

1. (a) **【5】**  $t = 3.76/1.88 = 2$ , 原假设下  $t$  统计量服从  $t_{n-7} = t_{50}$ .  
 (b) **【5】**  $F = \frac{n-p}{p-1} \times \frac{R^2}{1-R^2} = \frac{57-7}{7-1} \times \frac{0.81}{1-0.81} = 35.526$ . 分布为  $F_{p-1, n-p} = F_{6, 50}$ .  
 (c) **【5】**  $r_{y\hat{y}} = \sqrt{R^2} = 0.9$ .  
 (d) **【5】** 简单回归的  $b$  估计的方差  $v = \sigma^2/s_{xx}$ , 而全模型下的  $b$  估计的方差为  $v \times \frac{1}{1-R_x^2} = 1.88^2 = 3.534$ , 所以  $v = (1 - R_x^2) \times 4.534 = 1.767$ .

2. (a)  $z_i = 0$  时, 模型为  $y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \epsilon_i \sim (0, \sigma^2), i = 1, \dots, n_0$ , 此即第一个模型.  
 $z_i = 1$  时, 模型为  $y_i = (\beta_0 + \gamma_0) + (\beta_1 + \gamma_1)x_i + \epsilon_i, \epsilon_i \sim (0, \sigma^2), i = n_0 + 1, \dots, n$ , 此即第二个模型, 其中  $\alpha_0 = \beta_0 + \gamma_0, \alpha_1 = \beta_1 + \gamma_1$ .

(b) 由(a)知,  $\alpha_1 - \beta_1 = \gamma_1$ , 故  $H_0: \alpha_1 = \beta_1 \Leftrightarrow$  在合并后的模型中  $H_0: \gamma_1 = 0$

3. (a) **【10】** 给定  $X$  条件下  $E(\tilde{\beta}|X) = E((AX)^{-1}AY) = (AX)^{-1}AX\beta = \beta$ .

(b) **【10】** 由GM定理:

$$\text{var}(\tilde{Y}) = (AX)^{-1}AA'(AX)^{-1} \geq \text{var}(\hat{Y}) = \sigma^2 P_X$$

因为无偏,  $M = \text{var}$ ,

$$m(\tilde{Y}) = \text{tr}M(\tilde{Y}) = \text{tr}(\text{var}(\tilde{Y})) \geq \text{tr}(\text{var}(\hat{Y})) = m(\hat{Y}).$$

4. (a)  $M(\tilde{\beta}) = E(\tilde{\beta} - \beta)(\tilde{\beta} - \beta)' = \text{var}(\tilde{\beta}) + \mathbf{b}\mathbf{b}'$ , 其中  $\mathbf{b}$  为  $\tilde{\beta}$  的偏差:

$$\mathbf{b} = E(\tilde{\beta}) - \beta = \begin{pmatrix} E(\tilde{\beta}_1) - \beta_1 \\ E(\tilde{\beta}_2) - \beta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -\beta_2 \end{pmatrix},$$

其中我们用到了事实  $E(\tilde{\beta}_1) = E[(X_1'X_1)^{-1}X_1'Y] = E[X_1'X_1)^{-1}X_1'(X_1\beta_1 + X_2\beta_2) = \beta_1 + X_1'X_1)^{-1}X_1'X_2\beta_2 = \beta_1$  (因为  $X_1'X_2 = 0$ )。另外,  $\text{var}(\tilde{\beta}_1) = \text{var}(X_1'X_1)^{-1}X_1'Y) = \sigma^2(X_1'X_1)^{-1}, \tilde{\beta}_2 = 0$ , 故

$$\text{var}(\tilde{\beta}) = \text{var} \begin{pmatrix} \tilde{\beta}_1 \\ \tilde{\beta}_2 \end{pmatrix} = \begin{pmatrix} \sigma^2(X_1'X_1)^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

两者相加即得:

$$M(\tilde{\beta}) = \begin{pmatrix} \sigma^2(X_1'X_1)^{-1} & 0 \\ 0 & \beta_2\beta_2' \end{pmatrix}$$

- (b)  $M(\hat{\beta}) = \text{var}(\hat{\beta}) = \sigma^2(X'X)^{-1}$ , 因为  $X_1'X_2 = 0$ , 故

$$M(\hat{\beta}) = \sigma^2 \begin{pmatrix} X_1'X_1 & 0 \\ 0 & X_2'X_2 \end{pmatrix}^{-1} = \begin{pmatrix} \sigma^2(X_1'X_1)^{-1} & 0 \\ 0 & \sigma^2(X_2'X_2)^{-1} \end{pmatrix}$$

由引理, 条件  $\|X_2\beta_2\|^2 = \beta_2'X_2'X_2\beta_2 \leq \sigma^2$  等价于  $\beta_2\beta_2' \leq \sigma^2(X_2'X_2)^{-1}$ , 所以  $M(\tilde{\beta}) \leq M(\hat{\beta})$ 。

5. (a) 【8】  $Y = X\gamma + \epsilon, \epsilon \sim N(0, \sigma^2 I_4)$ , 其中  $X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{pmatrix}$

$$\hat{\alpha} = \frac{y_1+y_2+y_3+y_4}{4}, \hat{\beta} = \frac{y_1+y_2-y_3-y_4}{4}, \text{var}(\hat{\alpha}) = \text{var}(\hat{\beta}) = \frac{\sigma^2}{4}. \text{总方差} = \frac{\sigma^2}{2}.$$

(b) 【8】  $\hat{Y} = (\hat{\alpha} + \hat{\beta}, \hat{\alpha} + \hat{\beta}, \hat{\alpha} - \hat{\beta}, \hat{\alpha} - \hat{\beta})' = (\frac{y_1+y_2}{2}, \frac{y_1+y_2}{2}, \frac{y_3+y_4}{2}, \frac{y_3+y_4}{2})'$ .

原假设下  $\hat{\beta} = \frac{y_1+y_2}{2}$ , 拟合值  $\hat{Y}_0 = (\frac{y_1+y_2}{2}, \frac{y_1+y_2}{2}, 0, 0)'$ , 所以

$$F = \frac{\|\hat{Y} - \hat{Y}_0\|^2/1}{\|Y - \hat{Y}\|^2/(4-2)} = \frac{2(y_3 + y_4)^2}{(y_1 - y_2)^2 + (y_3 - y_4)^2}$$

$H_0$  下,  $F \sim F_{1,2}$ .

(c) 【4】 方案2总方差  $= (\frac{1}{m} + \frac{1}{n})\sigma^2 \leq \frac{\sigma^2}{2}$ , 因为  $(m+n)(1/m + 1/n) \geq 4$ , 所以  $m+n \geq 8$ .

6. (20分) (a) 模型写成向量形式  $Y = \mathbf{1}a + \mathbf{x}b + \mathbf{z}c + \epsilon$ , 令  $\mathbf{z}^\perp = \mathbf{z} - P_1\mathbf{z} = \mathbf{z} - \mathbf{1}\bar{z}$  以及

$$\mathbf{x}^\perp = \mathbf{x} - P_{(\mathbf{1}, \mathbf{z})}\mathbf{x} = \mathbf{x} - P_1\mathbf{x} - P_{\mathbf{z}^\perp}\mathbf{x} = \mathbf{x} - \mathbf{1}\bar{x} - P_{\mathbf{z}^\perp}\mathbf{x} = \mathbf{x} - \mathbf{1}\bar{x} - (\mathbf{z} - \mathbf{1}\bar{z})\hat{\gamma}$$

带入  $\hat{b} = \mathbf{x}^{\perp'}Y/\|\mathbf{x}^\perp\|^2$  即得。

(b) 由 (a) 的结论知

$$\text{var}(\hat{b}|x, z) = \sigma^2 / \sum_{i=1}^n (x_i - \bar{x} - (z_i - \bar{z})\hat{\gamma})^2$$

而由  $\hat{\gamma} = s_{xz}/s_{zz}$

$$\sum_{i=1}^n (x_i - \bar{x} - (z_i - \bar{z})\hat{\gamma})^2 = s_{xx} - 2\hat{\gamma}s_{xz} + \hat{\gamma}^2s_{zz} = s_{xx} - \frac{s_{xz}^2}{s_{zz}} = s_{xx}(1 - r_{xz}^2).$$

(c) 相比于独立情形,  $\hat{b}$  的方差增大了  $VIF = 1/(1 - r_{xz}^2)$  倍。

(d)  $x, z$  高度相关 (比如  $r_{xz}^2 \approx 1$ ) 而且误差方差较小时, 回归方程的显著性检验会比较显著, 而  $H_0: b = 0$  的检验  $t_x = \sqrt{s_{xx}(1 - r_{xz}^2)} \frac{\hat{b}}{\hat{\sigma}}$  可能会不显著, 同样  $H_0: c = 0$  的检验  $t_z = \sqrt{s_{zz}(1 - r_{xz}^2)} \frac{\hat{c}}{\hat{\sigma}}$  可能会不显著。