit the commutators between $\overline{D_t^+}$ and $\mathcal{H}, (1 - \overline{\Delta})$, the density function ρ. Indeed, later we will see that the concrete form of those omitted term is not important, and we just need to find out the major term as in [49, Appendix A]. Under this setting, we have

$$[\![\Lambda_{\varepsilon,b}(\nabla^{\varphi_0} \cdot v_0)]\!] \sim \left[\!\!\left[(\bar{b}_0 \cdot \overline{\nabla}) v_0 \cdot b_0\right]\!\!\right] - ([\![\bar{v}_0]\!] \cdot \overline{\nabla}) q_0^- + \sigma\overline{\Delta} v_{03}^+ \text{ on } \Sigma. \tag{D.3}$$

For higher-order compatibility conditions, we invoke the wave equation (7.11) to get (cf. [49, Appendix A.1])

$$(\overline{D_t})^2 q = \Lambda_{\varepsilon,b}\Delta^\varphi q + \mathcal{M}_0(v,b) + \mathcal{N}_0(v,b) \quad \text{on } \Sigma, \tag{D.4}$$

where

$$\mathcal{M}_0(v,b) = -(\bar{b} \cdot \overline{\nabla})^2 q + (\bar{b} \cdot \overline{\nabla})^2 b \cdot b + \mathcal{R}_0(v,b), \quad \mathcal{N}_0(v,b) = \partial_i^\varphi v^j \partial_j^\varphi v^i - \partial_i^\varphi b^j \partial_j^\varphi b^i$$

and $\mathcal{R}_0(v,b)$ only contains the first-order derivatives of b, v with the form

$$\mathcal{R}_0(v,b) = P_0(b)((\partial^{i_1} v)(\partial^{i_2} v) + (\partial^{j_1} b)(\partial^{j_2} b))$$

where $P_0(b)$ is a polynomial of b only containing cubic and quadratic terms and $(i_1, i_2, j_1, j_2) = (0,0,1,1)$ or $(1,1,0,0)$. Taking substraction between the equation of q^+ and the equation of q^-, we get

$$\left[\!\!\left[(\overline{D_t})^2 q\right]\!\!\right]|_{t=0} = [\![\Lambda_{\varepsilon,b_0}\Delta^{\varphi_0} q_0]\!] + [\![\mathcal{M}_0(v_0,b_0) + \mathcal{N}_0(v_0,b_0)]\!] \quad \text{on } \Sigma.$$

Then using $\overline{D_t^+} = \overline{D_t^-} + ([\![\bar{v}]\!] \cdot \overline{\nabla})$, we get

$$\left[\!\!\left[(\overline{D_t})^2 q\right]\!\!\right]|_{t=0} = (\overline{D_t^+})^2(\sigma\mathcal{H}(\psi))|_{t=0} + \mathcal{T}_{[\![v]\!]}^2 q^-|_{t=0},$$

where each $\mathcal{T}_{[\![v]\!]}$ represents either of $\overline{D_t^-}$ and $([\![\bar{v}]\!] \cdot \overline{\nabla})$. So, the second-order compatibility condition is reformulated as

$$\begin{aligned} [\![\Lambda_{\varepsilon,b_0}\Delta^{\varphi_0} q_0]\!] &= (\overline{D_t^+})^2(\sigma\mathcal{H}(\psi))|_{t=0} + \mathcal{T}_{[\![v]\!]}^2 q^-|_{t=0} - [\![\mathcal{M}_0(v_0,b_0) + \mathcal{N}_0(v_0,b_0)]\!] \\ &\sim -\sigma\overline{\Delta}\partial_3 q_0^+ + \sigma\overline{\Delta}(\bar{b}^+ \cdot \overline{\nabla}) b_{03}^+ + \mathcal{T}_{[\![v]\!]}^2 q^-|_{t=0} - [\![\mathcal{M}_0(v_0,b_0) + \mathcal{N}_0(v_0,b_0)]\!] \quad \text{on } \Sigma. \end{aligned} \tag{D.5}$$

Taking one more material derivative in the wave equation and again use the continuity equation, we get

$$(\overline{D_t})^3 q \sim -\Lambda_{\varepsilon,b}^2 \Delta^\varphi(\nabla^\varphi \cdot v) + \varepsilon^{-2}(\bar{b} \cdot \overline{\nabla})^2(\nabla^\varphi \cdot v) + \mathcal{M}_1(v,b,q) + \mathcal{N}_1(v,b,q) \tag{D.6}$$

where the concrete form of $\mathcal{M}_1, \mathcal{N}_1$ will be specified later. Recursively, after long and tedious calculations (cf. [49, (A.4)-(A.7)]), we find that the time-differentiated wave equation (restricted on $\{t=0\}\times\Sigma$) can be expressed as

$$m=2r+1,\quad -\Lambda_{\varepsilon,b_0}^{r+1}(\Delta^{\varphi_0})^r(\nabla^{\varphi_0}\cdot v_0)=(\overline{D_t})^{2r+1}q+\sum_{j=0}^{r}(\Delta^{\varphi_0})^j(\mathcal{M}_{2r-1-2j}(v_0,b_0,q_0)+\mathcal{N}_{2r-1-2j}(v_0,b_0,q_0))\text{ on }\Sigma, \tag{D.7}$$

$$m=2r,\ \ \Lambda_{\varepsilon,b_0}^{r}(\Delta^{\varphi_0})^r q_0=(\overline{D_t})^{2r}q+\sum_{j=0}^{r-1}(\Delta^{\varphi_0})^j(\mathcal{M}_{2r-2-2j}(v_0,b_0,q_0)+\mathcal{N}_{2r-2-2j}(v_0,b_0,q_0))\text{ on }\Sigma, \tag{D.8}$$

where $\mathcal{M}_{-1}(v_0,b_0):=-(\bar b_0\cdot\overline{\nabla})v_0\cdot b_0$ and $\mathcal{N}_{-1}:=0$, and for $r\geq 1$ we define

$$m=2r-1,\quad \mathcal{M}_{2r-1}(v_0,b_0,q_0)=(\bar b_0\cdot\overline{\nabla})^2(\Delta^{\varphi_0})^{r-1}(\nabla^{\varphi_0}\cdot v_0)+\sum_{l=2}^{r+1}\underbrace{b_0^{i_1}\cdots b_0^{i_{2l}}(\nabla^{2r+1}v_0)}_{<2^l\text{ terms}}+\mathcal{R}_{2r-1}(v_0,b_0,q_0), \tag{D.9}$$

$$\begin{aligned} m=2r,\quad \mathcal{M}_{2r}(v_0,b_0,q_0)=&-(\bar b_0\cdot\overline{\nabla})^2(\Delta^{\varphi_0})^r q_0+\mathcal{R}_{2r}(v_0,b_0,q_0),\\ &+\sum_{l=2}^{r+1}\underbrace{(\bar b_0\cdot\overline{\nabla})^{r+2}(\nabla^r b_0)b_0^{i_1}\cdots b_0^{i_{2l}}+(\bar b_0\cdot\overline{\nabla})^2(\nabla^{2r}q_0)b_0^{j_1}\cdots b_0^{j_{2l}}}_{<2^l\text{ terms}}; \end{aligned} \tag{D.10}$$

and the term $\mathcal{R}_m$, where every top-order term has $(m+1)$-th order derivative, has the following form

$$\mathcal{R}_m(v_0,b_0,q_0)=P_k(b_0)\left(C^m_{i_1\cdots i_p,j_1\cdots j_n,k_1\cdots k_l}(\nabla^{i_1}v_0)\cdots(\nabla^{i_p}v_0)(\nabla^{j_1}b_0)\cdots(\nabla^{j_n}b_0)(\nabla^{k_1}q_0)\cdots(\nabla^{k_l}q_0)\right),$$

where ∇ may represent either of ∇^{φ_0} or ∂, and $P_k(\cdot)$ is a polynomial of its arguments and the lowest power is 4 and the indices above satisfy

$$\begin{aligned} 1\leq i_1,\cdots,i_p,j_1,\cdots,j_n\leq k+1,\ 0\leq k_1,\cdots,k_l\leq m+1,\\ i_1+\cdots+i_p+j_1+\cdots+j_n+k_1+\cdots+k_l=m+1. \end{aligned}$$

The term $\mathcal{N}_m(v_0,b_0,q_0)$ has the following form

$$\begin{aligned} \mathcal{N}_m(v_0,b_0,q_0)=&P_{m,1}(b_0)(\nabla^{1+2\lfloor\frac{m}{2}\rfloor}v_0)(\nabla v_0)+P_{m,2}(b_0)(\nabla^{2\lceil\frac{m}{2}\rceil}q_0)(\nabla v_0)+P_{k,0}(b_0)(\nabla^{m+1}b_0)(\nabla v_0)\\ &+P'_m(b_0)D^m_{i_1\cdots i_p,j_1\cdots j_n,k_1\cdots k_l}\left((\nabla^{i_1}v_0)\cdots(\nabla^{i_p}v_0)(\nabla^{j_1}b_0)\cdots(\nabla^{j_n}b_0)(\nabla^{k_1}q_0)\cdots(\nabla^{k_l}q_0)\right), \end{aligned} \tag{D.11}$$

where $P_{m,1}(\cdot),P_{m,2}(\cdot),P'_m(\cdot)$ are polynomials of their arguments and $P_{m,0}(\cdot)$ is a polynomial of its arguments and the lowest power is 2. The indices above satisfy

$$\begin{aligned} 1\leq i_1,\cdots,i_p,j_1,\cdots,j_n\leq k,\ 0\leq k_1,\cdots,k_l\leq m,\\ i_1+\cdots+i_p+j_1+\cdots+j_n+k_1+\cdots+k_l=m+1. \end{aligned}$$

Next we take the difference between the equations (D.7)-(D.8) in Ω^+ and those in Ω^- and restrict the equation on $\{t=0\}\times\Sigma$ to get the jump condition in the m-th order compatibility conditions

$$\begin{aligned} m=2r+1,\quad -\left[\!\left[\Lambda_{\varepsilon,b_0}^{r+1}(\Delta^{\varphi_0})^r(\nabla^{\varphi_0}\cdot v_0)\right]\!\right]=&\left[\!\left[(\overline{D_t})^{2r+1}q\right]\!\right] \qquad\qquad (\text{D.12})\\ &+\sum_{j=0}^{r}\left[\!\left[(\Delta^{\varphi_0})^j(\mathcal{M}_{2r-1-2j}(v_0,b_0,q_0)+\mathcal{N}_{2r-1-2j}(v_0,b_0,q_0))\right]\!\right]\text{ on }\Sigma,\\ m=2r,\quad \left[\!\left[\Lambda_{\varepsilon,b_0}^{r}(\Delta^{\varphi_0})^r q_0\right]\!\right]=&\left[\!\left[(\overline{D_t})^{2r}q\right]\!\right] \qquad\qquad (\text{D.13})\\ &+\sum_{j=0}^{r-1}\left[\!\left[(\Delta^{\varphi_0})^j(\mathcal{M}_{2r-2-2j}(v_0,b_0,q_0)+\mathcal{N}_{2r-2-2j}(v_0,b_0,q_0))\right]\!\right]\text{ on }\Sigma. \end{aligned}$$

Then using $\overline{D_t^+} = \overline{D_t^-} + ([\![\bar{v}]\!] \cdot \overline{\nabla})$, we get

$$\left[\!\!\left[(\overline{D_t})^m q\right]\!\!\right] = (\overline{D_t^+})^m [\![q]\!] + \mathcal{T}_{[\![\bar{v}]\!]}^m q^-|_{t=0},$$

where each $\mathcal{T}_{[\![\bar{v}]\!]}$ represents either $(\overline{D_t^-})$ or $([\![\bar{v}]\!] \cdot \overline{\nabla})$. Using the jump condition for $[\![q]\!]$, we have

$$m = 2r : (\overline{D_t^+})^{2r} [\![q]\!] \sim \sigma\overline{\Delta}(\overline{D_t^+})^{2r-1} v_3^+ \sim \sigma\Lambda_{\varepsilon,b_0}^{r-1}\overline{\Delta}(\Delta^{\varphi_0})^{r-1}\partial_3 q_0^+ + \sigma\overline{\Delta}\mathcal{S}_{2r-1}(v_0^+, b_0^+, q_0^+) \tag{D.14}$$

$$m = 2r+1 : (\overline{D_t^+})^{2r+1} [\![q]\!] \sim \sigma\overline{\Delta}(\overline{D_t^+})^{2r} v_3^+ \sim -\sigma\Lambda_{\varepsilon,b_0}^{r}\overline{\Delta}(\Delta^{\varphi_0})^{r-1}\partial_3 (\nabla^{\varphi_0}\cdot v_0^+) + \sigma\overline{\Delta}\mathcal{S}_{2r}(v_0^+, b_0^+, q_0^+) \tag{D.15}$$

where the leading-order terms in $\mathcal{S}_m$ are

$$\mathcal{S}_{2r-1} \overset{L}{=} (\Lambda_{\varepsilon,b_0})^{r-2}(\bar{b}_0\cdot\overline{\nabla})^2(\Delta^{\varphi_0})^{r-2}\partial_3 q_0^+, \quad \mathcal{S}_{2r} \overset{L}{=} -(\Lambda_{\varepsilon,b_0})^{r-1}(\bar{b}_0\cdot\overline{\nabla})^2(\Delta^{\varphi_0})^{r-2}\partial_3(\nabla^{\varphi_0}\cdot v_0^+). \tag{D.16}$$

Thus, the compatibility conditions for the original current-vortex sheets system (1.33) are reformulated as

$$\begin{aligned} m = 2r+1,\quad & -\left[\!\!\left[\Lambda_{\varepsilon,b_0}^{r+1}(\Delta^{\varphi_0})^r(\nabla^{\varphi_0}\cdot v_0)\right]\!\!\right] \sim \sum_{j=0}^{r}\left[\!\!\left[(\Delta^{\varphi_0})^j(\mathcal{M}_{2r-1-2j}(v_0,b_0,q_0) + \mathcal{N}_{2r-1-2j}(v_0,b_0,q_0))\right]\!\!\right] \\ & + \mathcal{T}_{[\![\bar{v}]\!]}^{2r+1} q^-|_{t=0} - \sigma\Lambda_{\varepsilon,b_0}^{r}\overline{\Delta}(\Delta^{\varphi_0})^{r-1}\partial_3(\nabla^{\varphi_0}\cdot v_0^+) + \sigma\overline{\Delta}\mathcal{S}_{2r}(v_0^+, b_0^+, q_0^+) \quad \text{on } \Sigma, \end{aligned} \tag{D.17}$$

$$\begin{aligned} m = 2r,\quad & \left[\!\!\left[\Lambda_{\varepsilon,b_0}^{r}(\Delta^{\varphi_0})^r q_0\right]\!\!\right] \sim \sum_{j=0}^{r-1}\left[\!\!\left[(\Delta^{\varphi_0})^j(\mathcal{M}_{2r-2-2j}(v_0,b_0,q_0) + \mathcal{N}_{2r-2-2j}(v_0,b_0,q_0))\right]\!\!\right] \\ & + \mathcal{T}_{[\![\bar{v}]\!]}^{2r} q^-|_{t=0} + \sigma\Lambda_{\varepsilon,b_0}^{r-1}\overline{\Delta}(\Delta^{\varphi_0})^{r-1}\partial_3 q_0^+ + \sigma\overline{\Delta}\mathcal{S}_{2r-1}(v_0^+, b_0^+, q_0^+)) \quad \text{on } \Sigma. \end{aligned} \tag{D.18}$$

Note that the time-differentiated kinematic boundary condition is already implicitly used when deriving the above compatibility conditions. Similarly, the compatibility conditions for the κ-approximate problem (3.1) are reformulated as

$$\begin{aligned} m = 2r+1,\quad & -\left[\!\!\left[\Lambda_{\varepsilon,b_0}^{r+1}(\Delta^{\varphi_0})^r(\nabla^{\varphi_0}\cdot v_0^\kappa)\right]\!\!\right] \sim \sum_{j=0}^{r}\left[\!\!\left[(\Delta^{\varphi_0})^j(\mathcal{M}_{2r-1-2j}(v_0^\kappa,b_0^\kappa,q_0^\kappa) + \mathcal{N}_{2r-1-2j}(v_0^\kappa,b_0^\kappa,q_0^\kappa))\right]\!\!\right] \\ & + \mathcal{T}_{[\![\bar{v}]\!]}^{2r+1} q^-|_{t=0} - \sigma\Lambda_{\varepsilon,b_0}^{r}\overline{\Delta}(\Delta^{\varphi_0})^{r-1}\partial_3(\nabla^{\varphi_0}\cdot v_0^{\kappa,+}) + \kappa\Lambda_{\varepsilon,b_0}^{r}\overline{\Delta}^2(\Delta^{\varphi_0})^{r-1}\partial_3(\nabla^{\varphi_0}\cdot v_0^{\kappa,+}) + \kappa\Lambda_{\varepsilon,b_0}^{r}\overline{\Delta}(\Delta^{\varphi_0})^{r}\partial_3 q_0^+ \\ & + (\sigma\overline{\Delta} - \kappa\overline{\Delta}^2)\mathcal{S}_{2r}(v_0^{\kappa,+}, b_0^{\kappa,+}, q_0^{\kappa,+}) + \kappa\overline{\Delta}\mathcal{S}_{2r+1}(v_0^{\kappa,+}, b_0^{\kappa,+}, q_0^{\kappa,+}) \quad \text{on } \Sigma, \end{aligned} \tag{D.19}$$

$$\begin{aligned} m = 2r,\quad & \left[\!\!\left[\Lambda_{\varepsilon,b_0}^{r}(\Delta^{\varphi_0})^r q_0^\kappa\right]\!\!\right] \sim \sum_{j=0}^{r-1}\left[\!\!\left[(\Delta^{\varphi_0})^j(\mathcal{M}_{2r-2-2j}(v_0^\kappa,b_0^\kappa,q_0^\kappa) + \mathcal{N}_{2r-2-2j}(v_0^\kappa,b_0^\kappa,q_0^\kappa))\right]\!\!\right] \\ & + \mathcal{T}_{[\![\bar{v}]\!]}^{2r} q^-|_{t=0} + \sigma\Lambda_{\varepsilon,b_0}^{r-1}\overline{\Delta}(\Delta^{\varphi_0})^{r-1}\partial_3 q_0^+ - \kappa\Lambda_{\varepsilon,b_0}^{r-1}\overline{\Delta}^2(\Delta^{\varphi_0})^{r-1}\partial_3 q_0^+ - \kappa\Lambda_{\varepsilon,b_0}^{r}\overline{\Delta}(\Delta^{\varphi_0})^{r-1}\partial_3(\nabla^{\varphi_0}\cdot v_0^+) \\ & + (\sigma\overline{\Delta} - \kappa\overline{\Delta}^2)\mathcal{S}_{2r-1}(v_0^{\kappa,+}, b_0^{\kappa,+}, q_0^{\kappa,+}) + \kappa\overline{\Delta}\mathcal{S}_{2r}(v_0^{\kappa,+}, b_0^{\kappa,+}, q_0^{\kappa,+}) \quad \text{on } \Sigma. \end{aligned} \tag{D.20}$$

D.2 Construction of the converging initial data

Given initial data $(v_0^\pm, b_0^\pm, q_0^\pm, S_0^\pm, \psi_0)$ of (1.33) satisfying the compatibility conditions (D.17)-(D.18) up to 7-th order, we now construct the initial data $(v_0^{\kappa,\pm}, b_0^{\kappa,\pm}, q_0^{\kappa,\pm}, S_0^{\kappa,\pm}, \psi_0^\kappa)$ to (3.1) satisfying the compatibility conditions (D.19)-(D.20) up to 7-th order that converge to the given data as $\kappa \to 0_+$. To do this, we just need to **equally distribute the κ-term to the solution in Ω^+ and the solution in Ω^-**.

D.2.1 Recover the 0-th order and the 1-st order compatibility conditions

First, we pick $b_0^{\kappa,\pm} = b_0^\pm$, $\psi_0^\kappa = \psi_0$. We define $\partial_t\psi^\kappa|_{t=0} := v_0^\pm \cdot N_0$ and $\partial_t b^\pm|_{t=0} = (b_0^\pm\cdot\nabla^{\varphi_0})v_0^\pm - b_0^\pm(\nabla^{\varphi_0}\cdot v_0^\pm)$ in $\Omega^\pm$. Then the constraints for the magnetic field are automatically satisfied. Now, we construct $q_0^{(0)}$ such that $(v_0^\pm, b_0^\pm, q_0^{(0),\pm}, \psi_0)$ satisfies the 0-th order compatibility condition (D.20). The function $q_0^{(1),\pm}$ is set to be the solution to the poly-harmonic equation

$$\begin{cases}\Delta^2 q_0^{(0),\pm} = \Delta^2 q_0^\pm & \text{in } \Omega^\pm \\ q_0^{(0),\pm} = q_0^\pm \mp \frac{1}{2}\kappa\overline{\Delta}^2\psi_0 \pm \frac{1}{2}\kappa\overline{\Delta}(v_0^\pm\cdot N_0) & \text{on } \Sigma \\ \partial_3 q_0^{(0),\pm} = \partial_3 q_0^\pm & \text{on } \Sigma \\ \partial_3^j q_0^{(0),\pm} = \partial_3^j q_0^\pm,\ \ 0\leq j\leq 1 & \text{on } \Sigma^\pm. \end{cases} \tag{D.21}$$

Then for $s\geq 4$, we have

$$\|q_0^{(0),\pm} - q_0^\pm\|_{s,\pm} \lesssim \kappa|\overline{\Delta}^2\psi_0|_{s-0.5} + \kappa|\overline{\Delta}(v_0^\pm\cdot N_0)|_{s-0.5} \to 0 \quad \text{as } \kappa\to 0.$$

With this $q_0^{(0)}$, we define $\partial_t^2\psi|_{t=0} = \partial_t(v^\pm\cdot N)|_{t=0}$ via $(v_0^\pm, b_0^\pm, q_0^{(0),\pm}, \psi_0)$ on Σ. (Note that $\partial_t v\cdot N|_{t=0}$ already includes $\partial_3 q_0$. Only when we have $\partial_3 q_0^{(0),\pm} = \partial_3 q_0^\pm$ on Σ can we keep the jump condition $[\![\partial_t(v\cdot N)]\!] = 0$.) and also define the corresponding $\partial_t^2 b|_{t=0}$ in $\Omega^\pm$ via the evolution equation of b. Thus, the ∂_t-differentiated boundary constraint for $b\cdot N$ is also satisfied.

Now we introduce $v_0^{(0),\pm}$ such that $(v_0^{(0),\pm}, b_0^\pm, q_0^{(0),\pm}, \psi_0)$ satisfies the 1-st order compatibility condition (D.19). We define $\bar v_{0i}^{(0),\pm} = \bar v_{0i}^\pm$ for $i=1,2$ and define $v_{03}^{(0),\pm}$ via the following poly-harmonic equation

$$\begin{cases}\Delta^3 v_{03}^{(0),\pm} = \Delta^3 v_{03}^\pm & \text{in } \Omega^\pm \\ \Lambda_{\varepsilon,b_0}(\nabla^{\varphi_0}\cdot v_{03}^{(0),\pm}) = (\nabla^{\varphi_0}\cdot\Lambda_{\varepsilon,b_0} v_{03}^\pm) \mp \frac{1}{2}([\![\bar v_0]\!]\cdot\overline{\nabla})(q_0^{(0),\pm} - q_0^\pm) \mp \frac{\kappa}{2}\overline{\Delta}^2 v_{03}^+ \pm \frac{\kappa}{2}\overline{\Delta}\partial_3 q_0^{(0),+} & \text{on } \Sigma \\ v_{03}^{(0),\pm} = v_{03}^\pm, \quad \partial_3^2 v_{03}^{(0),\pm} = \partial_3^2 v_{03}^\pm & \text{on } \Sigma \\ \partial_3^j v_{03}^{(0),\pm} = \partial_3^j v_{03}^\pm,\ \ 0\leq j\leq 2 & \text{on } \Sigma^\pm. \end{cases} \tag{D.22}$$

It is also straightforward to see the convergece for $s\geq 6$

$$\|v_0^{(0),\pm} - v_0^\pm\|_{s,\pm} \lesssim |q_0^{(0),\pm} - q_0^\pm|_{s-0.5} + \kappa(|v_{03}^+|_{s+2.5} + |\partial_3 q_0^+|_{s+0.5}).$$

D.2.2 Higher-order compatibility conditions

For $r\geq 1$, we can inductively define $q_0^{(r),\pm}$ such that $(v_0^{(r-1),\pm}, b_0^\pm, q_0^{(r),\pm}, \psi_0)$ satisfies the compatibility condition up to $2r$-th order

$$\begin{cases}\Delta^{2r+2} q_0^{(r),\pm} = \Delta^{2r+2} q_0^{(r-1),\pm} & \text{in } \Omega^\pm \\ \Lambda_{\varepsilon,b_0}^r(\Delta^{\varphi_0})^r q_0^{(r),\pm} = \Lambda_{\varepsilon,b_0}^r(\Delta^{\varphi_0})^r q_0^{(r-1),\pm} & \\ \quad + \sum_{j=0}^{r-1}(\Delta^{\varphi_0})^j\Big((\mathcal{M}_{2r-2-2j} + \mathcal{N}_{2r-2-2j})(v_0^{(r-1),\pm}, b_0^\pm, q_0^{(r),\pm}) - (\mathcal{M}_{2r-2-2j} + \mathcal{N}_{2r-2-2j})(v_0^{(r-2),\pm}, b_0^\pm, q_0^{(r-1),\pm})\Big) & \\ \quad \pm\frac{1}{2}\Big((\mathcal{T}^{2r}_{[\![\bar v^{(r-1)}]\!]} q^{(r),-} - \mathcal{T}^{2r}_{[\![\bar v^{(r-2)}]\!]} q^{(r-1),-}) + \sigma\Lambda_{\varepsilon,b_0}^{r-1}\underbrace{\overline{\Delta}(\Delta^{\varphi_0})^{r-1}\partial_3(q_0^{(r),+} - q_0^{(r-1),+})}_{=0} & \\ \qquad + \sigma\overline{\Delta}(\mathcal{S}_{2r-1}(v_0^{(r-1),+}, b_0^+, q_0^{(r),+}) - \mathcal{S}_{2r-1}(v_0^{(r-2),+}, b_0^+, q_0^{(r-1),+}))\Big) & \\ \quad \mp\frac{\kappa}{2}\left(\Lambda_{\varepsilon,b_0}^{r-1}\underbrace{\overline{\Delta}^2(\Delta^{\varphi_0})^{r-1}\partial_3(q_0^{(r),+} - q_0^{(r-1),+})}_{=0} - \overline{\Delta}\Lambda_{\varepsilon,b_0}^r(\Delta^{\varphi_0})^{r-1}\partial_3\nabla^{\varphi_0}\cdot(v_0^{(r-1),+} - v_0^{(r-2),+})\right) & \\ \quad \mp\frac{\kappa}{2}\Big((\overline{\Delta}^2\mathcal{S}_{2r-1} - \overline{\Delta}\mathcal{S}_{2r})(v_0^{(r-1),+}, b_0^+, q_0^{(r),+}) - (\overline{\Delta}^2\mathcal{S}_{2r-1} - \overline{\Delta}\mathcal{S}_{2r})(v_0^{(r-2),+}, b_0^+, q_0^{(r-1),+})\Big) & \text{on } \Sigma \\ \partial_3^j q_0^{(r),\pm} = \partial_3^j q_0^{(r-1),\pm},\ \ 0\leq j\leq 2r+1,\ j\neq 2r & \text{on } \Sigma \\ \partial_3^j q_0^{(r),\pm} = \partial_3^j q_0^{(r-1),\pm},\ \ 0\leq j\leq 2r+1 & \text{on } \Sigma^\pm, \end{cases} \tag{D.23}$$

and define $\bar{v}_0^{(r),\pm} = \bar{v}_0^{(r-1),\pm}$ and $v_{03}^{(r),\pm}$ such that $(v_0^{(r),\pm}, b_0^\pm, q_0^{(r),\pm}, \psi_0)$ satisfies the compatibility condition up to $(2r+1)$-th order

$$\begin{cases}
\Delta^{2r+3} v_{03}^{(r),\pm} = \Delta^{2r+3} v_{03}^{(r-1),\pm} & \text{in } \Omega^\pm \\
-\Lambda_{\varepsilon,b_0}^r (\Delta^{\varphi_0})^r (\nabla^{\varphi_0}\cdot v_0^{(r),\pm}) = -\Lambda_{\varepsilon,b_0}^r (\Delta^{\varphi_0})^r (\nabla^{\varphi_0}\cdot v_0^{(r-1),\pm}) & \\
\quad + \sum_{j=0}^r (\Delta^{\varphi_0})^j \left((\mathcal{M}_{2r-1-2j} + \mathcal{N}_{2r-1-2j})(v_0^{(r),\pm}, b_0^\pm, q_0^{(r),\pm}) - (\mathcal{M}_{2r-1-2j} + \mathcal{N}_{2r-1-2j})(v_0^{(r-1),\pm}, b_0^\pm, q_0^{(r-1),\pm}) \right) & \\
\quad \pm \frac{1}{2}\Big((\mathcal{T}^{2r+1}_{[\![\bar{v}^{(r)}]\!]} q^{(r),-} - \mathcal{T}^{2r}_{[\![\bar{v}^{(r-1)}]\!]} q^{(r-1),-}) - \sigma \Lambda_{\varepsilon,b_0}^{r-1} \underbrace{\overline{\Delta}(\Delta^{\varphi_0})^{r-1}\partial_3 \nabla^{\varphi_0}\cdot (v_{03}^{(r),+} - v_{03}^{(r-1),+})}_{=0} & \\
\qquad + \sigma\overline{\Delta}(\mathcal{S}_{2r}(v_0^{(r),+}, b_0^+, q_0^{(r),+}) - \mathcal{S}_{2r}(v_0^{(r-1),+}, b_0^+, q_0^{(r-1),+}))\Big) & \\
\quad \pm \frac{\kappa}{2} \left(\overline{\Delta}^2 \Lambda_{\varepsilon,b_0}^{r-1} \underbrace{(\Delta^{\varphi_0})^{r-1}\partial_3\nabla^{\varphi_0}\cdot (v_0^{(r),+} - v_0^{(r-1),+})}_{=0} - \overline{\Delta}\Lambda_{\varepsilon,b_0}^r (\Delta^{\varphi_0})^{r-1}\partial_3 (q_0^{(r),+} - q_0^{(r-1),+}) \right) & \\
\quad \mp \frac{\kappa}{2} \left((\overline{\Delta}^2 \mathcal{S}_{2r} - \overline{\Delta}\mathcal{S}_{2r+1})(v_0^{(r),+}, b_0^+, q_0^{(r),+}) - (\overline{\Delta}^2 \mathcal{S}_{2r} - \overline{\Delta}\mathcal{S}_{2r+1})(v_0^{(r-1),+}, b_0^+, q_0^{(r-1),+}) \right) & \text{on } \Sigma \\
\partial_3^j v_{03}^{(r),\pm} = \partial_3^j v_{03}^{(r-1),\pm}, \quad 0 \le j \le 2r+2,\ j \neq 2r+1 & \text{on } \Sigma \\
\partial_3^j v_{03}^{(r),\pm} = \partial_3^j v_{03}^{(r-1),\pm}, \quad 0 \le j \le 2r+2 & \text{on } \Sigma^\pm.
\end{cases} \tag{D.24}$$

Since we require the compatibility conditions up to 7-th order, we can stop at $r = 3$ and define $(v_0^{\kappa,\pm}, b_0^{\kappa,\pm}, q_0^{\kappa,\pm}, S_0^{\kappa,\pm}, \psi_0^\kappa)$ to be $(v_0^{(3),\pm}, b_0^\pm, q_0^{(3),\pm}, S_0^\pm, \psi_0)$. It is also straightforward to see the convergence after long and tedious calculations: For $s \ge 2\times(2r+3) = 18$, we have the convergence

$$\left\|(v_0^{\kappa,\pm}, q_0^{\kappa,\pm}) - (v_0^\pm, q_0^\pm)\right\|_{s,\pm} \lesssim P(\|v_0^\pm, b_0^\pm, q_0^\pm, S_0^\pm\|_{s+1,\pm}) \left(\kappa|\psi_0|_{s+3.5} + \sum_{j=0}^r \kappa|(\Delta^{\varphi_0})^j v_0^\pm|_{s+1.5-2j} + \kappa|(\Delta^{\varphi_0})^{(j-1)_+}\partial_3 q_0^\pm|_{s+0.5-2j} \right)$$
$$\to 0 \qquad \text{as } \kappa \to 0,$$

provided that the given initial data is sufficiently regular. Specifically, picking $s = 18$, the given data is required to satisfy $\|(v_0^\pm, b_0^\pm, q_0^\pm, S_0^\pm)\|_{20,\pm} + |\psi_0|_{21.5} < +\infty$. We may assume the given data belongs to C^∞-class for convenience.

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