Classification of Groups of Order $n \leqslant 50$

Contents

1 Groups of order 8

For $|G| = 8 = 2^3$

1. G is abelian. (1) $G \cong \mathbb{Z}_8$ [8, 1]. (2) $G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4$ [8, 2]. (3) $G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ [8, 5].

2. G is non-abelian. There exists $a \in G$ and $o(a) = 4$, $\langle a\rangle \triangleleft G$. Let $b \in G \setminus \langle a\rangle$, $o(b) = 2$ or $o(b) = 4$.

Let $bab^{-1} = a^i \in \langle a\rangle$, then $o(a^i) = o(bab^{-1}) = o(a) = 4$, $i = 1$ or $i = 3$. G is non-abelian, so $i = 3$.

(1) $o(b) = 2$. $G = \langle a, b \mid a^4 = b^2 = 1, bab^{-1} = a^3\rangle \cong D_4$ [8, 3]. This is the dihedral group of order 8.

(2) $o(b) = 4$. $G = \langle a, b \mid a^4 = b^4 = 1, bab^{-1} = a^3\rangle \cong Q_8$ [8, 4]. This is the quaternion group.

$Q_8 := \{\pm I, \pm\left(\begin{smallmatrix} i & 0 \\ 0 & -i\end{smallmatrix}\right), \pm\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right), \pm\left(\begin{smallmatrix} 0 & i \\ i & 0\end{smallmatrix}\right)\}$. The isomorphism in 2(2) is given by $a \mapsto \left(\begin{smallmatrix} i & 0 \\ 0 & -i\end{smallmatrix}\right), b \mapsto \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)$.

element in D_4	1	a	a^2	a^3	b	ab	a^2b	a^3b
order	1	4	2	4	2	2	2	2

element in Q_8	1	a	a^2	a^3	b	ab	a^2b	a^3b
order	1	4	2	4	4	4	4	4

In summary, 1(1), 1(2), 1(3), 2(1), 2(2) give all **5** non-isomorphic groups of order **8**.

2 Groups of order 12

For $|G| = 12 = 2^2 \cdot 3$, $N(2) = 2k + 1 \mid 3$, $N(3) = 3l + 1 \mid 4$, hence $N(2) = 3$, $N(3) = 1$ or $N(2) = 1$, $N(3) = 4$.

1. $N(2) = 3$, $N(3) = 1$. G is a semidirect product of the Sylow 3-subgroup $\mathbb{Z}_3$ and a Sylow 2-subgroup of order 4.

(1) Sylow 2-subgroup is $\mathbb{Z}_4$. We have homomorphism $\varphi : \mathbb{Z}_4 \to \mathrm{Aut}(\mathbb{Z}_3) = \mathbb{Z}_2$.

(i) φ is trivial. $G \cong \mathbb{Z}_3 \oplus \mathbb{Z}_4 \cong \mathbb{Z}_{12}$ [12, 2].

(ii) φ is non-trivial. Let $\mathbb{Z}_3 = \langle x\rangle$, $\mathbb{Z}_4 = \langle y\rangle$. $G \cong \langle x, y \mid x^3 = y^4 = 1, yxy^{-1} = x^2\rangle$ [12, 1].

(2) Sylow 2-subgroup is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. We have homomorphism $\psi : \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_3) = \mathbb{Z}_2$.

(i) ψ is trivial. $G \cong \mathbb{Z}_3 \times (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_6$ [12, 5].

(ii) ψ is non-trivial. $G \cong (\mathbb{Z}_3 \rtimes \mathbb{Z}_2) \oplus \mathbb{Z}_2 \cong S_3 \oplus Z_2 \cong D_3 \oplus \mathbb{Z}_2 \cong D_6$ [12, 4].

2. $N(2) = 1$, $N(3) = 4$. Let P be a Sylow 3-subgroup. Action of G on G/P induces homomorphism $\varphi : G \to S_4$.

$\ker\varphi = \bigcap_{g\in G} gPg^{-1} \leqslant P$. If $\ker\varphi = P$, then $P \triangleleft G$, $N(3) = 1$, back to case 1.

If $\ker\varphi = 1$, then φ is injective, $[S_4 : \varphi(G)] = 2$. The only subgroup of S_4 of order 12 is A_4, so $G \cong A_4$ [12, 3].

Alternative method:

G is a semidirect product of the Sylow 2-subgroup of order 4 and a Sylow 3-subgroup $\mathbb{Z}_3$.

(1) Sylow 2-subgroup is $\mathbb{Z}_4$. We have trivial homomorphism $\mathbb{Z}_3 \to \mathrm{Aut}(\mathbb{Z}_4) = \mathbb{Z}_2$. $G \cong \mathbb{Z}_3 \oplus \mathbb{Z}_4 \cong \mathbb{Z}_{12}$.

(2) Sylow 2-subgroup is $\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle x\rangle \oplus \langle y\rangle$. Consider the nontrivial homomorphism $\varphi : \mathbb{Z}_3 \to \mathrm{Aut}(\mathbb{Z}_2 \oplus \mathbb{Z}_2) = S_3$.

Let $\mathbb{Z}_3 = \langle z\rangle$. $\varphi(z)$ maps $1, x, y, xy$ to $1, y, xy, x$, and $\varphi(z^2)$ maps $1, x, y, xy$ to $1, xy, x, y$ respectively.

$G \cong \langle x, y, z \mid x^2 = y^2 = z^3 = 1, zxz^{-1} = y, zyz^{-1} = xy, zxyz^{-1} = x, z^2xz^{-2} = xy, z^2yz^{-2} = x, z^2xyz^{-2} = y\rangle$

It can be reduced to $G \cong \langle x, z \mid x^2 = z^3 = 1, (zx)^3 = 1\rangle \cong A_4$ with isomorphism $z \mapsto (123)$ and $x \mapsto (12)(34)$.

In summary, 1(1)(i), 1(1)(ii), 1(2)(i), 1(2)(ii), 2(2) give all **5** non-isomorphic groups of order **12**.

3 Groups of order 18

For $|G| = 18 = 3^2 \cdot 2$, $N(3) = 1$. Sylow 3-subgroup is normal.

G is a semidirect product of the Sylow 3-subgroup of order 9 and a Sylow 2-subgroup $\mathbb{Z}_2$.

1. Sylow 3-subgroup is $\mathbb{Z}_9$. We have homomorphism $\varphi : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_9) \cong \mathbb{Z}_6$.

(1) φ is trivial. $G \cong \boldsymbol{\mathbb{Z}_2 \oplus \mathbb{Z}_9 \cong \mathbb{Z}_{18}}$ $[\mathbf{18, 2}]$. This is an abelian group.

(2) φ is non-trivial. Let $\mathbb{Z}_9 = \langle x \rangle$, $\mathbb{Z}_2 = \langle y \rangle$. $\varphi(y)$ is of order 2, so $\varphi(y)(x) = x^8 = x^{-1}$.

$G \cong \langle x, y \mid x^9 = y^2 = 1, yxy^{-1} = x^{-1} \rangle \cong \boldsymbol{D_9}$ $[\mathbf{18, 1}]$. This is an non-abelian group.

2. Sylow 3-subgroup is $\mathbb{Z}_3 \oplus \mathbb{Z}_3$. We have homomorphism $\varphi : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_3 \oplus \mathbb{Z}_3) \cong \mathrm{GL}_2(\mathbb{F}_3)$.

Consider $\mathbb{Z}_2 = \{\bar{0}, \bar{1}, +\}$. $\varphi(\bar{1})$ is of order 1 or 2 and can always be diagonalized. The same diagonalization yields the same homomorphism since it's equivalent to represent homomorphism φ with another basis of $\mathbb{Z}_3 \oplus \mathbb{Z}_3$.

(1) $\varphi(\bar{1})$ can be diagonalized to $\begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{pmatrix}$, i.e. φ is trivial. $G \cong \boldsymbol{\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_3 \oplus \mathbb{Z}_6}$ $[\mathbf{18, 5}]$.

(2) $\varphi(\bar{1})$ can be diagonalized to $\begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \overline{-1} \end{pmatrix} = \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{2} \end{pmatrix}$. $G \cong \mathbb{Z}_3 \oplus (\mathbb{Z}_3 \rtimes \mathbb{Z}_2) \cong \boldsymbol{S_3 \oplus \mathbb{Z}_3 \cong D_3 \oplus \mathbb{Z}_3}$ $[\mathbf{18, 3}]$.

(3) $\varphi(\bar{1})$ can be diagonalized to $\begin{pmatrix} \overline{-1} & \bar{0} \\ \bar{0} & \overline{-1} \end{pmatrix} = \begin{pmatrix} \bar{2} & \bar{0} \\ \bar{0} & \bar{2} \end{pmatrix}$. Let $\mathbb{Z}_3 \oplus \mathbb{Z}_3 = \langle x \rangle \oplus \langle y \rangle$, $\mathbb{Z}_2 = \langle z \rangle$.

$G \cong \boldsymbol{\langle x, y, z \mid x^3 = y^3 = z^2 = 1, xy = yx, zxz^{-1} = x^{-1}, zyz^{-1} = y^{-1} \rangle}$ $[\mathbf{18, 4}]$.

Groups in 1 and groups in 2 have different Sylow 3-subgroup.

Group in 1(1) is abelian while group in 1(2) is non-abelian.

Group in 2(1) is abelian while groups in 2(2) and 2(3) are non-abelian.

Group in 2(2) has 3 elements of order 2, while group in 2(3) has 9 ($x^p y^q z$, $p = 0, 1, 2$, $q = 0, 1, 2$) such elements.

In summary, 1(1), 1(2), 2(1), 2(2), 2(3) give all **5** non-isomorphic groups of order **18**.

4 Groups of order 20

For $|G| = 20 = 2^2 \cdot 5$, $N(5) = 1$.

G is a semidirect product of the Sylow 5-subgroup $\mathbb{Z}_5 = \langle x \rangle$ and a Sylow 2-subgroup H of order 4.

1. $H = \mathbb{Z}_4 = \langle y \rangle$. We have homomorphism $\varphi : H = \mathbb{Z}_4 \to \mathrm{Aut}(\mathbb{Z}_5) \cong \mathbb{Z}_4$.

(1) $\varphi(H) = 1$. $G \cong \mathbb{Z}_4 \times \mathbb{Z}_5 \cong \boldsymbol{\mathbb{Z}_4 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{20}}$ $[\mathbf{20, 2}]$.

(2) $\varphi(H) = \mathbb{Z}_2$. $G \cong \boldsymbol{\langle y, x \mid x^5 = y^4 = 1, yxy^{-1} = x^4 \rangle}$ $[\mathbf{20, 1}]$.

(3) $\varphi(H) = \mathbb{Z}_4$. $G \cong \boldsymbol{\langle y, x \mid x^5 = y^4 = 1, yxy^{-1} = x^2 \rangle}$ $[\mathbf{20, 3}]$.

Note that for group in 1(2), $xy^2 = y^2 x$, $\langle x, y^2 \rangle = \mathbb{Z}_{10}$. For group in 1(3), if it has a subgroup K of order 10, then K is normal and $K \cap H = \langle y^2 \rangle$. $D_5 \cong \langle x, y^2 \rangle \leqslant K$, so $K \cong D_5$ and groups in 1(2) and 1(3) are not isomorphic.

2. $H = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. We have homomorphism $\psi : H = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_5) \cong \mathbb{Z}_4$.

(1) $\psi(H) = 1$. $G \cong (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \times \mathbb{Z}_5 \cong \boldsymbol{\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{10}}$ $[\mathbf{20, 5}]$.

(2) $\psi(H) = \mathbb{Z}_2$. $G \cong (\mathbb{Z}_5 \rtimes \mathbb{Z}_2) \oplus \mathbb{Z}_2 \cong \boldsymbol{D_5 \oplus \mathbb{Z}_2 \cong D_{10}}$ $[\mathbf{20, 4}]$.

In summary, 1(1), 1(2), 1(3), 2(1), 2(2) give all **5** non-isomorphic groups of order **20**.

5 Groups of order 24

For $|G| = 24 = 2^3 \cdot 3$

1. $N(2) = 1$. G is a semidirect product of the Sylow 2-subgroup N of order 8 and a Sylow 3-subgroup $\mathbb{Z}_3$.

(1) $N = \mathbb{Z}_8$. We have trivial homomorphism $\mathbb{Z}_3 \to \mathrm{Aut}(\mathbb{Z}_8) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. $G \cong \mathbb{Z}_8 \times \mathbb{Z}_3 \cong \boldsymbol{\mathbb{Z}_3 \oplus \mathbb{Z}_8 \cong \mathbb{Z}_{24}}$ $\mathbf{[24, 2]}$.

(2) $N = \mathbb{Z}_2 \oplus \mathbb{Z}_4$. We have trivial homomorphism $\mathbb{Z}_3 \to \mathrm{Aut}(\mathbb{Z}_2 \oplus \mathbb{Z}_4) \cong D_4$.

$G \cong (\mathbb{Z}_2 \oplus \mathbb{Z}_4) \times \mathbb{Z}_3 \cong \boldsymbol{\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{12}}$ $\mathbf{[24, 9]}$.

(3) $N = D_4$. We have trivial homomorphism $\mathbb{Z}_3 \to \mathrm{Aut}(D_4) \cong D_4$. $G \cong D_4 \times \mathbb{Z}_3 \cong \boldsymbol{D_4 \oplus \mathbb{Z}_3}$ $\mathbf{[24, 10]}$.

(4) $N = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. We have homomorphism $\varphi : \mathbb{Z}_3 \to \mathrm{Aut}(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2) \cong \mathrm{GL}_3(\mathbb{F}_2)$.

(i) φ is trivial. $G \cong (\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2) \times \mathbb{Z}_3 \cong \boldsymbol{\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_6}$ $\mathbf{[24, 15]}$.

(ii) φ is non-trivial. Using rational canonical form, $\varphi(\bar{1})$ can be quasi-diagonalized to $\begin{pmatrix} \bar{1} & \bar{0} & \bar{0} \\ \bar{0} & \bar{0} & \bar{1} \\ \bar{0} & \bar{1} & \bar{1} \end{pmatrix}$.
Or equivalently, note that $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ has 7 non-trivial elements and $\varphi(\bar{1})$ is of order 3, so it must have a non-trivial fixed point and therefore fix one $\mathbb{Z}_2$ in $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. $G \cong ((\mathbb{Z}_2 \oplus \mathbb{Z}_2) \rtimes \mathbb{Z}_3) \oplus \mathbb{Z}_2 \cong \boldsymbol{A_4 \oplus \mathbb{Z}_2}$ $\mathbf{[24, 13]}$.

(5) $N = Q_8$. We have homomorphism $\psi : \mathbb{Z}_3 \to \mathrm{Aut}(Q_8) \cong S_4$.

(i) ψ is trivial. $G \cong Q_8 \times \mathbb{Z}_3 \cong \boldsymbol{Q_8 \oplus \mathbb{Z}_3}$ $\mathbf{[24, 11]}$.

(ii) ψ is non-trivial. Subgroups of order 3 in S_4 are conjugate, which corresponds to rename the generators of Q_8, so there's only one non-trivial action of $\mathbb{Z}_3$ on Q_8, given by $\mathrm{i} \mapsto \mathrm{j} \mapsto \mathrm{k} \mapsto \mathrm{i}$. $G \cong \boldsymbol{Q_8 \rtimes \mathbb{Z}_3 \cong \mathrm{SL}_2(\mathbb{Z}_3)}$ $\mathbf{[24, 3]}$.

2. $N(3) = 1$. G is a semidirect product of the Sylow 3-subgroup $\mathbb{Z}_3 = \langle x \rangle$ and a Sylow 2-subgroup H of order 8.

We only need to consider the non-trivial cases.

(1) $H = \mathbb{Z}_8 = \langle y \rangle$. $G \cong \boldsymbol{\langle x, y \mid x^3 = y^8 = 1, yxy^{-1} = x^2 \rangle}$ $\mathbf{[24, 1]}$.

(2) $H = \mathbb{Z}_2 \oplus \mathbb{Z}_4 = \langle y \rangle \oplus \langle z \rangle$. We have epimorphism $\varphi : \mathbb{Z}_2 \oplus \mathbb{Z}_4 \to \mathrm{Aut}(\mathbb{Z}_3) \cong \mathbb{Z}_2$.

(i) $\ker \varphi = \mathbb{Z}_4 = \langle z \rangle$. $\mathbb{Z}_4 = \langle z \rangle$ acts on $\mathbb{Z}_3$ trivially, and $\mathbb{Z}_2 = \langle y \rangle$ acts on $\mathbb{Z}_3$ non-trivially.

$G \cong (\mathbb{Z}_3 \rtimes \mathbb{Z}_2) \oplus \mathbb{Z}_4 \cong \boldsymbol{S_3 \oplus \mathbb{Z}_4 \cong D_3 \oplus \mathbb{Z}_4}$ $\mathbf{[24, 5]}$.

(ii) $\ker \varphi = \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle y \rangle \oplus \langle z^2 \rangle$. $\mathbb{Z}_2 = \langle y \rangle$ acts on $\mathbb{Z}_3$ trivially, and $\varphi(z)$ is of order 2 in $\mathrm{Aut}(\mathbb{Z}_3)$.

$G \cong \langle x, y, z \mid x^3 = y^2 = z^4 = 1, yz = zy, yxy^{-1} = x, zxz^{-1} = x^2 \rangle \cong \boldsymbol{\langle x, z \mid x^3 = z^4 = 1, zxz^{-1} = x^2 \rangle \oplus \mathbb{Z}_2}$ $\mathbf{[24, 7]}$.

(3) $H = D_4 = \langle y, z \mid y^2 = z^4 = 1, (yz)^2 = 1 \rangle$. We have epimorphism $\psi : D_4 \to \mathrm{Aut}(\mathbb{Z}_3) \cong \mathbb{Z}_2$.

(i) $\ker \psi = \mathbb{Z}_4 = \langle z \rangle$. $\mathbb{Z}_4 = \langle z \rangle$ acts on $\mathbb{Z}_3$ trivially, and $\mathbb{Z}_2 = \langle y \rangle$ acts on $\mathbb{Z}_3$ non-trivially.

$G \cong \langle x, y, z \mid x^3 = y^2 = z^4 = 1, (yz)^2 = 1, yxy^{-1} = x^2, zxz^{-1} = x \rangle$. Note that $\langle x, z \rangle \cong \mathbb{Z}_{12}$.

Let $x = w^4$, $z = w^3$, then it can be reduced to $G \cong \langle y, w \mid y^2 = w^{12} = 1, (yw)^2 = 1 \rangle \cong \boldsymbol{D_{12}}$ $\mathbf{[24, 6]}$.

(ii) $\ker \psi = \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle y \rangle \oplus \langle z^2 \rangle$. $\mathbb{Z}_2 = \langle y \rangle$ acts on $\mathbb{Z}_3$ trivially, and $\psi(z)$ is of order 2 in $\mathrm{Aut}(\mathbb{Z}_3)$.

$G \cong \boldsymbol{\langle x, y, z \mid x^3 = y^2 = z^4 = 1, (yz)^2 = 1, yxy^{-1} = x, zxz^{-1} = x^2 \rangle}$ $\mathbf{[24, 8]}$.

(4) $H = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. $G \cong (\mathbb{Z}_3 \rtimes \mathbb{Z}_2) \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong \boldsymbol{S_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong D_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong D_6 \oplus \mathbb{Z}_2}$ $\mathbf{[24, 14]}$.

(5) $H = Q_8 = \langle y, z \mid y^4 = z^4 = 1, zyz^{-1} = y^3 \rangle$. We have epimorphism $\psi : Q_8 \to \mathrm{Aut}(\mathbb{Z}_3) \cong \mathbb{Z}_2$.

Subgroups of order 4 in Q_8 are all isomorphic to $\mathbb{Z}_4$, so there's only one $\mathbb{Z}_3 \rtimes Q_8$ under isomorphism.

Let $\ker\varphi = \langle y\rangle$, then $\mathbb{Z}_3 \rtimes Q_8 = \langle x, y, z \mid x^3 = y^4 = z^4 = 1, zyz^{-1} = y^3, yxy^{-1} = x, zxz^{-1} = x^2\rangle$. $\langle x, y\rangle = \mathbb{Z}_{12}$.

Let $x = w^4, y = w^3$. $G \cong \langle w, z \mid w^{12} = z^4 = 1, zwz^{-1} = w^{11}\rangle \cong \boldsymbol{\langle w, z \mid w^{12} = 1, z^2 = w^6, zwz^{-1} = w^{-1}\rangle}$ **[24, 4]**.

This is the dicyclic group of order 24, also binary von Dyck group or binary triangle group with parameters $(6, 2, 2)$.

3. $N(2) = 3, N(3) = 4$. G has 4 Sylow 3-subgroups P_1, P_2, P_3, P_4, $|N_G(P_i)| = 24/4 = 6$.

Action of G on P_i by conjugation induces homomorphism $\varphi : G \to S_4$, $\ker\varphi = \bigcap_{i=1}^{4} N(P_i) \subseteq N_G(P_1)$.

(1) $\ker\varphi = N_G(P_1)$. $N_G(P_1) = N(P_2)$, $P_1 = P_2$. Contradiction.

(2) $\ker\varphi = P_1$. $P_1 \lhd G$. Contradiction.

(3) $\ker\varphi = 1$. $G \cong \boldsymbol{S_4}$ **[24, 12]**.

(4) $\ker\varphi \cong \mathbb{Z}_2$. $\mathbb{Z}_2 \lhd G$ and $\mathbb{Z}_2 \subseteq Z(G)$. G has 3 Sylow 2-subgroups Q_1, Q_2, Q_3 of order 8.

Action of G on Q_i by conjugation induces homomorphism $\psi : G \to S_3$. $|N_G(Q_i)| = 24/3 = 8$, $Q_i = N_G(Q_i)$.

Thus $\operatorname{im}\psi$ contains all transpositions and is an epimorphism, $\ker\psi \lhd G$ and $|\ker\psi| = 4$. $\mathbb{Z}_2 \subseteq Z(G) \subseteq \ker\psi$.

$|\varphi(\ker\psi)| = 2$ and $\varphi(\ker\psi) \lhd \operatorname{im}\varphi = A_4$, but A_4 has no normal subgroup of order 2. Contradiction.

In summary, 1(1), 1(2), 1(3), 1(4)(i), 1(4)(ii), 1(5)(i), 1(5)(ii), 2(1), 2(2)(i), 2(2)(ii), 2(3)(i), 2(3)(ii), 2(4), 2(5), 3(3) give all **15** non-isomorphic groups of order **24**.

$Q_8 \rtimes \mathbb{Z}_3 \cong \mathrm{SL}_2(\mathbb{Z}_3)$.

Proof:

$|\mathrm{SL}_2(\mathbb{Z}_3)| = (3^2 - 1) \cdot (3^2 - 3)/(3 - 1) = 24$. $Z(\mathrm{SL}_2(\mathbb{Z}_3)) = \left\{ \begin{pmatrix} \bar 1 & \bar 0 \\ \bar 0 & \bar 1 \end{pmatrix}, \begin{pmatrix} \bar 2 & \bar 0 \\ \bar 0 & \bar 2 \end{pmatrix} \right\} \lhd \mathrm{SL}_2(\mathbb{Z}_3)$. Denote $\mathrm{SL}_2(\mathbb{Z}_3)$ by G.

$\begin{pmatrix} \bar 1 & \bar 1 \\ \bar 0 & \bar 1 \end{pmatrix}$ and $\begin{pmatrix} \bar 1 & \bar 0 \\ \bar 1 & \bar 1 \end{pmatrix}$ are of order 3, so G has 4 Sylow 3-subgroups, denoted by P_1, P_2, P_3, P_4. $|N_G(P_i)| = 24/4 = 6$.

Action of G on P_i yields homomorphism $\varphi : G \to S_4$, $\ker\varphi = \bigcap_{i=1}^{4} N_G(P_i)$. $Z(G) \leqslant \ker\varphi \leqslant N_G(P_1)$.

If $\ker\varphi = N_G(P_1)$, then $N_G(P_1) = N_G(P_2)$, $P_1 = P_2$. Contradiction. Therefore $\ker\varphi = Z(G) \cong \mathbb{Z}_2$ and $G/\mathbb{Z}_2 \cong A_4$.

$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \lhd A_4$, $[A_4 : \mathbb{Z}_2 \oplus \mathbb{Z}_2] = 3$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ has 3 subgroups of index 2.

By the correspondence theorem, there exists $N \lhd G$ with the same property. $[G : N] = 3$, so $|N| = 8$.

$\begin{pmatrix} \bar 2 & \bar 0 \\ \bar 0 & \bar 2 \end{pmatrix} \in G$ is the unique element of order 2, so N has only one element of order 2. $N \cong \mathbb{Z}_8$ or $N \cong Q_8$.

N has 3 subgroups of index 2, so $N \cong Q_8$. $Q_8 \cong N \lhd G$, $\mathbb{Z}_3 \cong P_1 \leqslant G$, $P_1 \cap N = 1$, $G = P_1 N$, so $G \cong N \rtimes P_1$.

Therefore $\mathrm{SL}_2(\mathbb{Z}_3) \cong Q_8 \rtimes \mathbb{Z}_3$. $\square$

$\mathbb{Z}_3 \cong \left\langle \begin{pmatrix} \bar 1 & \bar 1 \\ \bar 0 & \bar 1 \end{pmatrix} \right\rangle$, $Q_8 \cong \left\langle \begin{pmatrix} \bar 0 & \bar 1 \\ \bar 2 & \bar 0 \end{pmatrix}, \begin{pmatrix} \bar 2 & \bar 2 \\ \bar 2 & \bar 1 \end{pmatrix} \right\rangle$.

$(\mathbb{Z}_2 \oplus \mathbb{Z}_2) \rtimes S_3 \cong S_4$ since $S_3 \lesssim S_4$ and $K_4 = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \lhd S_4$.

6 Groups of order 28

For $|G| = 28 = 2^2 \cdot 7$, $N(7) = 1$.

G is a semidirect product of the Sylow 7-subgroup $\mathbb{Z}_7 = \langle x \rangle$ and a Sylow 2-subgroup H of order 4.

1. $H = \mathbb{Z}_4 = \langle y \rangle$. We have homomorphism $\varphi : H = \mathbb{Z}_4 \to \mathrm{Aut}(\mathbb{Z}_7) \cong \mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$.

(1) $\varphi(H) = 1$. $G \cong \mathbb{Z}_4 \times \mathbb{Z}_7 \cong \boldsymbol{\mathbb{Z}_4 \oplus \mathbb{Z}_7 \cong \mathbb{Z}_{28}}$ $[\mathbf{28, 2}]$.

(2) $\varphi(H) = \mathbb{Z}_2$. $G \cong \boldsymbol{\langle x, y \mid x^7 = y^4 = 1, yxy^{-1} = x^6 \rangle}$ $[\mathbf{28, 1}]$.

2. $H = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. We have homomorphism $\psi : H = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_7) \cong \mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$.

(1) $\psi(H) = 1$. $G \cong (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \times \mathbb{Z}_7 \cong \boldsymbol{\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_7 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{14}}$ $[\mathbf{28, 4}]$.

(2) $\psi(H) = \mathbb{Z}_2$. $G \cong (\mathbb{Z}_7 \rtimes \mathbb{Z}_2) \oplus \mathbb{Z}_2 \cong \boldsymbol{D_7 \oplus \mathbb{Z}_2 \cong D_{14}}$ $[\mathbf{28, 3}]$.

In summary, 1(1), 1(2), 2(1), 2(2) give all **4** non-isomorphic groups of order **28**.

7 Groups of order 30

For $|G| = 30 = 2 \cdot 3 \cdot 5$, $N(3) = 1$ or $N(5) = 1$. $\mathbb{Z}_3$ and $\mathbb{Z}_5$ generate $\mathbb{Z}_3 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{15} \triangleleft G$.

G is a semidirect product of the normal subgroup $\mathbb{Z}_{15}$ and a Sylow 2-subgroup $\mathbb{Z}_2$.

We have homomorphism $\varphi : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_{15}) \cong \mathrm{Aut}(\mathbb{Z}_3) \times \mathrm{Aut}(\mathbb{Z}_5) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4$.

1. $\varphi(\bar{1}) = (\bar{0}, \bar{0})$. $\mathbb{Z}_2$ acts on $\mathbb{Z}_3$ and $\mathbb{Z}_5$ trivially. $G \cong \boldsymbol{\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{30}}$ $[\mathbf{30, 4}]$.

2. $\varphi(\bar{1}) = (\bar{1}, \bar{0})$. $\mathbb{Z}_2$ acts on $\mathbb{Z}_3$ non-trivially and acts on $\mathbb{Z}_5$ trivially. $G \cong (\mathbb{Z}_3 \rtimes \mathbb{Z}_2) \oplus \mathbb{Z}_5 \cong \boldsymbol{S_3 \oplus \mathbb{Z}_5}$ $[\mathbf{30, 1}]$.

3. $\varphi(\bar{1}) = (\bar{0}, \bar{2})$. $\mathbb{Z}_2$ acts on $\mathbb{Z}_3$ trivially and acts on $\mathbb{Z}_5$ non-trivially. $G \cong (\mathbb{Z}_5 \rtimes \mathbb{Z}_2) \oplus \mathbb{Z}_3 \cong \boldsymbol{D_5 \oplus \mathbb{Z}_3}$ $[\mathbf{30, 2}]$.

4. $\varphi(\bar{1}) = (\bar{1}, \bar{2})$. $\mathbb{Z}_2$ acts on $\mathbb{Z}_3$ and $\mathbb{Z}_5$ non-trivially. $G \cong (\mathbb{Z}_5 \oplus \mathbb{Z}_3) \rtimes \mathbb{Z}_2 \cong \boldsymbol{D_{15}}$ $[\mathbf{30, 4}]$.

In summary, 1, 2, 3, 4 give all **4** non-isomorphic groups of order **30**.

8 Groups of order 40

For $|G| = 40 = 2^3 \cdot 5$, $N(5) = 1$.

G is a semidirect product of the Sylow 5-subgroup $\mathbb{Z}_5 = \langle x \rangle$ and a Sylow 5-subgroup H of order 8.

1. $H = \mathbb{Z}_8 = \langle y \rangle$. We have homomorphism $\varphi : \mathbb{Z}_8 \to \mathrm{Aut}(\mathbb{Z}_5) \cong \mathbb{Z}_4$.

(1) $\varphi(H) = 1$. $G \cong \mathbb{Z}_5 \times \mathbb{Z}_8 \cong \mathbb{Z}_5 \oplus \mathbb{Z}_8 \cong \mathbb{Z}_{40}$ $[40, 2]$.

(2) $\varphi(H) = \mathbb{Z}_2$. $G \cong \langle x, y \mid x^5 = y^8 = 1, yxy^{-1} = x^4 \rangle$ $[40, 1]$.

(3) $\varphi(H) = \mathbb{Z}_4$. $G \cong \langle x, y \mid x^5 = y^8 = 1, yxy^{-1} = x^2 \rangle$ $[40, 3]$.

2. $H = \mathbb{Z}_2 \oplus \mathbb{Z}_4 = \langle y \rangle \oplus \langle z \rangle$. We have homomorphism $\varphi : \mathbb{Z}_2 \oplus \mathbb{Z}_4 \to \mathrm{Aut}(\mathbb{Z}_5) \cong \mathbb{Z}_4$.

(1) $\varphi(H) = 1$. $G \cong \mathbb{Z}_5 \times (\mathbb{Z}_2 \oplus \mathbb{Z}_4) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{20}$ $[40, 9]$.

(2) $\varphi(H) = \mathbb{Z}_2$.

(i) $\ker \varphi = \mathbb{Z}_4 = \langle z \rangle$. $\mathbb{Z}_4$ acts trivially on $\mathbb{Z}_5$, and $\mathbb{Z}_2$ acts non-trivially on $\mathbb{Z}_5$. $G \cong (\mathbb{Z}_5 \rtimes \mathbb{Z}_2) \times \mathbb{Z}_4 \cong D_5 \oplus \mathbb{Z}_4$ $[40, 5]$.

(ii) $\ker \varphi = \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle x \rangle \oplus \langle y^2 \rangle$. $G \cong \langle x, z \mid x^5 = z^4 = 1, zxz^{-1} = x^4 \rangle \oplus \mathbb{Z}_2$ $[40, 7]$.

(3) $\varphi(H) = \mathbb{Z}_4$, $\ker \varphi = \mathbb{Z}_2 = \langle y \rangle$. $G \cong \langle x, z \mid x^5 = z^4 = 1, zxz^{-1} = x^2 \rangle \oplus \mathbb{Z}_2$ $[40, 12]$.

3. $H = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. We have homomorphism $\varphi : \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_5) \cong \mathbb{Z}_4$.

(1) $\varphi(H) = 1$. $G \cong \mathbb{Z}_5 \times (\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{10}$ $[40, 14]$.

(2) $\varphi(H) = \mathbb{Z}_2$. $G \cong (\mathbb{Z}_5 \rtimes \mathbb{Z}_2) \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong D_5 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong D_{10} \oplus \mathbb{Z}_2$ $[40, 13]$.

4. $H = D_4 = \langle y, z \mid y^4 = z^2 = 1, (yz)^2 = 1 \rangle$. We have homomorphism $\varphi : D_4 \to \mathrm{Aut}(\mathbb{Z}_5) \cong \mathbb{Z}_4$.

(1) $\varphi(H) = 1$. $G \cong \mathbb{Z}_5 \times D_4 \cong D_4 \oplus \mathbb{Z}_5$ $[40, 10]$.

(2) $\varphi(H) = \mathbb{Z}_2$.

(i) $\ker \varphi = \mathbb{Z}_4 = \langle y \rangle$. $G \cong \langle x, y, z \mid x^5 = y^4 = z^2 = 1, (yz)^2 = 1, yxy^{-1} = x, zxz^{-1} = x^4 \rangle$. $\langle x, y \rangle \cong \mathbb{Z}_{20}$.

Let $x = w^4$, $y = w^5$. It can be reduced to $G \cong \langle w, z \mid w^{20} = z^2 = 1, (zw)^2 = 1 \rangle \cong D_{20}$ $[40, 6]$.

(ii) $\ker \varphi = \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle y^2 \rangle \oplus \langle z \rangle$. $G \cong \langle x, y, z \mid x^5 = y^4 = z^2 = 1, (yz)^2 = 1, yxy^{-1} = x^4, zxz^{-1} = x \rangle$ $[40, 8]$.

5. $H = Q_8$. We have homomorphism $\varphi : Q_8 \to \mathrm{Aut}(\mathbb{Z}_5) \cong \mathbb{Z}_4$.

(1) $\varphi(H) = 1$. $G \cong \mathbb{Z}_5 \times Q_8 \cong Q_8 \oplus \mathbb{Z}_5$ $[40, 11]$.

(2) $\varphi(H) = \mathbb{Z}_2$. We have a unique non-trivial semidirect product $G \cong \mathbb{Z}_5 \rtimes Q_8$ $[40, 4]$.

In summary, 1(1), 1(2), 1(3), 2(1), 2(2)(i), 2(2)(ii), 2(3), 3(1), 3(2), 4(1), 4(2)(i), 4(2)(ii), 5(1), 5(2) give all **14** non-isomorphic groups of order **40**.

9 Groups of order 42

For $|G| = 42 = 2 \cdot 3 \cdot 7$, $N(3) = 1$ or $N(3) = 7$. $N(7) = 1$.

1. $N(3) = 1$. Sylow 3-subgroup $\mathbb{Z}_3$ and Sylow 7-subgroup $\mathbb{Z}_7$ are normal, so they generate $\mathbb{Z}_3 \oplus \mathbb{Z}_7 \cong \mathbb{Z}_{21} \triangleleft G$.

G is a simidirect product of normal subgroup $\mathbb{Z}_{21}$ and a Sylow 2-subgroup $\mathbb{Z}_2$.

We have homomorphism $\varphi : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_{21}) \cong \mathrm{Aut}(\mathbb{Z}_3) \times \mathrm{Aut}(\mathbb{Z}_7) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_6$.

(1) $\varphi(\bar{1}) = (\bar{0}, \bar{0})$. $\mathbb{Z}_2$ acts on $\mathbb{Z}_3$ and $\mathbb{Z}_7$ trivially. $G \cong \mathbb{Z}_{21} \times \mathbb{Z}_2 \cong \boldsymbol{\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7 \cong \mathbb{Z}_{42}}$ $[\mathbf{42, 6}]$.

(2) $\varphi(\bar{1}) = (\bar{1}, \bar{0})$. $\mathbb{Z}_2$ acts on $\mathbb{Z}_3$ non-trivially and acts on $\mathbb{Z}_7$ trivially. $G \cong (\mathbb{Z}_3 \rtimes \mathbb{Z}_2) \oplus \mathbb{Z}_7 \cong \boldsymbol{S_3 \oplus \mathbb{Z}_7}$ $[\mathbf{42, 3}]$.

(3) $\varphi(\bar{1}) = (\bar{0}, \bar{3})$. $\mathbb{Z}_2$ acts on $\mathbb{Z}_3$ trivially and acts on $\mathbb{Z}_7$ non-trivially. $G \cong (\mathbb{Z}_7 \rtimes \mathbb{Z}_2) \oplus \mathbb{Z}_3 \cong \boldsymbol{D_7 \oplus \mathbb{Z}_3}$ $[\mathbf{42, 4}]$.

(4) $\varphi(\bar{1}) = (\bar{1}, \bar{3})$. $\mathbb{Z}_2$ acts on $\mathbb{Z}_3$ and $\mathbb{Z}_7$ non-trivially. $G \cong \mathbb{Z}_{21} \rtimes \mathbb{Z}_2 \cong \boldsymbol{D_{21}}$ $[\mathbf{42, 5}]$.

2. $N(3) = 7$. For any Sylow 3-subgroup $P \cong \mathbb{Z}_3$, let $H = N_G(P)$, $P \leqslant H$, $|H| = 42/7 = 6$.

G is a semidirect product of Sylow 7-subgroup $\mathbb{Z}_7 = \langle x \rangle$ and H. We have homomorphism $\psi : H \to \mathrm{Aut}(\mathbb{Z}_7) \cong \mathbb{Z}_6$.

If $\psi(P) = 1$, then $P \cong \mathbb{Z}_3$ acts on $\mathbb{Z}_7$ trivially and we have subgroup $\mathbb{Z}_7 \times \mathbb{Z}_3 \cong \mathbb{Z}_{21}$. This is case 1.

If $\psi(P) \neq 1$, then $\ker \psi = 1$ or $\ker \psi = \mathbb{Z}_2$, and $H = \mathbb{Z}_6 = \langle y \rangle$.

(1) $\ker \psi = \mathbb{Z}_2 = \langle y^3 \rangle$. $G \cong \boldsymbol{\langle x, y \mid x^7 = y^6 = 1, yxy^{-1} = x^2 \rangle \cong (\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \oplus \mathbb{Z}_2}$ $[\mathbf{42, 2}]$.

(2) $\ker \psi = 1$. $G \cong \boldsymbol{\langle x, y \mid x^7 = y^6 = 1, yxy^{-1} = x^3 \rangle}$ $[\mathbf{42, 1}]$.

Group in 2(1) has only 1 element of order 2, while group in 2(2) has 6 elements of order 2.

In summary, 1(1), 1(2), 1(3), 1(4), 2(1), 2(2) give all **6** non-isomorphic groups of order **42**.

10 Groups of order 44

For $|G| = 44 = 2^2 \cdot 11$, $N(11) = 1$.

G is a simidirect product of Sylow 11-subgroup $\mathbb{Z}_{11} = \langle x \rangle$ and a Sylow 2-subgroup H of order 4.

We have homomorphism $\varphi : H \to \mathrm{Aut}(\mathbb{Z}_{11}) = \mathbb{Z}_{10} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_5$.

1. $H = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

(1) $\varphi(H) = 1$. $G \cong \mathbb{Z}_{11} \times (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \cong \boldsymbol{\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{11} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{22}}$ $[\mathbf{44, 4}]$.

(2) $\varphi(H) = \mathbb{Z}_2$. $G \cong (\mathbb{Z}_{11} \rtimes \mathbb{Z}_2) \oplus \mathbb{Z}_2 \cong \boldsymbol{D_{11} \oplus \mathbb{Z}_2 \cong D_{22}}$ $[\mathbf{44, 3}]$.

2. $H = \mathbb{Z}_4 = \langle y \rangle$.

(1) $\varphi(H) = 1$. $G \cong \mathbb{Z}_{11} \times \mathbb{Z}_4 \cong \boldsymbol{\mathbb{Z}_4 \oplus \mathbb{Z}_{11} \cong \mathbb{Z}_{44}}$ $[\mathbf{44, 2}]$.

(2) $\varphi(H) = \mathbb{Z}_2$, $\ker \varphi = \langle y^2 \rangle$. $G \cong \boldsymbol{\langle x, y \mid x^{11} = y^4 = 1, yxy^{-1} = x^{10} \rangle}$ $[\mathbf{44, 1}]$.

In summary, 1(1), 1(2), 2(1), 2(2) give all **4** non-isomorphic groups of order **44**.

11 Groups of order 45

For $|G| = 45 = 3^2 \cdot 5$, $N(3) = N(5) = 1$.

G is a semidirect product of the Sylow 3-subgroup of order 9 and the Sylow 5-subgroup $\mathbb{Z}_5$.

1. Sylow 3-subgroup is $\mathbb{Z}_3 \oplus \mathbb{Z}_3$. $G(\mathbb{Z}_3 \oplus \mathbb{Z}_3) \times \mathbb{Z}_5 \cong\cong \boldsymbol{\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_3 \oplus \mathbb{Z}_{15}}$ $\mathbf{[45, 2]}$.

2. Sylow 3-subgroup is $\mathbb{Z}_9$. $G \cong \mathbb{Z}_9 \times \mathbb{Z}_5 \cong \boldsymbol{\mathbb{Z}_5 \oplus \mathbb{Z}_9 \cong \mathbb{Z}_{45}}$ $\mathbf{[45, 1]}$.

In summary, 1,2 gives all **2** non-isomorphic groups of order **45**.

12 Groups of order 50

For $|G| = 50 = 2 \cdot 5^2$, $N(5) = 1$.

G is a semidirect product of the Sylow 5-subgroup of order 25 and a Sylow 2-subgroup $\mathbb{Z}_2$.

1. Sylow 5-subgroup is $\mathbb{Z}_{25}$. We have homomorphism $\varphi : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_{25})$.

(1) φ is trivial. $G \cong \boldsymbol{\mathbb{Z}_2 \oplus \mathbb{Z}_{25} \cong \mathbb{Z}_{50}}$ $\mathbf{[50, 2]}$.

(2) φ is non-trivial. $G \cong \boldsymbol{D_{25}}$ $\mathbf{[50, 1]}$.

2. Sylow 5-subgroup is $\mathbb{Z}_5 \oplus \mathbb{Z}_5$. We have homomorphism $\psi : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_5 \oplus \mathbb{Z}_5) \cong \mathrm{GL}_2(\mathbb{F}_5)$.

(1) $\psi(\bar{1})$ can be diagonalized to $\begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{pmatrix}$. $G \cong \boldsymbol{\mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_5 \oplus \mathbb{Z}_{10}}$ $\mathbf{[50, 5]}$.

(2) $\psi(\bar{1})$ can be diagonalized to $\begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & -\bar{1} \end{pmatrix}$. $G \cong \mathbb{Z}_5 \oplus (\mathbb{Z}_5 \rtimes \mathbb{Z}_2) \cong \boldsymbol{D_5 \oplus \mathbb{Z}_5}$ $\mathbf{[50, 3]}$.

(3) $\psi(\bar{1})$ can be diagonalized to $\begin{pmatrix} -\bar{1} & \bar{0} \\ \bar{0} & -\bar{1} \end{pmatrix}$. $G \cong \boldsymbol{(\mathbb{Z}_5 \oplus \mathbb{Z}_5) \rtimes \mathbb{Z}_2}$ $\mathbf{[50, 4]}$.

In summary, 1(1), 1(2), 2(1), 2(2), 2(3) give all **5** non-isomorphic groups of order **50**.

13 Groups of order p

Suppose $|G| = p$ and p is an odd prime, then $G \cong \boldsymbol{\mathbb{Z}_p}$.

For $n = p \leqslant 50$, $n = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47$.

14 Groups of order $2p$

Suppose $|G| = 2p$ and p is an odd prime. $N(p) = 1$.

G is a semidirect product of the Sylow p-subgroup $\mathbb{Z}_p = \langle x \rangle$ and a Sylow 2-subgroup $\mathbb{Z}_2 = \langle y \rangle$.

We have homomorphism $\varphi : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_p) \cong \mathbb{Z}_{p-1}$.

1. φ is trivial. $G \cong \boldsymbol{\mathbb{Z}_2 \oplus \mathbb{Z}_p}$.

2. φ is non-trivial, $\varphi(y)(x) = x^{p-1}$. $G \cong \langle x, y \mid x^p = y^2 = 1, yxy^{-1} = x^{-1} \rangle \cong \boldsymbol{D_p}$.

For $n = 2p \leqslant 50$, $n = 6, 10, 14, 22, 26, 34, 38, 46$.

15 Groups of order p^2

Suppose $|G| = p^2$ and p is an odd prime, then $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$ or $G \cong \mathbb{Z}_{p^2}$.

For $n = p^2 \leqslant 50$, $n = 4, 9, 25, 49$.

16 Groups of order pq

Suppose $|G| = pq$ and p, q are odd primes, $p < q$. $N(q) = 1$.

G is a semidirect product of the Sylow q-subgroup $\mathbb{Z}_q = \langle a \rangle$ and a Sylow p-subgroup $\mathbb{Z}_p = \langle b \rangle$.

We have homomorphism $\varphi : \mathbb{Z}_p \to \mathrm{Aut}(\mathbb{Z}_q) \cong \mathbb{Z}_{q-1}$.

1. $\varphi(\langle a \rangle) = 1$ and G is abelian. $bab^{-1} = \varphi(b)(a) = a$. $G \cong \langle a, b \mid a^q = b^p = 1, bab^{-1} = a \rangle \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$.

2. $\varphi(\langle a \rangle) \neq 1$ and G is non-abelian. Let $bab^{-1} = \varphi(b)(a) = a^r$ for some $r \in \{2, \dots, q-1\}$.

$\varphi(b^p)(a) = a^{r^p} = a$, so $r^p \equiv 1 \bmod q$. $N(p) \mid q$ and $N(p) \equiv 1 \bmod p$, so $N(p) = kp + 1 = q$ and $p \mid q - 1$.

$G \cong \langle a, b \mid a^q = b^p = 1, bab^{-1} = a^r \rangle$, where $r^p \equiv 1 \bmod q$, $r \in \{2, \dots, q-1\}$ and $p \mid q - 1$.

Let $F = \langle x, y \rangle$ and define epimorphism $\pi : F \to G$ by $\pi(x) = a$, $\pi(y) = b$. $K := \langle x^q, y^p, yxy^{-1}x^{-r} \rangle \subseteq \ker \pi$.

$G \cong F/\ker \pi$. $|F/K| = |F/\ker \pi| \cdot |\ker \pi/K| = |G| \cdot |\ker \pi/K| = pq \cdot |\ker \pi/K|$.

Elements in F/K have form $\bar{x}^i \bar{y}^j$, $0 \leqslant i \leqslant q-1$, $0 \leqslant j \leqslant p-1$. $|F/K| \leqslant pq$, so $|\ker \pi/K| = 1$, $K = \ker \pi$.

This proof doesn't guarantee the existence of non-abelian group of order pq.

It only proves if such group exists, it can be characterized in this way.

For $n = pq \leqslant 50$, $n = 15, 21, 33, 35, 39$.

1. For $|G| = 15 = 3 \cdot 5$, $3 \nmid 5$. G is abelian and $G \cong \mathbb{Z}_3 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{15}$ [15, 1].

2. For $|G| = 21 = 3 \cdot 7$, $3 \mid 6$.

(1) G is abelian. $G \cong \mathbb{Z}_3 \oplus \mathbb{Z}_7 \cong \mathbb{Z}_{21}$ [21, 2].

(2) G is non-abelian. Let $G_1 = \langle a, b \mid a^7 = b^3 = 1, bab^{-1} = a^2 \rangle$, $G_2 = \langle a', b' \mid a'^7 = b'^3 = 1, b'a'b'^{-1} = a'^4 \rangle$.

$\pi : G_1 \to G_2, \pi(a) = a'^4$, $\pi(b) = b'^2$ is isomporphism. $G \cong \langle a, b \mid a^7 = b^3 = 1, bab^{-1} = a^2 \rangle$ [21, 1].

3. For $|G| = 33 = 3 \cdot 11$, $3 \nmid 10$. G is abelian. $G \cong \mathbb{Z}_3 \oplus \mathbb{Z}_{11} \cong \mathbb{Z}_{33}$ [33, 1].

4. For $|G| = 35 = 5 \cdot 7$, $5 \nmid 6$. G is abelian and $G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_7 \cong \mathbb{Z}_{35}$ [35, 1].

5. For $|G| = 39 = 3 \cdot 13$, $3 \mid 12$.

(1) G is abelian. $G \cong \mathbb{Z}_3 \oplus \mathbb{Z}_{13} \cong \mathbb{Z}_{39}$ [39, 2].

(2) G is non-abelian. Let $G_1 = \langle a, b \mid a^{13} = b^3 = 1, bab^{-1} = a^3 \rangle$, $G_2 = \langle a', b' \mid a'^{13} = b'^3 = 1, b'a'b'^{-1} = a'^9 \rangle$.

$\pi : G_1 \to G_2$, $\pi(a) = a'^9$, $\pi(b) = b'^2$ is isomporphism. $G \cong \langle a, b \mid a^{13} = b^3 = 1, bab^{-1} = a^3 \rangle$ [39, 1].

17 Groups of order p^3

Suppose $|G| = p^3$ and p is an odd prime.

1. G is abelian. $G \cong \mathbb{Z}_{p^3}$ or $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$ or $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$.

2. G is non-abelian. If G doesn't contain element of order p^2, then $G \cong \langle a, b \mid a^{p^2} = b^p = 1, b^{-1}ab = a^{1+p} \rangle$.

If G contains an element of order p^2, then $G \cong \langle a, b, c \mid a^p = b^p = c^p = 1, ac = ca, cb = bc, ab = bac \rangle$.

(1) G doesn't contain element of order p^2.

G is non-abelian, so $G/Z(G)$ is not cyclic and $G/Z(G) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$. Let $G/Z(G) = \langle \bar{a} \rangle \oplus \langle \bar{b} \rangle$, $a, b \in G \setminus Z(G)$.

By assumption, $o(a) = o(b) = p$, hence $c := a^{-1}b^{-1}ab \in Z(G)$. $a, b, Z(G)$ generate G, so $c \neq 1$ and $Z(G) = \langle c \rangle$.

a, b, c generate G, and $a^p = b^p = c^p = 1$, $ac = ca$, $bc = cb$, $ab = bac$.

Let $F = \langle x, y, z \rangle$ and define epimorphism $\pi : F \to G$ by $\pi(x) = a$, $\pi(y) = b$, $\pi(z) = c$.

$K := \langle x^p, y^p, z^p, xzx^{-1}z^{-1}, yzy^{-1}z^{-1}, x^{-1}y^{-1}xyz^{-1} \rangle \subset \ker \pi$ and $G \cong F/\ker \pi$.

$|F/K| = |F/\ker \pi| \cdot |\ker \pi/K| = p^3 \cdot |\ker \pi/K|$. Elements in F/K have form $\bar{x}^i \bar{y}^j \bar{z}^k$ where $i, j, k \in \{0, \ldots, p-1\}$.

Therefore, $|F/K| \leqslant p^3$ and $\ker \pi = K$. $G \cong \langle a, b, c \mid a^p = b^p = c^p = 1, ac = ca, cb = bc, ab = bac \rangle$.

(2) $a \in G$ and $o(a) = p^2$. $|G| = p^3$, $|\langle a \rangle| = p^2$, so $\langle a \rangle \lhd G$. Let $G/\langle a \rangle = \langle \bar{b} \rangle$. $|G/\langle a \rangle| = p$, so $b \notin \langle a \rangle$ and $b^p \in \langle a \rangle$.

Let $bab^{-1} = a^r$, $r \in \{1, \ldots, p^2 - 1\}$. G is a non-abelian group generated by a, b, so $r \neq 1$.

$b^{-i}ab^i = a^{r^i}$, $a = b^{-p}ab^p = a^{r^p}$, so $r^p \equiv 1 \pmod{p^2}$. $(r, p) = 1$, so $r^{p-1} \equiv 1 \pmod p$. Hence $r \equiv 1 \pmod p$.

Let $r = 1 + tp$, $t \in \{1, \cdots, p-1\}$. $(t, p) = 1$, and there exists j s.t. $jt \equiv 1 \pmod p$. $(j, p) = 1$, so $b^j \notin \langle a \rangle$.

$b^{-j}ab^j = a^{r^j} = a^{(1+tp)^j} = a^{1+jtp} = a^{1+p}$. Replace b^j by b, we have $b \notin \langle a \rangle$, $b^p \in \langle a \rangle$, $b^{-1}ab = a^{1+p}$.

Let $b^p = a^s$. By assumption, $o(b) = p$ or $o(b) = p^2$, so $b^p = a^s$ has order 1 or p, and hence $p \mid s$.

Let $s = pu$, $b^p = a^{up}$. From $a^ib = ba^{(1+p)i}$, we have $(ba^{-u})^p = b^p a^{-u[1+(1+p)+(1+p)^2+\cdots+(1+p)^{p-1}]}$.

$1 + (1+p) + (1+p)^2 + \cdots + (1+p)^{p-1} = \frac{(p+1)^p - 1}{p} \equiv p \pmod{p^2}$, so $(ba^{-u})^p = b^p a^{-up} = 1$.

Let $c = ba^{-u}$, then we have $c^p = 1$, $c \notin \langle a \rangle$, and $c^{-1}ac = a^u \left(b^{-1}ab\right) a^{-u} = a^{1+p}$.

G is generated by element a of order p^2 and c of order p, and $ac = ca^{p+1}$.

Let $F = \langle x, y \rangle$ and define epimorphism $\pi : F \to G$ by $\pi(x) = a$, $\pi(y) = c$. Similarly, we have $G \cong F/\ker \pi$.

$G \cong \langle a, b \mid a^{p^2} = b^p = 1, b^{-1}ab = a^{1+p} \rangle$.

This proof doesn't guarantee the existence of non-abelian group of order p^3.

It only proves if such group exists, it can be characterized in this way.

For $|G| = p^3 \leqslant 50$ and p is odd prime, $|G| = 27$.

1. G is abelian. (1) $G \cong \mathbb{Z}_{27}$ $[27, 1]$. (2) $G \cong \mathbb{Z}_3 \oplus \mathbb{Z}_9$ $[27, 2]$. (3) $G \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ $[27, 5]$.

2. G is non-abelian.

(1) $G \cong \langle a, b \mid a^9 = b^3 = 1, b^{-1}ab = a^4 \rangle$ $[27, 4]$.

(2) $G \cong \langle a, b, c \mid a^3 = b^3 = c^3 = 1, ac = ca, cb = bc, ab = bac \rangle$ $[27, 3]$.