Solutions to *Algebraic Topology* by Allen Hatcher

Contents

1 Chapter 0

Skipped for triviality: 1–3, 9–12, 14–15, 17, 19–22, 24–29.

4. f_1 is homotopy inverse for inclusion $i : A \hookrightarrow X$.

5. Suppose $f_t : X \to X$ is deformation retraction. $\mathrm{id}_X \overset{f_t}{\simeq} c_{x_0}$.

For each neighborhood $U \ni x$, there exists $t_0 \in (0,1)$ s.t. $f_t(X) \subseteq U$ for all $t \in [t_0, 1]$. Let $V = f_{t_0}(X)$.

$h_t = f_{t+(1-t)t_0} \circ f_{t_0}^{-1}$ is homotopy from inclusion $i : V \hookrightarrow U$ to constant map $V \to \{x_0\}$.

6. (a) First deformation retracts to the bottom line $[0,1] \times \{0\}$, then deformation retracts to a point.

X doesn't deformation retract to any other point because of Exercise **0.5**.

(b)(c) Y deformation retracts in the weak sense to the middle zigzag, so it's a homotopy equivalence.

The middle zigzag is homeomorphic to $\mathbb{R}^1$, which is contractible, so Y is contractible.

There's no true deformation retraction from Y to the zigzag, otherwise Y will deformation retract to a point.

7. X is union of infinite cones on the Cantor set arranged end-to-end and getting smaller and smaller.

The "baseline" of X is $[0,1)$. One-point compactification of $X \times \mathbb{R}$ is obtained by adding the endpoint 1 of $[0,1)$.

After one-point compactification, $\{0\} \times \mathbb{R}$ and additional point $\{1\} \times \{0\}$ become the boundary of D^2.

Y is obtained from one-point compactification of $X \times \mathbb{R}$ by wrapping one more cone on the Cantor set around the boundary of D^2. Y doesn't deformation retract to a point because of Exercise **0.5**.

X can deformation retract to baseline $[0,1)$ in the weak sense in the following way:

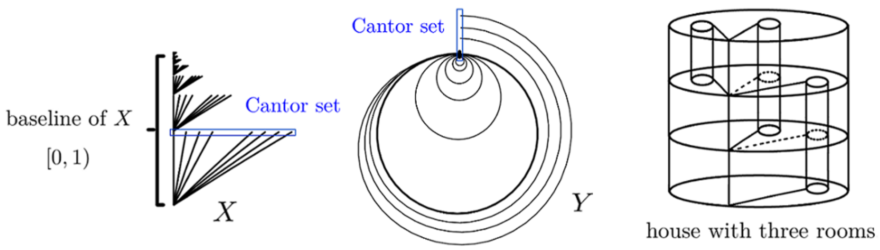
For $n \in \mathbb{N}$, point on $[1 - \frac{1}{2^{n+1}}, 1 - \frac{1}{2^{n+2}}]$ moves to $[1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}]$ alone $[0,1)$, and point on $[0, 1/2]$ moves to $\{0\}$.

The point on cones moves to $[0,1)$ in the similar way, so X deformation retract to $[0,1)$ in the weak sense, and one-point compactification of $X \times \mathbb{R}$ deformation retract to D^2 in the weak sense.

Y deformation retract to D^2 with a cone on the Cantor set around the boundary of D^2 in the weak sense.

This space can deformation retract to D^2 in the weak sense by moving points on cone and rotating D^2 clockwise.

Thus $D^2 \hookrightarrow Y$ is homotopy equivalence, D^2 is contractible, so Y is contractible.



8. The picture above is the house with three rooms. It's similar for the general case.

13. The desired r_t^s is given by $r_t^s = \begin{cases} r_t^0 \circ r_{2st}^1 & 0 \leqslant s \leqslant 1/2 \\ r_{t \cdot 2(1-s)}^0 \circ r_t^1 & 1/2 \leqslant s \leqslant 1 \end{cases}$

16. $S^\infty := \{(x_1, x_2, \dots, x_n, \dots) \mid \text{there exists } N \text{ s.t. } x_k = 0 \text{ for } k \geqslant N, \sqrt{\sum_{i=1}^\infty |x_i|^2} = 1\}$.

Let $T : S^\infty \to S^\infty$, $(x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)$, $f_t = (1-t)\mathrm{id}_{S^\infty} + tT \neq 0$ and $\tilde{f}_t = f_t / |f_t|$.

Let K be constant map $S^\infty \to (1, 0, \dots)$, $g_t = (1-t)T + tK \neq 0$ and $\tilde{g}_t = g_t / |g_t|$. $\mathrm{id}_{S^\infty} \overset{\tilde{f}_t}{\simeq} T \overset{\tilde{g}_t}{\simeq} K$.

18. Let $\pi_1 : S^m \times S^n \times \{0\} \to S^m$, $\pi_2 : S^m \times S^n \times \{1\} \to S^n$.

$S^m * S^n = (S^m \times S^n \times [0,1/2])/\pi_1 \cup_{S^m \times S^n \times \{1/2\}} (S^m \times S^n \times [1/2,1])/\pi_2$. $S^m \times S^n \times \{1/2\} \simeq \partial D^{m+1} \times \partial D^{n+1}$.

$S^m \times S^n \times [0,1/2]/\pi_1 \simeq S^m \times CS^n \simeq \partial D^{m+1} \times D^{n+1}$, $S^m \times S^n \times [1/2,1]/\pi_2 \simeq S^n \times CS^m \simeq D^{m+1} \times \partial D^{n+1}$.

$\partial D^{m+1} \times D^{n+1} \cup_{\partial D^{m+1} \times \partial D^{n+1}} D^{m+1} \times \partial D^{n+1} \simeq \partial(D^{m+1} \times D^{n+1}) \simeq \partial D^{m+n+2} \simeq S^{m+n+1}$.

23. Suppose X, Y and $A = X \cap Y$ are contractible. $(X, A), (Y, A), (X \cup Y, A)$ have HEP.

$X \cup Y \simeq (X \cup Y)/A \simeq (X/A) \vee (Y/A) \simeq X \vee Y \simeq \{*_1\} \vee \{*_2\} \simeq \{*\}$.

2 Section 1.1

Skipped for triviality: 1, 4, 6–8, 10–15, 17–20.

2. Show that $h_1 \simeq h_2$ iff change-of-basepoint homomorphism $\beta_{h_1} = \beta_{h_2}$.

3. ($\Rightarrow$) If $\pi_1(X)$ is abelian, h_1, h_2 are two paths from x_0 to x_1, $[f] \in \pi_1(X, x_1)$, then $[f][\overline{h}_2 \cdot h_1] = [\overline{h}_2 \cdot h_1][f]$.

($\Leftarrow$) For $[f], [g] \in \pi_1(X, x_0)$, let $g = g_1 \cdot \overline{g}_2$. $\beta_{\overline{g}_1} = \beta_{\overline{g}_2}$, $[g][f] = [f][g]$. X is path-connected, so $\pi_1(X)$ is abelian.

5. $f : X \to Y$ is nullhomotopic $\Leftrightarrow$ f can extend to CX. Let $\pi : X \times I \to CX$.

($\Rightarrow$) $F : X \times I \to Y$, $F|_{X \times \{0\}} = f$, $F(X \times \{1\}) = \{y_0\}$. F induces $\tilde{F} : CX = X \times I / X \times \{1\} \to Y$. $\tilde{F}|_{X \times \{0\}} = f$.

($\Leftarrow$) If $F : CX \to Y$ is extension of $f : X \to Y$, then $F \circ \pi : X \times I \to Y$ is the required homotopy.

(a) $\Leftrightarrow$ (b) since $CS^1 \simeq D^2$. (a) $\Leftrightarrow$ every loop in X is homotopic to constant loop $\Leftrightarrow$ (c).

9. For all $s \in S^2 \subseteq \mathbb{R}^3$, there exists unique plane $P_1^s \subseteq \mathbb{R}^3$ which divide A_1 into 2 pieces of equal measure.

Let $\overrightarrow{Os}$ be normal vector of P_1^s, then $B^s := \{v \in \mathbb{R}^3 \mid \text{for all } p \in P_1^s,\ \overrightarrow{pv} \cdot \overrightarrow{Os} \geqslant 0\}$ is half of $\mathbb{R}^3$.

Map $S^2 \to \mathbb{R}^2$, $s \mapsto (m(B^s \cap A_2), (m(B^s \cap A_3))$ is continuous. From Borsuk-Ulam theorem, there exists $s_0 \in S^2$ s.t.

$m(B^{s_0} \cap A_2) = m(B^{-s_0} \cap A_2), m(B^{s_0} \cap A_3) = m(B^{-s_0} \cap A_3))$. Hence $P_1^{s_0}$ is the required plane.

16. If $r : X \to A$ is retraction, then $i_* : \pi_1(A) \to \pi_1(X)$ induced by $A \hookrightarrow X$ is injection.

(c) $i_* = 0$. (f) $i_*(1) = 2$.

3 Section 1.2

Skipped for triviality: 1, 7, 16–17, 19.

Skipped for difficulty: 22.

2. Note that convex set is simply-connected, and intersection of two convex sets is still a convex set.

3. For $n \geqslant 3$, $\pi_1(\mathbb{R}^n - \bigcup_{i=1}^k \{x_i\}) = \pi_1(D^n - \bigcup_{i=1}^k \{x_i\}) = \pi_1(\bigvee_{i=1}^k S^{n-1}) = 0$. Generalization: Exercise **1.2.6**.

4. $\mathbb{R}^3 - X = \mathbb{R}^3 - \{0\} - X \simeq S^2 - X = S^2 - \bigcup_{i=1}^{2n}\{x_i\} \simeq \mathbb{R}^2 - \bigcup_{i=1}^{2n-1}\{x_i\} \simeq \bigvee_{i=1}^{2n-1} S^1$, so $\pi_1(\mathbb{R}^3 - X) \cong *_{i=1}^{2n-1}\mathbb{Z}$.

5. From Proposition **1A.1**, every connected graph contains a maximal tree, namely a contractible graph which contains all the vertices of the connected graph. Suppose X contains a maximal tree M.

If $X = M$, then X doesn't contain any loops and $\pi_1(X, x_0) = \pi_1(M, x_0) = 0$ for any $x_0 \in X$.

Now suppose $M \neq X$, there're finitely many edges $e_1, \ldots, e_n$ of X not in M.

Fix a basepoint x_0 in M. Note that each edge e_i corresponds to a loop based at x_0 in $M \cup e_i$.

$X = \bigcup_{i=1}^n (M \cup e_i)$. Any three intersection $(M \cup e_i) \cap (M \cup e_j) \cap (M \cup e_k)$ is path-connected.

For $i \neq j$, $(M \cup e_i) \cap (M \cup e_j) = M$ is contractible, so from van-Kampen's theorem, $\pi_1(X) = *_{i=1}^n \pi_1(M \cup e_i, x_0)$.

For each i, $\pi_1(X \cup e_i, x_0)$ is generated by a loop based at x_0 and goes around the bounded complementary region form by $X \cup e_i$, such loop doesn't go through any other e_j $(j \neq i)$.

6. If A is discrete subspace of X, then for each $x \in A$, there exists an open ball $B_x \subseteq \mathbb{R}^n$ s.t. $B_x \cap A = \{x\}$.

$\mathbb{R}^n - A$ deformation retracts to $X \coloneqq \mathbb{R}^n - \bigcup_{x \in A} B_x$. X is path-connected.

Let Y be space obtained by attaching n-cells to X via $\varphi_\alpha : \partial D^n \to \partial B_x$ for each $x \in A$, then $Y = \mathbb{R}^n$.

Attaching n-cells $(n \geqslant 3)$ doesn't change fundamental group, so $\pi_1(X) = \pi_1(Y) = 0$.

8. Two tori T_1, T_2. $\pi_1(T_1) = \mathbb{Z} \times \mathbb{Z} = \langle a \rangle \times \langle b \rangle$, $\pi_1(T_2) = \mathbb{Z} \times \mathbb{Z} = \langle c \rangle \times \langle d \rangle$.

$\pi_1(X) \cong \pi_1(T_1) * \pi_1(T_2)/N$, $N = \langle ac^{-1} \rangle$. $\pi_1(X) = \langle a, b, c, d \mid [a, b] = [c, d] = ac^{-1} = 1 \rangle \cong (\mathbb{Z} * \mathbb{Z}) \times \mathbb{Z}$.

9. (1) $\pi_1(M_h') = \langle a_1, b_1, \ldots, a_h, b_h, c \mid [a_1, b_1] \cdots [a_h, b_h] c^{-1} = 1 \rangle = \langle a_1, b_1, \ldots, a_h, b_h \rangle$. $\pi_1(C) = \langle c \rangle \cong \mathbb{Z}$.

If M_h' retracts to C, then $i_* : \pi_1(C) \to \pi_1(M_h')$ is injective. $i_*(c) = c = [a_1, b_1] \cdots [a_h, b_h]$ in $\pi_1(M_h')$.

Abelianization preserves injectivity, so $(i_*)_{ab} : \pi_1(C) \to \pi_1(M_h')_{ab}$ is injective. But $(i_*)_{ab}(c) = 0$. Contradiction.

In particular, there is no retraction $M_g \to C$, since such restriction would give a retraction $M_h' \to C$.

(2) CW complex structure on M_g consists of one 0-cell, $2g$ 1-cells $a_1, b_1, \ldots, a_g, b_g$ and one 2-cell.

The 1-skeleton is $\bigvee_{i=1}^g (S^1_{a_i} \vee S^1_{b_i})$, the attachment map of the 2-cell is $[a_1, b_1] \cdots [a_g, b_g]$.

Collapsing $\bigvee_{i=2}^g (S^1_{a_i} \vee S^1_{b_i})$ induces quotient map $q : M_g \to M_1 = S^1 \times S^1$.

$r : M_1 = S^1 \times S^1 \to S^1 \times \{s_0\} = C', s_0 \in S^1$, $(x, y) \mapsto (x, s_0)$ is retraction, so $r \circ q : M_g \to C'$ is a retraction.

10. $D^2 \times I - \{\alpha, \beta\} \simeq D^2 \times I - \{\text{two parallel lines}\} \simeq D^2 - \{x_0, y_0\}$. γ is the boundary circle, so it's not null-homotopic.

11. Suppose X is path-connected, $f : X \to X$ fixes basepoint $x_0 \in X$.

Bundle $X \hookrightarrow T_f \to T_f/X = S^1$ induces split short exact sequence

$1 = \pi_2(S^1, 1) \to \pi_1(X, x_0) \to \pi_1(T_f, x_0) \to \pi_1(S^1, 1) \to \pi_0(X, x_0) = 1.$

To show how $\pi_1(S^1,1)$ acts on $\pi_1(X,x_0)$, consider $[\alpha]\in\pi_1(X,x_0)$, $\beta(t)=[x_0,t]\in T_f$. $[\beta]\in\pi_1(S^1,1)$.

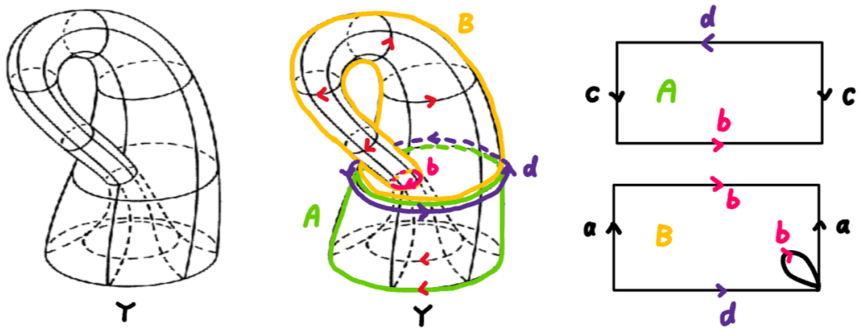
Define homotopy $H_s(t):I\to X\times[0,1]$:

$$H_s(t)=\begin{cases}(x_0,3ts), & t\in[0,1/3]\\ (f\circ\alpha(3t-1),s), & t\in[1/3,2/3]\\ (x_0,3(1-t)s), & t\in[2/3,1]\end{cases}$$

Let $\pi: X\times[0,1]\to T_f, (x,0)\sim(f(x),1)$, $\widetilde{H}_s=\pi\circ H_s: I\to T_f$, $\widetilde{H}_s(t)=[H_s(t)]$.

$\widetilde{H}_0\simeq f(\alpha)$, $\widetilde{H}_1=\beta\alpha\beta^{-1}$, so $[\beta][\alpha][\beta]^{-1}=f_*([\alpha])$. $\pi_1(T_f)\cong\pi_1(X)\rtimes_{f_*}\mathbb{Z}$.

12. From Exercise **0.20**, $X\simeq S^1\vee S^1\vee S^2$, so $\pi_1(X)\cong\mathbb{Z}*\mathbb{Z}$.



$\pi_1(Y)=\langle a,b,c,d\mid cbc^{-1}d=1, aba^{-1}b^{-1}d^{-1}=1\rangle=\langle a,b,c\mid aba^{-1}b^{-1}cbc^{-1}=1\rangle$, denoted by G.

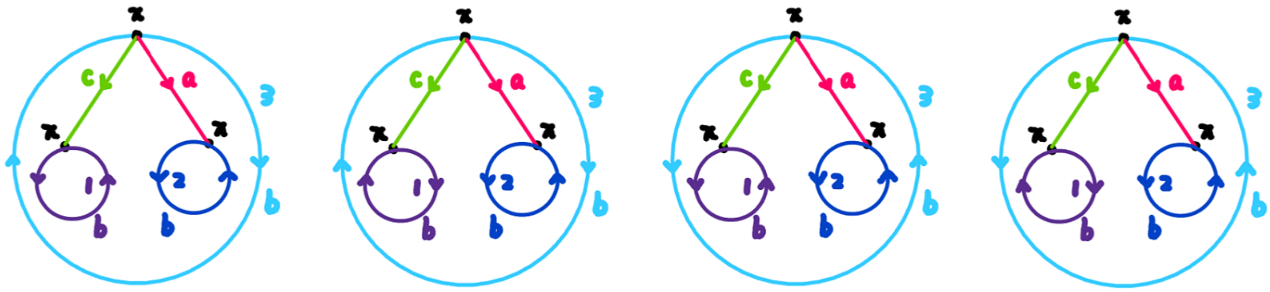
Replace c by ad, then a,b,d are generators of G and $aba^{-1}b^{-1}cbc^{-1}=1$ becomes $a^{-1}bab^{-1}db^{-1}d^{-1}=1$.

Replace d by c' and a^{-1} by a', then a',b,c' are generators of G, $a^{-1}bab^{-1}db^{-1}d^{-1}=1$ becomes $a'ba'^{-1}b^{-1}c'b^{-1}c'^{-1}=1$.

Therefore $\langle a,b,c\mid aba^{-1}b^{-1}cbc^{-1}=1\rangle\cong\langle a,b,c\mid aba^{-1}b^{-1}cb^{-1}c^{-1}=1\rangle$.

$\mathbb{R}^3-Z$ deformation retracts to Y, so $\pi_1(Y)\cong\pi_1(\mathbb{R}^3-Z)$.

13. Orientation of circle is represented by $+$ (clockwise) and $-$ (counter clockwise).



Case 1 Case 2 Case 3 Case 4

In case 1, orientation of circle $1,2,3$ is $(-,-,+)$, fundamental group is $G_1:=\langle a,b,c\mid aba^{-1}bcbc^{-1}\rangle$.

In case 2, orientation of circle $1,2,3$ is $(+,-,+)$, fundamental group is $G_2:=\langle a,b,c\mid aba^{-1}bcb^{-1}c^{-1}\rangle$.

In case 3, orientation of circle $1,2,3$ is $(-,-,-)$, fundamental group is $G_3:=\langle a,b,c\mid aba^{-1}b^{-1}cbc^{-1}\rangle$.

In case 4, orientation of circle $1,2,3$ is $(+,-,-)$, fundamental group is $G_4:=\langle a,b,c\mid aba^{-1}b^{-1}cb^{-1}c^{-1}\rangle$.

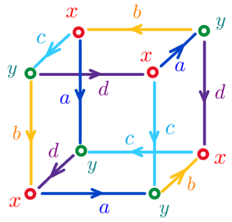
From Exercise **1.2.12**, $G_3\cong G_4$, case 3 and case 4 are equivalent.

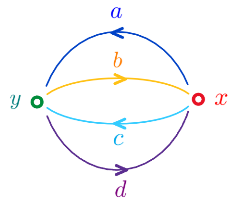
In $G_2:=\langle a,b,c\mid aba^{-1}bcb^{-1}c^{-1}\rangle$, replace a by c' and c by a', then a',b,c' are generators of G_2 and $aba^{-1}bcb^{-1}c^{-1}=1$ becomes $a'ba'^{-1}b^{-1}c'b^{-1}c'^{-1}=1$. $G_2\cong\langle a',b,c'\mid a'ba'^{-1}b^{-1}c'b^{-1}c'^{-1}=1\rangle\cong G_4$.

G_1 has abelianization $\mathbb{Z}_3\oplus\mathbb{Z}\oplus\mathbb{Z}$ while G_2,G_3,G_4 have abelianization $\mathbb{Z}\oplus\mathbb{Z}$, so cases 2, 3, 4 are equivalent.

14. Suppose the quotient space is X. It has two 0-cells, four 1-cells, three 2-cells and one 3-cell.

$X^1 \simeq \bigvee_{i=1}^3 S^1$, $\pi_1(X^1)$ is generated by $\alpha = ad, \beta = b^{-1}d, \gamma = cd$.





Attaching 2-cells gives the following relations

$abcd = \alpha\beta^{-1}\gamma = 1, ac^{-1}d^{-1}b = \alpha\gamma^{-1}\beta^{-1} = 1, adb^{-1}c^{-1} = \alpha\beta\gamma^{-1} = 1.$

Attaching 3-cells doesn't change fundamental group, so

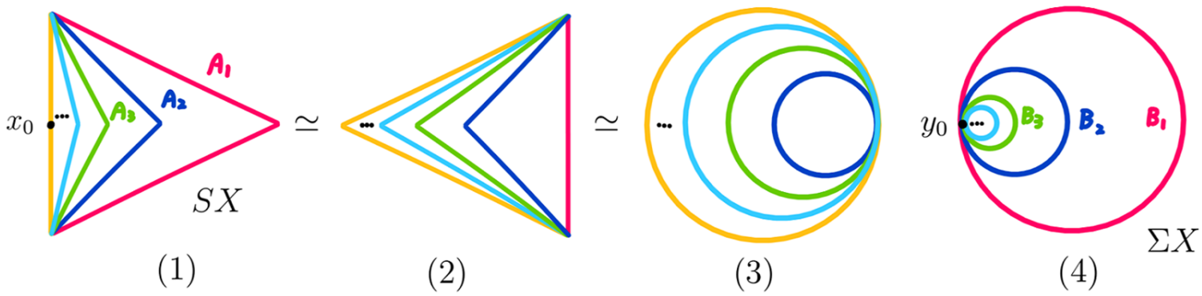
$\pi_1(X) = \langle \alpha, \beta, \gamma \mid \alpha\beta^{-1}\gamma = \alpha\gamma^{-1}\beta^{-1} = \alpha\beta\gamma^{-1} = 1\rangle.$

$\pi_1(X) \cong \langle \alpha, \beta \mid \alpha\beta\alpha = \beta, \alpha = \beta\alpha\beta\rangle \cong \langle \alpha, \beta \mid \alpha^4 = 1, \beta^2 = \alpha^2, \beta\alpha\beta^{-1} = \alpha^3\rangle \cong Q_8.$

15. Triangles in $L(X)$ is just triangulation of 2-cells in X, and this doesn't change homotopy type.

18. (a) $X = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots\right\}$, SX in fig(1) is homeomorphic to wedge sum of circles of radius $\frac{1}{\pi}\sqrt{\left(\frac{n}{n+1}\right)^2 + \left(\frac{1}{2}\right)^2}$ for $n = 1, 2, \cdots$ and circle of radius $\frac{1}{\pi}\sqrt{1^2 + \left(\frac{1}{2}\right)^2}$ in fig(3).

Note that fig(3) is also reduced suspension obtained from SX by collapsing segment $\{1\} \times I$, which indicates reduced suspension depends on the choice of basepoint.



(b) Region containing "$\cdots$" means there're countably many circles in it.

From outside to inside, circles in SX and ΣX are denoted by A_n and B_n with basepoint x_0 and y_0.

Retraction $r_i : SX \to A_i$ mapping A_j's to the left yellow segment for $j \neq i$ induces homomorphism $\phi : \pi_1(SX, x_0) \to \prod_{i=1}^\infty \pi_1(A_i, x_0) = \prod_\infty \mathbb{Z}$, $\phi(a) = ((r_1)_*(a), (r_2)_*(a), \ldots)$. $\operatorname{im}\phi = \bigoplus_\infty \mathbb{Z}$.

Retraction $s_i : \Sigma X \to B_i$ mapping B_j's to y_0 for $j \neq i$ induces homomorphism $\psi : \pi_1(\Sigma X, y_0) \to \prod_{i=1}^\infty \pi_1(B_i, y_0) = \prod_\infty \mathbb{Z}$, $\psi(b) = ((s_1)_*(b), (s_2)_*(b), \ldots)$. ψ is surjective.

For quotient map $q : SX \to \Sigma X$, $\psi \circ q_* = \phi$. Mapping cone $C = C(SX) \sqcup \Sigma X/\sim$, $(x, 1) \sim q(x)$ for $x \in SX$.

Write $C = U_1 \cup U_2$, where U_1 is space after removing the tip of mapping cone in C, and U_2 is $C(SX)$.

$U_1 = SX \times (0, 1] \sqcup \Sigma X/\sim$, $(x, 1) \sim q(x)$ for $x \in SX$. U_1 deformation retracts to ΣX. U_2 is contractible.

U_1 and U_2 are open in C. $U_1 \cap U_2 \simeq SX \times (0, 1] \simeq SX$ so $U_1 \cap U_2$ is path-connected.

From van Kampen's theorem, $\pi_1(C) \cong \pi_1(U_1) * \pi_1(U_2)/N$, N is normal subgroup generated by words of form $(i_1)_*(w)(i_2)_*(w^{-1})$ where $i_k : U_1 \cap U_2 \hookrightarrow U_k$, $k = 1, 2$ is inclusion and $w \in \pi_1(U_1 \cap U_2)$. $\pi_1(U_2) = 0$, $(i_2)_* = 0$.

$U_1 \cap U_2 \simeq SX$, $U_1 \simeq \Sigma X$,so $(i_1)_* : \pi_1(U_1 \cap U_2) \to \pi_1(U_1)$ corresponds to $q_* : \pi_1(SX) \to \pi_1(\Sigma X)$.

Hence $\pi_1(C) \cong \pi_1(\Sigma X)/N'$ where N' is normal subgroup generated by $\operatorname{im} q_*$.

For surjective homomorphism $\psi' : \pi_1(\Sigma X) \xrightarrow{\psi} \prod_\infty \mathbb{Z} \to \prod_\infty \mathbb{Z}/\bigoplus_\infty \mathbb{Z}$, $\psi' \circ q_* = 0$, $\operatorname{im} q_* \subseteq \ker\psi'$.

$\ker\psi'$ is normal, so $N' \subseteq \ker\psi'$ and ψ' induces surjective homomorphism $\pi_1(C) \cong \pi_1(\Sigma X)/N' \to \prod_\infty \mathbb{Z}/\bigoplus_\infty \mathbb{Z}$.

20. $X = \bigcup_{n=1}^{\infty} C_n$. Denote n-th circle in $\bigvee_\infty S^1$ by D_n and common point by x_0.

On each C_n and D_n, we can define a coordinate θ representing a from 0 to 2π.

Define $f : \bigvee_\infty S^1 \to X$, f maps point in D_n of coordinate θ to point in C_n of coordinate θ.

Define $g : X \to \bigvee_\infty S^1$, g maps point in C_n of coordinate θ to point in D_n of coordinate θ.

$f \circ g = \mathrm{id}_X$, $g \circ f = \mathrm{id}_{\bigvee_\infty S^1}$, so $X = \bigcup_{n=1}^{\infty} C_n \simeq \bigvee_\infty S^1$. X is closed subset in $\mathbb{R}^2$, so it's first countable.

$\bigvee_\infty S^1$ is not first countable,so it can't be embedded in any first countable space, especially $\mathbb{R}^2$.

Let $\{B_i\}_{i=1}^{\infty}$ be countable neighborhoods of x_0 in $\bigvee_\infty S^1$. Let $V_i \subseteq D_i$ be neighborhood of x_0 s.t. $V_i \subsetneq B_i \cap D_i$.

$\bigvee_{i=1}^{\infty} V_i$ is a neighborhood of x_0 and doesn't contain any B_i, so $\bigvee_\infty S^1$ is not first countable.

21. Let Y be path-connected. $X * Y := (X \times Y \times [0,1])/\sim$, where $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$.

Consider $U := (X \times Y \times [0,1))/\sim \ \simeq X \times CY$ and $V := (X \times Y \times (0,1])/\sim \ \simeq CX \times Y$.

$\pi_1(U \cap V) \cong \pi_1(X \times Y \times (0,1)) \cong \pi_1(X) \oplus \pi_1(Y)$. $\pi_1(U) \cong \pi_1(X)$. $\pi_1(V) \cong \pi_1(Y)$.

Inclusion $i_1 : U \cap V \hookrightarrow U$, $i_2 : U \cap V \hookrightarrow U$ induces $(i_1)_* : \pi_1(X) \oplus \pi_1(Y) \to \pi_1(X)$, $(i_2)_* : \pi_1(X) \oplus \pi_1(Y) \to \pi_1(Y)$.

From van Kampen's theorem, $\pi_1(X * Y) \cong \pi_1(X) * \pi_1(Y)/N$, N is generated by $(i_1)_*(a,b)(i_2)_*(a,b)^{-1} = ab^{-1}$ for all $(a,b) \in \pi_1(X) \oplus \pi_1(Y)$, so $N = \pi_1(X) * \pi_1(Y)$, $\pi_1(X * Y) = 0$, $X * Y$ is simply-connected.

Alternative proof: Let $(x_1, y_1, z_1), (x_2, y_2, z_2)$ be two points in $X * Y$, and $\alpha : [0,1] \to X$ be a path from x_1 to x_2.

$$\beta(t) = \begin{cases} (x_1, y_1, (1-3t)z_1) & 0 \leqslant t \leqslant 1/3 \\ (x_1, y_2, (3t-1)z_2) & 1/3 \leqslant t \leqslant 2/3 \\ (\alpha(3t-2), y_2, z_2) & 2/3 \leqslant t \leqslant 1 \end{cases}$$ is a path from (x_1, y_1, z_1) to (x_2, y_2, z_2). $X * Y$ is path-connected.

WLOG, let $\gamma : [0,1] \to X * Y$ be a loop with endpoint $\gamma(0) = \gamma(1) = (x, y, 0)$. Write $\gamma(t) = (x(t), y(t), z(t))$.

$F_s(t) := (x(t), y(t), sz(t)), s \in [0,1]$ is a homotopy between $\gamma(t)$ and $\gamma_1(t) = (x(t), y, 0)$.

$$G_s(t) := \begin{cases} (x, y, 2t), & 0 \leqslant t \leqslant s/2 \\ (x(\dfrac{t - s/2}{1-s}), y, s), & s/2 \leqslant t \leqslant 1 - s/2 \\ (x, y, 2-2t), & 1 - s/2 \leqslant t \leqslant 1 \end{cases}$$ is a homotopy between $\gamma_1(t)$ and $\gamma_2(t) = \begin{cases} (x, y, 2t), & 0 \leqslant t \leqslant 1/2 \\ (x, y, 2-2t), & 1/2 \leqslant t \leqslant 1 \end{cases}$.

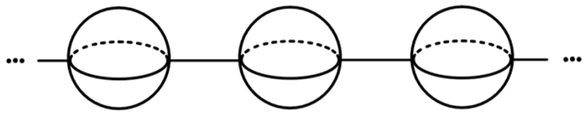
γ_2 is null-homotopic, so $\gamma \simeq \gamma_1 \simeq \gamma_2$ is null-homotopic, hence $X * Y$ is simply-connected.

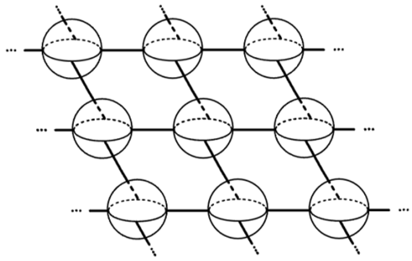
4 Section 1.3

Skipped for triviality: 1-3, 5, 16, 22, 28.

Skipped for difficulty: 33.

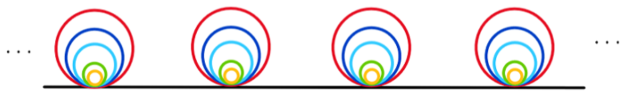
4.



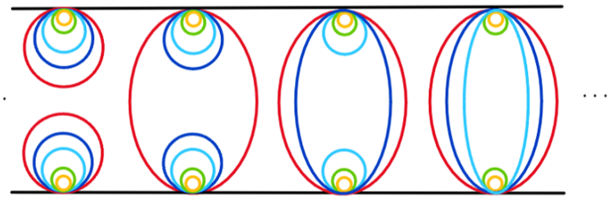


6. Let $p : Y \to \widetilde{X} \to X$, x_0 be the common point of shrinking wedge of circles X.

For any neighborhood U of x_0 in X, there exist a connected component $\widetilde{U}$ of $p^{-1}(U)$ which contains two points in $p^{-1}(x_0)$, so $p|_{\widetilde{U}} : \widetilde{U} \to U$ can't be homeomorphism.



covering space $\tilde{X}$ of shrinking wedge of circles



two-sheeted covering space $Y \to \tilde{X}$

7. $Y = \{(x, \sin(1/x) \mid 0 < x < 1\} \cup [-1, 1] \times \{0\} \cup C$ is quasi-circle circle, C is arc connecting $(0, 0)$ and $(1, \sin 1)$.

Let L be the segment $[-1, 1] \times \{0\}$ on the y-axis. $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Covering map $p : \mathbb{R} \to S^1$, $p(t) = e^{2\pi i t}$.

(1) WLOG suppose $f(L) = \{1\}$. Let $\tilde{f} : Y \to \mathbb{R}$ be the lift of $f : Y \to S^1$.

$\tilde{f}(Y - L)$ is connected and $\tilde{f}(Y - L) \subseteq p^{-1}(f(Y - L)) = \mathbb{R} - 2\pi\mathbb{Z}$. WLOG suppose $\tilde{f}(Y - L) \subseteq (0, 2\pi)$.

By surjectivity of f, $\tilde{f}(Y - L) = (0, 2\pi)$. Y is compact, $[0, 2\pi] = \overline{\tilde{f}(Y - L)} \subseteq \overline{\tilde{f}(Y)} = \tilde{f}(Y)$, so $\{0, 2\pi\} \subseteq \tilde{f}(L)$.

$\tilde{f}(L) \subseteq p^{-1}(f(L)) = p^{-1}(1) = 2\pi\mathbb{Z}$, so $\tilde{f}(L)$ is not connected. Contradiction.

This also shows quasi-circle Y is not contractible because f is not nullhomotopic.

Otherwise from homotopy lifting property f will have a lift, since any constant map $Y \to S^1$ has a lift $Y \to \mathbb{R}$.

(2) Note that there exists an open set $V \subseteq Y$ containing L with two path-components, $V_1 \supseteq L$ and V_2.

Let $g : I \to Y$ be a path. If $g(x) \in L$, then there's a path-connected open neighborhood $I_0 \ni x$ s.t. $g(I_0) \subseteq V_1$.

Thus $g^{-1}(L) \subseteq U$ for some open set U s.t. $g(U) \subseteq V_1$. $g(I - U)$ is compact set in $Y - L$, so it must be contained in $C \cup \{(x, \sin(1/x) \mid \varepsilon < x < 1\}$ for some $\varepsilon > 0$, and $g(I)$ is contained in $L \cup C \cup \{(x, \sin(1/x) \mid \varepsilon < x < 1\}$, which is contractible. Hence $g : I \to Y$ is nullhomotopic and $\pi_1(Y) = 0$.

8. For covering space $p : \widetilde{X} \to X$ and $q : \widetilde{Y} \to Y$ of locally path-connected space X and Y, $\widetilde{X}$ and $\widetilde{Y}$ are locally path-connected. Let $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ be a homotopy equivalence.

From lifting criterion, $f \circ p : \widetilde{X} \to Y$ has a lift $F : \widetilde{X} \to \widetilde{Y}$ w.r.t. $q : \widetilde{Y} \to Y$, i.e. $q \circ F = f \circ p$.

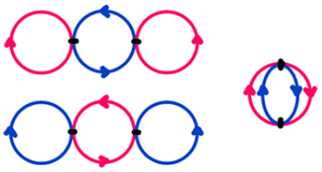
$g \circ q : \widetilde{Y} \to X$ has a lift $G : \widetilde{Y} \to \widetilde{X}$ w.r.t. $p : \widetilde{X} \to X$, i.e. $p \circ G = g \circ q$. $p \circ G \circ F \simeq p$, $q \circ F \circ G \simeq q$.

$p : \widetilde{X} \to X$ has a lift $\mathrm{id}_{\widetilde{X}}$ and $q : \widetilde{Y} \to Y$ has a lift $\mathrm{id}_{\widetilde{Y}}$, so $G \circ F \simeq \mathrm{id}_{\widetilde{X}}$ and $F \circ G \simeq \mathrm{id}_{\widetilde{Y}}$.

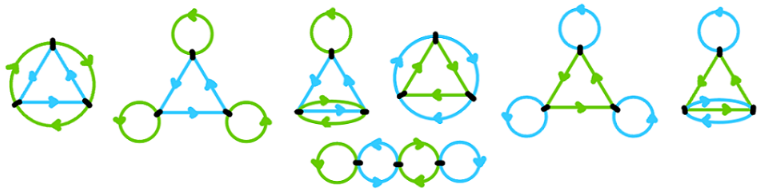
9. $f_* : \pi_1(X) \to \pi_1(S^1) \cong \mathbb{Z}$ induced by $f : X \to S^1$ is trivial, so it has a lift $\tilde{f} : X \to \mathbb{R}$.

$\mathbb{R}$ is contractible, so $\tilde{f} : X \to \mathbb{R}$ is nullhomotopic, $f = p \circ \tilde{f}$ is also nullhomotopic.

10.

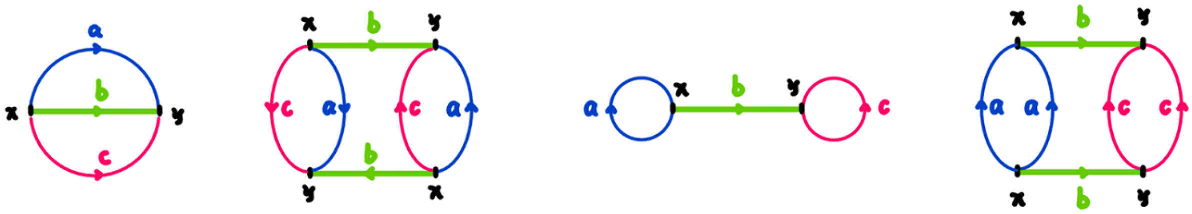


connected 2-sheeted covering space



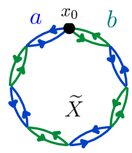
connected 3-sheeted covering space

11. X_1 and X_2 have 2 points and 3 edges, they can't be covering spaces of other space. $\widetilde{X}_1 = \widetilde{X}_2$.



X_1 2-sheeted covering space $\widetilde{X}_1$ X_2 2-sheeted covering space $\widetilde{X}_2$

12. Let N be normal subgroup generated by $a^2, b^2, (ab)^4$, $p : \widetilde{X} \to S_1 \vee S_1$ be covering space.



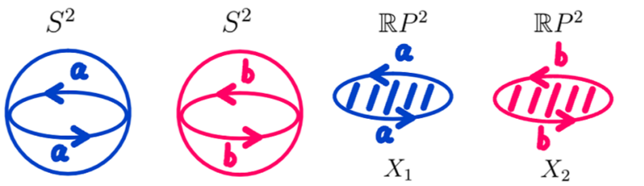
$N \subseteq \pi_1(\widetilde{X}, x_0)$. $\widetilde{X}$ is normal, so $p_*(\pi_1(\widetilde{X}, x_0))$ is normal.

p_* is injective, so $\pi_1(\widetilde{X}, x_0)$ is normal and $N = \pi_1(\widetilde{X}, x_0)$.

13. Let N be subgroup of $\mathbb{Z} * \mathbb{Z}$ generated by the cubes of elements. N is normal subgroup and $\mathbb{Z} * \mathbb{Z}/N$ is Burnside group $B(2,3)$ of order 27, so covering space of $S^1 \vee S^1$ corresponding to N is normal and 27-sheeted.

14. Let X_1 and X_2 denote the first and second copy of $\mathbb{R}P^2$, $\pi_1(X_1) = \mathbb{Z}_2 = \langle a \rangle$, $\pi_1(X_2) = \mathbb{Z}_2 = \langle b \rangle$.

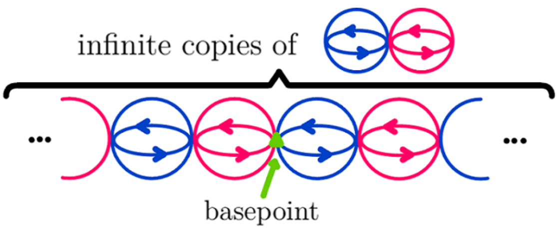
Covering map maps blue S^2 to X_1 and red S^2 to X_2. Consider subgroups of $\pi_1(X_1 \vee X_2) = \mathbb{Z}_2 * \mathbb{Z}_2 = \langle a \rangle * \langle b \rangle$



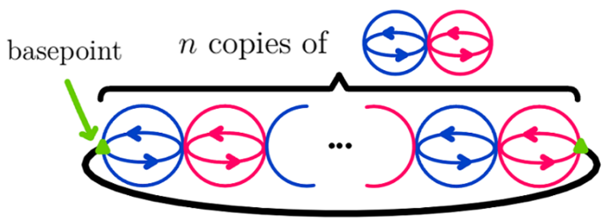
(1) For trivial subgroup 1, it corresponds to the the universal cover, i.e. the infinite chain of S^2.

(2) For subgroup isomorphic to infinite cyclic group $\mathbb{Z}$, it is generated by $(ab)^n$ or $(ba)^n$ of index $2n$ ($n \geqslant 1$).

It corresponds to a "necklace" of $2n$ copies of S^2.



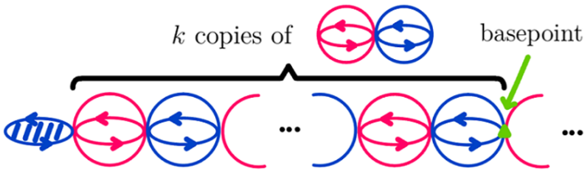
universal covering of $\mathbb{R}P^2 \vee \mathbb{R}P^2$



covering space corresponding to subgroup $\langle (ab)^n \rangle$ and $\langle (ba)^n \rangle$

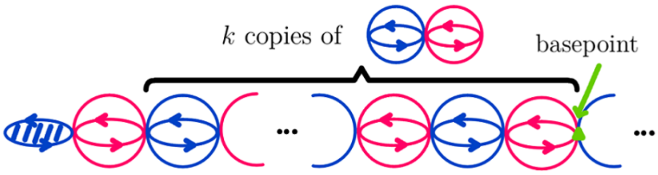
(3) For subgroup isomorphic to $\mathbb{Z}_2$, it's generated by $(ab)^m \cdot a$ or $(ba)^m \cdot b$ ($k \geqslant 0$).

It corresponds to $\mathbb{R}P^2$ attached to an infinite chain of S^2.



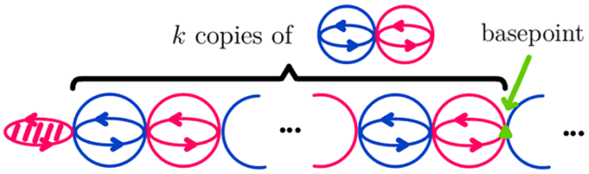
$(ab)^{2k} \cdot a = (ab)^k \cdot a \cdot (ba)^k$

covering space corresponding to subgroup $\langle (ab)^{2k} \cdot a \rangle (k \geqslant 0)$



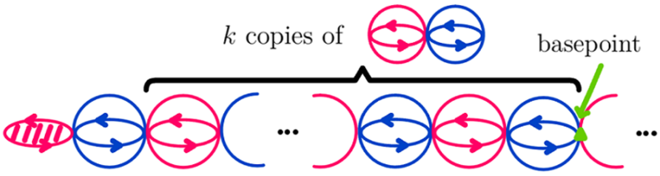
$(ba)^{2k+1} \cdot b = (ba)^k \cdot bab \cdot (ab)^k$

covering space corresponding to subgroup $\langle (ba)^{2k+1} \cdot b \rangle (k \geqslant 0)$



$(ba)^{2k} \cdot b = (ba)^k \cdot b \cdot (ab)^k$

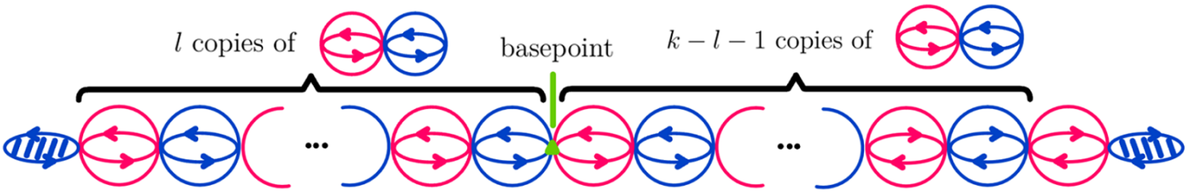
covering space corresponding to subgroup $\langle (ba)^{2k} \cdot b \rangle (k \geqslant 0)$



$(ab)^{2k+1} \cdot a = (ab)^k \cdot aba \cdot (ba)^k$

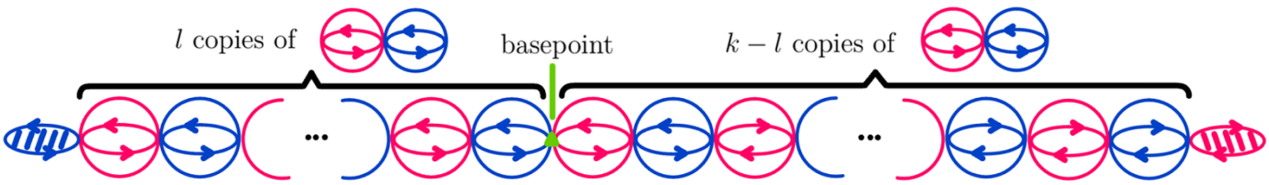
covering space corresponding to subgroup $\langle (ab)^{2k+1} \cdot a \rangle (k \geqslant 0)$

(4) For subgroup isomorphic to the infinite dihedral group $\mathbb{Z}_2 * \mathbb{Z}_2$, it's generated by $(ab)^n$ and $(ab)^m \cdot a$ $(m \leqslant n)$. It corresponds to a finite chain of S^2's with both ends attached an $\mathbb{R}P^2$.



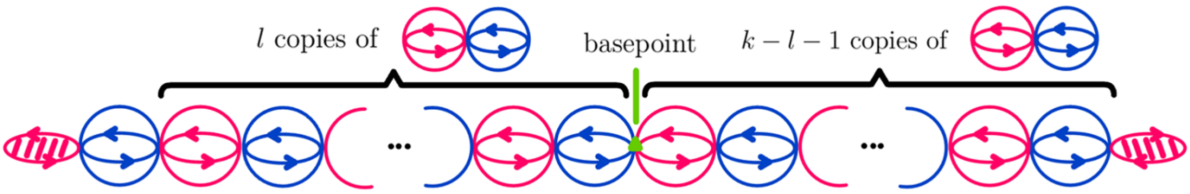
$(ab)^{2l} \cdot a = (ab)^l \cdot a \cdot (ba)^l$, $(ab)^{2k} = (ab)^l \cdot a \cdot (ba)^l \cdot (ba)^{k-l-1} \cdot bab \cdot (ab)^{k-l-1}$

covering space corresponding to subgroup $\langle (ab)^{2k}, (ab)^{2l} \cdot a \rangle$ $(k \geqslant l+1 \geqslant 1)$



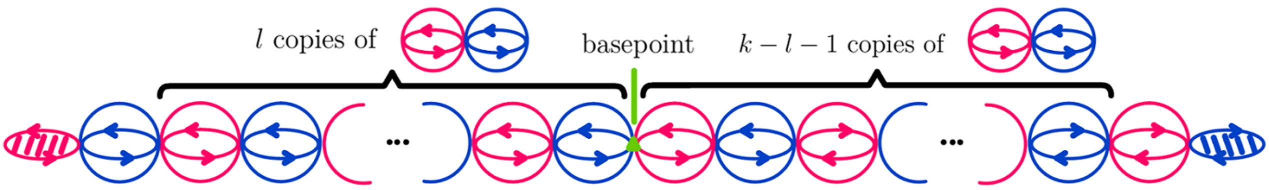
$(ab)^{2l} \cdot a = (ab)^l \cdot a \cdot (ba)^l$, $(ab)^{2k+1} = (ab)^l \cdot a \cdot (ba)^l \cdot (ba)^{k-l} \cdot b \cdot (ab)^{k-l}$

covering space corresponding to subgroup $\langle (ab)^{2k+1}, (ab)^{2l} \cdot a \rangle$ $(k \geqslant l \geqslant 0)$



$(ab)^{2l+1} \cdot a = (ab)^l \cdot aba \cdot (ba)^l$, $(ab)^{2k} = (ab)^l \cdot aba \cdot (ba)^l \cdot (ba)^{k-l-1} \cdot b \cdot (ab)^{k-l-1}$

covering space corresponding to subgroup $\langle (ab)^{2k}, (ab)^{2l+1} \cdot a \rangle$ $(k \geqslant l+1 \geqslant 1)$



$(ab)^{2l+1} \cdot a = (ab)^l \cdot aba \cdot (ba)^l$, $(ab)^{2k+1} = (ab)^l \cdot aba \cdot (ba)^l \cdot (ba)^{k-l-1} \cdot bab \cdot (ab)^{k-l-1}$

covering space corresponding to subgroup $\langle (ab)^{2k+1}, (ab)^{2l+1} \cdot a \rangle$ $(k \geqslant l+1 \geqslant 1)$

15. Choose basepoint $x_0 \in A$ with $\tilde{x}_0 \in \widetilde{A}$. Let $i : A \hookrightarrow X$, $i : \widetilde{A} \hookrightarrow \widetilde{X}$ be inclusions. $p|_{\widetilde{A}} : \widetilde{A} \to A$, $p : \widetilde{X} \to X$.

For $[f] \in \ker q_*$, $[f] = 0$ in $\pi_1(X, x_0)$ so f lifts to a loop $\tilde{f}$ in $\widetilde{X}$ (also in $\widetilde{A}$), $[f] = (p|_{\widetilde{A}})_*([\tilde{f}])$, $\ker q_* \subseteq \operatorname{im}(p|_{\widetilde{A}})_*$.

$i \circ p|_{\widetilde{A}} = p \circ i$, $i_* \circ (p|_{\widetilde{A}})_* = p_* \circ i_* = 0$, $\operatorname{im}(p|_{\widetilde{A}})_* \subseteq \ker q_*$. Thus $\operatorname{im}(p|_{\widetilde{A}})_* = \ker q_*$.

17. There's a 2-dimensional cell complex X s.t. $\pi_1(X) = G$ and a normal covering space $p : \widetilde{X} \to X$ s.t. $p_*(\pi_1(\widetilde{X})) \cong N$, $G(\widetilde{X}) \cong G/N$. p_* is injective, so $\pi_1(\widetilde{X}) \cong p_*(\pi_1(\widetilde{X})) \cong N$.

18. Suppose $\pi_1(X) = G$. $G' = [G, G] \triangleleft G$, there exists normal covering space $p : \widetilde{X} \to X$ s.t. $p_*(\pi_1(\widetilde{X})) \cong \pi_1(\widetilde{X}) \cong G'$. $G(\widetilde{X}) = G/G'$ is abelian, so $p : \widetilde{X} \to X$ is abelian covering space.

Suppose $q : \widetilde{X}' \to X$ is another abelian covering space, $q_*(\pi_1(\widetilde{X}')) \cong N \triangleleft G$ and $G(\widetilde{X}') = G/N$ is abelian, then $G' \subseteq \ker(G \to G/N) = N$, $p : \widetilde{X} \to X$ has a lift $\tilde{p} : \widetilde{X} \to \widetilde{X}'$ s.t. $q \circ \tilde{p} = p$.

$p : \widetilde{X} \to X$, $q : \widetilde{X}' \to X$ are covering spaces. From Exercise **1.3.16**, $\tilde{p} : \widetilde{X} \to \widetilde{X}'$ is a covering space.

Use unique lifting property, the 'universal' abelian covering is unique up to isomorphism.

For $X = S^1 \vee S^1$, its universal abelian covering space is $\{(x, y) \in \mathbb{R}^2, x \in \mathbb{Z} \text{ or } y \in \mathbb{Z}\}$.

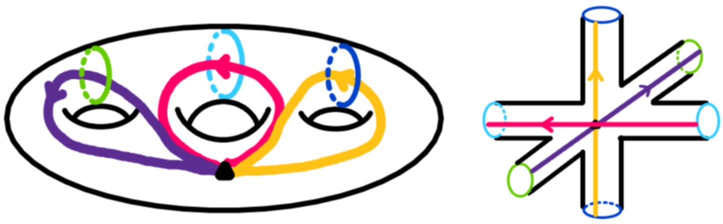
For $X = S^1 \vee S^1 \vee S^1$, its universal abelian covering space is $\{(x, y, z) \in \mathbb{R}^3, x \in \mathbb{Z} \text{ or } y \in \mathbb{Z} \text{ or } z \in \mathbb{Z}\}$.

19. Let $G = \pi_1(M_g) = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$.

Let $\widetilde{X}$ be universal abelian covering space, $G' = \pi_1(\widetilde{X}) = [G, G]$, $G(\widetilde{X}) \cong \pi_1(M_g)_{ab} \cong \mathbb{Z}^{2g}$.

For normal covering space X with $G(X) \cong \mathbb{Z}^n$, let $N' = \pi_1(X)$. $G' \subseteq N'$, $G(X) \cong G/N' \cong \dfrac{G/G'}{N'/G'} \cong \dfrac{\mathbb{Z}^{2g}}{N'/G'} \cong \mathbb{Z}^n$.

The picture below is the case for $n = 3$ and $g = 3$. It's similar for $g \geqslant 3$.



If such a covering space $Y \to M_g$ exists, we have an embedding $Y \to \mathbb{R}^3$ with $G(Y) = \mathbb{Z}^3$.

Taking the quotient yields embedding $M_g \to T^3$, which induces a surjection $\pi_1(M_g) \to \pi_1(T^3)$.

Suppose there's an embedding $i : M_g \to T^3$, let Y be covering space corresponding to $\ker(\pi_1(M_g) \to \pi_1(T^3))$.

Then $Y \to M_g \to T^3$ has a lift $\Phi : Y \to \mathbb{R}^3$ via covering map $\mathbb{R}^3 \to T^3$, and Φ is injective.

$Y \to M_g$ and $\mathbb{R}^3 \to T^3$ are local homeomorphisms, $M_g \to T^3$ is embedding, so $\Phi : Y \to \mathbb{R}^3$ is an embedding.

20. Fundamental group of Klein bottle is $\langle x, y \mid xyxy^{-1} = 1 \rangle$.

Non-normal covering space by a Klein bottle is corresponding to subgroup $\langle x^3, y \rangle$. $x^3 \cdot y \cdot x^3 \cdot y^{-1} = 1$.

Non-normal covering space by a torus is corresponding to subgroup $\langle x^3, xy^2 \rangle$. $x^3 \cdot xy^2 \cdot (x^3)^{-1} \cdot (xy^2)^{-1} = 1$.

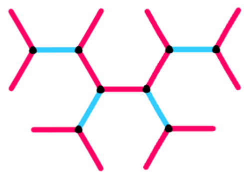
21. (1) Let M be Möbius band. $\pi_1(S^1\times S^1)=\langle a,b\mid ab=ba\rangle$, $\pi_1(M)=\langle c\rangle$. $\pi_1(X)=\langle a,b,c\mid ab=ba, a=c^2\rangle$.

$\pi_1(S^1\times S^1)\to\pi_1(X)$ and $\pi_1(M)\to\pi_1(X)$ induced by inclusions are injective, so universal cover $\mathbb{R}^2$ of $S^1\times S^1$ and universal cover $\mathbb{R}\times[0,1]$ of Möbius band embed into universal cover of X.

The construction is an example in Bass-Serre theory:

The universal cover of X is product $T\times\mathbb{R}$ where T is an infinite tree in which every vertex has valence 3.

The union of adjacent red edges crossed with $\mathbb{R}$ depicts $\mathbb{R}^2$,



and the blue edge crossed with $\mathbb{R}$ depicts $\mathbb{R}\times[0,1]$.

$\pi_1(X)=\langle a,b,c\mid ab=ba,a=c^2\rangle=\langle b,c\mid bc^2=c^2b\rangle$.

b acts on $T\times\mathbb{R}$ by translating T along the red direction by 1 unit.

c acts on $T\times\mathbb{R}$ by flipping T over a midpoint of a selected blue edge and translating along the $\mathbb{R}$ factor 1 unit.

(2) Let e^2 be 2-cell of $\mathbb{R}P^2$ and D be closed unit disk in $\mathbb{R}^2$.

Shrinking Möbius band to its central circle induces a homotopy from X to $S^1\cup_f e^2$, $f:\partial e^2=S^1\to S^1, z\mapsto z^4$.

Universal cover of $S^1\cup_f e^2$ is homeomorphic to $D\times\{1,2,3,4\}/\sim$, where $(x,i)\sim(y,j)$ iff $x=y\in\partial D$.

The universal cover of X is homeomorphic to the quotient of $D\times\{a,b,c,d\}\cup S^1\times[-1,1]/\sim$, where

$(x,a)\sim(x,c)\sim(x,1)$ for $x\in\partial D=S^1$, $(x,b)\sim(x,d)\sim(x,-1)$ for $x\in\partial D=S^1$.

$\pi_1(Y)=\langle x,y\mid x^2=1,y^2=x\rangle=\mathbb{Z}_4$ acts as follows:

$(re^{2\pi\mathrm{i}\theta},a)\mapsto(re^{2\pi\mathrm{i}(\theta+1/4)},b)\mapsto(re^{2\pi\mathrm{i}(\theta+1/2)},c)\mapsto(re^{2\pi\mathrm{i}(\theta+3/4)},d)\mapsto(re^{2\pi\mathrm{i}\theta},a)$ for points in disks $D\times\{a,b,c,d\}$,

$(e^{2\pi\mathrm{i}\theta},t)\mapsto(e^{2\pi\mathrm{i}(\theta+1/4)},-t)\mapsto(e^{2\pi\mathrm{i}(\theta+1/2)},t)\mapsto(e^{2\pi\mathrm{i}(\theta+3/4)},-t)\mapsto(e^{2\pi\mathrm{i}\theta},t)$ for points in $S^1\times[-1,1]$.

Covering map $\tilde{X}\to X$ maps the disks to $\mathbb{R}P^2$ and the cylinder to the Möbius band.

23. Fix $x\in X$ and neighborhood U of x s.t. $H=\{g\in G\mid U\cap g(U)\}$ is finite.

Let V_g be disjoint open sets of gx for $g\in H$, then $V=\bigcap_{g\in H}g^{-1}(V_g)$ is the desired neighborhood of x.

24. (a) For covering space $X\xrightarrow{\pi}Y\to X/G$, let $H=\{g\in G\mid\pi(x)=\pi(gx),\ \text{for all } x\in X\}$.

Y is isomorphic to X/H via $f_1:Y\to X/H$, $y\mapsto Hx$, $x\in\pi^{-1}(y)$ and $f_2:X/H\to Y$, $Hx\mapsto\pi(x)$.

(b) (i) Suppose $X\xrightarrow{p_1}X/H_1\xrightarrow{q_1}X/G$, $X\xrightarrow{p_2}X/H_2\xrightarrow{q_2}X/G$. Let $N_1=(q_1)_*(\pi_1(X/H_1))$, $N_2=(q_2)_*(\pi_1(X/H_1))$.

If $X/H_1\xrightarrow{q_1}X/G$, $X/H_2\xrightarrow{q_2}X/G$ are isomorphic, then $gN_1g^{-1}=N_2$ for some $g\in\pi_1(X/G)$.

Let $\Phi:\pi_1(X/G)\to G$ be surjection given by deck transformations on $X\to X/G$, then $\Phi(N_i)=H_i, i=1,2$.

For $[\alpha]\in\pi_1(X/H_1)$, α has a lift α in X from $\tilde{x}_0$ to $\tilde{x}_1$, with $\tilde{x}_1=h_1\tilde{x}_0$ for some $h_1\in H_1$ and $\Phi((q_1)_*([\alpha]))=h_1$.

For $h_1'\in H_1$, fix $x_0\in X$, let β be path from x_0 to $h_1'x_0$, then $[p_1(\beta)]\in\pi_1(X/H_1)$ and $\Phi((q_1)_*([p_1(\beta)]))=h_1'$.

From $gN_1g^{-1}=N_2$ for $g\in\pi_1(X/G)$ and $\Phi(N_i)=H_i, i=1,2$, $H_2=\Phi(g)\cdot H_1\cdot\Phi(g^{-1})$.

(ii) If $H_2=gH_1g^{-1}$ for some $g\in G$, then X/H_1 is isomorphic to X/H_2 via $f_1:X/H_1\to X/H_2$, $H_1x\mapsto H_2gx$ and $f_2:X/H_2\to X/H_1$, $H_2x\mapsto H_1g^{-1}x$.

(c) Let $p:X/H\to X/G$ be covering space.

(i) If $H\triangleleft G$, then for $Hx,Hgx\in p^{-1}(Gx)$, $Hgx=gHx$ where $g\in G$ is a deck transformation on $X\to X/G$.

This descends to deck transformation $X/H\to X/G$, so $p:X/H\to X/G$ is normal.

(ii) If $p: X/H \to X/G$ is normal, then $p_*(\pi_1(X/H))$ is normal in $\pi_1(X/G)$.

Let $\Phi : \pi_1(X/G) \to G$ be surjection in (b), then $\Phi(p_*(\pi_1(X/H))) = H$ is normal in G.

25. Non-Hausdorff: Orbit of $(0,1)$ contains $(0, 2^{-n})$ and orbit of $(0,-1)$ contains $(0,-2^{-n})$.

Let $p: X \to X/\mathbb{Z}$. Exact sequence $0 \to \pi_1(X) \xrightarrow{p_*} \pi_1(X/\mathbb{Z}) \to \mathbb{Z} \to 0$ right splits, so $\pi_1(X/\mathbb{Z}) \cong \pi_1(X) \oplus \mathbb{Z} = \mathbb{Z}^2$.

26. (a) Let $\mathcal{C}$ be connected components of $\widetilde{X}$ and $\pi : p^{-1}(x_0) \to \mathcal{C}$, $\tilde{x} \mapsto$ connected component $\widetilde{C}(\tilde{x}) \ni \tilde{x}$.

For $[\alpha] \in \pi_1(X, x_0)$, $[\alpha] \cdot \tilde{x}$ is the endpoint of lift of α starting at $\tilde{x}$. $\tilde{\pi} : p^{-1}(x_0)/\pi_1(X, x_0) \to \mathcal{C}$ is injective.

For $C \in \mathcal{C}$, $\tilde{\pi}^{-1}(C) = C \cap p^{-1}(x_0)$. Thus $\tilde{\pi}$ is 1-1.

(b) Suppose C is component of $\widetilde{X}$ containing a given lift $\tilde{x}_0$ of x_0. $\tilde{p} : C \to X$ is connected covering space.

Let H be stabilizer of $\tilde{x}_0$ for the action of $\pi_1(X, x_0)$, i.e. the subgroup of all $[\gamma] \in \pi_1(X, x_0)$ s.t. $[\gamma] \cdot \tilde{x}_0 = \tilde{x}_0$.

Let $N = \tilde{p}_*(\pi_1(C, \tilde{x}_0))$. We have $N = H$ by definition.

27. (Revised) For $[\gamma] \in \pi_1(X, x_0)$, $x_0 \in X$, $\tilde{x}_0 \in p^{-1}(x_0)$, suppose γ has lift γ_1 from $\tilde{x}_1$ to $\tilde{x}_0$, γ_1 from $\tilde{x}_0$ to $\tilde{x}_2$.

For universal cover $p : \widetilde{X} \to X$, $\pi_1(X) \cong G(\widetilde{X})$. $[\gamma]$ corresponds to deck transformation $\phi_{[\gamma]}$ taking $\tilde{x}_0$ to $\tilde{x}_2$.

Action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$ means a homomorphism $\pi_1(X, x_0) \to S_{p^{-1}(x_0)}$, where $S_{p^{-1}(x_0)}$ is the permutation group of $p^{-1}(x_0)$.

$\pi_1(X, x_0)$ acts on $p^{-1}(x_0)$ by lifting loops at x_0 (monodromy action) means $\Phi_1([\gamma])(\tilde{x}_0) = \tilde{x}_1$.

$\pi_1(X, x_0)$ acts on $p^{-1}(x_0)$ by restricting deck transformations to the fiber means $\Phi_2([\gamma])(\tilde{x}_0) = \phi_{[\gamma]}(\tilde{x}_0) = \tilde{x}_2$.

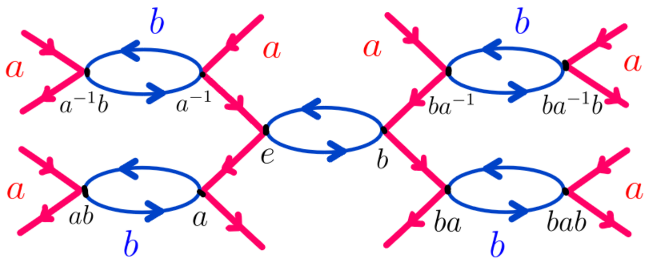
These two actions are the same when $\pi_1(X) = \mathbb{Z}_2$.

29. Let $\pi_1 : Y \to Y/G_1$, $Y \to Y/G_2$ be covering spaces.

If $\varphi : Y/G_1 \to Y/G_2$ is homeomorphism, there's a lift $\tilde{\varphi} : Y \to Y$ s.t. $\pi_2\tilde{\varphi} = \varphi\pi_2$ and $\tilde{\varphi}G_1\tilde{\varphi}^{-1} = G_2$.

If $hG_1g^{-1} = G_2$, then $h : Y \to Y$ induces a homeomorphism $\bar{h} : Y/G_1 \to Y/G_2$, $G_1y \mapsto G_2h(y)$.

30.



31. Suppose $X = \bigvee_{i=1}^n S^1$. Let $p : \widetilde{X} \to X$ be a normal cover and $N = p_*(\pi_1(\widetilde{X}))$. $N \triangleleft F_n = *_n\mathbb{Z}$.

We want to show $\widetilde{X}$ is the Cayley graph of $G = F_n/N$. Denote Cayley Graph of G by $C(G)$. $G(\widetilde{X}) = G$.

Fix basepoint $\tilde{x} \in \widetilde{X}$, there's a bijection Φ from the vertex set of $C(G)$ to vertex set of $\widetilde{X}$ given by $\Phi(g) = g \cdot \tilde{x}$.

If (v, w) is an edge in $C(G)$, there exists a generator $g \in G$ s.t. $w = gv$. $\Phi(w) = w \cdot \tilde{x} = g \cdot (v \cdot \tilde{x}) = g \cdot \Phi(v)$.

The edge $(\Phi(v), \Phi(w))$ is in $\widetilde{X}$, so Φ can extend to $\widetilde{\Phi} : C(G) \to \widetilde{X}$.

For vertex $v \in \widetilde{X}$, path γ from $\tilde{x}$ to v defines a word in F_n. For another path η from $\tilde{x}$ to v, $\bar{\eta} \cdot \gamma$ defines a word in N.

Hence we get an map $\widetilde{X} \to G = F_n/N \to C(G)$, which is the inverse of $\widetilde{\phi}$.

32. Let $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X$ be covering spaces where $\widetilde{X}_1, \widetilde{X}_2, X$ are CW complexes.

(a) If $\varphi : \widetilde{X}_1 \to \widetilde{X}_2$ is covering space isomorphism, then $\varphi(\widetilde{X}_1^1) = \widetilde{X}_2^1$, $\varphi|_{\widetilde{X}_1^1} : \widetilde{X}_2^1 \to \widetilde{X}_2^1$ is isomorphism.

Conversely, suppose $p_1|_{\widetilde{X}_1^1} : \widetilde{X}_1^1 \to X_1^1$, $p_2|_{\widetilde{X}_2^1} : \widetilde{X}_2^1 \to X_2^1$ are isomorphic via isomorphism $\varphi : \widetilde{X}_1^1 \to \widetilde{X}_2^1$.

Suppose φ is defined on $\widetilde{X}_1^{k-1}$ and $\phi : \partial e_k \to X_{k-1}$ is attaching map for X, we want to extend φ over $p^{-1}(e_k)$.

$p_1^{-1}(e_k)$ and $p_2^{-1}(e_k)$ are disjoint unions of k-cells mapping to e_k homeomorphically.

For every $e \in p_1^{-1}(e_k)$, there's some $e' \in p_2^{-1}(e_k)$ s.t. $\varphi(\partial e) = \partial e'$, so we can define $\varphi|_e = (p_2|_{e'})^{-1} \circ p_1|_e$.

(b) Deck transformation of $\widetilde{X} \to X$ restricting on $\widetilde{X}^1$ is deck transformation of $\widetilde{X}^1 \to X^1$.

Conversely, suppose by induction $\widetilde{X}^k \to X^k$ is normal cover, $x \in (k+1)$-cell $e \subseteq X$ and $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(x_0)$.

Let e_0, e_1 be $(k+1)$-cells in $\widetilde{X}$ containing $\tilde{x}_0$ and $\tilde{x}_1$ respectively.

For $y \in \partial e$, there's a path $\gamma \subseteq \overline{e}$ from x to y. γ has lifts $\gamma_i \subseteq \overline{e_i}$ in $\widetilde{X}^{k+1}$ from $\tilde{x}_i$ to $y_i \in \partial e_i \subseteq \widetilde{X}^k$ for $i = 0, 1$.

Deck transformation over $\widetilde{X}^k$ sending y_0 to $\to y_1$ extends to deck transformation on $\widetilde{X}^{k+1}$ sending $\tilde{x}_0$ to $\tilde{x}_1$.

Hence $\widetilde{X}^{k+1} \to X^{k+1}$ is a normal covering space and $\widetilde{X} \to X$ is normal.

(c) Deck transformation of $\widetilde{X} \to X$ restricting on $\widetilde{X}^1$ is deck transformation of $\widetilde{X}^1 \to X^1$, and a deck transformation of $\widetilde{X}^1 \to X^1$ extends uniquely to a deck transformation of $\widetilde{X} \to X$ from (b).

5 Section 1.A

Skipped for triviality: 6.

Skipped for difficulty: 11–13.

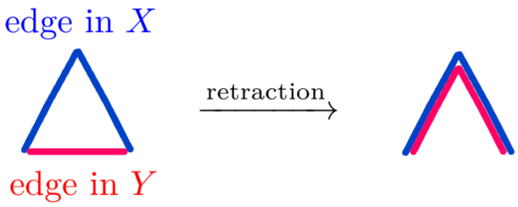
1. Note that a basis for weak topology of X consists of open intervals in the edges together with the path-connected neighborhood of the vertices. A neighborhood of the latter sort at vertex v is the union of connected open neighborhoods U_α of v in $\bar{e}_\alpha$ for all $\bar{e}_\alpha$ containing v. Such e_α is finite, so such U_α is open in canonical metric of $\mathbb{R}^2$.

Open interval is open in canonical metric of $\mathbb{R}^2$. Thus weak topology on X is a metric topology.

2. Denote the connected graph by X and its connected subgraph by Y.

If X is a tree, then Y is also a tree, and retraction maps $X - Y$ to vertices in $X \cap Y$.

If X contains a loop, then the retraction can be given via the following operation.



3. (1) A tree can be obtained from a vertex by attaching a vertex with an edge finite times, so $\chi(X) = 1$ for X a tree.

(2) Suppose T is maximal tree in X. Note that $\chi(T) - \chi(X) = 1 - \chi(X)$ is number of edges in $X - T$.

4. For any edge $e \subseteq Y$, $Y - e$ is a tree and contained in a maximal tree T.

$\pi_1(X, x_0)$ has a basis with one generator corresponding to $e \subseteq X - T$.

5. $g : S^1 \hookrightarrow S^1 \vee S^1$, $f : S^1 \vee S^1 \twoheadrightarrow S^1$ s.t. $f \circ g = 1$.

7. Let F be free group of n generators, $X = \bigvee_{i=1}^n S^1$ with wedge point x_0 and $\pi_1(X, x_0) = F$.

Let $p : \tilde{X} \to X$ be covering space corresponding to $N \triangleleft X$ and T be a maximal tree in $\tilde{X}$.

Suppose N is finitely generated, then $\tilde{X} - T$ contains finitely many edges and $V_0 = \{\text{vertex } x \mid x \in \bar{e}_\alpha \text{ for some edge } e_\alpha \subseteq \tilde{X} - T\}$ is finite. Let V_i be set of vertices of distance at most i from some vertex in V_0.

Each vertex intersects at most $2n$ closure of edges, so V_i is finite for each i.

If N is of infinite index, then $\tilde{X}$ contains infinitely many vertices. N is normal, so for any vertex $v \in \tilde{X}$, $p_*(\pi_1(\tilde{X}, v)) = N$.

Let γ be a non-trivial loop in X based at x_0 corresponding to an element in N, which is a reduced word of length k.

Choose vertex $v \in \tilde{X} - V_{k+1}$. Lift of γ at v, say $\tilde{\gamma}$, is a path of length k in $\tilde{X}$ and by definition $\tilde{\gamma} \subseteq T$, $[\tilde{\gamma}] = 0$.

$[\gamma] = p_*[\tilde{\gamma}] = 0$. Contradiction.

8. First prove the case of free groups, the general case follows since every group is a quotient group of a free group.

For finitely generated free group, its subgroup of finite index corresponds to a graph of finite vertices and edges, and there're finitely many possibilities for such graph and such subgroup.

9. (1) For given group G, there exist a 2-dimensional cell complex X s.t. $\pi_1(X, x_0) = G$ for some $x_0 \in X$.

Note that X is path-connected, locally path-connected and semilocally simply-connected, for subgroup $H \subseteq= G$, there exists a covering space $p : X_H \to X$ s.t. $p_*(\pi_1(X_H, \tilde{x}_0)) = H$ for some basepoint $\tilde{x}_0 \in p^{-1}(x_0)$.

Change basepoint $\tilde{x}_0$ within $p^{-1}(x_0)$ corresponds to changing H to its conjugate subgroup in G.

Since $[G : H] = \#\{p^{-1}(x_0)\} = n$, H has at most n conjugate subgroups in G.

(2) Consider homomorphism induced by group action $\rho : G \to S_{G/H}$, $\rho(g)(g'H) = (gg')H$.

$\ker \rho = \bigcap_{g \in G} gHg^{-1} \subseteq H$ and is normal in G of index $|S_{G/H}| = |S_n| = n!$.

10. This is Marshall Hall's Theorem in Stallings' article *Topology of finite graphs*.

See also: Projection between graphs extends to a covering space.

11. Why are free groups residually finite.

12. Exercise 1.A.12 in Hatcher's Algebraic Topology.

14. The following proof comes from "Infinite combinatorics: from finite to infinite", *Horizons of combinatorics*. Section 2.2 Spanning trees. Page 192 – 193.

($\Rightarrow$) Let $G = (V, E)$ be a graph and $\mathcal{T}$ be the family of subtrees of G. For $T, T' \in \mathcal{T}$, write $T \prec T'$ if $T \subseteq T'$.

Since $\mathcal{T}$ is closed under increasing union, $\langle \mathcal{T}, \prec \rangle$ has a maximal element $T = (V', E')$ by Zorn's Lemma.

Since there is no edge between V' and $V - V'$, we have $V = V'$. Hence T is a maximal tree.

($\Leftarrow$) Let $\mathcal{A} = \{A_i : i \in I\}$ be a family of non-empty sets. We want to find a choice function.

First assume the elements of $\mathcal{A}$ are pairwise disjoint. Construct a graph $G = (V, E)$ as follows:

Let $V = \{x\} \cup \{y_i, z_i : i \in I\} \cup \bigcup \{A_i : i \in I\}$, where $\{x\} \cup \{y_i, z_i : i \in I$ are new, pairwise different vertices.

Let $E = \{xy_i : i \in I\} \cup \bigcup_{i \in I} \{z_i a, ay_i : a \in A_i\}$. G is connected and by assumption has a maximal tree $T = (V, F)$.

Then we have

(1) $\{xy_i : i \in I\} \subseteq F$.

(2) For each $i \in I$, there is exactly one $a_i \in A_i$ s.t. $z_i a_i, a_i y_i \in F$.

(3) For each $a \in A_i - \{a_i\}$, we have $z_i a \in F$ iff $ay_i \notin F$.

Thus $f(i) = a_i$ is a choice function for $\mathcal{A}$ and f is definable using T.

6 Section 2.1

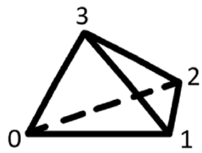
Skipped for triviality: 11, 13, 15, 22, 30.

Skipped for difficulty: 10, 21, 23–25, 28.

1. Möbius band.

2. Let $S = [012] \cup [123] \subseteq \Delta^3 = [0123]$, $[01] \sim [13]$ and $[02] \sim [23]$. $S/\sim$ is Klein bottle.

Deformation retraction $F : \Delta^3 \times I \to S$ induces continuous quotient map $\bar{F} : \Delta^3/\sim \times I \to S/\sim$.

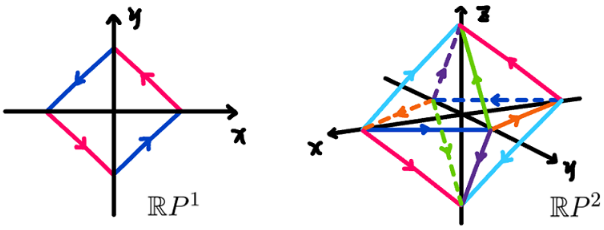


$[01] \sim [23]$, $[02] \sim [13]$ produces Δ-complex deformation retracting onto a torus T^2.

$[01] \sim [02]$, $[13] \sim [23]$ produces Δ-complex deformation retracting onto a 2-sphere S^2.

$[01] \sim -[23]$, $[02] \sim -[13]$ produces Δ-complex deformation retracting onto $\mathbb{R}P^2$.

3.



4. Denote this space by X. $H_0^\Delta(X) = \mathbb{Z}$. $H_1^\Delta(X) = \mathbb{Z} \oplus \mathbb{Z}$. $H_n^\Delta(X) = 0$ for $n \geqslant 2$.

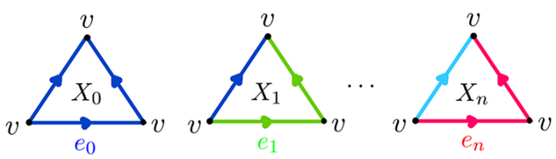
5. Denote Klein bottle by K. $H_0^\Delta(X) = \mathbb{Z}$. $H_1^\Delta(X) = \mathbb{Z} \oplus \mathbb{Z}_2$. $H_n^\Delta(X) = 0$ for $n \geqslant 2$.

6. Denote this space by X. $\Delta_0(X) = \langle v \rangle = \mathbb{Z}$.

$\Delta_1(X) = \langle e_0, \dots, c_n \rangle = \mathbb{Z}^{n+1}$. $\Delta_2(X) = \langle X_0, \dots X_n \rangle = \mathbb{Z}^{n+1}$.

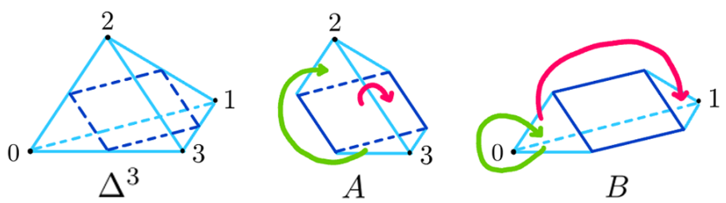
$\partial_2 X_0 = e_0$, $\partial_2 X_i = 2e_i - e_{i-1}$ for $i = 1, \dots, n$. $\ker \partial_1 = \Delta_1(X)$.

$H_0^\Delta(X) = \mathbb{Z}$. $H_1^\Delta(X) = \langle e_n \mid 2^n e_n \rangle = \mathbb{Z}_{2^n}$. $H_n^\Delta(X) = 0$ for $n \geqslant 2$.

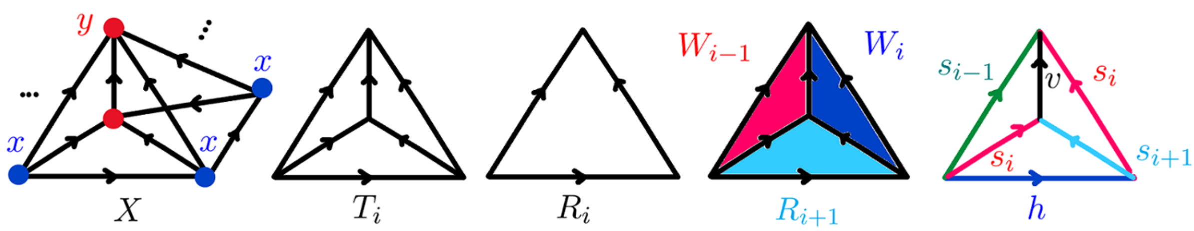


7. $\Delta^3 = [0123] = A \cup B$. $\partial[0123] = [123] - [023] + [013] - [012]$. Let $[123] \sim [023]$, $[013] \sim [012]$.

$A/\sim = \partial D^2 \times D^2$, $B/\sim = D^2 \times \partial D^2$. $S^3 = \partial D^4 = \partial(D^2 \times D^2) = \partial D^2 \times D^2 \cup D^2 \times \partial D^2 = A/\sim \cup B/\sim \ = \Delta^3/\sim$.



8. $\Delta_0(X) = \langle x, y \rangle$. $\Delta_1(X) = \langle s_1, \dots, s_n, v, h \rangle$. $\Delta_2(X) = \langle W_1, \dots, W_n, R_1, \dots, R_n \rangle$. $\Delta_3(X) = \langle T_1, \dots, T_n \rangle$.



$\partial_3 T_i = W_i - W_{i-1} + R_i - R_{i+1}$. $\partial_2 R_i = s_i - s_{i-1} + h$, $\partial_2 W_i = v - s_i + s_{i+1}$. $\partial_1 s_i = y - x$, $\partial_1 h = 0$, $\partial_1 v = 0$.

Note that $\partial_2 R_1 = s_1 - s_n + h = h + (s_1 - s_2) + \cdots + (s_{n-1} - s_n)$, $\partial_2 W_n = v - s_n + s_1 = v + (s_1 - s_2) + \cdots + (s_{n-1} - s_n)$.

$\ker\partial_1 = \langle s_1 - s_2, \dots, s_{n-1} - s_n, h, v\rangle$. $\ker\partial_2 = \operatorname{im}\partial_3$. $\ker\partial_3 = \langle T_1 + \cdots + T_n\rangle$.

$H_0^\Delta(X) = \ker\partial_0/\operatorname{im}\partial_1 = \langle x, y\rangle/\langle y - x\rangle = \mathbb{Z}$. $H_1^\Delta(X) = \ker\partial_1/\operatorname{im}\partial_2 = \langle h \mid nh = 0\rangle = \mathbb{Z}_n$. $H_2^\Delta(X) = 0$. $H_3^\Delta(X) = \mathbb{Z}$.

9. $\Delta_k(X) = \langle a_k\rangle = \mathbb{Z}$ for $k \leqslant n$. $\partial a_k = \sum_{i=0}^k (-1)^i a_{k-1} = a_{k-1}$ for k even and 0 for k odd.

12. For $f, g : X \to Y$ and chain maps $f_\#, g_\# : C_n(X) \to C_n(Y)$, $f_\#$ and $g_\#$ are chain homotopic means there exists prism operators $P : C_n(X) \to C_{n+1}(Y)$ s.t. $\partial P + P\partial = g_\# - f_\#$.

14. (0) Prerequisites: In Abelian category $\mathcal{A}$, suppose $b : B \to D$ is morphism, and $g : C \to D$ is epimorphism, then the followings are equivalent:

(i) $$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$ is pull-back.

(ii) $$\begin{tikzcd} 0 \arrow[r] & E \arrow[r] \arrow[d, equal] & A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r] \arrow[d, "b"] & 0 \\ 0 \arrow[r] & E \arrow[r] & C \arrow[r, "g"] & D \arrow[r] & 0 \end{tikzcd}$$ is commutative diagram with exact rows.

(iii) $0 \to A \xrightarrow{\binom{f}{a}} B \oplus C \xrightarrow{(b,-g)} D \to 0$ is exact.

(1) For abelian group A, $0 \to \mathbb{Z}_{p^m} \xrightarrow{f} A \xrightarrow{\pi_1} \mathbb{Z}_{p^n} \to 0$ is exact $\Leftrightarrow$ $A \cong \mathbb{Z}_{p^k} \times \mathbb{Z}_{p^{m+n-k}}$ where $0 \leqslant k \leqslant \min\{m, n\}$.

($\Rightarrow$) Suppose $\pi_2(a) = \bar{1}$ for some $a \in A$. Define $g : \mathbb{Z}_{p^m} \times \mathbb{Z} \to A$ by $g(x, y) = f(x) + y \cdot a$ for $x \in \mathbb{Z}_{p^m}, y \in \mathbb{Z}$.

The key point is there's a multiplication of elements in $\mathbb{Z}$ and A, which requires A to be a $\mathbb{Z}$-mod/abelian group.

Claim: $(\mathbb{Z}_{p^m} \times \mathbb{Z}, p, g)$ is pull-back of $\pi_1 : A \to \mathbb{Z}_{p^n}$ and $\pi_2 : \mathbb{Z} \to \mathbb{Z}_{p^n}$, where $p : \mathbb{Z}_{p^m} \times \mathbb{Z} \to \mathbb{Z}$ is projection.

Pull-back of $\pi_1 : A \to \mathbb{Z}_{p^n}$ and $\pi_2 : \mathbb{Z} \to \mathbb{Z}_{p^n}$ is (P, ψ_1, ψ_2), where $P = \{(m, n) \in A \times \mathbb{Z} \mid \pi_1(m) = \pi_2(n)\}$, $\psi_1 : P \to A$ is projection $A \times \mathbb{Z} \to \mathbb{Z}$ restricted on P, $\psi_2 : P \to \mathbb{Z}$ is projection $A \times \mathbb{Z} \to \mathbb{Z}$ restricted on P.

$h : \mathbb{Z}_{p^m} \times \mathbb{Z} \to P$ is defined by $h(x, y) = (g(x, y), y)$, $x \in \mathbb{Z}_{p^m}, y \in \mathbb{Z}$.

$k : P \to \mathbb{Z}_{p^m} \times \mathbb{Z}$ is defined by $k(m, n) = (f^{-1}(m - n \cdot a), n)$, $m \in A, n \in \mathbb{Z}$.

Note that $m - n \cdot a \in \ker\pi_1 = \operatorname{im} f$ and f is injective, we have commutative diagram (I) and enlarged commutative diagram (II) with exact rows and columns.

(I) $$\begin{tikzcd} P \arrow[dr, "k"] \arrow[drr, bend left, "\psi_2"] \arrow[ddr, bend right, "\psi_1"'] & & \\ & \mathbb{Z}_{p^m} \times \mathbb{Z} \arrow[r, "p"] \arrow[d, "g"] \arrow[ul, bend right, "h"'] & \mathbb{Z} \arrow[d, "\pi_2"] \\ & A \arrow[r, "\pi_1"] & \mathbb{Z}_p \end{tikzcd}$$

(II) $$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ & & \mathbb{Z} \arrow[r, equal] \arrow[d] & \mathbb{Z} \arrow[d] & \\ 0 \arrow[r] & \mathbb{Z}_{p^m} \arrow[r, "i"] \arrow[d, equal] & \mathbb{Z}_{p^m} \times \mathbb{Z} \arrow[r, "p"] \arrow[d, "g"] & \mathbb{Z} \arrow[r] \arrow[d, "\pi_2"] & 0 \\ 0 \arrow[r] & \mathbb{Z}_{p^m} \arrow[r, "f"] & A \arrow[r, "\pi_1"] \arrow[d] & \mathbb{Z}_{p^n} \arrow[r] \arrow[d] & 0 \\ & & 0 & 0 & \end{tikzcd}$$

The middle column is short exact sequence of form $0 \longrightarrow \mathbb{Z} \xrightarrow{\begin{pmatrix} r \\ p^n \end{pmatrix}} \mathbb{Z}_{p^m} \times \mathbb{Z} \longrightarrow A \longrightarrow 0$ for some $r \in \mathbb{N}$.

It's equivalent to short exact sequence $0 \longrightarrow \mathbb{Z} \times \mathbb{Z} \xrightarrow{\begin{pmatrix} p^m & r \\ 0 & p^n \end{pmatrix}} \mathbb{Z}_{p^m} \times \mathbb{Z} \longrightarrow A \longrightarrow 0$.

The integer matrix $\begin{pmatrix} p^m & r \\ 0 & p^n \end{pmatrix}$ is equivalent to $\begin{pmatrix} p^k & 0 \\ 0 & p^{m+n-k} \end{pmatrix}$ where $(p^k) = (p^m, p^n, r)$.

Thus $A \cong \mathbb{Z}_{p^k} \times \mathbb{Z}_{p^{m+n-k}}$ for $0 \leqslant k \leqslant \min\{m,n\}$.

($\Leftarrow$) For $m,n \in \mathbb{N}$, let $k \in \mathbb{N}$ s.t. $0 \leqslant k \leqslant \min\{m,n\}$, then $k \leqslant m \leqslant m+n-k$.

We have epimorphism $\alpha : \mathbb{Z}_{p^m} \to \mathbb{Z}_{p^k}$ and monomorphism $\beta : \mathbb{Z}_{p^m} \to \mathbb{Z}_{p^{m+n-k}}$. $\mathbb{Z}_{p^m} \xrightarrow{(\alpha,\beta)} \mathbb{Z}_{p^k} \times \mathbb{Z}_{p^{m+n-k}}$ is injective.

$\operatorname{coker}(\alpha,\beta) = \langle a, b \mid a^{p^k} = b^{p^{m+n-k}} = 1, ab = ba, ab^{p^{n-k}} = 1\rangle = \langle b \mid b^{p^n} = 1\rangle = \mathbb{Z}_{p^n}$.

Thus we have short exact sequence $0 \to \mathbb{Z}_{p^m} \xrightarrow{(\alpha,\beta)} \mathbb{Z}_{p^k} \times \mathbb{Z}_{p^{m+n-k}} \longrightarrow \mathbb{Z}_{p^n} \to 0$.

(2) For abelian group A, $0 \longrightarrow \mathbb{Z} \longrightarrow A \longrightarrow \mathbb{Z}_n \longrightarrow 0$ is exact $\Leftrightarrow A \cong \mathbb{Z}_d \times \mathbb{Z}$ where $d \mid n$.

($\Rightarrow$) We have short exact sequence of form $0 \longrightarrow \mathbb{Z} \xrightarrow{\binom{r}{n}} \mathbb{Z} \times \mathbb{Z} \longrightarrow A \longrightarrow 0$ for some $r \in \mathbb{Z}$.

The integer matrix $\binom{r}{n}$ is equivalent to $\binom{d}{0}$, where $d = (r,n)$. Thus $A \cong \mathbb{Z}_d \times \mathbb{Z}$.

($\Leftarrow$) If $d \mid n$, then $0 \longrightarrow \mathbb{Z} \xrightarrow{\binom{1}{n/d}} \mathbb{Z}_d \times \mathbb{Z} \longrightarrow \mathbb{Z}_n \longrightarrow 0$ is exact.

16. (a) $H_0(X,A) = 0 \Leftrightarrow H_0(A) \to H_0(X)$ is surjective iff A meets each path-component of X.

(b) $H_1(X,A) = 0 \Leftrightarrow H_1(A) \to H_1(X)$ is surjective and $H_0(A) \to H_0(X)$ is injective.

$H_0(A) \to H_0(X)$ is injective iff X each path-component of X contains at most one path-component of A.

17. Suppose A is k points in path-connected space X, then $X \cup CA \simeq X \vee (\bigvee_{i=1}^{k-1} S^1)$.

$H_n(X,A) \cong \widetilde{H}_n(X \cup CA) \cong \widetilde{H}_n(X \vee (\bigvee_{i=1}^{k-1} S^1)) \cong \widetilde{H}_n(X) \oplus (\bigoplus_{i=1}^{k-1} \widetilde{H}_n(S^1))$.

(a) $\widetilde{H}_2(S^2) = \mathbb{Z}$, $\widetilde{H}_n(S^2) = 0$ for $n \neq 2$. $\widetilde{H}_1(S^1 \times S^1) = \mathbb{Z} \oplus \mathbb{Z}$, $\widetilde{H}_2(S^1 \times S^1) = \mathbb{Z}$, $\widetilde{H}_n(S^1 \times S^1) = 0$ for $n \geqslant 3$.

$H_1(S^2, A) = \mathbb{Z}^{k-1}$, $H_2(S^2, A) = \mathbb{Z}$, $H_n(S^2, A) =$ for $n \geqslant 3$.

$H_1(S^1 \times S^1, A) = \mathbb{Z}^{k+1}$, $H_2(S^1 \times S^1, A) = \mathbb{Z}$, $H_n(S^1 \times S^1, A) = 0$ for $n \geqslant 3$.

(b) $X/A \simeq T^2 \vee T^2$. $H_n(X,A) \cong \widetilde{H}_n(X/A) = \widetilde{H}_n(T^2 \vee T^2) \cong \widetilde{H}_n(T^2) \oplus \widetilde{H}_n(T^2)$.

$X/B \simeq T^2/\{*_1, *_2\} \simeq T^2 \vee S^1$. $H_n(X,B) \cong \widetilde{H}_n(X/B) = \widetilde{H}_n(T^2 \vee S^1) \cong \widetilde{H}_n(T^2) \oplus \widetilde{H}_n(S^1)$.

18. $\widetilde{H}_1(\mathbb{R}) \to \widetilde{H}_1(\mathbb{R},\mathbb{Q}) \to \widetilde{H}_0(\mathbb{Q}) \to \widetilde{H}_0(\mathbb{R})$ is exact. $\widetilde{H}_1(\mathbb{R}) = 0 = \widetilde{H}_0(\mathbb{R})$, $\widetilde{H}_1(\mathbb{R},\mathbb{Q}) \cong \widetilde{H}_0(\mathbb{Q})$.

$0 \to \widetilde{H}_0(\mathbb{Q}) \to H_0(\mathbb{Q}) \xrightarrow{\varphi} \mathbb{Z} \to 0$ is exact, where $\varphi : H_0(\mathbb{Q}) \to \mathbb{Z}$ is induced by $\varepsilon : C_0(\mathbb{Q}) \to \mathbb{Z}$, $\varepsilon(\sum_i n_i \sigma_i) = \sum_i n_i$.

For $\sigma_q : \Delta^0 \to q \in \mathbb{Q}$ in $C_0(\mathbb{Q})$, $\{\sigma_q - \sigma_0 \mid q \in \mathbb{Q}\}$ is a basis for $\ker \varepsilon$, $\{[\sigma_q - \sigma_0] \mid q \in \mathbb{Q}\}$ is a basis for $\ker \varphi = \widetilde{H}_0(\mathbb{Q})$.

19. Denote this space by X. $H_0(X) = \mathbb{Z}$. $H_1(X) = \bigoplus_\infty \mathbb{Z}$. $H_n(X) = 0$ for $n \geqslant 2$.

20. Long exact sequence of triple $(CX, X, *)$ gives $H_{n+1}(CX, X) \cong H_n(X, *)$, thus $\widetilde{H}_{n+1}(SX) \cong \widetilde{H}_n(X)$.

$\widetilde{H}_{n+1}(\bigcup_{i=1}^k CX) = \widetilde{H}_{n+1}(\bigcup_{i=1}^{k-1} CX \cup CX) \cong H_{n+1}(\bigcup_{i=1}^{k-1} CX, X) = \widetilde{H}_{n+1}(\bigvee_{i=1}^{k-1} SX) = \bigoplus_{i=1}^{k-1} \widetilde{H}_{n+1}(SX) = \bigoplus_{i=1}^{k-1} \widetilde{H}_n(X)$.

21. Explicit isomorphism $\widetilde{H}_n(X) \cong \widetilde{H}_{n+1}(SX)$.

26. From section 2.A, for X path-connected, $\widetilde{H}_1(X) = H_1(X) \cong \pi_1(X)_{ab}$.

Note $H_1(X,A) \cong \widetilde{H}_1(X \cup CA)$. $X \cup CA$ is homotopic to $\bigvee_\infty S^1$, while X/A is homeomorphic to Hawaiian Earring.

$H_1(X,A) \cong \bigoplus_\infty \mathbb{Z}$. The singular homology of the Hawaiian Earring.

27. (a) By naturality, we have commutative diagram

$$\begin{array}{ccccccccc} H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,A) & \longrightarrow & H_{n-1}(A) & \longrightarrow & H_{n-1}(X) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ H_n(B) & \longrightarrow & H_n(Y) & \longrightarrow & H_n(Y,B) & \longrightarrow & H_{n-1}(B) & \longrightarrow & H_{n-1}(Y) \end{array}$$

$f : X \to Y$ and $f|_A : A \to B$ are homotopy equivalences, so from 5-lemma, $H_n(X,A) \cong H_n(Y,B)$.

(b) For any be continuous map $g : (D^n, D^n - \{0\}) \to (D^n, S^{n-1})$, $g(0) \subseteq S^{n-1}$, so $g : D^n \to S^{n-1}$ is nullhomotopic

29. $H_0(S^1 \times S^1) = H_0(S^1 \vee S^1 \vee S^2) = \mathbb{Z}$, $H_1(S^1 \times S^1) = H_1(S^1 \vee S^1 \vee S^2) = \mathbb{Z}^2$, $H_2(S^1 \times S^1) = H_2(S^1 \vee S^1 \vee S^2) = \mathbb{Z}$, $H_n(S^1 \times S^1) = H_n(S^1 \vee S^1 \vee S^2) = 0$ for $n \geqslant 3$.

Universal cover of $S^1 \times S^1$ is $\mathbb{R}^2$. It's contractible hence has homology group 0.

Universal cover of $S^1 \vee S^1 \vee S^2$ is universal cover of $S^1 \vee S^1$ with a S^2 attached at each vertex, denoted by X.

$X = X^2$, $X^1 = S^1 \vee S^1$ is contractible, so $H_2(X) = H_2(X^2) \cong H_2(X^2, X^1) \cong \widetilde{H}_2(X^2/X^1) = \widetilde{H}_2(\bigvee S^2) \neq 0$.

31.
$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\alpha"] & 0 \arrow[r] \arrow[d, "\beta"] & \mathbb{Z} \arrow[r] \arrow[d, "\gamma"] & \mathbb{Z} \arrow[r] \arrow[d, "\delta"] & 0 \arrow[d, "\varepsilon"] \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

7 Section 2.2

Skipped for triviality: 7, 15, 22, 37.

Skipped for difficulty: 16.

Note: Exercise 34 is deleted by the author — see the errata for comments.

1. For $f : D^n \to D^n$, $\tilde{f} : D^n_+ \cup D^n_- = S^n \to D^n_- \subseteq S^n$ is not surjective, $\deg \tilde{f} = 0$. $\tilde{f}$ has fixed point in D^n_-.

2. (1) For $f : S^{2n} \to S^{2n}$, if f has no fixed points, then $\deg f = -1$. If $-f$ has no fixed points, then $\deg f = 1$.

Thus either f or $-f$ must have a fixed point, i.e. there's some point $x \in S^{2n}$ s.t. $f(x) = x$ or $f(x) = -x$.

(2) For $g : \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$, quotient map $\pi : S^{2n} \to \mathbb{R}P^{2n}$, $g \circ \pi$ has a lift $\tilde{g} : S^{2n} \to S^{2n}$ s.t. $g \circ \pi = \pi \circ \tilde{g}$.

For $\tilde{g} : S^{2n} \to S^{2n}$, there exists point $x \in S^{2n}$ s.t. $\tilde{g}(x) = x$ or $\tilde{g}(x) = -x$, so $g(\pi(x)) = \pi(\tilde{g}(x)) = \pi(x)$.

(3) Consider linear transformation $T : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, $(x_1, x_2, \ldots, x_{2n}) \mapsto (-x_{2n}, x_1, x_2, \ldots, x_{2n-1})$. $T^{2n} = -\mathrm{id}_{2n}$.

$x^{2n} + 1$ is characteristic polynomial of T and has no real roots, so T has no real eigenvalues or eigenvectors.

Thus $T : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ induces a map $\mathbb{R}P^{2n-1} \to \mathbb{R}P^{2n-1}$ without eigenvectors.

3. (1) $\deg f = 0$, so f and $-f$ have fixed point(s).

(2) For non-vanishing vector field F, let $G = \dfrac{F(x)}{\|F(x)\|} : D^n \to S^{n-1}$ and $i : \partial D^n = S^{n-1} \hookrightarrow D^n$ be inclusion.

$G|_{\partial D^n} = G \circ i : S^{n-1} \to S^{n-1}$ satisfies $(G|_{\partial D^n})_* = 0$, so $\deg G|_{\partial D^n} = 0$.

4. $S^n \xrightarrow{\pi} D^n \xrightarrow{q} D^n/\partial D^n = S^n$. $\pi : S^n \to D^n$ given by $(x_1, \ldots, x_n, x_{n+1}) \mapsto (x_1, \ldots, x_n)$ is projection.

5. Let f_k be reflection of S^n across n-dimensional hyperplane with unit normal vector k. Treat k as a point on S^n.

For $x \in S^n$, we have $f_k(x) = x - 2\langle x, k\rangle k$. For different reflections f_a and f_b, let $\gamma : [0,1] \to S^n$ be a path from a to b.

Then $F : S^n \times [0,1]$, $F(x,t) = f_{\gamma(t)}(x)$ is the desired homotopy from f_a to f_b.

6. (1) Method 1: Suppose $f : S^n \to S^n$, $\deg f = k$. $g : S^1 \to S^1, z \mapsto z^k$ is of degree k and has fixed point x_0.

Suspension $Sg : S^2 \to S^2$ and $Sg|_{S^1} = g$, so $Sg(x_0) = g(x_0) = x_0$ and $\deg Sg = \deg g$.

By induction, $S^{n-1}g : S^n \to S^n$ and $S^{n-1}g|_{S^1} = g$, x_0 is fixed point of $S^{n-1}g$. $\deg S^{n-1}g = k = \deg f$, $S^{n-1}g \simeq f$.

(2) Method 2: WLOG suppose $f : S^n \to S^n$ has no fixed points, then f is homotopic to antipodal map.

When n is odd, the antipodal map is homotopic to identity map on S^n, so f is homotopic to identity map.

When n is even, let $n = 2m$. Consider homotopy $H(x,t) : S^{2m} \times [0,\pi] \to S^{2m}$ given by

$((x_1, x_2, \ldots, x_{2m+1}), t) \mapsto (x_1 \cos t - x_2 \sin t, x_2 \cos t + x_1 \sin t, \ldots, x_{2m-1} \cos t - x_{2m} \sin t, x_{2m} \cos t + x_{2m-1} \sin t, -x_{2m+1})$.

$H(x,t)$ is homotopy from $g : S^{2m} \to S^{2m}, (x_1, x_2, \ldots, x_{2m+1}) \mapsto (x_1, x_2, \ldots, -x_{2m+1})$ to antipodal map on S^{2m}.

Thus f is homotopic to g, which has fixed points $(x_1, x_2, \ldots, x_{2m}, 0) \in S^{2m}$.

8. First, ∞ is not a zero. Suppose $z_1, \ldots, z_k$ are the roots of f with multiplicities $n_1, \ldots, n_k$, then $\deg f = \sum\limits_{i=1}^{k} n_i$.

For appropriate local coordinate chart near z_i, f has form $w = z^{n_i} h(z)$, where $h(z)$ is a non-vanishing homomorphic function, thus $\deg \hat{f}|z_i = n_i$. $\deg \hat{f} = \sum\limits_i \deg \hat{f}|z_i = \sum\limits_{i=1}^{k} n_i = \deg f$.

9. (a) Let $X_1 = S^2/\{\{N\},\{S\}\} \simeq S^1 \vee S^2$. $H_n(X_1) = \mathbb{Z}$ for $n = 0,1,2$ or 0 for $n \geqslant 3$.

(b) Let $X_2 = S^1 \times (S^1 \vee S^1)$. $0 \to \mathbb{Z}\langle U, L\rangle \xrightarrow{d_2} \mathbb{Z}\langle a,b,c\rangle \xrightarrow{d_1} \mathbb{Z}\langle v\rangle \to 0$. $d_1 = 0$, $d_2 = 0$.

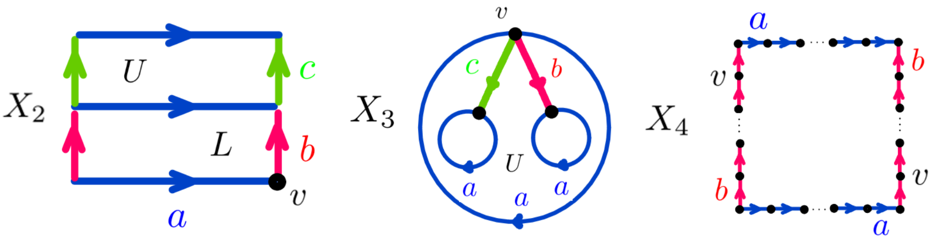
$H_0(X_2) = \mathbb{Z}$, $H_1(X_2) = \mathbb{Z}^3$, $H_2(X_2) = \mathbb{Z}^2$, $H_n(X_2) = 0$ for $n \geqslant 3$.

(c) Let the space be X_3. Attachment map of 2-cell U is $ca^{-1}c^{-1}ba^{-1}b^{-1}a$. $0 \to \mathbb{Z}\langle U\rangle \xrightarrow{d_2} \mathbb{Z}\langle a,b,c\rangle \xrightarrow{d_1} \mathbb{Z}\langle v\rangle \to 0$.

$d_1 = 0$, $d_2(U) = -a$. $H_0(X_3) = \mathbb{Z}$, $H_1(X_3) = \mathbb{Z}^2$, $H_n(X_3) = 0$ for $n \geqslant 2$.

(d) Let the space be X_4. Attachment map of 2-cell U is $a^n b^m a^{-n} b^{-m}$. $0 \to \mathbb{Z}\langle U\rangle \xrightarrow{d_2} \mathbb{Z}\langle a,b\rangle \xrightarrow{d_1} \mathbb{Z}\langle v\rangle \to 0$.

$d_1 = 0$, $d_2 = 0$. $H_0(X_4) = \mathbb{Z}$, $H_1(X_4) = \mathbb{Z}^2$, $H_2(X_4) = \mathbb{Z}$, $H_n(X_4) = 0$ for $n \geqslant 3$.



10. Let $\alpha_n : S^n \to S^n$ be antipodal map. $\deg \alpha_n = (-1)^{n+1}$.

(1) X has one 0-cell v, one 1-cell e, two 2-cells D_+, D_-. $0 \to \mathbb{Z}\langle D_+, D_-\rangle \xrightarrow{d_2} \mathbb{Z}\langle e\rangle \xrightarrow{d_1} \mathbb{Z}\langle v\rangle \to 0$.

$d_2(D_\pm) = (1 + \deg \alpha_1)e = 2e$, $d_1 e = 0$. $H_0(X) = H_2(X)\mathbb{Z}$, $H_1(X) = \mathbb{Z}^2$, $H_n(X) = 0$ for $n \geqslant 3$.

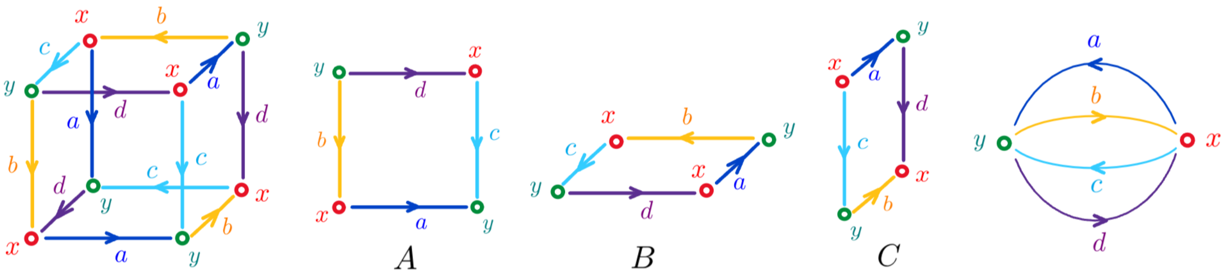
(2) $Y = S^3/\sim$ has one 0-cell v, one 1-cell e_1, one 2-cell e_2 and two 3-cells D_+, D_-. $d_3(D_\pm) = (1 + \deg \alpha_2)e = 0$.

$0 \to \mathbb{Z}\langle D_+, D_-\rangle \xrightarrow{d_3} \underbrace{\mathbb{Z}\langle e_2\rangle \xrightarrow{d_2} \mathbb{Z}\langle e_1\rangle \xrightarrow{d_1} \mathbb{Z}\langle v\rangle \to 0}_{\text{cellular chain complex of } \mathbb{R}P^2}$. $d_3 = 0$, $d_2 = 2$, $d_1 = 0$.

$H_0(Y) = \mathbb{Z}$, $H_1(Y) = \mathbb{Z}_2$, $H_2(Y) = 0$, $H_3(Y) = \mathbb{Z}^2$, $H_n(Y) = 0$ for $n \geqslant 3$.

11. Related: Exercise **1.2.14**

Suppose the quotient space is X. It has two 0-cells x, y, four 1-cells a,b,c,d, three 2-cells A,B,C and one 3-cell.



Faces of the 3-cell is identified via a twist, so $d_3 = 0$. $d_2(A) = a+b-c-d$, $d_2(B) = a+b+c+d$, $d_2(C) = a-b-c+d$.

Let $\alpha = a+d$, $\beta = -b+d$, $\gamma = c+d$. $d_2(A) = \alpha - \beta - \gamma$, $d_2(B) = \alpha - \beta + \gamma$, $d_2(C) = \alpha + \beta - \gamma$.

$C_1(X) = \mathbb{Z}\langle a,b,c,d\rangle = \mathbb{Z}\langle \alpha,\beta,\gamma,d\rangle = \mathbb{Z}\langle \alpha-\beta-\gamma,\beta,\gamma,d\rangle$. $C_2(X) = \mathbb{Z}\langle A,B,C\rangle = \mathbb{Z}\langle A, B-A, C-A\rangle$.

$d_2(A) = \alpha - \beta - \gamma$, $d_2(B-A) = 2\gamma$, $d_2(C-A) = 2\beta$. $d_1(\alpha) = 0$, $d_1(\beta) = 0$, $d_1(\gamma) = 0$, $d_1(d) = x - y$.

Cellular chain complex is $0 \to \mathbb{Z} \xrightarrow{d_3} \mathbb{Z}\langle A, B-A, C-A\rangle \xrightarrow{d_2} \mathbb{Z}\langle \alpha-\beta-\gamma,\beta,\gamma,d\rangle \xrightarrow{d_1} \mathbb{Z}\langle x,y\rangle \to 0$.

$H_0(X) = \mathbb{Z}\langle x,y\rangle/\mathbb{Z}\langle x-y\rangle \cong \mathbb{Z}$, $H_1(X) = \mathbb{Z}\langle \alpha-\beta-\gamma,\beta,\gamma\rangle/\mathbb{Z}\langle \alpha-\beta-\gamma, 2\beta, 2\gamma\rangle = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

$\ker d_2 = 0$, $H_2(X) = 0$. $d_3 = 0$, $H_3(X) = \mathbb{Z}$. $H_n(X) = 0$ for $n \geqslant 4$.

12. $H_2(S^1\vee S^1)\to H_2(S^1\times S^1)\to H_2(S^1\times S^1,S^1\vee S^1)\to H_1(S^1\vee S^1)\to H_1(S^1\times S^1)\to H_1(S^1\times S^1,S^1\vee S^1)$.

$H_2(S^1\vee S^1)=0=H_1(S^1\times S^1,S^1\vee S^1)$, $H_2(S^1\times S^1)=\mathbb{Z}=H_2(S^1\times S^1,S^1\vee S^1)$, $H_1(S^1\vee S^1)=\mathbb{Z}^2=H_1(S^1\times S^1)$.

For $f:S^2\to S^1\times S^1$ and universal cover $\pi:\mathbb{R}^2\to S^1\times S^1$, $\pi_1(S^2)=0$, so f has a lift $\tilde f:S^2\to\mathbb{R}^2$ s.t. $\pi\circ\tilde f=f$.

$\mathbb{R}^2$ is contractible, so $\tilde f$ is nullhomotopic, hence f is nullhomotopic.

13. Let $2,3:S^1\to S^1$ denote the attachment maps of degree 2 and 3 of 2-cells e_1^2 and e_2^2.

(a) $X=S^1\cup_2 e_1^2\cup_3 e_2^2=e_0\cup e_1\cup_2 e_1^2\cup_3 e_2^2$. Subcomplexes are $e_0,S^1,S^1\cup_2 e_1^2,S^1\cup_3 e_2^2$ and X.

$H_0(e_0)=\mathbb{Z}$, $H_n(e_0)=0$ for $n\geqslant 1$. $X/e_0=X$. $H_0(S^1)=H_1(S^1)=\mathbb{Z}$, $H_n(S^1)=0$ for $n\geqslant 2$. $X/S^1=S^2\vee S^2$.

$H_0(S^1\cup_2 e_1^2)=\mathbb{Z}$, $H_1(S^1\cup_2 e_1^2)=\mathbb{Z}_2$, $H_n(S^1\cup_2 e_1^2)=0$ for $n\geqslant 2$. $X/(S^1\cup_2 e_1^2)=S^2$.

$H_0(S^1\cup_3 e_2^2)=\mathbb{Z}$, $H_1(S^1\cup_3 e_2^2)=\mathbb{Z}_3$, $H_n(S^1\cup_3 e_2^2)=0$ for $n\geqslant 2$. $X/(S^1\cup_3 e_2^2)=S^2$.

$H_0(X)=\mathbb{Z}$, $H_1(X)=0$, $H_2(X)=\mathbb{Z}$, $H_n(X)=0$ for $n\geqslant 3$. $X/X=\{*\}$.

(b) (1) $\pi_1(S^1\cup_2 e_1^2,e^0)=\langle e^1\mid (e^1)^2\rangle$. Attachment map $3:S^1\to S^1\subseteq S^1\cup_2 e_1^2$ is an element in $\pi_1(S^1\cup_2 e_1^2)$.

$[3]=(e^1)^3=e^1$, so attachment map 3 is homotopic to attachment map $1:S^1\to S^1\subseteq S^1\cup_2 e_1^2$ of degree 1.

Note that $2:S^1\to S^1\subseteq D^2$ is nullhomotopic, so it's homotopic to constant map $0:S^1\to S^1,S^1\mapsto e^0$.

$X=S^1\cup_2 e_1^2\cup_3 e_2^2\simeq S^1\cup_2 e_1^2\cup_1 e_2^2=(S^1\cup_1 e_2^2)\cup_2 e_1^2=D^2\cup_2 e_1^2\simeq D^2\cup_0 e_1^2=D^2\vee S^2\simeq S^2$.

(2) $X\to X/e_0=X$ is a homotopy equivalence.

$X\to X/S^1=S^2\vee S^2$ is not a homotopy equivalence since $H_2(X)=\mathbb{Z}$ and $H_2(S^2\vee S^2)=\mathbb{Z}^2$.

Consider quotient map $q:X\to X/(S^1\cup_2 e_1^2)=e_0\cup e_2^2=S^2$. q is cellular and induces a cellular chain map.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}\langle e_1^2,e_2^2\rangle \arrow[r, "d_2"] \arrow[d, "q_\#"] & \mathbb{Z}\langle e^1\rangle \arrow[r, "d_1"] \arrow[d, "q_\#"] & \mathbb{Z}\langle e^0\rangle \arrow[r] \arrow[d, "q_\#"] & 0 \\ 0 \arrow[r] & \mathbb{Z}\langle e_2^2\rangle \arrow[r] & 0 \arrow[r] & \mathbb{Z}\langle e^0\rangle \arrow[r] & 0 \end{tikzcd}$$

$H_2(X)=\mathbb{Z}\langle 3e_1^2-2e_2^2\rangle$. $q_*(3e_1^2-2e_2^2)=-2e_2^2$.

$q_*:H_2(X)\to H_2(S^2)$ is not isomorphism, so $q:X\to X/(S^1\cup_2 e_1^2)=S^2$ is not a homotopy equivalence.

Similar argument shows quotient map $X\to X/(S^1\cup_2 e_2^2)=S^2$ is not a homotopy equivalence.

14. (1) Let $\alpha_n:S^n\to S^n$ be antipodal map. If $f:S^n\to S^n$ is even, then $f=f\circ\alpha_n$, $\deg f=\deg f\cdot(-1)^{n+1}$.

If n is even, then $\deg f=0$. Assume n is odd in the followings. Let $\pi:S^n\to\mathbb{R}P^n$ be quotient map.

For even map $f:S^n\to S^n$, define $g:\mathbb{R}P^n\to S^n$ by $[x]\mapsto f(x)$, then $f=g\circ\pi$.

Consider quotient map $q:\mathbb{R}P^n\to\mathbb{R}P^n/\mathbb{R}P^{n-1}=S^n$, $q\circ\pi:S^n\to S^n$, $\deg(q\circ\pi)=2$.

$H_n(\mathbb{R}P^{n-1})\to H_n(\mathbb{R}P^n)\xrightarrow{q_*}H_n(\mathbb{R}P^n/\mathbb{R}P^{n-1})\to H_{n-1}(\mathbb{R}P^{n-1})$. n is odd, $H_n(\mathbb{R}P^{n-1})=0=H_{n-1}(\mathbb{R}P^{n-1})$.

q_* is isomorphism, $\deg(q\circ\pi)=2$, so $\pi_*(1)=2$, $f_*(1)=g_*\circ\pi_*(1)=g_*(2)=2g_*(1)$. f is even.

(2) For any even $2k$, by Example 2.31 there exists map $g:S^n\to S^n$ of degree k, then $g\circ q\circ\pi$ is of degree $2k$.

17. Let X, Y be CW complexes and $f : X \to Y$ be cellular map.

By naturality of singular homology (boundary map ∂_n), we have the following commutative diagram, where f_* is induced by singular homology by f.

$$\begin{tikzcd}
H_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] \arrow[d, "f_*"] \arrow[rr, bend left, "d_n"] & H_{n-1}(X^{n-1}) \arrow[r, "j_{n-1}"] \arrow[d, "f_*"] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[d, "f_*"] \\
H_n(Y^n, Y^{n-1}) \arrow[r, "\partial_n"] \arrow[rr, bend right, "d_n"'] & H_n(Y^{n-1}) \arrow[r, "j_{n-1}"] & H_{n-1}(Y^{n-1}, Y^{n-2})
\end{tikzcd}$$

f induces a chain map $f_\#$ between cellular chain complexes of X and Y, i.e. the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[r, "d_{n+1}"] \arrow[d, "f_\#"] & H_n(X^n, X^{n-1}) \arrow[r, "d_n"] \arrow[d, "f_\#"] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] \arrow[d, "f_\#"] & \cdots \\
\cdots \arrow[r] & H_{n+1}(Y^{n+1}, Y^n) \arrow[r, "d_{n+1}"] & H_n(Y^n, Y^{n-1}) \arrow[r, "d_n"] & H_{n-1}(Y^{n-1}, Y^{n-2}) \arrow[r] & \cdots
\end{tikzcd}$$

$f_\#$ induces a map $f_*^{CW} : H_n^{CW}(X) \to H_n^{CW}(Y)$.

$$\begin{tikzcd}
 & & & & 0 & & \\
 & 0 \arrow[dr] & & H_n(X^{n+1}) \arrow[ur] \arrow[r, "\cong"] & H_n(X) & & \\
 & & H_n(X^n) \arrow[ur] \arrow[dr, "j_n"] & & & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"'] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
 & & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"'] & & \\
 & & & 0 \arrow[ur] & & &
\end{tikzcd}$$

j_n is injective, $\operatorname{im}\partial_{n+1} \cong j_n(\operatorname{im}\partial_{n+1}) = \operatorname{im} d_{n+1}$, $H_n(X^n) \cong j_n(H_n(X^n)) = \operatorname{im} j_n = \ker \partial_n = \ker d_n$.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im}\partial_{n+1} \arrow[r] \arrow[d, "\cong"', "j_n"] & H_n(X^n) \arrow[r] \arrow[d, "\cong"', "j_n"] & H_n(X) \arrow[r] \arrow[d, "\cong"', "\varphi_X"] & 0 \\
0 \arrow[r] & \operatorname{im} d_{n+1} \arrow[r] & \ker d_n \arrow[r] & H_n^{CW}(X) \arrow[r] & 0
\end{tikzcd}$$

Isomorphism $\varphi_X : H_n(X) \to H_n^{CW}(X)$ is induced by j_n.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Im}\partial_{n+1} \arrow[rr] \arrow[dr, "f_*"] \arrow[dd, "\cong"', "j_n"] & & H_n(X^n) \arrow[rr] \arrow[dr, "f_*"] \arrow[dd, "\cong"', "j_n"] & & H_n(X) \arrow[rr] \arrow[dr, "f_*"] \arrow[dd, "\cong"', "\varphi_X"] & & 0 & \\
 & 0 \arrow[r] & \operatorname{Im}\partial_{n+1} \arrow[rr] \arrow[dd, "\cong"', "j_n"] & & H_n(Y^n) \arrow[rr] \arrow[dd, "\cong"', "j_n"] & & H_n(Y) \arrow[rr] \arrow[dd, "\cong"', "\varphi_Y"] & & 0 \\
0 \arrow[r] & \operatorname{Im} d_{n+1} \arrow[rr] \arrow[dr, "f_\#"] & & \operatorname{Ker} d_n \arrow[rr] \arrow[dr, "f_\#"] & & H_n^{CW}(X) \arrow[rr] \arrow[dr, "f_*^{CW}"] & & 0 & \\
 & 0 \arrow[r] & \operatorname{Im} d_{n+1} \arrow[rr] & & \operatorname{Ker} d_n \arrow[rr] & & H_n^{CW}(Y) \arrow[rr] & & 0
\end{tikzcd}$$

The front, back, top, bottom and the left cube of this diagram commute, so the right cube must commute, i.e. the isomorphism between singular and cellular homology is natural.

18. Consider long exact sequences for good pairs $(X^n \cup A^{n+1}, X^{n-1} \cup A^n)$ and note that

$H_n(X^n \cup A^{n+1}, X^{n-1} \cup A^n) \cong \widetilde{H}_n(X^n \cup A^{n+1}/X^{n-1} \cup A^n) \cong \widetilde{H}_n(X^n/X^{n-1} \cup A^n) = \widetilde{H}_n(X^n, X^{n-1} \cup A^n)$.

$$\begin{tikzcd}
 & & H_{n-1}(X^{n-1}\cup A^n) \arrow[dr, "j_{n-1}"] & & \\
\cdots \arrow[r, "d_{n+1}"] & H_n(X^n \cup A^{n+1}, X^{n-1}\cup A^n) \arrow[ur, "\partial_n"] \arrow[rr, "d_n"] & & H_{n-1}(X^{n-1}\cup A^n, X^{n-2}\cup A^{n-1}) \arrow[r, "d_{n-1}"] & \cdots \\
\cdots \arrow[r, "d_{n+1}"] & H_n(X^n, X^{n-1}\cup A^n) \arrow[u, "\cong"'] \arrow[rr, "d_n"] & & H_{n-1}(X^{n-1}, X^{n-2}\cup A^{n-1}) \arrow[u, "\cong"'] \arrow[r, "d_{n-1}"] & \cdots
\end{tikzcd}$$

19. The standard CW structure of $\mathbb{R}P^n/\mathbb{R}P^m$ consists one k-cell for $m+1 \leqslant k \leqslant n$ and one 0-cell.

$0 \to \underbrace{\mathbb{Z} \xrightarrow{d_n} \cdots \xrightarrow{d_{m+2}} \mathbb{Z}}_{n-m} \xrightarrow{d_{m+1}} \underbrace{0 \xrightarrow{d_m} 0 \to \cdots \to 0}_{m} \xrightarrow{d_1} \mathbb{Z} \to 0.$

$d_k = 2$ for k even and $m+1 \leqslant k \leqslant n$, $d_k = 0$ otherwise.

$H_i(\mathbb{R}P^n/\mathbb{R}P^m) = \mathbb{Z}_2$ for i odd and $m+1 \leqslant i < n$, $H_i(\mathbb{R}P^n/\mathbb{R}P^m) = \mathbb{Z}$ for $i = 0$, n (n odd) and $m+1$ (m odd).

$H_i(\mathbb{R}P^n/\mathbb{R}P^m) = 0$ otherwise.

20. Let b_i^X, b_j^Y, c_k be Betti numbers of X, Y and $X \times Y$ respectively.

Note that each k-cell in $X \times Y$ is the product of an i-cell in X and j-cell in Y with $i+j=k$, so $c_k = \sum_{i+j=k} b_i^X b_j^Y$.

$\chi(X \times Y) = \sum_k (-1)^k c_k = \sum_k (-1)^k \sum_{i+j=k} b_i^X b_j^Y = \sum_{i,j} (-1)^{i+j} b_i^X b_j^Y = \sum_i (-1)^i b_i^X \cdot \sum_j (-1)^j b_j^Y = \chi(X)\chi(Y).$

21. Let b_n^X, b_n^A, b_n^B, $b_n^{A\cap B}$ be Betti numbers of X, A, B and $A \cap B$ respectively, then $b_n^X = b_n^A + b_n^B - b_n^{A\cap B}$.

$\chi(X) = \sum\limits_n (-1)^n b_n^X = \sum\limits_n (-1)^n b_n^A + \sum\limits_n (-1)^n b_n^B - \sum\limits_n (-1)^n b_n^{A\cap B} = \chi(A) + \chi(B) - \chi(A \cap B).$

23. M_g is compact, so $M_g \to M_h$ is finite sheeted. Let $M_g \to M_h$ be n-sheeted. $\chi(M_g) = 2-2g$, $\chi(M_h) = 2-2h$.

$\chi(M_g) = n\chi(M_h)$, $2-2g = n(2-2h)$, so $g = n(h-1)+1$.

24. (1) The first graph is K_5 with 5 vertices and 10 edges. $\chi(K_5) = -5$.

If K_5 is 1-skeleton of S^2, then from $\chi(S^2) = 2$, S^2 has 7 polygons with 20 edges in total.

Let $n_1, \ldots, n_7$ be the number of edges of polygons, $n_1 + \cdots + n_7 = 20$.

We must have $n_i = 2$ for some i, which means two of vertices of K_5 are connected by two edges. Contradiction.

(2) The second graph is $K_{3,3}$ with 6 vertices and 9 edges. $\chi(K_{3,3}) = -3$.

If $K_{3,3}$ is 1-skeleton of S^2, then S^2 has 5 polygons with 18 edges in total. Notice that a circle in $K_{3,3}$ contains at least 4 edges, so we need at least 4 edges to bound a polygon, and 5 polygons need 20 edges. Contradiction.

25. Existence: $\varphi_n(X) = n \cdot (\chi(X) - 1)$ has the desired properties.

Let φ_n denote the function φ for $n \in \mathbb{Z}$. For CW complex A and B, $(A \vee B)/A = B$, $\varphi_n(A \vee B) = \varphi_n(A) + \varphi_n(B)$.

For $S^{k-1} \subseteq S^k$ as equator, $S^k/S^{k-1} = S^k \vee S^k$, $\varphi_n(S^k) = \varphi_n(S^{k-1}) + 2\varphi_n(S^k)$, $\varphi_n(S^k) = -\varphi_n(S^{k-1}) = (-1)^k \cdot n$.

Suppose finite CW complex X has c_i i-cells, c_i is nonzero for finitely many i.

$\varphi_n(X^k) = \varphi_n(X^{k-1}) + \varphi_n(\bigvee_\alpha S_\alpha^k) = \varphi_n(X^{k-1}) + c_k \cdot \varphi_n(S^k) = \varphi_n(X^{k-1}) + n \cdot (-1)^k c_k$. $\varphi_n(\{*\}) = 0$.

By induction, we have $\varphi_n(X) = n \cdot (\chi(X) - 1)$. The uniqueness is guaranteed by property (b)(c) via calculation.

26. (a) ($\Rightarrow$) If $r : X \cup CA \to X \cup CA$ is retraction, then $f_t(a) = r([a,t]), a \in A, t \in I$ is the homotopy.

($\Leftarrow$) Define $r : X \cup CA \to X \cup CA$ by $r([a,t]) = f_t(a)$ for $a \in A, t \in I$, $r(x) = x$ for $x \in X$.

(b) If A is contractible in X, then we have retraction $r : X \cup CA \to X$. $\widetilde{H}_n(X \cup CA) \cong \widetilde{H}_n(X) \oplus \widetilde{H}_n(X \cup CA/X)$.

$\widetilde{H}_n(X \cup CA) \cong H_n(X,A)$. $(X \cup CA)/X \simeq SA$, $\widetilde{H}_n(X \cup CA/X) \cong \widetilde{H}_n(SA) \cong \widetilde{H}_{n-1}(A)$.

27. Given $A \subseteq X$, $C_n(X,A) := C_n(X)/C_n(A)$. $0 \to C_n(A) \to C_n(X) \to C_n(X,A) \to 0$ is exact by definition.

$0 \to H_n(A) \to H_n(X) \to H_n(X,A) \to 0$ is exact if boundary homomorphisms $\partial : H_n(X,A) \to H_{n-1}(X)$ are zero.

Let $C'_n(X,A)$ be subgroup of $C_n(X)$ generated by singular n-simplices $\sigma : \Delta^n \to X$ whose image isn't contained in A.

Every element σ in $C_n(X)$ has a unique decomposition $\sigma = \sigma_1 + \sigma_2$, where $\sigma_1 \in C_n(A)$ and $\sigma_2 \in C'_n(X,A)$.

$C_n(X) \cong C_n(A) \oplus C'_n(X,A)$, $C'_n(X,A) \cong C_n(X)/C_n(A)$ and we have isomorphism $\varphi : C_n(X,A) \xrightarrow{\cong} C'_n(X,A)$.

However, boundary map ∂ doesn't take $C'_n(X,A)$ to $C'_{n-1}(X,A)$, since for $\sigma \in C'_n(X,A)$, $\partial\sigma$ may have faces in A.

Thus $0 \to C_n(A) \to C_n(X) \to C_n(X,A) \to 0$ only splits as graded abelian groups, not as a chain complex, which is not enough to induce a split on homology.

28. Related: Exercise **1.3.21**

(a) Let X be the space in question, Y be the Möbius band and $N \simeq S^1$ be a neighborho od of the identified circle in X, then $A = T^2 \cup N \simeq T^2$, $B = Y \cup N \simeq Y \simeq S^1$, A, B are open in X and $X = A \cup B$.

Consider MV sequence for reduced homology groups:

$$\begin{tikzcd}
\widetilde{H}_2(N) \arrow[r] & \widetilde{H}_2(A) \oplus \widetilde{H}_2(B) \arrow[r, "\phi"] & \widetilde{H}_2(X) \arrow[r, "\psi"] & \widetilde{H}_1(N) \arrow[r, "\varphi"] & \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \arrow[r] & \widetilde{H}_1(X) \arrow[r] & 0 \\
\widetilde{H}_2(S^1) \arrow[u, "\cong"'] & \widetilde{H}_2(T^2) \oplus \widetilde{H}_2(S^1) \arrow[u, "\cong"'] & & \widetilde{H}_1(S^1) \arrow[u, "\cong"'] & \widetilde{H}_1(T^2) \oplus \widetilde{H}_1(S^1) \arrow[u, "\cong"'] & & \\
0 \arrow[u, "\cong"'] & \mathbb{Z}\langle a \rangle \arrow[u, "\cong"'] & & \mathbb{Z}\langle b \rangle \arrow[u, "\cong"'] & \mathbb{Z}\langle c, d \rangle \oplus \mathbb{Z}\langle e \rangle \arrow[u, "\cong"'] & &
\end{tikzcd}$$

$\phi(b) = c + 2e$, ϕ is injective, so $\psi = 0$, ϕ is isomorphism. $\widetilde{H}_2(X) \cong \mathbb{Z}$.

$\operatorname{im}\varphi = \mathbb{Z}\langle c+2e \rangle$, $\widetilde{H}_1(X) \cong \mathbb{Z}\langle c, d \rangle \oplus \mathbb{Z}\langle e \rangle / \mathbb{Z}\langle c+2e \rangle = \mathbb{Z}\langle c+2e, d, e \rangle / \mathbb{Z}\langle c+2e \rangle = \mathbb{Z}^2$.

X is path connected, so $H_0(X) = \mathbb{Z}$. $H_1(X) = \mathbb{Z}^2$, $H_2(X) = \mathbb{Z}$, $H_n(X) = 0$ for $n \geqslant 3$.

(b) Let X be the space in question, Y be the Möbius band and $N \simeq \mathbb{R}P^1 \simeq S^1$ be a neighborhood of the identified circle in X, then $A = \mathbb{R}P^2 \cup N \simeq \mathbb{R}P^2$, $B = Y \cup N \simeq Y \simeq S^1$, A, B are open in X and $X = A \cup B$.

Consider MV sequence for reduced homology groups:

$$\begin{tikzcd}
\widetilde{H}_2(N) \arrow[r] & \widetilde{H}_2(A) \oplus \widetilde{H}_2(B) \arrow[r] & \widetilde{H}_2(X) \arrow[r, "\psi"] & \widetilde{H}_1(N) \arrow[r, "\varphi"] & \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \arrow[r] & \widetilde{H}_1(X) \arrow[r] & 0 \\
\widetilde{H}_2(S^1) \arrow[u, "\cong"'] & H_2(\mathbb{R}P^2) \oplus H_2(S^1) \arrow[u, "\cong"'] & & \widetilde{H}_1(S^1) \arrow[u, "\cong"'] & \widetilde{H}_1(\mathbb{R}P^2) \oplus H_1(S^1) \arrow[u, "\cong"'] & & \\
0 \arrow[u, "\cong"'] & 0 \arrow[u, "\cong"'] & & \mathbb{Z}\langle a \rangle \arrow[u, "\cong"'] & \mathbb{Z}_2\langle b \rangle \oplus \mathbb{Z}\langle c \rangle \arrow[u, "\cong"'] & &
\end{tikzcd}$$

$\varphi(a) = b + 2c$, φ is injective, so $\psi = 0$, $\widetilde{H}_2(X) = 0$.

$\widetilde{H}_1(X) \cong \mathbb{Z}_2\langle b \rangle \oplus \mathbb{Z}\langle c \rangle / \mathbb{Z}\langle b + 2c \rangle = \langle b, c \mid b^2 = 1, bc = cb, bc^2 = 1 \rangle = \langle c \mid c^4 = 1 \rangle = \mathbb{Z}_4$.

X is path connected, so $H_0(X) = \mathbb{Z}$. $H_1(X) = \mathbb{Z}^4$, $H_n(X) = 0$ for $n \geqslant 2$.

29. (1) R deformation retracts to $\bigvee_g S^1$. Let two copies of R be R_1 and R_2. Let A, B be neighborhood of R_1 and R_2 s.t. A, B are open in X and deformation retract to R_1 and R_2 respectively. X is path-connected, $H_0(X) = \mathbb{Z}$. $A \simeq R_1 \simeq \bigvee_g S^1$, $B \simeq R_2 \simeq \bigvee_g S^1$. $A \cap B \simeq M_g$, $X = A \cup B$. We have MV sequence:

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{H}_3(A) \oplus \widetilde{H}_3(B) \arrow[r] & \widetilde{H}_3(X) \arrow[r, "\varphi"] & \widetilde{H}_2(A \cap B) \arrow[r] & \widetilde{H}_2(A) \oplus \widetilde{H}_2(B) \arrow[r] & \widetilde{H}_2(X) \arrow[r] & \cdots \\
& \widetilde{H}_3(R_1) \oplus \widetilde{H}_3(R_2) \arrow[u, "\cong"'] & & \widetilde{H}_2(M_g) \arrow[u, "\cong"'] & H_2(\bigvee_g S^1) \oplus H_2(\bigvee_g S^1) \arrow[u, "\cong"'] & & \\
& 0 \arrow[u, "\cong"'] & & \mathbb{Z} \arrow[u, "\cong"'] & 0 \arrow[u, "\cong"'] & &
\end{tikzcd}$$

φ is isomorphism, $\widetilde{H}_3(X) \cong \mathbb{Z}$.

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{H}_2(X) \arrow[r, "\psi"] & \widetilde{H}_1(A \cap B) \arrow[r, "\varphi"] & \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \arrow[r, "\gamma"] & \widetilde{H}_1(X) \arrow[r] & 0 \\
& & \widetilde{H}_1(M_g) \arrow[u, "\cong"'] & \widetilde{H}_1(\bigvee_g S^1) \oplus H_1(\bigvee_g S^1) \arrow[u, "\cong"'] & & \\
& & \mathbb{Z}\langle a_1, b_1, \ldots, a_g, b_g \rangle \arrow[u, "\cong"'] & \mathbb{Z}\langle c_1, \ldots, c_g \rangle \oplus \mathbb{Z}\langle d_1, \ldots, d_g \rangle \arrow[u, "\cong"'] & &
\end{tikzcd}$$

$\varphi(a_i) = c_i + d_i$, $\varphi(b_i) = 0$. $\widetilde{H}_2(X) \cong \psi(\widetilde{H}_2(X)) = \operatorname{im}\psi = \ker\varphi = \mathbb{Z}\langle b_1, \ldots, b_g \rangle = \mathbb{Z}^g$.

$\widetilde{H}_1(X) \cong \widetilde{H}_1(A) \oplus \widetilde{H}_1(B)/\ker\gamma = \widetilde{H}_1(A) \oplus \widetilde{H}_1(B)/\operatorname{im}\varphi \cong \mathbb{Z}\langle c_1, \ldots, c_g, d_1, \ldots, d_g \rangle / \mathbb{Z}\langle c_1 + d_1, \ldots, c_g + d_g \rangle = \mathbb{Z}^g$.

$H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}^g$, $H_2(X) = \mathbb{Z}^g$, $H_3(X) = \mathbb{Z}$, $H_n(X) = 0$ for $n \geqslant 4$.

(2) Consider long exact sequence for good pair (R, M_g).

$$\begin{tikzcd}
\widetilde{H}_3(R) \arrow[r] & \widetilde{H}_3(R, M_g) \arrow[r] & \widetilde{H}_2(M_g) \arrow[r] & \widetilde{H}_2(R) \arrow[r] & \widetilde{H}_2(R, M_g) \\
\widetilde{H}_3(\bigvee_g S^1) \cong 0 \arrow[u, "\cong"'] & & \mathbb{Z} \arrow[u, "\cong"'] & \widetilde{H}_2(\bigvee_g S^1) \cong 0 \arrow[u, "\cong"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\widetilde{H}_2(R, M_g) \arrow[r, "\varphi"] & \widetilde{H}_1(M_g) \arrow[r, "\psi"] & \widetilde{H}_1(R) \arrow[r] & \widetilde{H}_1(R, M_g) \arrow[r] & 0 \\
& \mathbb{Z}^{2g} \arrow[u, "\cong"'] & \widetilde{H}_1(\bigvee_g S^1) \cong \mathbb{Z}^g \arrow[u, "\cong"'] & &
\end{tikzcd}$$

$\widetilde{H}_3(R, M_g) \cong \widetilde{H}_2(M_g) = \mathbb{Z}$. ψ is surjective, $\ker\psi = \mathbb{Z}^g$, $\widetilde{H}_2(R, M_g) \cong \operatorname{im}\varphi = \ker\psi = \mathbb{Z}^g$.

$\widetilde{H}_1(R, M_g) \cong \widetilde{H}_1(R)/\operatorname{im}\psi = 0$. $H_0(R, M_g) \cong \widetilde{H}_0(R/M_g) = 0$.

$H_0(R, M_g) = 0$, $H_1(R, M_g) = 0$, $H_2(R, M_g) = \mathbb{Z}^g$, $H_3(R, M_g) = \mathbb{Z}$, $H_n(R, M_g) = 0$ for $n \geqslant 4$.

30. (0) Prerequisites: Suppose we have exact A-modules sequence $M_{i-2} \xrightarrow{f_{i-1}} M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \xrightarrow{f_{i+2}} M_{i+2}$.

$\operatorname{im} f_i \cong M_{i-1}/\ker f_i = M_{i-1}/\operatorname{im} f_{i-1} = \operatorname{coker} f_{i-1}$. $M_i/\operatorname{im} f_i = M_i/\ker f_{i+1} \cong \operatorname{im} f_{i+1} = \ker f_{i+2}$.

$0 \to \operatorname{im} f_i \hookrightarrow M_i \twoheadrightarrow M_i/\operatorname{im} f_i \to 0$ is exact, so $0 \to \operatorname{coker} f_{i-1} \to M_i \to \ker f_{i+2} \to 0$ is exact.

If in addition M_i's are $\mathbb{Z}$-modules / abelian groups and $\ker f_{i+2}$ is free, then $M_i \cong \operatorname{coker} f_{i-1} \oplus \ker f_{i+2}$.

(1) $H_0(S^2) = \mathbb{Z}$, $H_1(S^2) = 0$, $H_2(S^2) = \mathbb{Z}$, $H_n(S^2) = 0$ for $n \geqslant 3$.

$H_0(S^1 \times S^1) = \mathbb{Z}$, $H_1(S^1 \times S^1) = \mathbb{Z}^2$, $H_2(S^1 \times S^1) = \mathbb{Z}$, $H_n(S^1 \times S^1) = 0$ for $n \geqslant 3$.

$0 \to H_3(T_f) \to H_2(X) \xrightarrow{\mathrm{id}-f_*} H_2(X) \to H_2(T_f) \to H_1(X) \xrightarrow{\mathrm{id}-f_*} H_1(X) \to H_1(T_f) \to H_0(X)$ is exact.

(a) For reflection $f : S^2 \to S^2$, $\deg(\mathrm{id} - f_*) = 2$. $H_0(T_f) = \mathbb{Z}$, $H_1(T_f) = \mathbb{Z}$, $H_2(T_f) = \mathbb{Z}_2$, $H_n(T_f) = 0$ for $n \geqslant 3$.

(b) $f : S^2 \to S^2$ has degree 2. $H_0(T_f) = \mathbb{Z}$, $H_1(T_f) = \mathbb{Z}$, $H_n(T_f) = 0$ for $n \geqslant 3$.

(c) $f : S^1 \times S^1 \to S^1 \times S^1$ is given by matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. $\det A = -1$.

Similar to Exercise **2.2.7**, $\mathrm{id} - f_* : H_2(S^1 \times S^1) \to H_2(S^1 \times S^1)$ maps 1 to $1 - \operatorname{sign}(\det A) = 2$.

$H_0(T_f) = \mathbb{Z}$, $H_1(T_f) = \mathbb{Z}_2 \oplus \mathbb{Z}^2$, $H_2(T_f) = \mathbb{Z}_2 \oplus \mathbb{Z}$, $H_n(T_f) = 0$ for $n \geqslant 3$.

(d) $f : S^1 \times S^1 \to S^1 \times S^1$ is given by matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. $\mathrm{id} - f_* : H_2(S^1 \times S^1) \to H_2(S^1 \times S^1)$ maps 1 to $1 - \operatorname{sign}(\det A) = 0$.

$H_0(T_f) = \mathbb{Z}$, $H_1(T_f) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$, $H_2(T_f) = \mathbb{Z}$, $H_3(T_f) = \mathbb{Z}$, $H_n(T_f) = 0$ for $n \geqslant 4$.

(e) $f : S^1 \times S^1 \to S^1 \times S^1$ is given by matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. $\mathrm{id} - f_* : H_2(S^1 \times S^1) \to H_2(S^1 \times S^1)$ maps 1 to $1 - \operatorname{sign}(\det A) = 0$.

$H_0(T_f) = \mathbb{Z}$, $H_1(T_f) = \mathbb{Z}_2 \oplus \mathbb{Z}$, $H_2(T_f) = \mathbb{Z}$, $H_3(T_f) = \mathbb{Z}$, $H_n(T_f) = 0$ for $n \geqslant 4$.

31. Suppose for $x_0 \in U \subseteq X$, $y_0 \in V \subseteq Y$, $X \vee Y = X \coprod Y/(x_0 \sim y_0)$ and U, V deformation retract to x_0 and y_0.

Let $A = X \cup V$, $B = Y \cup U$, then $A \simeq X$, $B \simeq Y$ and $A \cup B = X \vee Y$, $A \cap B = U \vee V \simeq \{*\}$, $\widetilde{H}_n(A \cap B) = 0$.

MV sequence $\widetilde{H}_n(A \cap B) \to \widetilde{H}_n(A) \oplus \widetilde{H}_n(B) \to \widetilde{H}_n(A \cup B) \to \widetilde{H}_{n-1}(A \cap B)$ gives $\widetilde{H}_n(X) \oplus \widetilde{H}_n(Y) \cong \widetilde{H}_n(X \vee Y)$.

32. Suppose N is a neighborhood of X in SX that deformation retracts to X.

$SX = CX \cup CX$, let A be union of the first cone CX with N and B be union of the second cone CX with N.

$A \simeq CX$, $B \simeq CX$. $\widetilde{H}_n(A) = 0$, $\widetilde{H}_n(B) = 0$. $A \cup B = SX$, $A \cap B = N \simeq X$.

MV sequence $\widetilde{H}_n(A) \oplus \widetilde{H}_n(B) \to \widetilde{H}_n(A \cup B) \to \widetilde{H}_{n-1}(A \cap B) \to \widetilde{H}_{n-1}(A) \oplus \widetilde{H}_{n-1}(B)$ gives $\widetilde{H}_n(SX) \cong \widetilde{H}_{n-1}(X)$.

33. (1) Let $X_k = A_1 \cup \cdots \cup A_k$, $Y_k = A_k \cap \cdots \cap A_n$. Define $Y_{n+1} = X$.

Prove by induction on k that $\widetilde{H}_i(X_k \cap Y_{k+1}) = 0$ for $1 \leqslant k \leqslant n$ and $i \geqslant k - 1$.

By assumption, this holds for $k = 1$. Suppose it holds for $k = j - 1$, i.e. $\widetilde{H}_i(X_{j-1} \cap Y_j) = 0$ for $i \geqslant j - 2$.

$X_j \cap Y_{j+1} = (X_{j-1} \cup A_j) \cap Y_{j+1} = (X_{j-1} \cap Y_{j+1}) \cup (A_j \cap Y_{j+1}) = (X_{j-1} \cap Y_{j+1}) \cup Y_j$. $(X_{j-1} \cap Y_{j+1}) \cap Y_j = X_{j-1} \cap Y_j$.

$\widetilde{H}_i(X_{j-1} \cap Y_j) \to \widetilde{H}_i(X_{j-1} \cap Y_{j+1}) \oplus \widetilde{H}_i(Y_j) \to \widetilde{H}_i(X_j \cap Y_{j+1}) \to \widetilde{H}_{i-1}(X_{j-1} \cap Y_j)$ is exact. $\widetilde{H}_i(Y_j) = 0$ by assumption.

For $i \geqslant j - 1$, $\widetilde{H}_i(X_{j-1} \cap Y_j) = 0$, $\widetilde{H}_{i-1}(X_{j-1} \cap Y_j) = 0$, thus $\widetilde{H}_i(X_j \cap Y_{j+1}) \cong \widetilde{H}_i(X_{j-1} \cap Y_{j+1}) \cong \cdots \cong 0$.

The procedure above is also valid for $j = n$, so $\widetilde{H}_i(X_k \cap Y_{k+1}) = 0$ for $1 \leqslant k \leqslant n$ and $i \geqslant k - 1$.

Especially for $k = n$, $X_n \cap Y_{n+1} = X$, we have $\widetilde{H}_i(X) = 0$ for $i \geqslant n - 1$.

(2) Consider boundary of an $(n-1)$-simplex, which is homeomorphic to S^{n-2}. It has n faces of dimension $n - 2$.

Let n open sets be small neighborhood of these n faces respectively, then their non-empty intersections will be neighborhoods of lower dimensional faces which are contractible.

35. Suppose $H_1(X)$ contains torsion and X embeds into $\mathbb{R}^3$ s.t. N is a neighborhood of X and N is homeomorphic to M, where M is the mapping cylinder of $M_g \to X$, M_g is closed orientable surface of genus g.

M deformation retracts to X, so $N \simeq X$. Let $A = \mathbb{R}^3 - X$, $B = N$. $A \cap B = N - X \simeq M_g \times [0,1) \simeq M_g$, $A \cup B = \mathbb{R}^3$.

From the following reduced MV sequence, we have $\widetilde{H}_n(M_g) \cong \widetilde{H}_n(\mathbb{R}^3 - X) \oplus \widetilde{H}_n(X)$.

$$\begin{array}{ccccccc}
\widetilde{H}_{n+1}(\mathbb{R}^3) & \longrightarrow & \widetilde{H}_n(A\cap B) & \longrightarrow & \widetilde{H}_n(A) \oplus \widetilde{H}_n(B) & \longrightarrow & \widetilde{H}_n(\mathbb{R}^3) \\
\uparrow \cong & & \uparrow \cong & & \uparrow \cong & & \uparrow \cong \\
0 & & \widetilde{H}_n(M_g) & & \widetilde{H}_n(\mathbb{R}^3 - X) \oplus \widetilde{H}_n(X) & & 0
\end{array}$$

For $n=1$, $H_1(M_g) = \mathbb{Z}^{2g}$ but $H_1(X)$ has a torsion. Contradiction.

36. (1) Let $x_0 \in S^n$, $r : S^n \to \{x_0\}$ be retraction, then $\mathrm{id} \times r : X \times S^n \to X \times \{x_0\}$ is retraction and we have $H_i(X \times S^n) \cong H_i(X \times \{x_0\}) \oplus H_i(X \times S^n, X \times \{x_0\}) \cong H_i(X) \oplus H_i(X \times S^n, X \times \{x_0\})$.

(2) Let $A = X \times D^n_+$, $B = X \times D^n_-$ s.t. $x_0 \in D^n_+ \cap D^n_- = S^{n-1}$. Let $C = D = X \times \{x_0\}$, then $C \subseteq A$, $D \subseteq B$.

$A \simeq X$, $B \simeq X$. $A \cap B = X \times S^{n-1}$, $A \cup B = X \times S^n$. $C \cap D = C \cup D = X \times \{x_0\}$.

MV sequence $H_i(A,C) \oplus H_i(B,D) \to H_i(A\cup B, C \cup D) \to H_{i-1}(A \cap B, C \cap D) \to H_{i-1}(A,C) \oplus H_{i-1}(B,D)$ gives $H_i(X \times S^n, X \times \{x_0\}) \cong H_{i-1}(X \times S^{n-1}, X \times \{x_0\}) \cong \cdots \cong H_{i-n}(X \times S^0, X \times \{x_0\}) \cong H_{i-n}(X)$.

38. We have the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r] \arrow[d] & A_n \arrow[r, "f_1"] \arrow[d, equal] & B_n \arrow[r, "f_2"] \arrow[d, "g_1"'] & C_n \arrow[r] \arrow[d, "g_2"] & A_{n-1} \arrow[r, "f_3"] \arrow[d, equal] & B_{n-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & E_{n+1} \arrow[r, "h_1"'] & A_n \arrow[r] & D_n \arrow[r, "h_2"'] & E_n \arrow[r, "h_3"'] & A_{n-1} \arrow[r] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

This yields exact sequence $\cdots \to E_{n+1} \xrightarrow{f_1 \circ h_1} B_n \xrightarrow{\binom{f_2}{g_1}} C_n \oplus D_n \xrightarrow{(g_2, -h_2)} E_n \xrightarrow{f_3 \circ h_3} B_{n-1} \to \cdots$

The exactness of this sequence can be verified via diagram chasing.

39. Let $(X,Y) = (A \cup B, C \cup D)$ be CW pairs.

(1) For $A = B$, consider long exact sequences for triples $(A, C, C \cap D)$ and $(A, C \cup D, D)$. $H_i(C, C\cap D) \cong H_i(C \cup D, D)$.

$$\begin{array}{l}
\to H_{n+1}(A,C) \to H_n(C, C\cap D) \to H_n(A, C\cap D) \to H_n(A,C) \to H_{n-1}(C, C\cap D) \to H_{n-1}(A, C \cap D) \to \\
\to H_{n+1}(A, C\cup D) \to H_n(C \cup D, D) \to H_n(A,D) \to H_n(A, C \cup D) \to H_{n-1}(C \cup D, D) \to H_{n-1}(C \cup D, D) \to
\end{array}$$

From Exercise **2.2.38**, we have the following relative MV sequences for $(X,Y) = (A \cup B, C \cup D)$ with $A = B$.

$\cdots \to H_{n+1}(A, C \cup D) \to H_n(A, C \cap D) \to H_n(A,C) \oplus H_n(A,D) \to H_n(A, C \cup D) \to H_{n-1}(C, C \cap D) \to \cdots$

(2) For $C = D$, consider long exact sequences for triples $(A, A \cap B, C)$ and $(A \cup B, B, C)$. $H_i(A, A \cap B) \cong H_i(A \cup B, B)$.

$$\begin{array}{l}
\to H_{n+1}(A,C) \to H_{n+1}(A, A\cap B) \to H_n(A \cap B, C) \to H_n(A,C) \to H_n(A, A\cap B) \to H_{n-1}(A \cap B, C) \to \\
\to H_{n+1}(A \cup B, C) \to H_{n+1}(A \cup B, B) \to H_n(B,C) \to H_n(A \cup B, C) \to H_n(A \cup B, B) \to H_{n-1}(B,C) \to
\end{array}$$

We have the following relative MV sequences for $(X,Y) = (A \cup B, C \cup D)$ with $C = D$.

$\cdots \to H_{n+1}(A \cup B, C) \to H_n(A \cap B, C) \to H_n(A,C) \oplus H_n(B,C) \to H_n(A \cup B, C) \to H_{n-1}(A \cap B, C) \to \cdots$

40. (1) From chain complexes $0 \to C_i(X) \xrightarrow{n} C_i(X) \longrightarrow C_i(X;\mathbb{Z}_n) \to 0$, we have long exact sequence

$\cdots \to H_i(X) \xrightarrow{n} H_i(X) \longrightarrow H_i(X;\mathbb{Z}_n) \longrightarrow H_{i-1}(X) \xrightarrow{n} H_{i-1}(X) \to \cdots$

From prerequisite in Exercise **2.2.30**, $0 \to H_i(X)/nH_i(X) \to H_i(X;\mathbb{Z}_n) \to n\text{-Torsion}(H_{i-1}(X)) \to 0$ is exact, where $n\text{-Torsion}(G) = \ker(G \xrightarrow{n} G)$.

(2) ($\Rightarrow$) If $\widetilde{H}_i(X;\mathbb{Z}_n) = 0$ for all i and all primes p, then $\widetilde{H}_i(X) \xrightarrow{p} \widetilde{H}_i(X)$ is isomorphism.

For any $x \in \widetilde{H}_i(X)$ and $p/q \in \mathbb{Q}$, p, q primes, $px \in \widetilde{H}_i(X)$ and these exists unique $y \in \widetilde{H}_i(X)$ s.t. $qy = px$.

Let $p/q \cdot x$ be y. $\widetilde{H}_i(X)$ is abelian group and addition is already defined, thus $H_i(X)$ is vector space over $\mathbb{Q}$.

($\Leftarrow$) If $H_i(X)$ is vector space over $\mathbb{Q}$ for all i, then $\widetilde{H}_i(X) \xrightarrow{p} \widetilde{H}_i(X)$ is isomorphism for all i and all primes p.

$H_i(X)/pH_i(X) = 0$, $p\text{-Torsion}(H_{i-1}(X)) = 0$, so $\widetilde{H}_i(X;\mathbb{Z}_p) = 0$ for all i and all primes p.

41. For finite CW complex X, suppose c_i is the number of i-cells in X. We have the following cellular chain complex $0 \to H_n(X^n, X^{n-1};F) \to H_{n-1}(X^{n-1}, X^{n-2};F) \to \cdots \to 0$, where $H_i(X^i, X^{i-1};F) \cong F^{c_i}$.

$\chi(X) = \sum_i c_i = \sum_i \dim H_i(X^i, X^{i-1};F) = \sum_i \dim H_i^{CW}(X;F) = \sum_i \dim H_i(X;F)$.

Generalization: Suppose X has finite integral homology, i.e. finite number of nonzero homology groups, which are all finitely generated. Let n be the top dimension of non-vanishing homology, F be a field.

(1) $\operatorname{char} F = 0$. Let b_i be the i-th Betti number of X, i.e. $H_i(X;\mathbb{Z}) = \mathbb{Z}^{b_i} \oplus T$, where T is the torsion subgroup.

$\chi(X,\mathbb{Z}) = \sum\limits_i (-1)^i b_i$. From universal coefficient theorem, $H_i(X;F) = (H_i(X;\mathbb{Z}) \otimes F) \oplus \operatorname{Tor}(H_{i-1}(X;\mathbb{Z}), F)$.

$\operatorname{char} F = 0$, so Tor-term vanishes, $H_i(X;F) = F^{b_i}$. It follows that $\chi(X,\mathbb{Z}) = \chi(X,F)$.

(2) $\operatorname{char} F \neq 0$. Suppose $H_i(X;\mathbb{Z}) = \mathbb{Z}^{b_i} \oplus (\mathbb{Z}/p\mathbb{Z})^{c_i^p} \oplus T_i^p$, where T_i^p is the torsion part which is not p-torsion.

The universal coefficient theorem gives: $H_i(X;F) = \begin{cases} F^{b_0} & i = 0 \\ F^{b_i + c_i^p + c_{i-1}^p} & 1 \leqslant i \leqslant n \\ F^{c_n^p} & i = n+1 \end{cases}$

$\chi(X;F) = b_0 - (b_1 + c_1^p + c_0^p) + \cdots + (-1)^n (b_n + c_n^p + c_{n-1}^p) + (-1)^{n+1} c_n^p$.

Each c_i^p cancels with the one in the next factor, so all is left is $\chi(X;F) = \sum_i (-1)^i b_i = \chi(X,\mathbb{Z})$.

42. (1) $H_1(X;\mathbb{Z})$ is of rank $n > 1$, so $X \simeq \bigvee_n S^1$. Consider $X = \bigvee_n S^1$ first.

To show $\phi : G \to \mathrm{GL}_n(\mathbb{Z})$ is injective, suppose $g : X \to X$ is homeomorphism s.t. $\phi(g) = \mathrm{id}$, then g maps each S^1 to itself and fixes the wedge point x_0. Let $f = g|_{S^1} : S^1 \to S^1$, then f fixes x_0 and $f_* = \mathrm{id}$, so f preserves the orientation. G is finite group, so f is of finite order and there exists a smallest positive integer k s.t. $f^k = \mathrm{id}$.

Let $y \in S^1$, $f(y) \neq y$, then points $y, f(y), f^2(y), \cdots, f^k(y) = y$ are permuted in S^1 clockwise or counterclockwise since f preserves the orientation. Arc between $f^i(y)$ and $f^{i+1}(y)$ is mapped by f to the next one, and such arcs cover S^1, so one of these arcs contains x_0, but f fixes x_0. Contradiction. Thus $f = \mathrm{id} : S^1 \to S^1$ and $g = \mathrm{id} : \bigvee_n S^1 \to \bigvee_n S^1$.

(2) For general finite connected graph $X \simeq \bigvee_n S^1$ ($n \geqslant 2$), there exists a vertex x_0 of valence $\geqslant 3$.

x_0 belongs to different loops based at x_0, and g maps loops to themselves and preserves the orientation, so g fixes x_0, and the followings are the same as the situation for $\bigvee_n S^1$.

(3) For coefficient group $\mathbb{Z}_m$, $\phi : G \to \mathrm{GL}_n(\mathbb{Z}_m)$. Suppose $g : X \to X$ is homeomorphism s.t. $\phi(g) = \mathrm{id}$.

If $m > 2$, then g preserves the orientation in each loop since $-\overline{1} = \overline{m-1} \neq \overline{1}$. This doesn't hold for $m = 2$.

43. (a) Suppose $\mathcal{C} = (\cdots C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots)$ is chain complex of free abelian groups.

$\operatorname{im}\partial_n$ is submodule of free $\mathbb{Z}$-module C_{n-1}, so it's free and exact sequence $0 \to \ker\partial_n \hookrightarrow C_n \twoheadrightarrow \operatorname{im}\partial_n \to 0$ splits.

Let $K_n = \ker\partial_n$, $L_n = \operatorname{im}\partial_n$, then $C_n \cong K_n \oplus L_n$ and $\mathcal{D}_n = (0 \to L_{n+1} \to K_n \to 0)$ is subcomplex.

$\mathcal{C} = \bigoplus_n \mathcal{D}_n$, i.e. chain complex $\mathcal{C}$ splits as a direct sum of subcomplexes $\mathcal{D}_n$.

(b) Suppose groups C_n are finitely generated, then map $L_{n+1} \to K_n$ is a linear transformation between finite dimensional vector spaces. Note that $L_{n+1} = \operatorname{im}\partial_{n+1} \subseteq K_n = \ker\partial_n$, write $L_{n+1} = \mathbb{Z}^j$ and $K_n = \mathbb{Z}^k$ with $j \leqslant k$.

By change of basis properly, which is equivalent to elementary row and column operations on $L_{n+1} = \mathbb{Z}^j \to K_n = \mathbb{Z}^k$, map $L_{n+1} \to \mathbb{Z}^k$ takes each basis vector in $\mathbb{Z}^j$ to a multiple of a basis vector in $\mathbb{Z}^k$, which gives splitting of complex $0 \to L_{n+1} \to K_n \to 0$ into summands $0 \to 0 \to \mathbb{Z} \to 0$ and $0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to 0$.

(c) This is universal coefficient theorem for homology. Cellular chain complex has the following decomposition.

Sequence (1) corresponds to $\mathbb{Z}$ summand of $H_n(X;\mathbb{Z})$.

Sequence (2) corresponds to $\operatorname{im}(\mathbb{Z} \xrightarrow{m} \mathbb{Z}) = \mathbb{Z}_m$ summand of $H_n(X;\mathbb{Z})$.

(3) is irrelevant to $H_n(X;\mathbb{Z})$.

(4) corresponds to $\ker(\mathbb{Z} \xrightarrow{m} \mathbb{Z}) = 0$ summand of $H_n(X;\mathbb{Z})$ and $\operatorname{im}(\mathbb{Z} \xrightarrow{m} \mathbb{Z}) = \mathbb{Z}_m$ summand of $H_{n-1}(X;\mathbb{Z})$.

For $H_n(X^n, X^{n-1};\mathbb{Z})$ and $H_n(X^n, X^{n-1};G)$ where G is an abelian group, the numbers of summands are equal to the number of n-cells in X, so summands in $H_n(X;\mathbb{Z})$ and $H_n(X;G)$ are in 1-1 correspondence to each other.

$$\cdots \longrightarrow H_{n+1}(X^{n+1}, X^n;\mathbb{Z}) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1};\mathbb{Z}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2};\mathbb{Z}) \xrightarrow{d_{n-1}} H_{n-2}(X^{n-2}, X^{n-3};\mathbb{Z}) \longrightarrow \cdots$$

$$0 \longrightarrow L_{n+1} = \operatorname{im} d_{n+1} \longrightarrow K_n = \ker d_n \longrightarrow 0$$

$$0 \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \quad (1)$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow 0 \quad (2)$$

$$0 \longrightarrow L_n = \operatorname{im} d_n \longrightarrow K_n = \ker d_{n-1} \longrightarrow 0$$

$$0 \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \quad (3)$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow 0 \quad (4)$$