Evans Ch12 习题

本节习题中，除特别声明，我们均假设函数是光滑的，记号 $\square_d = \partial_t^2 - \Delta_d$

[12.1] 设 u 光滑，且是拟线性方程 $u_{tt} - \sum_{i=1}^{n} (L_{p_i}(\nabla u))_{x_i} = 0$ in $\mathbb{R}^d \times (0,\infty)$ 的解

请构造合适的能量 $E(t)$，使得 $E'(t) = 0$

证明：令 $E(t) = \frac{1}{2}\|u_t\|^2_{L^2(\mathbb{R}^d)} + \int_{\mathbb{R}^d} L(\nabla u)\,dx$.

$$\Rightarrow E'(t) = \int_{\mathbb{R}^d} u_t u_{tt}\,dx + \int_{\mathbb{R}^d} \sum_{i=1}^{n} L_{p_i}(\nabla u)\,\partial_t u_{x_i}\,dx$$

$$\overset{\text{分部积分}}{\Longrightarrow} E'(t) = \int_{\mathbb{R}^d} u_t u_{tt}\,dx - \sum_{i=1}^{n} (L_{p_i}(\nabla u))_{x_i}\, u_t\,dx$$

$$= \int_{\mathbb{R}^d} u_t \underbrace{\left(u_{tt} - \sum_{i=1}^{n} (L_{p_i}(\nabla u))_{x_i}\right)}_{\overset{\shortparallel}{0}} dx = 0.$$

$\square$

[12.2] 设 u 为 Klein-Gordon 方程 $\begin{cases} u_{tt} - \Delta u + m^2 u = 0 & \text{in } \mathbb{R}^d \times (0,\infty) \\ u(0) = g,\ u_t(0) = h \end{cases}$ 的解

求证：(1) $E(t) = \frac{1}{2}\int_{\mathbb{R}^d} u_t^2 + |\nabla u|^2 + m^2 u^2\,dx$ 关于 t 不变

(2) $\lim_{t\to\infty} \int_{\mathbb{R}^d} |\nabla u|^2 + m^2 u^2\,dx = E(0)$. (Hint: 模仿 §4.3.1)

证明：(1) $E'(t) = \int_{\mathbb{R}^d} u_t u_{tt} + \nabla u \cdot \nabla u_t + m^2 u \cdot u_t\,dx$

$$\overset{\text{第2项分部积分}}{=} \int_{\mathbb{R}^d} u_t \underbrace{(u_{tt} - \Delta u + m^2 u)}_{=0}\,dx = 0$$

(2) 不妨设 $m = 1$，令 $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$

原方程作 Fourier 变换可得：

$$\begin{cases} \hat{u}_{tt} + \langle\xi\rangle^2 \hat{u} + \hat{u} = 0 & \text{in } \mathbb{R}^d \times (0,\infty) \\ \hat{u}(0) = \hat{g},\ \hat{u}_t(0) = \hat{h} \end{cases}$$

$$\Rightarrow \hat{u}(\xi, t) = \hat{g}(\xi)\cos(t\langle\xi\rangle) + \frac{\hat{h}}{\langle\xi\rangle}\sin(t\langle\xi\rangle)$$

待估计的式子是 $I = \int_{\mathbb{R}^d} |\nabla u|^2 + \cdots$

由 Plancherel 恒等式:

$$I=\int_{\mathbb{R}^d}|\hat{u}(\xi)|^2\langle\xi\rangle^2\,d\xi \quad \leftarrow \text{不妨 } g,h \text{ 实值吧.}$$

$$=\int_{\mathbb{R}^d}\langle\xi\rangle^2|\hat{g}(\xi)|^2\cos^2(t\langle\xi\rangle)\,d\xi \quad \leftarrow I_1$$

$$+\int_{\mathbb{R}^d}|\hat{h}(\xi)|^2\sin^2(t\langle\xi\rangle)\,d\xi \quad \leftarrow I_2$$

$$\underbrace{+\,2\int_{\mathbb{R}^d}\cos(t\langle\xi\rangle)\sin(t\langle\xi\rangle)\,\langle\xi\rangle\hat{g}(\xi)\hat{h}(\xi)\,d\xi}_{=:J}$$

$\boxed{\text{Claim}}$ $J\longrightarrow 0$ as $t\longrightarrow+\infty$

$$J=\int_{\mathbb{R}^d}\sin(2t\langle\xi\rangle)f(\xi)\,d\xi. \quad \text{其中 } f(\xi)=\langle\xi\rangle\hat{g}(\xi)\hat{h}(\xi)\in S(\mathbb{R}^d).$$

$$=\int_0^\infty\sin\left(2t\sqrt{\rho^2+1}\right)\left(\int_{\partial B(0,\rho)}f\,dS\right)d\rho$$

令 $F(\rho):=\int_{\partial B(0,\rho)}f\,dS\in C^\infty(\mathbb{R})$.

而 $f\in S(\mathbb{R}^d)\Rightarrow F(\rho)\in L^1(\mathbb{R})$.

于是 $\exists\{F_n\}\subseteq C_c^\infty(0,+\infty)$ s.t. $F_n\longrightarrow F$ in $L^1(\mathbb{R})$

0的一个小邻域与 spt F_n = $\emptyset$

(相当于 F_n 的支集总离原点差一些)

对 F_n 而言.

令 $J_n=\int_0^\infty\sin\left(2t\sqrt{\rho^2+1}\right)F_n(\rho)\,d\rho$

$$\overset{u=\sqrt{\rho^2+1}}{=\!=}\int_1^\infty\frac{uF_n(\sqrt{u^2-1})}{\sqrt{u^2-1}}\sin(2tu)\,du.$$

$$=\int_{\text{spt}F_n}\boxed{\frac{u}{\sqrt{u^2-1}}F_n\left(\sqrt{u^2-1}\right)}\sin(2tu)\,du.$$

spt F_n 远离 $u=1$ (因 $\rho=0$ 远离 spt F_n) ~~$\Rightarrow\in L^1(\mathbb{R}^+)$~~

$$=\int_1^\infty\underbrace{\left(\chi_{\text{spt}F_n}\frac{u}{\sqrt{u^2-1}}F_n\left(\sqrt{u^2-1}\right)\right)}_{\in L^1(\mathbb{R}^+)}\sin(2tu)\,du$$

于是由Riemann-Lebesgue引理知，$J_n \to 0$ as $t\to\infty$

而 $|J-J_n| \underset{\text{sin 有界}}{\le} \|F_n-F\|_{L^1(\mathbb{R}^d)} \to 0$ as $n\to\infty$ $\forall t>0$ 均成立

故 $J=\int_{\mathbb{R}^d} \sin(2t\langle\xi\rangle) f(\xi)\, d\xi \to 0$ as $t\to\infty$.

~~Rmk~~ $\boxed{\text{Warning:}}$ ①这里不可使用Riemann-Lebesgue引理，因为sin里面是 $2t\langle\xi\rangle$，不是 $2t\xi$

②这里不可直接变量替换 $u=\sqrt{|\xi|^2+1}$，因f并不紧支于 $(0,+\infty)$，故端点0处有奇性

Claim 证毕.

对 $I_1 = \int_{\mathbb{R}^d} \langle\xi\rangle^2 |\hat{g}(\xi)|^2 \cos^2(t\langle\xi\rangle)\, d\xi$

$$= \frac{1}{2}\underbrace{\int_{\mathbb{R}^d} \langle\xi\rangle^2 |\hat{g}(\xi)|^2 \cos(2t\langle\xi\rangle)\, d\xi}_{\substack{\downarrow \text{ 同 } J\to 0 \text{ 的证明} \\ 0 \text{ as } t\to\infty}} + \frac{1}{2}\int_{\mathbb{R}^d} \langle\xi\rangle^2 |\hat{g}(\xi)|^2\, d\xi$$

$$\overset{t\to\infty}{=\!=} \frac{1}{2}\int_{\mathbb{R}^d} \langle\xi\rangle^2 |\hat{g}(\xi)|^2\, d\xi \overset{\text{Plancherel}}{=\!=} \frac{1}{2}\|\nabla g\|^2_{L^2(\mathbb{R}^d)}$$

$$I_2 \xrightarrow[t\to\infty]{\text{同 } I_1} \frac{1}{2}\|h\|^2_{L^2(\mathbb{R}^d)}$$

$$\therefore I \longrightarrow \frac{1}{2}\int_{\mathbb{R}^d} |\nabla g|^2 + |h|^2 \; dx = E(u) \quad \text{as } t\to\infty$$

$\square$

$\boxed{\text{Rmk}}$：此题也可直接用Stein的"Harmonic Analysis" Chapter 8.1 Prop 3（震荡积分渐近展开）去证明 $J \lesssim O(t^{-\frac{d}{2}})$ as $t\to\infty$.

$\square$

[12.3] 设 u 满足 [12.2] 中的方程.

$$\bar{u}(\bar{x},t) = u(x,t)\cos(m\,x_{d+1}).$$

(1) 求证：$\Box_{d+1}\bar{u} = 0 \quad \text{in } \mathbb{R}^{d+1}\times(0,\infty)$.

(2) $d=1$ 时. 求解 Klein-Gordon 方程.

证明：(1)

$$\Box_{d+1}\bar{u} = \partial_t^2\big(u(x,t)\big)\cos(m x_{d+1}) - \sum_{i=1}^{d}\partial_{x_i}^2\big(u(x,t)\big)\cos(m x_{d+1}) + m^2 u(x,t)\cos(m x_{d+1})$$

$$= \cos(m x_{d+1})\,\underbrace{(\Box_d u + m^2 u)}_{=0} = 0$$

(2) 实际上 (1) 是 "$\Longleftrightarrow$" 的.

于是考虑
$$\begin{cases} \Box_2 \bar{u} = 0 \\ \bar{u}(0) = \bar{g}(\bar{x}) = g(x_1)\cos(m x_2) \\ \bar{u}_t(0) = \bar{h}(\bar{x}) = h(x_1)\,\cancel{\cos}\,(m x_2). \end{cases}$$

由 Poisson 公式：

$$\bar{u}(x_1, x_2, t) = \frac{1}{2}\partial_t\left(t^2\int_{(\bar{x},t)}\frac{\bar{g}(y)}{(t^2+|y-x|^2)^{\frac{1}{2}}}\,dy\right) + \frac{t^2}{2}\int_{B(\bar{x},t)}\frac{\bar{h}(y)}{(t^2+|y-x|^2)^{\frac{1}{2}}}\,dy$$

$$u(x_1,t) = \bar{u}(x_1,0,t).$$

$\Box$

[12.4] 设 u 为 $\Box u + \lambda u_t = 0$ 在 $\mathbb{R}^d\times(0,\infty)$ 中的解. $\lambda>0$.
试求一个指数项，该项乘 u 之后给出 $\Box v + cv = 0$ 的解. 这个操作将 Klein-Gordon 方程的系数反号.

证明：令 $v = e^{\frac{\lambda}{2}t}u(x,t)$.

即 $u(x,t) = e^{-\frac{\lambda t}{2}}v(x,t)$.

$$\Box u + \lambda u_t \underset{\substack{\uparrow \\ \text{直接计算}}}{=} e^{-\lambda t/2}\left(\Box v - \frac{\lambda^2}{4}v\right) = 0$$

$$\Rightarrow \Box v - \frac{\lambda^2}{4}v = 0$$

$\Box$

[12.5] 证明：对任一给定的 $y\in\mathbb{R}^d\setminus\{0\}$，录 $u=e^{i(x\cdot y-\sigma t)}$ 为 Klein-Gordon 方程 $u_{tt}-\Delta u+m^2u=0$ 的解，$\sigma=\sqrt{|y|^2+m^2}$

证明：$u_{tt}=(-i\sigma)^2\ e^{i((x\cdot y)-\sigma t)}$

$$=-(|y|^2+m^2)\ e^{i(x\cdot y-\sigma t)}$$

$u_{x_jx_j}=(-iy_j)^2\,e^{i(x\cdot y-\sigma t)} \Rightarrow -\Delta u=|y|^2\,e^{i((x\cdot y)-\sigma t)}$

$\Rightarrow (\Box+m^2)u=0$.

□

[12.6] 设 $\begin{cases}\Box u=0 & \text{in } \mathbb{R}^3\times(0,\infty)\\ u(0)=g,\ u_t(0)=h & \text{on } \mathbb{R}^3\times\{0\}\end{cases}$ $\quad g,h\in C_c^\infty(\mathbb{R}^3)$.

求证：$\exists\, C>0$ s.t. $|u(x,t)|\le\frac{C}{t}$ ← 这应是 t 较大的时候

一般应是 $|u(x,t)|\lesssim\frac{C}{\langle t\rangle}$.

证明：由 Ch 2.4 的 Kirchoff 公式有

$$u(x,t)=\frac{1}{4\pi t^2}\int_{\partial B(x,t)} t h(y)+g(y)+\nabla g(y)\cdot(y-x)\,dS_y.$$

① $t<1$ 时，$|h(y)|\lesssim1$

$\therefore\left|\frac{1}{4\pi t^2}\int_{\partial B(x,t)} t\,h(y)\,dS_y\right|\lesssim|t|\le1$.

后面两项同理（注意 $|y-x|=t$）

② $t\ge1$ 时，由 $\mathrm{Spt}\,g$，$\mathrm{Spt}\,h$ 紧知，$\mathcal{H}^2(\partial B(x,t)\cap\mathrm{Spt}\,h)\lesssim1$.

~~而相关函数~~ 被积函数 $L^\infty_{\mathrm{Spt}\,g}$ norm $\lesssim t$

$\therefore |u(x,t)|\lesssim\frac{1}{t}$

□

[12.~~7~~] 设 $\begin{cases}\Box u=0 & \text{in } \mathbb{R}^2\times(0,\infty).\\ u(0)=g\quad u_t(0)=h & \text{on } \mathbb{R}^2\times\{0\}.\end{cases}$ $\quad g,h\in C_c^\infty$

求证：$|u(x,t)|\lesssim\frac{1}{\langle t\rangle^{\frac12}}$.

证明：由 Poisson 公式.

$$u(t,x)=\underbrace{\frac{1}{2\pi}\partial_t\int_{B(x,t)}\frac{g(y)}{\sqrt{t^2-|y-x|^2}}dy}_{I_1}+\underbrace{\frac{1}{2\pi}\int_{B(x,t)}\frac{h(y)}{\sqrt{t^2-|y-x|^2}}dy}_{I_2}.$$

~~证明~~ $I_2 = \frac{1}{2\pi}\int_{B(x,t)} \frac{h(y)}{\sqrt{t^2-|y-x|^2}}\,dy$

设 g.h 支于 $B(0,R)$ 中

则 $|I| \underset{\rho=|y-x|}{\le} \frac{\|h\|_{L^\infty}}{2\pi}\int_0^t\int_{\partial B(a,\rho)}\frac{dS}{\sqrt{t^2-\rho^2}}\,d\rho$

$\le \frac{\|h\|_{L^\infty}}{2\pi}\int_0^t \frac{\mathcal{H}^2(\partial B(x,\rho)\cap B(0,R))}{\sqrt{t^2-\rho^2}}\,d\rho$

$\lesssim \frac{1}{\sqrt{t}}\underbrace{\int_{\max\{0,|x|-R\}}^{\min\{t,|x|+R\}}\frac{4\pi R^2}{\sqrt{t-\rho}}\,d\rho}_{\lesssim R^{5/2}} \lesssim \frac{1}{\sqrt{t}}$

对 I_1：~~变量替换~~ 知

$\int_{B(x,t)}\frac{g(y)}{\sqrt{t^2-|y-x|^2}}\,dy \overset{y=x+tz}{=} t\int_{B(0,1)}\frac{g(x+tz)}{\sqrt{1-|z|^2}}\,dz$

$\Rightarrow I_1 = \frac{1}{2\pi}\partial_t\int_{B(x,t)}\frac{g(y)\,dy}{\sqrt{t^2-|y-x|^2}}\,dy$

$= \frac{1}{2\pi}\int_{B(0,1)}\frac{g(x+tz)}{\sqrt{1-|z|^2}}\,dz + \frac{t}{2\pi}\int_{B(0,1)}\frac{\nabla g(x+tz)\cdot z}{\sqrt{1-|z|^2}}\,dz$

换回 $\overset{y=x+tz}{=}$ $\frac{1}{2\pi t}\int_{B(x,y)}\frac{g(y)}{\sqrt{t^2-|y-x|^2}}\,dy$

$+\frac{1}{2\pi t}\int_{B(x,t)}\frac{|\nabla g(y)|\cdot(y-x)}{\sqrt{t^2-|y-x|^2}}\,dy$

第一项同 I_2

$\lesssim \frac{1}{t^{3/2}} + \frac{\overset{\le t}{\max|y-x|}}{t}\int_{B(x,t)}\frac{\|\nabla g\|_{L^\infty}}{\sqrt{t^2-|y-x|^2}}\,dy$

$\lesssim \frac{1}{t^{3/2}} + \int_{B(x,t)}\frac{dy}{\sqrt{t^2-|y-x|^2}}\,dy \lesssim \frac{1}{\sqrt{t}}$ $\quad\therefore I_1+I_2 \lesssim \frac{1}{\sqrt{t}}$ □

[12.8] 设 u^ε 为波方程 $\begin{cases} \Box_2 u^\varepsilon = 0 \quad u_t^\varepsilon(\cdot)=0 \\ u^\varepsilon(0) = g^\varepsilon(\cdot) = \begin{cases} e^{-\frac{\varepsilon(r-2)^2}{(r-1)(3-r)}} & 1<r<3 \quad r=|x| \\ 0 & \text{otherwise} \end{cases} \end{cases}$ 的解

求证: $\sup\limits_{\mathbb{R}^2\times[0,4]} |u^\varepsilon(x,t)| \longrightarrow \infty$ as $\varepsilon\to 0^+$

证明: 由Poisson公式

$$u(x,t) = \frac{1}{2\pi}\partial_t \int_{B(x,t)} \frac{g^\varepsilon(y)\,dy}{\sqrt{t^2-|y-x|^2}}$$

$$\Rightarrow u(0,t) = \frac{\partial_t}{2\pi}\int_{B(0,t)} \frac{g^\varepsilon(y)\,dy}{\sqrt{t^2-|y|^2}}.$$

$$= \frac{\partial_t}{2\pi}\int_1^3\int_{\partial B(0,\rho)} \frac{e^{-\varepsilon\frac{(\rho-2)^2}{(\rho-1)(3-\rho)}}}{\sqrt{t^2-\rho^2}}\,dS\,d\rho.$$

$$= \frac{1}{2\pi}\int_1^3 \frac{t\rho}{(t^2-\rho^2)^{\frac{3}{2}}} e^{-\frac{\varepsilon(\rho-2)^2}{(\rho-1)(3-\rho)}}\,d\rho.$$

$$\xrightarrow{\varepsilon\to 0^+} \frac{1}{2\pi}\int_1^3 \frac{t\rho}{(t^2-\rho^2)^{\frac{3}{2}}}\,d\rho = -t\left(\frac{1}{\sqrt{t^2-9}} - \frac{1}{\sqrt{t^2-1}}\right)$$

$$\longrightarrow \infty \quad \text{as } t\to 3.$$

□

*[12.9] 波方程Kelvin变换不变性

设 $u:\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$, 其双曲Kelvin变换为 $\bar{u}(x,t) = u(\bar{x},\bar{t})\left|t^2-|\bar{x}|^2\right|^{\frac{d-1}{2}}$

$$= u\left(\frac{x}{|x|^2-t^2}, \frac{t}{|x|^2-t^2}\right)\frac{1}{\left||x|^2-t^2\right|^{\frac{d-1}{2}}}$$

$\forall |x|^2\neq t^2$. $\bar{x} = \frac{x}{|x|^2-t^2}$ $\quad \bar{t} = \frac{t}{|x|^2-t^2}$

求证: $\Box u = 0 \Longrightarrow \Box\bar{u} = 0$.

证明: 设 $r=|x|$.

令 $X = \frac{x}{t^2-r^2}$, $T = \frac{t}{t^2-r^2}$. $\quad 0<r<t$.

则 $K: (\mathbb{R}^{1+d}, g=-dt^2+dx_1^2+\cdots+dx_d^2) \longrightarrow (\mathbb{R}^{1+d}, \bar{g} = -dT^2+dX_1^2+\cdots+dX_d^2)$

是共形变换 (直接验证将Minkowski space-time 映到Minkowski space-time)

且可记成极坐标. 那么 $K:(t,r,\omega)\to(T,R,\omega)$

$T = \frac{t}{t^2-r^2}$ $\quad r<t$ $\quad$ ($r>t$ 同理可算, 略)

$R = \frac{r}{t^2-r^2}$

$\omega = \omega$

① 我们先证

$$dT^2 - dR^2 = \Omega^2(dt^2 - dr^2),\ \Omega = \frac{1}{t^2-r^2}.$$

直接计算:

$$dT = \partial_t\left(\frac{t}{t^2-r^2}\right)dt + \partial_r\left(\frac{t}{t^2-r^2}\right)dr$$

$$= -\frac{r^2+t^2}{(t^2-r^2)^2}dt + \frac{2rt}{(t^2-r^2)^2}dr$$

$$dT^2 = \left(-\frac{(r^2+t)^2}{(t^2-r^2)^2}dt + \frac{2rt}{(t^2-r^2)^2}dr\right)$$

$$dR = \cancel{\left(-\frac{(r^2+t^2)}{(t^2-r^2)^2}\right)}\ \partial_t\left(\frac{r}{t^2-r^2}\right)dt + \partial_r\left(\frac{r}{t^2-r^2}\right)dr$$

$$= -\frac{2tr}{(t^2-r^2)^2}dt + \frac{t^2+r^2}{(t^2-r^2)^2}dr$$

$$\cancel{(dR)^2}\ dR^2 = \left(\frac{-2tr}{(t^2-r^2)^2}dt + \frac{t^2+r^2}{(t^2-r^2)^2}dr\right)^2$$

$$dT^2 - dR^2 \overset{\text{拆括号}}{=} \left(\frac{(r^2+t^2)^2}{(t^2-r^2)^4} - \frac{4r^2t^2}{(t^2-r^2)^4}\right)(dt^2 - dr^2)$$

$$= \Omega^2(dt^2 - dr^2) \qquad \Omega = \frac{1}{t^2-r^2}$$

$$\Rightarrow \bar{g} = \Omega^2 g$$

② 设R为数量曲率 (Scalar curvature)

则 $\Box_g u + \frac{d-1}{4d}Ru = 0$

在变换 $g \longmapsto \bar{g} := e^{2\Phi}g$. $p = -\frac{4}{d-1}$ 下不变

$u \longmapsto e^{\frac{2\Phi}{p}}u =: \bar{u}$

特别,由$\Omega>0$知,取 $e^{2\Phi} = \Omega^2$ 即契合原题

check:

~~$\Box_{\bar g}\bar{u} = \bar{g}^{\mu\nu}\partial_\mu\partial_\nu\bar{u} - \bar{g}^{\mu\nu}\Gamma^{\eta}_{\mu\nu}\partial_\eta\bar{u}$~~

注意 $\Box_g = g^{\mu\nu}\partial_\mu\partial_\nu$)

(直接check. ~~$= \exp\left(-\frac{d+3}{2}\Phi\right)\left(\Box_g u + \left(\frac{4}{p}\Delta\right.\right.$~~

证明见 Yvonne Choquet-Bruhat, Cecile Dewitt-Morrette: Analysis, Manifolds and Physics, Vol.1 Page 352.

~~而 $\tilde{g} = \Omega^{-2} g$~~

在③的证明中，最后的结果为

$$\Box_g u - \frac{d-1}{4d} R u = \Omega^{\cancel{\frac{3+d}{2}}} \left(\Box_{\bar g} \bar u - \frac{d-1}{4d} R \bar u \right) \cdot \Omega^{-\frac{3+d}{2}}$$

而 $\mathbb{R}^{1+d}$ 数量曲率为0，故 $\Box_g u = \Box_{\bar g} \bar u \cdot \Omega^{-\frac{3+d}{2}}$

~~i.e. $\Box \bar u = (t^2 - |x|^2)^{-\frac{d+}{2}}$~~

$\Rightarrow$ $\Box_g u = 0$ implies $\Box_{\bar g} \bar u = 0$

$\Box$

Rmk：证明参考了 Yvonne Choquet-Bruhat, Cecile Dewitt-~~M~~Morette, Analysis, Manifolds and Physics, Vol 2. Page 260

$\Box$

[2.10] 假设 u, v 满足

$$\begin{cases} (u-v)_t = 2a \sin\left(\frac{u+v}{2}\right) & a \neq 0 \\ (u+v)_x = \frac{2}{a} \sin\left(\frac{u-v}{2}\right) \end{cases}$$

求证：$w := u$, $w := v$ 均是 sine-Gordon 方程 $w_{xt} = \sin w$ 的解

并解释为何等价于 $\Box w = \sin w$

证明：(1) 第一个方程对x求导，第二个方程对t求导，可得：

$$u_{xt} - v_{xt} = 2a \cos\left(\frac{u+v}{2}\right)(u_x + v_x) = a\frac{2}{a} \cos\left(\frac{u+v}{2}\right) \sin\left(\frac{u-v}{2}\right)$$

$$u_{xt} + v_{xt} = \frac{1}{a} \cos\left(\frac{u-v}{2}\right)(u_t - v_t)$$

$$\overset{\text{代入第一个方程}}{=} \frac{1}{a} \cos\left(\frac{u-v}{2}\right) \cdot 2a \sin\left(\frac{u+v}{2}\right) = 2 \sin\left(\frac{u+v}{2}\right) \cos\left(\frac{u-v}{2}\right)$$

~~$= \sin(u+v)$~~

相加 $\Rightarrow$ $u_{xt} = \sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) + \cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right) = \sin u$.

相减 $\Rightarrow$ $v_{xt} = \sin v$

∴ 均为 $w_{xt} = \sin w$ 的解

(2) 为何等价于 $\Box w = \sin w$?

现有 $w_{xt} = \sin w$. ~~令 $x = \frac{\alpha+\beta}{2}$ $t = \frac{\alpha-\beta}{2}$~~

~~则 $\partial_{xt}\left(\frac{\alpha+\beta}{2}\right)$~~ ~~$\alpha = \frac{x+t}{2}$, $\beta = \frac{x-t}{2}$~~

~~$\partial_{xt}\left(w\left(\frac{\alpha+\beta}{2}, \frac{\alpha-\beta}{2}\right)\right) = \left(\frac{1}{2}\partial_\alpha + \frac{1}{2}\partial_\beta\right)\left(\frac{1}{2}\partial_\alpha - \frac{1}{2}\partial_\beta\right) w$~~

令 $x = \frac{\alpha+\beta}{2}$. $t = \frac{\alpha-\beta}{2}$ ~~$= \frac{1}{4}(\Box w)\left(\frac{\alpha+\beta}{2}, \frac{\alpha-\beta}{2}\right)$~~

则 $\alpha = \frac{x+t}{2}$. $\beta = \frac{x-t}{2}$ $\Box$

$\partial_x = \partial_\alpha + \partial_\beta$. $\partial_t = \partial_\alpha - \partial_\beta$ $\Rightarrow$ $\partial_x\partial_t w = \sin w$ 变成 $\Box_{\alpha\beta} w = \sin w$

[12.11] 接着T10，给定 sine-Gordon 方程的一个解 v. 我们可以通过解T10中的方程组得到另一个解u. 此过程称作 Bäcklund 变换。

从 v=0 开始. 用 Bäcklund 变换去计算，a 不同时. u=?

解. v=0 代入，便有

$$\begin{cases} u_t = 2a\sin\frac{u}{2} \\ u_x = \frac{2}{a}\sin\frac{u}{2}. \end{cases}$$

首先有：$u_t = a^2 u_x$，这是一个传输方程

所以 u 必有形式 $f(at+\frac{x}{a})$ （见课本 ch2.1）

而又注意到 $u_t = 2a\sin\frac{u}{2}$

$$= 4a\sin\frac{u}{4}\cos\frac{u}{4}$$

$$\Rightarrow \frac{u_t}{4\cos^2\frac{u}{4}} = a\tan\frac{u}{4}$$

$$\Rightarrow \partial_t\left(\tan\frac{u}{4}\right) = a\tan\frac{u}{4}$$

直接计算 $\Rightarrow \tan\frac{u}{4} = Ce^{at+\frac{x}{a}}$

$u = 4\arctan(Ce^{at+\frac{x}{a}})$ 为所求.

$\square$

[12.12] 求证 Sobolev 不等式：$\|\partial^{\beta_1}u_1\cdots\partial^{\beta_m}u_m\|_{L^2(\mathbb{R}^d)} \lesssim \prod_{j=1}^{m}\|u_j\|_{H^k(\mathbb{R}^d)}$，$k>\frac{d}{2}$

证明：我们先证：

① $\|\partial^\beta u\|_{L^\infty} \lesssim \|u\|_{H^k}$，若 $\frac{1}{2}-\frac{k}{d}+\frac{|\beta|}{d}<0$.

此为显见，因为此时 $k-|\beta|>\frac{d}{2}$. $H^{k-|\beta|}\hookrightarrow L^\infty$

$\Rightarrow \|\partial^\beta u\|_{L^\infty} \lesssim \|\partial^\beta u\|_{H^{k-|\beta|}} \lesssim \|u\|_{H^k(\mathbb{R}^d)}$.

② 若 $\frac{1}{2}-\frac{k-|\beta|}{d}=\frac{1}{p}>0$. 则 $\|\partial^\beta u\|_{L^p} \lesssim \|u\|_{H^k}$

这由 Gagliardo-Nirenberg-Sobolev 不等式是显然的

③ 若 $\frac{1}{2}=\frac{k-|\beta|}{d}$ 则 $\|\partial^\beta u\|_{L^p} \lesssim \|u\|_{H^k}$ $\forall\, 2\le p<\infty$

这仍是易见的. 因为只用证 $\|\partial^\beta u\|_{L^p} \lesssim \|u\|_{H^{\frac{d}{2}}}$.

$\forall\varepsilon>0$. $H^{\frac{d}{2}} \hookrightarrow H^{\frac{d}{2}-\varepsilon}$. 再由 GNS 不等式知.

$\forall p\in[2,\infty)$ $\exists\varepsilon$. s.t. $H^{\frac{d}{2}-\varepsilon}(\mathbb{R}^d)\hookrightarrow L^p(\mathbb{R}^d)$

故该式成立.

下面开始凑Hölder不等式

将 $\beta_1, \dots, \beta_m$ 分成三组: Hölder之后的可积指标为

$\lambda_1, \dots, \lambda_r$ 满足① L^∞

$\mu_1, \dots, \mu_s$ 满足② L^{p_i} $1 \le i \le s$. $2 \le p_i < \infty$ 待定

$\nu_1, \dots, \nu_t$ 满足③ L^{q_j} $1 \le j \le t$. $\frac{1}{q_j} = \frac{1}{2} - \frac{k - |\nu_j|}{d}$

若能做到

$$\|(\partial^{\lambda_1} u_{\lambda_1} \cdots \partial^{\lambda_r} u_{\lambda_r}) \cdot (\partial^{\mu_1} u_{\mu_1} \cdots \partial^{\mu_s} u_{\mu_s}) \cdot (\partial^{\nu_1} u_{\nu_1} \cdots \partial^{\nu_t} u_{\nu_t})\|_{L^2}$$

$$\lesssim \|\partial^{\lambda_1} u_{\lambda_1}\|_{L^\infty} \cdots \|\partial^{\lambda_r} u_{\lambda_r}\|_{L^\infty} \cdot \|\partial^{\mu_1} u_{\mu_1}\|_{L^{p_1}} \cdots \|\partial^{\mu_s} u_{\mu_s}\|_{L^{p_s}}$$

$$\cdot \|\partial^{\nu_1} u_{\nu_1}\|_{L^{q_1}} \cdots \|\partial^{\nu_t} u_{\nu_t}\|_{L^{q_t}}.$$

则再由①~③即有上式 $\lesssim \prod_{j=1}^m \|u_j\|_{H^k(\mathbb{R}^d)}$.

下面只用验证 Hölder 不等式的指标能做到.

i.e. 可选取 $p_i \in [2, +\infty)$. s.t. $\frac{1}{2} = \sum_{i=1}^s \frac{1}{p_i} + \sum_{j=1}^t \frac{1}{q_j} + \frac{1}{\infty} + \dots + \frac{1}{\infty}$

这因为

$$\sum_{j=1}^t \frac{1}{q_j} = \frac{t}{2} - \frac{kt}{d} + \sum_{j=1}^t \frac{|\nu_j|}{d}$$

$$\le \frac{t}{2} - \frac{kt}{d} + \cancel{\sum} \frac{k}{d}$$

而右边 $< \frac{1}{2} \iff \frac{t}{2} - \frac{(t-1)k}{d} < \frac{1}{2}$

$\iff \frac{t-1}{2} < \frac{(t-1)k}{d} \iff k > \frac{d}{2}$ ✓ 说明可以做到

于是可选取恰当的 $p_1, \dots, p_s$ s.t. $\frac{1}{2} = \frac{1}{\infty} + \dots + \frac{1}{\infty} + \frac{1}{p_1} + \dots + \frac{1}{p_s} + \frac{1}{q_1} + \dots + \frac{1}{q_t}$.

证毕! □

12.13] 光滑映射 $u: \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^m$. $u = (u^1, \dots, u^m)$ 被称作射到 S^{m-1} 的波映射(wave map into S^{m-1}), 是指 $|u| = 1$ in $\mathbb{R}^n \times [0, \infty)$.
$u_{tt} - \Delta u \perp S^{m-1}$ at u.

求证: $u_{tt} - \Delta u = (|\nabla u|^2 - |u_t|^2) u$.

证明: 显见 $u_{tt} - \Delta u \,/\!/\, u$. 因 $u_{tt} - \Delta u \perp S^{m-1}$ at $u \in S^{m-1}$.

而 $|u|^2 = 1 \Rightarrow \partial_t |u|^2 = 2u \cdot u_t = 0 \Rightarrow u u_t = 0$

$\Rightarrow u_{tt} \cdot u = \partial_t (u_t \cdot u) - u_t \cdot u_t = -|u_t|^2$.

~~∴ $(u_{tt} - \Delta u) \cdot u =$~~

同样. $\Delta u \cdot u = \sum_{j=1}^{n} u_{x_j x_j} \cdot u$

$$= \underbrace{\sum_{j=1}^{n} (u \cdot u_{x_j})_{x_j}}_{} - \sum_{j=1}^{n} u_{x_j}^2$$

↳ 因 $u_{x_j} \cdot u = \frac{\partial_{x_j} |u|^2}{2} = 0$

$$= 0 - |\nabla u|^2$$

故 $\begin{cases} (\Box u) \cdot u = |\nabla u|^2 - |u_t|^2 \\ \Box u // u \end{cases} \Rightarrow \Box u = (|\nabla u|^2 - |u_t|^2)\, u.$

$\square$

[12.14] ~~求证~~: 若u是射入 S^{m-1} 的波映射(如13所述), 且是有紧支集(关于x).

求证. $\frac{d}{dt}\int_{\mathbb{R}^n} |u_t|^2 + |\nabla u|^2 \, dx = 0$

证明: : $\frac{d}{dt}\int_{\mathbb{R}^n} |u_t|^2 + |\nabla u|^2 \, dx$

~~$= \int_{\mathbb{R}^n} u_t \cdot u_{tt} + \nabla u \cdot \nabla u_t$~~

$$= \sum_{i=1}^{m} \frac{d}{dt}\int_{\mathbb{R}^n} |u_t^i|^2 + |\nabla u_t^i|^2 \, dx$$

$$= \sum_{i=1}^{m} \int_{\mathbb{R}^n} u_t^i u_{tt}^i + \nabla u_t^i \cdot \nabla u^i \, dx$$

$$\overset{\text{分部积分}}{=} \sum_{i=1}^{m} \int_{\mathbb{R}^n} u_t^i (u_{tt}^i - \Delta u^i) \, dx$$

$$= \int_{\mathbb{R}^n} u_t \cdot \Box u \, dx.$$

$$= \int_{\mathbb{R}^n} (|\nabla u|^2 - |u_t|^2)\underbrace{(u \cdot u_t)}_{0} \, dx$$

$$= 0$$

15. 跳过. (书上 §12.4 明明有更强的结论).

$\square$

关于NLS:

[12.16] 设u为复值函数. 满足 NLS $\quad iu_t+\Delta u = f(|u|^2)u \quad$ in $\mathbb{R}^d\times(0,\infty)$. ---(*)

$f:\mathbb{R}\to\mathbb{R}$. 求证：若 $\xi\in\mathbb{R}^d$，则 $w(x,t):=e^{\frac{i}{4}(2\xi\cdot x-|\xi|^2t)}u(x-\xi t,t)$

也满足NLS.（称作 Galilean Invariance）.

证明：$\partial_t w = e^{\frac{i}{4}(2\xi\cdot x-|\xi|^2t)}\left(\frac{-i|\xi|^2}{4}\right)e^{-|\xi|^2t}u(x-\xi t,t)+e^{-|\xi|^2t}u_t(x-\xi t,t)-\sum_{j=1}^d u_{x_j}(x-\xi t,t)\cdot\xi_j$

$$\partial_{x_j}w = e^{\frac{i}{4}(2\xi\cdot x-|\xi|^2t)}\left(\frac{i}{2}\xi_j\, u(x-\xi t)+u_{x_j}(x-\xi t,t)\right).$$

$$\partial^2_{x_j}w = e^{\frac{i}{4}(2\xi\cdot x-|\xi|^2t)}\Big[\frac{i}{2}\xi_j\left(\frac{i}{2}\xi_j u(x-\xi t)+u_{x_j}(x-\xi t,t)\right)+\frac{i}{2}\xi_j u_{x_j}(x-\xi t,t)+u_{x_jx_j}(x-\xi t,t)\Big]$$

$$\Rightarrow \Delta w = e^{\frac{i}{4}(2\xi\cdot x-|\xi|^2t)}\left(-\frac{|\xi|^2}{4}u(x-\xi t)+\left(\sum_{j=1}^d i\cdot u_{x_j}(x-\xi t,t)\cdot\xi_j\right)+\Delta u(x-\xi t,t)\right)$$

$$\Rightarrow iw_t-\Delta w = e^{\frac{i}{4}(2\xi\cdot x-|\xi|^2t)}\underbrace{(iu_t-\Delta u)}_{}\qquad \text{“}f(|u|^2)u$$

$$\underset{|w|=|u|}{=} f(|w|^2)\cdot w$$

□.

[12.17-12.18] 设 u 为(*)的速降解.

求证：(1) $\frac{d}{dt}\int_{\mathbb{R}^d}|u|^2\,dx=0$ (质量守恒).

(2) $\frac{d}{dt}\int_{\mathbb{R}^d}|\nabla u|^2+F(|u|^2)\,dx=0$. $\quad F'=f$ (能量守恒).

(3) $\frac{d}{dt}\int_{\mathbb{R}^d}\frac{\bar u\nabla u-u\nabla\bar u}{2i}\,dx=0$ (动量守恒).

(4) $\frac{d^2}{dt^2}\int_{\mathbb{R}^d}|u|^2|x|^2\,dx = 8\int_{\mathbb{R}^d}|\nabla u|^2+4d\int_{\mathbb{R}^d}f(|u|^2)u^2-F(|u|^2)\,dx$

(5). 用(4)来证.(NLS) $iu_t+\Delta u=-|u|^2u$ in $\mathbb{R}^d\times(0,\infty)$ 在 $E(0)=\int_{\mathbb{R}^d}|\nabla u(\cdot,0)|^2-\frac{|u(\cdot,0)|^4}{2}\,dx<0$ 时无整体解. 其中 $d\geq 2$

证明：(1) $\frac{d}{dt}\int_{\mathbb{R}^d}|u|^2\,dx = 2\mathrm{Re}\int_{\mathbb{R}^d}\bar u u_t\,dx$

(*)两边乘以 $2\bar u$ 得. $2\bar u u_t - i\Delta u\cdot\bar u = -i\bar u u f(|u|^2) = -i|u|^2 f(|u|^2)$.

$$\therefore \frac{d}{dt}\int_{\mathbb{R}^d} = 2\mathrm{Re}\int_{\mathbb{R}^d} i\bar u\Delta u - \underbrace{i|u|^2f(|u|^2)}_{\text{实}}\,dx = 2\mathrm{Re}\int_{\mathbb{R}^d} i\bar u\Delta u$$

$$= -2\mathrm{Im}\int_{\mathbb{R}^d}\bar u\Delta u = \underset{=0}{2\mathrm{Im}}\int_{\mathbb{R}^d}|\nabla u|^2dx$$

(2) $\frac{d}{dt}\int_{\mathbb{R}^d} |\nabla u|^2 + F(|u|^2)\,dx$

$= \underbrace{\cancel{\frac{d}{dt}}\mathrm{Re}\int_{\mathbb{R}^d} 2\nabla u_t \cdot \nabla \bar{u}}_{I_1} + \underbrace{\frac{d}{dt}\int_{\mathbb{R}^d} F(|u|^2)\,dx}_{I_2}$

$I_1 = 2\,\mathrm{Re}\int_{\mathbb{R}^d} \nabla \bar{u} \cdot \nabla u_t \, dx$

$\overset{\text{分部积分}}{=} -2\,\mathrm{Re}\int_{\mathbb{R}^d} u_t \cdot \Delta \bar{u} \, dx$

$= -2\,\mathrm{Re}\int_{\mathbb{R}^d} \Delta \bar{u}\,(-i)\,(-\Delta u + f(|u|^2)u)\,dx$

$= -2\,\mathrm{Im}\int_{\mathbb{R}^d} \underbrace{|\Delta u|^2}_{\mathrm{Im}=0} + \Delta \bar{u} \cdot u \cdot f(|u|^2)$

$= 2\,\mathrm{Im}\int_{\mathbb{R}^d} \bar{u} \cdot \Delta u\, f(|u|^2)\,dx$

$I_2 = \int_{\mathbb{R}^d} f(|u|^2)\,\partial_t(|u|^2)$

$\overset{|u|^2 = u\bar{u}}{=} 2\,\mathrm{Re}\int_{\mathbb{R}^d} f(|u|^2)\,\bar{u}\,u_t\,dx$

代入 $iu_t + \Delta u = f(|u|^2)u$

$= 2\,\mathrm{Re}\int_{\mathbb{R}^d} f(|u|^2)\,\bar{u}\,(f(|u|^2)u - \Delta u)\cdot(-i)\,dx$

$= 2\,\mathrm{Im}\int_{\mathbb{R}^d} f(|u|^2)\,\bar{u}\,(f(|u|^2)\,u - \Delta u)\,dx$

$= 2\,\mathrm{Im}\int_{\mathbb{R}^d} \underbrace{|u|^2 f^2(|u|^2)}_{\mathrm{Im}=0} - f(|u|^2)\,\bar{u}\,\Delta u$

$= -2\,\mathrm{Im}\int_{\mathbb{R}^d} \bar{u}\,\Delta u\, f(|u|^2)\,dx = -I_1$

$\therefore I_1 + I_2 = 0 = \frac{d}{dt}\int_{\mathbb{R}^d} |\nabla u|^2 + F(|u|^2)\,dx$

(3) $\frac{d}{dt}\int_{\mathbb{R}^d} \frac{\bar{u}\nabla u - u\nabla\bar{u}}{2i}\,dx$

$= \frac{d}{dt}\int_{\mathbb{R}^d} \operatorname{Im}(\bar{u}\nabla u)\,dx = \operatorname{Im}\int_{\mathbb{R}^d} \frac{d}{dt}(\bar{u}\nabla u)\,dx$

$= \operatorname{Im}\int_{\mathbb{R}^d} \bar{u}_t\nabla u + u\nabla\bar{u}_t\,dx$

第2项分部积分 $\operatorname{Im}\int_{\mathbb{R}^d} \bar{u}_t\nabla u - \nabla u\cdot\bar{u}_t\,dx = 0.$

(4). 先求 $\frac{d}{dt}\int_{\mathbb{R}^d} |x|^2|u|^2\,dx$

$= \int_{\mathbb{R}^d} |x|^2\partial_t(u\bar{u})\,dx = 2\int_{\mathbb{R}^d}|x|^2\operatorname{Re}(\bar{u}u_t)$

$= 2\int_{\mathbb{R}^d}|x|^2\operatorname{Re}\left(\bar{u}(-i)\left(f(|u|^2)u - \Delta u\right)\right)$

$= 2\int_{\mathbb{R}^d} \underbrace{\operatorname{Re}\left(|x|^2|u|^2 f(|u|^2)\cdot(-i)\right)}_{\overset{\shortparallel}{0}\ \text{(因 f 实值)}} - |x|^2\operatorname{Re}\left((-i)\,\bar{u}\Delta u\right)\,dx.$

$= -2\int_{\mathbb{R}^d} |x|^2\operatorname{Im}(\bar{u}\Delta u)\,dx$

$= -2\int_{\mathbb{R}^d} |x|^2\operatorname{Im}(\bar{u}\Delta u + \underbrace{|\nabla u|^2}_{\text{实值, 故 Im}=0.})\,dx$

$= -2\int_{\mathbb{R}^d} |x|^2\operatorname{Im}\left(\sum_{i=1}^d \bar{u}\,\partial_i^2 u + \partial_i u\,\partial_i\bar{u}\right)\,dx.$

$= -2\int_{\mathbb{R}^d} |x|^2\operatorname{Im}\sum_{i=1}^d \partial_i(\bar{u}\,\partial_i u)\,dx$

$= -2\int |x|^2\,\nabla\cdot(\operatorname{Im}\bar{u}\nabla u)\,dx.$

(分部积分) $= +2\int(\nabla|x|^2)\cdot\operatorname{Im}(\bar{u}\nabla u)\,dx.$ $\quad \nabla|x|^2 = (2x_1,\dots,2x_d) = 2x$

$= 4\int x\cdot\operatorname{Im}(\bar{u}\nabla u)\,dx = 4\operatorname{Im}\int\bar{u}\,(x\cdot\nabla u)\,dx$

$= 4\operatorname{Im}\int \bar{u}\cdot|x|\left(\nabla u\cdot\frac{x}{|x|}\right)\,dx$ $\quad$ 令 $u_r = \nabla u\cdot\frac{x}{|x|}$

$= 4\operatorname{Im}\int r\bar{u}u_r\,dx.$ $\quad r = |x|$

再求一导:

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^d}|x|^2|u|^2\,dx$$

$$= \frac{d}{dt}\left(4\,\mathrm{Im}\int_{\mathbb{R}^d}\bar{u}(x\cdot\nabla u)\,dx\right)$$

$$= 4\,\mathrm{Im}\int_{\mathbb{R}^d}\bar{u}_t(x\cdot\nabla u)+\bar{u}(x\cdot\nabla u_t)\,dx$$

$$= 4\,\mathrm{Im}\int_{\mathbb{R}^d}\sum_{j=1}^{d}\bar{u}_t\,x_j\cdot u_{x_j}+\bar{u}\,x_j\cdot\partial_{x_j}u_t\,dx.$$

第2项分部积分

$$= 4\,\mathrm{Im}\int_{\mathbb{R}^d}\sum_{j=1}^{d}\bar{u}_t\,x_j u_{x_j}-(\bar{u}x_j)_{x_j}\,u_t\,dx$$

$$= 4\,\mathrm{Im}\int_{\mathbb{R}^d}\sum_{j=1}^{d}\bar{u}_t\,x_j u_{x_j}-\bar{u}_{x_j}x_j u_t-\bar{u}u_t\,dx$$

$$= \underbrace{-8\,\mathrm{Im}\int_{\mathbb{R}^d}\sum_{j=1}^{d}u_t\bar{u}_{x_j}x_j}_{J_1}\;-\;4d\,\mathrm{Im}\underbrace{\int_{\mathbb{R}^d}\bar{u}u_t\,dx}_{J_2}$$

$$J_2 = \mathrm{Im}\int_{\mathbb{R}^d}\bar{u}u_t\,dx = \mathrm{Re}\int_{\mathbb{R}^d}\bar{u}(-iu_t)\,dx$$

$$= \mathrm{Re}\int_{\mathbb{R}^d}\bar{u}\left(\Delta u-f(|u|^2)u\right)dx$$

$$= \mathrm{Re}\int_{\mathbb{R}^d}\bar{u}\Delta u-f(|u|^2)|u|^2\,dx.$$

(分部积分) $= \mathrm{Re}\int_{\mathbb{R}^d}-|\nabla u|^2-f(|u|^2)|u|^2\,dx$

$$\Rightarrow -4dJ_2 = 4d\int_{\mathbb{R}^d}|\nabla u|^2\,dx+4d\int_{\mathbb{R}^d}f(|u|^2)|u|^2\,dx$$

$$J_1 = -8\,\mathrm{Im}\int_{\mathbb{R}^d}\sum_{j=1}^{d}u_t\bar{u}_{x_j}x_j.$$

$$= \mathrm{Im}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\left(i\Delta u-iu f(|u|^2)\right)\bar{u}_{x_j}x_j\,dx$$

$$= \underbrace{-\mathrm{Re}\int_{\mathbb{R}^d}\Delta u\,(\nabla\bar{u}\cdot x)\,dx}_{K_1}+\underbrace{\mathrm{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^d}u\bar{u}_{x_j}x_j f(|u|^2)\,dx.}_{K_2}$$

$k_2 = \mathrm{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^d} u\bar{u}_{x_j}\, x_j\, f(|u|^2)\,dx$

$= \frac{1}{2}\mathrm{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^d} \underbrace{\partial_j(|u|^2)}_{=2\mathrm{Re}\,\bar{u}_{x_j}u}\cdot x_j f(|u|^2)\,dx.$

$F' = f$

$= \frac{1}{2}\mathrm{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^d} \partial_j(|u|^2)\, F'(|u|^2)\cdot x_j\,dx = \frac{1}{2}\mathrm{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\partial_j(F(|u|^2))\cdot x_j\,dx$

分部积分 $= -\frac{1}{2}\mathrm{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^d} F(|u|^2)\,\underbrace{\partial_j x_j}_{=1}\,dx = -\frac{d}{2}\int_{\mathbb{R}^d} F(|u|^2)\,dx.$

$k_1 = -\mathrm{Re}\int_{\mathbb{R}^d} \Delta u\,(\nabla\bar{u}\cdot x)\,dx.$

分部积分 $= \mathrm{Re}\int_{\mathbb{R}^d} \nabla u\cdot\nabla(\nabla\bar{u}\cdot x)\,dx$

$= \mathrm{Re}\int_{\mathbb{R}^d}\sum_j\sum_k u_{x_k}\overline{u_{x_j}}\,\delta_{jk} + u_{x_k}\overline{u_{x_jx_k}}\,x_j\,dx$

$= \mathrm{Re}\sum_k\int_{\mathbb{R}^d} u_{x_k}\overline{u_{x_k}}\,dx + \sum_j\sum_k\mathrm{Re}\int_{\mathbb{R}^d} u_{x_k}\overline{u_{x_jx_k}}\,x_j\,dx.$

$= \int_{\mathbb{R}^d}|\nabla u|^2dx + \frac{1}{2}\sum_j\sum_k\mathrm{Re}\int_{\mathbb{R}^d}\left(u_{x_k}\overline{u_{x_jx_k}} + u_{x_jx_k}\overline{u_{x_k}}\right)x_j\,dx.$

$= \int_{\mathbb{R}^d}|\nabla u|^2dx + \frac{1}{2}\sum_j\mathrm{Re}\int_{\mathbb{R}^d} x_j\,\partial_j(|\nabla u|^2)\,dx.$

分部积分 $= \int_{\mathbb{R}^d}|\nabla u|^2dx - \frac{1}{2}\mathrm{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^d}|\nabla u|^2dx$

$= (1-\frac{d}{2})\int_{\mathbb{R}^d}|\nabla u|^2dx$

$\therefore \frac{d^2}{dt^2}\int_{\mathbb{R}^d}|x|^2|u|^2dx = \cancel{-8\left(-\frac{d}{2}+(1-\frac{d}{2})\right)\int_{\mathbb{R}^d}|\nabla u|^2dx}$

$+8(1-\frac{d}{2})\int_{\mathbb{R}^d}|\nabla u|^2dx - 4d\int_{\mathbb{R}^d}F(|u|^2)\,dx$

$+4d\int_{\mathbb{R}^d}|\nabla u|^2dx + 4d\int_{\mathbb{R}^d}f(|u|^2)|u|^2dx$

$= 8\int_{\mathbb{R}^d}|\nabla u|^2dx + 4d\int_{\mathbb{R}^d}f(|u|^2)|u|^2 - F(|u|^2)\,dx$

(4) 得证

(5). 暂时加条件 $\operatorname{Im}\int r\bar{\varphi}\varphi_r\,dx < 0$ $\varphi = u(0,\cdot)$

φ为初值 $u(0)$

令 $I(t)=\int_{\mathbb{R}^d}|x|^2|u|^2\,dx$

由(4)证明 $I'(t) = 4\operatorname{Im}\int r\bar{u}u_r\,dx < 0$

知. $I(t)\le I(0) =: A_0$. $I(0)>0$.

$I''(t) \overset{(4)}{\underset{(2)}{=}} \cancel{(8-4d)} \overset{(4)}{=} 8\int_{\mathbb{R}^d}|\nabla u|_2^2\,dx - 4d\int_{\mathbb{R}^d}\frac{|u|^4}{2}\,dx$

$= (8-4d)\|\nabla u\|_{L^2}^2 + \underbrace{4dE(0)}_{<0}$

~~d≥3时~~ $\Rightarrow I''(t) \le (8-4d)\|\nabla u\|^2_{L^2}\le 0$

$\Rightarrow -I''(t)\ge (4d-8)\|\nabla u\|^2_{L^2}$

~~而 $I'(t)$~~ 先设 $d\ge 3$

$(I'(t))^2 = 4^2\left(\operatorname{Im}\int r\bar u u_r\,dx\right)^2$

$\le 16\left(\int |x|^2|u|^2\,dx\right)\cdot\|\nabla u\|^2_{L^2}$

$I''\le 0$. $I'(0)<0 \Rightarrow I'(t)<0$

$\Rightarrow I(t)<I(0)=A_0$

$\therefore (I'(t))^2 \le 16A_0\|\nabla u\|^2_{L^2}$

$\le \frac{-4A_0}{d-2}I''(t)$

$\Rightarrow I''(t) \le -\frac{d-2}{4A_0}(I'(t))^2$

$\Rightarrow \frac{d}{dt}\left(\frac{1}{I'(t)}\right) \ge \frac{d-2}{4A_0}$

$\Rightarrow \frac{1}{I'(t)} \ge \frac{1}{I'(0)} + \frac{d-2}{4A_0}t = \frac{4A_0+(d-2)tI'(0)}{4A_0I'(0)}$

$\Rightarrow I'(t)\le \frac{4A_0I'(0)}{4A_0+(d-2)I'(0)t}$ $\Rightarrow \|\nabla u\|_{L^2}\ge \frac{-\sqrt{A_0}\,I'(0)}{4A_0+(d-2)I'(0)t}$

$\ge -4\sqrt{A_0}\,\|\nabla u\|_{L^2}$

故 $t\to -\frac{4A_0}{(d-2)I'(0)}+0$ 时

$\|\nabla u\|_{L^2}\to+\infty$

$\Rightarrow$ 无整体解.

$d=2$时： $I''(t) = 8E(0)$

故 $I'(t) = 8E(0)t + I'(0)$.

$I(t) = 4E(0)t^2 + I'(0)t + I(0)$.

$E(0)<0$ 知. t充分大. $I(t)<0$. 这与 $I(t) = \int_{\mathbb{R}^d} |x|^2|u|^2 \geq 0$ 矛盾！

$\square$

Rmk：此题（根据书上Hint）参考了 R.T. Glassey：On the blowing up of solutions to the Cauchy problem for NLS, Journal of Mathematical Physics 18, 1794 (1977)

但原文方程为 $iu_t - \Delta u = F(|u|^2)u$. 之后符号与此题有出入.

同时.原文对(5)中添加的条件成立是假设. 但Evans书上没有、实际上.若 $\varphi(x) = u(x,0)$.

有形式 $\varphi(x) = e^{-i|x|^2}\underbrace{\psi(x)}_{\text{实值.非零}}$， 则所加条件满足.

$\square$