这个文档比较模糊，清晰版可以在我的知乎主页上找到（知乎搜索用户名yx3x）。我的原稿放在家里没带在身边。

Ch7 习题

1. $\begin{cases} u_t - \Delta u = f & \text{in } U_T \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \times [0,T] \\ u = g & \text{on } U \times \{t=0\} \end{cases}$ 至多一个光滑解

Proof: ~~全世~~ 设 u_1, u_2 为原方程2个光滑解. $v = u_1 - u_2$ 要证 $v=0$

v 满足 $\begin{cases} v_t - \Delta v = \cancel{f}\, 0 & \text{in } U_T \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial U \times [0,T] \\ v = 0 & \text{on } U \times \{t=0\} \end{cases}$

两边乘以v. 积分得.

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{L^2}^2 - \int_U v\,\Delta v = 0$$

分部积分

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|v(t)\|_{L^2}^2 = -\int_U |\nabla v|^2\,dx \le 0$$

又: $\|v(0)\|_{L^2}^2 = 0$ 故 $\forall t \in (0,T]$. $\|v(t)\|_{L^2}^2 \le 0$

$$\Rightarrow \|v(t)\|_{L^2} = 0 \quad \underset{v \in C^\infty}{\Rightarrow} \quad v = 0 \quad \text{in } [0,T]$$

□

2. 设 u 是如下方程的光滑解.

$\begin{cases} u_t - \Delta u = \cancel{f}\, 0 & \text{in } U \times (0,\infty) \\ u = 0 & \text{on } \partial U \times [0,\infty) \\ u = g & \text{on } \partial U \times \{t=0\} \end{cases}$ 证明: $\|u(\cdot,t)\|_{L^2(U)} \le e^{-\lambda_1 t}\|g\|_{L^2(U)}$

$\lambda_1 > 0$ 是 $-\Delta$ 的主特征值

Proof: 方程两边乘以u. 积分得

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 = -\int_U |\nabla u|^2\,dx$$

又: $\lambda_1 = \inf\limits_{\substack{u \in H_0^1(U) \\ u \ne 0}} \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^2}^2}$

故 $\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 \le -\lambda_1 \|u(t)\|_{L^2}^2$.

由 Gronwall 不等式

$$\|u(t)\|_{L^2}^2 \le e^{-2\lambda_1 t}\|u(0)\|_{L^2}^2 = e^{-2\lambda_1 t}\|g\|_2^2$$

□

3. 设 u 为
$$\begin{cases} \partial_t u + Lu = 0 & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = g & \text{on } U \times \{t=0\} \end{cases}$$
的光滑解，其中 L 为二阶椭圆算子

v 为
$$\begin{cases} \partial_t v - L^* v = 0 & \text{in } U_T \\ v = 0 & \text{on } \partial U \times [0,T] \\ v = h & \text{on } U \times \{t=T\} \end{cases}$$
的光滑解

求证：$\displaystyle\int_U g(x) v(x,0)\, dx = \int_U u(x,T) h(x)\, dx$.

证明：$\displaystyle\int_U u(x,T) h(x) - g(x) v(x,0)\, dx = \int_U u(x,T) v(x,T) - u(x,0) v(x,0)\, dx$

显见，这是关于 t 分部积分出来的边界项

$$\text{上式} = \int_0^T \int_U (\partial_t u)\cdot v + (\partial_t v)\cdot u \, dx\, dt.$$

$\langle Lu, v\rangle = \langle u, L^* v\rangle$

$$= \int_0^T \int_U (\partial_t u)\cdot v + Lu\cdot v - u L^* v + \partial_t v \cdot u \, dx\, dt$$

$$= \int_0^T \int_U \underbrace{(\partial_t u + Lu)}_{0} v + u\cdot \underbrace{(\partial_t v - L^* v)}_{0} \, dx\, dt = 0$$

□

4. (Galerkin Method for Poisson).

$f \in L^2(U)$. $u_m = \sum_{k=1}^m d_m^k w_k$ solves $\displaystyle\int_U Du_m \cdot Dw_k \, dx = \int_U f \cdot w_k \, dx$, $1 \le k \le m$.

证明：$\{u_m\}$ 存在子列，在 $H_0^1(U)$ 中弱收敛于 $\begin{cases} -\Delta u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$ 的弱解.

Pf. Step 1: $\{u_m\}$ 在 H_0^1 中一致有界

$\displaystyle\int Du_m \cdot Dw_k \, dx = \int_U f \cdot w_k \, dx$. 两边乘以 d_m^k，对 k 求和有，

$$\int \|Du_m\|_{L^2}^2 = \int_U f u_m \, dx$$

$$C\|u_m\|_{H_0^1}^2 \le \quad \le \|f\|_2 \|u_m\|_{H_0^1} \le \varepsilon \|u_m\|_{H_0^1}^2 + C(\varepsilon)\|f\|_{L^2}^2.$$

ε 充分小 $\Rightarrow \|u_m\|_{H_0^1}^2 \le C\|f\|_{L^2}^2$ ✓

Step 2: $\exists$ 子列 $u_{m_k} \rightharpoonup u$ in $H_0^1(U)$.

$Du_{m_k} \rightharpoonup v$ in $L^2(U)$

$v \overset{a.e.}{=} Du$?

$$\forall \varphi \in C_c^\infty \quad \underset{\displaystyle \shortparallel \atop \displaystyle -\int u_{m_k} \cdot D^2\varphi}{\int Du_{m_k} \cdot D\varphi} \;\to\; \int v \cdot D\varphi, \qquad \int Du_{m_k}\cdot D\varphi = \int f \varphi$$

$\downarrow k \to \infty$

$$-\int u \cdot D^2\varphi = \int Du \cdot D\varphi$$

~~同理可得 左 $= \int v D\varphi$~~

□

[7.5] 设

$$\begin{cases} \mathbf{u}_k \rightharpoonup \mathbf{u} \ \ in\, L^2(0,T;H_0^1(U)), \\ \mathbf{u}_k' \rightharpoonup \mathbf{v} \ \ in\, L^2(0,T;H^{-1}(U)). \end{cases}$$

证明:$\mathbf{u}' = \mathbf{v} \ \ in\, L^2(0,T;H^{-1}(U))$.

证明：我们断言：

Claim: 对任意$\phi \in C_c^\infty(0,T), w \in H_0^1(U)$, 成立:

$$\left\langle \int_0^T \phi'(t)\mathbf{u}(t)dt, w \right\rangle = \left\langle -\int_0^T \mathbf{v}(t)\phi(t)dt, w \right\rangle,$$

其中$\langle\cdot,\cdot\rangle$代表$H^{-1}(U), H_0^1(U)$中元素之间的作用(pairing).

若Claim获证，那么在$H^{-1}(U)$中(即作为$H_0^1(U)$上的连续线性泛函)成立:

$$\int_0^T \phi'(t)\mathbf{u}(t)dt = -\int_0^T \mathbf{v}(t)\phi(t)dt.$$

再由时间弱导数定义知

$$\int_0^T \phi'(t)\mathbf{u}(t)dt = -\int_0^T \mathbf{u}'(t)\phi(t)dt.$$

这样就有$\mathbf{u}' = \mathbf{v} \ \ in\, L^2(0,T;H^{-1}(U))$.

Claim的证明仍然由直接计算可得: 注意到$t \mapsto \pi(t)w \in L^2(0,T;H_0^1)$, 那么:

$$\begin{aligned}
\left\langle \int_0^T \phi'(t)\mathbf{u}(t)dt, w \right\rangle &= \int_0^T \langle \phi'(t)\mathbf{u}(t), w\rangle\, dt \\
&= \int_0^T \langle \mathbf{u}(t), \phi'(t)w\rangle\, dt \\
(\mathbf{u}_k \rightharpoonup \mathbf{u} \ \ in\, L^2(0,T;H_0^1(U))) &= \lim_{k\to\infty}\int_0^T \langle \mathbf{u}_k(t), \phi'(t)w\rangle\, dt \\
&= \lim_{k\to\infty}\int_0^T \langle \mathbf{u}_k(t)\phi'(t), w\rangle\, dt \\
&= \lim_{k\to\infty}\left\langle \int_0^T \mathbf{u}_k(t)\phi'(t)dt, w \right\rangle \\
&= -\lim_{k\to\infty}\left\langle \int_0^T \mathbf{u}_k'(t)\phi(t)dt, w \right\rangle \\
&= -\lim_{k\to\infty}\int_0^T \langle \mathbf{u}_k'(t)\phi(t), w\rangle\, dt \\
(\mathbf{u}_k' \rightharpoonup \mathbf{v} \ \ in\, L^2(0,T;H^{-1}(U))) &= -\int_0^T \langle \mathbf{v}(t), \phi(t)w\rangle\, dt \\
&= \left\langle -\int_0^T \mathbf{v}(t)\phi(t)dt, w \right\rangle
\end{aligned}$$

6. 设H是Hilbert空间，$u_k \to u$ in $L^2(0,T;H)$，$\underset{0\le t\le T}{\operatorname{ess\,sup}}\|u_k(t)\|\le C$ $\forall k\in\mathbb{Z}_+$

证明：$\underset{0\le t\le T}{\operatorname{ess\,sup}}\|u(t)\|\le C$

Proof：先证：$\forall\, 0\le a\le b\le T$，$v\in H$，有：

(hint). $\int_a^b (v, u_k(t))\,dt \le C\|v\|\cdot|b-a|$. (显然)

若hint成立，则

$$\frac{1}{b-a}\int_a^b (v,u_k(t))\,dt \le C\|v\|$$

这样 由Lebesgue微分定理，

t是[a,b]中点.

对a.e. $t\in[0,T]$ 成立：$|\langle u_k^{(t)}, v\rangle| \le C\|v\|$

$k\to\infty$，由弱收敛知，$|\langle u,v\rangle|\le C\|v\|$ a.e. $t\in[0,T]$

$\Rightarrow \underset{0\le t\le T}{\operatorname{ess\,sup}}\|u(t)\|\le C$.

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6. H为Hilbert空间. $u_k \to u$ in $L^2(0,T;H)$. 且 $\underset{0\le t\le T}{\operatorname{ess\,sup}}\|u_k(t)\|\le C$ $\forall k\in\mathbb{Z}_+$.

求证：$\underset{0\le t\le T}{\operatorname{ess\,sup}}\|u(t)\|\le C$

证明：设 $f_{ab}(v) = \int_a^b (v, u(t))\,dt$.

显然 $f_{ab}\in L^2(0,T;H)$.

故 $\lim\limits_{k\to\infty} f_{a,b}(u_k) = f_{a,b}(u) = \int_a^b (u(t),u(t))\,dt$.

而
$$f_{a,b}(u_k) = \int_a^b (u_k(t), u(t))\,dt \le C\int_a^b \|u(t)\|_H\,dt \le C\sqrt{b-a}\,\Big\|\|u(t)\|_H\Big\|_{L^2_t(a,b)}$$

$$\overset{k\to\infty}{\Longrightarrow} \int_a^b\|u(t)\|_H^2\,dt \le C^2(b-a) \quad \forall a,b\in[0,T]$$

$\Rightarrow \|u(t)\|_H \le C$ a.e. $t\in[0,T]$. □

除以b-a. 用Lebesgue微分定理即可

令$b\to a$

7. 设 u是光滑解：$\begin{cases} u_t - \Delta u + cu = 0 & \text{in } U\times(0,\infty) \\ u = 0 & \text{on } \partial U\times[0,\infty) \\ u = g & \text{on } U\times\{t=0\} \end{cases}$

且函数c 满足 $c \geq \gamma > 0$.

证明：$|u(x,t)| \leq Ce^{-\gamma t}$

Proof：设 $v = e^{\gamma t}u$.

则 $\partial_t v - \Delta v + cv = \gamma e^{\gamma t}u + e^{\gamma t}u_t - e^{\gamma t}\Delta u + ce^{\gamma t}u$

$$= \gamma v + (\underbrace{(\partial_t - \Delta + c)u}_{0})e^{\gamma t}$$

$$= \gamma v$$

$\Rightarrow \begin{cases} \partial_t v - \Delta v + \underbrace{(c-\gamma)}_{\geq 0} v = 0 & \text{in } U\times(0,\infty) \\ v = 0 & \text{on } \partial U\times[0,\infty) \\ v = g & \text{on } U\times\{t=0\} \end{cases}$

由弱极大值原理：

$\forall (x,t) \in U_T$,

$$|v(x,t)| = e^{\gamma t}|u(x,t)| \leq \sup_{\Gamma_T} |v(x,t)|$$

$$= \sup_{x\in U} |g(x)|.$$

$$\Rightarrow |u(x,t)| \leq e^{-\gamma t}\|g\|_{L^\infty}$$

8. 若u是7中方程的光滑解：$g \geq 0$，但 C 有界但不一定非负，证明 $u \geq 0$

Proof：令 $v = e^{-(\|c\|_{L^\infty}+1)t}u$.

$\overset{\text{同7.}}{\Longrightarrow} \begin{cases} \partial_t v - \Delta v + (c + \|c\|_{L^\infty} + 1)v = 0 & \text{in } U\times(0,\infty) \\ v = 0 & \text{on } \partial U\times[0,\infty) \\ v = g & \text{on } U\times\{t=0\} \end{cases}$

由弱极大值原理. $\min\limits_{\bar{U}_T} v \geq -\max\limits_{\Gamma_T} u^- = -\max\limits_{U} g^-$.

$g \geq 0 \Rightarrow g^- \equiv 0 \rightarrow \min\limits_{\bar{U}_T} v \geq 0 \Rightarrow \min\limits_{\bar{U}_T} u \geq 0.$ □

9. 证明 7.1.3 中的 (54).
7.2.3 中的 (59).

Proof:

(54) 是什么？ $\forall u\in H^2(U)\cap H_0^1(U)$ 成立： $\beta\|u\|_{H^2}^2 \le (Lu,-\Delta u)+\gamma\|u\|_{L^2(U)}^2$.

($\exists\beta>0,\ \gamma\ge 0$).

实际用上的只是对 Galerkin 逼近序列用此不等式.

在此，为了方便，我们设 $Lu=-\sum_{i,j=1}^{n}\partial_j(a^{ij}\partial_i u)$ 　　在实际构造中 $\underline{\Delta u_m=0 \text{ on } \partial U}$.

$\{u\in C^\infty,\ u|_{\partial U}=0,\ \Delta u|_{\partial U}=0\}$.

要证： $-\int_U\sum_{i,j}a^{ij}\partial_i u\,\partial_j(\Delta u)\,dx \ge \frac{\theta}{2}\int_U|D^2u|^2dx - C\int_U|u|^2dx$

① 先说二阶导，希望分部积分.

分部积分可得：

$$-\sum_{i,j}\int_U a^{ij}\partial_i u\,\partial_j(\Delta u) = -\sum_{i,j}\int_U a^{ij}\partial_i u\,\nabla\cdot(\nabla\partial_j u)\,dx$$

$$=\sum_{i,j}\int_U\nabla(a^{ij}\partial_i u)\cdot\nabla(\partial_j\nabla u)\,dx - \sum_{i,j}\int_{\partial U}a^{ij}\partial_i u\,(\partial_j\nabla u)\cdot\vec{n}\,dS$$

$$=\sum_{i,j}\left(\int_U a^{ij}\partial_{ik}u\,\partial_{jk}u\,dx+\int_U\partial_k a^{ij}\,\partial_i u\,\partial_{jk}u\,dx\right) - I.$$

$$I:=\sum_{i,j,k}\int_{\partial U}a^{ij}\partial_i u\,\underline{\partial_{jk}u\cdot\cos(\vec{n},e_k)}_{=\frac{\partial(\partial_j u)}{\partial\vec{n}}}\,dS$$

希望对 I 有何控制？

问题：

· 计算出 u 的所有二阶导数 (along ∂U).

· 如何处理 $\frac{\partial u}{\partial x_n}$ (边界的法向导数？)？

手段：边界拉直 (on 单边法向定义)，写成局部坐标.

· 直接计算. 先求 $\partial_{nn}u$. 再求 $\partial_\alpha\partial_n u$. 再求 $\partial_\alpha\partial_\beta u$.

· 希望的结果. $|I|\le\varepsilon\int_U|D^2u|^2+C_\varepsilon\int_U|u|^2dx$.

$\xLeftarrow{\text{由 Young}}$ $|I|\lesssim\int_U|Du|^2dx$

因 I 本身是 ∂U 的积分，若转化为 U 中的积分，只有靠“迹定理”

$\xLeftarrow{\text{迹定理}}$ $|I|\lesssim\int_{\partial U}\left|\frac{\partial u}{\partial n}\right|^2\underline{dS}$.

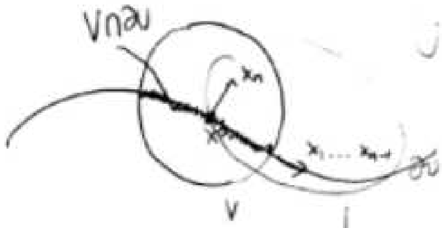
$\xLeftarrow{\text{局部坐标下，法向为 } e_n}$ $|I|\lesssim\sum\int_\Sigma(\partial_{x_n}u)^2\,dx^{n-1}$ ← 于是这成为了我们的目标.

即：设法用 $\partial_{x_n}u$ 等，来给出 I 中各个导数 (尤其是二阶的) 的估计.

$\partial_{x_n}\partial_n u$.

于是，我们现在要做的是，将全部∂U上的各阶导数用 $\partial_{x_n}u$，$\partial\partial_{x_n}^2 u$ 表示出来。

Step 1：由单位分解（因∂U紧），可以假设 a^{ij} 的支集包含于某一点 $x_0\in\partial U$ 的邻域 V 内.



不妨设：x^0 是 ~~原点~~ 原点，
∂U 在 x^0 处的法向是 x_n 轴 ($\vec{e}_n$).

Σ = $V\cap\partial U$ 在 x_n 轴的投影.

记 $x_n = w(x')$. $x'=(x_1,\ldots,x_{n-1})\in\Sigma$

$w\in C^2(\Sigma)$.

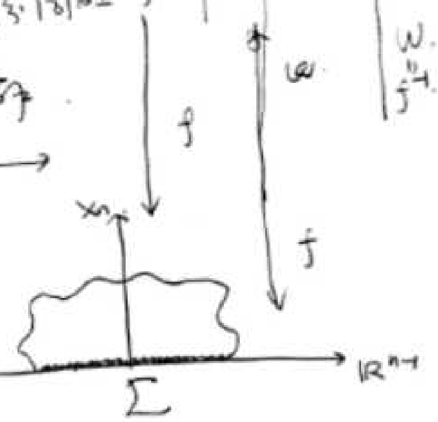
局部微分同胚于上半平面

→ 这一段文字其实可以用 带边流形之语言来理解

如同 则存在 ~~坐标~~ 微分同胚 f：

w 即是"该微分同胚之逆"

坐标映射之逆



$(\partial_\alpha = \frac{\partial}{\partial x_\alpha})$.

Step 2：计算 $\partial_\alpha\partial_\beta u$. $(\partial_\alpha = \frac{\partial}{\partial x_\alpha})$.

(1°) 令 $v(x') = \partial_n u(x', w(x'))$ → 即 $V\cap\partial U$ 任一点，用局部坐标可写成 $(x', w(x'))$

求 $\partial_n^2 u$.

对上式求导 $(\alpha=1,2,\ldots,n-1)$.

有：$\partial_\alpha v = \partial_\alpha\partial_n u + \partial_n^2 u\cdot\partial_\alpha w$. … ①

又：$u|_{\partial U}=0$. 故. $u(x', w(x'))=0$. $\forall x'\in\Sigma$.

对 x_α 求导：$\partial_\alpha u + \partial_n u\,\partial_\alpha w = \partial_\alpha u + v\cdot\partial_\alpha w = 0$. on $V\cap\partial U$.

对 x_β 求导. $\partial_\alpha\partial_\beta u + \partial_\alpha\partial_n u\,\partial_\beta w + \partial_\beta v\,\partial_\alpha w + v\cdot\partial_\alpha\partial_\beta w = 0$. …②

$1\le\alpha,\beta\le n-1$.

取 $\alpha=\beta$ 有：$\partial_\alpha^2 u + \partial_\alpha\partial_n u\,\partial_\alpha w + \partial_\alpha v\,\partial_\alpha w + v\cdot\partial_\alpha^2 w = 0$ …③

对 α 从 $1\sim n-1$ 求和：

注意到 $\Delta u\big|_{\partial u}=0 \Rightarrow -\partial_n^2 u = \sum\limits_{1\le\alpha\le n-1} \partial_\alpha^2 u$.

有：$-\partial_n^2 u + \sum\limits_{1\le\alpha\le n-1} \partial_\alpha\partial_n u \cdot \partial_\alpha w + \partial_\alpha v\, \partial_\alpha w + v\,\partial_\alpha^2 w = 0$ …④.

①代入④．有：（重复指标代表求和）．

$$-\partial_n^2 u + (\partial_\alpha v - \partial_n^2 u\, \partial_\alpha w)\partial_\alpha w + \partial_\alpha v\,\partial_\alpha w + v\,\partial_\alpha^2 w = 0$$

$$\Rightarrow \partial_n^2 u\left(1+\sum_\alpha(\partial_\alpha w)^2\right) = v\Delta_{n-1} w + 2\partial_\alpha v\,\partial_\alpha w$$

$$\Rightarrow \partial_n^2 u = \frac{2\partial_\alpha w}{\sqrt{1+|\nabla_{n-1}w|^2}}\partial_\alpha v + \frac{\Delta w}{\sqrt{1+|\nabla_{n-1}w|^2}} v$$

这样．对 $\partial_n^2 u$，我们达到了目的，即用 $\partial_\alpha v$．v (i.e. $\partial_\alpha\partial_n u$. $\partial_n u$) 表示．

方便起见．令．

$$\sigma_{nn}^\alpha = \frac{2\partial_\alpha w}{\sqrt{1+|\nabla_{n-1}w|^2}} \in C^1(\Sigma)$$

$$T_{nn} = \frac{\Delta w}{\sqrt{1+|\nabla_{n-1}w|^2}} \in C(\Sigma).$$

$$\Rightarrow \partial_n^2 u = \sigma_{nn}^\alpha \partial_\alpha v + T_{nn} v \quad \cdots ⑤$$

$\boxed{\text{上下指标表示求和}}$

(2) 求 $\partial_\alpha\partial_n u$.

⑤代入①．即有：

α 换成 β．因这里与①中 α 重复．

$$\partial_\alpha v = \partial_\alpha\partial_n u + \partial_\alpha w\left(\sigma_{nn}^\beta \partial_\beta v + T_{nn} v\right).$$

$$\Rightarrow \partial_\alpha\partial_n u = \partial_\alpha v - \partial_\alpha w\left(\sigma_{nn}^\beta \partial_\beta v + T_{nn} v\right).$$

$$= \partial_\beta v\left(\delta_\alpha^\beta - \sigma_{nn}^\beta \partial_\alpha w\right) - T_{nn}\partial_\alpha w \cdot v.$$

令．$T_{\alpha n} = T_{nn}\partial_\alpha w \in C(\Sigma)$　$\sigma_{\alpha n}^\beta = \delta_\alpha^\beta - \sigma_{nn}^\beta \partial_\alpha w \in C^1(\Sigma)$

有：$\partial_\alpha\partial_n u = \sigma_{\alpha n}^\beta \partial_\beta v - T_{\alpha n} v$ …⑥

(3°). 求 $\partial_\beta\partial_\alpha u$.

①代入② 有:

$$\partial_{\alpha\beta} u + (\sigma_{\alpha n}^{\nu}\partial_\nu v + \tau_{\alpha n} v)\partial_\beta w + \partial_\beta v\,\partial_\alpha w + v\,\partial_\alpha\partial_\beta w = 0$$

$$\Rightarrow \partial_{\alpha\beta} u = -(\sigma_{\alpha n}^{\nu}\partial_\nu v + \tau_{\alpha n} v)\partial_\beta w - \partial_\beta v\,\partial_\alpha w - \partial_{\alpha\beta} w\cdot v$$

$$= \partial_\nu v\left(-\sigma_{\alpha n}^{\nu}\partial_\beta w - \delta_\beta^{\nu}\partial_\alpha w\right)$$

$$+ v\left(-\partial_{\alpha\beta} w - \tau_{\alpha n}\partial_\beta w\right)$$

令 $\sigma_{\alpha\beta}^{\nu} = -\sigma_{\alpha n}^{\nu}\partial_\beta w - \delta_\beta^{\nu}\partial_\alpha w$

$\tau_{\alpha\beta} = -\partial_{\alpha\beta} w - \tau_{\alpha n}\partial_\beta w$

$$\Rightarrow \partial_{\alpha\beta} u = \sigma_{\alpha\beta}^{\nu}\partial_\nu v + \tau_{\alpha\beta} v \qquad \sigma_{\alpha\beta}^{\nu} \in C^1(\Sigma),\ \tau_{\alpha\beta} \in C(\Sigma)$$

Step3: 完成估计:

如今. 可以用一些 $C^1(\Sigma)$ 上的 g^α 与 $C^0(\Sigma)$ 上的 h.

表示 I 如下:

$$I = \int_\Sigma v(g^\alpha\partial_\alpha v + hv)\, d\mathcal{H}^{n-1}\cdot \underbrace{|\det w|}_{=1}$$

$$|I| = \cancel{\left|\int_\Sigma \frac{v(g^\alpha\partial_\alpha v^2 + hv^2)}{2}\right|}$$

$$\left|\int_\Sigma \left(\frac{1}{2}g^\alpha\partial_\alpha(v^2) + hv^2\right) d\mathcal{H}^{n-1}\right|.$$

$$\overset{\text{分部积分}}{=} \left|\int_\Sigma \left(h - \frac{1}{2}\partial_\alpha g^\alpha\right) v^2\, d\mathcal{H}^{n-1}\right|.$$

$$\lesssim \int_\Sigma v^2\, d\mathcal{H}^{n-1} \lesssim \int_{\partial U}\left|\frac{\partial u}{\partial n}\right|^2 dS \overset{\text{迹定理}}{\lesssim} \int_U |Du|^2 dx$$

再由 Ch5. T9 有 $\int_U |Du|^2 dx \leq \|u\|_{L^2}\|D^2u\|_{L^2} \lesssim \varepsilon\|D^2u\|_{L^2}^2 + C(\varepsilon)\|u\|_{L^2}^2$ □

10. 求证：如下方程至多一个光滑解 ~~$d>0$ ← 书上应该漏条件~~

$$\begin{cases} u_{tt} + d u_t - u_{xx} = f & \text{in } (0,1)\times(0,T) \\ u = 0 & \text{on } (\{0\}\times[0,T])\cup(\{1\}\times[0,T]) \\ u(0)=g \quad u_t(0)=h \end{cases}$$

证明：若有2个光滑解 u_1, u_2，令 $v = u_1 - u_2$，

则 v 满足

$$\begin{cases} v_{tt} + d v_t - v_{xx} = 0 & \text{in } (0,1)\times(0,T) \\ v = 0 & \text{on } (\{0\}\times[0,T])\cup(\{1\}\times[0,T]) \\ v(0)=g. \quad v_t(0)=h. \end{cases}$$

令 $E(t) = \frac{1}{2}\|v_t\|_{L^2}^2 + \frac{1}{2}\|v_x\|_2^2$

则 $E'(t) = \int_0^1 v_t v_{tt} + v_x v_{xt}\, dx$

第2项分部积分.

$= \int_0^1 v_t v_{tt} - v_{xx} v_t\, dx$

$= \int_0^1 (-d v_t) v_t\, dx$

$= -d \int_0^1 v_t^2 \leq 0$ ← $d \geq 0$

又 $E(0)=0$，$\therefore E(t) \equiv 0 \Rightarrow v(t) = \text{const}.$

又 $v(0)=0$ $\therefore v \equiv 0$

d<0也是可以做的。当d<0时，上面不等式最后一步可以被-d∫(v_t)^2+(v_x)^2 控制。也就是说我们有E'(t)≤-2dE(t)
也就是d/dt(e^{2dt}E(t))≤0. 而E(0)=0，那么必有E(t)=0. □

11. $\begin{cases} \partial_t^2 u + \partial_x^4 u = 0 & \text{in } (0,1)\times(0,T). \\ u = \partial_x u = 0 & \text{on } (\{0\}\times[0,T]) \cup (\{1\}\times[0,T]) \\ u = g,\ u_t = h. & \text{on } [0,1]\times\{t=0\} \end{cases}$

存在至多一个光滑解

Proof:

$V = u_1 - u_2$.

$\Rightarrow \partial_t^2 V + \partial_x^4 V = 0.$

乘以 V_t $\Rightarrow$ $\partial_t V \partial_t^2 V + \partial_t V \partial_x^4 V = 0.$

而: $\partial_t (\partial_t V)^2 = 2\partial_t^2 V \partial_t V.$

$\partial_t (\partial_x^2 V) = 2(\partial_x^2 V \cdot \partial_t \partial_x^2 V)$

所以上式 $\Rightarrow$ $\frac{1}{2}\partial_t \left(\|V_t\|_{L^2}^2 + \|V_{xx}\|_{L^2}^2 \right) = 0$

积分.

(分部2次, 对 x).

$\Rightarrow \|V_t\|^2 + \|V_{xx}\|^2 = const. =$ $t=0$ 时的值 $= 0$

$\Rightarrow V = 0.$

□.

12: 设 A 为实Banach空间 X 上的闭算子, 定义域为 $D(A)$ ~~预解式为ρ(A)~~ 若 $\lambda, \nu \in \rho(A)$

求证: (1) $R_\lambda - R_\nu = (\nu - \lambda) R_\lambda R_\nu$

(2) $R_\lambda R_\nu = R_\nu R_\lambda$.

证明: (1) 不妨 $\lambda \neq \nu$

$R_\lambda - R_\nu = R_\lambda \cdot \underbrace{(\nu I - A) R_\nu}_{= Id} - \underbrace{(\lambda I - A) R_\lambda}_{Id} R_\nu$

$= (\nu - \lambda) R_\lambda R_\nu$

(2) 调换 λ, ν. 有 $R_\nu R_\lambda (\lambda - \nu) = R_\nu - R_\lambda$.

$= -(R_\lambda - R_\nu)$

$= -(\nu - \lambda) R_\lambda R_\nu$

约掉 $\lambda - \nu$

$\Rightarrow R_\lambda R_\nu = R_\nu R_\lambda$

□

13. Justify the equality

$$A \int_0^\infty e^{-\lambda t} S(t)u \, dt = \int_0^\infty e^{-\lambda t} AS(t)u \, dt$$

used in (16) of §7.4.1. (Hint: Approximate the integral by a Riemann sum and recall A is a closed operator.)

先证明：对 $\int_0^M$. A 可与之换序：

$\forall M \in \mathbb{R}_+$. 对 $[0, M]$ 作 k 等分：

$$[0, M] = \bigcup_{j=0}^{2^k-1} \left[\frac{j}{2^k}M, \frac{j+1}{2^k}M\right]$$

由 $I_k(t) := \sum_{j=1}^{k} e^{-\lambda t_j} S(t_j)u \, \chi_{t \in [t_{j-1}, t_j]} \rightrightarrows e^{-\lambda t} S(t)u$ as $k \to \infty$

↑simple functions

及 A 闭 知：

$$A \int_0^M e^{-\lambda t} S(t)u \, dt = A\left(\lim_{k\to\infty} \int_0^M I_k(t)\, dt\right)$$

$$\overset{A\text{闭}}{=} \lim_{k\to\infty} A \int_0^M I_k(t)\, dt$$

$$\underset{I_k \text{ simple}}{=} \lim_{k\to\infty} \int_0^M A I_k(t)\, dt$$

$$\underset{MCT}{\overset{A\text{闭}}{=}} \int_0^M \lim_{k\to\infty} A I_k(t)\, dt$$

$$= \int_0^M A e^{-\lambda t} S(t)u \, dt \overset{A \text{ linear}}{=\!=} \int_0^M e^{-\lambda t} AS(t)u \, dt$$

又：$e^{-\lambda t}$ rapidly decays，$\|S(t)\| \le 1$，A 闭

$$\Rightarrow \int_0^M e^{-\lambda t} S(t)u \, dt \xrightarrow[\exists]{M\to\infty} \int_0^{+\infty} e^{-\lambda t} S(t)u \, dt.$$

$$\Rightarrow A\left(\int_0^M e^{-\lambda t} S(t)u \, dt\right) \xrightarrow{M\to\infty} A\left(\int_0^\infty e^{-\lambda t} S(t)u \, dt\right)$$

therefore "||" holds.

|| 刚刚已证.

$$\int_0^M e^{-\lambda t} AS(t)u \, dt \xrightarrow{M\to\infty} \int_0^\infty e^{-\lambda t} AS(t)u \, dt.$$

||

$$\int_0^M e^{-\lambda t} S(t)Au \, dt \xrightarrow{M\to\infty} \int_0^\infty e^{-\lambda t} S(t)Au \, dt$$

□.

14.

设 ϕ 是热方程的基本解，即 $\phi(x,t)=\frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}}$.

$\forall t>0$，令

$$[S(t)g](x)=\int_{\mathbb{R}^n}\phi(x-y,t)g(y)\,dy. \qquad x\in\mathbb{R}^n.$$

$$g:\mathbb{R}^n\longrightarrow\mathbb{R}.$$

$$S(0)g=g.$$

则：$\{S(t)\}_{t\geq 0}$ 是 $L^2(\mathbb{R}^n)$ 上的压缩半群

不是 $L^\infty(\mathbb{R}^n)$ 上的压缩半群

· $S(t)$ 是 L^2 上的压缩半群

· $\|S(t)g\|_{L^2}=\|\phi*g\|_{L^2}$

$$\leq\|\phi\|_{L^1}\|g\|_{L^2}=\|g\|_{L^2} \Rightarrow \|S(t)\|\leq 1.$$

· $\widehat{S(t+s)g}=\widehat{\phi(x-y,\,t+s)*g}(\xi)$.

$$\hat\phi(\xi,t+s)\,\hat g(\xi)$$

$$=\hat\phi(\xi,t)\,\hat\phi(\xi,s)\,\hat g(\xi)=\widehat{S(t)S(s)g}$$

· $t\mapsto S(t)g$ 连续性：

$$\|S(t+h)g-S(t)g\|_{L^2}\leq\|S(h)g-g\|_{L^2}.$$

$$=\Big\|\int_{\mathbb{R}^n}\phi(x-y,h)g(y)\,dy-g(x)\Big\|_{L^2_x}$$

$$=\Big\|\int_{\mathbb{R}^n}\phi(x-y,h)(g(y)-g(x))\,dy\Big\|_{L^2_x}$$

$$=\Big\|\int_{\mathbb{R}^n}\phi(y,h)(g(x-y)-g(x))\,dy\Big\|_{L^2_x}$$

$$\leq\int_{\mathbb{R}^n}\phi(y,h)\,\underbrace{\|g(x-y)-g(x)\|_{L^2_x}}_{\downarrow 0\ \text{(平移连续性)}}\,dy$$

$$\overset{DCT}{=}0.$$

·$S(t)$不是L^∞上的压缩半群，因 在$t=0$处不连续

令. ~~$g(x,y)=\chi_{\{x>0,\ y>0\}}$~~ $g(x)=\chi_{\{x>0\}}$. $g(0)=0$.

$(S(t)g)(x)=\int_{-\infty}^{x}\frac{e^{-\frac{y^2}{4t}}}{\sqrt{4\pi t}}dy$. $\quad (S(t)g)(0)=\frac{1}{2}$ $\quad \forall t>0$

$\Rightarrow \|S(t)g-g\|_{L^\infty}\geq\frac{1}{2}$ □

15. 设X上有以A为无穷小生成元的压缩半群$\{S(t)\}_{t\geq0}$

归纳地定义 $D(A^k):=\{u\in D(A^{k-1}) \mid A^{k-1}u\in D(A)\}$ $(k\geq2)$

证明：若$\exists k$. $u\in D(A^k)$, 则 $\forall t\geq0$ $S(t)u\in D(A^k)$

Pf. $\forall j\in\{1,2,\dots,k-1\}$. 要证: $A^jS(t)u\in D(A)$

i.e. $\lim_{s\to0^+}\frac{S(s)A^jS(t)u-A^jS(t)u}{s}$ $\exists$

$$\frac{S(s)A^jS(t)u-A^jS(t)u}{s}$$
$$=\frac{S(s)S(t)A^ju-S(t)A^ju}{s}$$
$$=\frac{S(t+s)(A^ju)-S(t)(A^ju)}{s}$$

归纳假设 $A^ju\in D(A)$

$\overset{s\to0}{=}$ exists $\quad$ (以及 $S(0)=Id$). □

16. 用T15证明：若u是 $\begin{cases}u_t-\Delta u=0 & \text{in } U_T\\ u=0 & \text{on } \partial U\times[0,T]\\ u=g & \text{on } U\times\{t=0\}\end{cases}$ 在$X=L^2(U)$中的半群解

且 $g\in C_c^\infty(U)$, 则 $u(\cdot,t)\in C^\infty(U)$, $0\leq t\leq T$

Pf. 先证$\{S(t)\}_{t\geq0}$压缩，此为显见. 因 在方程两边同乘u. 分部积分有

$\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2)+\|\nabla u\|^2=0$. $\Rightarrow \frac{d}{dt}\|u\|_{L^2}^2\leq0$

$\Rightarrow \|u(t)\|_{L^2}^2\leq\|u(0)\|_{L^2}^2$

$\therefore$ $\{S(t)\}$压缩 $\quad \Rightarrow \|S(t)g\|_{L^2}\leq\|g\|_{L^2}$ $\forall g\in L^2$.

而 $g\in C_c^\infty(U)\subset H^{2k}(U)\cap H_0^{2k-1}(U)=D(\Delta^k)$

$k=1$ 是显然的. 由归纳即可.

由T15知. $\forall t\geq0$

$u(\cdot,t)=S(t)g\in H^{2k}(U)\cap H_0^{2k-1}(U)$. $\forall k\in\mathbb{Z}_+$.

由Sobolev嵌入知 $u(\cdot,t)\in C^\infty(U)$.

□