Ch 6 习题：本节假设 L 一致椭圆. $U\subset\mathbb{R}^n$ 为有界开集. $\partial U\in C^\infty$.

[6.1] 考虑带位势 c 的 Laplace 方程 $-\Delta u+cu=0$ ⋯(*)

和散度形式的方程 $-\mathrm{div}(aDv)=0$ $a\geq 0$.

(1) 证明：若 u 为(*)的解，$w>0$ 也是(*)的解，则 $v=\frac{u}{w}$ 是(**)的解 $(a=w^2)$

(2) 反之，若 v 是(**)的解，则 $u=va^{\frac12}$ 是(*)的解，(对某个位势 c).

证明：(1). $-\Delta u+cu=0$, $-\Delta w+cw=0$

$v=\frac{u}{w}$

$$\Rightarrow \partial_i v=\frac{\partial_i u\cdot w-u\cdot\partial_i w}{w^2}\qquad \cancel{\Rightarrow a\partial_i v=\frac{a\partial_i u\, w-au\partial_i w}{w^2}}$$

$$\cancel{\mathrm{div}(aDv)=\sum_{i=1}^n\partial_i(a\partial_i v)}$$

$$\cancel{=\sum_{i=1}^n\partial_i a\,\partial_i v+\sum_{i=1}^n a\cdot\partial_i\partial_i v}$$

$$\cancel{=\sum_{i=1}^n\partial_i a\cdot\frac{\partial_i u\, w-u\partial_i w}{w^2}+\sum_{i=1}^n a\cdot\frac{\partial_i(\partial_i u\cdot w-u\partial_i w)w^2-2w\partial_i w(\partial_i u\cdot w-u\partial_i w)}{w^4}}$$

$a=w^2 \Rightarrow a\partial_i v=\partial_i u\cdot w-\partial_i w\cdot u$

$$\mathrm{div}(aDv)=\sum_{i=1}^n\partial_i(\partial_i u\, w-\partial_i w\cdot u)$$

$$=\sum_{i=1}^n(\partial_i\partial_i u\cdot w+\partial_i u\partial_i w-\partial_i w\partial_i u-\partial_i\partial_i w\cdot u)$$

$$=\Delta u\cdot w-\Delta w\cdot u$$

$$=cuw-cuw=0.$$

□.

(2). 若 $-\mathrm{div}(aDv)=0$.

则 $\sum_{i=1}^n\partial_i(a\partial_i v)=0\Rightarrow\sum_{i=1}^n\partial_i a\cdot\partial_i v+a\sum_{i=1}^n\partial_i\partial_i v=0$

$$-\Delta u+cu\underset{u=a^{\frac12}v}{=}-\sum_{i=1}^n\partial_i\partial_i(va^{\frac12})+cva^{\frac12}$$

$$=cva^{\frac12}-\sum_{i=1}^n\partial_i\left(\partial_i v\cdot a^{\frac12}+\frac12 a^{-\frac12}\partial_i a\cdot v\right)$$

$$=cva^{\frac12}-\sum_{i=1}^n\Big(\partial_i\partial_i v\cdot a^{\frac12}+\partial_i v\cdot\frac12 a^{-\frac12}\partial_i a+\frac12 a^{-\frac12}\partial_i a\,\partial_i v+(\partial_i\partial_i(a^{\frac12}))v\Big)$$

$$=cva^{\frac12}-a^{-\frac12}\cdot\underbrace{\Big(\sum_{i=1}^n\partial_i a\,\partial_i v+a\partial_i\partial_i v\Big)}-\sum_{i=1}^n(\partial_i\partial_i a^{\frac12})v$$

$$=v(ca^{\frac12}-\Delta\sqrt a)\overset{\text{!}}{=}0$$

取 $c=\frac{\Delta\sqrt a}{\sqrt a}$ 即可 □

[6.2]. 设 $Lu = -\sum_{i,j=1}^{n} \partial_j(a^{ij}\partial_i u) + cu$.

证明：存在常数 $\mu > 0$ s.t. 在 $c(x) \geq -\mu$ $(x \in U)$ 的条件下，$B[\cdot,\cdot]$ 满足 Lax-Milgram 定理条件

证明： $B[u,v] = \sum_{i,j=1}^{n} \int_U a^{ij}\partial_i u \partial_j v + cuv \quad \forall u,v \in H_0^1(U)$

① $|B[u,v]| \leq \|a^{ij}\|_{L^\infty} \sum_{i,j=1}^{n} \int_U |\partial_i u||\partial_j v| + \|c\|_{L^\infty} \int_U |u|\cdot|v| \, dx$

$\overset{\text{Hölder}}{\leq} C\left(\|Du\|_{L^2}\|D_j v\|_{L^2} + \|u\|_{L^2}\|v\|_{L^2}\right)$

$\leq C\left(\|u\|_{H_0^1}\|v\|_{H_0^1}\right)$

② $|B[u,u]| = \int \sum_{i,j=1}^{n} a^{ij}\partial_i u \partial_j u + cu^2$

$\overset{L\text{一致椭圆}}{\geq} \theta\|Du\|_{L^2}^2 + \int cu^2$

~~$u \in H_0^1$，由 Poincaré 不等式 $\|u\|_{L^2} \leq C'\|Du\|_{L^2}$ (for some $C' > 0$).~~

$\cancel{= \theta\|Du\|_{L^2}^2 + (c+\mu)\|u\|_{L^2}^2 - (\mu)\|u\|_{L^2}^2}$

~~Poincaré 不等式：因 $u \in H_0^1(U)$，故 $\exists C' > 0$，$\|u\|_{L^2} \leq C\|Du\|_{L^2}$~~

$\cancel{\geq (\theta - c^2\mu - c^2\varepsilon_0)\|Du\|_{L^2}^2 + (c+\mu+\varepsilon_0)\|u\|_{L^2}^2}$

~~取 μ 满足 $\theta - c^2\mu - c^2\varepsilon_0 > \varepsilon_0$ 即 $\mu \leq \theta - (1+\frac{1}{c^2})\varepsilon_0$ 即可.~~

$\cancel{\geq \theta\|Du\|_{L^2}^2 - c(C')^2\|Du\|_{L^2}^2}$

$= \theta\|Du\|_{L^2}^2 + \int c(+\mu+\varepsilon)u^2 - (\mu+\varepsilon)\int u^2$

Poincare: $\forall u \in H_0^1(U)$, then $\exists C' > 0$, $\|u\|_{L^2} \leq C'\|Du\|_{L^2}$.

$\geq \theta\|Du\|_{L^2}^2 + \int (c+\mu+\varepsilon)u^2 - (\mu+\varepsilon)(C')^2\|Du\|_{L^2}^2$

取 $(\mu+\varepsilon)(C')^2 = \frac{\theta}{2}$ (ε 足够小) 于是

上式 $\geq \frac{\theta}{2}\|Du\|_{L^2}^2 = \frac{\theta}{4}\|Du\|_{L^2}^2 + \frac{\theta}{4}\|Du\|_{L^2}^2$

$\overset{\text{Poincare}}{\geq} \frac{\theta}{4}\|Du\|_{L^2}^2 + \frac{\theta}{4}\frac{1}{C'}\|u\|_{L^2}^2$

$\geq \min\left\{\frac{\theta}{4}, \frac{\theta}{4}\frac{1}{C'}\right\}\|u\|_{H_0^1}^2$ □

[6.3] $u\in H_0^2(U)$ 是如下边值问题 $\begin{cases}\Delta^2 u = f & \text{in } U \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U\end{cases}$ 的弱解，若 $\int_U \Delta u \Delta v\, dx = \int_U fv\, dx \quad \forall v\in H_0^2(U)$

今给定 $f\in L^2(U)$，证明该方程存在唯一弱解

证明：令 $B[u,v] = \int_U \Delta u \Delta v\, dx$

① $|B[u,v]| = \int_U |\Delta u|\cdot|\Delta v|\, dx$

$$\underset{\text{Hölder}}{\leq} C\|D^2 u\|_{L^2}\|D^2 v\|_{L^2}$$

$$\underset{u,v\in H_0^2(U)}{\leq} C'\|u\|_{H_0^2}\|v\|_{H_0^2}$$

由 Poincaré 知. $\|u\|_2$, $\|Du\|_2$ 由 $\|D^2u\|_2$ 控制)

② $B[u,u] \overset{\text{先设 } u\in C_c^\infty(U)}{=\!=} \int_U \underset{\cancel{|\Delta u|\cdot|\Delta u|}}{\Delta u \Delta u}\, dx$

$$= \sum_{j,k=1}^n \int \partial_j^2 u\, \partial_k^2 u\, dx$$

$$\overset{\text{分部积分}}{=\!=} -\sum_{\substack{j=1\\k=1}}^n \int \partial_j u \cdot \partial_j \partial_k^2 u\, dx.$$

$$\overset{\text{再分部积分}}{\underset{\partial_j\partial_k^2 = \partial_k(\partial_j\partial_k)}{=\!=}} \sum_{j,k=1}^n \int (\partial_j\partial_k u)^2\, dx = \|D^2 u\|_{L^2}^2$$

$$\underset{\text{Poincaré}}{\geq} c\left(\|D^2 u\|_{L^2}^2 + \|u\|_{L^2}^2 + \|Du\|_{L^2}^2\right)$$

$$= c\|u\|_{H_0^2(U)}^2.$$

~~对一般的 $u\in H_0^2(U)$ 可以找到一列 $u_n\in C_c^\infty(U)$ s.t. $\|u_n - u\|_{H_0^2(U)} \to 0$ as $n\to\infty$.~~

对一般的 $u\in H_0^2(U)$, $\exists \{u_n\}\subset C_c^\infty(U)$ s.t. $\|u_n - u\|_{H_0^2(U)} \to 0$

故 $\|u_n\|_{H_0^2} \to \|u\|_{H_0^2}$.

$$\left|\|\Delta u_n\|_{L^2}^2 - \|\Delta u\|_{L^2}^2\right| \leq \|\Delta(u_n - u)\|_{L^2}^2 \leq C\|D^2(u_n-u)\|_{L^2}^2 \to 0$$

$\therefore B[u,u] \geq c\|u\|_{H_0^2(U)}^2$ 对 $u\in H_0^2(U)$ 成立.

那么由①②，据 Lax-Milgram 定理知. $\exists!\, u\in H_0^2(U)$

s.t. $\forall v\in H_0^2(U)$, $B[u,v] = (f,v)_{L^2}$. given $f\in L^2$. □

[6.4] 称 $u\in H^1(U)$ 是 Neumann 边值问题 $\begin{cases} -\Delta u = f & \text{in } U \\ \dfrac{\partial u}{\partial \nu}=0 & \text{on } \partial U \end{cases}$ 的弱解，是指：

设 U 连通

$\forall v\in H^1(U)$，成立：$\int_U \nabla u\cdot\nabla v\,dx = \int_U fv\,dx$.

现设 $f\in L^2(U)$. 证明：上述方程弱解存在 $\iff \int_U f\,dx=0$.

证明：$\Rightarrow$) 令 $v=1$

$\Leftarrow$) 令 $B[u,v]=\int_U \nabla u\cdot\nabla v$.

$$H_\sigma^1(U)=\{u\in H^1(U) \mid \int u\,dx=0\}$$

Step 1: $H_\sigma^1(U)$ 为 Hilbert 空间，内积为 $B[\cdot,\cdot]$

实际上，$\ell: H^1(U)\to\mathbb{R}$，$u\mapsto \int_U u\,dx$ 作为 $H^1(U)$ 上的连续线性泛函，满足：

$H_\sigma^1(U)=\ell^{-1}(0)$

而 $\{0\}\subseteq\mathbb{R}$ 闭 $\therefore$ $\ell^{-1}(\{0\})$ 闭 $\Rightarrow$ $H_\sigma^1(U)$ 为 $H^1(U)$ 的闭子空间，从而是 Hilbert 空间

$B[\cdot,\cdot]$ 为内积？check：双线性易见。

只用证 $B[u,u]=0 \iff u=0$ in H_σ^1

实际上，由 U 连通，据 Poincaré 不等式：

$$\|u-\langle u\rangle_U\|_{L^2}\le \|\nabla u\|_{L^2} = \sqrt{B[u,u]}=0$$

$\overset{\langle u\rangle_U=0}{\Longrightarrow} u=0$ a.e. ✓

Step 2: 由 Riesz 表示定理. $\forall f\in L^2(U)$，$\int_U f=0$，$\exists!\, u_f\in H_\sigma^1(U)$，

s.t. $\forall v\in H_\sigma^1(U)$，$\int_U \nabla u_f\cdot\nabla v\,dx = B[u_f,v]=(f,v)$

Step 3: $\forall v\in H^1(U)$，$v-\langle v\rangle_U\in H_\sigma^1(U)$. 故由 Step 2 知，

给定 $f\in L^2$，$\exists!\, u_f\in H_\sigma^1\subset H^1$

s.t. $(f, v-\langle v\rangle_U)=\int_U\nabla u_f\cdot\nabla(v-\langle v\rangle_U)\,dx=\int_U \nabla u_f\cdot\nabla v\,dx$

而 $\int_U f=0$ $\therefore$ $\int fv=\int\nabla u_f\cdot\nabla v\,dx$. 证毕！

□

[6.5]

设 $\begin{cases} -\Delta u = f & \text{in } U \\ u + \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases}$ 如何定义该方程的 $H^1(U)$ 弱解？
若给定 $f \in L^2(U)$，如何证明解的存在唯一性？

证明：令 $B[u,v] = \int_U \nabla u \cdot \nabla v \, dx + \int_{\partial U} Tr\, u \cdot Tr\, v \, d\mathcal{H}^{n-1}$ $\forall u, v \in H^1(U)$

* 为何如此定义？若 $u, v \in C^\infty(U)$，则

$$\int -\Delta u \cdot v \underset{\text{分部积分}}{=} \int_U \nabla u \cdot \nabla v \, dx - \int_{\partial U} v \cdot \nabla u \cdot \vec{\nu} \, d\mathcal{H}^{n-1}$$

$$= \int_U \nabla u \cdot \nabla v \, dx - \int_{\partial U} v \cdot \frac{\partial u}{\partial \nu} \, d\mathcal{H}^{n-1}$$

$$\overset{-\frac{\partial u}{\partial \nu} = u \text{ on } \partial U}{=} \int_U \nabla u \cdot \nabla v \, dx + \int_{\partial U} u \cdot v \, d\mathcal{H}^{n-1}$$ 符合我们的定义.

下面 check Lax-Milgram 定理的条件即可.

① $|B[u,v]| \le C \|u\|_{H^1} \|v\|_{H^1}$ 显见（$\int_{\partial U}$ 项用迹定理即可）

② 下证：$B[u,u] = \int_U \nabla u \cdot \nabla u \, dx + \int_{\partial U} (Tr\, u)^2 \, d\mathcal{H}^{n-1} \underset{\exists \beta > 0}{\geq} \beta \|u\|^2_{H^1(U)}$.

若不然，则 $\forall n \in \mathbb{Z}_+$，$\exists u_n \in H^1(U)$ with $\|u_n\|_{H^1(U)} = 1$.

s.t. $n B[u_n, u_n] < \|u_n\|^2_{H^1(U)} = 1$.

$\Rightarrow B[u_n, u_n] < \frac{1}{n}$.

由于 $\{u_n\} \subset H^1(U)$ 一致有界，由 Banach-Alaoglu 定理

$\exists$ 子列 $u_{n_k} \rightharpoonup$ some u $\boxed{\in H^1(U)}$ in $H^1(U)$. （由 $H^1(U)$ 自反保证.）

这在此时即是指 $u_{n_k} \rightharpoonup u$ in L^2，$\nabla u_{n_k} \rightharpoonup \nabla u$ in L^2.

而 $H^1(U) \hookrightarrow\hookrightarrow L^2(U)$ 故 $u_{n_k} \to u$ in $L^2(U)$.

但 $\|\nabla u_{n_k}\|^2_{L^2} \le B[u_{n_k}, u_{n_k}] \le \frac{1}{n_k} \to 0$ as $k \to \infty$

$\|\nabla u\|^2_{L^2} \le \liminf_{k \to \infty} \|\nabla u_{n_k}\|^2_{L^2} = 0$.

~~u = const.~~ $\Rightarrow \nabla u = 0$ a.e.

①②验证后，由 Lax-Milgram 定理即可得到证明.

故现在有 $u_{n_k} \to u$ in H^1 $\Rightarrow \|u\|_{H^1} = 1$.

而 $\nabla u = 0$ 表明 u 在 U 的每个连通分支中 const.

但 $\|Tr\, u\|_{L^2(\partial U)} \le \|Tr\,(u - u_{n_k})\|_{L^2(\partial U)} + \|Tr\, u_{n_k}\|_{L^2(\partial U)}$

$\le \|Tr\| \cdot \|u - u_{n_k}\|_{H^1} + \sqrt{\frac{1}{n_k}} \to 0$ as $k \to \infty$ （迹定理）

这是因为对 $u = 0$ on ∂U 用零迹定理知 $u \in H^1_0(U)$，再用 Poincaré 不等式 $\|u\|_{L^2} \le C \|\nabla u\|_{L^2} = 0$

$\Rightarrow u = 0$ on ∂U $\Rightarrow u = 0$ in U 与 $\|u\|_{H^1} = 1$ 矛盾！

□

[6.6]. 设U连通. $\partial U=\Gamma_1 \sqcup \Gamma_2$. Γ_i为不交闭集.

请定义如下问题 $\begin{cases} -\Delta u=f & \text{in } U. \\ u=0 & \text{on } \Gamma_1 \\ \frac{\partial u}{\partial \nu}=0 & \text{on } \Gamma_2 \end{cases}$ 的弱解, 并讨论 $\exists!$ 性.

注：此题解法有小问题，应将函数限制为H1(U)中全体在Γ1上的迹为0的函数
即H:={u∈H1(U): Tr u|Γ1=0}.
可以证明H空间上的Poincare不等式，即对任意u属于H,有||u||L2≤C||D u||L2

Pf: 猜形式: 先设 $u, v\in C^\infty(U)$.

$$-\Delta u=f \Rightarrow \int_U -\Delta u\cdot v\,dx=\int_U fv\,dx$$

左边分部积分可得.

$$\int_U fv\,dx=-\int_{\partial U}\nabla u\cdot\vec{\nu}\,v\,dx+\int_U \nabla u\cdot\nabla v\,dx$$

$$=\int_U \nabla u\cdot\nabla v\,dx-\int_{\partial U}\frac{\partial u}{\partial \nu}v\,dx.$$

$\partial U=\Gamma_1\cup\Gamma_2$. Γ_1上: $u=0$ 故分部积分时边界项消失

Γ_2上. $\frac{\partial u}{\partial \nu}=0$

∴ 应定义为 $\int_U \nabla u\cdot\nabla v\,dx=\int_U fv\,dx \quad \forall u,v\in H^1(U)$.

$\exists!$ 性与 [6.5] 类似, 略. $\square$.

[6.7]. $u\in H^1(\mathbb{R}^n)$ 紧支, 且是 $-\Delta u+c(u)=f$ in $\mathbb{R}^n$ 的弱解. $f\in L^2(\mathbb{R}^n)$.

$c(u)\in L^2(\mathbb{R}^n)$. $c:\mathbb{R}\to\mathbb{R}$ smooth. $c(0)=0$. $c'(x)\geq 0$. 证明: $\|D^2u\|_{L^2}\leq C\|f\|_{L^2}$.

Pf: u 为 $-\Delta u+c(u)=f$ 弱解 $\Rightarrow \forall v\in H^1(\mathbb{R}^n)$. $\int_U Du\cdot Dv+c(u)\cdot v\,dx=\int_U fv\,dx$

令 $v=-D_k^{-h}D_k^h u \quad 0<|h|<1$. 则 $v\in H^1(\mathbb{R}^n)$ 且紧支.

代入有: $\int_U Du\cdot(-DD_k^{-h}D_k^h u)\,dx-\int_U c(u)D_k^{-h}D_k^h u\,dx=-\int fD_k^{-h}D_k^h u\,dx$.

由"D与D_k^h可交换"与"差商形式分部积分"

$$\Rightarrow \int_U \underbrace{|DD_k^h u|^2}_{①}+\underbrace{D_k^h c(u)\cdot D_k^h u}_{②}\,dx=-\underbrace{\int fD_k^{-h}D_k^h u}_{③}.$$

$$② = \frac{c(u(x+he_k))-c(u(x))}{h}D_k^h(u(x)) \xlongequal[\exists\xi\in\mathbb{R}.]{\text{中值}} c'(\xi)\left(\frac{u(x+he_k)-u(x)}{h}\right)^2=c'(\xi)|D_k^h u|^2\geq 0$$

$$\therefore \int_U|D_k^h Du|^2\leq -\int fD_k^{-h}D_k^h u$$

$$\leq \left|\int fD_k^{-h}D_k^h u\right|^2\leq C\|f\|_{L^2}^2+\varepsilon\|D_k^{-h}D_k^h u\|_{L^2}^2$$

$$\leq C\|f\|_{L^2}^2+\varepsilon C\|D_k^h Du\|_{L^2}^2$$

取 $C\varepsilon<\frac{1}{2}$ 即有 $\int_U|D_k^h Du|^2dx\lesssim\|f\|_{L^2}^2$

$$\Rightarrow D^2u\in L^2.\quad \|D^2u\|_{L^2}\lesssim\|f\|_{L^2}$$

$\square$

[6.8]. 设 $u\in C^\infty(U)$ 为 $Lu=-\sum_{i,j} a^{ij}(x)u_{x_ix_j}=0$ 的解. a^{ij} 导数均有界.

求证: $\|\nabla u\|_{L^\infty(U)} \le C\left(\|\nabla u\|_{L^\infty(\partial U)}+\|u\|_{L^\infty(\partial U)}\right)$

证明: 令 $v=|\nabla u|^2+\lambda u^2$. 若能将 $\lambda>0$ 选取合适, 使 $Lv\le 0$, 那么对 v 用弱极大值原理即得.

直接计算: • $\partial_{x_ix_j}(u^2)=\partial_{x_i}(2uu_{x_j})$

$$=2u_{x_i}u_{x_j}+2uu_{x_ix_j}$$

• $|\nabla u|^2=\sum_k (u_{x_k})^2$

• $\partial_{x_j}|\nabla u|^2=\sum_k 2u_{x_k}u_{x_kx_j}$

• $\partial_{x_i}\partial_{x_j}|\nabla u|^2 = 2\sum_k\left(u_{x_kx_i}u_{x_kx_j}+u_{x_k}u_{x_kx_jx_i}\right)$

$\Longrightarrow Lv = -a^{ij}\left(|\nabla u|^2\right)_{x_ix_j} - \lambda a^{ij}(u^2)_{ij}$ 上下指标表示Einstein求和.

$$=\sum_k\left(-2u_{x_k}a^{ij}(u_{x_k})_{x_ix_j} - 2a^{ij}u_{x_kx_i}u_{x_kx_j}\right) - 2\lambda a^{ij}u_{x_i}u_{x_j} - 2\lambda u\underbrace{a^{ij}u_{x_ix_j}}_{0}$$

$$=-2\sum_{i,j}a^{ij}\left(\sum_k u_{x_kx_i}u_{x_kx_j}+\lambda u_{x_i}u_{x_j}\right) - 2\sum_{k=1}^n u_{x_k}\sum_{i,j=1}^n a^{ij}u_{x_kx_ix_j}$$

$$= -2\sum_{i,j}a^{ij}(\nabla u)_{x_i}\cdot(\nabla u)_{x_j} - 2\lambda\sum_{i,j}a^{ij}u_{x_i}u_{x_j}$$

$$-2\sum_{k=1}^n u_{x_k}\cdot\left(\Big(\underbrace{\sum_{i,j=1}^n a^{ij}u_{x_ix_j}}_{0}\Big)_{x_k} - \sum_{i,j=1}^n a^{ij}_{x_k}u_{x_ix_j}\right).$$

L-一致椭圆

$$\le -2\theta\sum_{k=1}^n|\nabla u_{x_k}|^2 - 2\lambda\theta|\nabla u|^2$$

$$+2\sum_k\sum_{i,j} u_{x_k}a^{ij}_{x_k}u_{x_ix_j}.$$

$$\le -2\theta\sum_k|\nabla u_{x_k}|^2 - 2\lambda\theta|\nabla u|^2 + c\left|\sum_{i,j,k}u_{x_ix_j}u_{x_k}\right|$$

$$\le -2\theta\sum_k|\nabla u_{x_k}|^2 - 2\lambda\theta|\nabla u|^2 + \frac{c}{2\varepsilon}|\nabla^2 u|^2+\frac{C_\varepsilon}{2}|\nabla u|^2.$$

取 $\varepsilon=\frac{c}{4\theta}$. 上式 $Lv\le\left(-2\lambda\theta+\frac{c^2}{8\theta}\right)|\nabla u|^2<0$ 取 λ 充分大即可

□

[6.9] 设u是 $\begin{cases} Lu = -\sum_{i,j=1}^{n} a^{ij}\partial_{ij} u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$ 的光滑解.

f 有界

固定 $x^0 \in \partial U$. 定义 $w \in C^2$ 为 x^0处的闸函数(barrier) 是指

$$\begin{cases} Lw \geq 1 & \text{in } U \\ w(x^0)=0 & \\ w \geqslant 0 & \text{on } \partial U \end{cases}$$

证明: $|\nabla u(x^0)| \leq C\left|\frac{\partial w}{\partial n}(x^0)\right|$

<u>证明</u>: 先对w用$\overset{弱}{\vee}$极大值原理. $\min_{\bar U} w = \min_{\partial U} w = w(x^0)$.

令 $V_1 = u + w\|f\|_{L^\infty}$ $\quad Lv_1 \geq 0$ 再用弱极大值原理知 $\min_{\bar U} V_1 = \min_{\partial U} V_1 = \|f\|_{L^\infty} w(x^0) = V_1(x^0)$.

$V_2 = u - w\|f\|_{L^\infty}$ $\quad Lv_2 \leq 0$ $\quad \max_{\bar U} V_2 = v_2(x^0)$.

据Hopf 引理: $0 \geqslant \frac{\partial V_1}{\partial \nu}(x^0) = \frac{\partial u}{\partial \nu}(x^0) + \|f\|_{L^\infty}\frac{\partial w}{\partial \nu}(x^0)$.

$$0 \leq \frac{\partial V_2}{\partial \nu}(x^0) = \frac{\partial u}{\partial \nu}(x^0) - \|f\|_{L^\infty}\frac{\partial w}{\partial \nu}(x^0)$$

而 $u=0$ on ∂U 所以 $\nabla u // \nu$ $\Rightarrow |\nabla u(x^0)| = \left|\frac{\partial u}{\partial \nu}(x^0)\right| \leq \|f\|_{L^\infty}\left|\frac{\partial w}{\partial \nu}(x^0)\right|$

□

[6.10] U连通. 分别用能量法、极大值原理

证明: $\begin{cases} -\Delta u = 0 & \text{in } U \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases}$ 的唯一光滑解为 $u = const$.

<u>证明</u>: (1) 能量法: 找能量泛函 $I[w] = \frac{1}{2}\int_U |\nabla w|^2 dx$ 的极小化子.

而 $u = const$ 恰好使 I 极小 ($I=0$ 了都)

$I=0 \Rightarrow \nabla w = 0 \xRightarrow{U连通} w = const$ $\quad$ ∴只有常值解

(2) 极大值原理法:

若 u在U内部达极大值, 由U连通, 据强极大值原理即可

若 $x^0 \in \partial U$ 使得 $u(x^0) = \sup_{\bar U} u(x)$.

且 $\forall x \in U$. $u(x^0) > u(x)$

则 Hopf 引理 $\Rightarrow \frac{\partial u}{\partial \nu}(x^0) > 0$. 矛盾!

□

[6.11] 设 $u\in H^1(U)$ 为 $-\sum_{i,j=1}^{n}\partial_j(a^{ij}\partial_i u)=0$ in U 的有界弱解

$\phi:\mathbb{R}\to\mathbb{R}$ 凸，C^∞，$w=\phi(u)$

求证：$\forall v\in H_0^1(U)$ 且 $v\geq 0$，都有 $B[w,v]\leq 0$

证明：$B[u,v]=\int_U\sum_{i,j}a^{ij}\partial_i u\,\partial_j v\,dx$ $\quad u\in H^1(U)$, $v\in H_0^1(U)$

由习题5.17知 $\phi(u)\in H^1(U)$.

为了避免不能分部积分的Sobolev函数麻烦，我们先设 $v\geq 0$，$v\in C_c^\infty(U)$.

$$B[\phi(u),v]=\int_U\sum_{i,j}a^{ij}\,\partial_i(\phi(u))\,\partial_j v\,dx$$

$$=\int_U\sum_{i,j}\phi'(u)\,\partial_i u\,\partial_j v\,dx$$

注意到：$\phi'(u)\partial_j v=\partial_j(\phi'(u)v)-\phi''(u)\partial_j u\cdot v$ （当一个是Sobolev，另一个 C_c^∞ 时，Leibniz Rule 成立）

于是 $$=\underbrace{\int_U\sum_{i,j}a^{ij}\partial_i u\,\partial_j(\underbrace{\phi'(u)v}_{\in H_0^1(U)})\,dx}_{=0\ \text{注意原方程弱解定义}}-\int_U\sum_{i,j}a_{ij}\phi''(u)\,\partial_i u\,\partial_j u\cdot v\,dx$$

L-一致椭圆

$$\leq -\underbrace{\theta}_{\geq 0}\int\underbrace{\phi''(u)}_{\geq 0\ (\text{因}\phi\text{凸})}|\nabla u|^2\cdot\underbrace{v}_{\geq 0}\,dx$$

$$\leq 0$$

□

[6.12] $Lu = -\sum_{i,j=1}^{n} a^{ij}\partial_{ij}u + \sum_{i=1}^{n} b^i \partial_i u + cu$.

设 $\exists v \in C^2(U)\cap C(\bar{U})$ 使 $\begin{cases} Lv \geq 0 & \text{in } U \\ v > 0 & \text{on } \partial U \end{cases}$

求证：$\forall u \in C^2(U)\cap C(\bar{U})$，只要 $\begin{cases} Lu \leq 0 & \text{in } U \\ u \leq 0 & \text{on } \partial U \end{cases}$，就有 $u \leq 0$ in U.

证明：设 $u \in C^2(U)\cap C(\bar{U})$，$Lu \leq 0$ in U，$u \leq 0$ on ∂U

令 $w = \frac{u}{v} \in C^2(U)\cap C(\bar{U})$.

如今，我们希望构造一个椭圆算子 M，s.t. $Mw \leq 0$ on $\{x\in\bar{U} \mid u>0\} \subseteq U$.

若能证此，则进一步假设 $A = \{x \in \bar{U} \mid u>0\} \neq \phi$. 由弱极大值原理

$0 < \sup_{A} w = \sup_{\partial A} w = \frac{0}{v} = 0$. 这不可能　所以 $A = \phi \Rightarrow u \leq 0$ in U.

先求 $-a^{ij}\partial_{ij}w$，以方便确定 M

$$-a^{ij}\partial_{ij}\left(\frac{u}{v}\right) = -a^{ij}\partial_i\left(\frac{\partial_j u\cdot v - \partial_j v\cdot u}{v^2}\right) = -a^{ij}\left(\partial_i\left(\frac{\partial_j u}{v}\right) - \partial_i\left(\frac{\partial_j v\cdot u}{v^2}\right)\right)$$

$$= -a^{ij}\left(\frac{\partial_i\partial_j u\cdot v - \partial_j u\,\partial_i v}{v^2} - \frac{-2uv\,\partial_i v\,\partial_j v + v^2\partial_i\partial_j v\cdot u + v^2\partial_j v\cdot\partial_i u}{v^4}\right)$$

$$= -\frac{a^{ij}\partial_{ij}u\cdot v + a^{ij}\partial_{ij}v\cdot u}{v^2} + \frac{a^{ij}\partial_i v\cdot\partial_j u - a^{ij}\partial_i u\cdot\partial_j v}{v^2} + a^{ij}\cdot\frac{2}{v}\cdot\underbrace{\frac{\partial_i v - v\partial_i u}{v^2}}_{\partial_i w}\cdot\partial_j v$$

对 i,j 求和，上式第2项消失

$$-\sum_{i,j} a^{ij}\partial_{ij}\left(\frac{u}{v}\right) \underset{a^{ij}=a^{ji}}{=} \frac{(Lu - b^i\partial_i u - cu)\,v + (-Lv + b^i\partial_i v + cv)\,u}{v^2} + a^{ij}\cdot\frac{2}{v}\,\partial_j v\,\partial_i w$$

上、下指标代表求和

$$= \frac{Lu}{v} - \frac{uLv}{v^2} - b^i\partial_i w + \frac{2}{v}a^{ij}\partial_j v\,\partial_i w$$

$\therefore$ 令 $Mw = \sum_{i,j} a^{ij}\partial_{ij}w + \partial_i w\left(b^i - a^{ij}\partial_j v\cdot\frac{2}{v}\right)$

$$= \frac{Lu}{v} - \frac{uLv}{v^2} \leq 0 \quad \text{on } \{x\in\bar{U} \mid u>0\} \subset U$$

此时 $\begin{matrix} Lu\leq 0 \\ v>0\end{matrix}$　$\begin{matrix} u\leq 0 \\ Lv>0\end{matrix}$

而 M 显见是一致椭圆的

□

[6.13] (柯朗极大极小原理)

设 $Lu=-\sum_{i,j=1}^{n}\partial_j(a^{ij}\partial_i u)$　　$a^{ij}=a^{ji}$　　对零边值问题. 设 L 有特征值 $0<\lambda_1<\lambda_2\le\cdots$

求证: $\lambda_k=\sup\limits_{S\in\Sigma_{k-1}}\ \inf\limits_{\substack{u\in S^{\perp}\\ \|u\|_{L^2}=1}} B[u,u]$　　$k\in\mathbb{Z}_+$

其中 Σ_{k-1} 是 $H_0^1(U)$ 全体 $(k-1)$ 维子空间~~全体~~

证明: 先证 $\lambda_k=\sup\limits_{S\in E_{k-1}}\ \inf\limits_{u\in S^{\perp},\ \|u\|_{L^2}=1} B[u,u]$

E_{k-1}: $L^2(U)$ 全体 $(k-1)$ 维子空间

作 $A=L^{-1}: L^2\longrightarrow H_0^1(U)\hookrightarrow L^2(U)$　　则 A 为 $L^2(U)\to L^2(U)$ 紧算子

形式上的逆 $f\longmapsto u\longmapsto u$.

设 λ_k 对应特征元素为 w_k. $\|w_k\|_{L^2}=1$, $\langle w_i,w_j\rangle_{L^2}=\delta_{ij}$

则 $Lw_k=\lambda_k w_k\Rightarrow Aw_k=\frac{1}{\lambda_k}w_k$ ∴ A的特征值为 $\lambda_1^{-1}>\lambda_2^{-1}\ge\cdots\ge\cdots>0$

由 Hilbert-Schmidt 定理. A关于 λ_k^{-1} 有特征向量 e_k. $\|e_k\|_{L^2(U)}=1$

则 $\forall f\in L^2(U)$. $f=\sum_i(f,e_i)e_i$　　$\{e_k\}_{k\in\mathbb{Z}_+}$ 为 $L^2(U)$ 的标准正交基

$\Rightarrow B[u,u]=\langle Lu,u\rangle=\sum_{i=1}^{\infty}\lambda_i(u,e_i)^2$.　　$\forall u\in L^2$, $\|u\|_{L^2}=1$

① $\forall S\in E_{k-1}$, $\exists u_k\in \mathrm{Span}\{e_1,\dots,e_k\}$ s.t. $u_k\perp S$ (by Hilbert-Schmidt thm).

$\Rightarrow \inf\limits_{\substack{\|u\|_{L^2}=1\\ u\in S^{\perp}}} B[u,u]\le B[u_k,u_k]=\sum_{i=1}^{k}\lambda_i(u,e_i)^2\le\lambda_k$.

② 取 $S=\mathrm{Span}\{e_1,\dots,e_{k-1}\}$ 则 $\forall u\in S^{\perp}$.　$\lambda_k=B[e_k,e_k]\le\sum_{j\ge k}\lambda_j(u,e_j)^2=B[u,u]$

①② 推出 $\lambda_k=\sup\limits_{S\in E_{k-1}}\ \inf\limits_{\substack{\|u\|_{L^2}=1\\ u\in S^{\perp}}} B[u,u]$

由 $H_0^1(U)\subseteq L^2(U)$ 知, $\lambda_k\ge\sup\limits_{S\in\Sigma_{k-1}}\ \inf\limits_{\substack{\|u\|_{L^2}=1\\ u\in S^{\perp}}} B[u,u]$.

为证 $\le$. 取 $S=\mathrm{Span}\left\{\frac{e_1}{\sqrt{\lambda_1}},\dots,\frac{e_k}{\sqrt{\lambda_k}}\right\}$　(由课本6.5节知. $\left\{\frac{e_j}{\sqrt{\lambda_j}}\right\}_{j\in\mathbb{Z}_+}$ 为 $H_0^1(U)$ 标正基, 内积为 $B[\cdot,\cdot]$)

从而 $\forall u\in S^{\perp}$ 且 $\|u\|_{L^2}=1$　设 $u=\sum_{j>k}(a_j\sqrt{\lambda_j})\frac{e_j}{\sqrt{\lambda_j}}$

$\Rightarrow B[u,u]=\sum_{j>k}a_j^2\lambda_j\ge\lambda_k$　　证毕!

□

14. λ_1是如下椭圆算子主特征值.

$$Lu = -\sum_{i,j} a^{ij}\partial_{ij}u + \sum_{i} b^i \partial_i u + cu.$$

则 $\lambda_1 = \sup\limits_{\substack{u\in C^\infty(\bar U),\\ u>0 \text{ in } U \\ u=0 \text{ on } \partial U}} \inf\limits_{x\in U} \dfrac{Lu(x)}{u(x)}.$

Pf

14题前半部分修正:

① 令$X = \{u \in C^\infty(\bar U) : u > 0 \text{ in } U,\ u|_{\partial U} = 0\}$. 则由6.5节定理3， 存在$w_1 \in X$作为$L$关于$\lambda_1$的特征向量。注意，这个特征函数不仅仅是$H_0^1(U)$函数。事实上，由于Evans第六章习题假设了椭圆算子L的系数均为光滑函数，且区域U有界且具有光滑边界，所以可以不断使用椭圆正则性定理，直接证得$w_1 \in C^\infty(\bar U)$. 从而$Lw_1 = \lambda_1 w_1$在U中逐点成立。这样，就得到想要的不等式。

$$\inf_{x\in U}\frac{Lw_1}{w_1} = \lambda_1 \le \sup_{u\in X}\inf_{x\in U}\frac{Lu}{u}.$$

$\lambda_1 \ge \sup\limits_{u\in X}\inf\limits_{x\in U}\dfrac{Lu}{u}$ 的证明仍然同之前的答案。

② $\forall u \in X$. $\quad \inf\limits_{x\in U}\dfrac{Lu}{u} \le \lambda_1$

$$\Longleftrightarrow \inf_{x\in U}(Lu - \lambda_1 u) \le 0.$$

Consider. $L^* w_1^* = \lambda_1 w_1^*$. $\quad w_1^* > 0$为L^*关于λ_1的特征向量.

$$\Longrightarrow (L^* w_1^*, u) = (\lambda_1 w_1^*, u)$$

$$\Longleftrightarrow \langle Lu, w_1^*\rangle = \langle \lambda_1 u, w_1^*\rangle$$

$$\Longleftrightarrow \langle Lu - \lambda_1 u, w_1^*\rangle = 0$$

$$\Longrightarrow \inf_x (Lu - \lambda_1 u) \le 0$$

check: λ_1为L^*主特征值 (设为λ_1^*)

由: $\lambda_1^* (w_1^*, w_1)_{L^2} = \langle L^* w_1^*, w_1\rangle_{L^2}$

$= \langle w_1^*, Lw_1\rangle_{L^2}$

$= \lambda_1 \langle w_1^*, w_1\rangle_{L^2}$

$\Rightarrow \lambda_1^* = \lambda_1$. □

[6.15] $U(\tau)\subseteq\mathbb{R}^n$，$\partial U(\tau)$ 速度为 v. $\forall\tau$. 考虑特征值问题 ~~$\lambda=\lambda(\tau)$~~

$\downarrow$ 关于 τ C^∞.

$$\begin{cases} -\Delta w=\lambda w & \text{in } U(\tau) \\ w=0 & \text{on } \partial U(\tau) \end{cases}$$

$\|w\|_{L^2(U(\tau))}=1$

$\lambda=\lambda(\tau)\in C^\infty$

$w=w(x,\tau)\in C^\infty_{\tau,x}$

利用 $$\frac{d}{d\tau}\int_{U(\tau)} f\,dx=\int_{\partial U(\tau)} f(\vec{v}\cdot\nu)\,dS+\int_{U(\tau)}\partial_\tau f\,dx$$

去证明 Hadamard 变分公式 $$\dot\lambda=-\int_{\partial U(\tau)}\left|\frac{\partial w}{\partial\nu}\right|^2(\vec{v}\cdot\nu)\,dS$$

证明：$-\Delta w=\lambda w \Rightarrow$ $$-\int_{U(\tau)} w\cdot\Delta w=\lambda\int_{U(\tau)}w^2=\lambda$$

$\parallel$ 分部积分

$$\int_{U(\tau)}|\nabla w|^2\,dx-\int_{\partial U(\tau)}\underbrace{w}_{=0}\cdot\frac{\partial w}{\partial\nu}\,dS=\int_{U(\tau)}|\nabla w|^2\,dx$$

$\Rightarrow \lambda=\int_{U(\tau)}|\nabla w|^2\,dx$ 对 τ 求导. 有 $\dot\lambda=\int_{\partial U(\tau)}|\nabla w|^2(\vec{v}\cdot\nu)\,dS+\int_{U(\tau)}\partial_\tau|\nabla w|^2\,dx$

由 $w=0$ on $\partial U(\tau)$ 知. $\nabla w \cancel{\nu}=\frac{\partial w}{\partial\nu}$

$$\therefore\ \dot\lambda=\int_{\partial U(\tau)}\left|\frac{\partial w}{\partial\nu}\right|^2(\vec{v}\cdot\nu)\,dS+\int_{U(\tau)}\partial_\tau|\nabla w|^2\,dx \quad \cdots ①$$

① 第2项 $=\int_{U(\tau)}2\nabla w\cdot\nabla(\partial_\tau w)\,dx$. $\qquad \int_{U(\tau)}2w\cdot\partial_\tau(\lambda w)\,dx$

$$\overset{\text{分部积分}}{\underset{w|_{\partial U(\tau)}=0}{=}}\int_{U(\tau)}2w(-\Delta\partial_\tau w)\,dx \overset{-\Delta w=\lambda w}{=\!=} \cancel{\int_{U(\tau)}2\Delta w\cdot\partial_\tau w}$$

$$=\int_{U(\tau)}\cancel{\quad}\,2\lambda w\cdot\partial_\tau w\,dx+\underbrace{2\dot\lambda\int_{U(\tau)}w^2\,dx}_{=2\dot\lambda}$$

$$=2\dot\lambda+2\lambda\int_{U(\tau)}\partial_\tau w^2\,dx$$

$$=2\dot\lambda+2\lambda\left(\underbrace{\frac{d}{d\tau}\int_{U(\tau)}w^2\,dx}_{\text{1 求导}=0}-\int_{\partial U(\tau)}\underbrace{w^2}_{0}(\vec{v}\cdot\nu)\,dS\right)$$

$$=2\dot\lambda$$

于是 ① $\Rightarrow$ $$\dot\lambda=-\int_{\partial U(\tau)}\left|\frac{\partial w}{\partial\nu}\right|^2(\vec{v}\cdot\nu)\,dS.$$

$\square$.