Ch 6 习题：本节假设 L 一致椭圆，$U\subset\mathbb{R}^n$ 为有界开集，$\partial U\in C^\infty$.

[6.1] 考虑带位势 c 的 Laplace 方程 $-\Delta u+cu=0 \quad \cdots(*)$

和散度形式的方程 $-\mathrm{div}(aDv)=0 \qquad a>0.$

(1) 证明：若 u 为 $(*)$ 的解，$w>0$ 也是 $(*)$ 的解，则 $v=\frac{u}{w}$ 是 $(**)$ 的解 $(a=w^2)$

(2) 反之，若 v 是 $(**)$ 的解，则 $u=va^{\frac12}$ 是 $(*)$ 的解（对某个位势 c）.

证明：(1). $\quad -\Delta u+cu=0, \quad -\Delta w+cw=0$

$$v=\frac{u}{w}$$

$$\Rightarrow \partial_i v=\frac{\partial_i u\cdot w-u\cdot\partial_i w}{w^2} \qquad \cancel{\Rightarrow a\partial_i v=\frac{a\partial_i u\, w-au\partial_i w}{w^2}}$$

$$\cancel{\mathrm{div}(aDv)}=\sum_{i=1}^n\partial_i(a\partial_i v)$$

$$\cancel{=\sum_{i=1}^n\partial_i a\,\partial_i v+\sum_{i=1}^n a\,\partial_i\partial_i v}$$

$$\cancel{=\sum_{i=1}^n\partial_i a\cdot\frac{\partial_i u\, w-u\partial_i w}{w^2}+\sum_{i=1}^n a\cdot\frac{\partial_i(\partial_i u\cdot w-u\partial_i w)w^2-2w\partial_i w(\partial_i u\cdot w-u\partial_i w)}{w^4}}$$

$$a=w^2 \Rightarrow a\partial_i v=\partial_i u\cdot w-\partial_i w\cdot u$$

$$\mathrm{div}(aDv)=\sum_{i=1}^n \partial_i(\partial_i u\, w-\partial_i w\cdot u)$$

$$=\sum_{i=1}^n(\partial_i\partial_i u\cdot w+\partial_i u\partial_i w-\partial_i w\partial_i u-\partial_i\partial_i w\cdot u)$$

$$=\Delta u\cdot w-\Delta w\cdot u$$

$$=cuw-cuw=0.$$

(2). 若 $-\mathrm{div}(aDv)=0$.

则 $\sum_{i=1}^n\partial_i(a\partial_i v)=0 \Rightarrow \sum_{i=1}^n\partial_i a\cdot\partial_i v+a\sum_{i=1}^n\partial_i\partial_i v=0$

$$-\Delta u+cu \underset{u=a^{\frac12}v}{=} -\sum_{i=1}^n\partial_i\partial_i(va^{\frac12})+cva^{\frac12}$$

$$=cva^{\frac12}-\sum_{i=1}^n\partial_i\left(\partial_i v\cdot a^{\frac12}+\frac12 a^{-\frac12}\partial_i a\cdot v\right)$$

$$=cva^{\frac12}-\sum_{i=1}^n\Big(\partial_i\partial_i v\cdot a^{\frac12}+\partial_i v\cdot\frac12 a^{-\frac12}\partial_i a+\frac12 a^{-\frac12}\partial_i a\,\partial_i v$$
$$+(\partial_i\partial_i(a^{\frac12}))v\Big)$$

$$=cva^{\frac12}-a^{-\frac12}\cdot\underbrace{\left(\sum_{i=1}^n\partial_i a\,\partial_i v+a\,\partial_i\partial_i v\right)}-\sum_{i=1}^n(\partial_i\partial_i a^{\frac12})v$$

$$=v\left(ca^{\frac12}-\Delta\sqrt{a}\right)\overset{\text{“}}{=}0$$

取 $c=\dfrac{\Delta\sqrt{a}}{\sqrt{a}}$ 即可 □

[6.2]. 设 $Lu = -\sum_{i,j=1}^{n} \partial_j(a^{ij}\partial_i u) + cu$.

证明: 存在常数 $\mu>0$ s.t. 在 $c(x) \geq -\mu$ $(x\in U)$ 的条件下, $B[\cdot,\cdot]$ 满足 Lax-Milgram 定理条件

证明: $B[u,v] = \sum_{i,j=1}^{n}\int_U a^{ij}\partial_i u \partial_j v + cuv \quad \forall u,v\in H_0^1(U)$

① $$|B[u,v]| \leq \|a^{ij}\|_{L^\infty}\sum_{i,j=1}^{n}\int_U |\partial_i u||\partial_j v| + \|c\|_{L^\infty}\int_U |u|\cdot|v| \, dx$$

$$\overset{\text{Hölder}}{\leq} C\left(\|Du\|_{L^2}\|D_j v\|_{L^2} + \|u\|_{L^2}\|v\|_{L^2}\right)$$

$$\leq C\left(\|u\|_{H_0^1}\|v\|_{H_0^1}\right)$$

② $$|B[u,u]| = \int\sum_{i,j=1}^{n} a^{ij}\partial_i u\partial_j u + cu^2$$

$$\overset{L\text{一致椭圆}}{\geq} \theta\|Du\|_{L^2}^2 + \int cu^2$$

~~$u\in H_0^1$, 由 Poincaré 不等式 $\|u\|_{L^2} \leq C'\|Du\|_{L^2}$ (for some $C'>0$).~~

$$\cancel{= \theta\|Du\|_{L^2}^2 + (c+\mu+\varepsilon_0)\|u\|_{L^2}^2 - (\mu+\varepsilon_0)\|u\|_{L^2}^2}$$

~~Poincaré不等式: 因 $u\in H_0^1(U)$. 故 $\exists C'>0$, $\|u\|_{L^2} \leq C'\|Du\|_{L^2}$~~

$$\cancel{\geq (\theta - c^2\mu - c^2\varepsilon_0)\|Du\|_{L^2}^2 + (c+\mu+\varepsilon_0)\|u\|_{L^2}^2}$$

~~取 μ 满足 $\theta - c^2\mu - c^2\varepsilon_0 \geq \varepsilon_0$ 即 $\mu \leq \frac{\theta}{c^2} - (1+\frac{1}{c^2})\varepsilon_0$ 即可.~~

$$\cancel{\geq \theta\|Du\|_{L^2}^2 - c(C')^2\|Du\|_{L^2}^2}$$

$$= \theta\|Du\|_{L^2}^2 + \int (c+\mu+\varepsilon)u^2 - (\mu+\varepsilon)\int u^2$$

Poincaré: $\forall u\in H_0^1(U)$, then $\exists C'>0$, $\|u\|_{L^2} \leq C'\|Du\|_{L^2}$.

$$\geq \theta\|Du\|_{L^2}^2 + \int (c+\mu+\varepsilon)u^2 - (\mu+\varepsilon)(C')^2\|Du\|_{L^2}^2$$

取 $(\mu+\varepsilon)(C')^2 = \frac{\theta}{2}$ (ε 足够小), 于是

上式 $$\geq \frac{\theta}{2}\|Du\|_{L^2}^2 = \frac{\theta}{4}\|Du\|_{L^2}^2 + \frac{\theta}{4}\|Du\|_{L^2}^2$$

$$\overset{\text{Poincaré}}{\geq} \frac{\theta}{4}\|Du\|_{L^2}^2 + \frac{\theta}{4}\frac{1}{C'}\|u\|_{L^2}^2$$

$$\geq \min\left\{\frac{\theta}{4}, \frac{\theta}{4}\frac{1}{C'}\right\}\|u\|_{H_0^1}^2$$

□

[6.3] $u\in H_0^2(U)$ 是如下边值问题 $\begin{cases}\Delta^2 u = f & \text{in } U\\ u=\frac{\partial u}{\partial \nu}=0 & \text{on } \partial U\end{cases}$ 的弱解，若 $\int_U \Delta u \Delta v\,dx = \int_U fv\,dx \quad \forall v\in H_0^2(U)$

今给定 $f\in L^2(U)$，证明该方程存在唯一弱解

证明：令 $B[u,v]=\int_U \Delta u\Delta v\,dx$

① $$|B[u,v]| = \int_U |\Delta u|\cdot|\Delta v|\,dx \overset{\text{Hölder}}{\le} C\|D^2u\|_{L^2}\|D^2v\|_{L^2} \underset{u,v\in H_0^2(U)}{\le} C'\|u\|_{H_0^2}\|v\|_{H_0^2}$$

(由Poincaré知. $\|u\|_2$, $\|Du\|_2$ 由 $\|D^2u\|_2$ 控制)

② $$B[u,u] \overset{\text{先设 } u\in C_c^\infty(U)}{=} \int_U \Delta u\Delta u \cancel{|\Delta u|\cdot|\Delta u|}\,dx$$

$$= \sum_{j,k=1}^n \int \partial_j^2 u\,\partial_k^2 u\,dx$$

$$\overset{\text{分部积分}}{=} -\sum_{j,k=1}^n \int \partial_j u \cdot \partial_j\partial_k^2 u\,dx$$

$$\overset{\text{再分部积分}}{=} \sum_{j,k=1}^n \int (\partial_j\partial_k u)^2\,dx = \|D^2u\|_{L^2}^2$$

$\partial_j\partial_k^2=\partial_k(\partial_j\partial_k)$

$$\underset{\text{Poincaré}}{\ge} c\left(\|D^2u\|_{L^2}+\|u\|_{L^2}^2+\|Du\|_{L^2}^2\right) = c\|u\|^2_{H_0^2(U)}$$

~~对一般的 $u\in H_0^2(U)$，可以找到一列 $u_n\in C_c^\infty(U)$ s.t. $\|u_n-u\|_{H_0^2(U)}\to 0$ as $n\to\infty$.~~

对一般的 $u\in H_0^2(U)$，$\exists\{u_n\}\subset C_c^\infty(U)$ s.t. $\|u_n-u\|_{H_0^2(U)}\to 0$

故 $\|u_n\|_{H_0^2}\to\|u\|_{H_0^2}$.

$$\left|\|\Delta u_n\|_{L^2}^2-\|\Delta u\|_{L^2}^2\right| \le \|\Delta(u_n-u)\|_{L^2}^2 \le C\|D^2(u_n-u)\|_{L^2}^2\to 0$$

$\therefore B[u,u]\ge c\|u\|^2_{H_0^2(U)}$ 对 $u\in H_0^2(U)$ 成立.

那么~~据~~由①②，据 Lax-Milgram 定理知. $\exists!\, u\in H_0^2(U)$

s.t. $\forall v\in H_0^2(U)$，$B[u,v]=(f,v)_{L^2}$. given $f\in L^2$. □

[6.4] 称 $u\in H^1(U)$ 是 Neumann 边值问题的 $\begin{cases} -\Delta u = f & \text{in } U \\ \frac{\partial u}{\partial \nu}=0 & \text{on } \partial U \end{cases}$ 的弱解，是指：

设 U连通

$\forall v\in H^1(U)$，成立：$\int_U \nabla u\cdot \nabla v \, dx=\int_U fv\, dx$.

现设 $f\in L^2(U)$. 证明：上述方程弱解存在 $\Longleftrightarrow \int_U f\,dx=0$.

证明：

$\Rightarrow$) 令 $v=1$

$\Leftarrow$) 令 $B[u,v]=\int_U \nabla u\cdot\nabla v$.

$$H_\sigma^1(U)=\{u\in H^1(U) \mid \int u\,dx=0\}$$

Step 1: $H_\sigma^1(U)$ 为 Hilbert 空间，内积为 $B[\cdot,\cdot]$

实际上，$l: H^1(U)\to\mathbb{R}$，$u\mapsto\int_U u\,dx$ 作为 $H^1(U)$ 上的连续线性泛函，满足：

$H_\sigma^1(U)=l^{-1}(0)$

而 $\{0\}\subseteq\mathbb{R}$ 闭 $\therefore l^{-1}(\{0\})$ 闭 $\Rightarrow H_\sigma^1(U)$ 为 $H^1(U)$ 的闭子空间，从而是 Hilbert 空间

$B[\cdot,\cdot]$ 为内积？check：双线性易见.

只用证 $B[u,u]=0 \Leftrightarrow u=0$ in H_σ^1

实际上，由U连通，据 Poincaré 不等式：

$$\|u-\langle u\rangle_U\|_{L^2}\le\|\nabla u\|_{L^2}=\sqrt{B[u,u]}=0$$

$$\overset{\langle u\rangle_U=0}{\Longrightarrow} u=0 \quad \text{a.e.} \quad ✓$$

Step 2: 由Riesz 表示定理. $\forall f\in L^2(U)$, $\int_U f=0$, $\exists!\, u_f\in H_\sigma^1(U)$,

s.t. $\forall v\in H_\sigma^1(U)$, $\int_U \nabla u_f\cdot\nabla v\,dx=B[u_f,v]=(f,v)$

Step 3: $\forall v\in H^1(U)$, $v-\langle v\rangle_U\in H_\sigma^1(U)$. 故由 Step 2 知，

给定 $f\in L^2$, $\exists!\, u_f\in H_\sigma^1\subset H^1$

s.t. $(f, v-\langle v\rangle_U)=\int_U\nabla u_f\cdot\nabla(v-\langle v\rangle_U)\,dx=\int_U\nabla u_f\cdot\nabla v\,dx$

而 $\int_U f=0$ $\therefore$ $\int fv=\int\nabla u_f\cdot\nabla v\,dx$. 证毕！

□

[6.5]

设 $\begin{cases} -\Delta u = f & \text{in } U \\ u + \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases}$ 如何定义该方程的 $H^1(U)$ 弱解？若给定 $f \in L^2(U)$，如何证明解的存在唯一性？

证明：令 $B[u,v] = \int_U \nabla u \cdot \nabla v \, dx + \int_{\partial U} Tr\, u \cdot Tr\, v \, d\mathcal{H}^{n-1}$ $\forall u, v \in H^1(U)$

* 为何如此定义？若 $u, v \in C^\infty(U)$，则

$$\int -\Delta u \cdot v \underset{\text{分部积分}}{=} \int_U \nabla u \cdot \nabla v \, dx - \int_{\partial U} v \, \nabla u \cdot \vec{\nu} \, d\mathcal{H}^{n-1}$$

$$= \int_U \nabla u \cdot \nabla v \, dx - \int_{\partial U} v \frac{\partial u}{\partial \nu} \, d\mathcal{H}^{n-1}$$

$$\overset{-\frac{\partial u}{\partial \nu} = u \text{ on } \partial U}{=} \int_U \nabla u \cdot \nabla v \, dx + \int_{\partial U} u \cdot v \, d\mathcal{H}^{n-1}$$ 符合我们的定义。

下面 check Lax-Milgram 定理的条件即可。

① $|B[u,v]| \le C \|u\|_{H^1} \|v\|_{H^1}$ 显见（$\int_{\partial U}$ 项用迹定理即可）

② 下证：$B[u,u] = \int_U \nabla u \cdot \nabla u \, dx + \int_{\partial U} (Tr\, u)^2 \, d\mathcal{H}^{n-1} \underset{\exists \beta > 0}{\ge} \beta \|u\|^2_{H^1(U)}$

若不然，则 $\forall n \in \mathbb{Z}_+$，$\exists u_n \in H^1(U)$ with $\|u_n\|_{H^1(U)} = 1$

s.t. $n B[u_n, u_n] < \|u_n\|^2_{H^1(U)} = 1$.

$\Rightarrow B[u_n, u_n] < \frac{1}{n}$.

由于 $\{u_n\} \subset H^1(U)$ 一致有界，由 Banach-Alaoglu 定理（由 $H^1(U)$ 自反保证）

$\exists$ 子列 $u_{n_k} \rightharpoonup$ some $u \in H^1(U)$ in $H^1(U)$.

这在此处是指 $u_{n_k} \rightharpoonup u$ in L^2，$\nabla u_{n_k} \rightharpoonup \nabla u$ in L^2.

而 $H^1(U) \hookrightarrow\hookrightarrow L^2(U)$ 故 $u_{n_k} \to u$ in $L^2(U)$.

但 $\|\nabla u_{n_k}\|^2_{L^2} \le B[u_{n_k}, u_{n_k}] \le \frac{1}{n_k} \to 0$ as $k \to \infty$

$\|\nabla u\|^2_{L^2} \le \liminf_{k\to\infty} \|\nabla u_{n_k}\|^2_{L^2} = 0$.

~~u = const.~~ $\Rightarrow \nabla u = 0$ a.e.

①②验证后，由 Lax-Milgram 定理即可得到证明.

故现在有 $u_{n_k} \to u$ in H^1 $\Rightarrow \|u\|_{H^1} = 1$.

而 $\nabla u = 0$ 表明 u 在 U 的每个连通分支中 const.

但 $\|Tr\, u\|_{L^2(\partial U)} \le \|Tr\,(u - u_{n_k})\|_{L^2(\partial U)} + \|Tr\, u_{n_k}\|_{L^2(\partial U)}$

$\le \|Tr\| \cdot \|u - u_{n_k}\|_{H^1} + \underbrace{\sqrt{\tfrac{1}{n_k}}}_{\text{由迹定理}} \to 0$ as $k \to \infty$

这是因为对 $u=0$ on ∂U 用迹定理知 $u \in H^1_0(U)$，再用 Poincaré 不等式 $\|u\|_{L^2} \le C \|\nabla u\|_{L^2} = 0$

$\Rightarrow u = 0$ on ∂U $\Rightarrow u = 0$ in U 与 $\|u\|_{H^1} = 1$ 矛盾！ □

[6.6]. 设U连通. $\partial U = \Gamma_1 \sqcup \Gamma_2$. Γ_1为非闭集.

请定义如下问题 $\begin{cases} -\Delta u = f & \text{in } U, \\ u = 0 & \text{on } \Gamma_1 \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_2 \end{cases}$ 的弱解, 并讨论 ∃! 性.

注：此题解法有小问题，应将函数限制为$H^1(U)$中全体在Γ_1上的迹为0的函数，其范数应该设计为u的H_0^1范数+$Tr_{\Gamma_2}u$的L^2范数。

Pf: 猜形式: 先设 $u, v \in C^\infty(U)$.

$$-\Delta u = f \Rightarrow \int_U -\Delta u \cdot v\, dx = \int_U f v\, dx$$

左边分部积分可得.

$$\int_U f v\, dx = -\int_{\partial U} \nabla u \cdot \vec{\nu}\, v\, dx + \int_U \nabla u \cdot \nabla v\, dx$$

$$= \int_U \nabla u \cdot \nabla v\, dx - \int_{\partial U} \frac{\partial u}{\partial \nu} v\, dx.$$

$\partial U = \Gamma_1 \cup \Gamma_2$. Γ_1上: $u=0$ 故分部积分时边界项消失

Γ_2上. $\frac{\partial u}{\partial \nu} = 0$

∴ 应定义为 $\int_U \nabla u \cdot \nabla v\, dx = \int_U f v\, dx \quad \forall u, v \in H^1(U)$.

∃! 性与 [6.5] 类似. 略.

□.

[6.7]. $u \in H^1(\mathbb{R}^n)$ 紧支. 且是 $-\Delta u + c(u) = f$ in $\mathbb{R}^n$ 的弱解. $f \in L^2(\mathbb{R}^n)$.

$c(u) \in L^2(\mathbb{R}^n)$. $c: \mathbb{R} \to \mathbb{R}$ smooth. $c(0)=0$. $c'(x) \geq 0$. 证明: $\|D^2 u\|_{L^2} \leq C\|f\|_{L^2}$.

Pf: u 为 $-\Delta u + c(u) = f$ 弱解 $\Rightarrow \forall v \in H^1(\mathbb{R}^n)$. $\int_U \nabla u \cdot \nabla v + c(u) \cdot v\, dx = \int_U f v\, dx$

令 $v = -D_k^{-h} D_k^h u \quad 0<|h|<1$. 则 $v \in H^1(\mathbb{R}^n)$ 且紧支.

代入有: $\int_U Du \cdot (-D D_k^{-h} D_k^h u)\, dx - \int_U c(u) D_k^{-h} D_k^h u\, dx = -\int f D_k^{-h} D_k^h u\, dx$.

由"D与D_k^h可交换"与"差商形式分部积分"

$$\Rightarrow \int_U \underbrace{|D D_k^h u|^2}_{①} + \underbrace{D_k^h c(u) \cdot D_k^h u}_{②}\, dx = -\int f D_k^{-h} D_k^h u.$$

$$② = \frac{c(u(x+he_k)) - c(u(x))}{h} D_k^h(u(x)) \xlongequal[\exists \xi \in \mathbb{R}]{\text{中值}} c'(\xi) \left(\frac{u(x+he_k)-u(x)}{h}\right)^2 \cdot |D_k^h u(x)|$$

$$= c'(\xi) |D_k^h u|^2 \geq 0$$

$$\therefore \int_U |D_k^h Du|^2 \leq -\int f D_k^{-h} D_k^h u$$

$$\leq \left|\int f D_k^{-h} D_k^h u\right|^2 \leq C\|f\|_{L^2}^2 + \varepsilon \|D_k^{-h} D_k^h u\|_{L^2}^2$$

$$\leq C\|f\|_{L^2}^2 + \varepsilon C \|D_k^h Du\|_{L^2}^2$$

取 $C\varepsilon < \frac{1}{2}$ 即有 $\int_U |D_k^h Du|^2\, dx \lesssim \|f\|_{L^2}^2$

$$\Rightarrow D^2 u \in L^2. \quad \|D^2 u\|_{L^2} \lesssim \|f\|_{L^2}$$

□

[6.8]. 设 $u\in C^\infty(U)$ 为 $Lu=-\sum_{i,j} a^{ij}(x)\,u_{x_ix_j}=0$ 的解. a^{ij} 导数均有界.

求证: $\|\nabla u\|_{L^\infty(U)} \le C\left(\|\nabla u\|_{L^\infty(\partial U)}+\|u\|_{L^\infty(\partial U)}\right)$

证明: 令 $v=|\nabla u|^2+\lambda u^2$. 若能将 $\lambda>0$ 选取合适, 使 $Lv\le 0$, 那么对 v 用弱极大值原理即得.

直接计算:

- $\partial_{x_ix_j}(u^2)=\partial_{x_i}(2u\,u_{x_j})$
  $= 2u_{x_i}u_{x_j}+2u\,u_{x_ix_j}$.
- $|\nabla u|^2=\sum_k (u_{x_k})^2$
- $\partial_{x_j}|\nabla u|^2=\sum_k 2u_{x_k}u_{x_kx_j}$
- $\partial_{x_i}\partial_{x_j}|\nabla u|^2 = 2\sum_k\left(u_{x_kx_i}u_{x_kx_j}+u_{x_k}u_{x_kx_jx_i}\right)$

$\Longrightarrow Lv = -a^{ij}\left(|\nabla u|^2\right)_{x_ix_j} - \lambda a^{ij}(u^2)_{ij}$ 上下指标表示Einstein求和.

$$=\sum_k\left(-2u_{x_k}a^{ij}(u_{x_k})_{x_ix_j} - 2a^{ij}u_{x_kx_i}u_{x_kx_j}\right) - 2\lambda a^{ij}u_{x_i}u_{x_j} - 2\lambda u\cdot\underbrace{a^{ij}u_{x_ix_j}}_{0}$$

$$=-2\sum_{i,j}a^{ij}\left(\sum_k u_{x_kx_i}u_{x_kx_j}+\lambda u_{x_i}u_{x_j}\right) - 2\sum_{k=1}^n u_{x_k}\sum_{i,j=1}^n a^{ij}u_{x_kx_ix_j}$$

$$= -2\sum_{i,j}a^{ij}(\nabla u)_{x_i}\cdot(\nabla u)_{x_j} - 2\lambda\sum_{i,j}a^{ij}u_{x_i}u_{x_j}$$
$$-2\sum_{k=1}^n u_{x_k}\cdot\Big(\big(\underbrace{\sum_{i,j=1}^n a^{ij}u_{x_ix_j}}_{0}\big)_{x_k} - \sum_{i,j=1}^n a^{ij}_{x_k}u_{x_ix_j}\Big).$$

L-一致椭圆

$$\le -2\theta\sum_{k=1}^n|\nabla u_{x_k}|^2 - 2\lambda\theta|\nabla u|^2$$
$$+2\sum_k\sum_{i,j} u_{x_k}a^{ij}_{x_k}u_{x_ix_j}.$$

$$\le -2\theta\sum_k|\nabla u_{x_k}|^2 - 2\lambda\theta|\nabla u|^2 + C\Big|\sum_{i,j,k}u_{x_ix_j}u_{x_k}\Big|$$

$$\le -2\theta\sum_k|\nabla u_{x_k}|^2 - 2\lambda\theta|\nabla u|^2 + \frac{C}{2\varepsilon}|\nabla^2u|^2 + \frac{C\varepsilon}{2}|\nabla u|^2.$$

取 $\varepsilon=\frac{C}{4\theta}$. 上式 $Lv\le\left(-2\lambda\theta+\frac{C^2}{8\theta}\right)|\nabla u|^2<0$ 取 λ 充分大即可

□

[6.9] 设u是 $\begin{cases} Lu = -\sum\limits_{i,j=1}^{n} a^{ij}\partial_{ij} u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$ 的光滑解.

f有界

固定 $x^0 \in \partial U$. 定义 $w \in C^2$ 为 x^0处的闸函数(barrier) 是指

$$\begin{cases} Lw \geq 1 & \text{in } U \\ w(x^0) = 0 & \\ w \geq 0 & \text{on } \partial U \end{cases}$$

证明: $\left|\nabla u(x^0)\right| \leq C\left|\frac{\partial w}{\partial n}(x^0)\right|$

证明: 先对w用弱极大值原理. $\min\limits_{\bar U} w = \min\limits_{\partial U} w = w(x^0)$.

令 $v_1 = u + w\|f\|_{L^\infty}$ $\qquad Lv_1 \geq 0$ $\qquad$ 再用弱极大值原理知 $\min\limits_{\bar U} v_1 = \min\limits_{\partial U} v_1 = \|f\|_{L^\infty} w(x^0) = v_1(x^0)$.

$v_2 = u - w\|f\|_{L^\infty}$ $\qquad Lv_2 \leq 0$ $\qquad \max\limits_{\bar U} v_2 = v_2(x^0)$.

据Hopf引理: $0 \geq \frac{\partial v_1}{\partial \nu}(x^0) = \frac{\partial u}{\partial \nu}(x^0) + \|f\|_{L^\infty}\frac{\partial w}{\partial \nu}(x^0)$

$$0 \leq \frac{\partial v_2}{\partial \nu}(x^0) = \frac{\partial u}{\partial \nu}(x^0) - \|f\|_{L^\infty}\frac{\partial w}{\partial \nu}(x^0)$$

而 $u=0$ on ∂U 所以 $\nabla u /\!/ \nu \Rightarrow \left|\nabla u(x^0)\right| = \left|\frac{\partial u}{\partial \nu}(x^0)\right| \leq \|f\|_{L^\infty}\left|\frac{\partial w}{\partial \nu}(x^0)\right|$

□

[6.10] U连通. 分别用能量法、极大值原理

证明: $\begin{cases} -\Delta u = 0 & \text{in } U \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases}$ 的唯一光滑解为 $u = \text{const}$.

证明: (1) 能量法: 寻找能量泛函 $I[w] = \frac{1}{2}\int_U |\nabla w|^2 dx$ 的极小化子.

而 $u=\text{const}$ 恰好使 I 极小 ($I=0$) 都

$I = 0 \Rightarrow \nabla w = 0 \xRightarrow{U\text{连通}} w = \text{const}$ $\qquad$ ∴只有常值解

(2) 极大值原理法:

若u在U内部达极大值, 由U连通, 据强极大值原理即可

若 $x^0 \in \partial U$ 使得 $u(x^0) = \sup\limits_{\bar U} u(x)$.

且$\forall x \in U$ $u(x^0) > u(x)$

则 Hopf引理 $\Rightarrow \frac{\partial u}{\partial \nu}(x^0) > 0$. 矛盾!

□

[6.11] 设 $u\in H^1(U)$ 为 $-\sum_{i,j=1}^{n}\partial_j(a^{ij}\partial_i u)=0$ in U 的有界弱解

$\phi:\mathbb{R}\to\mathbb{R}$ 凸, C^∞. $w=\phi(u)$

求证：$\forall v\in H_0^1(U)$ 且 $v\geq 0$. 都有 $B[w,v]\leq 0$

证明：$B[u,v]=\int_U\sum_{i,j}a^{ij}\partial_i u\,\partial_j v\,dx$ $\quad u\in H^1(U)$, $v\in H_0^1(U)$

由习题5.17知 $\phi(u)\in H^1(U)$.

为了避免Sobolev函数不能分部积分的麻烦，我们先设 $v\geq 0$. $v\in C_c^\infty(U)$.

$$B[\phi(u),v]=\int_U\sum_{i,j}a^{ij}\,\partial_i(\phi(u))\,\partial_j v\,dx$$

$$=\int_U\sum_{i,j}\phi'(u)\,\partial_i u\,\partial_j v\,dx$$

注意到：$\phi'(u)\partial_j v=\partial_j(\phi'(u)v)-\phi''(u)\partial_j u\cdot v$ （当一个函数Sobolev, 另一个 C_c^∞ 时, Leibniz Rule 成立）.

于是 $$=\underbrace{\int_U\sum_{i,j}a^{ij}\partial_i u\,\partial_j(\underbrace{\phi'(u)v}_{\in H_0^1(U)})\,dx}_{=0\ (\text{注意原方程弱解定义})}-\int_U\sum_{i,j}a_{ij}\phi''(u)\,\partial_i u\,\partial_j u\cdot v\,dx$$

L一致椭圆

$$\leq -\underbrace{\theta}_{\geq 0}\int\underbrace{\phi''(u)}_{\geq 0\ (\text{因}\phi\text{凸})}|\nabla u|^2\cdot\underbrace{v}_{\geq 0}\,dx$$

$$\leq 0$$

□

[6.12] $Lu = -\sum_{i,j=1}^{n} a^{ij}\partial_{ij}u + \sum_{i=1}^{n} b^i\partial_i u + cu$.

设 $\exists v \in C^2(U)\cap C(\bar U)$ 使 $\begin{cases} Lv \geq 0 & \text{in } U \\ v > 0 & \text{on } \partial U \end{cases}$

求证：$\forall u \in C^2(U)\cap C(\bar U)$，只要 $\begin{cases} Lu \leq 0 & \text{in } U \\ u \leq 0 & \text{on } \partial U \end{cases}$，就有 $u \leq 0$ in U.

证明：设 $u\in C^2(U)\cap C(\bar U)$，$Lu\leq 0$ in U，$u\leq 0$ on ∂U.

令 $w = \frac{u}{v} \in C^2(U)\cap C(\bar U)$.

如今，我们希望构造一个椭圆算子 M，s.t. $Mw \leq 0$ on $\{x\in\bar U \mid u>0\} \subseteq U$.

若能证此，则进一步假设 $A=\{x\in\bar U \mid u>0\} \neq \phi$. 由弱极大值原理

$0 < \sup_{A} w = \sup_{\partial A} w = \frac{0}{v} = 0$. 这不可能 所以 $A=\phi \Rightarrow u\leq 0$ in U.

先求 $-a^{ij}\partial_{ij}w$，以方便确定 M

$$-a^{ij}\partial_{ij}\left(\frac{u}{v}\right) = -a^{ij}\partial_i\left(\frac{\partial_j u\cdot v - \partial_j v\cdot u}{v^2}\right) = -a^{ij}\left(\partial_i\left(\frac{\partial_j u}{v}\right) - \partial_i\left(\frac{\partial_j v\cdot u}{v^2}\right)\right)$$

$$= -a^{ij}\left(\frac{\partial_i\partial_j u\cdot v - \partial_j u\,\partial_i v}{v^2} - \frac{-2uv\,\partial_i v\,\partial_j v + v^2\partial_i\partial_j v\cdot u + v^2\partial_j v\cdot\partial_i u}{v^4}\right)$$

$$= -\frac{a^{ij}\partial_{ij}u\cdot v + a^{ij}\partial_{ij}v\cdot u}{v^2} + \frac{a^{ij}\partial_i v\cdot\partial_j u - a^{ij}\partial_i u\cdot\partial_j v}{v^2} + a^{ij}\frac{2}{v}\cdot\underbrace{\frac{u\partial_i v - v\partial_i u}{v^2}}_{\partial_i w}\cdot\partial_j v$$

对 i,j 求和，上式第2项消失

$$-\sum_{i,j} a^{ij}\partial_{ij}\left(\frac{u}{v}\right) \underset{a^{ij}=a^{ji}}{=} \frac{(Lu - b^i\partial_i u - cu)\,v + (-Lv + b^i\partial_i v + cv)\,u}{v^2} + a^{ij}\frac{2}{v}\partial_j v\,\partial_i w$$

上、下指标代表求和

$$= \frac{Lu}{v} - \frac{uLv}{v^2} - b^i\partial_i w + \frac{2}{v}a^{ij}\partial_j v\,\partial_i w$$

$\therefore$ 令 $Mw = \sum_{i,j} a^{ij}\partial_{ij}w + \partial_i w\left(b^i - a^{ij}\partial_j v\cdot\frac{2}{v}\right)$

$$= \frac{Lu}{v} - \frac{uLv}{v^2} \leq 0 \quad \text{on } \{x\in\bar U \mid u>0\} \subset U.$$

此时 $\begin{matrix} Lu\leq 0 \\ v>0 \end{matrix}$ $\quad \begin{matrix} u\leq 0 \\ Lv>0 \end{matrix}$

而 M 显见是一致椭圆的

□

[6.13] (柯朗极大极小原理)

设 $Lu=-\sum_{i,j=1}^{n}\partial_j(a^{ij}\partial_i u)$ $a^{ij}=a^{ji}$ 对零边值问题. 设 L 有特征值 $0<\lambda_1<\lambda_2\le\cdots$

求证: $\lambda_k=\sup\limits_{S\in\Sigma_{k-1}}\ \inf\limits_{\substack{u\in S^\perp\\ \|u\|_{L^2}=1}} B[u,u]$ $k\in\mathbb{Z}_+$.

其中 Σ_{k-1} 是 $H_0^1(U)$ 全体 $(k-1)$ 维子空间~~全体~~

证明: 先证 $\lambda_k=\sup\limits_{S\in \underbrace{E_{k-1}}_{L^2(U)\text{全体}(k-1)\text{维子空间}}}\ \inf\limits_{u\in S^\perp,\ \|u\|_{L^2}=1} B[u,u]$

作 $A=\underbrace{L^{-1}}_{\text{形式上的逆}}: L^2\longrightarrow H_0^1(U)\hookrightarrow L^2(U)$ 则 A 为 $L^2(U)\to L^2(U)$ 紧算子

$f\longmapsto u\longmapsto u$.

设 λ_k 对应特征函数为 w_k. $\|w_k\|_{L^2}=1$ $\langle w_i,w_j\rangle_{L^2}=\delta_{ij}$

则 $Lw_k=\lambda_k w_k\Rightarrow Aw_k=\frac{1}{\lambda_k}w_k$ ∴ A的特征值为 $\lambda_1^{-1}>\lambda_2^{-1}\ge\cdots\ge\cdots>0$

由 Hilbert-Schmidt 定理. A关于 λ_k^{-1} 有特征向量 e_k. $\|e_k\|_{L^2(U)}=1$

则 $\forall f\in L^2(U)$. $f=\sum_i(f,e_i)e_i$ $\{e_k\}_{k\in\mathbb{Z}_+}$ 为 $L^2(U)$ 的标准正交基

$\Rightarrow B[u,u]=\langle Lu,u\rangle=\sum_{i=1}^{\infty}\lambda_i(u,e_i)^2$. $\forall u\in L^2,\ \|u\|_{L^2}=1$

① $\forall S\in E_{k-1}$, $\exists u_k\in\mathrm{Span}\{e_1,\dots,e_k\}$ s.t. $u_k\perp S$ (by Hilbert-Schmidt thm).

$\Rightarrow \inf\limits_{\substack{\|u\|_{L^2}=1\\ u\in S^\perp}} B[u,u]\le B[u_k,u_k]=\sum_{i=1}^{k}\lambda_i(u,e_i)^2\le\lambda_k$.

② 取 $S=\mathrm{Span}\{e_1,\dots,e_{k-1}\}$ 则 $\forall u\in S^\perp$. $\lambda_k=B[e_k,e_k]\le\sum_{j\ge k}\lambda_j(u,e_j)^2=B[u,u]$

①② 推出 $\lambda_k=\sup\limits_{S\in E_{k-1}}\ \inf\limits_{\substack{\|u\|_{L^2}=1\\ u\in S^\perp}} B[u,u]$

由 $H_0^1(U)\subseteq L^2(U)$ 知, $\lambda_k\ge\sup\limits_{S\in\Sigma_{k-1}}\ \inf\limits_{\substack{\|u\|_{L^2}=1\\ u\in S^\perp}} B[u,u]$.

为证 $\le$. 取 $S=\mathrm{Span}\{\frac{e_1}{\sqrt{\lambda_1}},\dots,\frac{e_k}{\sqrt{\lambda_k}}\}$ (由课本6.5节知 $\{\frac{e_j}{\sqrt{\lambda_j}}\}_{j\in\mathbb{Z}_+}$ 为 $H_0^1(U)$ 标正基, 内积为 $B[\cdot,\cdot]$)

从而 $\forall u\in S^\perp$ 且 $\|u\|_{L^2}=1$ 设 $u=\sum_{j\ge k}(a_j\sqrt{\lambda_j})\frac{e_j}{\sqrt{\lambda_j}}$

$\Rightarrow B[u,u]=\sum_{j\ge k}a_j^2\lambda_j\ge\lambda_k$ 证毕!

□

14. λ_1是如下椭圆算子主特征值.

$$Lu = -\sum_{i,j} a^{ij}\partial_{ij}u + \sum_{i} b^i \partial_i u + cu.$$

则 $$\lambda_1 = \sup_{\substack{u\in C^\infty(\bar U),\\ u>0 \text{ in } U\\ u=0 \text{ on } \partial U}} \ \inf_{x\in U} \frac{Lu(x)}{u(x)}.$$

Proof: $X = \{u\in C^\infty(\bar U) \mid u>0 \text{ in } U,\ u=0 \text{ on } \partial U\}$.

① 由 Thm 6.5.3 $\exists\, w_1>0$ $w_1\in H^1(U)$. 为 L 关于 λ_1 的特征向量.

从而 $\exists \{u_n\}\subset X$. $u_n \to w_1$ in H^1.

$$\Rightarrow \sup_X \inf_x \frac{Lu}{u} \geq \inf_x \frac{Lu_n}{u_n} \underset{n\to\infty}{=} \lambda_1.$$

② $\forall u\in X$. $\inf_{x\in U} \frac{Lu}{u} \leq \lambda_1$

$$\Longleftrightarrow \inf_{x\in U}(Lu-\lambda_1 u) \leq 0.$$

Consider. $L^* w_1^* = \lambda_1 w_1^*$. $w_1^*>0$ 为 L^* 关于 λ_1 的特征向量.

$$\Longleftrightarrow (L^* w_1^*, u) = (\lambda_1 w_1^*, u).$$

$$\Longleftrightarrow \langle Lu, w_1^*\rangle = \langle \lambda_1 u, w_1^*\rangle$$

$$\Longleftrightarrow \langle Lu-\lambda_1 u, w_1^*\rangle = 0$$

$$\Longrightarrow \inf_x (Lu-\lambda_1 u) \leq 0$$

check: λ_1为L^*主特征值. (假设为λ_1^*)

由: $\lambda_1^*(w_1^*, w_1)_{L^2} = \langle L^* w_1^*, w_1\rangle_{L^2}$

$= \langle w_1^*, Lw_1\rangle_{L^2}$

$= \lambda_1 \langle w_1^*, w_1\rangle_{L^2}$

$\Rightarrow \lambda_1^* = \lambda_1$. □

[6.15] $U(\tau)\subseteq\mathbb{R}^n$，$\partial U(\tau)$ 速度为 v. $\forall\tau$. 考虑特征值问题 ~~$\lambda=\lambda(\tau)$~~

$U(\tau)$ 关于 τ C^∞.

$$\begin{cases} -\Delta w=\lambda w & \text{in } U(\tau) \\ w=0 & \text{on } \partial U(\tau) \end{cases}$$

$\|w\|_{L^2(U(\tau))}=1$

$\lambda=\lambda(\tau)\in C^\infty$

$w=w(x,\tau)\in C^\infty_{\tau,x}$

利用 $\displaystyle\frac{d}{d\tau}\int_{U(\tau)} f\,dx=\int_{\partial U(\tau)} f(\vec{v}\cdot\nu)\,dS+\int_{U(\tau)}\partial_\tau f\,dx$

去证明 Hadamard 变分公式

$$\dot\lambda=-\int_{\partial U(\tau)}\left|\frac{\partial w}{\partial\nu}\right|^2(\vec{v}\cdot\nu)\,dS$$

证明：$-\Delta w=\lambda w\Rightarrow$

$$-\int_{U(\tau)} w\cdot\Delta w=\lambda\int_{U(\tau)} w^2=\lambda$$

$\|$ 分部积分

$$\int_{U(\tau)}|\nabla w|^2\,dx-\int_{\partial U(\tau)}\underbrace{w}_{=0}\cdot\frac{\partial w}{\partial\nu}\,dS=\int_{U(\tau)}|\nabla w|^2\,dx$$

$\Rightarrow \lambda=\int_{U(\tau)}|\nabla w|^2\,dx$ 对 τ 求导. 有 $\dot\lambda=\int_{\partial U(\tau)}|\nabla w|^2(\vec{v}\cdot\nu)\,dS+\int_{U(\tau)}\partial_\tau|\nabla w|^2\,dx$

由 $w=0$ on $\partial U(\tau)$ 知. $\nabla w=\frac{\partial w}{\partial\nu}$

$$\therefore\ \dot\lambda=\int_{\partial U(\tau)}\left|\frac{\partial w}{\partial\nu}\right|^2(\vec{v}\cdot\nu)\,dS+\int_{U(\tau)}\partial_\tau|\nabla w|^2\,dx\quad\cdots①$$

① 第2项 $=\displaystyle\int_{U(\tau)}2\nabla w\cdot\nabla(\partial_\tau w)\,dx$

$$\underset{w|_{\partial U(\tau)}=0}{\overset{\text{分部积分}}{=}}\int_{U(\tau)}2w(-\Delta\partial_\tau w)\,dx\overset{-\Delta w=\lambda w}{=\!=}\int_{U(\tau)}2w\cdot\partial_\tau(\lambda w)\,dx\qquad \cancel{\int_{U(\tau)}2\Delta w\cdot\partial_\tau w}$$

$$=\int_{U(\tau)}2\lambda w\cdot\partial_\tau w\,dx+\underbrace{2\dot\lambda\int_{U(\tau)}w^2\,dx}_{=2\dot\lambda}$$

$$=2\dot\lambda+2\lambda\int_{U(\tau)}\partial_\tau w^2\,dx$$

$$=2\dot\lambda+2\lambda\Big(\underbrace{\frac{d}{d\tau}\int_{U(\tau)}w^2\,dx}_{\overset{\|}{1}\ \text{求导}=0.}-\int_{\partial U(\tau)}\underbrace{w^2}_{=0}(\vec{v}\cdot\nu)\,dS\Big)$$

$$=2\dot\lambda$$

于是 ① $\Rightarrow$ $\displaystyle\dot\lambda=-\int_{\partial U(\tau)}\left|\frac{\partial w}{\partial\nu}\right|^2(\vec{v}\cdot\nu)\,dS$.

$\square$.