Evans Chapter 5 习题 ($U\subset\mathbb{R}^n$ 开, $\partial U\in C^\infty$) ($B(x,r)$ 表示闭球)

[5.1] 设 $k\in\mathbb{Z}_+$, $0<\gamma\le 1$. 证明: $C^{k,\gamma}(\bar U)$ 是 Banach 空间.

证明: Step 1: 验证 $\|\cdot\|_{C^{k,\gamma}(\bar U)}$ 是范数. $\|u\|_{C^{k,\gamma}(\bar U)}=\sum_{|\alpha|\le k}\|D^\alpha u\|_{C(\bar U)}+\sum_{|\alpha|=k}[D^\alpha u]_{C^{0,\gamma}(\bar U)}$

① 正定性:

$\|u\|_{C^{k,\gamma}(\bar U)}\ge 0$ 为显见.

若 $\|u\|_{C^{k,\gamma}(\bar U)}=0$ 则 $\|D^\alpha u\|_{C(\bar U)}=0 \quad \forall|\alpha|\le k$

$\Rightarrow \|u\|_{C(\bar U)}=0 \Rightarrow u=0$ in $\bar U$.

② 齐次性. $\|\lambda u\|_{C^{k,\gamma}(\bar U)}=|\lambda|\cdot\|u\|_{C^{k,\gamma}(\bar U)}$ $\forall\lambda\in\mathbb{C}$ 显见

③ 三角不等式. 设 $u,v\in C^{k,\gamma}(\bar U)$

$$\|u+v\|_{C^{k,\gamma}(\bar U)}=\sum_{|\alpha|\le k}\|D^\alpha(u+v)\|_{C(\bar U)}+\sum_{|\alpha|=k}[D^\alpha(u+v)]_{C^{0,\gamma}(\bar U)}$$

$$\overset{\|\cdot\|_{C(\bar U)}\text{是范数}}{\le}\sum_{|\alpha|\le k}\left(\|D^\alpha u\|_{C(\bar U)}+\|D^\alpha v\|_{C(\bar U)}\right)+\sum_{|\alpha|=k}\sup_{\substack{x\ne y\\ x,y\in\bar U}}\frac{|u(x)+v(x)-u(y)-v(y)|}{|x-y|^\gamma}$$

$$\le\sum_{|\alpha|\le k}\|D^\alpha u\|_{C(\bar U)}+\sum_{|\alpha|\le k}\|D^\alpha v\|_{C(\bar U)}+\sum_{|\alpha|=k}\sup_{\substack{x\ne y\\ x,y\in\bar U}}\frac{|u(x)-u(y)|}{|x-y|^\gamma}+\sum_{|\alpha|=k}\sup_{\substack{x\ne y\\ x,y\in\bar U}}\frac{|v(x)-v(y)|}{|x-y|^\gamma}$$

$$=\|u\|_{C^{k,\gamma}(\bar U)}+\|v\|_{C^{k,\gamma}(\bar U)}.$$

Step 1 证毕!

Step 2: $(C^{k,\gamma}(\bar U),\|\cdot\|_{C^{k,\gamma}(\bar U)})$ Banach.

设 $\{u_n\}$ 为 $C^{k,\gamma}(\bar U)$ 中的柯西列. 则 ~~$\|u_n-u_m\|_{C(\bar U)}$~~ $\sum_{|\alpha|\le k}\|D^\alpha u_n-D^\alpha u_m\|_{C(\bar U)}\to 0$

~~由 $C^k(\bar U)$ 完~~ $\sum_{|\alpha|=k}[D^\alpha u_n-D^\alpha u_m]_{C^{0,\gamma}(\bar U)}\to 0$

特别地. $\underbrace{\sup_{|\alpha|\le k}\sup_{x\in\bar U}|D^\alpha(u_n-u_m)(x)|}_{=\|u_n-u_m\|_{C^k(\bar U)}}\to 0$ as $m,n\to\infty$

由 $(C^k(\bar U),\|\cdot\|_{C^k(\bar U)})$ Banach 知. $\exists u\in C^k(\bar U)$. $u_n\to u$ in $C^k(\bar U)$.

下面先证 $[D^\alpha u_n-D^\alpha u]_{C^{0,\gamma}(\bar U)}\to 0$ as $n\to\infty$

上式 $=\sup_{\substack{x\ne y\\ x,y\in\bar U}}\dfrac{|D^\alpha u_n(x)-D^\alpha u(x)-(D^\alpha u_n(y)-D^\alpha u(y))|}{|x-y|^\gamma}$

这两步应该调换一下顺序，先证明u在$C^{k,\gamma}$里面，再证明收敛性

$\to 0$ as $n\to\infty$ (因 $D^\alpha u_n$ 一致收敛于 $D^\alpha u$).

于是, 只欠证 $u\in C^{k,\gamma}(\bar U)$, 这只需要 $\forall|\alpha|=k$, $[D^\alpha u]_{C^{0,\gamma}(\bar U)}<+\infty$

事实上 $\forall x,y\in\bar U$. $x\ne y$. $\dfrac{|D^\alpha u(x)-D^\alpha u(y)|}{|x-y|^\gamma}\le\limsup_{n\to\infty}\dfrac{|D^\alpha u_n(x)-D^\alpha u_n(y)|}{|x-y|^\gamma}$

$\le\limsup_{n\to\infty}[D^\alpha u_n]_{C^{0,\gamma}(\bar U)}<+\infty$ (柯西列必有界) □

[5.2] $0<\beta<\gamma\le 1$ 时，证明：

$$\|u\|_{C^{0,\gamma}(\bar U)} \le \|u\|_{C^{0,\beta}(\bar U)}^{1-\gamma/1-\beta}\ \|u\|_{C^{0,1}(\bar U)}^{\frac{\gamma-\beta}{1-\beta}}.$$

证明：

$$\|u\|_{C^{0,\gamma}(\bar U)} = \|u\|_{C(\bar U)} + \sup_{\substack{x\neq y\\ x,y\in\bar U}} \frac{|u(x)-u(y)|}{|x-y|^{\gamma}}$$

$$\le \|u\|_{C(\bar U)}^{\frac{1-\gamma}{1-\beta}}\ \|u\|_{C(\bar U)}^{\frac{\gamma-\beta}{1-\beta}} + \sup_{\substack{x\neq y\\ x,y\in\bar U}} \frac{|u(x)-u(y)|^{\frac{1-\gamma}{1-\beta}}}{|x-y|^{\beta\cdot\frac{1-\gamma}{1-\beta}}}\ \sup_{\substack{x\neq y\\ x,y\in\bar U}} \frac{|u(x)-u(y)|^{\frac{\gamma-\beta}{1-\beta}}}{|x-y|^{1\cdot\frac{\gamma-\beta}{1-\beta}}}$$

离散Hölder

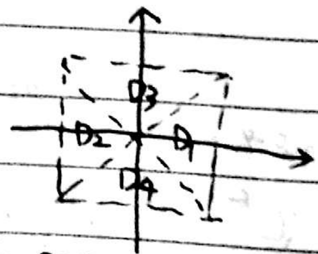
$$\le \left(\|u\|_{C(\bar U)} + \sup_{\substack{x\neq y\\ x,y\in\bar U}} \frac{|u(x)-u(y)|}{|x-y|^{\beta}}\right)^{\frac{1-\gamma}{1-\beta}} \cdot \left(\|u\|_{C(\bar U)} + \sup_{x\neq y} \frac{|u(x)-u(y)|}{|x-y|}\right)^{\frac{\gamma-\beta}{1-\beta}}$$

$$= \|u\|_{C^{0,\beta}(\bar U)}^{1-\gamma/1-\beta}\ \|u\|_{C^{0,1}(\bar U)}^{\gamma-\beta/1-\beta}.$$

$\square$

[5.3] $U=(-1,1)\times(-1,1)\subset\mathbb{R}^2$

令 $u(x)=\begin{cases} 1-x_1 & x_1>0 \quad |x_2|<x_1 \to D_1\\ 1+x_1 & x_1<0 \quad |x_2|<-x_1 \to D_2\\ 1-x_2 & x_2>0 \quad |x_1|<x_2 \to D_3\\ 1+x_2 & x_2<0 \quad |x_1|<-x_2 \to D_4\end{cases}$



问哪些 $p\in[1,+\infty]$，使 $u\in W^{1,p}(U)$.

证明： $u\in L^p(U)$ 为显见 $\forall 1\le p\le+\infty$，下面先求 u 的弱导数

Claim: $V\ \cancel{\text{函}} = \begin{cases} (-1,0) & \text{in } D_1\\ (1,0) & \text{in } D_2\\ (0,-1) & \text{in } D_3\\ (0,1) & \text{in } D_4.\end{cases}$ 是 u 的一阶弱导数 Du

Check：$\forall\varphi\in C_c^\infty(U)$.

$$\int_U V\cdot\varphi = \sum_{i=1}^4\int_{D_i} V_i\varphi \xlongequal[\varphi\in C_c^\infty]{} \int_{D_1}(-1,0)\varphi + \int_{D_2}(1,0)\varphi + \int_{D_3}(0,-1)\varphi + \int_{D_4}(0,1)\varphi$$

$$= \sum_{i=1}^4 \int_{D_i} \underbrace{\nabla u}_{\text{强导数}}\varphi \xlongequal{\text{分部积分}} \sum_{i=1}^4 -\int_{D_i} u\nabla\varphi + \underbrace{\int_{\partial D_i} u\varphi\, n_i}_{0}$$

(∂D_i 在U内部抵消，$\varphi|_{\partial U}=0$)

$= -\int_U u \cdot \nabla\varphi \, dx$ · 从而 v的确是u的一个弱导数.

$v \in L^p \ \forall 1\le p\le +\infty$ 知 $u\in W^{1,p}(\bar U)$ $\forall 1\le p\le+\infty$

注意：Cantor-Lebesgue函数没有弱导数。事实上如果该函数存在弱导数，可以通过定义证明弱导数必须a.e.=0. 之后用下面的引理得出f(x)=const a.e.(这里是指f a.e.等于同一个常数)。因此Cantoe-Lebesgue函数不能成为推翻结论的反例

[5.4]. 设 $n=1$. $u\in W^{1,p}(0,1)$. $1\le p<+\infty$

(1) 证明 u a.e. 等于一个绝对连续函数 $u^* \in L^p(0,1)$.

(2) $1<p<\infty$ 时. $|u(x)-u(y)| \le |x-y|^{1-\frac{1}{p}}\left(\int_0^1 |u'|^p dt\right)^{\frac{1}{p}}$

证明(1)

Lemma(周民强实变函数解题指南 P256) 设 $f\in L^1[a,b]$, $\forall \varphi\in C_c^1[a,b]$, 若有 $\int_a^b f(x)\varphi(x)=0$

则 $f(x)=c$. a.e

Proof: 设 g是任一紧支于(a,b)的连续函数.

h是 - - - - — — — — — , $\int_a^b h(x)\,dx=1$.

令 $\varphi(x)=\int_a^x g(t)dt - \int_a^x h(t)dt\cdot\int_a^b g(t)\,dt$. $x\in[a,b]$

则 $\varphi\in C_c^1(a,b)$.

$\varphi'(x)=g(x)-h(x)\int_a^b g(t)dt$. $\forall x\in[a,b]$

从而 $0=\int f(x)\varphi'(x)\,dx$

$=\int_a^b f(x)\left(g(x)-\int_a^b g(t)dt\cdot h(x)\right)dx$

$=\int_a^b f(x)g(x) - \int_a^b f(x)h(x)dx\cdot\int_a^b g(x)dx = \int_a^b \left(f(x)-\int_a^b f(t)h(t)dt\right)g(x)dx$

于是: $f(x)-\int_a^b f(t)h(t)dt=0$ a.e. $\Rightarrow f(x)=c$ a.e.

注:这用到了 $f\in L^1(\mathbb{R}^d)$ 若 $\forall\varphi\in C_c(a,b)$. $\int f\varphi=0$ 则 $f=0$ a.e.

该命题可由反证法得出: 否则设 $m(E)>0$, $f(x)>0$ in E. (E有界)

则 ∃ 紧支连续函数 $\{\varphi_k\}$. $\begin{cases} \|\varphi_k-\chi_E\|_{L^1}\to 0 \\ |\varphi_k|\le 1 \end{cases}$ $\varphi_k\to\chi_E$ a.e. in E.

由 $|f\varphi|\le|f|$ $\forall x\in E$

$\therefore 0<\int_E f(x)\,dx = \int_{\mathbb{R}^d} f(x)\chi_E(x)\,dx \overset{DCT}{=} \lim_{k\to\infty}\int_{\mathbb{R}^d} f(x)\varphi_k(x)\,dx=0$. 矛盾

引理证毕. #

回到原题. 令 $u^*=\int_0^x u'(t)\,dt$, 其中 u' 为u的弱导数.

则 u^* 绝对连续. 下证 $u=u^*+\text{const}$ a.e.

$\forall \varphi \in C_c^\infty(0,1)$，则 $\varphi \in C_c^1(0,1)$.

$$\int_0^1 (u^* - u)\varphi' dx = \int_0^1 \int_0^x u' dt \cdot \varphi' dx - \int_0^1 u \varphi' dx$$

第一项重积分第二个积分的范围是t到1

$$= \int_0^1 \int_0^t \varphi'(x) dx \cdot u'(t) dt + \int_0^1 u' \varphi(x) dx$$

$$= \int_0^1 (\underbrace{\varphi(1)}_{=0} - \varphi(t))\, u'(t) dt + \int_0^1 u'(x) \varphi(x) dx$$

$$= 0$$

再由引理即得.

(2). 由 u a.e.=绝对连续函数知

$$|u(x) - u(y)| = \left| \int_0^1 \chi_{\{x \le t \le y\}}(t)\, u'(t)\, dt \right|$$

a.e. $x, y \in [0,1]$ 不妨 $x \le y$

$$\le |x-y|^{\frac{1}{p'}} \left(\int_x^y |u'|^p dt \right)^{\frac{1}{p}} \qquad \frac{1}{p} + \frac{1}{p'} = 1$$

Hölder.

这里应该是0到1的积分

□

5. U 有界，U, V 开. $V \subset\subset U$ 证明：$\exists \zeta \in C^\infty(U)$. s.t. $\zeta \begin{cases} \equiv 1 & \text{on } V \\ = 0 & \text{near } \partial U \end{cases}$

证明：取开集 W. $V \subset\subset W \subset\subset U$

令 $\zeta(x) = (\chi_W * \eta_\varepsilon)(x)$ $\qquad |\varepsilon| < \frac{1}{2}\min\{\text{dist}(\partial V, \partial W), \text{dist}(\partial W, \partial U)\}$

则在 V 上. $\zeta(x) = \int_{\mathbb{R}^n} \chi_W(y)\, \eta_\varepsilon(x-y)\, dy$

$$= \int_{B(0,\varepsilon)} \eta_\varepsilon(y) \cdot \chi_W(x-y)\, dy.$$

$x \in V$ 时. $\forall y \in B(0,\varepsilon)$ $\quad |x-y| \le |x| + |y| \le |x| + \frac{1}{2}\text{dist}(\partial V, \partial W)$

$$\Rightarrow x - y \in W \quad \Rightarrow \chi_W(x-y) = 1$$

$$\Rightarrow \zeta(x) = 1 \quad \forall x \in V$$

同理. 令 $U_\varepsilon = \{x \in U \mid \text{dist}(x, \partial U) < \frac{\varepsilon}{3}\}$

可证，$\zeta(x) = 0$ in U_ε. $\Rightarrow \zeta = 0$ near ∂U.

$\zeta \in C^\infty$ 由 mollifier 性质易得

□

[5.6] U 有界，$\{V_i\}_1^N$ 是 $\mathbb{R}^n$ 中的开集，$U \subset\subset \bigcup_{i=1}^n V_i$，证明：存在 C^∞ 函数 $\{\zeta_i\}_1^N$

s.t. $\begin{cases} 0 \le \zeta_i \le 1 \\ Spt\,\zeta_i \subset V_i \quad 1 \le i \le N \\ \sum_{i=1}^N \zeta_i = 1 \quad \text{in } U. \end{cases}$

证明：对 $\bar{U} \subset \bigcup_{i=1}^n V_i$.

~~对每个 V_i，由 [5.5] 知 存在 C^∞ 函数 η_i $(1 \le i \le N)$ s.t.~~ ~~$\begin{cases} 0 < \eta_i \le 1 \\ Spt\,\eta_i \subseteq V_i \\ \sum_{i=1}^N \eta_i(x) = 1 \end{cases}$~~

$\forall x \in \bar{U}$，存在以 x 为中心的闭球 $B(x) \subseteq V_i$ (for some i) ~~$\forall x \in \bar{U}$~~

因 $\bar{U}$ 紧 $\bar{U} \subseteq \bigcup_{x \in \bar{U}} \mathring{B}(x)$. 故存在有限子覆盖 $\bigcup_{j=1}^m B(x_j)$.

对任何 $i \in \{1, 2, \dots, N\}$. 令 $U_i = \bigcup_{\{j : \mathring{B}(x_j) \subseteq O_i\}} \mathring{B}(x_j)$ 则 $\bar{U} \subset \bigcup_{i=1}^N U_i$

由上一题，$\exists \varphi_i \in C^\infty$，$0 \le \varphi_i \le 1$ $\varphi_i = 1$ in $\bar{U}$

$Spt\,\varphi_i \subseteq V_i$

令 $\eta_1 = \varphi_1$，$\eta_2 = \varphi_2(1-\varphi_1)$，$\dots$，$\eta_N = \varphi_N(1-\varphi_1)\cdots(1-\varphi_N)$.

则 $Spt\,\eta_i \subset V_i$.

$\eta_1 + \cdots + \eta_N = 1 - \prod_{i=1}^N (1-\varphi_i)$. 因 $\forall x \in \bar{U}$，总有一个 φ_i 是 1. 故 $\eta_1 + \cdots + \eta_N = 1$

□

[5.7] U 有界，且存在 C^∞ 向量场 $\vec{\alpha}$，使 $\vec{\alpha} \cdot \vec{\nu} \ge 1$ along ∂U ($\vec{\nu}$ 为 ∂U 单位外法向).

$1 \le p < +\infty$. 请对 $\int_{\partial U} \vec{\alpha} \cdot \vec{\nu}\, |u|^p\, ds$ 用 Gauss-Green 公式 证明：$\forall u \in C^1(\bar{U})$.

$$\int_{\partial U} |u|^p\, dS \le C \int_U |Du|^p + |u|^p\, dx$$

证明：

$$\int_{\partial U} |u|^p ds \le \int_{\partial U} (|u|^p \vec{\alpha}) \cdot \vec{\nu}\, ds \xlongequal{\text{Gauss-Green}} \int_U \operatorname{div}(|u|^p \vec{\alpha})\, dx$$

$$= \sum_{i=1}^n \int_U \partial_i (|u|^p \alpha_i)\, dx \qquad (\vec{\alpha} = (\alpha_1 \dots \alpha_n))$$

$$= \sum_{i=1}^n \int_U \partial_i |u|^p\, \alpha_i\, dx + \sum_{i=1}^n \int_U |u|^p\, \partial_i \alpha_i\, dx$$

$\vec{\alpha} \in C^\infty$ ∂U 紧 故 $\partial_i \vec{\alpha}$、$\vec{\alpha}$ 一致有界

$$\le C \sum_{i=1}^n \int_U \partial_i |u|^p + C \int_U |u|^p\, dx$$

$$\le C \sum_{i=1}^n \int_U p|u|^{p-1} \cdot |\partial_i u| + C \int_U |u|^p\, dx$$

$$\le C \left(\int_U |u|^p + \int_U |u|^{p-1} |Du|\, dx \right).$$

Young 不等式

$$\le C \int_U (|u|^p + |Du|^p)\, dx$$

□

[5.8] U有界，$\partial U\in C^1$. 证明：不存在 $T: L^p(U)\to L^p(\partial U)$ 为有界线性算子，使 $Tu = u|_{\partial U}$,
$\forall u\in C(\bar{U})\cap L^p(\partial U)$

证明：反设存在这样的T. 则 取 $u_m = \max\{0,\ 1-m\,\mathrm{dist}(x,\partial U)\}$

则 $Tu_m\equiv 1$ on ∂U $\quad \|u_m\|_{L^p(\partial U)} = \int_{\partial U} 1\, d\mathcal{H}^{n-1} = \underbrace{\mathcal{H}^{n-1}}(\partial U) > 0$

∂U的$n-1$维 Hausdorff 测度 (Lebesgue)

但$\|u_m\|_{L^p(U)}\to 0$ as $m\to\infty$.

check: $\displaystyle\int_U |u_m|^p\,dx \xrightarrow[\text{DCT}]{u_m\to 0 \text{ as } m\to\infty \text{ in } U,\ |u_m|\le 1 \text{ in } L^p(U)} 0$.

故 $\|T\|$ ~~$=\sup$~~ $\geq \limsup\limits_{m\to\infty} \dfrac{\|Tu_m\|_{L^p(\partial U)}}{\|u_m\|_{L^p(U)}} = \limsup\limits_{m\to\infty} \dfrac{H^{n-1}(\partial U)}{0} = +\infty$

与T有界矛盾！

$\square$

[5.9] (1) 分部积分证明：$\|Du\|_{L^2}\le C\|u\|_{L^2}^{1/2}\|D^2u\|_{L^2}^{1/2}$. $\quad\forall u\in C_c^\infty(U)$.

(2) U有界，$\partial U\in C^\infty$ 证明 上述不等式 对$u\in H_0^1(U)\cap H^2(U)$成立.

证明：(1) $u\in C_c^\infty(U)$ 则

$$\|Du\|_{L^2}^2 = \int_U |Du|^2\,dx = \sum_{i=1}^n\int_U(\partial_i u)^2\,dx$$

$$\xlongequal[u|_{\partial U}=0]{\text{分部积分}} -\sum_{i=1}^n\int_U u\cdot\partial_{ii}u\,dx$$

$$= -\int_U u\cdot\Delta u\,dx \le \int_U |u|\cdot|\Delta u|\,dx$$

$$\le C\int_U |u|\cdot|D^2u|\,dx$$

$$\le C\|u\|_{L^2}\|Du\|_{L^2}.$$

(2) $\forall u\in H_0^1(U)\cap H^2(U)$

存在一列$\{v_n\}\subset C_c^\infty(U)$ $\quad v_n\to u$ in $H_0^1(U)$

$\{w_n\}\subset C^\infty(U)$ $\quad w_n\to u$ in $H^2(U)$

则 $$\int_U Dv_k\cdot Dw_k = \sum_{i=1}^n\int_U \partial_i v_k\cdot\partial_i w_k\,dx$$

$$\xlongequal{\text{分部积分}} -\sum_{i=1}^n\int_U v_k\,\partial_{ii}w_k\,dx$$

$$= -\int_U v_k\Delta w_k\,dx \le C\int_U |v_k||D^2w_k|\,dx$$

$$\le C\|v_k\|_2\|D^2w_k\|_2. \quad \cdots(*)$$

再由：$\|v_k\|_2 \to \|u\|_2$.

$\|D^2 w_k\|_2 \to \|D^2 u\|_2$.

故 $\|v_k\|_2 \cdot \|D^2 w_k\|_2 \to \|u\|_2 \|D^2 u\|_2$.

而 $\int Du \cdot Du - \int Dv_k \cdot Dw_k$

$= \int Du \cdot (Du - Dw_k)\,dx + \int Dw_k \cdot (Du - Dv_k)\,dx$

$\le \|Du\|_2 \|Du - Dw_k\|_2 + \underbrace{\|Dw_k\|_2}_{\text{一致有界(因}\{Dw_k\}\text{为}L^2\text{中柯西列)}} \|Du - Dv_k\|_2$

$\to 0$ as $k \to \infty$

于是 (*) 两边 $k\to\infty$ 即可 □

[5.10]. (1) $\forall u \in C_c^\infty(U)$. $2 \le p < \infty$ $\|Du\|_p \le C\|u\|_p^{1/2} \|D^2u\|_p^{1/2}$.

(2). $\forall u \in C_c^\infty(U)$. $1 \le p < \infty$. $\|Du\|_{2p} \le C\|u\|_\infty^{1/2} \|D^2 u\|_p^{1/2}$.

证明：(1) $\|Du\|_p^p = \int_U |Du|^p\,dx$

$= \int_U |Du|^{p-2} \cdot |Du|^2\,dx = \sum_{i=1}^n \int_U \partial_i u (\partial_i u\, |Du|^{p-2})\,dx$

$\underset{u\in C_c^\infty(U),\ u|_{\partial U}=0}{\overset{\text{分部积分}}{=\!=}} -\sum_{i=1}^n \int_U u\, \partial_i(\partial_i u \cdot |Du|^{p-2})\,dx$.

$= \underbrace{-\int_U u \cdot \Delta u \cdot |Du|^{p-2}\,dx}_{I_1} \underbrace{- \sum_{i=1}^n \int u\, \partial_i u \cdot \partial_i |Du|^{p-2}\,dx}_{I_2}$

$I_1 = -\int_U u \cdot \Delta u \cdot |Du|^{p-2} dx$

$\le C \int_U |u| \cdot |D^2 u| \cdot |Du|^{p-2}\,dx.$ $\overset{\text{三元Hölder}\ \frac{1}{p}+\frac{1}{p}+\frac{p-2}{p}=1}{\le} C\|u\|_p \|D^2u\|_p \left(\int |Du|^{p-2\cdot\frac{p}{p-2}}dx\right)^{\frac{p-2}{p}}$

$= C\|u\|_p \|D^2 u\|_p \cdot \|Du\|_p^{p-2}$.

$I_2 = -\sum_{i=1}^n \int u \cdot \partial_i u \cdot \partial_i |Du|^{p-2}\,dx$

$= -\sum_{i=1}^n \int u \cdot \partial_i u \cdot \partial_i \left(\sum_{j=1}^n (\partial_j u)^2\right)^{\frac{p-2}{2}}\,dx.$

$= -\sum_{i=1}^n \int u \cdot \partial_i u \cdot (p-2) \cdot \sum_{j=1}^n \partial_i \partial_j u \cdot \partial_j u\, |Du|^{p-4}\,dx$

$= -(p-2) \sum_{i=1}^n \int u \cdot |Du|^{p-4} \sum_{i,j=1}^n \partial_i u (\partial_i \partial_j u)\, \partial_j u\,dx$

$$\leq C\int_U |u|\cdot|Du|^{p-4}\cdot\left|(\partial_1 u,\dots,\partial_n u)\cdot\begin{pmatrix}\partial_{11}u & \cdots & \partial_{1n}u\\ \vdots & & \vdots\\ \partial_{n1}u & \cdots & \partial_{nn}u\end{pmatrix}\begin{pmatrix}\partial_1 u\\ \vdots\\ \partial_n u\end{pmatrix}\right|dx$$

$$\leq C\int_U |u|\cdot|Du|^{p-4}\ |Du|\cdot|D^2u|\cdot|Du|\ dx$$

$$= C\int_U |u|\cdot|Du|^{p-2}\cdot|D^2u|\,dx \overset{\text{三元Hölder，同}I_1}{\leq} C\|u\|_p\|Du\|_p^{p-2}\|D^2u\|_p$$

$$\therefore \int|Du|^p dx \leq C\int_U |u|\cdot|Du|^{p-2}|D^2u|\,dx$$

$$\leq C\|u\|_p\|Du\|_p^{p-2}\|D^2u\|_p$$

两边开2次方即可.

(2) 同(1)有 $\|Du\|_{2p}^{2p} \leq C\int_U |u|\cdot|Du|^{2p-2}|D^2u|\,dx$

$$\leq C\|u\|_\infty\int_U |Du|^{2p-2}|D^2u|\,dx$$

$$\overset{\text{Hölder}}{\leq} C\|u\|_\infty\|Du\|_{2p}^{2p-2}\|D^2u\|_{2p}.$$

开平方即可.

$\square$

[5.11] U连通，$u\in W^{1,p}(U)$，$Du=0$ a.e. in U 则 $u=\text{const}$ a.e. in U.

证明：此题不能用Poincaré不等式 $\|u-(u)_U\|_p\leq C\|Du\|_p$ 因为该题结论被用于证明Poincaré不等式

令 $u^\varepsilon=\eta^\varepsilon * u$. $V\subset\subset U$.

则 ε充分小时. $Du^\varepsilon=\eta^\varepsilon * Du=(Du)^\varepsilon$ in V.

$u^\varepsilon\in C^\infty(V) \Rightarrow \exists$常数$C_\varepsilon$ s.t. $u^\varepsilon=C_\varepsilon$ in V.

而 $\|u^\varepsilon\|_{L^p}=\|\eta^\varepsilon * u\|_{L^p}\leq\|\eta^\varepsilon\|_{L^1}\|u\|_{L^p}=\|u\|_{L^p}<\infty$ uniformly on ε.

$\Rightarrow \{C_\varepsilon\}_{\varepsilon>0}$ 一致有界 故有收敛子列 $C_{\varepsilon_i}\to C\in\mathbb{R}$.

因 $C\in L^p$，由控制收敛定理易有

$\|u^{\varepsilon_i}-C\|_{L^p}\to 0$ as $i\to\infty$

又 $\|u^{\varepsilon_i}-u\|_{L^p}\to 0$ as $i\to\infty$ $\therefore u=C$ a.e. in any $V\subset\subset U$

故 $u=C$ a.e. in U

$\square$

[5.12] 举例说明. 若 $\|D^h u\|_{L^1(V)} \le C$ $\forall\, 0<|h|<\frac{1}{2}\mathrm{dist}(V,\partial U)$. 则 并不能推出 $u\in W^{1,1}(V)$.

证明: 令 ~~U = [-2017, 2017]~~ $U=(-0.001,\ 1.001)^n$

$V=(0,1)^n$.

$$u(x)=\begin{cases}1 & 0<x_1<\frac{1}{2}\\ 0 & \text{否则}.\end{cases}$$

$u(x)\in L^\infty(V)$

$$\|D^h u\|_{L^1(V)}=\int_V \underbrace{|D^h u|}_{\le\frac{1}{h}}\,dx \le \int_{\frac{1}{2}-h}^{\frac{1}{2}}\int_0^1\cdots\int_0^1 \left|\frac{1}{h}\right| dx_n\cdots dx_1$$

$$=1$$

但 $u\notin W^{1,1}(V)$. 否则 设 $\partial_{x_1}u$ 为 u 的 x_1 方向弱偏导. 且 $\partial_{x_1}u\in L^1(V)$.

$\forall\phi\in C_c^\infty(V)$ $\int_V \partial_{x_1}u\,\phi\,dx=0$ (~~因 $\partial_{x_1}u=0$ a.e.~~ (因 u 只取0,1值))

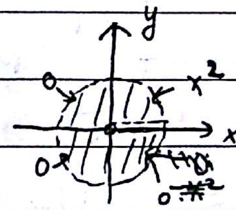
$$-\int_V u\cdot\partial_{x_1}\phi \overset{\|}{=} -\int_{V\cap\{x_1>\frac{1}{2}\}}\partial_{x_1}\phi\,dx$$

这不可能. □

[5.13] 找一个 $U\subset\mathbb{R}^n$ 开, $u\in W^{1,\infty}(U)$ 但 u 不是 U上的 Lipschitz 连续函数.

证明: 令 $U=\mathring{B}(0,1)-\{(x,y)\in\mathring{B}(0,1)\mid x\ge 0,\ y=0\}$

$u(x)=\cancel{\mathrm{sgn}(y)}\cdot(\max\{0,x_0\})^2\cdot\max\{\mathrm{sgn}\,y,0\}$



则 $u(x)$ 在U中可微.

$\partial_{x_1}u(x)=2\max\{\mathrm{sgn}\,y,0\}\max\{0,x\}$.

$\partial_{x_2}u(x)=0$.

$\Rightarrow u\in W^{1,\infty}(U)$.

但u 并不Lip. 因 $\forall\varepsilon>0$ $u(\frac{1}{2},\varepsilon)-u(\frac{1}{2},-\varepsilon)=\frac{1}{2}$

$$\Rightarrow \mathrm{Lip}(u)\ge\frac{\frac{1}{2}}{2\varepsilon}=\frac{1}{4\varepsilon}\to\infty \quad \text{as } \varepsilon\to 0^+.$$

□

[5.14] 证明: $U=\mathring{B}(0,1)$ 时 ($n>1$), $u=\log\log(1+\frac{1}{|x|})\in W^{1,n}(U)$

证明: $\partial_i u(x) = \dfrac{1}{\log(1+\frac{1}{|x|})}\cdot\dfrac{1}{1+\frac{1}{|x|}}\left(-\dfrac{1}{|x|^2}\cdot \operatorname{sgn} x_i\right).$

$$= -\frac{1}{\log(1+\frac{1}{|x|})}\cdot\frac{x_i}{|x|^3}\cdot\frac{1}{1+\frac{1}{|x|}}$$

$$= -\frac{1}{\log(1+\frac{1}{|x|})}\cdot\frac{x_i}{|x|^2}\cdot\frac{1}{|x|+1}$$

$\Rightarrow |Du| \leq C\cdot\dfrac{1}{|x|}\cdot\dfrac{1}{\log(1+\frac{1}{|x|})}$

$$\int_{B(0,1)}|Du|^n\,dx \leq C\int_0^1\left(\frac{1}{\log(1+\frac{1}{|\rho|})}\right)^n\cdot\frac{1}{\rho^n}\cdot\rho^{n-1}\,d\rho.$$

写成极坐标形式

$$\overset{z=\log(1+\frac{1}{\rho})}{\leq} C\int_{\log 2}^{\infty}\frac{1}{z^n}\,dz < +\infty$$

$$\int_{B(0,1)}|u|^n\,dx = \int_0^1\left|\log\log(1+\frac{1}{\rho})\right|^n\rho^{n-1}\,d\rho$$

$$= \int_0^{\frac{1}{e-1}}\left|\log\log(1+\frac{1}{\rho})\right|^n\rho^{n-1}\,d\rho \quad I_1$$

$$+\int_{\frac{1}{e-1}}^1 \log\log(1+\frac{1}{\rho})^n\,\rho^{n-1}\,d\rho \quad I_2$$

$$I_2 \leq \int_{\frac{1}{e-1}}^1\left(\log(1+\frac{1}{\rho})\right)^n\rho^{n-1}\,d\rho$$

$$\leq \int_{\frac{1}{e-1}}^1(\log 2)^n\rho^{n-1}\,d\rho < \infty$$

$$\cancel{I_1 = -\int_0^{\frac{1}{e-1}}\log\log(1+\frac{1}{\rho})^n\rho^{n-1}\,d\rho}$$

$$\cancel{\leq -\int_0^1\left(\log\frac{\rho}{\rho+1}\right)^n\rho^{n-1}\,d\rho}$$

$$\cancel{= -\int_0^1\log\left(1-\frac{1}{\rho+1}\right)^n\rho^{n-1}\,d\rho.}$$

$$= (-1)^{n+1}\int_0^1 \log\left(1+\frac{1}{\rho}\right)^n \rho^{n-1}\, d\rho .$$

$$\cancel{\le \alpha \int_0^1 \rho}$$

$$I_1 \overset{z=\frac{1}{\rho}}{=} \int_{e-1}^{\infty} \log\log(1+z)^n \cdot \frac{dz}{z^{n+1}}$$

$$\le C\int_{e-1}^{\infty} \frac{dz}{z^{n+1/2}} < \infty \qquad 故.\ u\in W^{1,n}(U)$$

□

[5.15] Fix $\alpha>0$. $U=\mathring{B}(0,1)$. 证明：存在常数 $C(n,\alpha)$，使

$\int_U u^2\,dx \le C\int_U |Du|^2\,dx$. 其中已知 $\mathcal{L}^n\{x\in U \mid u(x)=0\} \ge \alpha$. $u\in H^1(U)$

证明: $$\int_U u^2\,dx \overset{令\ \langle u\rangle = \frac{1}{|U|}\int_U u\,dx}{=} \int_{U-A}\left(u-\langle u\rangle+\langle u\rangle\right)^2 dx \qquad 其中\ A=\{x \mid u(x)=0\}$$

$$= \int_{U-A}(u-\langle u\rangle)^2\,dx + 2\underbrace{\int_{U-A}(u-\langle u\rangle)\,dx}_{=0}\cdot\langle u\rangle + \int_{U-A}\langle u\rangle^2\,dx$$

这个等于0是错的！可以直接从上一步用$(X+Y)^2\leqslant 2X^2+2Y^2$进行放缩到下一步，之后follow原有答案

Poincaré 不等式 U连通有界 $\|u-\langle u\rangle\|_p \le C\|Du\|_p$

$$\le C\|Du\|_{L^2}^2 + \int_{U-A}\langle u\rangle^2$$

$$= C\|Du\|_{L^2}^2 + |\langle u\rangle^2|\cdot|U-A|$$

$$= C\|Du\|_{L^2}^2 + \frac{1}{|U|^2}\left(\int_{U-A}|u|\,dx\right)^2\cdot|U-A|$$

Hölder
$$\le C\|Du\|_{L^2}^2 + \frac{1}{|U|^2}\cdot\left(\int_{U-A}|u|^2\,dx\right)\cdot|U-A|^2 .$$

$$\overset{令\ 1-C_0=\frac{|U-A|^2}{|U|^2}}{=} C\|Du\|_{L^2}^2 + (1-C_0)\|u\|_{L^2}^2 .$$

$\Rightarrow \exists C'>0:\ \int_U u^2\,dx \le C'\int_U |Du|^2\,dx$

□

[5.16] 证明：$\forall n\geq 3$，$\exists$ const. C. $\int_{\mathbb{R}^n}\frac{u^2}{|x|^2}dx\leq C\int_{\mathbb{R}^n}|Du|^2dx$，$\forall u\in H^1(\mathbb{R}^n)$

证明：先设 $u\in C_c^\infty(\mathbb{R}^n)$ 设 $F(x)=\frac{x}{|x|^2}$.

则.
$$\int_{\mathbb{R}^n}u^2\,\mathrm{div}F\,dx=-\int_{\mathbb{R}^n}D(u^2)\cdot F(x)\,dx.$$
$$=-2\int_{\mathbb{R}^n}uD(u)\cdot F(x)\,dx$$
$$=-2\int_{\mathbb{R}^n}Du\cdot(uF)\,dx$$
$$\Rightarrow\left|\int_{\mathbb{R}^n}u^2\,\mathrm{div}F\,dx\right|=2\left|\int_{\mathbb{R}^n}Du\cdot uF\,dx\right|$$
$$\leq 2\,\|Du\|_2\,\|uF\|_{L^2}.$$

由 $\mathrm{div}F(x)=\frac{n-2}{|x|^2}$. $|F(x)|^2=\frac{1}{|x|^2}$ 代入.有：
$$\frac{(n-2)^2}{4}\left(\int_{\mathbb{R}^n}\frac{u^2}{|x|^2}dx\right)^2\leq\int|Du|^2dx\cdot\int_{\mathbb{R}^n}\frac{u^2}{|x|^2}dx$$
$$\Rightarrow\frac{(n-2)^2}{4}\int_{\mathbb{R}^n}\frac{u^2}{|x|^2}dx\leq\int_{\mathbb{R}^n}|Du|^2dx.$$

易证 $H^1(U)=H_0^1(U)\Leftarrow U=R^d$

对一般的 $u\in H^1(\mathbb{R}^n)$. 由于 $H^1(\mathbb{R}^n)=H_0^1(\mathbb{R}^n)$.

故 $\exists u_k\in C_c^\infty(\mathbb{R}^n)$. $u_k\to u$ in $H^1(\mathbb{R}^n)$.

从而 $\int_{\mathbb{R}^n}|Du_k|^2dx\longrightarrow\int_{\mathbb{R}^n}|Du|^2dx$
$$\geq\frac{(n-2)}{4}\int_{\mathbb{R}^n}\frac{u_k^2}{|x|^2}dx.$$

$\Rightarrow\frac{u_k}{|x|}\in L^2(\mathbb{R}^n)$. 由 $u_k\to u$ in L^2

故存在子列 $u_{k_j}\to u$ a.e.
$$\Rightarrow\left(\frac{u_{k_j}}{|x|}\right)^2\to\left(\frac{u}{|x|}\right)^2\text{ a.e.}$$
$$\Rightarrow\frac{(n-2)^2}{4}\int_{\mathbb{R}^n}\frac{u^2}{|x|^2}dx\underset{\text{Fatou引理}}{\leq}\liminf_{j\to\infty}\frac{(n-2)^2}{4}\int_{\mathbb{R}^n}\frac{u_{k_j}^2}{|x|^2}dx\leq\liminf_{j\to\infty}\int_{\mathbb{R}^n}|Du_{k_j}|^2dx$$
$$=\int_{\mathbb{R}^n}|Du|^2dx$$

[5.17] (链式法则) $F:\mathbb{R}\to\mathbb{R}$. 是 C^1 的. 且 F' 有界 $u\in W^{1,p}(U)$. $1\le p\le\infty$.

证明：(1) 若 ~~U 有界~~ $\mathcal{L}^n(U)<\infty$. 则 $v:=F(u)\in W^{1,p}(U)$, $\partial_i v=F'(u)\partial_i u$. $1\le i\le n$

(2) 若 $\mathcal{L}^n(U)=+\infty$，但 $F(0)=0$ 则 (1) 结论也对

Rmk：证明过程中会体现. (2) 中 $F(0)=0$ 是必须的.

Proof：由 $|F(u)-F(0)|\le\|F'\|_{L^\infty}|u|\in L^p$ （[illegible] 中值定理）.

故 $F(u)-F(0)\in L^p(U)$

若 $\mathcal{L}^n(U)<\infty$. 则 $F(0)\in L^p(U)\Rightarrow F(u)\in L^p(U)$.

若 $\mathcal{L}^n(U)=+\infty$. 则 $F(0)=0\Leftrightarrow F(u)\in L^p(U)$.

$F(u)-F(0)\in L^p(U)$

~~[illegible]~~ $F'(u)\partial_i u\in L^p(U)$ 显然，因 $F'\in L^\infty(U)$. $\partial_i u\in L^p(U)$.

下面证明 $\partial_i F(u)=F'(u)\partial_i u$ $i=1,2,\dots,n$.

~~令 $V\subset\subset U$. $u^\varepsilon=\eta_\varepsilon*u$. 使得 $u^\varepsilon\in C^\infty(V_\varepsilon)$~~

$\forall\phi\in C_c^\infty(U)$ 设. $\operatorname{Spt}\phi\subset V\subset\subset U$. $u^\varepsilon=\eta_\varepsilon*u$.

要证：$\int_U F(u)\partial_i\phi\,dx=-\int_U F'(u)\partial_i u\cdot\phi\,dx$.

$$左=\int_V F(u)\partial_i\phi\,dx\overset{①}{=}\lim_{\varepsilon\to0^+}\int_V F(u^\varepsilon)\partial_i\phi\,dx.$$

$$\underset{\substack{因 u^\varepsilon\in C^\infty(U_\varepsilon)\\ 链式法则可用}}{\overset{分部积分}{=}}-\lim_{\varepsilon\to0^+}\int_V F'(u^\varepsilon)\partial_i u^\varepsilon\cdot\phi\,dx \qquad \text{充分 (设}\varepsilon\text{足够小)}$$

$$\overset{②}{=}-\int_V F'(u)\partial_i u\cdot\phi\,dx=-\int_U F'(u)\partial_i u\cdot\phi\,dx$$

check ①: $\int_V|F(u)-F(u^\varepsilon)|\cdot|\partial_i\phi|\,dx\le\int_V|u-u^\varepsilon|\cdot|\partial_i\phi|\,dx\cdot\|F'\|_{L^\infty}$

$$\le\|F'\|_{L^\infty(U)}\|u-u^\varepsilon\|_{L^p(V)}\|\partial_i\phi\|_{L^{p'}(U)}$$

$\to0$ as $\varepsilon\to0^+$ (因 $u^\varepsilon\to u$ in $L^p_{loc}(U)$)

②: $\left|\int_V F'(u)\partial_i u\,\phi\,dx-\int_V F'(u^\varepsilon)\partial_i u^\varepsilon\cdot\phi\,dx\right|$

$$\le\underbrace{\int_V|F'(u)-F'(u^\varepsilon)||\partial_i u||\phi|\,dx}_{A}+\underbrace{\int_V|F'(u^\varepsilon)|\cdot|\partial_i u-\partial_i u^\varepsilon|\cdot|\phi|\,dx}_{B}.$$

对 $B=\int_V |F'(u^\varepsilon)|\cdot|\partial_i u-\partial_i u^\varepsilon|\cdot|\phi|\,dx$

$\le \|F'\|_{L^\infty(U)}\cdot\|\partial_i u-\partial_i u^\varepsilon\|_{L^p(V)}\cdot\|\phi\|_{L^{p'}(V)}$.

$\longrightarrow 0$ as $\varepsilon\to 0^+$ （因 $u^\varepsilon\longrightarrow u$ in $W^{1,p}_{loc}$）

$A=\int_V |F'(u)-F'(u^\varepsilon)|\cdot|\partial_i u|\cdot|\phi|\,dx$

由 $u^\varepsilon\longrightarrow u$ a.e. in V （光滑化性质）.

F'连续 $\Rightarrow F'(u)\to F'(u^\varepsilon)$ a.e. in V

又：$|F'(u)-F'(u^\varepsilon)|\cdot|\partial_i u|\cdot|\phi|\le 2\|F'\|_{L^\infty}|\partial_i u|\cdot|\phi|\in L^1$ （由Hölder即得）

故由控制收敛定理. $A\longrightarrow 0$ as $\varepsilon\to 0^+$.

证毕! $\square$

[5.18] $1\le p\le\infty$. U有界.

(1) 证明：若 $u\in W^{1,p}(U)$. 则 $|u|\in W^{1,p}(U)$.

(2) 若 $u\in W^{1,p}(U)$. 则 $u^+,u^-\in W^{1,p}(U)$.

$Du^+=\begin{cases}Du & \mathcal{L}^n\text{-a.e. on }\{u>0\}\\ 0 & \mathcal{L}^n\text{-a.e. on }\{u\le 0\}\end{cases}$ $\quad Du^-=\begin{cases}0 & \mathcal{L}^n\text{-a.e. on }\{u\ge 0\}\\ -Du & \mathcal{L}^n\text{-a.e. on }\{u<0\}\end{cases}$

(3). $u\in W^{1,p}(U)$. 则 $Du=0$ a.e. on $\{u=0\}$

Proof：只用证(2). (2) $\Rightarrow$ (1) 显然

若(2)对， 则 $Du=Du^+-Du^-=0$ on $\{u=0\}$ $\mathcal{L}^n$-a.e.

下证(2). 令 $F_\varepsilon(r)=(\sqrt{r^2+\varepsilon^2}-\varepsilon)\chi_{\{r\ge 0\}}\in C^1(\mathbb{R})$

且 $F_\varepsilon'(r)\in L^\infty(\mathbb{R})$. （Fix $\varepsilon>0$）.

由门题 $\int_U F_\varepsilon(u)\partial\phi\,dx=-\int_U F_\varepsilon'(u)\partial_i u\cdot\phi\,dx$ ……(*)

注意到. $u^+=\lim_{\varepsilon\to 0^+}F_\varepsilon(u)$.

(*)左边令 $\varepsilon\to 0$. 极限为 $\int_U \lim_{\varepsilon\to 0^+}F_\varepsilon(u)\,\partial_i\phi\,dx=\int_U u^+\partial_i\phi\,dx$

这由控制收敛即得. (*)右边同理 $\longrightarrow -\int \partial_i u\,\chi_{\{u>0\}}\phi\,dx$

故 $\partial_i u^+ = \partial u \, \chi_{\{u>0\}}$ a.e. ~~in {u>0}~~ 同理 u^- 有类似结果，因 $u^- = (-u)^+$.

□

[5.19] 设 $u \in H^1(U)$ 按书上 Hint 证明 $Du = 0$ a.e. in $\{u=0\}$

证：取 ϕ 是 C^∞、有界、不减光滑，ϕ' 有界. $\phi(z) = z$ $\forall |z| \le 1$.

令 $u^\varepsilon(x) = \varepsilon \phi(\frac{u}{\varepsilon})$

① claim $u^\varepsilon \to 0$ in $L^2(U)$.

$\forall \varphi \in C_c^\infty(U)$. $\int_U u^\varepsilon \varphi \, dx = \varepsilon \int_U \phi(\frac{u}{\varepsilon}) \varphi \, dx$

$\le \varepsilon \cdot \|\phi\|_{L^\infty} \underbrace{\|\varphi\|_{L^1}}_{<+\infty} \to 0$ as $\varepsilon \to 0^+$.

$\therefore \forall \varphi \in C_c^\infty(U)$. $\langle \varphi, u^\varepsilon \rangle \to 0$ as $\varepsilon \to 0^+$.

~~$\mathrm{X}\ C_c^\infty(U) \overset{dense}{\subseteq}$~~

又 $\|u^\varepsilon\|_{L^2}^2 = \varepsilon^2 \int_U |\phi(\frac{u}{\varepsilon})|^2$ ~~$\le \|\phi\|_{L^\infty}^2$~~

因 $\left.\begin{matrix}\phi' \text{有界} \\ \phi(0)=0\end{matrix}\right\}$ 故 $\forall x \in \mathbb{R}$. $|\phi(x)| \le \|\phi'\|_{L^\infty} |x|$.

$\therefore \|u^\varepsilon\|_2^2 \le \varepsilon^2 \int_U \frac{u^2}{\varepsilon^2} \|\phi'\|_\infty^2 dx = \|u\|_{L^2}^2 < \infty$

$\therefore \left\{\begin{matrix}\|u^\varepsilon\|_{L^2} \text{ 一致有界} \\ \forall \varphi \in C_c^\infty(U) \overset{dense}{\subseteq} (L^2(U))^* = L^2(U). \ \langle \varphi, u^\varepsilon \rangle \to 0\end{matrix}\right. \Rightarrow u^\varepsilon \rightharpoonup 0$ in $L^2(U)$.

② $\partial_i u^\varepsilon \rightharpoonup 0$ in $L^2(U)$.

$\|\partial_i u^\varepsilon\|_{L^2}^2 = \int |\partial_i u^\varepsilon|^2 = \int |\phi'(\frac{u}{\varepsilon}) \cdot \partial_i u|^2 dx \le \|\phi'\|_{L^\infty}^2 \|\partial_i u\|_{L^2}^2$

$< \infty$ (一致).

又 $\forall \varphi \in C_c^\infty(U)$,

$\langle \partial_i u^\varepsilon, \varphi \rangle = \int \partial_i u^\varepsilon \cdot \varphi = -\int u^\varepsilon \cdot \underbrace{\partial_i \varphi}_{\in L^2 = (L^2)^*} \to 0$ (因 $u^\varepsilon \rightharpoonup 0$ in L^2)

$\therefore \partial_i u^\varepsilon \rightharpoonup 0$ in $L^2(U)$.

如今 $\int Du^{\varepsilon}\cdot Du\,dx = \sum_{i=1}^{n}\int \partial_i u^{\varepsilon}\,\partial_i u\,dx$

$\to 0$ (因 $\partial_i u \in L^2$. $\partial_i u^{\varepsilon} \rightharpoonup 0$ in L^2)

又: $\int Du^{\varepsilon}\cdot Du = \sum_{i=1}^{n}\int \partial_i u^{\varepsilon}\,\partial_i u\,dx$

$= \sum_{i=1}^{n}\int \partial_i u\cdot \phi'\left(\frac{u}{\varepsilon}\right)\partial_i u\,dx$

$= \int |Du|^2\,\phi'\left(\frac{u}{\varepsilon}\right)dx$

限制在 $\{u=0\}$ 上. 令 $\varepsilon\to 0^+$ 即有 $Du=0$ a.e. on $\{u=0\}$

□

Rmk: 19 不需 u 有界. 因 $\phi(0)=0$, 17(2) 生效.

[5.20]. 若 $u\in H^s(\mathbb{R}^d)$, $s>\frac{d}{2}$. 则 $u\in L^\infty(\mathbb{R}^d)$. 且 $\|u\|_{L^\infty(\mathbb{R}^d)}\le C\|u\|_{H^s(\mathbb{R}^d)}$.

证明: ~~$u\in H^s(\mathbb{R}^d)$ 则 $u\in L^2(\mathbb{R}^d)$~~ $(s>\frac{d}{2})$

$\cancel{|u(x)| \le \lim_{N\to\infty}\int_{|x|<N}\left|\hat{u}(\xi)e^{2\pi i x\cdot\xi}\right|d\xi}$.

$\cancel{\le \lim_{N\to\infty}\int_{|x|<N}|\hat{u}(\xi)|\langle\xi\rangle^s\frac{1}{\langle\xi\rangle^s}d\xi}$.

Claim: $S(\mathbb{R}^d)$ 在 $H^s(\mathbb{R}^d)$ 中稠密

若 ~~这个~~ claim 对, 则我们只用对 $u\in S(\mathbb{R}^d)$ 证明即可 (再延拓)

$u\in S(\mathbb{R}^d)$ $|u(x)| = |(\hat{u}(\xi))^{\vee}(x)|$

$= \left|\int_{\mathbb{R}^d}\hat{u}(\xi)e^{2\pi i x\cdot\xi}d\xi\right|$

$\le \int_{\mathbb{R}^d}|\hat{u}(\xi)|\,d\xi$

$\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$ $= \int_{\mathbb{R}^d}|\hat{u}(\xi)|\,\langle\xi\rangle^s\frac{1}{\langle\xi\rangle^s}d\xi$

$\leq \| \frac{1}{\langle\xi\rangle^s} \|_{L^2} \| \langle\xi\rangle^s \hat{u} \|_{L^2}$.

$= C_{s,d} \|u\|_{H^s(\mathbb{R}^d)}$. ✓

再证 Claim：因 $S(\mathbb{R}^d) \overset{\text{dense}}{\subset} L^2(\mathbb{R}^d)$，故 $\exists\, v_k \in S(\mathbb{R}^d)$ $\forall u \in H^s(\mathbb{R}^d)$

$v_k \to \langle\xi\rangle^s \hat{u}$ in $L^2(\mathbb{R}^d)$.

令 $u_k = (\langle\xi\rangle^{-s} v_k)^\vee$ 这里合理的，因 $v_k \langle\xi\rangle^{-s} \in S(\mathbb{R}^d)$

故 $\|u_k - u\|_{H^s} = \|(\hat{u}_k - \hat{u}) \langle\xi\rangle^s\|_{L^2}$

$= \|(\langle\xi\rangle^{-s} v_k - \hat{u}) \langle\xi\rangle^s\|_{L^2}$.

$= \|v_k - \hat{u}\langle\xi\rangle^s\|_{L^2} \to 0$ as $k \to \infty$. $\square$

21题表明s>d/2时，H^s是一个Banach代数，也就是对乘积封闭；
一般地，对s≥0，结论是$\|uv\|_{H^s} \leq C(\|u\|_{L^\infty}\|v\|_{H^s}+\|v\|_{L^\infty}\|u\|_{H^s})$,证明需要用到调和分析里面的Littlewood-Paley分解，在此略去。借此（称作Moser不等式），再由20题便可得到21题结论。

[5.21] 若 $u, v \in H^s(\mathbb{R}^d)$，$s > \frac{d}{2}$，则 $uv \in H^s(\mathbb{R}^d)$

且 $\|uv\|_{H^s(\mathbb{R}^d)} \leq C_{s,d} \|u\|_{H^s} \|v\|_{H^s}$ 故 $s > \frac{d}{2}$ 时 H^s是代数.

证明：

$\|uv\|_{H^s(\mathbb{R}^d)} = \|\widehat{uv}\langle\xi\rangle^s\|_{L^2}$.

这个其实也应该先对Schwartz函数证明，我偷个懒。

$= \|(\hat{u} * \hat{v}) \langle\xi\rangle^s\|_{L^2}$.

$= \| \int \hat{u}(\xi-\eta)\hat{v}(\eta) \langle\xi\rangle^s \, d\eta \|_{L^2_\xi}$

这儿 $\langle\xi\rangle^s = 1 + |\xi|^s$.（与20题那个其实等价）.

~~$\leq (1+|\xi|^s)(1+|\eta|^s)$~~.

~~$= \langle\xi\rangle =$~~

① $|\xi| < \frac{|\eta|}{2}$ or $|\xi| > 2|\eta|$ 时. 若 $|\xi| < \frac{|\eta|}{2}$

$\langle\xi\rangle^s = 1+|\xi|^s \begin{cases} \leq 1 + \frac{|\eta|^s}{2^s} \leq 1+|\eta|^s \leq (1+|\xi-\eta|^s)(1+|\eta|^s) \\ \text{若 } |\xi| > 2|\eta| \text{ 则 } \left|\frac{\xi}{\eta}\right| > 2 \end{cases}$

$\Rightarrow 3\xi^2 - 8\xi\eta + 4\eta^2 > 0$

$\Rightarrow |\xi| \leq 2|\xi-\eta|$

$\Rightarrow \langle\xi\rangle^s \leq C(\langle\xi-\eta\rangle^s + \langle\eta\rangle^s)$.

② $\frac{|\eta|}{2} \le |\xi| \le 2|\eta|$ 时.

$1+|\xi|^s \le 2^s(1+|\eta|^s)(1+|\xi-\eta|^s)$.

故 $\langle\xi\rangle^s \le C_s\left(\langle\xi-\eta\rangle^s+\langle\eta\rangle^s\right)$.

代入有:

$$\|uv\|_{H^s} \lesssim_s \left\|\int \hat u(\xi-\eta)\hat v(\eta)\langle\xi-\eta\rangle^s \, d\eta\right\|_{L^2_\xi} + \left\|\int \hat u(\xi-\eta)\hat v(\eta)\langle\eta\rangle^s \, d\eta\right\|_{L^2_\xi}$$

$$= \cancel{\|\hat u\langle\xi\rangle} \ \|(\hat u\langle\cdot\rangle^s)*\hat v\|_{L^2_\xi} + \|\hat u*(\langle\cdot\rangle^s\hat v)\|_{L^2}.$$

$\|f*g\|_{L^2}\le\|f\|_{L^1}\|g\|_{L^2}$.

$$\le \|\hat u\langle\xi\rangle^s\|_{L^2}\|\hat v\|_{L^1} + \|\hat v\langle\xi\rangle^s\|_{L^2}\|\hat u\|_{L^1}$$

$$= \|u\|_{H^s}\|\hat v\langle\xi\rangle^s\langle\xi\rangle^{-s}\|_{L^1} + \|v\|_{H^s}\|\hat u\langle\xi\rangle^s\langle\xi\rangle^{-s}\|_{L^1}$$

$$\le \|u\|_{H^s}\|\hat v\langle\xi\rangle^s\|_{L^2}\|\langle\xi\rangle^{-s}\|_{L^2} + \|v\|_{H^s}\|\hat u\langle\xi\rangle^s\|_{L^2}\|\langle\xi\rangle^{-s}\|_{L^2}$$

$$\lesssim_s \|u\|_{H^s}\|v\|_{H^s}$$

□

注: