偏微分方程讲义 (II): 近代偏微分方程基础

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Preface

This lecture notes were written for the course 'Differential Equations II"(微分方程II) at University of Science and Technology of China (中国科学技术大学/USTC), which were extended and revised from the ones for the course "MA5213-Advanced Partial Differential Equations" at National University of Singapore (NUS). Both courses are considered as the second course in PDE for junior, senior undergraduate students and fresh graduate students who want to develop further in analysis and PDEs and related areas. The pre-requisites are listed in the Appendix, basically including the real analysis (Lebesgue's theory of measures, integrals and differentiation), L^p spaces and interpolation theorem, Fourier transforms and linear functional analysis (including Hilbert spaces, Hahn-Banach Theorem, weak and weak-* convergence, spectrum theory of compact operators).

The aim of these two courses is to present the fundamental theory of PDE tools and techniques, developed in the past several decades, that help students get in touch with frontier PDE research. Unlike the undergraduate PDE courses (MA4221 at NUS or 微分方程I at USTC), we no longer try to find explicit formulas for the smooth solutions to linear PDEs. Instead, we wish to develop the theory of linear PDEs with rough variable coefficients and hope to utilize the theory together with tools arising from real analysis, functional analysis and Fourier analysis to study the solutions to nonlinear PDEs that usually arise from various physical and realistic models.

The first chapter is devoted to the integer-order Sobolev spaces $W^{k,p}(U)$ where $k \in \mathbb{N}$, $1 \le p \le \infty$ and $U \subset \mathbb{R}^d$ is an open set with a sufficiently smooth (at least Lipschitzian) boundary ∂U. One of the major advantages of Sobolev spaces is that they capture the features of both integrability and differentiability of certain locally integrable functions. The Sobolev embedding theorems give quantitative relations among Sobolev spaces, L^p spaces and H'older continuous spaces $C^{k,\alpha}$. Also, Sobolev functions can be approximated by smooth functions in certain ways via the convolution with standard mollifiers.

It should also be noted that the concept of "weak solution" is introduced together with Sobolev spaces. To be honest, this may be one of the most important concepts in the study of PDEs. In practice, we usually have to analyze the behaviors of solutions to various nonlinear PDEs, and their coefficients are usually also not quite regular. It is then difficult to directly prove the existence of classical solutions. Instead, we can seek for the weak solution (usually in the sense of distribution) at first, which requires less regularity and one can use nice functions as test functions to describe the behaviors of the weak solutions. After obtaining the weak solutions, we can try to enhanced the regularity of the weak solutions such that they finally coincide with the desired classical solutions, provided that the coefficients and the initial data are sufficiently regular.

In Chapter 2 and 3 we use different methods to study the existence of different types of linear second-order PDEs: elliptic PDEs and parabolic PDEs. The existence theorems of elliptic PDEs are obtained by using the Lax-Milgram Theorem and the Fredholm Alternative in linear functional analysis. The existence of parabolic equations is obtained by using Galerkin's approximation, which is mathematically a generalization of "separation of variables" and is also frequently used in numerical works. The major reference of Chapter 1-Chapter 3 is Evans' famous PDE book [6, Chapter 5, 6, 7.1]. We also introduce the vanishing viscosity method to solve the linear symmetric hyperbolic system in $\mathbb{R}^d$, referred to Evans [6, Chap. 7.3.1].

For hyperbolic equations, in Chapter 4, we introduce the local existence of linear and quasi-linear wave equations instead of their global-in-time dynamics (which requires tools in Lorentzian geometry), and the major reference is the lecture notes by Jonathan (Winghong) Luk [12, Chap. 4-6]. In Chapter 5, we introduce several types of fundamental waves (mostly in 1D) arising from hyperbolic conservation laws: shocks, contact discontinuities and rarefaction waves, and the major reference is [11, Chap. 5].

Finally, the methods from Fourier analysis take up a large part of the lecture notes. This is because the Fourier transform allows us to establish much more refined descriptions of the differentiability. In particular, the derivatives of any order $s \in \mathbb{R}$ can be easily defined when $U = \mathbb{R}^d$ by using Fourier transform. In constrast, the fractional-order derivatives in a domain with boundary are usually characterized via the differential quotients (Sobolev-Slobodeckiĭ norms). Moreover, we also introduce the Littlewood-Paley projections as an efficient tool to localize different frequncy bands of a given function. The Littlewood-Paley theory is significant when we establish the Leibniz rule and the chain rule for fractional-order derivatives. Lots of powerful tools and refined inequalities are established with the help of Littlewood-Paley theory. The major references are Bahouri-Chemin-Danchin [2, Chap. 1] and Tao [17, Appendix A].

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If you spot any mistakes and have any suggestions, please let me know as soon as possible. Due to the shortage of time of one semester at NUS (12 weeks × 2 lectures/week × 1.5 hours/lecture), my lectures only covered Chapter 1–4 and 6 of this notes and many tedious proofs which are not related to PDEs are skipped. There are still several chapters and sections to be added, which aim to introduce more advanced techniques or theory, such as De Giorgi-Nash-Moser iteration, hyperbolic conservation laws, calculus of variations, etc.

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Contents

Chapter 1 Sobolev spaces

The first chapter mostly develops the basic theory of Sobolev spaces. During the past a few decades, Sobolev spaces have been proven to be extremely useful in the analysis of numerous problems of various types of PDEs. In fact, in many models arising from physics or other areas, it is almost impossible to carefully characterize the pointwise behaviors of the solutions. An example is the motion of water waves: it is hard to describe the status of each liquid particle, especially the vorticity of the fluid is nonzero. Thus, one has to find alternative ways to prove the existence of solutions to PDEs and analyze qualitative and quantitative properties of solutions, such as the energy method, calculus of variation, Fourier analysis, etc. Among these tools, the characterization of both differentiability and integrability of functions becomes significantly important. On the other hand, one of the major advantanges of Sobolev spaces $W^{k,p}$ is that this type of function spaces take into account of both the integrability and the differentiability of functions, while the Hölder-continuous function spaces $C^{k,\alpha}$ only takes in into account of the pointwise behaviors. What's more, one can "trade differentiability for enhanced integrability" and build quantitative relations between Sobolev spaces and L^p spaces or $C^{k,\alpha}$ spaces.

Before turning to Sobolev spaces, we first introduce the function spaces $C^{k,\alpha}$. They are actually more useful than the classical continuous function spaces C^k when establishing pointwise estimates, especially the Schauder estimates for elliptic PDEs.

Throughout the lecture notes, we assume $U \subset \mathbb{R}^d$ to be an open set. Assume also the index $\alpha \in (0, 1]$. We say a function u is **Hölder continuous with exponent** α if these exists a constant $C > 0$ such that

$$|u(\boldsymbol{x}) - u(\boldsymbol{y})| \leq C|\boldsymbol{x} - \boldsymbol{y}|^\alpha \quad \forall \boldsymbol{x}, \boldsymbol{y} \in U.$$

In particular, when $\alpha = 1$, we say u is **Lipschitz continuous in** U. Based on this, we introduce the space $C^{k,\alpha}$ as below.

Definition 1.0.1. Given a bounded continuous function $u : U \to \mathbb{R}$, we define

- The uniform norm: $\|u\|_{C(\overline{U})} := \sup_{\boldsymbol{x}\in U} |u(\boldsymbol{x})|$.
- The α^{th}-Hölder semi-norm:

$$[u]_{C^{0,\alpha}(\overline{U})} := \sup_{\substack{\boldsymbol{x}\neq\boldsymbol{y} \\ \boldsymbol{x},\boldsymbol{y}\in U}} \frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|}{|\boldsymbol{x} - \boldsymbol{y}|^\alpha}.$$

- The α^{th}-Hölder semi-norm: $\|u\|_{C^{0,\alpha}(\overline{U})} := \|u\|_{C(\overline{U})} + [u]_{C^{0,\alpha}(\overline{U})}$.

- The Hölder space $C^{k,\alpha}(\overline{U})$ ($k \in \mathbb{N}$):

$$C^{k,\alpha}(\overline{U}) = \left\{ u \in C^k(\overline{U}) \middle| \|u\|_{C^{k,\alpha}(\overline{U})} := \sum_{|\alpha|\leq k} \|\partial^\alpha u\|_{C(\overline{U})} + \sum_{|\alpha|=k} [\partial^\alpha u]_{C^{0,\alpha}(\overline{U})} < \infty \right\}.$$

One can check that $C^{k,\alpha}(\overline{U})$ equipped with $\|\cdot\|_{C^{k,\alpha}(\overline{U})}$ norm is a Banach space.

1.1 Weak derivative and Sobolev spaces

We usually cannot make good enough analytic estimates to show that the solutions we construct for some PDEs belong to $C^{k,\alpha}$ as they require very high pointwise regularity. To overcome such difficulty, people have found that Sobolev spaces are good choices to construct a "rough" solution, and in the construction procedures, the "weak derivative" plays an important role.

1.1.1 Weak derivative

Let us begin with a simple example. Given a bounded domain $U \subset \mathbb{R}^d$ and a function $f \in L^2(U)$, we consider Poisson's equation

$$-\Delta u = f \text{ in } U, \quad u = 0 \text{ on } \partial U.$$

Note that the source term f may be a very rough function, which makes it difficult to directly prove the existence of a solution that can be differentiated twice. On the other hand, if $u \in C^2(U)$, then by Gauss-Green formula we get

$$\int_U f\varphi \,\mathrm{d}\boldsymbol{x} = \int_U \nabla u \cdot \nabla\varphi \,\mathrm{d}\boldsymbol{x}, \quad \forall \varphi \in C_c^\infty(U). \tag{1.1.1}$$

Note that the fulfillment of the above integral equality only requires $u \in H_0^1(U)$, that is, $\nabla u \in L^2(U)$ and $u|_{\partial U} = 0$. Thus, we can alternatively define the "weak solution" to Poisson's equation by u satisfying the identity (1.1.1). The existence of such weak solutions is much easier to prove by using Riesz representation theorem for Hilbert spaces, and the term ∇u is actually the "weak derivative" of u, as it only belongs to $L^2(U)$ instead of $L^\infty(U)$.

Motivated by the above example, we introduce the concept of "weak derivative".

Definition 1.1.1. Suppose $u, v \in L^1_{\mathrm{loc}}(U)$ and $\alpha = (\alpha_1, \cdots, \alpha_d)$ is a multi-index. We say that v is the α^{th}-weak partial derivative of u, denoted by $\partial^\alpha u = v$, provided

$$\int_U u\partial^\alpha\varphi \,\mathrm{d}\boldsymbol{x} = (-1)^{|\alpha|} \int_U v\varphi \,\mathrm{d}\boldsymbol{x} \quad \forall \varphi \in C_c^\infty(U). \tag{1.1.2}$$

If such v does not exist, then we say u does not possess an α^{th}-weak partial derivative.

Remark 1.1.1. It should be noted that the above definition is slightly different from the distributional derivative in functional analysis (cf. Folland [8, Chapter 9]), as our definition requires the weak derivative to be a locally Lebesgue integrable function but the distributional derivative is only required to belong to $\mathcal{D}'(U) := (C_c^\infty(U))'$ and is not necessarily a function (e.g., the Dirac delta δ).

Proposition 1.1.1 (Uniqueness of weak derivatives). An α^{th}-weak partial derivative of u, if exists, is uniquely defined up to a set of measure zero.

Proof. Let $v_1, v_2 \in L^1_{\text{loc}}(U)$ be two α^{th}-weak partial derivatives of u. By definition, they satisfy

$$\int_U (v_1 - v_2)\varphi \,\mathrm{d}\boldsymbol{x} = 0 \quad \forall \varphi \in C_c^\infty.$$

Then the desired conclusion immediately follows from the following lemma. □

Lemma 1.1.2. If $w \in L^1_{\text{loc}}(U)$ satisfies $\int_U w\varphi \,\mathrm{d}\boldsymbol{x} = 0$ for all $\varphi \in C_c^\infty(U)$, then $w = 0$ a.e. in U.

Proof of Lemma 1.1.2. Let $\{\eta_\varepsilon\}_{\varepsilon>0}$ be a family of mollifiers defined in Appendix C.2 satisfying

$$\eta \in C_c^\infty(B(\mathbf{0},1)), \quad 0 \le \eta \le 1, \quad \int_{\mathbb{R}^d} \eta = 1, \quad \eta_\varepsilon(\boldsymbol{x}) = \frac{1}{\varepsilon^d}\eta\left(\frac{\boldsymbol{x}}{\varepsilon}\right).$$

Then we also have $\int_{\mathbb{R}^d} \eta_\varepsilon = 1$ for any $\varepsilon > 0$. Based on this, we have

$$\begin{aligned} w(\boldsymbol{x}) &= \int_U w(\boldsymbol{x})\eta_\varepsilon(\boldsymbol{y}-\boldsymbol{x})\,\mathrm{d}\boldsymbol{y} = \int_U (w(\boldsymbol{x}) - w(\boldsymbol{y}))\eta_\varepsilon(\boldsymbol{y}-\boldsymbol{x})\,\mathrm{d}\boldsymbol{y} + \underbrace{\int_U w(\boldsymbol{y})\eta_\varepsilon(\boldsymbol{y}-\boldsymbol{x})\,\mathrm{d}\boldsymbol{y}}_{=0 \text{ by assumption}} \\ &= \int_{U\cap B(\boldsymbol{x},\varepsilon)} (w(\boldsymbol{x}) - w(\boldsymbol{y}))\eta_\varepsilon(\boldsymbol{y}-\boldsymbol{x})\,\mathrm{d}\boldsymbol{y}. \end{aligned}$$

Thus, we have

$$\begin{aligned} |w(\boldsymbol{x})| &\le \frac{1}{\varepsilon^d}\int_{B(\boldsymbol{x},\varepsilon)} |w(\boldsymbol{x}) - w(\boldsymbol{y})|\eta(\frac{\boldsymbol{y}-\boldsymbol{x}}{\varepsilon})\,\mathrm{d}\boldsymbol{y} \\ (0 \le \eta \le 1) \qquad &\le \alpha(d) \fint_{B(\boldsymbol{x},\varepsilon)} |w(\boldsymbol{x}) - w(\boldsymbol{y})|\,\mathrm{d}\boldsymbol{y}, \end{aligned}$$

where $\alpha(d)$ is the volume of the unit ball in $\mathbb{R}^d$ and $\fint$ represents the volume-mean of the integral. The right side converges to 0 for a.e. $\boldsymbol{x} \in U$ as $\varepsilon \to 0$ thanks to Lebesgue Differentiation Theorem. □

Next, we introduce two examples of weak derivatives.

Example 1.1.1. Let $d = 1, U = (0, 2)$ and $u(x) = \begin{cases} x & 0 < x \leq 1 \\ 1 & 1 \leq x < 2 \end{cases}$. Define $v(x) = \begin{cases} 1 & 0 < x \leq 1 \\ 0 & 1 \leq x < 2 \end{cases}$. We next show that $u' = v$ in the weak sense. Choose any $\varphi \in C_c^\infty(U)$, we need to verify

$$\int_0^2 u(x)\varphi'(x)\,\mathrm{d}x = -\int_0^2 v(x)\varphi(x)\,\mathrm{d}x.$$

For the left side, we easily compute that

$$\begin{aligned}\int_0^2 u(x)\varphi'(x)\,\mathrm{d}x &= \int_0^1 x\varphi'(x)\,\mathrm{d}x + \int_1^2 \varphi'(x)\,\mathrm{d}x = -\int_0^1 \varphi(x)\,\mathrm{d}x + \varphi(1) - \varphi(0) + \varphi(2) - \varphi(1) \\ &= -\int_0^2 v(x)\varphi(x)\,\mathrm{d}x,\end{aligned}$$

where we use $\varphi \in C_c^\infty \Rightarrow \varphi(0) = \varphi(2) = 0$.

Example 1.1.2. Let $d = 1, U = (-1, 1)$ and $u(x) = \begin{cases} 0 & -1 < x \leq 0 \\ 1 & 0 \leq x < 1 \end{cases}$. We assert that u' does not exist in the weak sense, that is, there does not exist any $v \in L^1_{\mathrm{loc}}(U)$ satisfying

$$\int_{-1}^1 u(x)\varphi'(x)\,\mathrm{d}x = -\int_{-1}^1 v(x)\varphi(x)\,\mathrm{d}x \quad \forall \varphi \in C_c^\infty(U).$$

Suppose, to the contrary, that there exists some $v \in L^1_{\mathrm{loc}}(U)$ satisfying the above equality. Then we invoke the definition of u to get

$$-\int_{-1}^1 v(x)\varphi(x)\,\mathrm{d}x = \int_{-1}^1 u(x)\varphi'(x)\,\mathrm{d}x = \int_0^1 \varphi'(x)\,\mathrm{d}x = -\varphi(0), \quad \forall \varphi \in C_c^\infty(-1, 1).$$

Now, we pick a sequence of $\{\varphi_m(x)\} \subset C_c^\infty(-1, 1)$ satisfying $0 \leq \varphi_m \leq 1$, $\varphi_m(0) = 1$ and $\varphi_m(x) \to 0$ for all $x \neq 0$ as $m \to \infty$. Replacing φ by φ_m and letting $m \to \infty$, we find (by using Dominated Convergence Theorem) that

$$-1 = \lim_{m\to\infty} (-\varphi_m(0)) = -\lim_{m\to\infty} \int_{-1}^1 v(x)\varphi_m(x)\,\mathrm{d}x = -\int_{-1}^1 v(x) \lim_{m\to\infty} \varphi_m(x)\,\mathrm{d}x = 0,$$

a contracdiction!

Remark 1.1.2. The function u is called Heaviside function if we extend it to $\mathbb{R}$ with $u|_{x<1} = 0$ and $u|_{x>1} = 1$. The distributional derivative of u is exactly the Dirac mass at the origin $\delta_0 \in \mathcal{D}'$ which is not a function.

1.1.2 Sobolev space $W^{k,p}(U)$

Next, we introduce the definition of Sobolev spaces.

Definition 1.1.2 (Sobolev spaces). Given $k \in \mathbb{N}$ and $1 \le p \le \infty$, the Sobolev space $W^{k,p}(U)$ is defined by

$$W^{k,p}(U) := \left\{ f \in L^p(U) \middle| \sum_{|\alpha| \le k} \|\partial^\alpha f\|_{L^p(U)} < \infty \right\}.$$

That is, $W^{k,p}(U)$ consists of all locally Lebesgue integrable functions $f : U \to \mathbb{R}$ of which all weak derivatives up to k-th order are $L^p(U)$ functions. Moreover, $W^{k,p}(U)$ equipped with the norm $\|\cdot\|_{W^{k,p}(U)}$ norm is a Banach space, where the norm is defined by

$$\|f\|_{W^{k,p}(U)} = \sum_{|\alpha| \le k} \|\partial^\alpha f\|_{L^p(U)}, \quad 1 \le p \le \infty.$$

Remark 1.1.3. Since L^p norm contains $1/p$-th power of an integral. We sometimes also use the equivalent norm (defined below) when $1 \le p < \infty$

$$\|f\|_{W^{k,p}(U)} = \left(\sum_{|\alpha| \le k} \int_U |\partial^\alpha f|^p \,\mathrm{d}\boldsymbol{x} \right)^{\frac{1}{p}}.$$

When $p = 2$, we denote $H^k(U) := W^{k,2}(U)$.

Definition 1.1.3. Let $\{f_m\}, f$ belong to $W^{k,p}(U)$. We say $f_m \to f$ in $W^{k,p}(U)$ if $\|f_m - f\|_{W^{k,p}(U)} \to 0$ as $m \to \infty$. We say $f_m \to f$ in $W^{k,p}_{\mathrm{loc}}(U)$ if $\|f_m - f\|_{W^{k,p}(V)} \to 0$ for any $V \Subset U$.

Definition 1.1.4. We denote by $W^{k,p}_0(U)$ the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$. Thus, $f \in W^{k,p}_0(U)$ if and only if there exist functions $f_m \in C_c^\infty(U)$ such that $f_m \to f$ in $W^{k,p}(U)$. In fact, we have a further interpretation

$$f \in W^{k,p}_0(U) \Leftrightarrow f \in W^{k,p}(U) \text{ and } \partial^\alpha f = 0 \text{ on } \partial U \ \ \forall |\alpha| \le k-1.$$

However, the proof is highly nontrivial and we refer to Chapter 1.3 for details.

Example 1.1.3. Take $U = B(\mathbf{0}, 1) \subset \mathbb{R}^d$ and $u(\boldsymbol{x}) = |\boldsymbol{x}|^{-a}$ $(\boldsymbol{x} \neq \mathbf{0})$. Given $d \in \mathbb{N}^*$ and $p \in [1, \infty)$, we want to find $a > 0$ such that $u \in W^{1,p}(U)$.

First, we know that u is smooth away from the origin and so we can compute its classical derivative $\partial_i u(\boldsymbol{x}) = \frac{-a x_i}{|\boldsymbol{x}|^{a+2}}$ for $\boldsymbol{x} \neq 0$. Next, we verify this is also the weak derivative in U. To see this, fix an $\varepsilon > 0$ and pick an arbitrary $\varphi \in C_c^\infty$, we compute that

$$\int_{U \setminus B(\mathbf{0},\varepsilon)} u \partial_i \varphi \,\mathrm{d}\boldsymbol{x} = -\int_{U \setminus B(\mathbf{0},\varepsilon)} \partial_i u \,\varphi \,\mathrm{d}\boldsymbol{x} + \int_{\partial B(\mathbf{0},\varepsilon)} u \varphi \nu_i \,\mathrm{d}S_{\boldsymbol{x}},$$

where $\nu = -\frac{\boldsymbol{x}}{|\boldsymbol{x}|}$ is the unit inward normal vector on $\partial B(\mathbf{0}, \varepsilon)$. Then we verify that the boundary term vanishes as $\varepsilon \to 0$ in order to check $\partial_i u$ is also a weak derivative of u in U

$$\left| \int_{\partial B(\mathbf{0},\varepsilon)} u\varphi \nu_i \, \mathrm{d}S_{\boldsymbol{x}} \right| \le \|\varphi\|_{L^\infty} \mathrm{Area}(\partial B(\mathbf{0}, \varepsilon))\varepsilon^{-a} \le C_d \varepsilon^{d-1-a}$$

which converges to 0 if $a + 1 < d$. This also implicitly requires $d \ge 2$.

Next, we also need to verify that $\partial_i u \in L^p(U)$ for any $1 \le i \le d$. Since U is a bounded domain including the origin and $|\nabla u(\boldsymbol{x})| = \frac{|a|}{|\boldsymbol{x}|^{a+1}}$ for $\boldsymbol{x} \neq 0$, we know $|\nabla u(\boldsymbol{x})| \in L^p(U)$ if and only if $(a+1)p < d$. Thus, we conclude that $u \in W^{1,p}(U)$ if and only if $a < \frac{d-p}{p}$. In particular, $u \notin W^{1,p}(U)$ if $p \ge d$.

Example 1.1.4. Take $U = \mathbb{R}^d \setminus \overline{B(\mathbf{0}, 1)}$ and $u(\boldsymbol{x}) = |\boldsymbol{x}|^{-a}$ $(\boldsymbol{x} \neq \mathbf{0})$. Given $d \in \mathbb{N}^*$ and $p \in [1, \infty)$, we want to find $a > 0$ such that $u \in W^{1,p}(U)$.

In this case, we no longer need $a + 1 < d$ to ensure that the classical derivative $\partial_i u$ is also a weak derivative, because the singularity of $\partial_i u$ is $\boldsymbol{x} = \mathbf{0} \notin U$. As for the integrability, $|\boldsymbol{x}|^{-(a+1)} \in L^p(U)$ now requires $(a + 1)p > d$, that is, $a > \frac{d-p}{p}$. In particular, if $p \ge d$, then $u \in W^{1,p}(U)$ holds for any $a > 0$. Also, we no longer need $d \ge 2$.

Exercise 1.1

Exercise 1.1.1. Let $d = 1$ and $f \in W^{1,p}(0, 1)$, $1 \le p < +\infty$. Show that

(1) f a.e. agrees with an absolutely continuous function $f^* \in L^p(0, 1)$.

(2) When $p > 1$, we have $|f(x) - f(y)| \le |x - y|^{1-\frac{1}{p}} (\int_0^1 |f'(t)|^p \, \mathrm{d}t)^{\frac{1}{p}}$.

(Hint: Consider $f^*(x) = \int_0^x f'(t) \, \mathrm{d}t$. Here f' is the weak derivative of f in (0.1).)

Exercise 1.1.2. Let $\{\boldsymbol{x}_k\}_{k \in \mathbb{N}^*}$ be a countable, dense subset of $U = B(\mathbf{0}, 1) \subset \mathbb{R}^d$ and

$$u(\boldsymbol{x}) = \sum_{k=1}^{\infty} \frac{1}{2^k} |\boldsymbol{x} - \boldsymbol{x}_k|^{-a}.$$

Find the range for $a \in \mathbb{R}$ such that $u \in W^{1,p}(U)$.

Exercise 1.1.3. Let $\zeta \in C_c^\infty(U)$ and $u \in W^{k,p}(U)$. Prove that $\zeta u \in W^{k,p}(U)$ and the classical Leibniz's formula holds

$$\partial^\alpha (\zeta u) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \partial^\beta \zeta \, \partial^{\alpha - \beta} u.$$

1.2 Smooth approximation and basic calculus of Sobolev functions

We find that it is rather technical if we continue to take the weak derivative by its definition. It is natural to ask if we can approximate Sobolev functions by more regular functions (such as smooth functions) in a certain way and inherit the "good properties" of smooth functions to Sobolev functions. This can be done by means of mollification, introduced in Appendix C.2. Now we fix $k \in \mathbb{N}$, $1 \le p \le \infty$ and an open set $U \subset \mathbb{R}^d$. For each $\varepsilon > 0$, we define $U_\varepsilon = \{\boldsymbol{x} \in U | \operatorname{dist}(\boldsymbol{x}, \partial U) > \varepsilon\}$.

1.2.1 Local smooth approximation

The first theorem shows that any Sobolev function with $1 \le p < \infty$ has a smooth approximation in the interior.

Theorem 1.2.1 (Local smooth approximation). Assume $f \in W^{k,p}(U)$ for some $1 \le p < \infty$ and set $f_\varepsilon = \eta_\varepsilon * f$ in U_ε. Then $f_\varepsilon \in C^\infty(U_\varepsilon)$ for each $\varepsilon > 0$ and $f_\varepsilon \to f$ in $W^{k,p}_{\mathrm{loc}}(U)$ as $\varepsilon \to 0$.

Proof. The smoothness has been proven in Theorem C.2.1(1), so we skip it here. It remains to prove the approximation property. The key step is to verify

$$\partial^\alpha f_\varepsilon = \eta_\varepsilon * \partial^\alpha f \text{ in } U_\varepsilon, \quad \forall |\alpha| \le k.$$

Once this is true, then by Theorem C.2.1(4) we know $\partial^\alpha f^\varepsilon \to \partial^\alpha f$ in L^p_{loc} for each $|\alpha| \le k$, which is exactly the desired conclusion.

To confirm the key step, we fix $\varepsilon > 0$ and an $\boldsymbol{x} \in U_\varepsilon$ to compute that

$$\begin{aligned}\partial^\alpha f_\varepsilon(\boldsymbol{x}) &= \partial^\alpha \int_U \eta_\varepsilon(\boldsymbol{x}-\boldsymbol{y}) f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = \int_U \partial^\alpha_{\boldsymbol{x}} \eta_\varepsilon(\boldsymbol{x}-\boldsymbol{y}) f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \\ &= (-1)^{|\alpha|} \int_U \partial^\alpha_{\boldsymbol{y}} \eta_\varepsilon(\boldsymbol{x}-\boldsymbol{y}) f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}.\end{aligned}$$

Note that for each fixed $\boldsymbol{x} \in U_\varepsilon$, the function $\eta_\varepsilon(\boldsymbol{x}-\boldsymbol{y})$ (as a function of $\boldsymbol{y}$) belongs to $C^\infty_c(U)$. Thus, by definition of the weak derivative, we have

$$\int_U \partial^\alpha_{\boldsymbol{y}} \eta_\varepsilon(\boldsymbol{x}-\boldsymbol{y}) f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = (-1)^{|\alpha|} \int_U \eta_\varepsilon(\boldsymbol{x}-\boldsymbol{y}) \partial^\alpha_{\boldsymbol{y}} f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y},$$

and so

$$\partial^\alpha f_\varepsilon(\boldsymbol{x}) = \underbrace{(-1)^{2|\alpha|}}_{=1} \int_U \eta_\varepsilon(\boldsymbol{x}-\boldsymbol{y}) \partial^\alpha f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = (\eta_\varepsilon * \partial^\alpha f)(\boldsymbol{x}).$$

□

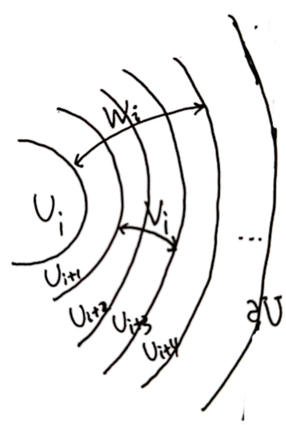
1.2.2 Global smooth approximation with or without boundary

In view of Theorem 1.2.1, it is natural to ask if we can approximate a given Sobolev function in the whole domain U instead of a compact subset. Furthermore, we ask if it is possible to extend the smooth approximation to the boundary. The answers are both positive and such approximations can be achieved by using the partition of unity, but we need extra assumption on the smoothness of ∂U.

Theorem 1.2.2 (Global smooth approximation). Assume U is bounded and $f \in W^{k,p}(U)$ for some $1 \leq p < \infty$. Then there exist a sequence of functions $f_m \in C^\infty(U) \cap W^{k,p}(U)$ such that $f_m \to f$ in $W^{k,p}(U)$.

Proof. Let $U_i = \{x \in U | \text{dist}\,(\boldsymbol{x}, \partial U) > 1/i\}$ for $i \in \mathbb{N}^*$ and then we have $U = \bigcup\limits_{i=1}^{\infty} U_i$, that is, the open set U is exhausted by a sequence of open subsets $\{U_i\}$. In Theorem 1.2.2, we have already constructed smooth approximations of u in each U_i and now we need to "glue" these approximate functions together. The problem is that this will be an *infinite sum* of smooth functions which may be no longer smooth. To ensure the smoothness, it suffices to make the infinite sum "*locally finite*", that is, for each fixed $\boldsymbol{x} \in U$, there are only finitely many nonzero terms in a small neighborhood of this $\boldsymbol{x}$. This can be achieved by the so-called **partition of unity**.

Speficially, for each $i \in \mathbb{N}^*$, we define $V_i := U_{i+3} \backslash \overline{U_{i+1}}$ and $W_i := U_{i+4} \backslash \overline{U_i}$ as in the picture below.



Let $\{\zeta_i\}_{i\in\mathbb{N}^*}$ be a smooth partition of unity subordinate to the open sets $\{V_i\}$, that is, suppose

- $0 \leq \zeta_i \leq 1$ and $\zeta_i \in C_c^\infty(V_i)$.
- $\sum\limits_{i=1}^{\infty} \zeta_i = 1$ in U.

By definition, we see that for each $\boldsymbol{x} \in U$, there are only finitely many i's such that $\zeta_i(\boldsymbol{x}) \neq 0$. Also, by Exercise 1.1.3, each $\zeta_i u$ belongs to $W^{k,p}(U)$ and $\text{Spt}\,(\zeta_i f) \subseteq V_i$.

Now we define $f^i := \eta_{\varepsilon_i} * (\zeta_i f)$ to be the smooth approximation of f in each slice V_i. Fix $\delta > 0$, the parameter ε_i is chosen sufficiently small such that $\|f^i - \zeta_i f\|_{W^{k,p}(U)} < \frac{\delta}{2^i}$. Also, we have $\text{Spt}\, f^i \subseteq W_i$. Note that it is necessary to introduce such W_i, as doing convolution with mollifier may enlarge the support of the given function (from V_i to W_i).

Now, we define $F := \sum\limits_{i=1}^{\infty} f^i$. By the locally finite property, we know there are only finitely many nonzero terms in the summation and thus $F \in C^\infty(U)$. On the other hand, we know $f = \sum\limits_{i=1}^{\infty} \zeta_i f$. Thus for any $V \Subset U$, we have

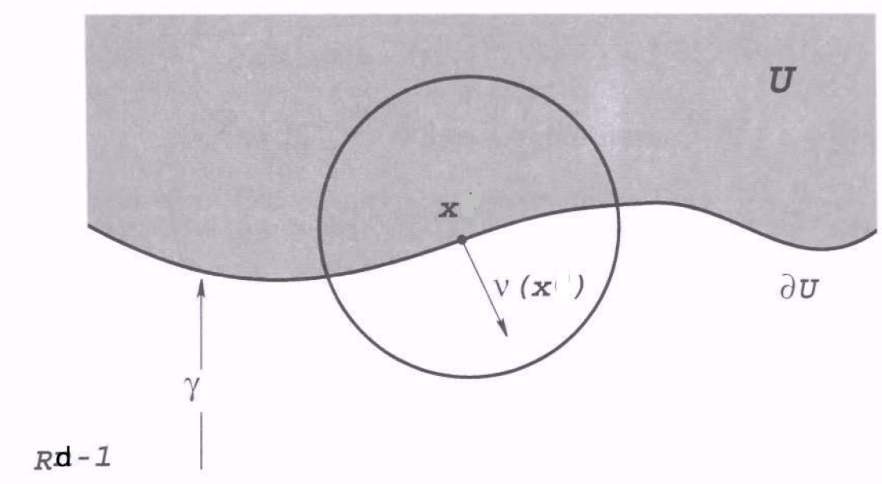
$$\|F - f\|_{W^{k,p}(V)} \leq \sum_i \|f^i - \zeta_i f\|_{W^{k,p}(U)} = \sum_i \|\eta_{\varepsilon_i} * (\zeta_i f) - \zeta_i f\|_{W^{k,p}(U)} < \sum_{i=1}^{\infty} 2^{-i}\delta = \delta.$$

This δ is independent of the choice of V, thus we get $\|F - f\|_{W^{k,p}(U)} \leq \delta$. In particular, let $\delta = 1, \frac{1}{2}, \frac{1}{3}, \cdots$ and we get a sequence $\{f_m\}$ as desired. □

Our next intention is to approximate a Sobolev function by functions smooth all the way up to the boundary. Here we need to assume the boundary ∂U is Lipschitz continuous.

Definition 1.2.1. We say the boundary ∂U is **Lipschitz continuous** if for each point $\boldsymbol{x} \in \partial U$, there exist $r > 0$ and a Lipschitz continuous mapping $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ such that, upon rotating and relabeling the coordinate axes if necessary, we have

$$U \cap B(\boldsymbol{x}, r) = \{\boldsymbol{y} | y_d > \gamma(y_1, \cdots, y_{d-1})\} \cap B(\boldsymbol{x}, r).$$



The boundary of U

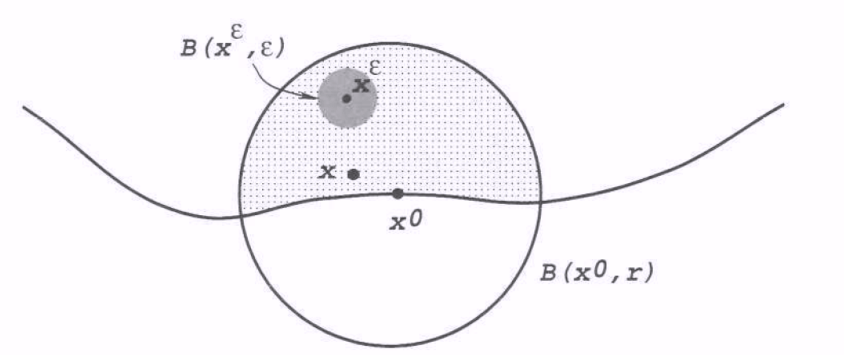
Theorem 1.2.3 (Global smooth approximation up to the boundary)**.** Let $U \subset \mathbb{R}^d$ be a bounded open set with a Lipschitz boundary ∂U. Suppose $f \in W^{k,p}(U)$ for some $1 \leq p < \infty$. Then there exist functions $f_m \in C^\infty(\overline{U})$ such that $f_m \to f$ in $W^{k,p}(U)$.

Proof. In view of Theorem 1.2.2, it remains to find smooth approximations to f near the boundary ∂U. Note that the boundedness of U leads to the compactness of ∂U, and thus the boundary ∂U can be covered by *finitely many* open sets, say $V_1, \cdots, V_N$. After constructing the covering of the boundary, it suffices to cover the remaining interior part by an open set $V_0 \subset U$. The approximation in V_0 has been studied in Theorem 1.2.1, so it suffices to construct the smooth approximation in each open cover of ∂U.

Fix $\boldsymbol{x}^0 \in \partial U$ and there exist $r > 0$ and a Lipschitz continuous function $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ such that

$$U \cap B(\boldsymbol{x}^0, r) = \{\boldsymbol{x} | x_d > \gamma(x_1, \cdots, x_{d-1})\} \cap B(\boldsymbol{x}^0, r).$$

We also write $V = U \cap B(\boldsymbol{x}^0, \frac{r}{2})$.



Given $\boldsymbol{x} \in V$, we define the shifted point $\boldsymbol{x}^\varepsilon := \boldsymbol{x} + \lambda\varepsilon e_d$ ($\varepsilon > 0$). For a fixed, suitably large $\lambda > 0$ (for example, $\lambda > \mathrm{Lip}(\gamma) + 2$), we know the ball $B(\boldsymbol{x}^\varepsilon, \varepsilon) \subseteq U \cap B(\boldsymbol{x}^0, r)$ is always true for any $\boldsymbol{x} \in V$ and any sufficiently small $\varepsilon > 0$. Next, we define the approximation. Let $f^\varepsilon(\boldsymbol{x}) := f(\boldsymbol{x}^\varepsilon)$ and define its approximation by $F^\varepsilon := \eta_\varepsilon * f^\varepsilon$. Then it is easy to see $F^\varepsilon \in C^\infty(\overline{V})$. We now make a claim.

Claim. $F^\varepsilon \to f$ in $W^{k,p}(V)$.

Before going to the proof of the claim, we would like to add some remarks.

1. **The construction of such approximation.** One may ask why we need to introduce the "shifted point" x^ε and pick $\lambda > 0$ sufficiently large. The reason is that an arbitrary point $\boldsymbol{x} \in V$ may be very close to the boundary. If we directly mollify the function f in a neighborhood of $\boldsymbol{x}$, then the convolution may enlarge the support such that $B(\boldsymbol{x}, \varepsilon) \cap U^c \neq \emptyset$. On the other hand, the Lipschitz continuity of the boundary guarantees that the boundary never "highly oscillates" (Lipschitz continuity implictly gives an upper bound for the first-order derivative), so if we pick $\lambda > \mathrm{Lip}(\gamma) + 2$, then such a shift of size $O(\lambda\varepsilon)$ ensures that $B(\boldsymbol{x}, \varepsilon) \subset U$ and thus provides enough room for the convolution with mollifier.
2. **How the claim implies the theorem?** In fact, if the claim is true, then it suffices to do a partition of unity for $\overline{U}$ to finish the proof of this theorem. Specifically, we fix a $\delta > 0$. Since ∂U is compact, we can find *finitely many* points $\boldsymbol{x}_i^0 \in \partial U$ ($1 \leq i \leq N$ and $r_i > 0$ such that $\partial U \subset \bigcup_{i=1}^{N} B(\boldsymbol{x}_i^0, r_i/2)$. Denote $V_i := U \cap B(\boldsymbol{x}_i^0, r_i/2)$ and then for each i, there exists a smooth function $f_i \in C^\infty(\overline{V_i})$ such that
$$\|f_i - f\|_{W^{k,p}(V_i)} < \delta, \quad 1 \leq i \leq N.$$

After this, we then find $V_0 \Subset U$ such that

$$U \subset \bigcup_{i=0}^{N} V_i \quad \text{and} \quad \exists f_0 \in C^\infty(\overline{V_0}), \text{ such that } \|f_0 - f\|_{W^{k,p}(U)} < \delta.$$

At this step, we already construct a *finite* open cover of $\overline{U}$ consisting of $\{V_0, B(\boldsymbol{x}_1^0, r_1/2), \cdots, B(\boldsymbol{x}_N^0, r_N/2)\}$. Let $\{\zeta_i\}_{i=0}^N$ be a partition of unity of $\overline{U}$ subordinate to this open cover and let $F := \sum_{i=0}^{N} \zeta_i f_i \in C^\infty(\overline{U})$. By definition of partition of unity, we have $f = \sum_{i=0}^{N} \zeta_i f$, so for any $|\alpha| \le k$

$$\|\partial^\alpha f - \partial^\alpha F\|_{L^p(U)} \le \sum_{i=0}^{N} \|\partial^\alpha(\zeta_i(f_i - f))\|_{L^p(V_i)} \le C \sum_{i=0}^{N} \|f_i - f\|_{W^{k,p}(V_i)} \le C(N+1)\delta.$$

Finally, it remains to prove the claim. Recall that F^ε is the mollification of the shifted version of f, so we shall split $F^\varepsilon - f$ into $F^\varepsilon - f^\varepsilon$ and $f^\varepsilon - f$ to control the gaps. For simplicity, we only prove the $L^p(U)$ convergence, and the convergence in $W^{k,p}$ norm follows in the same way. We have

$$\|F^\varepsilon - f\|_{L^p(V)} \le \|F^\varepsilon - f^\varepsilon\|_{L^p(V)} + \|f^\varepsilon - f\|_{L^p(V)}.$$

The convergence toward 0 of the second term immediately follows from the *translation continuity of L^p norm*, so it remains to prove the convergence to 0 for the first term. By definition, we have

$$\begin{aligned}
F^\varepsilon(\boldsymbol{x}) - f^\varepsilon(\boldsymbol{x}) = F^\varepsilon(\boldsymbol{x}) - f(\boldsymbol{x}^\varepsilon) &= \frac{1}{\varepsilon^d} \int_{B(\mathbf{0},\varepsilon)} \eta(\frac{\mathbf{w}}{\varepsilon})(f(\boldsymbol{x} + \lambda\varepsilon e_d - \mathbf{w}) - f(\boldsymbol{x} + \lambda\varepsilon e_d))\,\mathrm{d}\mathbf{w} \\
&\overset{\boldsymbol{z} := \frac{\mathbf{w}}{\varepsilon}}{=\!=\!=} \int_{B(\mathbf{0},1)} \eta(\boldsymbol{z})(f(\boldsymbol{x} + \lambda\varepsilon e_d - \varepsilon\boldsymbol{z}) - f(\boldsymbol{x} + \lambda\varepsilon e_d))\,\mathrm{d}\boldsymbol{z}.
\end{aligned}$$

So the $L_{\boldsymbol{x}}^p$ norm of this integral is controlled by using Minkowski's inequality for integrals

$$\begin{aligned}
\|F^\varepsilon - f^\varepsilon\|_{L^p(V)} &= \|F^\varepsilon - f^\varepsilon\|_{L_{\boldsymbol{x}}^p(U \cap B(\boldsymbol{x}^0, \frac{r}{2}))} \\
&= \left\| \|\eta(\boldsymbol{z})(f(\boldsymbol{x} + \lambda\varepsilon e_d - \varepsilon\boldsymbol{z}) - f(\boldsymbol{x} + \lambda\varepsilon e_d))\|_{L_{\boldsymbol{z}}^1(B(\mathbf{0},1))} \right\|_{L_{\boldsymbol{x}}^p(U \cap B(\boldsymbol{x}^0, \frac{r}{2}))} \\
\text{(Minkowski's inequality)} \quad &\le \left\| \|\eta(\boldsymbol{z})(f(\boldsymbol{x} + \lambda\varepsilon e_d - \varepsilon\boldsymbol{z}) - f(\boldsymbol{x} + \lambda\varepsilon e_d))\|_{L_{\boldsymbol{x}}^p(U \cap B(\boldsymbol{x}^0, \frac{r}{2}))} \right\|_{L_{\boldsymbol{z}}^1(B(\mathbf{0},1))} \\
&= \int_{B(\mathbf{0},1)} |\eta(\boldsymbol{z})| \|f(\cdot + \lambda\varepsilon e_d - \varepsilon\boldsymbol{z}) - f(\cdot + \lambda\varepsilon e_d))\|_{L_{\boldsymbol{x}}^p(V)}\,\mathrm{d}\boldsymbol{z}.
\end{aligned}$$

When $\varepsilon \to 0$, by the translation continuity of L^p norm, we know the integrand $\|f(\cdot + \lambda\varepsilon e_d - \varepsilon\boldsymbol{z}) - f(\cdot +$

$\lambda\varepsilon e_d))\|_{L^p_{\boldsymbol{x}}(V)}$ converges to 0. Also we have

$$|\eta(\boldsymbol{z})|\|f(\cdot+\lambda\varepsilon e_d-\varepsilon\boldsymbol{z})-f(\cdot+\lambda\varepsilon e_d))\|_{L^p_{\boldsymbol{x}}(V)}\le 2\|f\|_{L^p(U)}\in L^1_{\boldsymbol{z}}(B(\mathbf{0},1)),$$

and the dominant function does not depend on $\boldsymbol{z}$. By Dominated Convergence Theorem, we know

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{B(\mathbf{0},1)}|\eta(\boldsymbol{z})|\|f(\cdot+\lambda\varepsilon e_d-\varepsilon\boldsymbol{z})-f(\cdot+\lambda\varepsilon e_d))\|_{L^p_{\boldsymbol{x}}(V)}\,\mathrm{d}\boldsymbol{z}\\ =&\int_{B(\mathbf{0},1)}|\eta(\boldsymbol{z})|\lim_{\varepsilon\to 0}\|f(\cdot+\lambda\varepsilon e_d-\varepsilon\boldsymbol{z})-f(\cdot+\lambda\varepsilon e_d))\|_{L^p_{\boldsymbol{x}}(V)}\,\mathrm{d}\boldsymbol{z}=0.\end{aligned}$$

□

1.2.3 Basic calculus of functions in Sobolev spaces

In view of the smooth approximation, we expect to establish many of the usual calculus rules for Sobolev functions.

Proposition 1.2.4 (Calculus rules for Sobolev functions). Assume $1\le p<\infty$.

(1) If $f,g\in W^{1,p}(U)\cap L^\infty(U)$, then $fg\in W^{1,p}(U)\cap L^\infty(U)$ and $\partial_i(fg)=(\partial_i f)g+f(\partial_i g)$ holds a.e. in U for $i=1,\cdots,d$.

(2) If $f\in W^{1,p}(U)$ and $F\in C^1(\mathbb{R})$, $F'\in L^\infty(\mathbb{R})$, $F(0)=0$, then $F(f)\in W^{1,p}(U)$ and $\partial_i(F(f))=F'(f)\partial_i f$ a.e. in U for $i=1,\cdots,d$. Moreover, if U has finite Lebesgue measure in $\mathbb{R}^d$, then $F(0)=0$ is not necessary.

(3) If $f\in W^{1,p}(U)$, then $f^+,f^-,|f|\in W^{1,p}(U)$ and

$$\partial f^+=\begin{cases}\partial f & \text{a.e. on }\{f>0\}\\ 0 & \text{a.e. on }\{f\le 0\},\end{cases}$$

$$\partial f^-=\begin{cases}0 & \text{a.e. on }\{f\ge 0\}\\ -\partial f & \text{a.e. on }\{f<0\},\end{cases}$$

$$\partial|f|=\begin{cases}\partial f & \text{a.e. on }\{f>0\}\\ 0 & \text{a.e. on }\{f=0\}\\ -\partial f & \text{a.e. on }\{f<0\}.\end{cases}$$

In particular $\partial f=0$ a.e. on $\{f=0\}$.

Proof. We only prove (1) and the first equality in (3). The proof of (2) is left as an exercise.

For (1), choose $\varphi \in C_c^\infty(U)$ with $\operatorname{Spt}\varphi \subset V \Subset U$ for some open subset V. Let $f_\varepsilon := \eta_\varepsilon * f$ and $g_\varepsilon := \eta_\varepsilon * g$. Then, first we have

$$\int_U fg(\partial_i\varphi)\,\mathrm{d}\boldsymbol{x} = \int_V fg(\partial_i\varphi)\,\mathrm{d}\boldsymbol{x} = \lim_{\varepsilon\to 0}\int_V f_\varepsilon g_\varepsilon(\partial_i\varphi)\,\mathrm{d}\boldsymbol{x}.$$

Here we can directly verify how to commute the limit with the integral. In fact, using Hölder's inequality,

$$\begin{aligned}\int_V f_\varepsilon g_\varepsilon(\partial_i\varphi)\,\mathrm{d}\boldsymbol{x} - \int_V fg(\partial_i\varphi)\,\mathrm{d}\boldsymbol{x} &= \int_V f_\varepsilon(g_\varepsilon - g)(\partial_i\varphi)\,\mathrm{d}\boldsymbol{x} + \int_V (f_\varepsilon - f)g\partial_i\varphi\,\mathrm{d}\boldsymbol{x}\\ \le \|g_\varepsilon - g\|_{L^p(V)}\underbrace{\|f_\varepsilon\|_{L^\infty(V)}}_{\le\|f\|_{L^\infty(V)}}&\|\partial_i\varphi\|_{L^{p'}(V)} + \|g\|_{L^\infty}\|f_\varepsilon - f\|_{L^p(V)}\|\partial_i\varphi\|_{L^{p'}(V)}\\ \to 0 \text{ as } \varepsilon \to 0.&\end{aligned}$$

Here we note that the assumption $f, g \in L^\infty$ is necessary, and the convergence of $f_\varepsilon, g_\varepsilon$ follows from Theorem C.2.1.

The proof of (3) is slightly tricky. Given $\varepsilon > 0$, we define $F_\varepsilon(r) = \sqrt{r^2+\varepsilon^2} - \varepsilon$ for $r \ge 0$ and $F_\varepsilon(r) = 0$ for $r < 0$. Then it is easy to see $F_\varepsilon \in C^1(\mathbb{R})$ and $F_\varepsilon' \in L^\infty(\mathbb{R})$ is uniformly bounded in ε. Now we apply (2) to F_ε here and integrate by parts to get

$$\int_U F_\varepsilon(f)\partial_i\varphi\,\mathrm{d}\boldsymbol{x} = -\int_U F_\varepsilon'(f)\partial_i f\varphi\,\mathrm{d}\boldsymbol{x}, \quad \forall\varphi \in C_c^\infty(U).$$

Since $f \in W^{1,p}$ and $F_\varepsilon(f) \xrightarrow{a.e.} f^+$ as $\varepsilon \to 0$, the Dominated Convergence Theorem implies that

$$-\lim_{\varepsilon\to 0}\int_U F_\varepsilon'(f)\partial_i f\varphi\,\mathrm{d}\boldsymbol{x} = -\int_U \lim_{\varepsilon\to 0} F_\varepsilon'(f)\partial_i f\varphi\,\mathrm{d}\boldsymbol{x} = -\int_{U\cap\{f>0\}}\partial_i f\varphi.$$

Thus we get the expression for ∂f^+. Then using $f^- = (-f)^+$ and $|f| = f^+ + f^-$ leads to the rest three formulas. □

Exercise 1.2

Exercise 1.2.1. Let U, V be open sets with $V \Subset U$. Show that there exists $\zeta \in C^\infty(U)$ such that $\zeta = 1$ in V and $\zeta = 0$ near ∂U. (Hint: Take $V \Subset W \Subset U$ and mollify χ_W.)

Exercise 1.2.2. Assume $U \subset \mathbb{R}^d$ is bounded and $U \Subset \bigcup_{i=1}^N V_i$. Show that there exists $\zeta_i \in C^\infty(U)$, $i = 1, \cdots, N$, such that

$$0 \le \zeta_i \le 1,\ \operatorname{Spt}\zeta_i \subset V_i,\ \sum_{i=1}^N \zeta_i = 1 \text{ in } U.$$

(Hint: For each i, using Exercise 1.2.1 to construct $\varphi_i \in C^\infty(U)$ satisfying $\varphi_i = 1$ in $\overline{W_i}$, $W_i \Subset V_i$ and $\operatorname{Spt}\varphi_i \subset V_i$. Then let $\zeta_1 = \varphi_1$, $\zeta_2 = \varphi_2(1-\varphi_1), \cdots, \zeta_N = \varphi_N(1-\varphi_1)\cdots(1-\varphi_{N-1})$. Note that this is not the only choice.)

Exercise 1.2.3. Prove Proposition 1.2.4(2).

Exercise 1.2.4. Prove that $\|\nabla u\|_{L^2(U)}^2 \leq C\|u\|_{L^2(U)}\|\partial^2 u\|_{L^2(U)}$ for any $u \in C_c^\infty(U)$. Then extend this conclusion to $u \in H_0^1(U) \cap H^2(U)$ when U is bounded and ∂U is smooth.

Exercise 1.2.5. Assume $u \in C_c^\infty(U)$. Prove the following two inequalities.

(1) $\|\nabla u\|_{L^p(U)} \leq C\|u\|_{L^p(U)}^{\frac{1}{2}}\|\partial^2 u\|_{L^p(U)}^{\frac{1}{2}}$ for $2 \leq p < \infty$.

(2) $\|\nabla u\|_{L^{2p}(U)} \leq C\|u\|_{L^\infty(U)}^{\frac{1}{2}}\|\partial^2 u\|_{L^p(U)}^{\frac{1}{2}}$ for $1 \leq p < \infty$.

(Hint: You may need to use Hölder's inequality for three functions with index $(p, p, \frac{p}{p-2})$.)

Exercise 1.2.6. Let $U \subset \mathbb{R}^d$ be a *connected* open set and $f \in W^{1,p}(U)$ satisfies $\nabla f = \mathbf{0}$ a.e. in U. Prove that f agrees with a constant a.e. in U.

(Hint: You CANNOT use Poincaré's inequality to prove this, as this inequality is used in the proof of Poincaré's inequality. Consider $f_\varepsilon := \eta_\varepsilon * f$ in $V \Subset U$ and prove that $f_\varepsilon = C_\varepsilon$ a.e. in V for each sufficiently small $\varepsilon > 0$. Then use Theorem C.2.1 to show that $\|f_\varepsilon\|_{L^p}$ is uniformly bounded in ε and so is $\{C_\varepsilon\}$, which then implies a subsequence of $\{f_\varepsilon\}$ converge to a constant.)

1.3 Traces and extension

If $u \in C(U)$, then clearly we can define the pointwise value of u on the boundary ∂U. However, if $u \in W^{1,p}(U)$ is only a Sobolev function, then we can modify its value in a set of measure zero. In particular, the boundary ∂U also has zero d-dimensional Lebesgue measure. Thus, it is worth discussing the possibility of assigning "boundary values" along the boundary ∂U for a Sobolev function. Throughout this section, we assume $1 \leq p < \infty$.

Theorem 1.3.1 (Trace Theorem)**.** Let $U \subset \mathbb{R}^d$ be a bounded open set and ∂U be Lipschitz continuous. Then

(1) There exists a bounded linear operator $\mathrm{Tr} : W^{1,p}(U) \to L^p(\partial U; \mathrm{d}S)$ such that $\mathrm{Tr}\, f = f$ on ∂U for all $f \in W^{1,p}(U) \cap C(\overline{U})$ and
$$\|\mathrm{Tr}\, f\|_{L^p(\partial U)} \leq C\|f\|_{W^{1,p}(U)}$$
for each $f \in W^{1,p}(U)$ with the constant $C > 0$ depending only on p, U. Here $\mathrm{d}S = \mathcal{H}^{d-1}|_{\partial U}$ is the $(d-1)$-dimensional Hausdorff measure on ∂U (also interpreted as the surface measure).

(2) (Integration by parts) For any $\boldsymbol{\phi} \in C^1(\mathbb{R}^d \to \mathbb{R}^d)$ and $f \in W^{1,p}(U)$, there holds
$$\int_U f \operatorname{div} \boldsymbol{\phi} \,\mathrm{d}\boldsymbol{x} = -\int_U \nabla f \cdot \boldsymbol{\phi} \,\mathrm{d}\boldsymbol{x} + \int_{\partial U} (\boldsymbol{\phi} \cdot N)\mathrm{Tr}\, f \,\mathrm{d}S_{\boldsymbol{x}},$$

where N denotes the unit outer normal vector to ∂U.

Remark 1.3.1. The function $\operatorname{Tr} f$ is called the **trace** of f on ∂U. It is uniquely defined up to a set of $\mathcal{H}^{d-1}|_{\partial U}$-measure zero. We interpret $\operatorname{Tr} f$ as providing the "boundary values" of f on ∂U. In fact, it further satisfies that

$$\lim_{r\to 0} \fint_{U\cap B(\boldsymbol{x},r)} |f(\boldsymbol{y}) - \operatorname{Tr} f(\boldsymbol{x})| \,\mathrm{d}\boldsymbol{y} = 0, \quad \mathcal{H}^{d-1}\text{-a.e. } \boldsymbol{x} \in \partial U$$

and so

$$\operatorname{Tr} f(\boldsymbol{x}) = \lim_{r\to 0} \fint_{U\cap B(\boldsymbol{x},r)} f(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y}.$$

Proof. First, we assume $f \in C^1(\overline{U})$ and try to prove the case of $p = 1$, that is, $\int_{\partial U} |f| \,\mathrm{d}S_{\boldsymbol{x}} \le C \int_U |\nabla f| \,\mathrm{d}\boldsymbol{x}$. Since ∂U is Lipschitz continuous, for any $\boldsymbol{x}^0 \in \partial U$, we can find $r > 0$ and a Lipschitz continuous function $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ such that, upon rotating and relabeling the coordinate axes (if necessary),

$$U \cap B(\boldsymbol{x}^0, r) = \{\boldsymbol{x} | x_d > \gamma(x_1, \cdots, x_{d-1})\} \cap B(\boldsymbol{x}^0, r).$$

We write $B := B(\boldsymbol{x}^0, r)$ and suppose temporarily $f = 0$ in $U \backslash B$, that is, f is localized near the intersection of ∂U and B. Then, for the unit outer normal vector N of ∂U, we have

$$-e_d \cdot N = \cos\langle -e_d, N\rangle = \frac{1}{\sqrt{1 + \tan^2\langle -e_d, N\rangle}} \ge \frac{1}{\sqrt{1 + (\operatorname{Lip} \gamma)^2}} \quad \mathcal{H}^{d-1}\text{-a.e. on } B \cap \partial U.$$

Now, fix $\varepsilon > 0$ and set $\beta_\varepsilon(t) = \sqrt{t^2 + \varepsilon^2} - \varepsilon$ for $t \in \mathbb{R}$. Then, using Gauss-Green Theorem, we know that

$$\begin{aligned} \int_{\partial U} \beta_\varepsilon(f) \,\mathrm{d}S_{\boldsymbol{x}} &= \int_{B\cap\partial U} \beta_\varepsilon(f) \,\mathrm{d}S_{\boldsymbol{x}} \le C \int_{B\cap\partial U} \beta_\varepsilon(f)(-e_d \cdot N) \,\mathrm{d}S_{\boldsymbol{x}} \\ &\le -C \int_{B\cap U} \partial_{x_d}(\beta_\varepsilon(f)) \,\mathrm{d}\boldsymbol{x} \le C \int_{B\cap U} |\beta_\varepsilon'(f)||\nabla f(\boldsymbol{x})| \,\mathrm{d}\boldsymbol{x} \le C \int_U |\nabla f| \,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

Here $C > 0$ is a positive constant arising from the estimate of $(-e_d \cdot N)$ and we also use the fact that $|\beta_\varepsilon'| \le 1$. Now, let $\varepsilon \to 0$ (and also use Proposition 1.2.4(2)) to get

$$\int_{\partial U} |f| \,\mathrm{d}S_{\boldsymbol{x}} \le C \int_U |\nabla f| \,\mathrm{d}\boldsymbol{x}. \tag{1.3.1}$$

Do note that the approximation $\beta_\varepsilon(f) \to |f|$ is necessary, because $f \in C^1$ does not necessarily imply $|f|$ is continuously differentiable everywhere.

Next, we want to extend (1.3.1) to the case that $f \not\equiv 0$ in $U \backslash B$. In fact, this can be done by covering ∂U (which is compact) by a finite number of such balls $B(\boldsymbol{x}_i^0, r_i)$, $i = 1, \cdots, m$ and use a partition of unity as in the proof of Theorem 1.2.3 to obtain

$$\int_{\partial U} |f| \, \mathrm{d}S_{\boldsymbol{x}} \le C \int_U |\nabla f| + |f| \, \mathrm{d}\boldsymbol{x}, \quad \forall f \in C^1(\overline{U}). \tag{1.3.2}$$

Specifically, denote $B_i := B(\boldsymbol{x}_i^0, r_i)$ for $1 \le i \le m$ and $C_i = \sqrt{1 + (\mathrm{Lip}\, \gamma_i)^2}$ with $C = \max\{C_i\}$. Let $V, B_1, \cdots, B_m$ be a finite open covering of $\overline{U}$ with the partition of unity $\{\zeta_0, \zeta_1, \cdots \zeta_m\}$. Then mimicing the above proof, we have

$$\begin{aligned}
\int_{\partial U} \beta_\varepsilon(f) \, \mathrm{d}S_{\boldsymbol{x}} &\le C \sum_{i=1}^m \int_{B_i \cap \partial U} \zeta_i \beta_\varepsilon(f)(-e_d \cdot N) \, \mathrm{d}S_{\boldsymbol{x}} \\
&= C \sum_{i=1}^m \int_{B_i \cap U} \partial_{x_d}(\zeta_i \beta_\varepsilon(f)) \, \mathrm{d}\boldsymbol{x} \\
&= C \sum_{i=1}^m \int_{B_i \cap U} (\partial_{x_d} \zeta_i) \beta_\varepsilon(f) \, \mathrm{d}\boldsymbol{x} + \int_{B_i \cap U} \zeta_i \partial_{x_d}(\beta_\varepsilon(f)) \, \mathrm{d}\boldsymbol{x} \\
&\le C' \int_U |\beta_\varepsilon(f)| + |\beta_\varepsilon'(f)| |\nabla f| \, \mathrm{d}\boldsymbol{x} \qquad \exists C' > 0.
\end{aligned}$$

Then letting $\varepsilon \to 0$ leads to (1.3.2).

The next step is to extend this conclusion to a general $p \in (1, \infty)$. Fix such an index p, we replace $|f|$ by $|f|^p$ in (1.3.2) and use the same strategy to get

$$\int_{\partial U} |f|^p \, \mathrm{d}S_{\boldsymbol{x}} \le C \int_U |f|^p + |\nabla f| |f|^{p-1} \, \mathrm{d}\boldsymbol{x}.$$

Then use Young's inequality $ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'}$ with $p^{-1} + (p')^{-1} = 1$ and $a = |\nabla f|$, $b = |f|^{p-1}$ to get

$$\int_{\partial U} |f|^p \, \mathrm{d}S_{\boldsymbol{x}} \le C' \int_U |f|^p + |\nabla f|^p \, \mathrm{d}\boldsymbol{x} \qquad \forall f \in C^1(\overline{U}),$$

where this $C' > 0$ depends on p. Consequently, if we write $\mathrm{Tr}\, f := f|_{\partial U}$, then we obtain $\|\mathrm{Tr}\, f\|_{L^p(\partial U)} \le C\|f\|_{W^{1,p}(U)}$ for all $f \in C^1(\overline{U})$, for some $C > 0$ depending on p. Also, it is easy to verify the integration by parts formula (2) for $f \in C^1(\overline{U})$.

Finally, it remains to prove the same conclusion for $f \in W^{1,p}(U)$. Given such an f, we know, by Theorem 1.2.3, there exist functions $f_n \in C^\infty(\overline{U})$ converging to f in $W^{1,p}(U)$. Then we have

$$\|\mathrm{Tr}\, f_k - \mathrm{Tr}\, f_l\|_{L^p(\partial U)} \le C\|f_k - f_l\|_{W^{1,p}(U)}$$

which shows that $\{\mathrm{Tr}\, f_n\}$ is a Cauchy sequence in $L^p(\partial U)$ and thus produces a limit $\mathrm{Tr}\, f \in L^p(\partial U)$ defined by $\lim\limits_{n\to\infty} \mathrm{Tr}\, f_n$. Also this limit does not depend on the choices of smooth approximation of f. Now, given an $f \in W^{1,p}(U) \cap C(\overline{U})$, we can also use the global smooth approximation (up to the boundary) to define $\mathrm{Tr}\, f = f|_{\partial U}$. Since $W^{1,p}(U)\cap C(\overline{U})$ is dense in $W^{1,p}(U)$, we can extend the operator Tr , by the B.L.T. theorem, to a bounded linear operator from $W^{1,p}(U)$ to $L^p(\partial U)$ as desired. As for the integration by parts formula, we can also use the global smooth approximation (up to the boundary) to finish the proof and we do not repeat the details here. □

We next prove a further result about trace-zero functions. This conclusion is rather important, but the proof seems to be a bit too technical and the beginners can skip it.

Theorem 1.3.2. Assume $U \subset \mathbb{R}^d$ is a bounded open set and ∂U is Lipschitz continuous and $f \in W^{1,p}(U)$. Then

$$f \in W_0^{1,p}(U) \text{ if and only if } \mathrm{Tr}\, f = 0 \text{ on } \partial U.$$

Proof. The "only if" part is easy to prove. In fact, it is just a simple corollary of the smooth approximation. Given $f \in W_0^{1,p}(U)$, there exist a sequence $\{f_n\} \subset C_c^\infty(U)$ such that $f_n \to f$ in $W^{1,p}(U)$. Since $\mathrm{Tr}\, f_n = 0$ and $\mathrm{Tr} : W^{1,p}(U) \to L^p(\partial U)$ is a bounded linear operator, we get $\mathrm{Tr}\, f = 0$ on ∂U.

The converse is difficult. Given $f \in W^{1,p}(U)$ with $\mathrm{Tr}\, f = 0$ on ∂U, we need to construct a sequence $\{f_n\} \subset C_c^\infty(U)$ such that $\|f_n - f\|_{W^{1,p}(U)} \to 0$. We know Appendix C.2 that it is easy to construct a sequence of C_c^∞ functions to approach a Sobolev function in $L^p(U)$ norm. However, we also need to ensure the convergence of the first-order derivatives. Recall that the Global Smooth Approximation Theorem no longer ensures the boundary values of the smooth approximate functions are zero. In order to *simultanously obtain the convergence and the vanishing boundary values of the smooth approximation*, we can "truncate" f away from the boundary (with a distance that finally convergers to 0) and then mollify the "truncated" approximate function.

For technical simplicity, we assume $f \in W^{1,p}(\mathbb{R}^d_+)$, $\mathrm{Spt}\, f$ is compact and $\mathrm{Tr}\, f = 0$ on $\partial\mathbb{R}^d_+ = \mathbb{R}^{d-1}$. This can be achieved by doing a partition of unity of ∂U and flattening the graph of γ thanks to the Lipschitz continuity of ∂U.

Step 1: Construction of the truncation. We define a smooth cut-off function $\zeta \in C^\infty(\mathbb{R}_+)$ by

$$\zeta(x_d) = \begin{cases} 1 & x_d \in [0,1], \\ \text{decreasing, takes value in } [0,1] & x_d \in [1,2], \\ 0 & x_d \in [2,\infty). \end{cases}$$

Then for each $m \in \mathbb{N}^*$, we define $\zeta_m(\boldsymbol{x}_d) := \zeta(m x_d)$ for $\boldsymbol{x} \in \mathbb{R}^d_+$ and

$$w_m(\boldsymbol{x}) := f(\boldsymbol{x})(1 - \zeta_m(\boldsymbol{x})).$$

That is, w_m is a smooth truncation in $\mathbb{R}^{d-1} \times [\frac{1}{m}, \infty)$ and coincide with f in $\mathbb{R}^{d-1} \times [\frac{2}{m}, \infty)$. It is easy to see that $\|w_m - f\|_{L^p(U)} \to 0$ as $m \to \infty$.

Step 2: Convergence of the first-order derivatives. We can compute that

$$\partial_{x_d} w_m = \partial_{x_d} f(1 - \zeta_m(\boldsymbol{x})) - m f(\boldsymbol{x})\zeta'(x_d) \quad \text{and} \quad \partial_{x_i} w_m = \partial_{x_i} f(\boldsymbol{x})(1 - \zeta_m(\boldsymbol{x})) \;\; 1 \le i \le d-1.$$

Thus, we compute that

$$\int_{\mathbb{R}^d_+} |\nabla w_m - \nabla f|^p \, \mathrm{d}\boldsymbol{x} \le C \int_{\mathbb{R}^d_+} |\zeta_m|^p |\nabla f|^p \, \mathrm{d}\boldsymbol{x} + Cm^p \int_0^{\frac{2}{m}} \int_{\mathbb{R}^{d-1}} |f(\boldsymbol{x}', x_d)|^p \, \mathrm{d}\boldsymbol{x}' \, \mathrm{d}x_d. \tag{1.3.3}$$

Again by definition of ζ_m and the Dominated Convergence Theorem, it is easy to see that the first integral converges to 0

$$\int_{\mathbb{R}^d_+} |\zeta_m|^p |\nabla f|^p \, \mathrm{d}\boldsymbol{x} \to 0 \quad \text{as } m \to \infty. \tag{1.3.4}$$

As for the second term, noticing that the normal component $x_d \in [0, \frac{2}{m}]$ is very close to the boundary, we shall express $f(\boldsymbol{x}', x_d)$ as the sum of the boundary value of f and the integral of $\partial_{x_d} f$, i.e., an analogue of the fundamental theorem of calculus. However, here we only know f is a Sobolev function, but the fundamental theorem of calculus is only valid for absolutely continuous functions, so we have to introduce a suitable approximation to f. Since $\mathrm{Tr}\, f = 0$ on $\mathbb{R}^{d-1} \times \{x_d = 0\}$, there exists a sequence of functions $\{u_k\} \subset C^\infty(\mathbb{R}^d_+)$ such that $u_k \to f$ in $W^{1,p}(\mathbb{R}^d_+)$ and thus by Trace Theorem $\mathrm{Tr}\, u_k = u_k|_{x_d=0} \to 0$ in $L^p(\mathbb{R}^{d-1})$. For each u_k, we have $u_k(\boldsymbol{x}', x_d) = u_k(\boldsymbol{x}', 0) + \int_0^{x_d} \partial_{x_d} u_k(\boldsymbol{x}', t) \, \mathrm{d}t$. Thus, we have

$$\int_{\mathbb{R}^{d-1}} |u_k(\boldsymbol{x}', x_d)|^p \, \mathrm{d}\boldsymbol{x}' \le \int_{\mathbb{R}^{d-1}} |u_k(\boldsymbol{x}', 0)|^p \, \mathrm{d}\boldsymbol{x}' + \int_{\mathbb{R}^{d-1}} \left(\int_0^{x_d} 1 \cdot |\partial_{x_d} u_k(\boldsymbol{x}', t)| \, \mathrm{d}x \right)^p \mathrm{d}t,$$

in which the first term converges to 0 as $k \to \infty$ because of $\mathrm{Tr}\, u_k = u_k|_{x_d=0} \to 0$ in $L^p(\mathbb{R}^{d-1})$. As for the second term, we use Minkowski's inequality for integrals and Hölder's inequality to get

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}} \left(\int_0^{x_d} 1 \cdot |\partial_{x_d} u_k(\boldsymbol{x}', t)| \, \mathrm{d}\boldsymbol{x}' \right)^p \mathrm{d}t = \left\| \|\partial_{x_d} u_k\|_{L^1_{x_d}(0,x_m)} \right\|^p_{L^p_{\boldsymbol{x}'}(\mathbb{R}^{d-1})} \\
&\le \left\| \|\partial_{x_d} u_k\|_{L^p_{\boldsymbol{x}'}(\mathbb{R}^{d-1})} \right\|^p_{L^1_{x_d}(0,x_m)} = \left[\int_0^{x_d} 1 \cdot \underbrace{\left(\int_{\mathbb{R}^{d-1}} |\partial_{x_d} u_k(\boldsymbol{x}', t)|^p \, \mathrm{d}\boldsymbol{x}' \right)^{\frac{1}{p}}}_{\text{as a function of } t} \mathrm{d}t \right]^p \\
&\le \left[\left(\int_0^{x_d} 1^{p'} \right)^{\frac{1}{p'}} \cdot \left(\int_0^{x_d} \left(\int_{\mathbb{R}^{d-1}} |\partial_{x_d} u_k(\boldsymbol{x}', t)|^p \, \mathrm{d}\boldsymbol{x}' \right)^{\frac{1}{p} \cdot p} \mathrm{d}t \right)^{\frac{1}{p}} \right]^p \\
&\le C x_d^{p-1} \int_0^{x_d} \int_{\mathbb{R}^{d-1}} |\nabla u_k(\boldsymbol{x}', t)|^p \, \mathrm{d}\boldsymbol{x}' \, \mathrm{d}t.
\end{aligned}$$

Then taking $k \to \infty$, we get the analogue of the fundamental theorem of calculus for f

$$\int_{\mathbb{R}^{d-1}} |f(\boldsymbol{x}', x_d)|^p \, \mathrm{d}\boldsymbol{x}' \le C x_d^{p-1} \int_0^{x_d} \int_{\mathbb{R}^{d-1}} |\nabla f(\boldsymbol{x}', t)|^p \, \mathrm{d}\boldsymbol{x}' \, \mathrm{d}t, \quad \text{a.e. } x_d > 0. \tag{1.3.5}$$

Now, plugging (1.3.5) into the second term of (1.3.3), we obtain

$$\begin{aligned}
& m^p \int_0^{\frac{2}{m}} \int_{\mathbb{R}^{d-1}} |f(\boldsymbol{x}', x_d)|^p \, \mathrm{d}\boldsymbol{x}' \, \mathrm{d}x_d \\
& \le C m^p \int_0^{\frac{2}{m}} x_d^{p-1} \left(\int_0^{x_d} \int_{\mathbb{R}^{d-1}} |\nabla f(\boldsymbol{x}', t)|^p \, \mathrm{d}\boldsymbol{x}' \, \mathrm{d}t \right) \mathrm{d}x_d \\
& \le C m^p \left(\int_0^{\frac{2}{m}} x_d^{p-1} \, \mathrm{d}x_d \right) \cdot \sup_{x_d \in [0, \frac{2}{m}} \int_0^{x_d} \int_{\mathbb{R}^{d-1}} |\nabla f(\boldsymbol{x}', t)|^p \, \mathrm{d}\boldsymbol{x}' \, \mathrm{d}t \\
& = C m^p \cdot \frac{2^p}{m^p p} \cdot \int_0^{\frac{2}{m}} \int_{\mathbb{R}^{d-1}} |\nabla f(\boldsymbol{x}', t)|^p \, \mathrm{d}\boldsymbol{x}' \, \mathrm{d}t = C_p \int_0^{\frac{2}{m}} \int_{\mathbb{R}^{d-1}} |\nabla f(\boldsymbol{x}', t)|^p \, \mathrm{d}\boldsymbol{x}' \, \mathrm{d}t
\end{aligned} \tag{1.3.6}$$

which converges to 0 as $m \to \infty$. Thus, the right side of (1.3.3) converges to 0, that is $\nabla w_m \to \nabla f$ in $L^p(\mathbb{R}^d_+)$. So, we finish the proof of $w_m \to f$ in $W^{1,p}(\mathbb{R}^d_+)$.

Step 3: Mollification of the truncated approximate functions. The sequence $\{w_m\}$ can approximate f in $W^{1,p}(\mathbb{R}^d_+)$ norm, but these functions may not belong to $C_c^\infty(\mathbb{R}_d^+)$. Note that w_m vanishes in $\mathbb{R}^{d-1} \in [0, \frac{1}{m}]$, so we have enough room to mollify each w_m in x_d-direction and squeeze a subsequence by the diagonal argument.

Specifically, given $k \in \mathbb{N}^*$, step 2 shows that there exist a subsequence w_{m_k} satisfyting

$$\|w_{m_k} - f\|_{W^{1,p}(U)} < \frac{1}{k}.$$

Then for each k, we define $w_{m_k}^n := \eta_{\frac{1}{2mn}} * w_{m_k}$ with $\eta(x_d)$ the mollifier in x_d-direction. We know $w_{m_k}^n$ is **compactly supported** in $\mathbb{R}^{d-1} \times [\frac{1}{m}(1 - \frac{1}{2n}), \infty)$. By Theorem 1.2.1, we know for each $j \in \mathbb{N}^*$, we can find n_j (increasingly go to ∞) such that

$$\|w_{m_k}^{n_j} - w_{m_k}\|_{W^{1,p}(\mathbb{R}^d_+)} < \frac{1}{j}.$$

Thus, setting $f_k := w_{m_k}^{n_k} \in C_c^\infty(\mathbb{R}^d_+)$, we obtain the convergence

$$\|f_k - f\|_{W^{1,p}(\mathbb{R}^d_+)} \to 0 \quad \text{as } k \to \infty.$$

□

We end this section by an extension theorem. The conclusion is easy to understand, but the proof is rather technically complicated. In particular, for different differentiability index $k \in \mathbb{N}^*$, the proof also becomes rather different. Thus, we only list the conclusion and omit the proof here.

Theorem 1.3.3 (Sobolev Extension Theorem). Let $1 \leq p \leq \infty$ and $U \subset \mathbb{R}^d$ be a bounded open set with $\partial U \in C^k$ ($k \in \mathbb{N}^*$) (When $k = 1$, we only need ∂U is Lipschitzian). Assume $G \subset \mathbb{R}^d$ is an open set with $U \Subset G$. Then there exists a constant $C > 0$ depending on d, k, U, G and a bounded linear mapping $\mathbf{E} : W^{k,p}(U) \to W^{k,p}(\mathbb{R}^d)$ such that for any $f \in W^{k,p}(\Omega)$, it holds

(1) $\mathbf{E}f = f$ a.e. in U.
(2) $\operatorname{Spt} \mathbf{E}f \Subset G$.
(3) $\|\mathbf{E}f\|_{W^{k,p}(G)} \leq C\|f\|_{W^{k,p}(U)}$.

Exercise 1.3

Exercise 1.3.1. Let $U \subset \mathbb{R}^d$ be bounded with a Lipschitz continuous boundary. Prove that there does NOT exist a bounded linear operator $\operatorname{Tr} : L^p(U) \to L^p(\partial U)$ such that $\operatorname{Tr} f = f|_{\partial U}$ for all $f \in C(\overline{U}) \cap L^p(U)$.

(Hint: Try to construct a sequence $\{f_m\}$ satisfying $\|f_m\|_{L^p(U)} \to 0$ but $\operatorname{Tr} f_m \equiv 1$ on ∂U.)

Exercise 1.3.2. Let $f_\pm \in H^1(\mathbb{R}^d_\pm)$ with $\operatorname{Tr} f_- = \operatorname{Tr} f_+$ on $\mathbb{R}^{d-1} \times \{x_d = 0\}$. Define $f \in L^2(\mathbb{R}^d)$ by

$$f = \begin{cases} f_+ & x_d > 0 \\ f_- & x_d < 0 \end{cases}.$$

Prove that $f \in H^1(\mathbb{R}^d)$.

Exercise 1.3.3. Assume $f \in H^1_0(\mathbb{R}^d_+) \cap H^2(\mathbb{R}^d_+)$. Show that for any $1 \leq i \leq d-1$, the partial derivative $\partial_{x_i} f$ also belongs to $H^1_0(\mathbb{R}^d_+)$.

1.4 Sobolev embeddings

The goal of this section is to discover embeddings of Sobolev spaces into others, such as L^p spaces, $C^{k,\alpha}$ spaces and so on. In particular, we will mostly consider the embedding for $W^{1,p}(U)$, as we can do induction on k to find the correct spaces for embeddings if $k \geq 2$. The results are different according as the relation between p and the dimension d.

1.4.1 Gagliardo-Nirenberg-Sobolev inequality

The first case is $1 \leq p < d$. We want to ask if we can establish an estimate of the form

$$\|f\|_{L^q(\mathbb{R}^d)} \leq C\|\nabla f\|_{L^p(\mathbb{R}^d)}, \quad \forall f \in C_c^\infty(\mathbb{R}^d)$$

for certain $C > 0$, $q \geq 1$. The point is that the constant C should not depend on u. So, we may consider the **scaling invariance.** That is, given $f \in C_c^\infty(\mathbb{R}^d)$, $\lambda > 0$, we define the rescaled function $f_\lambda(\boldsymbol{x}) := f(\lambda \boldsymbol{x})$ for $\boldsymbol{x} \in \mathbb{R}^d$. For f_λ, we also want to have a similar inequality and the constant C does not depend on λ. We compute that

$$\int_{\mathbb{R}^d} |f_\lambda(\boldsymbol{x})|^q \,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d} |f(\lambda \boldsymbol{x})|^q \,\mathrm{d}\boldsymbol{x} = \lambda^{-d} \int_{\mathbb{R}^d} |f(\boldsymbol{y})|^q \,\mathrm{d}\boldsymbol{y}$$

and

$$\int_{\mathbb{R}^d} |\nabla f_\lambda|^p \,\mathrm{d}\boldsymbol{x} = \lambda^p \int_{\mathbb{R}^d} |\nabla f(\lambda \boldsymbol{x})|^p \,\mathrm{d}\boldsymbol{x} = \lambda^{p-d} \int_{\mathbb{R}^d} |\nabla f(\boldsymbol{y})|^p \,\mathrm{d}\boldsymbol{y}.$$

Plugging these back to the inequality for f_λ, we obtain

$$\|f\|_{L^q(\mathbb{R}^d)} \leq C \lambda^{1-d(\frac{1}{p}-\frac{1}{q})} \|\nabla f\|_{L^p(\mathbb{R}^d)}.$$

Thus, we must have $1 - d(\frac{1}{p} - \frac{1}{q}) = 0$, otherwise letting $\lambda \to 0$ or ∞ leads to a contradiction. This q satisfies $\frac{1}{q} = \frac{1}{p} - \frac{1}{d}$ or equivalently $q = \frac{dp}{d-p}$. We now denote it by

$$p^* := \frac{dp}{d-p}.$$

Theorem 1.4.1 (Gagliardo-Nirenberg-Sobolev inequality). Assume $1 \leq p < d$. There exists a constant $C > 0$ only depending on p, d, such that

$$\|f\|_{L^{p^*}(\mathbb{R}^d)} \leq C \|\nabla f\|_{L^p(\mathbb{R}^d)}, \quad \forall f \in W^{1,p}(\mathbb{R}^d).$$

Proof. In view of Theorem 1.2.1, we may assume $f \in C_c^1(\mathbb{R}^d)$. It should be noted that the constant C does not depend on the size of Spt f. The goal is to use the derivative of f to control f itself, so a natural idea is to use the fundamental theorem of calculus to compute f. For simplicity, we first consider the case $p = 1$ and then $1^* = \frac{d}{d-1}$.

$$\forall i = 1, \cdots, d, \quad f(\boldsymbol{x}) = \int_{-\infty}^{x_i} \partial_{x_i} f(x_1, \cdots, t_i, \cdots, x_d) \,\mathrm{d}t_i,$$

and so

$$|f(\boldsymbol{x})| \leq \int_{-\infty}^{+\infty} |\nabla f|(x_1, \cdots, t_i, \cdots, x_d) \,\mathrm{d}t_i.$$

Then

$$|f(\boldsymbol{x})|^{\frac{d}{d-1}} = (|f(\boldsymbol{x})|^d)^{\frac{1}{d-1}} \leq \prod_{i=1}^d \left(\int_{-\infty}^{+\infty} |\nabla f|(x_1, \cdots, t_i, \cdots, x_d) \,\mathrm{d}t_i \right)^{\frac{1}{d-1}}.$$

Note that the i-th term in the product does not depend on x_i. Now, we integrate with respect to x_1 to get

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} |f|^{1^*}\,\mathrm{d}x_1 \\
\le &\left(\int_{-\infty}^{+\infty} |\nabla f|(t_1, x_2 \cdots, x_d)\,\mathrm{d}t_i.\right)^{\frac{1}{d-1}} \cdot \int_{-\infty}^{+\infty} \prod_{i=2}^{d} \left(\int_{-\infty}^{+\infty} |\nabla f|(x_1, \cdots, t_i, \cdots, x_d)\,\mathrm{d}t_i\right)^{\frac{1}{d-1}}\,\mathrm{d}x_1.
\end{aligned}
$$

Using Hölder's inequality for $(d-1)$ functions, we know

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} |f|^{1^*}\,\mathrm{d}x_1 \\
\le &\left(\int_{-\infty}^{+\infty} |\nabla f|(t_1, x_2 \cdots, x_d)\,\mathrm{d}t_i\right)^{\frac{1}{d-1}} \cdot \left(\prod_{i=2}^{d} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\nabla f|(x_1, \cdots, t_i, \cdots, x_d)\,\mathrm{d}x_1\,\mathrm{d}t_i\right)^{\frac{1}{d-1}}.
\end{aligned}
$$

Then we integrate with respect to x_2 and use Hölder's inequality to find

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f|^{1^*}\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
\le &\left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\nabla f|(x_1, t_2, \cdots, x_d)\,\mathrm{d}x_1\,\mathrm{d}t_2\right)^{\frac{1}{d-1}} \times \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\nabla f|(t_1, x_2, \cdots, x_d)\,\mathrm{d}t_1\,\mathrm{d}x_2\right)^{\frac{1}{d-1}} \\
&\times \prod_{i=3}^{d} \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\nabla f|(x_1, \cdots, t_i, \cdots, x_d)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}t_i\right)^{\frac{1}{d-1}}.
\end{aligned}
$$

Repeat this step and eventually we have

$$
\begin{aligned}
\int_{\mathbb{R}^d} |f|^{1^*}\,\mathrm{d}\boldsymbol{x} &\le \prod_{i=1}^{d} \left(\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} |\nabla f|(x_1, \cdots, t_i, \cdots, x_d)\,\mathrm{d}x_1 \cdots \mathrm{d}t_i \cdots \mathrm{d}x_d\right)^{\frac{1}{d-1}} \\
&= \left(\int_{\mathbb{R}^d} |\nabla f|\,\mathrm{d}\boldsymbol{x}\right)^{\frac{d}{d-1}},
\end{aligned}
$$

which is exactly the GNS inequality for $p = 1$

$$
\|f\|_{L^{1^*}(\mathbb{R}^d)} \le \|\nabla f\|_{L^1(\mathbb{R}^d)}.
$$

For $1 < p < d$, we replace f by $g = |f|^\gamma$ with $\gamma > 1$ to be determined. Replacing f by g, we get

$$\left(\int_{\mathbb{R}^d} |f|^{\frac{\gamma d}{d-1}}\,\mathrm{d}\boldsymbol{x}\right)^{\frac{d-1}{d}} \le \gamma \int_{\mathbb{R}^d} |f|^{\gamma-1}|\nabla f|\,\mathrm{d}\boldsymbol{x}$$
$$\le \gamma \|\nabla f\|_{L^p(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} |f|^{(\gamma-1)p'}\right)^{\frac{1}{p'}}, \quad p' = \frac{p}{p-1}.$$

In order for a cancellation, we choose γ such that

$$\frac{\gamma d}{d-1} = \frac{(\gamma-1)p}{p-1} \Rightarrow \gamma = \frac{p(d-1)}{d-p} \Rightarrow \frac{\gamma d}{d-1} = p^* = \frac{(\gamma-1)p}{p-1}.$$

Thus, we obtain

$$\left(\int_{\mathbb{R}^d} |f|^{p^*}\,\mathrm{d}\boldsymbol{x}\right)^{\frac{d-1}{d}} \le C\|\nabla f\|_{L^p(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} |f|^{p^*}\right)^{\frac{1}{p'}} \Rightarrow \|f\|_{L^{p^*}(\mathbb{R}^d)} \le C\|\nabla f\|_{L^p(\mathbb{R}^d)}.$$

□

Theorem 1.4.2 (Sobolev embedding theorem). Let $U \subset \mathbb{R}^d$ be a bounded open set with a Lipschitz boundary and $1 \le p < d$.

(1) Any $f \in W^{1,p}(U)$ belongs to $L^{p*}(U)$ with the estimate $\|f\|_{L^{p^*}(U)} \le C\|f\|_{W^{1,p}(U)}$ where the constant C depends only on d, p, U.
(2) Any $f \in W_0^{1,p}(U)$ satisfies the estimate $\|f\|_{L^q(U)} \le C\|\nabla f\|_{L^p(U)}$ for all $q \in [1, p^*]$ where the constant C depends only on d, p, q, U.

In view of the second estimate, the norm $\|\nabla f\|_{L^p(U)}$ on $W_0^{1,p}(U)$ is equivalent to $\|f\|_{W^{1,p}(U)}$ if U is bounded.

Proof. Since ∂U is Lipschitz, we can extend $f \in W^{1,p}(U)$ to $\bar{f} \in W^{1,p}(\mathbb{R}^d)$ which satisfies $\bar{f} = f$ a.e. in U, $\operatorname{Spt}\bar{f}$ is compact and $\|\bar{f}\|_{W^{1,p}(\mathbb{R}^d)} \le C\|f\|_{W^{1,p}(U)}$. Since $\bar{f}$ has compact support, there exist a sequence of functions $u_m \in C_c^\infty(\mathbb{R}^d)$ such that $u_m \to \bar{f}$ in $W^{1,p}(\mathbb{R}^d)$. Using Theorem 1.4.1, we get

$$\|u_k - u_l\|_{L^{p^*}(\mathbb{R}^d)} \le C\|\nabla u_k - \nabla u_l\|_{L^p(\mathbb{R}^d)} \Rightarrow u_m \to \bar{f} \text{ in } L^{p^*}(\mathbb{R}^d).$$

Also by GNS inequality, we have $\|u_m\|_{L^{p^*}(\mathbb{R}^d)} \le C\|\nabla u_m\|_{L^p(\mathbb{R}^d)}$. Letting $m \to \infty$ and using the convergence result, we find

$$\|\bar{f}\|_{L^{p^*}(\mathbb{R}^d)} \le C\|\nabla \bar{f}\|_{L^p(\mathbb{R}^d)}.$$

This finishes the proof of (1). Assertation (2) follows in the same way by extending f to be 0 in $\mathbb{R}^d \backslash \overline{U}$ and using the boundedness of U and Hölder's inequality $\|f\|_{L^q(U)} \le \|1\|_{L^r(U)}\|f\|_{L^{p^*}(U)}$ with $\frac{1}{q} = \frac{1}{r} + \frac{1}{q^*}$. □

Remark 1.4.1. As p increasingly approaches d, the Sobolev conjugate $p^* = \frac{dp}{d-p} \to +\infty$. We expect that $W^{1,d}$ funcitons belong to L^∞, but this is false when $d \geq 2$. Instead, $W^{1,d}$ is embedding into a BMO-type space. See Exercise 1.4.10.

The conclusion of Theorem 1.4.2(2) also indicates the Poincaré's inequality for $W_0^{1,p}$ functions for any $1 \leq p \leq \infty$.

Corollary 1.4.3. Let $U \subset \mathbb{R}^d$ be bounded and $1 \leq p \leq \infty$. There exists $C > 0$ such that

$$\|f\|_{L^p(U)} \leq C\|\nabla f\|_{L^p(U)}$$

holds for all $f \in W_0^{1,p}(U)$.

1.4.2 Compact embedding

We have already seen that the GNS inequality implies to embedding $W^{1,p}(U) \hookrightarrow L^{p^*}(U)$ for $1 \leq p < d$ and $p^* = \frac{dp}{d-p}$. However, the boundedness only implies the weak-* convergence (weak convergence for reflexive spaces). In order for the strong convergence, we shall require the embedding map to be a compact operator. In this section, we demonstrate that $W^{1,p}(U)$ is *compactly* embedded in $L^q(U)$ for $1 \leq q < p^*$.

Definition 1.4.1 (Compact embedding). Let X, Y be two Banach spaces satisfying $X \subset Y$. We say that X **is compactly embedded in** Y, denoted by $X \hookrightarrow\hookrightarrow Y$, provided that

- (Boundedness) $\|f\|_Y \leq C\|f\|_X$ for some constant $C > 0$.
- (Compactness) Each bounded sequence in X is precompact in Y, that is, has a *convergent subsequence* in Y.

Theorem 1.4.4 (Rellich-Kondrachov). Let $U \subset \mathbb{R}^d$ be a bounded open set with a Lipschitz boundary ∂U. Suppose $1 \leq p < d$. Then $W^{1,p}(U) \hookrightarrow\hookrightarrow L^q(U)$ for all $1 \leq q < p^*$.

Proof. The embedding is guaranteed by the GNS inequality and $\mathcal{L}^d(U) < \infty$. It suffices to prove the compactness. Assume $\{f_m\}$ is uniformly bounded in $W^{1,p}(U)$ with

$$\sup_m \|f_m\|_{W^{1,p}(U)} \leq M$$

and we need to construct a subsequence $\{f_{m_k}\}$ that converges in $L^q(U)$. A good choice to "construct the compactness" is the Arzelà-Ascoli lemma, which asserts that a sequence of uniformly bounded and equicontinuous functions must admit a convergent subsequence in $L^\infty(U)$ norm when U is bounded. Since $L^\infty \subset L^q$ due to $\mathcal{L}^d(U) < \infty$, we then expect to get the strong convergence in $L^q(U)$.

Step 1: Mollification. In order to apply the Arzelà-Ascoli lemma, we need to mollify f_m to get the continuity. In view of Sobolev extension theorem (Theorem 1.3.3), we may assume $f_m \in W^{1,p}(\mathbb{R}^d)$ have compact support in some bounded open set $V \subset \mathbb{R}^d$. Also, we assume $\sup_m \|f_m\|_{W^{1,p}(V)} < \infty$.

Given $\varepsilon > 0$, we define $f_m^\varepsilon = \eta_\varepsilon * f_m$. We want to prove $\|f_m^\varepsilon - f_m\|_{L^q(V)} \to 0$ holds **uniformly** in m. If f_m is C^1, then by fundamental theorem of calculus, we get

$$\begin{aligned}|f_m^\varepsilon - f_m| &\le \int_{B(\mathbf{0},\varepsilon)} \eta_\varepsilon(\boldsymbol{y})|f_m(\boldsymbol{x}-\boldsymbol{y}) - f_m(\boldsymbol{x})|\,\mathrm{d}\boldsymbol{y} \\ &\le \int_{B(\mathbf{0},\varepsilon)} \eta_\varepsilon(\boldsymbol{y})\left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f_m(\boldsymbol{x}-t\boldsymbol{y})\right|\mathrm{d}\boldsymbol{y} \\ &\le \int_0^1\int_{B(\mathbf{0},\varepsilon)} |\eta_\varepsilon(\boldsymbol{y})|\cdot|\boldsymbol{y}|\cdot|\nabla f_m(\boldsymbol{x}-t\boldsymbol{y})|\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}t.\end{aligned}$$

Taking $L^1(V)$ norm and using Tonelli's lemma, we get

$$\begin{aligned}\|f_m^\varepsilon - f_m\|_{L^1(V)} &\le \int_0^1\int_{B(\mathbf{0},\varepsilon)} |\eta_\varepsilon(\boldsymbol{y})|\cdot|\boldsymbol{y}|\cdot\|\nabla f_m(\cdot - t\boldsymbol{y})\|_{L^1(V)}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}t \\ &\le \|\nabla f_m\|_{L^1(V)}\int_{B(0,\varepsilon)} |\eta_\varepsilon(\boldsymbol{y})|\cdot|\boldsymbol{y}|\,\mathrm{d}\boldsymbol{y} = \varepsilon\|\nabla f_m\|_{L^1(V)} \\ &\le \varepsilon\|1\|_{L^{p'}(V)}\|\nabla f_m\|_{L^p(V)} = C\varepsilon\|\nabla f_m\|_{L^p(V)} \le CM\varepsilon.\end{aligned}$$

This also holds for $f_m \in W^{1,p}(\mathbb{R}^d)$ by the approximation (Theorem 1.2.1). Now we replace $L^1(V)$ by $L^q(V)$. Let $\theta \in (0,1)$ satisfy $\frac{\theta}{1} + \frac{1-\theta}{p^*} = \frac{1}{q}$. Then by Hölder's inequality, we have

$$\begin{aligned}\|f_m^\varepsilon - f_m\|_{L^q(V)} &\le \|f_m^\varepsilon - f_m\|_{L^1(V)}^\theta\|f_m^\varepsilon - f_m\|_{L^{p^*}(V)}^{1-\theta} \le C(M\varepsilon)^\theta\|f_m^\varepsilon - f_m\|_{L^{p^*}(V)}^{1-\theta} \\ &\le C(M\varepsilon)^\theta(\|f_m^\varepsilon\|_{L^{p^*}(V)} + \|f_m\|_{L^{p^*}(V)})^{1-\theta} \le C'\varepsilon^\theta,\end{aligned}$$

where we use $\sup_m \|f_m\|_{L^{p^*}(V)} < \infty$ thanks to the GNS inequality. Therefore, given any $\delta > 0$, we have the convergence (uniform in m) $\|f_m^\varepsilon - f_m\|_{L^q(V)} < \delta$ as $\varepsilon \to 0$.

Step 2: The uniform (in m, not in ε!) boundedness of $\{f_m^\varepsilon\}$ <u>for fixed $\varepsilon > 0$.</u> This is quite straightforward by definition of mollification and Hölder's inequality

$$\begin{aligned}|f_m^\varepsilon(\boldsymbol{x})| &\le \int_{B(\boldsymbol{x},\varepsilon)} \eta_\varepsilon(\boldsymbol{x}-\boldsymbol{y})|f_m(\boldsymbol{y})|\,\mathrm{d}\boldsymbol{y} \le \|\eta_\varepsilon\|_{L^\infty}\|f_m\|_{L^1(V)} \\ &\le C\varepsilon^{-d}\|f_m\|_{L^q(V)} \le C'\varepsilon^{-d} < \infty.\end{aligned}$$

Step 3: The equi-continuity of $\{f_m^\varepsilon\}$ <u>for fixed $\varepsilon > 0$.</u> Again, by definition, we obtain

$$|\nabla f_m^\varepsilon(\boldsymbol{x})| \le \|\nabla\eta_\varepsilon\|_{L^\infty}\|f_m\|_{L^1(V)} \le C\varepsilon^{-d-1} < \infty.$$

Now, for each fixed $\varepsilon > 0$, by Arzelà-Ascoli lemma, there exists a subsequence $\{f_{m_k}^\varepsilon\}$ converging in

$L^\infty(V)$ norm and so

$$\limsup_{k,l\to\infty} \|f^\varepsilon_{m_k} - f^\varepsilon_{m_l}\|_{L^q(V)} \le \limsup_{k,l\to\infty} \|f^\varepsilon_{m_k} - f^\varepsilon_{m_l}\|_{L^\infty(V)}\|1\|_{L^q(V)} = 0.$$

Step 4: Strong convergence for the original sequence. Now, we need to combine step 1 and the convergence result for $\{f^\varepsilon_m\}$ to squeeze a convergent subsequence of $\{f_m\}$. This is done by a standard diagonal argument. Given $\delta > 0$ and sufficiently large $k, l \in \mathbb{N}^*$, we have

$$\begin{aligned}\|f_{m_k} - f_{m_l}\|_{L^q(V)} &\le \|f_{m_k} - f^\varepsilon_{m_k}\|_{L^q(V)} + \|f^\varepsilon_{m_k} - f^\varepsilon_{m_l}\|_{L^q(V)} + \|f^\varepsilon_{m_l} - f_{m_l}\|_{L^q(V)}\\ &\le 2\delta + \|f^\varepsilon_{m_k} - f^\varepsilon_{m_l}\|_{L^q(V)}.\end{aligned}$$

Note that we cannot directly taking $\delta \to 0$ together with $\limsup_{k,l}$ because *the subsequence we choose in step 3 may vary in* $\varepsilon > 0$. Thus, we must apply the diagonal argument here. Speficially, we take $\delta_1 = 1$ and by step 1 and 3, we can find a small $\varepsilon_1 > 0$ and a subsequence $\{f_{m_k,(1)}\}$ such that

$$\forall 0 < \varepsilon < \varepsilon_1, \quad \limsup_{k,l\to\infty} \|f^\varepsilon_{m_k,(1)} - f^\varepsilon_{m_l,(1)}\|_{L^q(V)} = 0,$$

and thus

$$\limsup_{k,l\to\infty} \|f_{m_k,(1)} - f_{m_l,(1)}\|_{L^q(V)} \le 2.$$

Next, taking $\delta_2 = \frac{1}{2}$ and we can find $\varepsilon_2 \in (0, \varepsilon_1)$ and $\underline{\text{a subsequence } \{f_{m_k,(2)}\} \subset \{f_{m_k,(1)}\}}$ such that

$$\forall 0 < \varepsilon < \varepsilon_2, \quad \limsup_{k,l\to\infty} \|f^\varepsilon_{m_k,(2)} - f^\varepsilon_{m_l,(2)}\|_{L^q(V)} = 0 \Rightarrow \limsup_{k,l\to\infty} \|f_{m_k,(2)} - f_{m_l,(2)}\|_{L^q(V)} \le 1.$$

Repeatedly taking $\delta_n = \frac{1}{n}$, for each $n \in \mathbb{N}^*$, there exist $\varepsilon_n \in (0, \varepsilon_{n-1})$ and $\{f_{m_k,(n)}\} \subset \{f_{m_k,(n-1)}\}$ such that

$$\forall 0 < \varepsilon < \varepsilon_n, \quad \limsup_{k,l\to\infty} \|f^\varepsilon_{m_k,(n)} - f^\varepsilon_{m_l,(n)}\|_{L^q(V)} = 0 \Rightarrow \limsup_{k,l\to\infty} \|f_{m_k,(n)} - f_{m_l,(n)}\|_{L^q(V)} \le \frac{1}{2^{n-1}}.$$

Finally, for each $k \in \mathbb{N}^*$, we let $g_{m_k} := f_{m_k,(k)}$ be the k-th member of our desired subsequence, then we get

$$\limsup_{k,l\to\infty} \|g^\varepsilon_{m_k} - g^\varepsilon_{m_l}\|_{L^q(V)} = \limsup_{k,l\to\infty} \|f^\varepsilon_{m_k,(k)} - f^\varepsilon_{m_l,(l)}\|_{L^q(V)} = 0.$$

□

Remark 1.4.2. Recall that $p^* = \frac{dp}{d-p}$ goes to $+\infty$ as p approaches d, we then expect $W^{1,p}(U) \hookrightarrow\hookrightarrow L^p(U)$ when U is bounded **for all** $1 \le p \le \infty$. In fact, when $p > d$, the proof requires Morrey's inequality and Arzelà-Ascoli lemma. We leave the proof in Exercise 1.4.1. Note also that $W^{1,p}_0(U) \hookrightarrow\hookrightarrow L^p(U)$ even if the boundary is not Lipschitz.

Remark 1.4.3 (Boundedness of U). It should be noted that the boundedness assumption of U is rather important. In fact, if U is unbounded, e.g., U is a strip $\mathbb{R}^{d-1} \times (-\varepsilon, \varepsilon) \subset \mathbb{R}^d$, $d \ge 2$, we may consider

$f_m(\boldsymbol{x}) = f(\boldsymbol{x} + me_1)$ for a given $f \in W^{1,p}(U)$. One can show the weak convergence $f_m \rightharpoonup 0$ in L^p, but its L^q norm is always equal to $\|f\|_{L^q}$. It should also be noted that the boundedness assumption of U usually CANNOT be replaced by $\mathcal{L}^d(U) < \infty$. There do exist some unbounded domains with finite volume in which the compact embedding still holds, but there are also other requirements on the shape of the domain and we refer to Adams-Fournier [1, Chapter 6] for details.

1.4.3 Poincaré's inequality

We now illustrate how the compactness argument can be used to generate new inequalities.

Notation 1.4.1. Given $f : U \to \mathbb{R}$, we define $(f)_U := \fint_U f \,\mathrm{d}\boldsymbol{y} = \frac{1}{\mathcal{L}^d(U)} \int_U f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$ when $\mathcal{L}^d(U) < \infty$.

The following theorem, known as the Poincaré's inequality, is one of the most important conclusions in PDE studies. It asserts that the derivative of a Sobolev function with zero mean must dominate the function itself.

Theorem 1.4.5 (Poincaré's inequality). Let $U \subset \mathbb{R}^d$ be a *bounded, connected* open set with a Lipschitz boundary ∂U. Assume $1 \le p \le \infty$. Then there exists a constant $C > 0$ depending only on d, p, U such that

$$\|f - (f)_U\|_{L^p(U)} \le C\|\nabla f\|_{L^p(U)}, \quad \forall f \in W^{1,p}(U). \tag{1.4.1}$$

Proof. We prove the inequality by contradiction. If the inequality were not true, then for each $k \in \mathbb{N}^*$, we can find a function $f_k \in W^{1,p}(U)$ such that

$$\|f_k - (f_k)_U\|_{L^p(U)} > k\|\nabla f_k\|_{L^p(U)}.$$

We then renormalize $\{f_k - (f_k)_U\}$ in $L^p(U)$ by introducing

$$g_k := \frac{f_k - (f_k)_U}{\|f_k - (f_k)_U\|_{L^p(U)}} \Rightarrow (g_k)_U = 0,\ \|g_k\|_{L^p(U)} = 1 \Rightarrow \|\nabla g_k\|_{L^p(U)} < \frac{1}{k}.$$

By Exercise 1.4.1, we know $W^{1,p}(U) \hookrightarrow\hookrightarrow L^p(U)$, so there exist a subsequence $\{g_{k_j}\}$ and g belonging to $L^p(U)$ such that $\|g_{k_j} - g\|_{L^p(U)} \to 0$. From the definition of g_k, we know $(g)_U = 0$ and $\|g\|_{L^p(U)} = 1$. But, on the other hand, we can prove that $\nabla g = \boldsymbol{0}$ a.e., which together with the connectedness of g and Exercise 1.2.6 implies g is a constant a.e. in U and thus $(g)_U = 0$ forces $g = 0$ a.e. This is definitely a contraction to $\|g\|_{L^p(U)} = 1$.

It remains to verify $\nabla g = \boldsymbol{0}$ a.e. in U. Indeed, for any $\varphi \in C_c^\infty(U)$, we have

$$\int_U g\partial_i\varphi\,\mathrm{d}\boldsymbol{x} = \lim_{j\to\infty}\int_U g_{k_j}\partial_i\varphi\,\mathrm{d}\boldsymbol{x} = -\lim_{j\to\infty}\int_U \partial_i g_{k_j}\varphi\,\mathrm{d}\boldsymbol{x} = 0,$$

which leads to $\nabla g = \boldsymbol{0}$ a.e. in U. □

Remark 1.4.4. This Poincaré's inequality is slightly different from the one in Theorem 1.4.2. Here we drop the vanishing boundary condition $f|_{\partial U} = 0$ but must require U to be bounded (otherwise the compact embedding becomes invalid). When $u \in W_0^{1,p}(U)$, the inequality $\|u\|_{L^p(U)} \le C\|\nabla u\|_{L^p(U)}$, also named after Poincaré, still holds in some unbounded domains (in fact, we only need U is bounded in one direction, such as a strip $\mathbb{R}^{d-1} \times (-1,1)$).

1.4.4 Morrey's embedding

The Sobolev embedding theorem gives the embedding of $W^{1,p}(U)$ into $L^q(U)$ with higher integrability $q \le p^* := \frac{dp}{d-p}$ when $p < d$. In Exercise 1.4.10, we will see $W^{1,d}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ is embedded into a BMO-type space. When $p > d$ and U is bounded, we will see in the section that the a Sobolev function f in $W^{1,p}(U)$ agrees with a $C^{0,\alpha}(\overline{U})$-Hölder continuous function a.e.. with index $\alpha = 1 - \frac{d}{p}$. When $p = \infty$, $C^{0,\alpha}$ is replaced by Lipschitz continuity. Moreover, the continuous version, which are differentiable a.e. due to the absolute continuity, must be the *precise representative* of the given function f defined by $f^*(\boldsymbol{x}) := \lim_{r\to 0}(f)_{\boldsymbol{x},r}$.

We begin with Morrey's inequality.

Theorem 1.4.6. Assume $d < p < \infty$. Define $\alpha := 1 - \frac{d}{p}$. There exists a constant $C > 0$ depending on p, d, such that

$$\|f\|_{C^{0,\alpha}(\mathbb{R}^d)} \le C\|f\|_{W^{1,p}(\mathbb{R}^d)}, \quad \forall f \in C_c^1(\mathbb{R}^d).$$

Proof. There are two inequalities to prove

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})| \le C|\boldsymbol{x}-\boldsymbol{y}|^\alpha \|f\|_{W^{1,p}(\mathbb{R}^d)},$$
$$|f(\boldsymbol{x})| \le C\|f\|_{W^{1,p}(\mathbb{R}^d)}.$$

Let us prove the first inequality and we will see the treatment for the second one is essentially similar.

Fix $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$, we write $r := |\boldsymbol{x}-\boldsymbol{y}|$ and $W := B(\boldsymbol{x}, r) \cap B(\boldsymbol{y}, r)$. Then we have

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})| = \fint_W |f(\boldsymbol{x}) - f(\boldsymbol{y})|\,\mathrm{d}\boldsymbol{z} \le \fint_W |f(\boldsymbol{x}) - f(\boldsymbol{z})|\,\mathrm{d}\boldsymbol{z} + \fint_W |f(\boldsymbol{x}) - f(\boldsymbol{z})|\,\mathrm{d}\boldsymbol{z}.$$

Since $\boldsymbol{z} \in B(\boldsymbol{x}, r)$, we can write $\boldsymbol{z} = \boldsymbol{x} + t\mathbf{w}$ with $\mathbf{w} \in \partial B(\mathbf{0}, 1)$ and $0 \le t < r$ in the first integral. So, we have

$$\begin{aligned}
\fint_W |f(\boldsymbol{x}) - f(\boldsymbol{z})| \,\mathrm{d}\boldsymbol{z} &= \frac{\mathcal{L}^d(B(\boldsymbol{x},r))}{\mathcal{L}^d(W)} \frac{1}{\mathcal{L}^d(B(\boldsymbol{x},r))} \int_{B(\boldsymbol{x},r)} |f(\boldsymbol{x}) - f(\boldsymbol{z})| \,\mathrm{d}\boldsymbol{z} \\
&\le \frac{C}{\mathcal{L}^d(B(\boldsymbol{x},r))} \int_{B(\boldsymbol{x},r)} |f(\boldsymbol{x}) - f(\boldsymbol{z})| \,\mathrm{d}\boldsymbol{z} \\
&= \frac{C}{\mathcal{L}^d(B(\boldsymbol{x},r))} \int_0^r \int_{\partial B(\mathbf{0},1)} |f(\boldsymbol{x}) - f(\boldsymbol{x} + t\mathbf{w})| t^{d-1} \,\mathrm{d}S_{\mathbf{w}} \,\mathrm{d}t \\
&= \frac{C}{\mathcal{L}^d(B(\boldsymbol{x},r))} \int_0^r \int_{\partial B(\mathbf{0},1)} \left| \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} f(\boldsymbol{x} + s\mathbf{w}) \,\mathrm{d}s \right| t^{d-1} \,\mathrm{d}S_{\mathbf{w}} \,\mathrm{d}t.
\end{aligned}$$

Next, we enlarge t to r and convert $\mathbf{w}$ back to $\boldsymbol{z} := \boldsymbol{x} + s\mathbf{w}$ to get

$$\begin{aligned}
\fint_W |f(\boldsymbol{x}) - f(\boldsymbol{z})| \,\mathrm{d}\boldsymbol{z} &\le \frac{C}{\mathcal{L}^d(B(\boldsymbol{x},r))} \int_0^r \left(\underline{\int_{\partial B(\mathbf{0},1)} \int_0^r} \frac{|\nabla f(\boldsymbol{x} + s\mathbf{w})|}{s^{d-1}} \underline{s^{d-1} \,\mathrm{d}s \,\mathrm{d}S_{\mathbf{w}}} \right) t^{d-1} \,\mathrm{d}t \\
&\le \frac{C}{\mathcal{L}^d(B(\boldsymbol{x},r))} \int_0^r \int_{B(\boldsymbol{x},r)} \frac{|\nabla f(\boldsymbol{z})|}{|\boldsymbol{x} - \boldsymbol{z}|^{d-1}} \,\mathrm{d}\boldsymbol{z} \; t^{d-1} \,\mathrm{d}t \\
&= \frac{Cr^d}{d\mathcal{L}^d(B(\boldsymbol{x},r))} \int_{B(\boldsymbol{x},r)} \frac{|\nabla f(\boldsymbol{z})|}{|\boldsymbol{x} - \boldsymbol{z}|^{d-1}} \,\mathrm{d}\boldsymbol{z} \\
&\le C' \int_{B(\boldsymbol{x},r)} \frac{|\nabla f(\boldsymbol{z})|}{|\boldsymbol{x} - \boldsymbol{z}|^{d-1}} \,\mathrm{d}\boldsymbol{z} \le C' \|\nabla f\|_{L^p(B(\boldsymbol{x},r))} \left\| |\boldsymbol{x} - \cdot|^{1-d} \right\|_{L^{p'}(B(\boldsymbol{x},r))}.
\end{aligned}$$

The last term can be directly computed by using polar coordinates

$$\left\| |\boldsymbol{x} - \cdot|^{1-d} \right\|_{L^{p'}(B(\boldsymbol{x},r))} = \left(\int_0^r \int_{\partial B(\mathbf{0},1)} \rho^{d-1} \frac{1}{\rho^{(d-1)p'}} \,\mathrm{d}S \,\mathrm{d}\rho \right)^{\frac{1}{p'}}$$

which is finite if and only if $(d-1)(p'-1) < 1$, which is equivalent to $p > d$. When $p > d$, it is equal to $C_d r^{1-\frac{d}{p}}$. Similar estimate holds if we replace $\boldsymbol{x}$ by $\boldsymbol{y}$. Thus, we have

$$\fint_W |f(\boldsymbol{x}) - f(\boldsymbol{y})| \,\mathrm{d}\boldsymbol{z} \le C'' r^{1-\frac{d}{p}} \|\nabla f\|_{L^p(\mathbb{R}^d)}.$$

Next, we prove the estimate for $|f(\boldsymbol{x})|$. We have

$$|f(\boldsymbol{x})| = \fint_{B(\boldsymbol{x},1)} |f(\boldsymbol{x})| \,\mathrm{d}\boldsymbol{y} \le C \left(\int_{B(\boldsymbol{x},1)} |f(\boldsymbol{x}) - f(\boldsymbol{z})| \,\mathrm{d}\boldsymbol{z} + \int_{B(\boldsymbol{x},1)} |f(\boldsymbol{z})| \,\mathrm{d}\boldsymbol{z} \right).$$

The first integral is controlled in the same way as above

$$\int_{B(\boldsymbol{x},1)} |f(\boldsymbol{x}) - f(\boldsymbol{z})| \,\mathrm{d}\boldsymbol{z} \le C \int_{B(\boldsymbol{x},1)} \frac{|\nabla f(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{z}|^{d-1}} \,\mathrm{d}\boldsymbol{z} \le C \|\nabla f\|_{L^p(\mathbb{R}^d)}.$$

The second integral is bounded by using Hölder's inequality

$$\int_{B(\boldsymbol{x},1)} |f(\boldsymbol{z})| \,\mathrm{d}\boldsymbol{z} \le \|1\|_{L^{p'}(B(\boldsymbol{x},r))} \|f\|_{L^p(B(\boldsymbol{x},r))} \le C \|f\|_{L^p(B(\boldsymbol{x},r))}.$$

Combining the above estimates, we obtain the conclusion of Morrey's inequality. □

We then conclude the following embedding theorem.

Theorem 1.4.7 (Morrey's embedding theorem). Let $U \subset \mathbb{R}^d$ be a bounded open set with a Lipschitz boundary. Assume $d < p < \infty$ and $f \in W^{1,p}(U)$. Then f coincide with its *precise representative* $f^*(\boldsymbol{x}) := \lim_{r\to 0}(f)_{\boldsymbol{x},r}$ a.e. in U. And $f^* \in C^{0,\alpha}(\overline{U})$ with index $\alpha = 1 - \frac{d}{p}$.

Proof. By Theorem 1.3.3 (Sobolev extension theorem), we may assume $f \in W^{1,p}(\mathbb{R}^d)$ is supported in a compact set. Since $d < p < \infty$, we can find a sequence of functions $f_m \in C_c^\infty(\mathbb{R}^d)$ such that $f_m \to f$ in $W^{1,p}(\mathbb{R}^d)$. Using Morrey's inequality, we know that $\{f_m\}$ is also a Cauchy sequence in $C^{0,\alpha}(\mathbb{R}^d)$. Thus, there exists a function $\bar{f} \in C^{0,\alpha}(\mathbb{R}^d)$ such that

$$f_m \to \bar{f} \quad \text{in } C^{0,\alpha}(\mathbb{R}^d).$$

By definition of f_m, we know the limit function $\bar{f}$ coincides with f a.e. in U. Also, applying Morrey's inequality to f_m and taking the limit $m \to \infty$, we get $\bar{f} \in C^{0,\alpha}(\mathbb{R}^d)$ with

$$\|\bar{f}\|_{C^{0,\alpha}(\mathbb{R}^d)} \le C \|\bar{f}\|_{W^{1,p}(\mathbb{R}^d)}.$$

Finally, this $\bar{f}$ must agree with the precise representative f^* everywhere in U. In fact, recall that

$$f^*(\boldsymbol{x}) = \lim_{r\to 0} \frac{1}{\mathcal{L}^d(B(\boldsymbol{x},r))} \int_{B(\boldsymbol{x},r)} f(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y} \overset{f=\bar{f}\text{ a.e.}}{=\!=\!=} \lim_{r\to 0} \frac{1}{\mathcal{L}^d(B(\boldsymbol{x},r))} \int_{B(\boldsymbol{x},r)} \bar{f}(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y}.$$

Since $\bar{f}$ is continuous, we know $f^* = \bar{f}$ everywhere in U thanks to Lebesgue Differentiation Theorem. □

1.4.5 Lipschitz continuity and differentiability

Indeed, the above theorem also holds for $p = \infty$, but the proof appears to be different, as C_c^∞ is not dense in L^∞. In this section, we aim to prove that f is locally Lipschitz continuous in U (not necessarily bounded) if and only if $f \in W^{1,\infty}_{\mathrm{loc}}(U)$.

Theorem 1.4.8 (Lipschitz continuity and $W^{1,\infty}$). Let $U \subset \mathbb{R}^d$ be an open set and $f : U \to \mathbb{R}$ be given. Then

$$f \text{ is locally Lipschitz in } U \qquad \text{if and only if} \qquad f \in W^{1,\infty}_{\text{loc}}(U).$$

Here we say "f is locally Lipschitz in U" to mean f is Lipschitz continuous in any compact subset of U.

Proof. We first prove the "only if" part. Assume f is locally Lipschitz in U, we need to prove that for each $i \in \{1, \cdots, d\}$, the weak ∂_i-derivative of f exists and is a.e. bounded in any compact subset of U. For each $V \Subset W \Subset U$, we choose $0 < h < \text{dist}\,(V, \partial W)$ and define the differential quotient

$$\forall \boldsymbol{x} \in V,\ \ D_i^h(f)(\boldsymbol{x}) := \frac{f(\boldsymbol{x} + he_i) - f(\boldsymbol{x})}{h}.$$

Notice that $\sup\limits_{h>0} |D_i^{-h}(f)| \le \text{Lip}(f|_W) < \infty$. The uniform boundedness and the fact that an L^∞ function must belong to L^p_{loc} for any $1 \le p < \infty$, so there exists a subsequence $h_j \to 0$ and a function $v_i \in L^\infty_{\text{loc}}(U)$ such that

$$D_i^{-h_j}(f) \rightharpoonup v_i \quad \text{weakly in } L^p_{\text{loc}}(U)\ \ 1 < p < \infty.$$

Here we note that $v_i \in L^\infty_{\text{loc}}(U)$ is not a straightforward corollary from the L^p_{loc}-weak convergence. Instead, it can be proved by using the definition of $L^\infty(W)$ norm ($W \Subset U$) and the *uniform-in-p* boundedness of $\|D_i^{-h}(f)\|_{L^p(W)}$. To see this, we denote $L := \text{Lip}(f|_W)$ and $A := \{\boldsymbol{x} \in W : v_i(\boldsymbol{x}) \ge L + \varepsilon\}$ for any fixed $\varepsilon > 0$. Since $1_A \in L^2(W)$, by the weak convergence, we know

$$\int_A D_i^{-h_j}(f)\,\mathrm{d}\boldsymbol{x} = \int_W D_i^{-h_j}(f) 1_A\,\mathrm{d}\boldsymbol{x} \to \int_A v_i\,\mathrm{d}\boldsymbol{x}.$$

Due to $\|D_i^{-h_j}(f)\|_{L^\infty(W)} \le L$, we have $\int_A D_i^{-h_j}(f)\,\mathrm{d}\boldsymbol{x} \le L \cdot \mathcal{L}^d(A)$. On the other hand, $v_i \ge L + \varepsilon$ in A implies $\int_A v_i\,\mathrm{d}\boldsymbol{x} \ge (L + \varepsilon) \cdot \mathcal{L}^d(A)$. This forces $\mathcal{L}^d(A) = 0$, that is, $v_i \le L$ a.e. in W. Similarly, we can prove $v_i \ge -L$ a.e. in W, and therefore we conclude $v_i \in L^\infty(W)$ for any $W \Subset U$.

Now we prove this v_i is exactly the ∂_i-weak derivative. For any $\varphi \in C_c^\infty(V)$, we have

$$\int_U f(\boldsymbol{x}) \frac{\varphi(\boldsymbol{x} + he_i) - \varphi(\boldsymbol{x})}{h}\,\mathrm{d}\boldsymbol{x} = -\int_U D_i^{-h}(f)(\boldsymbol{x})\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

Setting $h = h_j$ and $j \to \infty$, we deduce that

$$\int_U f \partial_i \varphi\,\mathrm{d}\boldsymbol{x} = -\int_U v_i \varphi\,\mathrm{d}\boldsymbol{x}.$$

Next, we prove the "if" part. Given $f \in W^{1,\infty}_{\text{loc}}(U)$ and $\varepsilon_0 > 0$, we can find bounded subsets $V \Subset W \Subset U$ such that $f \in W^{1,\infty}(W)$ and $\text{dist}\,(W, \partial U), \text{dist}\,(V, \partial W) > \varepsilon_0$. The first step is to mollify f. Define $f_\varepsilon := f * \eta_\varepsilon$ for $0 < \varepsilon < \varepsilon_0$.

Claim. The following statements hold.

- $\{f_\varepsilon\}$ uniformly converges in V as $\varepsilon \to 0$.
- Let the limit function be F. Then F is Lipschitz continuous in V.
- $F = f$ a.e. in any bounded $V \Subset U$ (and so F is the precise representative of f).

The proof of the claim is a little bit technical, as the smooth approximation usually does not hold in L^∞. At this step, we do not know the expected limit function for the uniform convergence, so we need to prove $\{f_\varepsilon\}$ is a Cauchy sequence in $L^\infty(V)$. For $\varepsilon, \delta \in (0, \varepsilon_0)$ and $\boldsymbol{x} \in V$, we have

$$\begin{aligned}|f_\varepsilon(\boldsymbol{x}) - f_\delta(\boldsymbol{x})| &= \left|\frac{1}{\varepsilon^d}\int_{B(\mathbf{0},\varepsilon)} \eta\left(\frac{\boldsymbol{y}}{\varepsilon}\right) f(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} - \frac{1}{\delta^d}\int_{B(\mathbf{0},\delta)} \eta\left(\frac{\boldsymbol{y}}{\delta}\right) f(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right| \\ &= \left|\int_{B(\mathbf{0},1)} \eta(\boldsymbol{y})(f(\boldsymbol{x}-\varepsilon\boldsymbol{y}) - f(\boldsymbol{x}-\delta\boldsymbol{y}))\,\mathrm{d}\boldsymbol{y}\right| \le \int_{B(\mathbf{0},1)} \eta(\boldsymbol{y})|f(\boldsymbol{x}-\varepsilon\boldsymbol{y}) - f(\boldsymbol{x}-\delta\boldsymbol{y})|\,\mathrm{d}\boldsymbol{y}.\end{aligned}$$

At this step, we cannot directly commute $\lim\limits_{\varepsilon,\delta\to 0}$ with $\int_{B(\mathbf{0},1)}$ because an L^∞ function may not be continuous a.e. Instead, we shall use $f \in W^{1,\infty}(W) \subset W^{1,p}(W)$ for $1 < p < \infty$ to deduce that there exists $f^* \in C^{0,\alpha}(\overline{W})$ that coincides with f a.e. in W. Then we replace f by f^* under the integral.

$$|f_\varepsilon(\boldsymbol{x}) - f_\delta(\boldsymbol{x})| \le \int_{B(\mathbf{0},1)} \eta(\boldsymbol{y})|f^*(\boldsymbol{x}-\varepsilon\boldsymbol{y}) - f^*(\boldsymbol{x}-\delta\boldsymbol{y})|\,\mathrm{d}\boldsymbol{y}.$$

Now, we use Dominated Convergence Theorem (the dominant function is $2\eta(\boldsymbol{y})\|f\|_{L^\infty(W)}$) to see that

$$\limsup_{\varepsilon,\delta\to 0} |f_\varepsilon(\boldsymbol{x}) - f_\delta(\boldsymbol{x})| \le \int_{B(\mathbf{0},1)} \limsup_{\varepsilon,\delta\to 0} \eta(\boldsymbol{y})|f^*(\boldsymbol{x}-\varepsilon\boldsymbol{y}) - f^*(\boldsymbol{x}-\delta\boldsymbol{y})|\,\mathrm{d}\boldsymbol{y} = 0.$$

Let the limit function be $F(\boldsymbol{x})$. By the uniform convergence $f_\varepsilon \rightrightarrows F$ in V, we know $F \in C(V)$. We now verify the Lipschitz continuity. For any $\boldsymbol{x}, \boldsymbol{y} \in V$ and $\boldsymbol{x} \neq \boldsymbol{y}$, we have

$$|F(\boldsymbol{x}) - F(\boldsymbol{y})| \le |F(\boldsymbol{x}) - f_\varepsilon(\boldsymbol{x})| + |f_\varepsilon(\boldsymbol{x}) - f_\varepsilon(\boldsymbol{y})| + |f_\varepsilon(\boldsymbol{y}) - F(\boldsymbol{y})|.$$

Thus, it remains to check for each $\varepsilon > 0$, the function f_ε is Lipschitz in V with a uniform-in-ε Lipschitz constant. We check that

$$|f_\varepsilon(\boldsymbol{x}) - f_\varepsilon(\boldsymbol{y})| = \left|\int_0^1 \nabla f_\varepsilon(t\boldsymbol{x} + (1-t)\boldsymbol{y}) \cdot (\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}t\right| \le \|\nabla f_\varepsilon\|_{L^\infty(V)}|\boldsymbol{x}-\boldsymbol{y}|.$$

Since $\|\nabla f_\varepsilon\|_{L^\infty(V)} \le \|\eta_\varepsilon\|_{L^1}\|\nabla f\|_{L^\infty} = \|\nabla f\|_{L^\infty}$, we get f_ε is Lipschitz in V with a uniform-in-ε upper bound for its Lipschitz constant. So, we also get $|F(\boldsymbol{x}) - F(\boldsymbol{y})| \le \|\nabla f\|_{L^\infty(V)}|\boldsymbol{x}-\boldsymbol{y}|$ for any $\boldsymbol{x}, \boldsymbol{y} \in V$. Finally, $F = f^*$ everywhere in V due to the continuity. □

Remark 1.4.5. It should be noted that one cannot assert that f is Lipschitz continuous in U if and only if f agrees with a Lipschitz continuous function in U. That is, the word "locally" cannot be removed, even if U is bounded. See Exercise 1.4.11 for a counterexample. In fact, $W^{1,\infty}(U) = C^{0,1}(U)$ holds for any *quasi-convex* domain U, that is, any two points $a, b \in U$ can be joined by a curve γ of length at most $M|a-b|$ for some $M > 0$ independent of a, b.

Exercise 1.4

Exercise 1.4.1. Prove that $W^{1,p}(U)$ is compactly embedded in $L^p(U)$ for any $1 \le p \le \infty$.

Exercise 1.4.2. Let $d \ge 2$. Verify that $u(\boldsymbol{x}) := \ln\ln(1 + \frac{1}{|\boldsymbol{x}|})$ belongs to $W^{1,d}(B)$ for $B =$ unit ball, but this u is definitely unbounded.

Exercise 1.4.3. Let $U \subset \mathbb{R}^d$ be a bounded *domain* and $u \in H^1(U)$ satisfy that the set $Z := \{\boldsymbol{x} \in U | u(\boldsymbol{x}) = 0\}$ satisfies $\mathcal{L}^d(Z) \ge \alpha \mathcal{L}^d(U)$ for some $\alpha \in (0,1)$. Prove that there exists a constant $C > 0$ only depending on d, α such that

$$\int_U u^2 \,\mathrm{d}\boldsymbol{x} \le C \int_U |\nabla u|^2 \,\mathrm{d}\boldsymbol{x}.$$

(Hint: In $U \backslash Z$, write $u^2 = (u - (u)_U + (u)_U)^2$ and apply Poincaré's inequality to $(u - (u)_U)^2$. The contribution of $(u)_U^2$ will be absorbed by the left side of the desired inequality because the measure of $U \backslash Z$ is strictly less than that of U.)

Exercise 1.4.4. Let $d \ge 3$ and $r > 0$, $B_r := B(\mathbf{0}, r)$. Suppose $u \in H^1(B_r)$. Prove that $\frac{u}{|\boldsymbol{x}|} \in L^2(B_r)$ with the estimate

$$\int_{B_r} \frac{u^2}{|\boldsymbol{x}|^2} \,\mathrm{d}\boldsymbol{x} \le C \int_{B_r} |\nabla u|^2 + \frac{u^2}{r^2} \,\mathrm{d}\boldsymbol{x}.$$

(Hint: First use $\nabla(|\boldsymbol{x}|^{-1}) = -\frac{\boldsymbol{x}}{|\boldsymbol{x}|^3}$ and integrate by parts. Then use $r \int_{\partial B_r} u^2 \,\mathrm{d}S = \int_{B_r} \nabla \cdot (\boldsymbol{x} u^2) \,\mathrm{d}\boldsymbol{x}$ to control the boundary term generated when integrating by parts.)

Exercise 1.4.5. Prove the following version of Hardy's inequality.

(1) Assume $u \in C_c^\infty(\mathbb{R}^d)$ and $\mathbf{F} : \mathbb{R}^d \to \mathbb{R}^d$ is a vector field and it is C^1 in $\mathbb{R}^d \backslash \{\mathbf{0}\}$. Prove that

$$\int_{\mathbb{R}^d} u^2 \mathrm{div}\, \mathbf{F} \,\mathrm{d}\boldsymbol{x} = -2 \int_{\mathbb{R}^d} \nabla u \cdot (u\mathbf{F}) \,\mathrm{d}\boldsymbol{x}.$$

(2) Let $d \ge 3$, $F(\boldsymbol{x}) = \frac{\boldsymbol{x}}{|\boldsymbol{x}|^2}$ and $f \in H^1(\mathbb{R}^d)$. Prove that

$$\frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{f^2}{|\boldsymbol{x}|^2} \,\mathrm{d}\boldsymbol{x} \le \int_{\mathbb{R}^d} |\nabla f|^2 \,\mathrm{d}\boldsymbol{x}.$$

Exercise 1.4.6. Let $U \subset \mathbb{R}^d$ be a bounded domain with smooth boundary and let $B \Subset U$ be an open ball. For every $\varepsilon \in (0,1)$, we assume u_ε is a smooth solution to the equation

$$\begin{cases} -\Delta u_\varepsilon + \varepsilon^{-1}(u_\varepsilon - f)\mathbf{1}_B = 0 & \text{in } U, \\ u_\varepsilon = 0 & \text{on } \partial U, \end{cases} \tag{1.4.2}$$

where $f \in H_0^1(U)$ is a given function. $\mathbf{1}_B$ is defined by $\mathbf{1}_B(\boldsymbol{x}) = 1$ if $\boldsymbol{x} \in B$ and $\mathbf{1}_B(\boldsymbol{x}) = 0$ if $\boldsymbol{x} \notin B$.

(1) Show that $\|\nabla u_\varepsilon\|_{L^2(U)}$ is bounded uniformly in ε.
(2) Prove that $u_\varepsilon \to f$ in $L^2(B)$.

Exercise 1.4.7. Let $\Omega := \{(x,y) \in \mathbb{R}^2 | 0 < x < 1, 0 < y < x^4\}$. Show that the function $f(x,y) = \frac{1}{x}$ belongs to $H^1(\Omega)$ but does not belong to $L^5(\Omega)$. Is this consistent with the Sobolev embedding theorem? Justify your answer.

Exercise 1.4.8. Let $U \subset \mathbb{R}^d$ be a bounded domain with smooth boundary. Prove that $H^2(U)$ is compactly embedded into $H^1(U)$. And for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$\|\nabla u\|_{L^2(U)} \le \varepsilon \|u\|_{H^2(U)} + C_\varepsilon \|u\|_{L^2(U)}, \quad \forall u \in H^2(U).$$

Exercise 1.4.9. Assume $1 \le p \le \infty$. Given $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, we define $(f)_{\boldsymbol{x},r} := \fint_{B(\boldsymbol{x},r)} f$. Prove that there exists a constant $C > 0$ depending on d, p, such that for each ball $B(\boldsymbol{x}, r) \subset \mathbb{R}^d$ and each $f \in W^{1,p}(B(\boldsymbol{x}, r))$, the following inequality holds

$$\|f - (f)_{\boldsymbol{x},r}\|_{L^p(B(\boldsymbol{x},r))} \le Cr \|\nabla u\|_{L^p(B(\boldsymbol{x},r))}.$$

(Hint: First prove the conclusion for the unit ball. For a general ball, consider $v(\boldsymbol{y}) := f(\boldsymbol{x} + r\boldsymbol{y})$ for $\boldsymbol{y} \in B(\mathbf{0}, 1)$.)

Exercise 1.4.10. Let $f \in W^{1,d}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Prove that $f \in BMO(\mathbb{R}^d)$ with the estimate

$$\fint_{B(\boldsymbol{x},r)} |f - (f)_{\boldsymbol{x},r}| \, \mathrm{d}\boldsymbol{y} \le C \|\nabla u\|_{L^d(\mathbb{R}^d)},$$

where $BMO(\mathbb{R}^d)$ is the function space of bounded mean oscillation with the seminorm

$$[u]_{BMO(\mathbb{R}^d)} := \sup_{B(\boldsymbol{x},r)} \fint_{B(\boldsymbol{x},r)} |f - (f)_{\boldsymbol{x},r}| \, \mathrm{d}\boldsymbol{y} < \infty.$$

Exercise 1.4.11. Let $U = B(\mathbf{0}, 1) \backslash \{(x, y) \in (\mathbf{0}, 1) | x \ge 0, \ y = 0\}$ be an open unit disk in $\mathbb{R}^2$ with a slit in positive x-axis. Define

$$u(x, y) = (\max\{0, x\})^2 \max\{\operatorname{sgn} y, 0\}.$$

Prove that $u \in W^{1,\infty}(U)$ but is not Lipschitz continuous in U.

Exercise 1.4.12. Assume $1 \le p \le \infty$. Let $U \subset \mathbb{R}^d$ be a bounded *domain* with a Lipschitz boundary, $S \subset \partial U$ is a part of the boundary with positive $(d-1)$-dimensional Hausdorff measure. Then there exists a constant $C > 0$ depending on p, S, U, such that

$$\|u\|_{L^p(U)} \le C \|\nabla u\|_{L^p(U)}$$

holds for all $u \in W^{1,p}(U)$ with $\operatorname{Tr} u|_S = 0$.

Exercise 1.4.13. Consider the eigenvalue problem for p-Laplace equation

$$\begin{cases} -\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u & x \in U \\ u = 0 & x \in \partial U. \end{cases}$$

Here $1 < p < \infty$, $\lambda \in \mathbb{R}$ is a parameter and $U \subset \mathbb{R}^d$ is a bouned domain. Prove that if this problem has a nonzero solution u, then the corresponding eigenvalue λ satisfies the estimate

$$\lambda \geq C\mathcal{L}^d(U)^{-\frac{p}{d}},$$

where $C > 0$ only depends on p, d. (Hint: Discuss 3 different cases: $p < d$, $p \geq d \geq 2$ and $p \geq d = 1$.)

1.5 General Sobolev inequalities

Finally, we record the general Sobolev inequalities for $W^{k,p}(U)$ ($k \in \mathbb{N}^*$). The proof is based on induction on k and a parallel argument to either Theorem 1.4.1 or Theorem 1.4.6 and we skip it here.

Notation 1.5.1. For $x \in \mathbb{R}$, we define $[x]$ to be the largest integer less than or equal to x and define $\{x\} := x - [x]$ to be its decimal part.

Theorem 1.5.1 (General Sobolev inequalities). Let $U \subset \mathbb{R}^d$ be a bounded open set with a Lipschitz boundary and $f \in W^{k,p}(U)$.

(1) If $k < \frac{d}{p}$, then $f \in L^q(U)$ with $\frac{1}{q} = \frac{1}{p} - \frac{k}{d}$ (equivalently, $q = \frac{dp}{d-kp}$. Also, we have the estimate

$$\|f\|_{L^q(U)} \leq C(d,k,p,U)\|f\|_{W^{k,p}(U)}.$$

(2) If $k > \frac{d}{p}$, then f coincides with its precise representative f^* a.e. in U and $f^* \in C^{k-[\frac{d}{p}]-1,\alpha}(\overline{U})$ with

$$\alpha = \begin{cases} 1 - \left\{\frac{d}{p}\right\} & \frac{d}{p} \notin \mathbb{Z}, \\ \text{any real number in } (0,1) & \frac{d}{p} \in \mathbb{Z}. \end{cases}$$

Also, we have the estimate

$$\|f^*\|_{C^{k-[\frac{d}{p}]-1,\alpha}(\overline{U})} \leq C(d,k,p,\alpha,U)\|f\|_{W^{k,p}(U)}$$

It should be noted that a special case of (2) is $H^k(U) \subset L^\infty(U)$, $\quad k > \frac{d}{2}$ with the estimate

$$\|f\|_{L^\infty(U)} \leq C\|f\|_{H^k(U)}. \tag{1.5.1}$$

which will be repeatedly used throughout this lecture notes.We will revisit and refine this conclusion and Exercise 1.5.1 in Chapter 6.

Exercise 1.5

Exercise 1.5.1. Let $U \subset \mathbb{R}^d$ be a bounded open set with a Lipschitz boundary and $f, g \in H^k(U)$ for a positive integer $k > \frac{d}{2}$. Prove that there exists a constant $C > 0$ depending on k, d, U such that

$$\|fg\|_{H^k(U)} \leq C\|f\|_{H^k(U)}\|g\|_{H^k(U)}. \tag{1.5.2}$$

Remark 1.5.1. For a general $k \in \mathbb{N}$ and $f, g \in H^k(U) \cap L^\infty(U)$, we actually have Moser-type inequality (also known as a special case of Kato-Ponce type inequality in "fractional Leibniz rule"):

$$\|fg\|_{H^k(U)} \leq C\left(\|f\|_{H^k(U)}\|g\|_{L^\infty(U)} + \|g\|_{H^k(U)}\|f\|_{L^\infty(U)}\right). \tag{1.5.3}$$

The estimates (1.5.1)-(1.5.3) also hold in unbounded domains or $\mathbb{R}^d$ and k is not necessarily an integer, but the proofs heavily rely on Fourier analysis. We will revisit this conclusion in Chapter 6.

Chapter 2 Linear Elliptic PDEs

In this chapter, we consider the boundary-value problem

$$Lu = f \text{ in } U, \qquad u = 0 \text{ on } \partial U. \tag{2.0.1}$$

Here $U \subset \mathbb{R}^d$ is a bounded open set, $u : \overline{U} \to \mathbb{R}$ is the unknown. The function $f : U \to \mathbb{R}$ is given, and L is a second-order partial differential operator having either the divergence form (mostly used to prove the existence)

$$Lu = -\sum_{i,j=1}^{d} \partial_j(a^{ij}(\boldsymbol{x})\partial_i u) + \sum_{i=1}^{d} b^i(\boldsymbol{x})\partial_i u + c(\boldsymbol{x})u \tag{2.0.2}$$

or the non-divergence form (mostly used in the maximum principle)

$$Lu = -\sum_{i,j=1}^{d} a^{ij}(\boldsymbol{x})\partial_i\partial_j u + \sum_{i=1}^{d} b^i(\boldsymbol{x})\partial_i u + c(\boldsymbol{x})u \tag{2.0.3}$$

for given coefficient functions $a^{ij}, b^i, c,\ (1 \le i, j \le d)$. The matrix $[a_{ij}]$ is assumed to be symmetric, that is, $a^{ij} = a^{ji}$. The boundary condition $u = 0$ on ∂U is called **Dirichlet boundary condition.**

Definition 2.0.1. We say the differential operator L defined in either (2.0.2) or (2.0.3) is **(uniformly) elliptic** if there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{d} a^{ij}(\boldsymbol{x})\xi_i\xi_j \ge \theta|\boldsymbol{\xi}|^2 \quad \text{a.e. } \boldsymbol{x} \in U,\ \forall \boldsymbol{\xi} \in \mathbb{R}^d. \tag{2.0.4}$$

Ellipticity means that for each $\boldsymbol{x} \in U$, the matrix $[a^{ij}(\boldsymbol{x})]$ is positive definite with smallest eigenvalue greater than or equal to θ. An obvious example is $a^{ij} = \delta^{ij}$ and $b^i = c = 0$ in which case the operator L is $-\Delta$.

From now on, we will follow Einstein's summation convention, that is, repeated indices represent the summation over them. For example, (2.0.2)-(2.0.3) are now written as

$$Lu = -\partial_j(a^{ij}\partial_i u) + b^i\partial_i u + cu \text{ or } Lu = -a^{ij}\partial_i\partial_j u + b^i\partial_i u + cu.$$

2.1 Weak solution and Sobolev space H^{-1}

2.1.1 Definition of weak solution

We assume $a^{ij}, b^i, c \in L^\infty(U)$ and $f \in L^2(U)$ in (2.0.1). As stated in the beginning of Chapter 1, it is usually not easy to directly prove the existence of the classical solution when the coefficients and the source term are not so regular. Instead, we alternatively solve the equation (2.0.1) *in the weak sense.* Specifically, let $v \in C_c^\infty(U)$ and u be a smooth solution. Then integration by parts leads to

$$\int_U a^{ij}\partial_i u \partial_j v + b^i \partial_i u v + cuv \,\mathrm{d}\boldsymbol{x} = \int_U fv \,\mathrm{d}\boldsymbol{x}. \tag{2.1.1}$$

By using smooth approximation, we can prove the same identity holds for any $v \in H_0^1(U)$. Also, this identity makes sense if $u \in H_0^1(U)$. Thus, we may find a "weak" solution to equation (2.0.1) in $H_0^1(U)$.

Definition 2.1.1. The **bilinear form** $B[\cdot,\cdot]$ associated with the elliptic operator in the divergence form (2.0.2) is

$$B[u,v] := \int_U a^{ij}\partial_i u \partial_j v + b^i \partial_i u v + cuv \,\mathrm{d}\boldsymbol{x}, \quad \forall u, v \in H_0^1(U). \tag{2.1.2}$$

We say that $u \in H_0^1(U)$ is a weak solution to (2.0.1) if

$$B[u,v] = (f,v)_{L^2(U)}, \quad \forall v \in H_0^1(U), \tag{2.1.3}$$

where $(\cdot,\cdot)_{L^2(U)}$ denotes the inner product in $L^2(U)$.

More generally, we will also encounter the case that $f \in H^{-1}(U)$ (the dual space of $H_0^1(U)$). In this case, we shall replace the right side of (2.1.3) by $\langle f, v\rangle$ where $\langle\cdot,\cdot\rangle$ is the pairing of $H^{-1}(U)$ and $H_0^1(U)$.

2.1.2 Sobolev space $H^{-1}(U)$

Now, we introduce the Sobolev space $H^{-1}(U)$.

Definition 2.1.2. We define $H^{-1}(U)$ to be the dual space of $H_0^1(U)$. That is, $f \in H_0^1(U)$ means f is a bounded linear functional on $H_0^1(U)$. The norm is defined by

$$\|f\|_{H^{-1}(U)} := \sup\left\{\langle f, u\rangle \middle| u \in H_0^1(U),\ \|u\|_{H_0^1(U)} \le 1\right\}.$$

It should be noted that we DO NOT identify $H_0^1(U)$ with its dual as in the Riesz Representation Theorem for Hilbert spaces. Actually, we have

$$H_0^1(U) \subset L^2(U) \subset H^{-1}(U)$$

and the first inclusion can be replaced by the compact embedding if U is bounded.

We have the following theorem.

Theorem 2.1.1 (Characterization of H^{-1}). Assume $f \in H^{-1}(U)$. Then there exist a sequence of functions $f^0, f^1, \cdots, f^d \in L^2(U)$ such that

$$\langle f, v\rangle = \int_U f^0 v + f^i \partial_i v \, \mathrm{d}\boldsymbol{x} \qquad \forall v \in H_0^1(U). \tag{2.1.4}$$

We write $f = f^0 - \sum_{i=1}^{d} \partial_i f^i$ whenever (2.1.4) holds. Furthermore, we have

$$\|f\|_{H^{-1}(U)} = \inf \left\{ \left(\int_U \sum_{i=0}^{d} |f^i|^2 \, \mathrm{d}\boldsymbol{x} \right)^{\frac{1}{2}} \middle| f \text{ satisfies (2.1.4)} \right\}.$$

In particular, we have

$$(v, u)_{L^2(U)} = \langle v, u\rangle$$

for all $u \in H_0^1(U)$ if we identify $v \in L^2(U)$ as an element in $H^{-1}(U)$.

Proof. Given $f \in H^{-1}(U)$, we shall construct $f^0, f^1, \cdots, f^d$ satisfying (2.1.4). This is essentially the application of Riesz Representation Theorem. Since f is a bounded linear functional on $H_0^1(U)$ and $H_0^1(U)$ is a Hilbert space, we know there exists an element $u \in H_0^1(U)$ such that $(u, v)_{H_0^1(U)} = \langle f, v\rangle$ holds for all $v \in H_0^1(U)$. Recall that the inner product of $H_0^1(U)$ can be defined by

$$(u, v) = \int_U \nabla u \cdot \nabla v + uv \, \mathrm{d}\boldsymbol{x}.$$

Thus, setting $f^0 = u$ and $f^i = \partial_i u$ ($1 \le i \le d$) leads to the desired result.

To prove the equivalent definition of $\|f\|_{H^{-1}(U)}$ for a given $f \in H^{-1}(U)$, we assume $g^0, g^1, \cdots, g^d \in L^2(U)$ satisfy

$$\langle f, v\rangle = \int_U g^0 v + g^i \partial_i v \, \mathrm{d}\boldsymbol{x}.$$

Setting $v = u$ in the previously defined $H_0^1(U)$ inner product, we obtain

$$\int_U |\nabla u|^2 + u^2 \, \mathrm{d}\boldsymbol{x} \le \int_U \sum_i |g^i|^2 \, \mathrm{d}\boldsymbol{x},$$

and thus

$$\int_U \sum_{i=0}^{d} |f^i|^2 \, \mathrm{d}\boldsymbol{x} \le \int_U \sum_{i=0}^{d} |g^i|^2 \, \mathrm{d}\boldsymbol{x}.$$

Finally, if $\|v\|_{H_0^1(U)} \le 1$, then $|\langle f, v\rangle|$ is bounded by the left side of the last inequality. Taking supremum over all $v \in H_0^1(U)$, we obtain that

$$\|f\|_{H^{-1}(U)}^2 \le \sum_i \int_U |f^i|^2 \, \mathrm{d}\boldsymbol{x}.$$

On the other hand, setting $v = \frac{u}{\|u\|_{H_0^1(U)}}$ allows us to reach the equality. □

Exercise 2.1

Exercise 2.1.1. Let $U \subset \mathbb{R}^d$ be a bounded open set with a Lipschitz boundary.

(1) Let $\{v_n\} \subset H_0^1(U)$ satisfy $\|v_n\|_{H_0^1(U)} \le 1$. Prove that there exist a subsequence $\{v_{n_k}\}$ and $v \in H_0^1(U)$ such that $\|v_{n_k} - v\|_{H^{-1}(U)} \to 0$.

(2) Let $v \in H_0^1(U)$, $\|v\|_{H_0^1(U)} = 1$. Prove that $v \in H^{-1}(U)$ and for any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ depending on ε such that

$$\|v\|_{L^2(U)} \le \varepsilon + C(\varepsilon)\|v\|_{H^{-1}(U)}.$$

Exercise 2.1.2. Is it true that $H^{-1}(\mathbb{R}^d) = (H^1(\mathbb{R}^d))'$? Here X' is the dual space of the Banach space X.

2.2 Existence theorem 1: Lax-Milgram theorem

In view of Definition 2.1.1, given $f \in L^2(U)$, we shall prove that there exists a unique $u \in H_0^1(U)$ such that (2.1.3) holds for all $v \in H_0^1(U)$. This motivates us to adopt the Lax-Milgram Theorem.

Theorem 2.2.1. Let H be a given Hilbert space with inner product $(\cdot,\cdot)$, norm $\|\cdot\|$ and pairing (with its dual space) $\langle\cdot,\cdot\rangle$. Assume $B : H \times H \to \mathbb{R}$ is a bilinear mapping and there exist constants $\alpha, \beta > 0$ such that

- (Boundedness) $|B[u, v]| \le \alpha\|u\|\|v\|$ for all $u, v \in H$;
- (Coercivity) $|B[u, u]| \ge \beta\|u\|^2$ for all $u \in H$.

Let $f : H \to \mathbb{R}$ be a bounded linear functional on H. Then there exists a unique $u \in H$ such that $B[u, v] = \langle f, v\rangle$ holds for all $v \in H$.

Proof. **Rough idea:** The desired conclusion is somewhat similar to the Riesz Representation Theorem. In fact, since f is a bounded linear functional on H, the Riesz Representation Theorem shows that there exists some $w \in H$ such that

$$\langle f, v\rangle = (w, v) \quad \forall v \in H.$$

On the other hand, we may define a linear operator $A : H \to H$ by $B[u, v] = (Au, v)$. If we can prove $A : H \to H$ is one-to-one and onto, then given f, we can produce the element u by defining $u = A^{-1}w$ as desired.

Step 1: Define the map A. By the boundedness assumption, we know that for any fixed element $u \in H$, the mapping $v \longmapsto B[u, v]$ is a bounded linear functional on H. Thus, using the Riesz Rrepresentation Theorem, we know there exists a unique element $w \in H$ such that $B[u, v] = (w, v)$ holds for any $v \in H$. So, we define $A : H \to H$ by $Au := w$, that is, $B[u, v] = (Au, v)$ for $u, v \in H$.

Step 2: A is a bounded linear operator. It is straightforward to see that A is linear and we skip the details here. The boundedness also follows from that of $B[\cdot, \cdot]$:

$$\|Au\|^2 = (Au, Au) = B[u, Au] \leq \alpha\|u\|\|Au\| \Rightarrow \|Au\| \leq \alpha\|u\| \quad \forall u \in H.$$

Step 3: A is 1-1 and onto. From the coercivity assumption, we have

$$\beta\|u\|^2 \leq B[u, u] = (Au, u) \leq \|Au\|\|u\| \Rightarrow \beta\|u\| \leq \|Au\|.$$

Thus, A is 1-1 and $R(A)$ (the range of A) is closed in H. To prove $R(A) = H$, it suffices to verify $(R(A))^\perp = 0$. In fact, if there exists a nonzero element $w \in (R(A))^\perp$, then we would obtain that $\beta\|w\|^2 \leq B[w, w] = (Aw, w) = 0$. Thus, we prove that A is 1-1 and onto on H.

Step 4: The existence of u. We now go back to the proof of this theorem. Given $f \in \mathcal{L}(H)$, by Riesz Representation Theorem, there exists some $w \in H$ satisfying $\langle f, v\rangle = (w, v)$ for all $v \in H$. Now, we can construct the desired u by defining $Au = w$ (or $u := A^{-1}w$). Then we get

$$B[u, v] = (Au, v) = (w, v) = \langle f, v\rangle, \quad \forall v \in H.$$

The uniqueness is easy to prove thanks to coercivity and we leave it to readers. □

Now, we wish to apply Lax-Milgram Theorem to the elliptic equation (2.0.1). By straightforward analysis, we can establish the following energy estimates.

Theorem 2.2.2 (Energy estimates). For the elliptic equation (2.0.1) and its associated bilinear form (2.1.2), we can establish the following estimates: There exist constants $\alpha, \beta > 0$ and $\gamma \geq 0$ such that

- $|B[u, v]| \leq \alpha\|u\|_{H_0^1(U)}\|v\|_{H_0^1(U)}$.
- $\beta\|u\|^2_{H_0^1(U)} \leq B[u, u] + \gamma\|u\|^2_{L^2(U)}$.

Proof. Let us recall the concrete form of $B[u, v]$.

$$B[u, v] = \int_U a^{ij}\partial_i u\partial_j v + b^i\partial_i uv + cuv\, \mathrm{d}\boldsymbol{x}, \quad u, v \in H_0^1(U).$$

So, we get

$$\begin{aligned}|B[u, v]| &\leq \|a^{ij}\|_{L^\infty(U)}\|\partial_i u\|_{L^2(U)}\|\partial_j v\|_{L^2(U)} + \|b^i\|_{L^\infty(U)}\|\partial_i u\|_{L^2(U)}\|v\|_{L^2(U)} + \|c\|_{L^\infty(U)}\|u\|_{L^2(U)}\|v\|_{L^2(U)} \\ &\leq C(\|u\|_{L^2(U)} + \|\nabla u\|_{L^2(U)})(\|v\|_{L^2(U)} + \|\nabla v\|_{L^2(U)}) \leq \alpha\|u\|_{H_0^1(U)}\|v\|_{H_0^1(U)}.\end{aligned}$$

The second inequality is established in a similar manner.

$$\begin{aligned}
B[u,u] &= \int_U a^{ij}\partial_i u \partial_j u + b^i \partial_i u\, u + cu^2 \,\mathrm{d}\boldsymbol{x} \\
&\geq \theta \int_U |\nabla u|^2 \,\mathrm{d}\boldsymbol{x} - \|b^i\|_{L^\infty} \|\partial_i u\|_{L^2(U)} \|u\|_{L^2(U)} - \|c\|_{L^\infty(U)} \|u\|^2_{L^2(U)} \\
&\geq \theta \int_U |\nabla u|^2 \,\mathrm{d}\boldsymbol{x} - \varepsilon \|\nabla u\|^2_{L^2(U)} - \left(\frac{C_1}{\varepsilon} + C_2\right) \|u\|^2_{L^2(U)}.
\end{aligned}$$

Here, we use the uniform ellipticity condition and Young's inequality. Taking $\varepsilon = \frac{\theta}{2}$, we obtain that

$$B[u,u] \geq \beta \|u\|^2_{H^1_0(U)} - \gamma \|u\|^2_{L^2(U)}$$

for some $\beta > 0$ and $\gamma \geq 0$. Note that $\gamma = 0$ can indeed be attained (e.g., $b^i = c = 0$). □

Note that if $\gamma > 0$, then $B[\cdot,\cdot]$ does not necessarily satisfy the assumptions of Lax-Milgram Theorem. When we use Lax-Milgram Theorem to prove the existence of equation (2.0.1), there should be some extra constraints imposed on the elliptic operator.

Theorem 2.2.3 (First existence theorem for weak solutions). There is a number $\gamma \geq 0$ (the one obtained in Theorem 2.2.2) such that for each $\mu \geq \gamma$ and each $f \in L^2(U)$, there exists a unique weak solution $u \in H^1_0(U)$ to the boundary-value problem

$$Lu + \mu u = f \text{ in } U, \qquad u = 0 \text{ on } \partial U. \tag{2.2.1}$$

Proof. For equation (2.2.1), we shall define its bilinear form (corresponding to $L_\mu := L + \mu I$) by

$$B_\mu[u,v] := B[u,v] + \mu(u,v)_{L^2(U)} \quad \forall u, v \in H^1_0(U).$$

Then by $\mu \geq \gamma$ and Theorem 2.2.2, we know $B_\mu[\cdot,\cdot]$ satisfies the assumption of Lax-Milgram Theorem.

Given $f \in L^2(U)$, we can identify f as an element in $H^{-1}(U)$ (the dual space of $H^1_0(U)$) and so the pairing $\langle f, v\rangle$ is equal to the inner product $(f,v)_{L^2(U)}$. By Lax-Milgram Theorem, there exists a unique function $u \in H^1_0(U)$ satisfying $B_\mu[u,v] = \langle f, v\rangle$ for all $v \in H^1_0(U)$, that is, u is the unique weak solution to (2.2.1). □

Remark 2.2.1. We can similarly prove the existence of the weak solution to $Lu+\mu u = f$ (with Dirichlet boundary condition) for $f \in H^{-1}(U)$, as it is enough to note $\langle f, v\rangle = \int_U f^0 v + f^i \partial_i v \,\mathrm{d}\boldsymbol{x}$ is a bounded linear functional on $H^1_0(U)$. In particular, this existence theorem shows that the mapping

$$L_\mu := L + \mu I \;:\; H^1_0(U) \to H^{-1}(U) \quad (\mu \geq \gamma)$$

is an isomorphism.

Exercise 2.2

In this section, we assume $U \subset \mathbb{R}^d$ to be a bounded open set with a smooth boundary. The coefficients a^{ij}, b^i, c are smooth and satisfy the uniform ellipticity.

Exercise 2.2.1. Let $Lu = -\partial_j(a^{ij}\partial_i u) + cu$. Prove that there exists a constant $\mu > 0$ such that the corresponding bilinear form $B[\cdot,\cdot]$ satisfies the hypothesis of Lax-Milgram Theorem whenever $c(\boldsymbol{x}) \geq -\mu$ for $\boldsymbol{x} \in U$.

Exercise 2.2.2. A function $u \in H_0^2(U)$ is a weak solution to the biharmonic equation

$$\Delta^2 u = f \text{ in } U, \qquad u = \frac{\partial u}{\partial N} = 0 \text{ on } \partial U$$

provided that $\int_U \Delta u \Delta v \, \mathrm{d}\boldsymbol{x} = \int_U f v \, \mathrm{d}\boldsymbol{x}$ holds for all $v \in H_0^2(U)$. Given $f \in L^2(U)$, prove the existence and the uniqueness of the weak solution.

Exercise 2.2.3. Let U be connected. A function $u \in H^1(U)$ is a weak solution to the Poisson's equation with Neumann boundary condition

$$-\Delta u = f \text{ in } U, \qquad \frac{\partial u}{\partial N} = 0 \text{ on } \partial U$$

provided that $\int_U \nabla u \cdot \nabla v \, \mathrm{d}\boldsymbol{x} = \int_U f v \, \mathrm{d}\boldsymbol{x}$ holds for all $v \in H^1(U)$. Given $f \in L^2(U)$, prove that there exists a weak solution if and only if $\int_U f \, \mathrm{d}\boldsymbol{x} = 0$.

Exercise 2.2.4. Consider the Poisson's equation with Robin boundary condition

$$-\Delta u = f \text{ in } U, \qquad u + \frac{\partial u}{\partial N} = 0 \text{ on } \partial U$$

Please define the weak solution $u \in H^1(U)$ to this problem and dicuss the existence and uniqueness for a given $f \in L^2(U)$.

(Hint: To prove the coercivity of the bilinear form, we may use trick similar to the proof of Poincaré's inequality in Theorem 1.4.5.)

Exercise 2.2.5. Let U be connected and assume ∂U consists of two disjoint closed sets Γ_1, Γ_2. Consider the Poisson's equation with mixed Dirichlet-Neumann boundary condition

$$-\Delta u = f \text{ in } U, \qquad u = 0 \text{ on } \Gamma_1, \quad \frac{\partial u}{\partial N} = 0 \text{ on } \Gamma_2.$$

Please define the weak solution $u \in H^1(U)$ to this problem and dicuss the existence and uniqueness for a given $f \in L^2(U)$.

(Hint: Choose test functions from $H := \{v \in H^1(U) | \operatorname{Tr} v|_{\Gamma_1} = 0\}$.)

Exercise 2.2.6. Let $u \in H^1(U)$ be a bounded weak solution to $-\partial_j(a^{ij}\partial_i u) = 0$ in U. Let $\phi : \mathbb{R} \to \mathbb{R}$ be convex and smooth and set $w = \phi(u)$. Prove that w is a weak subsolution, that is, $B[w, v] \leq 0$ for any $v \in H_0^1(U)$, $v \geq 0$.

Exercise 2.2.7. Given a variational proof for the existence of weak solution to $Lu = 0$ in U, $u = g$ on ∂U with L defined in (2.0.2), $b^i = c = 0$ and $g \in L^2(\partial U)$. Let

$$I[w] := \int_U \frac{1}{2} a^{ij} \partial_i w \partial_j w \, \mathrm{d}\boldsymbol{x}$$

for $w \in \mathcal{A}$ where

$$\mathcal{A} := \{w \in H^1(U) | \mathrm{Tr}\, w = g \text{ on } \partial U\}.$$

(1) Let $\{u_n\} \subset H^1(U)$ be a sequence weakly converging to u in $H^1(U)$ and $\ell := \liminf_{n\to\infty} I[u_n]$. Show that there exists a subsequence $\{u_{n_k}\}$ such that $\ell = \lim_{k\to\infty} I[u_{n_k}]$ and $u_{n_k} \to u$ in $L^2(U)$.
(2) Given any $\varepsilon > 0$ sufficiently small, prove that there exists a subset $G_\varepsilon \subset U$ such that u_{n_k} uniformly converges to u in G_ε, $|u(\boldsymbol{x})| + |\nabla u(\boldsymbol{x})| \le \varepsilon^{-1}$ in G_ε and $\mathcal{L}^d(U \backslash G_\varepsilon) < \varepsilon$.
(3) Prove that $\ell \ge I[u]$. This actually shows that $I[\cdot]$ is *weakly lower semi-continuous* on $H^1(U)$, that is, given any $\{u_n\} \subset H^1(U)$ weakly converging to u in $H^1(U)$, I satisfies $I[u] \le \liminf_{k\to\infty} I[u_n]$.
(4) Let $m := \inf_{w\in\mathcal{A}} I[w] < \infty$. Mimicing (1)-(3) to prove that there exists $u \in \mathcal{A}$ such that $I[u] = m$.
(5) Prove the uniqueness of the minimizer u in $\mathcal{A}$.
(Hint: If u_1, u_2 are two *different* minimizers, then define $\bar{u} = (u_1 + u_2)/2$ and prove $I[\bar{u}] < (I[u_1] + I[u_2])/2$.)
(6) Prove that the minimizer u is exactly the weak solution to $Lu = 0$ in U, $u = g$ on ∂U (in the sense of trace).

2.3 Existence theorem 2: Fredholm alternative

In view of Theorem 2.2.3, it is natural to ask if we can establish any existence result for the elliptic equation without the term μu in (2.2.1). The answer is definitely yes, but the uniqueness depends on whether the homogeneous equation $Lu = 0$ (with Dirichlet boundary condition) has nonzero solution or not. Furthermore, we can prove that if the homogeneous equation admits nonzero solution, then the solution space must be finite-dimensional and the dimension is equal to that of the corresponding "dual problem". This seems to be similar with the classification of solutions to a linear system $\mathbf{A}\boldsymbol{x} = \mathbf{b}$ in $\mathbb{R}^n$. But, for elliptic PDEs, we shall adopt the Fredholm theory of compact operators to establish analogous conclusions.

2.3.1 Properties of compact operators

First, we recall some basic properties of compact operators.

Definition 2.3.1. Let X, Y be Banach spaces. We say a bounded linear operator $K : X \to Y$ is a **compact operator** if $K(B)$ is precompact in Y (that is, $\overline{K(B)}$ is compact in Y) for any bounded set $B \subset X$. We denote by $K \in \mathfrak{C}(X, Y)$.

It is easy to see that K is a compact operator if and only if for any bounded sequence $\{x_n\} \subset X$, $\{Kx_n\}$ has a convergent subsequence in Y. Based on this, it is straightforward to get

Proposition 2.3.1. Let X, Y, Z be Banach spaces.

(1) If $K \in \mathfrak{C}(X, Y)$ and $x_n \rightharpoonup x$ weakly in X, then $Kx_n \to Kx$ in Y.
(2) If one of the two bounded linear operators $K_1 : X \to Y$ and $K_2 : Y \to Z$ is a compact operator, then the composition $K_2 \circ K_1 : X \to Z$ is a compact operator.
(3) If $K \in \mathfrak{C}(X, Y)$, then its adjoint $K^* \in \mathfrak{C}(Y', X')$ where X', Y' are the dual spaces of X, Y respectively.

Do note that (1) implies that the identity operator $I \in \mathfrak{C}(X)$ if and only if $\dim X < \infty$.

We also recall the Fredholm Alternative Theorem.

Theorem 2.3.2 (Fredholm alternative)**.** Let X be a Banach space, $K \in \mathfrak{C}(X)$. Then

(1) $\dim N(I - K) < \infty$, where $N(I - K) = \{x \in X | (I - K)x = 0\}$.
(2) $R(I - K)$ is closed.
(3) $R(I - K) = N(I - K^*)^\perp$ and $R(I - K^*) = {}^\perp N(I - K)$.
(4) $N(I - K) = \{0\}$ if and only if $R(I - K) = X$.
(5) $\dim N(I - K) = \dim N(I - K^*)$.

Here, for $M \subset X$, $F \subset X'$, we denote

$$^\perp M := \{f \in X' | \langle f, x\rangle = 0,\ \forall x \in M\}, \quad F^\perp := \{x \in X | \langle f, x\rangle = 0,\ \forall f \in X'\}.$$

A rather special example is to consider a linear system $\mathbf{A}\boldsymbol{x} = \mathbf{b}$ in $\mathbb{R}^n$. We know that this linear system has a solution $\boldsymbol{x}$ if and only if $\mathbf{b}$ can be written as the linear combination of $A_j := (a_{1j}, \cdots, a_{nj})^\top$, that is, $\mathbf{b} = \sum_{j=1}^{n} x_j A_j$. This is also equivalent to

$$\boldsymbol{z} \perp \mathbf{b} \Leftrightarrow \sum_{i=1}^{n} a_{ij} z_i = 0 \ \ (\boldsymbol{z} \perp A_j) \quad \forall j = 1, \cdots, n.$$

Thus, we know

- Given any $\mathbf{b} \in \mathbb{R}^n$, $\mathbf{A}\boldsymbol{x} = \mathbf{b}$ has a solution if and only if $\boldsymbol{z} \perp \mathbf{b}$ for any $\boldsymbol{z} \in \ker \mathbf{A}^\top$.
- There are only two cases:
  1. Either, given $\mathbf{b} \in \mathbb{R}^n$, the system $\mathbf{A}\boldsymbol{x} = \mathbf{b}$ has a unique solution $\boldsymbol{x} \in \mathbb{R}^n$;
  2. Or, $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ has nonzero solution, and $\dim \ker \mathbf{A} = \dim \ker \mathbf{A}^*$.

Now, let $\mathbf{A} = I - K$ for some matrix $K \in \mathbb{R}^{n\times n}$. Then the first conclusion coincides with Theorem 2.3.2(3). The second conclusion coincides with Theorem 2.3.2(4)-(5).

We finally recall the spectrum theorem of compact operators.

Definition 2.3.2. Let X be a Banach space and $A : X \to X$ is a bounded linear operator.

- The resolvent set of A is defined by $\rho(A) := \{\eta \in \mathbb{R} | A - \eta I \text{ is 1-1 and onto}\}$.
- The spectrum of A is defined by $\sigma(A) := \mathbb{R} \backslash \rho(A)$.

Given $\eta \in \rho(A)$, by the Closed Graph Theorem, we know $(A - \eta I)^{-1}$ is a bounded linear operator on X.

- We say $\lambda \in \sigma(A)$ is an eigenvalue of A if $N(A - \eta I) \neq \{0\}$. The set of all eigenvalues is denoted by $\sigma_p(A)$, called "point spectrum".
- If λ is an eigenvalue with $Aw = \lambda w$ for some $w \neq 0$, then we say w is an eigenvector of A associated with λ.

We now have

Theorem 2.3.3 (Riesz-Schauder). Let X be a Banach space and $K \in \mathfrak{C}(X)$. Then

- $0 \in \sigma(K)$ unless $\dim X < \infty$.
- $\sigma(K) \backslash \{0\} = \sigma_p(K) \backslash \{0\}$.
- The accumulation point of $\sigma_p(K)$, if exists, must be 0.

2.3.2 Fredholm alternative applied to elliptic PDEs

Given an elliptic operator L of divergence form (2.0.2) with $b^i \in C^1(\overline{U})$, we define its adjoint operator L^* by

$$L^* v := \partial_i (a^{ij} \partial_j v) - b^i \partial_i v + (c - \partial_i b^i) v.$$

The adjoint bilinear form $B^* : H_0^1(U) \times H_0^1(U) \to \mathbb{R}$ is defined by $B^*[v, u] := B[u, v]$ for $u, v \in H_0^1(U)$. We say $v \in H_0^1(U)$ is a **weak solution to the adjoint problem**

$$L^* v = f \text{ in } U, \quad u = 0 \text{ on } \partial U,$$

provided that $B^*[v, u] = (f, u)_{L^2(U)}$ holds for all $u \in H_0^1(U)$.

Remark 2.3.1. It should be noted that the concrete form of L^* is naturally deduced from the definition of adjointness, that is, $\langle Lu, v \rangle = \langle u, L^* v \rangle$. Let us temporarily assume u, v are smooth functions. Plugging the concrete form of L into the left side and integrating by parts, we get

$$\begin{aligned}
\langle Lu, v \rangle &= \int_U a^{ij} \partial_i u \partial_j v - \partial_i (b^i v) u + cuv \, \mathrm{d}\boldsymbol{x} = - \int_U \partial_i (a^{ij} \partial_j v) u - \partial_i (b^i v) u + cuv \, \mathrm{d}\boldsymbol{x} \\
&= \left\langle u, -\partial_i (a^{ij} \partial_j v) - b^i \partial_i v + (c - \partial_i b^i) v \right\rangle =: \langle u, L^* v \rangle.
\end{aligned}$$

Accordingly, we can then derive the concrete form of B^*.

In this section, we aim to use Theorem 2.3.2 (Fredholm Alternative) to establish the existence theory of the weak solution to equation (2.0.1).

Theorem 2.3.4 (Second existence theorem for weak solutions). Precisely one of the following two statements holds:

(A) For any $f \in L^2(U)$, equation (2.0.1), namely $Lu = f$ in U with $u = 0$ on ∂U, admits a unique weak solution $u \in H_0^1(U)$.

(B) The homogeneous equation, namely $Lu = 0$ in U with $u = 0$ on ∂U, admits nonzero weak solutions $u \in H_0^1(U)$.

Moreover, should (B) hold, then the solution space of the homogeneous equation, denoted by N, is a finite-dimensional subspace of $H_0^1(U)$ and $\dim N = \dim N^*$. Here N^* is the solution space of the adjoint homogeneous equation, namely $L^*v = 0$ in U with $v = 0$ on ∂U.

Finally, equation (2.0.1) has a weak solution if and only if $(f, v)_{L^2(U)} = 0$ for any $v \in N^*$.

Proof. The proof is divided into four steps.

Step 1: Formal construction of the solution. Let us recall that the First Existence Theorem, obtained by Lax-Milgram Theorem, shows that given $g \in L^2(U)$, the equation $L_\gamma u = g$ in U with $u = 0$ on ∂U has a unique weak solution in $H_0^1(U)$ where $L_\gamma := L + \gamma I$ and γ is the constant in Theorem 2.2.3. In such a case, we write $u = L_\gamma^{-1} g$.

Next, we go back to the equation $Lu = f$. For this equation, we know $u \in H_0^1(U)$ is a weak solution if and only if $u \in H_0^1(U)$ is a weak solution to $L_\gamma u = f + \gamma u$, that is,

$$B_\gamma[u, v] = \langle \gamma u + f, v \rangle \quad \forall v \in H_0^1(U),$$

which is further equivalent to

$$u = L_\gamma^{-1}(\gamma u + f) \Leftrightarrow (I - \gamma L_\gamma^{-1})u = L_\gamma^{-1} f.$$

Now, let $Ku = \gamma L_\gamma^{-1} u$ and $h = L_\gamma^{-1} f$ and then

$$u \text{ is a weak solution to (2.0.1)} \Leftrightarrow (I - K)u = h.$$

Step 2: Verify K is a compact operator on $L^2(U)$. Recall that L_γ satisfies the assumption of Lax-Milgram Theorem. From the coercivity, we know if $L_\gamma v = g$ for some $v \in H_0^1(U)$ and $g \in L^2(U)$, that is, $B_\gamma[v, \varphi] = (g, \varphi)_{L^2(U)}$ holds for all $\varphi \in H_0^1(U)$, then

$$\beta \|v\|^2_{H_0^1(U)} \le B_\gamma[v, v] = (g, v)_{L^2(U)} \le \|g\|_{L^2(U)} \|v\|_{L^2(U)} \le \|g\|_{L^2(U)} \|v\|_{H_0^1(U)},$$

which then gives

$$\|Kg\|_{H_0^1(U)} = \gamma \|v\|_{H_0^1(U)} \le C \|g\|_{L^2(U)} \quad \text{for some } C > 0.$$

So, $K : L^2(U) \to H_0^1(U)$ is a bounded linear operator. On the other hand, we have the compact embedding $H_0^1(U) \hookrightarrow\hookrightarrow L^2(U)$, so K, as a bounded linear operator from $L^2(U)$ to $L^2(U)$, is also a compact operator on $L^2(U)$. This is achieved by Proposition 2.3.1(2).

Step 3: Application of the Fredholm Alternative. Now we set $X = L^2(U)$, $K = \gamma L_\gamma^{-1}$ in Theorem 2.3.2 and get two possibilities.

- Case 1: $N(I - K) = \{0\}$. In this case, given any $h \in L^2(U)$, the equation $(I - K)u = h$ has a unique solution in $L^2(U)$. Then according to Step 1, we know this u also gives a weak solution to (2.0.1).

- Case 2: $N(I-K) \neq \{0\}$. In this case, we must have $\gamma \neq 0$. By Theorem 2.3.2, we know the homogeneous equation $u - Ku = 0$ has nonzero solutions in $L^2(U)$ and $\dim N(I-K) = \dim N(I-K^*)$.

Step 4: Verify the existence of solution $\Leftrightarrow (f, v) = 0\ \forall v \in N^*$. Furthermore, let v be the weak solution to $L^* v = 0$ (or equivalently $v - K^* v = 0$) in U with $v = 0$ on ∂U. We have

$$(h, v) = \gamma^{-1}(Kf, v) = \gamma^{-1}(f, K^* v) \overset{v=K^*v}{=\!=\!=} \gamma^{-1}(f, v).$$

So, $(I-K)u = h$ has a unique solution $\Leftrightarrow \langle h, v\rangle = 0$ for all $v \in N^* := N(I-K^*) \Leftrightarrow (f, v) = 0$. $\square$

Now, we conclude the following existence theorem

Theorem 2.3.5 (Third existence theorem for weak solutions). There exists an at most countable set $\Sigma \subset \mathbb{R}$ such that the boundary-value problem

$$Lu = \lambda u + f \text{ in } U, \quad u = 0 \text{ on } \partial U$$

has a unique weak solution for each $f \in L^2(U)$ if and only if $\lambda \notin \Sigma$. Moreover, if Σ is an infinite set, then $\Sigma = \{\lambda_k\}_{k \in \mathbb{N}^*}$, the values of a nondecreasing sequence with $\lambda_k \to +\infty$.

We call Σ the (real) spectrum of the operator L. The above theorem shows that the boundary-value problem

$$Lu = \lambda u \text{ in } U, \quad u = 0 \text{ on } \partial U$$

has a nontrivial solution $w \not\equiv 0$ if and only if $\lambda \in \Sigma$, in which case λ is called an **eigenvalue of** L and w is called a corresponding **eigenfunction**. Theorem 2.3.5 actually implies the eigenvalues of L must be a non-decreasing sequence going to $+\infty$.

Proof. Let γ be the constant from Theorem 2.2.3 and assume $\lambda > -\gamma$. Also we assume without loss of generality that $\gamma > 0$. By the Fredholm Alternative, the boundary-value problem

$$Lu = \lambda u + f \text{ in } U, \quad u = 0 \text{ on } \partial U$$

admits a unique weak solution for each $f \in L^2(U)$ if and only if the homogeneous problem

$$Lu = \lambda u \text{ in } U, \quad u = 0 \text{ on } \partial U$$

only has zero solution. This is true if and only if $u = 0$ is the only weak solution to

$$Lu + \gamma u = (\lambda + \gamma)u \text{ in } U, \quad u = 0 \text{ on } \partial U.$$

The last equation holds exactly when $u = L_\gamma^{-1}(\gamma + \lambda)u = \frac{\gamma+\lambda}{\gamma} Ku$ with $Ku := \gamma L_\gamma^{-1} u$. We know if $u = 0$ is the only weak solution, then $\frac{\gamma}{\gamma+\lambda}$ is not an eigenvalue of K. Consequently, we see that the equation

$Lu = \lambda u + f$ in U with $u = 0$ on ∂U has a unique solution if $\frac{\gamma}{\gamma+\lambda}$ is not an eigenvalue of K. Since $K \in \mathfrak{C}(L^2(U))$, by Theorem 2.3.3, we know the eigenvalues of K consist of either a finite set or else the values of a sequence converging to 0. This is equivalently to say $\lambda_k \to +\infty$ when the number of eigenvalues is infinite, as $\gamma > 0$ is given and λ only appears in the denominator of $\frac{\gamma}{\gamma+\lambda}$. □

Exercise 2.3

Exercise 2.3.1. Let Σ be the set of eigenvalues of L defined above. Given a real number $\lambda \notin \Sigma$ and a function $f \in L^2(U)$, we define $u \in H_0^1(U)$ to be the unique weak solution to $Lu = \lambda u + f$ in U with $u|_{\partial U} = 0$. Prove that there exists a constant $C > 0$ such that $\|u\|_{L^2(U)} \leq C\|f\|_{L^2(U)}$. (Hint: Prove this by contradiction.)

2.4 The eigenvalue problem of linear elliptic operators

In this section, we consider the eigenvalue problem for an elliptic operator

$$Lw = \lambda w \text{ in } U, \quad w = 0 \text{ on } \partial U. \tag{2.4.1}$$

Here $U \subset \mathbb{R}^d$ is a bounded domain (which implies connectedness) with a smooth boundary ∂U. For simplicity, we assume

$$Lu = -\partial_j(a^{ij}\partial_i u), \quad a^{ij} = a^{ji}, \quad a^{ij} \in C^\infty(\overline{U}). \tag{2.4.2}$$

Thus, the corresponding bilinear form is also symmetric, that is, $B[u,v] = B[v,u]$ holds for any $u, v \in H_0^1(U)$.

2.4.1 Orthogonality of eigenfunctions

The first theorem that we aim to prove in this section is stated as follows

Theorem 2.4.1 (Eigenvalues of symmetric elliptic operators). Each eigenvalue of L is a real number. Futhermore, if we repeat each eigenvalue according to its (finite) multiplicity, we have $\Sigma = \{\lambda_k\}_{k\in\mathbb{N}^*}$ where

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots, \qquad \lim_{k\to\infty} \lambda_k = +\infty.$$

Finally, there exists an orthonormal basis $\{w_k\}_{k\in\mathbb{N}^*}$ of $L^2(U)$ where $w_k \in H_0^1(U)$ is an eigenfunction corresponding to λ_k for each $k \in \mathbb{N}^*$:

$$Lw_k = \lambda_k w_k \text{ in } U, \quad w_k = 0 \text{ on } \partial U.$$

Remark 2.4.1. According to the regularity theory in Chapter 2.5, the smoothness of a^{ij} implies that $w_k \in C^\infty(U)$ and furthermore $w_k \in C^\infty(\overline{U})$ (this would require ∂U is also C^∞).

Before proving this theorem, let us briefly review the spectrum theory of symmetric compact operators.

Spectrum theory of symmetric compact operators

Let H be a complex Hilbert space.

Definition 2.4.1. We say a bounded linear operator $A : H \to H$ is **symmetric** if $(Ax, y) = (x, Ay)$ holds for all $x, y \in H$. Here $(\cdot, \cdot)$ is the inner product of H. It is easy to see that A is symmetric if and only if $A = A^*$.

Proposition 2.4.2. Let $A : H \to H$ be a bounded, linear operator. Then A is symmetric if and only if $(Ax, x) \in \mathbb{R}$ for any $x \in H$. In this case, we further have

(1) $\sigma(A) \subset \mathbb{R}$ and $\|(\lambda I - A)^{-1}x\| \le \frac{\|x\|}{|\mathbf{Im}\,\lambda|}$ for any $x \in H$, $\lambda \in \mathbb{C}$ with $\mathbf{Im}\,\lambda \neq 0$.
(2) Let $H_1 \subset H$ be an A-invariant closed subspace of H, then $A|_{H_1}$ is also symmetric on H_1.
(3) For any $\lambda, \lambda' \in \sigma_p(A)$ with $\lambda \neq \lambda'$, we have $N(\lambda I - A) \perp N(\lambda' I - A)$.
(4) $\|A\| = \sup_{\|x\|=1} |(Ax, x)|$.

On Hilbert spaces, the spectrum and structure of symmetric compact operators are quite similar to those of real symmetric matrices in Euclidean spaces. In particular, we recall that any real symmetric matrix is diagonalizable and the elements on the diagonal are exactly the eigenvalues, which also implies that the eigenvectors of a real symmetric matrix gives an orthogonal (actually orthonormal after normalization) basis of the Euclidean spaces. Also, the critical value of quadratic form is also an eigenvalue. These properties also hold for symmetric compact operators on Hilbert spaces.

Proposition 2.4.3. Let $A \in \mathfrak{C}(H)$ be symmetric. Then there exists an $x_0 \in H$, $\|x_0\| = 1$, such that

$$\lambda := |(Ax_0, x_0)| = \sup_{\|x\|=1} |(Ax, x)|, \quad Ax_0 = \lambda x_0.$$

Proposition 2.4.4. Let $A \in \mathfrak{C}(H)$ be symmetric. Then there is an at most countable sequence of real numbers $\{\lambda_k\}_{k \in \mathbb{N}^*}$ whose only possible accumulation point (if exists) is 0, such that $\{\lambda_k\}$ are exactly the eigenvalues of A. Also, there exists an orthonormal basis $\{e_k\}$ of H such that

$$x = \sum_{k \ge 1} (x, e_k) e_k, \quad Ax = \sum_k \lambda_k (x, e_k) e_k.$$

Proposition 2.4.5 (Courant minimax characterization). Let $A \in \mathfrak{C}(H)$ be symmetric and have eigenvalues $\lambda_1^+ \ge \lambda_2^+ \ge \cdots \ge 0 > \cdots \ge \lambda_2^- \ge \lambda_1^-$. Then

$$\lambda_n^+ = \inf_{E_{n-1}} \sup_{\substack{x \in E_{n-1}^\perp \\ x \neq 0}} \frac{(Ax, x)}{(x, x)}, \qquad \lambda_n^- = \sup_{E_{n-1}} \inf_{\substack{x \in E_{n-1}^\perp \\ x \neq 0}} \frac{(Ax, x)}{(x, x)}.$$

Here E_{n-1} can be any $(n-1)$-dimensional closed subspace of H.

We now turn to prove Theorem 2.4.1.

Proof of Theorem 2.4.1. Let $S = L^{-1} : L^2(U) \to L^2(U)$. In previous sections, we already prove that $S \in \mathfrak{C}(L^2(U))$. It suffices to verify S is symmetric on $L^2(U)$. In fact, pick $f, g \in L^2(U)$ and set $u := Sf, v := Sg$ and then $u, v \in H_0^1(U)$ are weak solutions to $Lu = f$ and $Lv = g$ (with Dirichlet boundary conditions) respectively. Thus, we have

$$(Sf, g)_{L^2(U)} = (u, g)_{L^2(U)} = B[v, u] = B[u, v] = (f, v)_{L^2(U)} = (f, Sg)_{L^2(U)}.$$

Also, we have $(Sf, f) = (u, f) = B[u, u] \geq 0$ for any $f \in L^2(U)$. Thus by Proposition 2.4.4, we know the eigenvalues of S are all *positive real numbers* and the corresponding eigenfunctions form an orthonormal basis of $L^2(U)$. For any eigenvalue of S, say $\eta > 0$, assume $Sw = \eta w$ for some $0 \neq w \in H_0^1(U)$. Then this is equivalent to $Lw = \lambda w$ with $\lambda = \eta^{-1}$. Thus, the conclusions in Theorem 2.4.1 are established. □

It should be noted that the study of the distributions of eigenvalues of elliptic operators and the behaviors of eigenfunctions is extremely important in mathematical physics. There are still many unsolved problems till now. Among all the previous studies, a landmark is due to H. Weyl: In a bounded domain $U \subset \mathbb{R}^d$ with a smooth boundary, the eigenvalues of the Laplacian operator in U (with vanishing Dirichlet boundary conditions) satisfy

$$\lim_{k \to \infty} \frac{\lambda_k^{d/2}}{k} = \frac{(2\pi)^d}{\mathcal{L}^d(U)\alpha(d)}.$$

In the next section, we will give a description of the smallest eigenvalue λ_1 of L. $\lambda_1 > 0$ is also called *principal eigenvalue*.

2.4.2 Variational principle of the principal eigenvalue

In this section, we aim to prove the following theorem.

Theorem 2.4.6 (Variational principle for the principal eigenvalue). Let $\lambda_1 > 0$ be the principal eigenvalue of the elliptic operator L with vanishing Dirichlet boundary condition (as defined in (2.4.1)-(2.4.2)). Then

(1) We have

$$\lambda_1 = \min\{B[u, u] | u \in H_0^1(U),\ \|u\|_{L^2(U)} = 1\}. \tag{2.4.3}$$

(2) The above minimum is attained for a smooth function w_1 that does not change sign within U. Also, w_1 solves the eigenvalue problem

$$Lw_1 = \lambda_1 w_1 \text{ in } U, \quad w_1 = 0 \text{ on } \partial U.$$

(3) Finally, if $u \in H_0^1(U)$ is any weak solution to

$$Lu = \lambda_1 u \text{ in } U, \quad u = 0 \text{ on } \partial U,$$

then u is a multiple of w_1. This implies that λ_1 must be simple. In particular, we have $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$

Proof. Let us first recall that the eigenfunctions of L, say $\{w_k\}$, form an orthonormal basis of $L^2(U)$. So, we have $(w_k, w_l)_{L^2(U)} = \delta_{kl}$. Also, notice that

$$B[w_k, w_l] = \lambda_k (w_k, w_l) = \lambda_k \delta_{kl} \qquad k, l \in \mathbb{N}^*,$$

which shows that $\{w_k\}$ forms an orthogonal subset of $H_0^1(U)$ (with inner product defined by $(\cdot,\cdot)_{H_0^1(U)} = B[\cdot,\cdot]$). To prove (1), it suffices to prove that $\{w_k\}$ also give an orthogonal basis of $H_0^1(U)$.

Claim. $\{w_k/\sqrt{\lambda_k}\}$ form an orthonormal basis of $H_0^1(U)$.

To prove the claim, it suffices to prove that $B[w_k, u] = 0$ for all $k \in \mathbb{N}^*$ implies $u = 0$. This is rather easy, as any $u \in H_0^1(U)$ also belongs to $L^2(U)$. So, u can be written as $\sum_{j=1}^{\infty} d_j w_j$ and so

$$B[w_k, u] = \sum_{j=1}^{\infty} B[w_k, d_j w_j] = d_k \lambda_k (w_k, w_k)_{L^2(U)} = 0 \Rightarrow d_k = 0 \qquad \forall k \in \mathbb{N}^*.$$

Without loss of generality, we assume $\|u\|_{L^2(U)} = 1$ and so we get $\sum_{j=1}^{\infty} d_j^2 = 1$. Also, we have

$$u = \sum_{j=1}^{\infty} d_j \sqrt{\lambda_j} \frac{w_j}{\sqrt{\lambda_j}},$$

which absolutely converges in $H_0^1(U)$.

With this claim, it is easy to see that given any u that has the above expansion in $H_0^1(U)$, we have

$$B[u,u] = \sum_{j=1}^{\infty} d_j^2 \lambda_j \geq \lambda_1 \sum_{j=1}^{\infty} d_j^2 = \lambda_1,$$

and the equality holds if and only if $u = w_1$. This finishes the proof of (1).

The proof of (2) is a bit tricky and relies on the Strong Maximum Principle (Theorem 2.6.4). First, we make the following claim:

Claim. Let $u \in H_0^1(U)$ satisfy $\|u\|_{L^2(U)} = 1$. Then u is the weak solution to

$$Lu = \lambda_1 u \text{ in } U, \quad u = 0 \text{ on } \partial U,$$

if and only if $B[u,u]=\lambda_1$.

For this claim, the "only if" part is trivial if we invoke the definition of $B[u,v]$. For the "if" part, assume $u\in H_0^1(U)$ satisfies $\|u\|_{L^2(U)}=1$ and $B[u,u]=\lambda_1$. Then we can expand u to be $\sum\limits_{k=1}^{\infty} d_k w_k$ with $d_k=(u,w_k)_{L^2(U)}$ and $\sum_k d_k^2=1$. Now, we compute $B[u,u]$

$$\lambda_1\sum_{k=1}^{\infty} d_k^2=\lambda_1=B[u,u]=\sum_{k=1}^{\infty} d_k^2\lambda_k \Rightarrow \sum_{k=1}^{\infty}(\lambda_k-\lambda_1)d_k^2=0,$$

which immediately leads to $d_k=0$ for all k satisfying $\lambda_k>\lambda_1$. Because the multiplicity of λ_1 is finite, we know

$$u=\sum_{k=1}^{m}(u,w_k)_{L^2(U)}w_k \qquad \text{for some } m\in\mathbb{N}^*, \quad Lw_k=\lambda_1 w_k \quad \forall 1\le k\le m.$$

This gives us $Lu=\lambda_1 u$ and completes the proof of the claim.

With this claim, we start to prove (2). Assume now u is a weak solution to

$$Lu=\lambda_1 u \text{ in } U, \quad u=0 \text{ on } \partial U,$$

and u is not identically zero. Then we need to prove either $u>0$ in U or $u<0$ in U. Without loss of generality, we again assume $\|u\|_{L^2(U)}=1$ and let

$$\alpha:=\int_U (u^+)^2\,\mathrm{d}\boldsymbol{x}, \qquad \beta:=\int_U (u^-)^2\,\mathrm{d}\boldsymbol{x}$$

where $u^+:=\max\{0,u\}$ and $u^-:=\max\{0,-u\}$ satisfy $u=u^+-u^-$ and $|u|=u^++u^-$. Then $\int_U u^2\,\mathrm{d}\boldsymbol{x}=1$ implies $\alpha+\beta=1$. By Proposition 1.2.4, we know that $u^\pm$ also belongs to $H_0^1(U)$ and

$$\partial u^+=\begin{cases}\partial u & \text{a.e. on } \{u>0\}\\ 0 & \text{a.e. on } \{u\le 0\},\end{cases}$$

$$\partial u^-=\begin{cases}0 & \text{a.e. on } \{u\ge 0\}\\ -\partial u & \text{a.e. on } \{u<0\},\end{cases}$$

which then implies

$$B[u^+,u^-]=\int_U a^{ij}\partial_i u^+\partial_j u^-\,\mathrm{d}\boldsymbol{x}=0 \qquad \text{at least one of } \partial u^+,\partial u^- \text{ are zero.}$$

Thus, by the claim and the bi-linearity of $B[\cdot,\cdot]$, we have

$$\lambda_1=B[u,u]=B[u^+-u^-,u^+-u^-]=B[u^+,u^+]+B[u^-,u^-]\ge\lambda_1\alpha+\lambda_1\beta=\lambda_1.$$

Here the last inequality holds true due to (1) and $u^\pm \in H_0^1(U)$. Therefore, the inequality above becomes an equality, that is, $B[u^\pm, u^\pm] = \lambda_1 \|u^\pm\|^2_{L^2(U)}$.

By the claim, we know $\underline{u^\pm \text{ are also eigenfunctions corresponding to the principal eigenvalue } \lambda_1 \text{ of } L}$. Since $a^{ij} \in C^\infty(\overline{U})$, we know $u^\pm \in C^\infty(U)$ by the interior regularity property (see Chapter 2.5). So, we have $Lu^+ = \lambda_1 u^+ \geq 0$ in U (this is a classical solution now, not only a weak solution!). Since U is assumed to be connected, by Strong Maximum Principle (Theorem 2.6.4), we have

$$\text{either } u^+ > 0 \text{ in } U, \qquad \text{or } u^+ = 0 \text{ in } U.$$

The desired conclusion follows from the argument below.

- If $u^+ > 0$ in U, then there is nothing to prove because $u^+ > 0$ already implies $u > 0$ in U.
- If $u^+ = 0$ in U, then that means $u \leq 0$ in U. There are two further possibilities:
  - If $u < 0$ always holds in U, then again we have nothing to prove.
  - If there exists an $\boldsymbol{x}_0 \in U$ such that $u(\boldsymbol{x}_0) = 0$, then that also means u^- attains its minimum in U. Since u^- is also an eigenfunction, we have $Lu^- = \lambda_1 u^- \geq 0$ in U. Again, by the Strong Maximum Principle, u^- must be a constant and this constant must be zero (because of $u^-(\boldsymbol{x}_0) = u(\boldsymbol{x}_0) = 0$). This together with $u^+ = 0$ in U implies $u = 0$ in U, which violates $\|u\|_{L^2(U)} = 1$.

Finally, we prove (3). If $u, \tilde{u}$ are both nonzero weak solutions to

$$Lu = \lambda_1 u \text{ in } U, \quad u = 0 \text{ on } \partial U.$$

Then by (2), we know $\int_U \tilde{u} \,\mathrm{d}\boldsymbol{x} \neq 0$ and there exists some $C \in \mathbb{R}$ such that

$$\int_U u - C\tilde{u} \,\mathrm{d}\boldsymbol{x} = 0.$$

But $u - C\tilde{u}$ is also an eigenfunction corresponding to λ_1, then (2) implies that $u - C\tilde{u} = 0$ in U. So, $\lambda_1 > 0$ must be a simple eigenvalue. □

Remark 2.4.2. The conclusion of (1) can also be written as

$$\lambda_1 = \min_{\substack{u \in H_0^1(U) \\ u \not\equiv 0}} \frac{B[u,u]}{\|u\|^2_{L^2(U)}}.$$

We end this section by stating the main result about the principal eigenvalue of non-symmetric elliptic operators. Assume $a^{ij}, b^i, c \in C^\infty(\overline{U})$ where U is a bounded domain with smooth boundary, $[a^{ij}]$ is symmetric and $c \geq 0$ in U. Then we have

Theorem 2.4.7 (Principal eigenvalue for non-symmetric elliptic operators). Define L by $Lu = -a^{ij}\partial_i\partial_j u + b^i \partial_i u + cu$ with a^{ij}, b^i, c satisfying the above conditions. Then

(1) There exists a real eigenvalue λ_1 for the operator L (with Dirichlet boundary condition) such that if $\lambda \in \mathbb{C}$ is any other eigenvalue, we must have $\mathbf{Re}(\lambda) \geq \lambda_1$.
(2) There exists a corresponding eigenfunction w_1, which is positive within U.
(3) The eigenvalue λ_1 is simple.

Exercise 2.4

Exercise 2.4.1 (Courant minimax principle). Let $Lu = -\partial_j(a^{ij}\partial_i u)$ be a symmetric, uniformly elliptic operator. Assume L, with zero boundary conditions, has eigenvalues $0 < \lambda_1 < \lambda_2 \leq \cdots$. Prove that

$$\lambda_k = \max_{S \in \Sigma_{k-1}} \min_{\substack{u \in S^\perp \\ \|u\|_{L^2(U)}=1}} B[u,u] \quad \forall k \in \mathbb{N}^*,$$

where Σ_{k-1} denotes the collection of $(k-1)$-dimensional subspaces of $H_0^1(U)$.

Exercise 2.4.2. Let $Lu = -\partial_j(a^{ij}\partial_i u) + b^i\partial_i u + cu$ be a symmetric, uniformly elliptic operator with principal eigenvalue $\lambda_1 > 0$ taken with zero boundary conditions. Prove the max-min representation formula

$$\lambda_1 = \sup_{\substack{u \in C^\infty(\overline{U}) \\ u>0 \text{ in } U \\ u=0 \text{ on } \partial U}} \inf_{x \in U} \frac{Lu(\boldsymbol{x})}{u(\boldsymbol{x})} \quad \forall k \in \mathbb{N}^*,$$

(Hint: Consider the eigenfunction w_1^* corresponding to λ_1 for the adjoint operator L^*. You may need Theorem 2.4.7.)

Exercise 2.4.3. Consider a family of smooth bounded domains $U(\tau) \subset \mathbb{R}^d$ that depends smoothly on the parameter $\tau \in \mathbb{R}$. As τ changes, each point on $\partial U(\tau)$ moves with velocity $\mathbf{v}$. For each τ, we consider eigenvalues $\lambda = \lambda(\tau)$ and corresponding eigenfunctions $w = w(\boldsymbol{x};\tau)$ defined by

$$-\Delta w = \lambda w \text{ in } U(\tau), \quad w|_{\partial U(\tau)} = 0$$

with $\|w\|_{L^2(U(\tau))} = 1$. Suppose that λ, ω are smooth functions of $\tau, \boldsymbol{x}$. Show that

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} = -\int_{\partial U(\tau)} \left|\frac{\partial w}{\partial N}(\boldsymbol{x};\tau)\right|^2 (\mathbf{v} \cdot N)\, \mathrm{d}S_{\boldsymbol{x}}$$

where $\mathbf{v} \cdot N$ is the normal velocity of the boundary $\partial\Omega(\tau)$.

(Hint: The variational principle gives $\lambda(\tau) = \int_{U(\tau)} |\nabla w(\boldsymbol{x})|^2\, \mathrm{d}\boldsymbol{x}$. Then compute $\lambda'(\tau)$ and then it remains to prove $\int_{U(\tau)} \partial_\tau |\nabla w(\boldsymbol{x};\tau)|^2 = 2\lambda'(\tau)$, in which you will need to use $\|w\|_{L^2(U(\tau))} \equiv 1$ to show that $\frac{\mathrm{d}}{\mathrm{d}\tau}\|w\|^2_{L^2(U(\tau))} = 0$.)

Remark 2.4.3. The conclusion of this problem shows that the principal eigenvalue of $-\Delta$ gets smaller as the domain U is enlarged.

Exercise 2.4.4. Given a variational proof of Theorem 2.4.6 (1). Define the functional

$$I[w] = \frac{1}{2}\int_U |\nabla w|^2 \,\mathrm{d}\boldsymbol{x}, \quad w \in \mathcal{A} := \{w \in H_0^1(U) | \|w\|_{L^2(U)} = 1 \text{ in } U\}.$$

(1) Pick a sequence $\{u_n\} \subset \mathcal{A}$ such that $I[u_n] \to m := \inf_{w\in\mathcal{A}} I[w]$. Prove that $\{u_n\}$ has a subsequence $\{u_{n_k}\}$ that weakly converges to some u in $H_0^1(U)$ and $I[u] \le m$.

(2) Prove that $u \in \mathcal{A}$ and then u is the desired minimizer satisfying $I[u] = m$. (Hint: Use the compact embedding $H_0^1(U) \hookrightarrow\hookrightarrow L^2(U)$.)

(3) Fix $v \in H_0^1(U)$ and choose $w \in H_0^1(U)$ with $\int_U uw \,\mathrm{d}\boldsymbol{x} \neq 0$. Consider the perturbation of $\|u\|_{L^2(U)}^2$ defined by $j(\tau,\sigma) := \int_U (u + \tau v + \sigma w)^2 \,\mathrm{d}\boldsymbol{x} - 1$. Prove that there exists $\phi \in C^1(\mathbb{R})$ such that $\phi(0) = 0$ and $j(\tau, \phi(\tau)) = 0$ for all sufficiently small $|\tau|$. Then verify that

$$\phi'(0) = -\frac{\int_U uv \,\mathrm{d}\boldsymbol{x}}{\int_U uw \,\mathrm{d}\boldsymbol{x}}.$$

(4) Set $w(\tau) := \tau v + \phi(\tau) w$ and $i(\tau) := I[u + w(\tau)]$ for sufficiently small $|\tau|$. Use $i'(0) = 0$ to prove that there exists $\lambda \in \mathbb{R}$ such that

$$\int_U \nabla u \cdot \nabla v \,\mathrm{d}\boldsymbol{x} = \lambda \int_U uv \,\mathrm{d}\boldsymbol{x} \qquad \forall v \in H_0^1(U).$$

(5) Prove that λ in (4) is exactly λ_1, the principal eigenvalue of $-\Delta$ in U with vanishing Dirichlet boundary condition.

2.5 Elliptic regularity

We now address the question as to whether a weak solution u to $Lu = f$ in U is "sufficiently regular". Assume $f \in L^2(U)$, then the solution u is expected to be second-order differentiable in a suitable sense, as L is a second-order differential operator. However, we still have to pick a suitable type of function spaces to achieve this second-order differentiability. In fact, we have

- $f \in L^2(U) \Rightarrow u \in H^2(U)$.
- $f \in C(\overline{U}) \not\Rightarrow u \in C^2(\overline{U})$. The counterexample is referred to Exercise 2.5.2.
- For $\alpha \in (0,1)$, $f \in C^{0,\alpha}(\overline{U}) \Rightarrow u \in C^{2,\alpha}(\overline{U})$.

In this section, we study the regularity of the $H_0^1(U)$-weak solution to (2.0.1) with L defined by (2.0.2). When $f \in L^2(U)$, the first step is to enhance the differentiability of u to second-order. In fact, let us consider $-\Delta u = f$ in $\mathbb{R}^d$ and assume u is a smooth solution and decays rapidly to 0 as $|\boldsymbol{x}| \to \infty$.

We then compute

$$\int_{\mathbb{R}^d} f^2 \,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d} (\Delta u)^2 \,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d} (\partial_i\partial_i u)(\partial_j\partial_j u) \,\mathrm{d}\boldsymbol{x} = -\int_{\mathbb{R}^d} \partial_j\partial_i\partial_i u \,\partial_j u \,\mathrm{d}\boldsymbol{x}$$
$$= \int_{\mathbb{R}^d} (\partial_i\partial_j u)^2 = \int_{\mathbb{R}^d} |\nabla^2 u|^2 \,\mathrm{d}\boldsymbol{x}.$$

Moreover, if $f \in H^m$ for some $m \in \mathbb{N}^*$, the regularity of u is expected to be H^{m+2} and finally we aim to prove the C^∞ regularity of u provided that f and coefficients of L are all C^∞. The existence of second-order (weak) derivative of an $H_0^1(U)$-weak solution is guaranteed by the properties of differential quotients discussed in the next section.

2.5.1 Difference quotient of Sobolev functions

In classical calculus, when we want to prove a function $f(\boldsymbol{x})$ has ∂_i derivative, it suffices to verify the limit of difference quotient exists, that is,

$$\lim_{h\to 0} \frac{f(\boldsymbol{x}+he_i) - f(\boldsymbol{x})}{h} \text{ exists.}$$

For Sobolev functions, we want to establish analogous arguments as above, but the existence of such limit is obtained by weak convergence instead of classical pointwise convergence.

Let $f : U \to \mathbb{R}$ be a locally integrable function in U and let $V \Subset U$.

Definition 2.5.1. The i-th difference quotient of size h is defined by

$$D_i^h f(\boldsymbol{x}) := \frac{f(\boldsymbol{x}+he_i) - f(\boldsymbol{x})}{h}, \quad 1 \le i \le d$$

for $\boldsymbol{x} \in V$ and $h \in \mathbb{R}$, $0 < |h| < \operatorname{dist}(V, \partial U)$. $D^h f := (D_1^h f, \cdots, D_d^h f)$.

Proposition 2.5.1 (Difference quotients and weak derivatives)**.** The following two assertions hold.

(1) Suppose $\underline{1 \le p < \infty}$ and $f \in W^{1,p}(U)$. Then for each $V \Subset U$,

$$\|D^h f\|_{L^p(V)} \le C\|\nabla f\|_{L^p(U)}$$

holds for some constant $C > 0$ and all $0 < |h| < \frac{1}{2}\operatorname{dist}(V, \partial U)$.

(2) Suppose $\underline{1 < p < \infty}$, $f \in L^p(V)$ and there exists a constant $C > 0$ such that

$$\|D^h f\|_{L^p(V)} \le C, \ \ \forall 0 < |h| < \frac{1}{2}\operatorname{dist}(V, \partial U).$$

Then

$$f \in W^{1,p}(V), \ \ \|\nabla f\|_{L^p(V)} \le C.$$

It should also be noted that $V \Subset U$ is redundant if we consider the corresponding conclusions for *tangential derivatives*.

Proof. Since $p < \infty$, we may assume f is smooth without loss of generality (otherwise we use the smooth approximation). Then for each $\boldsymbol{x} \in V$, $1 \le i \le d$ and $0 < |h| < \frac{1}{2}\mathrm{dist}\,(V, \partial U)$, we have

$$|f(\boldsymbol{x} + he_i) - f(\boldsymbol{x})| \le |h| \int_0^1 |\nabla f(\boldsymbol{x} + the_i)|\,\mathrm{d}t.$$

Thus, we have

$$\int_V |D^h f|^p\,\mathrm{d}\boldsymbol{x} \le C \sum_{i=1}^d \int_V \int_0^1 |\nabla f(\boldsymbol{x} + the_i)|^p\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x} = C \sum_{i=1}^d \int_0^1 \int_V |\nabla f(\boldsymbol{x} + the_i)|^p\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

This immediately leads to $\|D^h f\|_{L^p(V)} \le C\|\nabla f\|_{L^p(U)}$.

For (2), let $f \in L^p(U)$. We just notice that the following "integration by parts" formula holds for the difference quotient (actually, it is just as a result of change of variables)

$$\int_V f(D_i^h \varphi)\,\mathrm{d}\boldsymbol{x} = -\int_V (D_i^{-h} f)\varphi\,\mathrm{d}\boldsymbol{x} \qquad \forall \varphi \in C_c^\infty(V),\ 1 \le i \le d. \tag{2.5.1}$$

Since $\|D_i^{-h} f\|_{L^p(V)}$ is uniformly bounded in h and $1 < p < \infty$, we know there exists a subsequence $h_k \to 0$ such that

$$D_i^{-h_k} f \rightharpoonup v_i \qquad \text{weakly in } L^p(V)$$

for some $v_i \in L^p(V)$. Notice that $1 < p < \infty$ is necessary here, otherwise L^p space is not reflexive. Plugging this back to the "integration by parts" formula and taking the limit $h_k \to 0$, we get

$$\int_U f \partial_i \varphi\,\mathrm{d}\boldsymbol{x} = \int_V f \partial_i \varphi\,\mathrm{d}\boldsymbol{x} = -\lim_{h_k \to 0} \int_V D_i^{-h_k} f \varphi\,\mathrm{d}\boldsymbol{x} = -\int_V v_i \varphi\,\mathrm{d}\boldsymbol{x} = -\int_U v_i \varphi\,\mathrm{d}\boldsymbol{x}.$$

Thus, v_i is exactly the ∂_i-weak derivative of f and so $\nabla f \in L^p(V)$, $f \in W^{1,p}(V)$. □

2.5.2 Interior regularity

In this section, we aim to prove the interior regularity result for second-order elliptic equations.

Theorem 2.5.2 (Interior elliptic regularity). Assume $a^{ij} \in C^1(U), b^i, c \in L^\infty(U)$ and $f \in L^2(U)$. Suppose further $u \in H^1(U)$ is a weak solution to $Lu = f$ in U. Then $u \in H^2_{\mathrm{loc}}(U)$ and for each open set $V \Subset U$ we have the estimate

$$\|u\|_{H^2(V)} \le C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}), \tag{2.5.2}$$

where $C > 0$ depends only on V, U and the coefficients of L.

Do note that the interior regularity has nothing to do with the boundary value of u and thus there is no need to assume $u \in H_0^1(U)$. Also, since $u \in H^2_{\mathrm{loc}}(U)$, we have $Lu = f$ a.e. in U. Thus, u actually solves the PDE at least for a.e. points in U.

By induction on the differentiability order of f and the coefficients of L, it is easy to prove the followng two corollaries and we skip the proof here.

Corollary 2.5.3 (High-order interior elliptic regularity). Assume $m \in \mathbb{N}$, $a^{ij}, b^i, c \in C^{m+1}(U)$ and $f \in H^m(U)$. Suppose further $u \in H^1(U)$ is a weak solution to $Lu = f$ in U. Then $u \in H^{m+2}_{\mathrm{loc}}(U)$ and for each open set $V \Subset U$ we have the estimate

$$\|u\|_{H^{m+2}(V)} \leq C(\|f\|_{H^m(U)} + \|u\|_{L^2(U)}), \tag{2.5.3}$$

where $C > 0$ depends only on m, V, U and the coefficients of L.

Corollary 2.5.4 (C^∞ interior elliptic regularity). Assume $a^{ij}, b^i, c, f \in C^\infty(U)$. Suppose further $u \in H^1(U)$ is a weak solution to $Lu = f$ in U. Then $u \in C^\infty(U)$.

Proof of Theorem 2.5.2. Without loss of generality, we assume $b^i = c = 0$, otherwise we can move these lower-order terms to the right side of the equation. Recall that $u \in H_0^1(U)$ is a weak solution if

$$\int_U a^{ij} \partial_i u \partial_j v \, \mathrm{d}\boldsymbol{x} = \int_U f v \, \mathrm{d}\boldsymbol{x} \tag{2.5.4}$$

holds for any $v \in H_0^1(U)$. We now need to pick a suitable v such that

- The left side, after possibly integrating by parts, gives the $L^2(U)$ norm of $D^h(\nabla u)$.
- v vanishes on the boundary. That is, we have to "localize" all estimates away from ∂U.

It is not difficult to achieve the first one, but for the second one, we may have to insert some cut-off function in v. Fix an open subset $V \Subset U$ and choose an open set W such that $V \Subset W \Subset U$. Then we select a *smooth cut-off function* ζ satisfying

$$\zeta = 1 \text{ in } V, \quad \zeta = 0 \text{ in } \mathbb{R}^d \backslash W, \quad 0 \leq \zeta \leq 1.$$

This is necessary to localize the estimates away from the boundary ∂U.

Now, let $|h| > 0$ be small, choose $1 \leq k \leq d$ and define $v := -D_k^{-h}(\zeta^2 D_k^h u)$. We have

$$\begin{aligned}
\int_U a^{ij} \partial_i u \partial_j v \, \mathrm{d}\boldsymbol{x} &= -\int_U a^{ij} \partial_i u \partial_j (D_k^{-h}(\zeta^2 D_k^h u)) \, \mathrm{d}\boldsymbol{x} = -\int_U a^{ij} \partial_i u D_k^{-h}(\partial_j(\zeta^2 D_k^h u)) \, \mathrm{d}\boldsymbol{x} \\
\text{use (2.5.1)} \qquad &= \int_U D_k^h(a^{ij} \partial_i u) \partial_j (\zeta^2 D_k^h u) \, \mathrm{d}\boldsymbol{x} \\
&= \int_U a^{ij}(\boldsymbol{x} + h e_k) \, (D_k^h \partial_i u) \zeta^2 (D_k^h \partial_j u) \, \mathrm{d}\boldsymbol{x} + A_1
\end{aligned}$$

where

$$A_1 := \int_U a^{ij}(\boldsymbol{x}+he_k)\,\partial_j(\zeta^2)\,(D_k^h\partial_i u)(D_k^h u) + (D_k^h a^{ij})\partial_i u(\zeta^2 D_k^h\partial_j u + \partial_j(\zeta^2)\,D_k^h u)\,\mathrm{d}\boldsymbol{x}.$$

Since L is uniformly elliptic, we have

$$\int_U a^{ij}(\boldsymbol{x}+he_k)\,(D_k^h\partial_i u)\zeta^2(D_k^h\partial_j u)\,\mathrm{d}\boldsymbol{x} \geq \theta\int_U \zeta^2|D_k^h\nabla u|^2\,\mathrm{d}\boldsymbol{x}$$

which gives the $L^2(U)$ norm of $\zeta|D_k^h\nabla u|$ as desired. The term A_1 is directly controlled as follows

$$\begin{aligned}|A_1| &\leq C\|a^{ij}\|_{C^1(U)}\left(\|\zeta D_k^h\nabla u\|_{L^2(U)}\|D_k^h u\|_{L^2(U)} + \|D_k^h\nabla u\|_{L^2(U)}\|\nabla u\|_{L^2(U)} + \|D_k^h u\|_{L^2(U)}\|\nabla u\|_{L^2(U)}\right)\\ &\leq \varepsilon\|\zeta D_k^h\nabla u\|_{L^2(U)}^2 + \frac{C'}{\varepsilon}\left(\|D_k^h u\|_{L^2(U)}^2 + \|\nabla u\|_{L^2(U)}^2\right)\end{aligned}$$

for any $\varepsilon > 0$ suitably small. Now, we pick $\varepsilon \in (0, \frac{\theta}{2})$ such that $\varepsilon\|\zeta D_k^h\nabla u\|_{L^2(U)}^2$ can be absorbed by $\theta\|\zeta D_k^h\nabla u\|_{L^2(U)}^2$, we have

$$\int_U a^{ij}\partial_i u\partial_j v\,\mathrm{d}\boldsymbol{x} \geq \frac{\theta}{2}\|\zeta D_k^h\nabla u\|_{L^2(U)}^2 - C\|\nabla u\|_{L^2(U)}^2. \tag{2.5.5}$$

On the other hand, we have

$$\begin{aligned}\|D_k^{-h}(\zeta^2 D_k^h u)\|_{L^2(U)}^2 \leq \|\nabla(\zeta^2 D_k^h u)\|_{L^2(U)}^2 &\leq C\left(\int_W |\zeta^2\nabla D_k^h u|^2\,\mathrm{d}\boldsymbol{x} + \int_W |\nabla(\zeta^2)||D_k^h u|^2\,\mathrm{d}\boldsymbol{x}\right)\\ &\leq C\left(\|\nabla u\|_{L^2(U)}^2 + \|\zeta(D_k^h\nabla u)\|_{L^2(U)}^2\right).\end{aligned}$$

Again, by Young's inequality with ε, we have

$$\int_U fv\,\mathrm{d}\boldsymbol{x} \leq \varepsilon\|v\|_{L^2(U)}^2 + \frac{C}{\varepsilon}\|f\|_{L^2(U)}^2 \leq C\varepsilon\left(\|\nabla u\|_{L^2(U)}^2 + \|\zeta(D_k^h\nabla u)\|_{L^2(U)}^2\right) + C''\|f\|_{L^2(U)}^2.$$

Choosing $\varepsilon \in (0, \frac{\theta}{4C})$, we get

$$\int_U fv\,\mathrm{d}\boldsymbol{x} \leq \frac{\theta}{4}\|\zeta D_k^h\nabla u\|_{L^2(U)}^2 + C\left(\|\nabla u\|_{L^2(U)}^2 + \|f\|_{L^2(U)}^2\right) \tag{2.5.6}$$

Combining (2.5.4), (2.5.5) and (2.5.6), we get

$$\int_V |D_k^h\nabla u|^2\,\mathrm{d}\boldsymbol{x} \leq \int_U \zeta^2|D_k^h\nabla u|^2\,\mathrm{d}\boldsymbol{x} \leq C\int_U |f|^2 + |\nabla u|^2\,\mathrm{d}\boldsymbol{x}.$$

So, we get $\nabla u \in H^1_{\text{loc}}(U)$ and $u \in H^2_{\text{loc}}(U)$ with the estimate

$$\|u\|_{H^2(V)} \le C(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}).$$

The final step is to replace $\|u\|_{H^1(U)}$ by $\|u\|_{L^2(U)}$. This is rather easy if we set $v := \zeta^2 u$ in (2.5.4). Mimicing the steps above, we can prove

$$\|u\|_{H^1(V)} \le C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}).$$

Thus, we conclude that

$$\|u\|_{H^2(V)} \le C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}).$$

□

2.5.3 *Global regularity

Now we extend the estimates in Theorem 2.5.2 to study the regularity of weak solutions up to the boundary. In this section, we aim to prove the following conclusion

Theorem 2.5.5 (Elliptic regularity up to the boundary). Let U be a bounded open set with a C^2 boundary ∂U. Assume $a^{ij} \in C^1(\overline{U})$, $b^i, c \in L^\infty(U)$ and $f \in L^2(U)$. Suppose that $u \in H^1_0(U)$ is a weak solution to (2.0.1)

$$Lu = f \text{ in } U, \qquad u = 0 \text{ on } \partial U.$$

Then $u \in H^2(U)$ with the estimate

$$\|u\|_{H^2(U)} \le C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}). \tag{2.5.7}$$

Here the constant $C > 0$ depends on U and the coefficients of L.

Remark 2.5.1. If $u \in H^1_0(U)$ is the unique weak solution to (2.0.1), then by Exercise 2.3.1, the above estimate can be simplified to be

$$\|u\|_{H^2(U)} \le C\|f\|_{L^2(U)}. \tag{2.5.8}$$

Again by induction on the differentiability order of f and the coefficients of L, it is easy to prove the followng two corollaries.

Corollary 2.5.6 (High-order global elliptic regularity). Assume $m \in \mathbb{N}$, $a^{ij}, b^i, c \in C^{m+1}(\overline{U})$ and $f \in H^m(U), \partial U \in C^{m+2}$. Suppose further $u \in H^1_0(U)$ is a weak solution to

$$Lu = f \text{ in } U, \qquad u = 0 \text{ on } \partial U.$$

Then $u \in H^{m+2}(U)$ with the estimate

$$\|u\|_{H^{m+2}(U)} \le C(\|f\|_{H^m(U)} + \|u\|_{L^2(U)}), \tag{2.5.9}$$

where $C > 0$ depends only on m, U and the coefficients of L. If $u \in H_0^1(U)$ is the unique weak solution, the above estimate can be simplified to be

$$\|u\|_{H^{m+2}(U)} \le C\|f\|_{H^m(U)}. \tag{2.5.10}$$

Corollary 2.5.7 (C^∞ global elliptic regularity). Assume $a^{ij}, b^i, c, f \in C^\infty(\overline{U})$ and $\partial U \in C^\infty$. Suppose further $u \in H_0^1(U)$ is a weak solution to

$$Lu = f \text{ in } U, \qquad u = 0 \text{ on } \partial U.$$

Then $u \in C^\infty(\overline{U})$.

Proof of Theorem 2.5.5. First, the local regularity (Theorem 2.5.2) indicates that $Lu = f$ a.e. in U instead of only being in the weak sense. Indeed, Theorem 2.5.2 shows that for any $\varphi \in C_c^\infty(U)$, $B[u,\varphi] = (f,\varphi)$ and so $(Lu - f, \varphi)_{L^2(U)} = 0$. Lemma 1.1.2 then yields $Lu - f = 0$ a.e. in U.

Given $\boldsymbol{x}^0 \in \partial U$, there exists $r > 0$ and a C^2 function $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ such that

$$U \cap B(\boldsymbol{x}^0, r) = \{\boldsymbol{x} \in B(\boldsymbol{x}^0, r) | x_d > \gamma(x_1, \cdots, x_{d-1})\}.$$

Also, there exists a diffeomorphism Φ and sufficiently small $s > 0$ such that $\boldsymbol{y} = \Phi(\boldsymbol{x})$ and

$$U' := B(\boldsymbol{0}, s) \cap \{y_d > 0\} \subset \Phi(U \cap B(\boldsymbol{x}^0, r)).$$

Since U is bounded, we can cover the boundary ∂U by a finite family of such open balls and flatten the curved boundary of each open set via the above diffeomorphisms. Thus, we may assume $U = B(\boldsymbol{0}, 1) \cap \mathbb{R}_+^d$ for simplicity. In this case, $\partial_1, \cdots, \partial_{d-1}$ are tangential derivatives, so the tangential regularity already satisfies the conclusion in Theorem 2.5.2 and it remains to compute the normal derivatives involving ∂_d.

Let $U = B(\boldsymbol{0}, 1) \cap \mathbb{R}_+^d$ and $V = B(\boldsymbol{0}, \frac{1}{2}) \cap \mathbb{R}_+^d$. Choose a cut-off function $\zeta \in C^\infty(U)$ such that

$$\zeta = 1 \text{ in } B(\boldsymbol{0}, \frac{1}{2}), \quad \operatorname{Spt}\zeta \subset B(\boldsymbol{0}, 1), \quad 0 \le \zeta \le 1.$$

Now u is a weak solution to (2.0.1), so we get by definition that

$$\int_U a^{ij}\partial_i u \partial_j v \, \mathrm{d}\boldsymbol{x} = \int \left(f - b^i\partial_i u - cu\right) v \, \mathrm{d}\boldsymbol{x} \qquad \forall v \in H_0^1(U). \tag{2.5.11}$$

Now, for $1 \le k \le d-1$, taking $v = -D_k^h(\zeta^2 D_k^h u)$ in (2.5.11), we get as in Theorem 2.5.2

$$\int_V |D_k^h \nabla u|^2 \, \mathrm{d}\boldsymbol{x} \le C \int_U |f|^2 + |u|^2 + |\nabla u|^2 \, \mathrm{d}\boldsymbol{x} \qquad \forall 1 \le k \le d-1. \tag{2.5.12}$$

This actually gives us

$$\|\partial_i\partial_j u\|^2_{L^2(V)} \le C\left(\|f\|^2_{L^2(U)} + \|u\|^2_{L^2(U)} + \|\nabla u\|^2_{L^2(U)}\right), \qquad \forall i+j<2d. \tag{2.5.13}$$

It remains to control $\partial_d^2 u$. We do not have to compute the difference quotient as above. Instead, we can directly express the second-order normal derivative in terms of $\partial_i\partial_j u$ for $i+j<2d$ (at least one tangential derivative) which has been controlled in (2.5.13). Specifically, since $Lu = f$ a.e. in U (this step is necessary!), we have

$$a^{dd}\partial_d^2 u = -\sum_{i+j<2d} \partial_j(a^{ij}\partial_i u) + b^i\partial_i u + cu - f - \partial_d u \partial_d a^{dd}.$$

Also, the uniform ellipticity condition implies $a^{dd} \ge \theta$. Thus, we have

$$\|\partial_d^2 u\|^2_{L^2(V)} \le C\left(\|f\|^2_{L^2(U)} + \|u\|^2_{H^1(U)}\right). \tag{2.5.14}$$

Mimicing the last step in the proof of Theorem 2.5.2, we can improve the right side

$$\|\partial_d^2 u\|^2_{L^2(V)} \le C\left(\|f\|^2_{L^2(U)} + \|u\|^2_{L^2(U)}\right), \tag{2.5.15}$$

which together with (2.5.13) leads to our desired estimate

$$\|u\|^2_{H^2(V)} \le C\left(\|f\|^2_{L^2(U)} + \|u\|^2_{L^2(U)}\right). \tag{2.5.16}$$

□

Exercise 2.5

Exercise 2.5.1. Let $u \in H^1(\mathbb{R}^d)$ have compact support and be a weak solution to $-\Delta u + c(u) = f$ in $\mathbb{R}^d$. Here $f \in L^2(\mathbb{R}^d)$ and $c : \mathbb{R} \to \mathbb{R}$ is smooth with $c(0) = 0$, $c' \ge 0$. Prove that $u \in H^2(\mathbb{R}^d)$.

(Hint: Mimic the proof of the interior regularity theorem but without using the cut-off function ζ.)

Exercise 2.5.2. Let $B_R = B(\mathbf{0}, R) \subset \mathbb{R}^2$ with $R < 1$. Denote $\boldsymbol{x} = (x_1, x_2)$. Consider the equation

$$\Delta u = \frac{x_2^2 - x_1^2}{2|\boldsymbol{x}|^2}\left(\frac{4}{\sqrt{-\ln|\boldsymbol{x}|}} + \frac{1}{2(-\ln|\boldsymbol{x}|)^{\frac{3}{2}}}\right)$$

where the right side is continuous in $\overline{B_R}$ if we set it equal to 0 at the origin. Define

$$u(\boldsymbol{x}) := \sqrt{-\ln|\boldsymbol{x}|}(x_1^2 - x_2^2).$$

Prove that

(1) $u \in C(\overline{B_R}) \cap C^\infty(\overline{B_R}\backslash\{\mathbf{0}\})$ satisfies the above equation in $B_R\backslash\{\mathbf{0}\}$ with the boundary condition $u = \sqrt{-\ln R}(x_1^2 - x_2^2)$ on ∂B_R.
(2) $\lim\limits_{|\boldsymbol{x}|\to 0} \partial_1^2 u = \infty$ which then leads to $u \notin C^2(B_R)$.

Exercise 2.5.3. Give a counterexample of $f \in L^1(U)$ such that $\|D^h f\|_{L^1(V)} \le C$ for all $0 < |h| < \frac{1}{2}\text{dist}\,(V, \partial U)$ but $f \notin W^{1,1}(V)$.

Exercise 2.5.4. This exercise aims to complete the proof of Theorem 2.5.5 when $U \ne B(\mathbf{0}, 1) \cap R^d_+$. As in the proof, given $\boldsymbol{x}^0 \in \partial U$, there exists $r > 0$ and a C^2 function $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ such that

$$U \cap B(\boldsymbol{x}^0, r) = \{\boldsymbol{x} \in B(\boldsymbol{x}^0, r) | x_d > \gamma(x_1, \cdots, x_{d-1})\}.$$

Also, there exists a diffeomorphism Φ and sufficiently small $s > 0$ such that $\boldsymbol{y} = \Phi(\boldsymbol{x})$ (or $\boldsymbol{x} = \Psi(\boldsymbol{y})$) and

$$U' := B(\mathbf{0}, s) \cap \{y_d > 0\} \subset \Phi(U \cap B(\boldsymbol{x}^0, r)).$$

Denote also $V' := B(\mathbf{0}, \frac{s}{2}) \cap \{y_d > 0\}$.

(1) Prove that $|\det(\nabla\Phi)| = 1$.
(2) Define $u'(\boldsymbol{y}) := u(\Psi(\boldsymbol{y}))$. Prove that $u' \in H_0^1(U')$ is the weak solution to

$$L'u' = f' \text{ in } U', \qquad u' = 0 \text{ on } \partial U.$$

Here $f'(\boldsymbol{y}) := f(\Psi(\boldsymbol{y}))$ and

$$L'u' := -\partial_{y_l}({a'}^{kl}\partial_{y_k} u') + {b'}^k \partial_{y_k} u' + c(\Psi(\boldsymbol{y}))u'(\Psi(\boldsymbol{y}))$$

is still uniformly elliptic with coefficients

$${a'}^{kl} := a^{ij}(\Psi(\boldsymbol{y}))\frac{\partial\Phi^k}{\partial x_i}(\Psi(\boldsymbol{y}))\frac{\partial\Phi^l}{\partial x_j}(\Psi(\boldsymbol{y})), \quad {b'}^k := b^i(\Psi(\boldsymbol{y}))\frac{\partial\Phi^k}{\partial x_i}(\Psi(\boldsymbol{y})).$$

(3) Use (2) and the proof of Theorem 2.5.5 to show that

$$\|u'\|^2_{H^2(U')} \le C\left(\|f\|^2_{L^2(U)} + \|u\|^2_{L^2(U)}\right). \tag{2.5.17}$$

2.6 Maximum principle

In this section, we introduce one of the most important tools in the study of elliptic PDEs, namely the Maximum Principles. The Maximum Principle is based on a simple observation: if $u \in C^2$ attains its maximum over an open set U at a point $\boldsymbol{x}_0 \in U$, then

$$\nabla u(\boldsymbol{x}) = 0, \qquad \nabla^2 u(\boldsymbol{x}_0) \le 0.$$

It should be noted that this method no longer based on the L^2-based "energy method" as in previous sections. Instead, the deductions based on the Maximum Principles are mostly *pointwise* in character and so we have to require the solution belongs to C^2 (classical solution). For technical simplicity, we would assume the elliptic operator has the non-divergence form (2.0.3) with *continuous* coefficients throughout this section.

One may also ask if there is any *pointwise estimates* for the weak solution to elliptic PDEs, especially when the coefficients are not so regular as stated above. The answer is yes, but the proof and the conclusion are both rather different from what we are going to discuss in this section. In section 2.7, we will introduce the so-called *De Giorgi-Moser iteration* which gives the L^∞ estimates for the weak solutions to elliptic PDEs with rough coefficients.

2.6.1 Weak maximum principle

Assume $U \subset \mathbb{R}^d$ is a bounded open set (not necessarily connected). We have

Theorem 2.6.1 (Weak Maximum Principle). Assume $u \in C^2(U) \cap C(\overline{U})$ and $c = 0$ in U. Then $Lu \le 0$ (≥ 0, resp.) in U implies $\max_{\overline{U}} u = \max_{\partial U} u$ ($\min_{\overline{U}} u = \min_{\partial U} u$, resp.). In such case, we call u is a *subsolution* (*supersolution*, resp.). In particular, if $Lu = 0$ in U, then we have $\max_{\overline{U}} |u| = \max_{\partial U} |u|$.

Proof. The proof relies on a perturbation trick. Assume the conclusion already holds for u satisfying $Lu < 0$ in U. Then for the case $Lu \le 0$ in U, we consider $u^\varepsilon(\boldsymbol{x}) := u(\boldsymbol{x}) + \varepsilon e^{\lambda x_1}$ for $\boldsymbol{x} \in U$. We compute that

$$\begin{aligned} Lu^\varepsilon &= Lu + \varepsilon L(e^{\lambda x_1}) \le 0 + \varepsilon e^{\lambda x_1}(-\lambda^2 a^{11} + \lambda b^1) \\ &\le \varepsilon e^{\lambda x_1}(-\lambda^2 \theta + \lambda \|b\|_{L^\infty(U)}) < 0 \qquad \text{in } U, \end{aligned}$$

if we choose $\lambda > 0$ sufficiently large. By the assumption, we know $\max_{\overline{U}} u^\varepsilon = \max_{\partial U} u^\varepsilon$, and so

$$\max_{\overline{U}} u \le \max_{\overline{U}} (u + \varepsilon e^{\lambda x_1}) = \max_{\overline{U}} u^\varepsilon = \max_{\partial U} u^\varepsilon \le \max_{\partial U} u + \max_{\partial U} \varepsilon e^{\lambda x_1}.$$

Let $\varepsilon \to 0_+$ and we get the desired conclusion $\max_{\overline{U}} u = \max_{\partial U} u$.

Now, it remains to prove $\max_{\overline{U}} u = \max_{\partial U} u$ holds for all u such that $Lu < 0$ in U. Suppose, to the contrary, that there exists $\boldsymbol{x}_0 \in U$ with $u(\boldsymbol{x}_0) = \max_{\overline{U}} u$. At this interior point, we have

$$\partial_i u(\boldsymbol{x}_0) = 0 \ (1 \le i \le d), \quad \nabla^2 u(\boldsymbol{x}_0) \le 0 \text{ (negative semi-definite)} \Rightarrow \partial_i^2 u(\boldsymbol{x}_0) \le 0,\ 1 \le i \le d.$$

If a^{ij} is a diagonal matrix, then we already obtain our conclusion, as in this case each $a^{ii} \ge \theta$ and so

$$Lu(\boldsymbol{x}_0) = -a^{ii} \partial_i^2 u \ge 0$$

which leads to a contradiction with $Lu < 0$ in U. In general, $\{a^{ij}\}$ is strictly positive definite and symmetric, so there exists an orthogonal matrix $\mathbf{O} = \{o_{ij}\}$ such that

$$\mathbf{O}A\mathbf{O}^\top = \Lambda, \qquad \Lambda = \mathrm{diag}\,(\lambda_1, \cdots, \lambda_d), \;\; \lambda_i \geq \theta, \; 1 \leq i \leq d.$$

Also, we write $\boldsymbol{y} = \boldsymbol{x}_0 + \mathbf{O}(\boldsymbol{x} - \boldsymbol{x}_0)$ and then

$$\partial_{x_i} u = \sum_k o_{ki} \partial_{y_k} u, \qquad \partial_{x_i}\partial_{x_j} u = \sum_{k,l} o_{ki}(\partial_{y_k y_l} u) o_{lj}.$$

Hence, at $\boldsymbol{x}_0$, we compute that

$$Lu(\boldsymbol{x}_0) = -a^{ij}\partial_{x_i}\partial_{x_j} u = -(o_{ki}a^{ij}o_{lj})\partial_{y_k}\partial_{y_l} u = -\sum_{k=1}^d \lambda_k \partial_{y_k}^2 u \geq 0,$$

again a contradiction. □

When $c \geq 0$ in the coefficients of L, we can still establish a similar conclusion for positive maximum and negative minimum.

Theorem 2.6.2 (Weak Maximum Principle for $c \geq 0$). Assume $u \in C^2(U) \cap C(\overline{U})$ and $c \geq 0$ in U. Then $Lu \leq 0$ (≥ 0, resp.) in U implies $\max_{\overline{U}} u \leq \max_{\partial U} u^+$ ($\min_{\overline{U}} u \geq -\max_{\partial U} u^-$, resp.). In particular, if $Lu = 0$ in U, then again we have $\max_{\overline{U}} |u| = \max_{\partial U} |u|$.

Proof. First, we assume u satisfy $Lu < 0$ in U. If the *positive maximum* of u is attained at $\boldsymbol{x}_0 \in U$, then following the proof of Theorem , we have

$$Lu(\boldsymbol{x}_0) = -a^{ij}(\boldsymbol{x}_0)\partial_i\partial_j u(\boldsymbol{x}_0) + cu(\boldsymbol{x}_0) \geq 0 + cu(\boldsymbol{x}_0) \geq 0$$

which leads to a contradiction with $Lu < 0$ in U. Thus, when $Lu < 0$ in U, u must attain its positive maximum on the boundary ∂U.

In general, if $Lu \leq 0$ in U, we again do the perturbation as in the proof of Theorem 2.6.1 and no longer repeat the proof here. □

2.6.2 Hopf's lemma and strong maximum principle

In many cases, the weak maximum principle is already enough for us to control the pointwise bound of a (sub-)solution to an elliptic PDE. In this section, we substantially strengthen the conclusions of the maximum principle, by showing that a subsolution in a *connected* region cannot attain its maximum in the interior unless it is a constant. The proof depends on some subtle analysis of the outward normal derivative $\frac{\partial u}{\partial N}$ at a boundary maximal.

Lemma 2.6.3 (Hopf's lemma). Assume $u \in C^2(U) \cap C^1(\overline{U})$ and $c = 0$ in U. Suppose also

- $Lu \leq 0$ in U.
- There exists $\boldsymbol{x}^0 \in \partial U$ such that $u(\boldsymbol{x}^0) > u(\boldsymbol{x})$ for all $\boldsymbol{x} \in U$.
- U satisfies the *interior ball condition* at $\boldsymbol{x}^0$, that is, there exists an open ball $B \subset U$ with $\boldsymbol{x}^0 \in \partial B$.

Then we assert that

$$\frac{\partial u}{\partial N}(\boldsymbol{x}^0) > 0$$

where N is the outward unit normal vector to B at $\boldsymbol{x}^0$. If $c \geq 0$, then the same conclusion holds if $u(\boldsymbol{x}^0) \geq 0$.

Hopf's lemma then easily leads to the following Strong Maximum Principle.

Theorem 2.6.4 (Strong Maximum Principle). Let $U \subset \mathbb{R}^d$ be a bounded domain. Assume $u \in C^2(U) \cap C(\overline{U})$ and $c = 0$ in U. If $Lu \leq 0$ (≥ 0, resp.) in U and u attains its maximum (minimum, resp.) over $\overline{U}$ at an interior point, then

$$u \text{ is constant within } U.$$

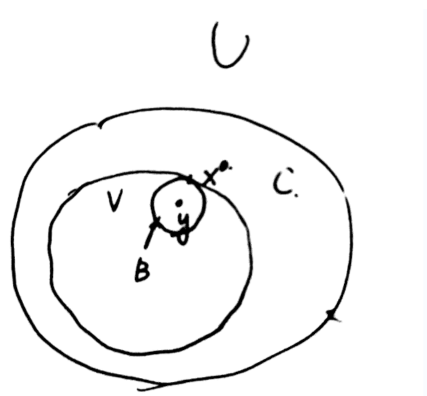
Simiarly for $c \geq 0$, we have

Theorem 2.6.5 (Strong Maximum Principle for $c \geq 0$). Let $U \subset \mathbb{R}^d$ be a bounded domain. Assume $u \in C^2(U) \cap C(\overline{U})$ and $c \geq 0$ in U. If $Lu \leq 0$ (≥ 0, resp.) in U and u attains its non-negative maximum (non-positive minimum, resp.) over $\overline{U}$ at an interior point, then

$$u \text{ is constant within } U.$$

First, we prove the Strong Maximum Principle ($c = 0$) provided that the Hopf's lemma holds.

Proof. Assume $Lu \leq 0$ in U. Write $M := \max_{\overline{U}} u$ and $C := \{\boldsymbol{x} \in U | u(\boldsymbol{x}) = M\}$. Suppose, to the contrary, the set $V := \{\boldsymbol{x} \in U | u(\boldsymbol{x}) < M\} \neq \emptyset$. Choose a point $\boldsymbol{y} \in V$ with dist $(\boldsymbol{y}, C) <$ dist $(\boldsymbol{y}, \partial U)$ and let B be the largest ball with center $\boldsymbol{y}$ whose interior lies in V. Then, there exists $\boldsymbol{x}^0 \in C$ such that $\boldsymbol{x}^0 \in \partial B$.



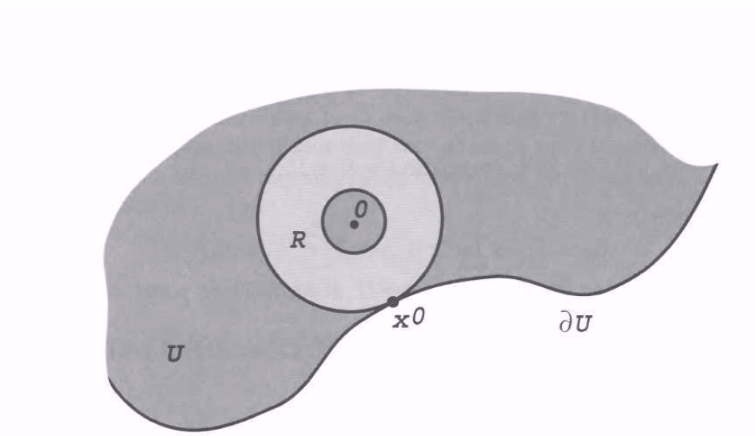
Do note that the connectedness of U is necessary here, otherwise dist (V, C) may be strictly positive so that $\boldsymbol{x}^0 \in C$ may not exist.

Now, V satisfies the interior ball condition. By Hopf's lemma, we get $\frac{\partial u}{\partial N}(\boldsymbol{x}^0) > 0$ with N the outward unit normal vector to ∂B. This leads to a contradiction: u attains its maximum at $\boldsymbol{x}_0$, which gives $\nabla u(\boldsymbol{x}^0) = \mathbf{0}$. □

Now, it remains to prove the Hopf's lemma. We only consider the case $c \geq 0$ (with $u(\boldsymbol{x}^0) \geq 0$) and the case $c = 0$ (without $u(\boldsymbol{x}^0) \geq 0$) can be proved in the same way.

Proof of Hopf's lemma. First, it is easy to see $\frac{\partial u}{\partial N}(\boldsymbol{x}^0) \geq 0$ as the value of u is non-decreasing when $\boldsymbol{x} \in B$ approaches $\boldsymbol{x}^0$ along the direction of N. To prove the strict inequality, we again consider adding a perturbation term to u.

WLOG the ball B is $B(\mathbf{0}, r)$.



In view of the assumptions of Hopf's lemma, we want to seek for a "nice" function v such that for $0 < \varepsilon \ll 1$:

(1) $u(\boldsymbol{x}_0) \geq u(\boldsymbol{x}) + \varepsilon v(\boldsymbol{x})$ for $x \in \partial B(\mathbf{0}, r)$ and $\partial \overline{B(\mathbf{0}, r/2)}$.
(2) $L(u + \varepsilon v) \leq 0$ in the annulus $A := B(\mathbf{0}, r) \backslash \overline{B(\mathbf{0}, r/2)}$.
(3) $v|_{\partial A} = 0$ and $\frac{\partial v}{\partial N}(\boldsymbol{x}_0) < 0$

Once we achieve this, we use the weak maximum principle to find that $u(\boldsymbol{x}) + \varepsilon v(\boldsymbol{x}) - u(\boldsymbol{x}_0) \leq 0$ in A and $= 0$ at $\boldsymbol{x}_0$. So, the normal derivative of $u + \varepsilon v - u(\boldsymbol{x}_0)$ must be nonnegative at $\boldsymbol{x}_0$ and thus $\frac{\partial u}{\partial N}(\boldsymbol{x}_0) \geq -\varepsilon \frac{\partial v}{\partial N}(\boldsymbol{x}_0) > 0$ as desired.

The "nice" function v is chosen to be $v(\boldsymbol{x}) = e^{-\lambda|\boldsymbol{x}|^2} - e^{-\lambda r^2}$ for $\boldsymbol{x} \in B(\mathbf{0}, r)$, where $\lambda > 0$ is a large constant to be determined. We compute that

$$\begin{aligned} Lv &= -a^{ij}\partial_i\partial_j v + b^i\partial_i v + cv = e^{-\lambda|\boldsymbol{x}|^2}\left(a^{ij}(-4\lambda^2 x_i x_j + 2\lambda\delta_{ij}) - 2\lambda b^i x_i + c(1 - e^{-\lambda(r^2 - |\boldsymbol{x}|^2)})\right) \\ &\leq e^{-\lambda|\boldsymbol{x}|^2}(-4\theta\lambda^2|\boldsymbol{x}|^2 + 2\lambda\sum_i a^{ii} + 2\lambda|b||\boldsymbol{x}| + c). \end{aligned}$$

In the annulus A, we have

$$Lv \leq e^{-\lambda|\boldsymbol{x}|^2}(-\theta\lambda^2 r^2 + 2\lambda\sum_i a^{ii} + 2\lambda|b|r + c)$$

which is ≤ 0 for $\boldsymbol{x} \in A$ if we pick $\lambda > \frac{d}{2r^2}$ large enough. The other desired properties can also be verified. $\square$

2.6.3 *Harnack's inequality: logarithmic gradient estimates

The maximum principles give the pointwise upper bound for a (sub-)solution to an elliptic PDE. We next want to ask if the "oscillation" of the solution can be controlled or not in a suitable way. The answer is given by the Harnack's inequality: at least in any subregion away from the boundary, the values of a nonnegative solution are comparable.

Theorem 2.6.6 (Harnack's inequality). Assume $u \geq 0$ is a C^2 solution to $Lu = 0$ in U with L defined by (2.0.3) and suppose $V \Subset U$ is connected. Then there exists a constant C such that

$$\sup_V u \leq C \inf_V u,$$

where $C > 0$ depends on V and the coefficients of L.

The proof technique is called the "logarithmic gradient estimates". We may assume $u > 0$, otherwise we just consider $u + \varepsilon > 0$ for small $\varepsilon > 0$. Given $V \Subset U$, we want to prove there exists $C > 0$ such that $u(\boldsymbol{x}) \leq Cu(\boldsymbol{y})$ for any $\boldsymbol{x}, \boldsymbol{y} \in V$, that is, $|\ln \frac{u(\boldsymbol{x})}{u(\boldsymbol{y})}| \leq C'$ for some $C' > 0$. The logarithemic term can be written as

$$\ln \frac{u(\boldsymbol{x})}{u(\boldsymbol{y})} = \ln u(\boldsymbol{x}) - \ln u(\boldsymbol{y}) = (\boldsymbol{x} - \boldsymbol{y}) \cdot \int_0^1 \nabla \ln u(t\boldsymbol{x} + (1-t)\boldsymbol{y})\, \mathrm{d}t,$$

and so it suffices to prove $\sup_V |\nabla \ln u| \leq C''$ for some $C'' > 0$.

One may ask why we consider converting the estimates to logarithemic functions. In fact, if we consider a special case: harmonic function ($L = -\Delta$), then we can easily get $-\Delta v = |\nabla v|^2$ for $v = \ln u$ and $\Delta w + 2\nabla w \cdot \nabla v = 2|\nabla^2 v|^2$ for $w = |\nabla v|^2$. Thus the problem is reduced to the interior gradient estimates of v and one can use the techniques in Exercise 2.6.8 to finish the proof.

Below we prove the case of $b^i = c = 0$ and $u > 0$. There are some tricky construction of auxiliary functions and we will briefly explain why we make such a choice by computing the case of harmonic functions after proving Harnack's inequality.

Proof. Suppose $b^i = c = 0$ and $u > 0$ as stated above. Let $v = \ln u$, we compute that

$$u = e^v, \;\; \partial_i u = e^v \partial_i v, \;\; \partial_i \partial_j v = e^v(\partial_i v \partial_j v + \partial_i \partial_j v),$$

which together with $Lu = 0$ gives us

$$a^{ij}(\partial_i \partial_j v + \partial_i v \partial_j v) = 0 \text{ in } U.$$

Now set $w := a^{ij}\partial_i v \partial_j v$ and then $w = -a^{ij}\partial_i \partial_j v$. We now claim

Claim. Let $b^k := -2a^{kl}\partial_l v$. Then

$$-a^{kl}\partial_k\partial_l w + b^k\partial_k w \le -\frac{\theta^2}{2}|\nabla^2 v|^2 + C|\nabla v|^2. \tag{2.6.1}$$

The proof of this claim is based on lots of tedious computations. We would like to see what happens if this claim holds true. We define $\zeta \in C_c^\infty(U)$ to be a cut-off function with $0 \le \zeta \le 1$ and $\zeta = 1$ in V. Then define $z = \zeta^4 w$ to localize the estimates within V.

Assume z attains its maximum at some $\boldsymbol{x}_0 \in U$. Then $\partial_k w(\boldsymbol{x}_0) = 0$ and so

$$0 = \partial_k z = \zeta^4\partial_k w + 4\zeta^3 w\partial_k\zeta \Rightarrow \zeta\partial_k w + 4(\partial_k\zeta)w = 0,$$

and

$$\partial_k\partial_l z = \zeta^4\partial_k\partial_l w + 4\zeta^3\partial_l\zeta\partial_k w + 12\zeta^2\partial_l\zeta + \partial_k\zeta w + 4\zeta^3\partial_k\partial_l\zeta w + 4\zeta^3\partial_k\zeta\partial_l w.$$

We compute

$$\begin{aligned}
&- a^{kl}\partial_k\partial_l z + b^k\partial_k z\\
=&\zeta^4\left(-a^{kl}\partial_k\partial_l w + b^k\partial_k w\right) - 12a^{kl}(\zeta^2\partial_l\zeta\partial_k\zeta)w - 4a^{kl}(\zeta^3\partial_k\zeta)\partial_l w - 4a^{kl}\zeta^3\partial_l\partial_k\zeta\, w + 4b^k\zeta^3\partial_k\zeta\, w\\
=&\zeta^4\left(-a^{kl}\partial_k\partial_l w + b^k\partial_k w\right) + O\left(\zeta^3|\nabla w| + \zeta^2 w + |\nabla v|\zeta^3 w\right)
\end{aligned}$$

where $|\nabla v|$ in the last term is obtained by $|b^k| \le C|\nabla v|$.

At $\boldsymbol{x}_0 \in U$, we have $\partial_k z = 0$ and $-a^{kl}\partial_k\partial_l z \ge 0$ (the Hessian matrix $\nabla^2 z$ is negative semi-definite), so we get

$$0 \le \zeta^4\left(-a^{kl}\partial_k\partial_l w + b^k\partial_k w\right) + C'\left(\zeta^3|\nabla w| + \zeta^2|w| + |\nabla v|\zeta^3|w|\right). \tag{2.6.2}$$

Combining (2.6.1) and (2.6.2) and the uniform ellipticity condition, we get

$$0 \le \zeta^4(-\frac{\theta^2}{2}|\nabla^2 v|^2 + C|\nabla v|^2) + C'\left(\zeta^3|\nabla w| + \zeta^2|w| + |\nabla v|\zeta^3|w|\right), \quad \text{at } \boldsymbol{x}_0.$$

Since $w = -a^{ij}\partial_i\partial_j v$, we actually get

$$\zeta^4 w^2 \le C''\left(\zeta^4|\nabla v|^2 + \zeta^3|\nabla w| + \zeta^2 w + \zeta^3|\nabla v|w\right), \quad \text{at } \boldsymbol{x}_0 \tag{2.6.3}$$

Next we analyze the terms on the right side of the above inequality.

- $\zeta^3|\nabla v|w = (\zeta^2|\nabla v|)\zeta w \le \varepsilon w\zeta^4|\nabla v|^2 + wC_\varepsilon\eta^2 \le \frac{\varepsilon}{\theta}\zeta^4 w^2 + C(\varepsilon)w\zeta^2$. Here we use ε-Young's inequality and $\theta|\nabla v|^2 \le w$ which is obtained by $w = a^{ij}\partial_i v\partial_j v$.
- $\zeta^4|\nabla w|$. Recall that $\zeta\partial_k w + 4\partial_k\zeta w = 0$. This gives us $|\zeta\nabla w| \le C|w|$ and so $\zeta^3|\nabla w| \le C\zeta^2|w|$.
- $\zeta^4|\nabla v|^2 \le \zeta^2|\nabla v|^2 \le \dfrac{\zeta^2 w}{\theta}$.

Plugging these estimates back to (2.6.3), we get

$$\zeta^4 w^2 \le C_1 \varepsilon \zeta^4 w^2 + C_2 \zeta^2 w, \quad \text{at } \boldsymbol{x}_0$$

for some $C_1, C_2 > 0$ and $\varepsilon > 0$. Choosing $\varepsilon < \frac{1}{2C_1}$, we get

$$\zeta^4 w^2 \le 2C_2 \zeta^2 w \Rightarrow z = \zeta^4 w \le 2C_2 \zeta^2 \le 2C_2, \quad \text{at } \boldsymbol{x}_0.$$

Since $z = \zeta^4 w$ attains its maximum at $\boldsymbol{x}_0$ and $\zeta = 1$ in V, by using $w \ge \theta |\nabla v|^2$, we get

$$|\nabla v| \le C_0$$

for some $C_0 > 0$ as desired.

It remains to verify the claim. By direct computation, we obtain

$$\begin{aligned} \partial_l w &= \partial_l a^{ij} (\partial_i v \partial_j v) + 2a^{ij} \partial_l \partial_i v \, \partial_j v \\ \partial_k \partial_l w &= 2a^{ij} \partial_l \partial_i v \partial_k \partial_j v + 2a^{ij} \partial_k \partial_l \partial_i v \, \partial_j v + R \end{aligned}$$

with R defined by

$$R := \partial_k \partial_l a^{ij} \, \partial_i v \, \partial_j v + 2\partial_l a^{ij} \partial_k \partial_i v \, \partial_j v + 2\partial_k a^{ij} \partial_l \partial_i v \partial_j v$$

and $|R| \le C(|\nabla v|^2 + |\nabla v||\nabla^2 v|) \le \varepsilon |\nabla^2 v|^2 + C(\varepsilon)|\nabla v|^2$ for $\varepsilon > 0$. Thus, we obtain that

$$-a^{kl} \partial_k \partial_l w = -R - 2a^{kl} a^{ij} (\partial_l \partial_i v)(\partial_j \partial_k v) - 2a^{kl} a^{ij} \partial_k \partial_l \partial_i v \, \partial_j v. \tag{2.6.4}$$

The first term is controlled via the uniform ellipticity. Since $\{a^{ij}\} = P^\top P$ for some matrix P, we then get

$$a^{kl} a^{ij} (\partial_l \partial_i v)(\partial_j \partial_k v) = (\nabla^2 v \cdot P) \cdot (\nabla^2 v \cdot P)^\top \ge \theta^2 |\nabla^2 v|^2.$$

The second term in (2.6.4) contains third-order derivative and we shall use $w = a^{kl} \partial_k \partial_l v$ to reduce the order:

$$\begin{aligned} -a^{kl} a^{ij} \partial_k \partial_l \partial_i v \, \partial_j v &= - a^{ij} \partial_j v a^{lk} \partial_i \partial_k \partial_l v = -a^{ij} \partial_j v \left(\partial_i (a^{kl} \partial_k \partial_l v) - \partial_i a^{lk} \partial_k \partial_l v \right) \\ &= - a^{ij} \partial_j v (\partial_i w - \partial_i a^{lk} \partial_k \partial_l v) = -\frac{1}{2} b^i \partial_i w + a^{ij} \partial_i a^{lk} \partial_j v \partial_k \partial_l v. \end{aligned}$$

Inserting these two terms back to (2.6.4) and using the estimate of $|R|$, we get

$$\begin{aligned} & - a^{kl} \partial_k \partial_l w + b^k \partial_k w \le |R| - \theta^2 |\nabla v|^2 + |a^{ij} \partial_i a^{lk} \partial_j v \partial_k \partial_l v| \\ & \le |R| - \theta^2 |\nabla^2 v|^2 + C |\nabla v||\nabla^2 v| \le \varepsilon |\nabla^2 v|^2 + C(\varepsilon)|\nabla v|^2 - \theta^2 |\nabla^2 v|^2 \\ & \le -\frac{\theta^2}{2} |\nabla^2 v|^2 + C(\varepsilon)|\nabla v|^2, \end{aligned}$$

where we use Young's inequality and take $\varepsilon \in (0, \frac{\theta^2}{2})$ to absorb the ε-term. The proof is completed. □

Remark 2.6.1 (The choice of $\zeta^4 w$). One may ask why we choose the power of ζ to be 4 instead of 2 or other positive numbers. In fact, we can consider the case of $L = -\Delta u$ and $U = B(\mathbf{0}, 1)$, $V = B(\mathbf{0}, \frac{1}{2})$. Given a cut-off function φ with $0 \le \varphi \le 1$ and $\varphi = 1$ in V, we compute the equation of $\Delta(\varphi w)$ to get

$$\begin{aligned}\Delta(\varphi w) + 2\nabla w \cdot \nabla(\varphi w) &= 2\varphi|\nabla^2 v|^2 + 2(\nabla\varphi)\cdot(\nabla^2 v)\cdot(\nabla v)^\top + 2w\nabla\varphi\cdot\nabla v + (\Delta\varphi)w \\ &\ge \varphi|\nabla^2 v|^2 - 2|\nabla\varphi||\nabla v|^3 + \left(\Delta\varphi - \frac{4|\nabla\varphi|^2}{\varphi}\right)|\nabla v|^2.\end{aligned}$$

If we pick $\varphi = \zeta^4$ and use

$$|\nabla^2 v|^2 \ge \sum_{i=1}^d \partial_i^2 v \ge \frac{1}{d}(\Delta v)^2 = \frac{|\nabla v|^4}{d} = \frac{w^2}{d},$$

then we get a lower bound that does not depend on v (notice that the right side below is a nonnegative quantity plus a quartic polynomial with positive leading-order coefficient)

$$\begin{aligned}\Delta(\zeta^4 w) + 2\nabla w\cdot\nabla(\zeta^4 w) \ge& \frac{1}{d}\zeta^4|\nabla v|^4 - 8\zeta^3|\nabla\zeta||\nabla v|^3 + 4\zeta^2(\zeta\Delta\zeta - 13|\nabla\zeta|^2)|\nabla v|^2 \\ \overset{t=\zeta|\nabla v|}{=\!=\!=}& \frac{1}{2d}\zeta^4 w^2 + \frac{t^4}{2d} - 8|\nabla\zeta|t^3 + 4(\zeta\Delta\zeta - 13|\nabla\zeta|^2)t^2 \ge -C' \quad \forall t\in\mathbb{R}.\end{aligned}$$

We keep the term $\frac{1}{2d}\zeta^4 w^2$ as we need to estimate the maximum of $\zeta^4 w$. Assume $\zeta^4 w$ attains its maximum at $\boldsymbol{x}_0 \in B_1$. Then $\nabla(\zeta^4 w) = 0$ and $\Delta(\zeta^4 w) \le 0$ at $\boldsymbol{x}_0$. Hence, $\zeta^4 w^2(\boldsymbol{x}_0) \le 2C'd$. This C' now depends on d and ζ. If $w(\boldsymbol{x}_0) \ge 1$, then $\zeta^4 w^2(\boldsymbol{x}_0) \le 2C'd$; otherwise $\zeta^4 w(\boldsymbol{x}_0) \le \zeta^4(\boldsymbol{x}_0)$. So, there exists $C > 0$ such that $\zeta^4 w \le C$ in B_1.

If we now replace $\varphi = \zeta^4$ by ζ^2, then the above inequality becomes

$$\begin{aligned}&\Delta(\zeta^2 w) + 2\nabla w\cdot\nabla(\zeta^2 w) \\ \ge& \frac{1}{d}\zeta^2|\nabla v|^4 - 4\zeta|\nabla\zeta||\nabla v|^3 + 2\zeta\Delta\zeta|\nabla v|^2 \underline{-16|\nabla\zeta|^2|\nabla v|^2}.\end{aligned}$$

The right side is no longer a polynomial of $t' := \sqrt{\zeta}|\nabla v|$ or analogous quantity because of the appearance of the underlined term. Thus, we may not guarantee that the right hand side has a lower bound that is independent of v.

Exercise 2.6

Exercise 2.6.1. Let u be a smooth solution to $-a^{ij}\partial_i\partial_j u = 0$ in U and $a^{ij} \in C^1(\overline{U})$. Prove that

$$\|\nabla u\|_{L^\infty(U)} \le C(\|\nabla u\|_{L^\infty(\partial U)} + \|u\|_{L^\infty(\partial U)}).$$

(Hint: Let $v = |\nabla u|^2 + \lambda u^2$. Pick sufficiently large λ such that $Lv \le 0$ in U.)

Exercise 2.6.2. Let u be a smooth solution to $-a^{ij}\partial_i\partial_j u = f$ in U and $u|_{\partial U} = 0$ where f is bounded. Fix $\boldsymbol{x}^0 \in \partial U$. We say a C^2 function w is a barrier at $\boldsymbol{x}^0$ if

$$Lw \geq 1 \text{ in } U, \quad w(\boldsymbol{x}^0) = 0, \quad w \geq 0 \text{ on } \partial U.$$

Show that if w is a barrier at $\boldsymbol{x}^0$, there exists a constant $C > 0$ such that

$$|\nabla u(\boldsymbol{x}^0)| \leq C|\frac{\partial w}{\partial N}(\boldsymbol{x}^0)|.$$

Exercise 2.6.3. Let $Lu = -a^{ij}\partial_i\partial_j u + b^i\partial_i u + cu$. If there exists $v \in C^2(U) \cap C(\overline{U})$ such that $Lv \geq 0$ in U and $v > 0$ on $\overline{U}$. Prove that any $u \in C^2(U) \cap C(\overline{U})$ satisfying $Lu \leq 0$ in U and $u|_{\partial U} \leq 0$ must be non-positive in U.

(Hint: Let $w = \frac{u}{v}$ and consider $\tilde{L}w = -a^{ij}\partial_i\partial_j w + \partial_i w(b^i - a^{ij}\partial_j v \cdot \frac{2}{v})$.)

Exercise 2.6.4 (Removable singularity of harmonic functions)**.** Assume u is harmonic in the punctured ball $\check{B}_R := B(\mathbf{0}, R)\backslash\{\mathbf{0}\} \subset \mathbb{R}^d$ $(d \geq 2)$ and satisfies

$$u(\boldsymbol{x}) = \begin{cases} o(|\boldsymbol{x}|^{2-d}) & d \geq 3 \\ o(\ln|\boldsymbol{x}|) & d = 2, \end{cases} \quad \text{as } |\boldsymbol{x}| \to 0.$$

Prove that u can be defined at $\boldsymbol{x} = \mathbf{0}$, that is, u is harmonic in $B(\mathbf{0}, R)$.

Exercise 2.6.5. For $\boldsymbol{x} \in \mathbb{R}^d\backslash\{\mathbf{0}\}$ $(d \geq 2)$, we define its inversion point with respect to the unit sphere by $\boldsymbol{x}^* := \frac{\boldsymbol{x}}{|\boldsymbol{x}|^2}$. Define the Kelvin transform of $u(\boldsymbol{x})$ by $(\mathcal{K}u)(\boldsymbol{x}) = u(\boldsymbol{x}^*)|\boldsymbol{x}^*|^{d-2} = u(\frac{\boldsymbol{x}}{|\boldsymbol{x}|^2})|\boldsymbol{x}|^{2-d}$. By following the steps below, prove that if u is harmonic, then so is $\mathcal{K}u$.

(1) For any $1 \leq i, j \leq d$, $\frac{\partial x_j^*}{\partial x_i} = \frac{\delta_{ij}}{|\boldsymbol{x}|^2} - \frac{2x_i x_j}{|\boldsymbol{x}|^4}$. Here $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Then conclude that $\nabla\boldsymbol{x}^*(\nabla\boldsymbol{x}^*)^\top = |\boldsymbol{x}|^{-4}I_d$, where I_d is the $d \times d$ identity matrix.
(2) Use (1) to show that $\Delta(\boldsymbol{x}^*) = 2(2-d)\frac{\boldsymbol{x}}{|\boldsymbol{x}|^4}$.
(3) Verify that $\Delta(\mathcal{K}u(\boldsymbol{x})) = \Delta(u(\frac{\boldsymbol{x}}{|\boldsymbol{x}|^2})|\boldsymbol{x}|^{2-d}) = 0$.
(4) Prove that if $u(\boldsymbol{x})$ is harmonic in the exterior of the unit ball, then $(\mathcal{K}u)(\boldsymbol{x})$ is a harmonic function inside the punctured unit ball.

Exercise 2.6.6. Let $U = \mathbb{R}^d\backslash\overline{B_1}$ and $u \in C^2(U) \cap C(\overline{U})$ be the solution to $\Delta u = 0$ in U and $u = 0$ on $\partial B(\mathbf{0}, 1)$. Assume u also satisfies $\lim_{|\boldsymbol{x}|\to\infty} u(\boldsymbol{x})/\ln|\boldsymbol{x}| = 0$.

(1) Show that $u \equiv 0$ in U when $d = 2$.
(2) Given a counterexample to show that (1) is not true for $d \geq 3$.
(3) When $d \geq 3$, if we alternatively assume $\lim_{|\boldsymbol{x}|\to\infty} u(\boldsymbol{x}) = 0$. Prove that $u \equiv 0$ in U.

(Hint: Use Exercise 2.6.4 and Exercise 2.6.5(4).)

Exercise 2.6.7. Prove that the PDE defined in Exercise 2.5.2 has no classical solution in $C^2(B_R)$.

(Hint: Suppose, to the contrary, v is such a solution. Consider $w = u - v$ where u is the solution defined in Exercise 2.5.2. Then w is harmonic and bounded in $B_R \backslash \{\mathbf{0}\}$. Exercise 2.6.4 then implies that w can be extended to a harmonic function B_R and therefore belongs to $C^2(B_R)$. This then leads to a contradiction with Exercise 2.5.2(2).)

Exercise 2.6.8. Let $u \in C^3(B_1) \cap C^1(\overline{B_1})$ be a harmonic function in B_1. Prove (without using the mean-value property) that

(1) $|\nabla u|$ attains its maximum on ∂B_1. (Hint: Compute $\Delta(|\nabla u|^2)$)
(2) There exists a constant $C > 0$ such that

$$\max_{B_{\frac{1}{2}}} |\nabla u| \le C \max_{\partial B_1} |u|.$$

(Hint: Find a suitable cut-off function $\varphi \in C_1^2(B_1)$ such that $\Delta(\varphi|\nabla u|^2) \ge -C'|\nabla u|^2$ for some $C' > 0$. Then use $\Delta(u^2) = 2|\nabla u|^2$ to obtain the desired conclusion.)

2.7 De Giorgi-Nash-Moser iteration (TBA)

Consider the equation $Lu = -\partial_j(a^{ij}\partial_i u + d^j u) + (b^i \partial_i u + cu) = f + \partial_i f^i$ in U. The coefficients satisfy $a^{ij} \in L^\infty(U)$ and there exist constants λ, Λ such that

$$\lambda|\boldsymbol{\xi}|^2 \le a^{ij}(\boldsymbol{x})\xi_i\xi_j \le \Lambda|\boldsymbol{\xi}|^2, \quad \forall \boldsymbol{\xi} \in \mathbb{R}^d,\ \boldsymbol{x} \in U;$$
$$\sum_i \|b^i\|_{L^d(\Omega)} + \sum_i \|d^i\|_{L^d(\Omega)} + \|c\|_{L^{\frac{d}{2}}(U)} \le \Lambda.$$

Assume also $c - \partial_i d^i \ge 0$ in the weak sense, that is, $\int_U (c\varphi + d^i\partial_i\varphi)\,\mathrm{d}\boldsymbol{x} \ge 0$ for any $\varphi \in C_c^\infty(U)$ with $\varphi \ge 0$. Let $u \in H^1(U)$ is a weak subsolution, that is $B[u,v] \le (\ge, =,$ resp.$)(f,v)_{L^2} - (f^i, \partial_i v)$ for any $v \in C_c^\infty(U)$ with $v \ge 0$, where $B[\cdot,\cdot]$ is the corresponding bilinear form. Let $p > d$, $l = \sup_{\partial U} u^+$. and $v := (u-k)_+$ for a given. Prove that

$$\frac{\lambda}{2}\|\nabla v\|_{L^2(U)}^2 - C\lambda\|v\|_{L^2(U)}^2 \le B[u,v]$$

Lemma 2.7.1. Let $\varphi : [k_0, \infty) \to [0,\infty)$ be a non-negative, non-increasing function for some $k_0 \in \mathbb{R}$. For $h > k \ge k_0$, it satisfies

$$\varphi(h) \le \frac{C}{(h-k)^\alpha}(\varphi(k))^\beta$$

for some $\alpha > 0, \beta > 1$. Prove that $\varphi(k_0 + D) = 0$ with $D := C^{\frac{1}{\alpha}}(\varphi(k_0))^{\frac{\beta-1}{\alpha}} 2^{\frac{\beta}{\beta-1}}$.

(Hint: Let $k_N := k_0 + d(1 - 2^{-N})$. Choose $r > 0$ such that $\varphi(k_N) \le \varphi(k_0) r^{-N}$.)

Chapter 3 Linear Parabolic Equations

Throughout this chapter, we always assume U to be an open bounded set in $\mathbb{R}^d$ and set $U_T := (0,T]\times U$ for some fixed time $T>0$. In this chapter, we study the initial-boundary-value problem

$$\begin{cases} \partial_t u + Lu = f & \text{in } U_T, \\ u = 0 & \text{on } [0,T]\times\partial U, \\ u = g & \text{on } \{t=0\}\times U. \end{cases} \tag{3.0.1}$$

Here $f : U_T \to \mathbb{R}$ and $g : U \to \mathbb{R}$ are given and $u : \overline{U_T} \to \mathbb{R}$ is the unknown $u = u(t,\boldsymbol{x})$. L denotes a second-order partial differential operator for each time t, having either the *divergence form*

$$Lu = -\sum_{i,j=1}^{d} \partial_j(a^{ij}(t,\boldsymbol{x})\partial_i u) + \sum_{i=1}^{d} b^i(t,\boldsymbol{x})\partial_i u + c(t,\boldsymbol{x})u \tag{3.0.2}$$

or else the *non-divergence form*

$$Lu = -\sum_{i,j=1}^{d} a^{ij}(t,\boldsymbol{x})\partial_i\partial_j u + \sum_{i=1}^{d} b^i(t,\boldsymbol{x})\partial_i u + c(t,\boldsymbol{x})u \tag{3.0.3}$$

for given coefficients a^{ij}, b^i, c, $1\le i,j\le d$.

Definition 3.0.1. We say the differential operator $\partial_t + L$ defined in either (3.0.2) or (3.0.3) is **(uniformly) parabolic** if there exists a constant $\theta>0$ such that

$$\sum_{i,j=1}^{d} a^{ij}(\boldsymbol{x})\xi_i\xi_j \ge \theta|\boldsymbol{\xi}|^2 \tag{3.0.4}$$

holds for all $(t,\boldsymbol{x})\in U_T$ and $\boldsymbol{\xi}\in\mathbb{R}^d$. In particular, for each fixed $t\in[0,T]$, the operator L is uniformly elliptic.

A simple example is $a^{ij}=\delta^{ij}$ and $b^i=c=0$, that is, the heat equation. We will see that the solutions to general second-order parabolic PDEs are similar in many ways to solutions to the heat equation. For general second-order parabolic PDEs, the second-order part $a^{ij}(t,\boldsymbol{x})\partial_i\partial_j u$ describes diffusion, the first-order part $b^i\partial_i u$ describes transport and the zero-th order term cu describes creation or depletion.

There have been many other (nonlinear) parabolic equations or systems, such as Navier-Stokes equations, Keller-Segel equations, Fokker-Planck equations, the Black-Scholes equation (a backward parabolic equation) and so on, that are frequently used in fluid dynamics, mathematical biology, kinetic theory, finance and many other areas.

3.1 Space-time Sobolev spaces

Before going to the study of linear parabolic equations, we must introduce the basic settings of Sobolev spaces involving the time variable. In particular, for a function $u : [0,T]\times U \to \mathbb{R}$ that maps $(t,\boldsymbol{x}) \in [0,T]\times U$ to $u(t,\boldsymbol{x}) \in \mathbb{R}$, we would associate u with a mapping

$$\mathbf{u} : [0,T] \to X$$

defined by

$$[\mathbf{u}(t)](\boldsymbol{x}) := u(t,\boldsymbol{x}), \ \ \boldsymbol{x} \in U, \ \ 0 \le t \le T.$$

Here X is a real Banach space with norm $\|\cdot\|$. In other words, we are going to consider u as a mapping $\mathbf{u}$ of t into a certain Banach space X which consists of some functions of $\boldsymbol{x}$. This point of view would clarify the presentation of weak solutions to evolutionary PDEs.

3.1.1 Banach space-valued functions

First, let us introduce the Banach space-valued functions. Let X be a real Banach space with norm $\|\cdot\|$. Let $\boldsymbol{f} : [0,T] \to X$ is a Banach space-valued function with $T > 0$.

Definition 3.1.1. We now extend some concepts in Lebesgue measure theory to Banach space-valued functions.

- (Simple functions) We say $\mathbf{s} : [0,T] \to X$ is a **simple function** if $\mathbf{s}(t) = \sum_{i=1}^{m} \chi_{E_i}(t)u_i$ holds for $t \in [0,T]$. Here all $u_i \in X$ and all E_i's are Lebesgue-measurable subsets of $[0,T]$.
- (Strongly measurable functions) We say $\boldsymbol{f} : [0,T] \to X$ is **strongly measurable** if there exists a sequence of simple functions $\mathbf{s}_k : [0,T] \to X$ such that $\mathbf{s}_k(t) \to f(t)$ holds for a.e. $t \in [0,T]$.
- (Weakly measurable functions) We say $\boldsymbol{f} : [0,T] \to X$ is **weakly measurable** if for any $u^* \in X'$ (X' the dual space of X), the mapping $t \longmapsto \langle u^*, \boldsymbol{f}(t)\rangle$ is Lebesgue measurable.
- We say $\boldsymbol{f} : [0,T] \to X$ is **almost separable valued** if there exists a null set $N \subset [0,T]$ such that $\{\boldsymbol{f}(t) | t \in [0,T]\backslash N\}$ is separable.

Theorem 3.1.1 (Pettis' lemma). $\boldsymbol{f} : [0,T] \to X$ is strongly measurable if and only if $\boldsymbol{f}$ is weakly measurable and almost separable valued.

Next, we define the integral of Banach space-valued functions.

Definition 3.1.2.

- Let $\mathbf{s}(t) = \sum_{i=1}^{m} \chi_{E_i}(t) u_i$ be a simple function, then

$$\int_0^T \mathbf{s}(t)\,\mathrm{d}t := \sum_{i=1}^{m} \mathcal{L}^1(E_i) u_i.$$

- We say a strongly measurable function $\boldsymbol{f} : [0,T] \to X$ is Bôchner integrable if there exist a sequence of simple functions $\{\mathbf{s}_k\}$ such that

$$\int_0^T \|\mathbf{s}_k(t) - \boldsymbol{f}(t)\|\,\mathrm{d}t \to 0.$$

- If $\boldsymbol{f} : [0,T] \to X$ is Bôchner integrable, then

$$\int_0^T \boldsymbol{f}(t)\,\mathrm{d}t := \lim_{k\to\infty} \int_0^T \mathbf{s}_k(t)\,\mathrm{d}t.$$

Theorem 3.1.2 (Bôchner's lemma). A strongly measurable function $\boldsymbol{f} : [0,T] \to X$ is Bôchner integrable if and only if the map $t \longmapsto \|\boldsymbol{f}(t)\|$ is Lebesgue integrable in $[0,T]$. In this case, we have

$$\left\| \int_0^T \boldsymbol{f}(t)\,\mathrm{d}t \right\| \le \int_0^T \|\boldsymbol{f}(t)\|\,\mathrm{d}t,$$

and

$$\left\langle u^*, \int_0^T \boldsymbol{f}(t)\,\mathrm{d}t \right\rangle = \int_0^T \langle u^*, \boldsymbol{f}(t)\rangle\,\mathrm{d}t$$

holds for each $u^* \in X'$.

3.1.2 Sobolev spaces involving the time variable

Now, we can introduce the Sobolev spaces involving the time variable. Let X be a real Banach space with norm $\|\cdot\|$.

Definition 3.1.3. Let $T > 0$ be given.

- We define $L^p(0,T;X)$ to be set consisting of all strongly measurable functions $\mathbf{u} : [0,T] \to X$ with

$$\|\mathbf{u}\|_{L^p(0,T;X)} := \left(\int_0^T \|\mathbf{u}(t)\|^p\,\mathrm{d}t \right)^{\frac{1}{p}} < \infty \qquad 1 \le p < \infty,$$

  and

$$\|\mathbf{u}\|_{L^\infty(0,T;X)} := \operatorname*{ess\,sup}_{0\le t\le T} \|\mathbf{u}(t)\| < \infty.$$

- We define $C([0,T];X)$ to be the set consisting of all continuous functions $\mathbf{u} : [0,T] \to X$ with

$$\|\mathbf{u}\|_{C([0,T];X)} := \max_{0\le t\le T} \|\mathbf{u}(t)\| < \infty.$$

Definition 3.1.4. Let $T > 0$ be given.

- (Weak derivative) Let $\mathbf{u} \in L^1(0,T;X)$. We say $\mathbf{v} \in L^1(0,T;X)$ is the **weak (time) derivative of u,** written $\mathbf{u}' = \mathbf{v}$, if

$$\int_0^T \varphi'(t)\mathbf{u}(t)\,\mathrm{d}t = -\int_0^T \varphi(t)\mathbf{v}(t)\,\mathrm{d}t$$

holds for all $\varphi \in C_c^\infty(0,T)$.
- (Sobolev spaces) We define $W^{1,p}(0,T;X)$ to be the set consisting of all functions $\mathbf{u} \in L^p(0,T;X)$ whose weak (time) derivatives $\mathbf{u}'$ exists and belongs to $L^p(0,T;X)$. The norm is defined by

$$\|\mathbf{u}\|_{W^{1,p}(0,T;X)} := \begin{cases} (\int_0^T \|\mathbf{u}(t)\|^p + \|\mathbf{u}'(t)\|^p\,\mathrm{d}t)^{\frac{1}{p}} & 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{0\le t\le T}(\|\mathbf{u}(t)\| + \|\mathbf{u}'(t)\|) & p = \infty. \end{cases}$$

Functions in $W^{1,p}(0,T;X)$ satisfy several basic properties of calculus.

Proposition 3.1.3 (Calculus in Sobolev spaces involving time)**.** Let $\mathbf{u} \in W^{1,p}(0,T;X)$ for some $1 \le p \le \infty$. Then

(1) $\mathbf{u} \in C([0,T];X)$ after possible being redefined on a set of measure zero.
(2) $\mathbf{u}(t) = \mathbf{u}(s) + \int_s^t \mathbf{u}'(\tau)\,\mathrm{d}\tau$ for all $0 \le s \le t \le T$.
(3) Furthermore, we have the estimate

$$\max_{0\le t\le T} \|\mathbf{u}(t)\| \le C\|\mathbf{u}\|_{W^{1,p}(0,T;X)},$$

where the constant $C > 0$ depends only on T.

The following conclusion will be useful when proving the existence and regularity of weak solutions to linear parabolic equations. Let $U \subset \mathbb{R}^d$ be an bounded open set.

Proposition 3.1.4. Suppose $\mathbf{u} \in L^2(0,T;H_0^1(U))$ with $\mathbf{u}' \in L^2(0,T;H^{-1}(U))$.

- Then $\mathbf{u} \in C([0,T];L^2(U))$ after possible being redefined on a set of measure zero.
- The mapping $t \longmapsto \|\mathbf{u}(t)\|^2_{L^2(U)}$ is absolutely continuous with

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}(t)\|^2_{L^2(U)} = 2\langle \mathbf{u}'(t), \mathbf{u}(t)\rangle$$

for a.e. $0 \le t \le T$.
- Futhermore, we have the estimate

$$\max_{0\le t\le T} \|\mathbf{u}(t)\|_{L^2(U)} \le C\left(\|\mathbf{u}\|_{L^2(0,T;H_0^1(U))} + \|\mathbf{u}'\|_{L^2(0,T;H^{-1}(U))}\right),$$

where the constant $C > 0$ depends only on T.

Proposition 3.1.5. Let $U \subset \mathbb{R}^d$ be an bounded open set with a smooth boundary ∂U. Take $m \in \mathbb{N}$. Suppose $\mathbf{u} \in L^2(0,T;H^{m+2}(U))$ with $\mathbf{u}' \in L^2(0,T;H^m(U))$.

- Then $\mathbf{u} \in C([0,T];H^{m+1}(U))$ after possible being redefined on a set of measure zero.
- Futhermore, we have the estimate

$$\max_{0\le t\le T} \|\mathbf{u}(t)\|_{H^{m+1}(U)} \le C\left(\|\mathbf{u}\|_{L^2(0,T;H^{m+2}(U))} + \|\mathbf{u}'\|_{L^2(0,T;H^m(U))}\right),$$

where the constant $C > 0$ depends only on T.

Exercise 3.1

Exercise 3.1.1. Assume we have the weak convergence

$$\begin{aligned} \mathbf{u}_k &\rightharpoonup \mathbf{u} && \text{in } L^2(0,T;H_0^1(U)),\\ \mathbf{u}_k' &\rightharpoonup \mathbf{v} && \text{in } L^2(0,T;H^{-1}(U)). \end{aligned}$$

Prove that $\mathbf{v} = \mathbf{u}'$.

Exercise 3.1.2. Let H be a real Hilbert space and assume we have the weak convergence $\mathbf{u}_k \rightharpoonup \mathbf{u}$ in $L^2(0,T;H)$. Assume also we have

$$\operatorname*{ess\,sup}_{0\le t\le T} \|\mathbf{u}_k(t)\| \le C, \quad \forall k \in \mathbb{N}^*$$

for some constant $C > 0$. Prove that

$$\operatorname*{ess\,sup}_{0\le t\le T} \|\mathbf{u}(t)\| \le C.$$

3.2 Existence of weak solutions: Galerkin's method

Following the arguments in Section 2.1, we shall first discuss the existence theory of linear parabolic equations when the coefficients, the source term f and the initial data g are not regular enough. We consider system (3.0.1) with L defined by the divergence form (3.0.2), that is,

$$\begin{cases} \partial_t u + Lu = f & \text{in } U_T,\\ u = 0 & \text{on } [0,T]\times\partial U,\\ u = g & \text{on } \{t=0\}\times U, \end{cases}$$

with

$$Lu = -\sum_{i,j=1}^{d} \partial_j(a^{ij}(t,\boldsymbol{x})\partial_i u) + \sum_{i=1}^{d} b^i(t,\boldsymbol{x})\partial_i u + c(t,\boldsymbol{x})u$$

being symmetric and uniformly elliptic for each $t \in [0,T]$. The source term $f \in L^2(U_T)$, the initial data $g \in L^2(U)$ and the coefficients $a^{ij}, b^i, c \in L^\infty(U_T)$ are given.

3.2.1 Definition of weak solutions

If the function $u(t,\boldsymbol{x})$ is a smooth solution to (3.0.1), we associate u with a mapping

$$\mathbf{u} : [0,T] \to H_0^1(U), \qquad t \longmapsto \mathbf{u}(t)$$

defined by $[\mathbf{u}(t)](\boldsymbol{x}) := u(t,\boldsymbol{x})$. In other words, when defining the weak solution, we are going to consider u as a mapping $\mathbf{u}$ of t into the space $H_0^1(U)$. Similarly, we define $\boldsymbol{f} : [0,T] \to L^2(U)$ by $[\boldsymbol{f}(t)](\boldsymbol{x}) := f(t,\boldsymbol{x})$. Thus, if we fix a function $v \in H_0^1(U)$, then we can multiply the parabolic equation with v and integrate by parts to get

$$(\mathbf{u}', v)_{L^2(U)} + B[\mathbf{u}, v; t] = (\boldsymbol{f}, v)_{L^2(U)}, \qquad \forall 0 \le t \le T.$$

Here the bilinear form is defined by

$$B[\mathbf{u}, v; t] := \int_U a^{ij}(t,\boldsymbol{x})\partial_i u \partial_j v + b^i(t,\boldsymbol{x})\partial_i u v + c(t,\boldsymbol{x}) u v \,\mathrm{d}\boldsymbol{x}, \;\; u, v \in H_0^1(U), \text{ a.e. } 0 \le t \le T.$$

We shall also investigate which function spaces does the time derivative $\mathbf{u}'$ belong to. In fact, from the parabolic equation we have

$$\partial_t u = \underbrace{(f - b^i\partial_i u - cu)}_{=:g^0} + \partial_j(\underbrace{a^{ij}\partial_i u}_{=:g^j}) = g^0 + \sum_{j=1}^{d} \partial_j g^j,$$

where each g^j $(0 \le j \le d)$ belongs to $L^2(U)$. Thus, the time derivative $\partial_t u(t,\cdot)$ belongs to $H^{-1}(U)$ with the estimates

$$\|\partial_t u\|_{H^{-1}(U)} \le \left(\sum_{j=0}^{d} \|g^j\|_{L^2(U)}^2 \right)^{\frac{1}{2}} \le C(\|u\|_{H_0^1(U)} + \|f\|_{L^2(U)}).$$

This estimate suggest that the weak time derivative $\mathbf{u}'$ should belong to $H^{-1}(U)$ for a.e. $t \in [0,T]$, and so the term $(\mathbf{u}', v)_{L^2(U)}$ should be replaced by $\langle \mathbf{u}', v \rangle$ with $\langle \cdot, \cdot \rangle$ being the pairing of $H^{-1}(U)$ and $H_0^1(U)$.

Now, we can introduce the definition of weak solutions to (3.0.1) with L defined by (3.0.2).

Definition 3.2.1 (Weak solution). We say a function $\mathbf{u} \in L^2(0,T; H_0^1(U))$ with $\mathbf{u}' \in L^2(0,T; H^{-1}(U))$ is a weak solution to (3.0.1) with L defined by (3.0.2), if

- $\langle \mathbf{u}', v\rangle + B[\mathbf{u}, v; t] = (\boldsymbol{f}, v)_{L^2(U)}$ holds for each $v \in H_0^1(U)$ and a.e. $t \in [0,T]$,
- $\mathbf{u}(0) = g$.

It should be noted that $\mathbf{u}(0) = g$, as a pointwise value of the function $\mathbf{u} : [0,T] \to L^2(U)$, can be defined thanks to $\mathbf{u} \in C([0,T]; L^2(U))$ in Proposition 3.1.4(1).

3.2.2 Motivation: separation of variables

Let us recall how to solve 1D heat equation with Dirichlet boundary condition in an interval $[0,\pi]$, namely we consider the classical solutions to

$$\begin{cases} \partial_t u - \partial_x^2 u = f(t,x) & t > 0,\ 0 < x < \pi \\ u(0,x) = g(x) & 0 \le x \le \pi \\ u(t,0) = 0,\ u(t,\pi) = 0 & t \ge 0, \end{cases} \tag{3.2.1}$$

where f, g are sufficiently regular.

This equation is solved by using separation of variables. First, we assume $f = 0$ and $u(t,x) = T(t)X(x)$ and then we can find (from the boundary condition) that

$$\lambda_n = n^2,\ X_n(t) = \sin nx,\ T_n(t) = A_n \cos nt + B_n \sin nt \Rightarrow u(t,x) = \sum_{n=1}^{\infty} T_n(t) X_n(x),$$

where the coefficients A_n, B_n are determined by the initial data $g(x)$. In general, when $f \not\equiv 0$, we just expand u, f in the basis $\{\sin nx\}$:

$$u(t,x) = \sum_{n=1}^{\infty} T_n(t) \sin nx, \quad f(t,x) = \sum_{n=1}^{\infty} f_n(t) \sin nx,$$

and solve the ODE $T_n''(t) + n^2 T_n(t) = f_n(t)$ to determine $T_n(t)$.

The principle behind the method of separation of variables is that $\{\sin nx\}$ exactly give an orthogonal basis of $L^2((0,\pi))$. Meanwhile, we also find that $\{\sin nx\}$ are exactly the eigenfunctions of $-\Delta = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ (with zero Dirichlet boundary condition) in $(0,\pi)$, corresponding to the eigenvalue n. In order to "generalize" this idea to general dimensions, we may consider

- The set U must be bounded, which guarantees that the spectrum of L must be discrete.
- We may expand the solution and the source term in the orthonormal basis $\{w_n\}$ given by the eigenfunctions of some symmetric elliptic operators.

However, we are now faced with a PDE whose coefficients, initial data and source term are all *rough*. We must prove that the infinite sum, obtained by the analogue of separation of variables, really converges in $L^2(0,T; H_0^1(U))$ with time derivative in $L^2(0,T; H^{-1}(U))$. Thus, it is reasonable to first do a "truncation" on the dimension of the eigenspaces. This method is called "Galerkin's approximation".

3.2.3 Galerkin approximation: Existence and uniqueness

Let $\{w_k\}$ be an orthonormal basis of $L^2(U)$ and orthogonal basis of $H_0^1(U)$ and each w_k is smooth. For example, we can pick $\{w_k\}$ to be the normalized eigenfunctions of $-\Delta$ in $H_0^1(U)$. The process of Galerkin's approximation is

1. **Construction of finite-dimensional truncation.** For $m \in \mathbb{N}^*$, we look for a sequence of functions $\mathbf{u}_m : [0,T] \to H_0^1(U)$ of the form

$$\mathbf{u}_m(t) := \sum_{k=1}^{m} d_m^k(t) w_k \tag{3.2.2}$$

where we hope to determine the coefficients $d_m^k(t)$ such that

$$d_m^k(0) = (g, w_k), \tag{3.2.3}$$

$$(\mathbf{u}_m', w_k)_{L^2(U)} + B[\mathbf{u}_m, w_k; t] = (\boldsymbol{f}, w_k)_{L^2(U)}. \tag{3.2.4}$$

hold for $1 \le k \le m$ and $0 \le t \le T$. This, roughly speaking, indicates that $\mathbf{u}_m$ approximates the projection of $\mathbf{u}$ onto the subspace spanned by $\{w_k\}_{1\le k\le m}$.
2. **Uniform(-in-m) energy estimates for the approximate sequence.** The weak limit, whose existence is ensured by Eberlein-Šmulian theorem, is expected to be the desired weak solution.
3. **Verify the weak limit is exactly the unique weak solution.** We will use Exercise 3.1.1 to verify that the weak limit of the time derivative $\mathbf{u}_k'$ is exactly the time deriative of the weak limit of $\mathbf{u}_k$.

Step 1: Construction of the sequence of approximate solutions

First, we shall prove the existence of the sequence of approximate solutions.

Theorem 3.2.1. For each $m \in \mathbb{N}$, there exists a unique function $\mathbf{u}_m$ of the form

$$\mathbf{u}_m(t) := \sum_{k=1}^{m} d_m^k(t) w_k \tag{3.2.5}$$

satisfying

$$d_m^k(0) = (g, w_k), \tag{3.2.6}$$

$$(\mathbf{u}_m', w_k)_{L^2(U)} + B[\mathbf{u}_m, w_k; t] = (\boldsymbol{f}, w_k)_{L^2(U)}, \quad 0 \le t \le T,\ 1 \le k \le m. \tag{3.2.7}$$

Proof. Because of the orthogonality, it is straightforward to see

$$(\mathbf{u}_m'(t), w_k) = \frac{\mathrm{d}}{\mathrm{d}t} d_m^k(t), \ \ B[\mathbf{u}_m, w_k; t] = \sum_{l=1}^{m} e^{kl}(t) d_m^l(t)$$

for $e^{kl}(t)=B[w_l,w_k,t]$, if assuming $\mathbf{u}_m$ has the form $\sum\limits_{k=1}^m d_m^k(t)w_k$. Let $f^k(t):=(\boldsymbol{f}(t),w_k)$. Then, we get the following linear system of ODEs

$$\frac{\mathrm{d}}{\mathrm{d}t}d_m^k(t)+\sum_{l=1}^m e^{kl}(t)d_m^l(t)=f^k(t),\ \ 1\le k\le m \tag{3.2.8}$$

with initial data $d_m^k(0)=(g,w_k)$. Since the coefficients $e^{kl}(t)$ are sufficiently regular, the standard existence theory for ODEs shows that there exists a unique, absolutely continuous function $\mathbf{d}_m(t)=(d_m^1(t),\cdots,d_m^m(t))$ satisfying the initial data and the ODE system for a.e. $0\le t\le T$. Then $\mathbf{u}_m$ defined in the theorem automatically satisfies the desired equality. □

Step 2: Uniform energy estimates of the approximate solutions

Theorem 3.2.2. There exists a constant $C>0$ depending only on U,T and the coefficients of L, such that for each $m\in\mathbb{N}$ there holds

$$\sup_{0\le t\le T}\|\mathbf{u}_m(t)\|_{L^2(U)}^2+\|\mathbf{u}_m\|_{L^2(0,T;H_0^1(U))}^2+\|\mathbf{u}_m'\|_{L^2(0,T;H^{-1}(U))}^2\le C\left(\|\boldsymbol{f}\|_{L^2(0,T;L^2(U))}^2+\|g\|_{L^2(U)}^2\right). \tag{3.2.9}$$

Before going to the proof, we must emphasize that the most important step to establish estimates for an energy functional $E(t)$ is to establish the Grönwall-type inequality

$$E(t)\le C\left(E(0)+\int_0^t E(\tau)\,\mathrm{d}\tau\right)$$

which is usually derived by the differentiated version $E'(t)\le AE(t)$ for some $A>0$.

Proof. Given $m\in\mathbb{N}$, we already have $(\mathbf{u}_m',w_k)_{L^2(U)}+B[\mathbf{u}_m,w_k;t]=(\boldsymbol{f},w_k)_{L^2(U)}$ for each $1\le k\le m$. Then we multiply $d_m^k(t)$ and take sum over $1\le k\le m$ to get

$$\underbrace{(\mathbf{u}_m',\mathbf{u}_m)_{L^2(U)}}_{=\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}_m(t)\|_{L^2(U)}^2}+B[\mathbf{u}_m,\mathbf{u}_m;t]=(\boldsymbol{f},\mathbf{u}_m)_{L^2(U)}. \tag{3.2.10}$$

In fact, this step is exactly the analogue of multiplying u on both sides of the heat equation $\partial_t u-\Delta u=f$ and integrate over U. Now, we obtain

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}_m(t)\|_{L^2(U)}^2+\int_U a^{ij}\partial_i u_m\,\partial_j u_m\,\mathrm{d}\boldsymbol{x}=(f,\mathbf{u}_m)-\int_U b^i\partial_i u_m\,u_m\,\mathrm{d}x-\int_U cu^2\,\mathrm{d}\boldsymbol{x}.$$

Using Hölder's inequality, the uniform ellipticity of L and Young's inequality, we get for any $\delta > 0$

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}_m\|^2_{L^2(U)} + \theta\|\nabla\mathbf{u}_m\|^2_{L^2(U)} &\le C\Big(\|f\|_{L^2(U)}\|\mathbf{u}_m\|_{L^2(U)} + \|\nabla\mathbf{u}_m\|_{L^2(U)}\|b\|_{L^\infty(U)}\|\mathbf{u}_m\|_{L^2(U)} \\ &\qquad + \|c\|_{L^\infty}\|\mathbf{u}_m\|^2_{L^2(U)}\Big) \\ &\le \delta\|\nabla\mathbf{u}_m\|^2_{L^2(U)} + C\Big(\|f\|^2_{L^2(U)} + \|\mathbf{u}_m\|^2_{L^2(U)}\Big). \end{aligned} \tag{3.2.11}$$

Pick $\delta \in (0, \theta/2)$ such that $\delta\|\nabla\mathbf{u}_m\|^2_{L^2(U)}$ can be absorbed by the left side. We know that there exists some $C > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathbf{u}_m(t)\|^2_{L^2(U)} + \theta\int_0^t \|\nabla\mathbf{u}_m(\tau)\|^2_{L^2(U)}\right) \le C\Big(\|f\|^2_{L^2(U)} + \|\mathbf{u}_m\|^2_{L^2(U)}\Big). \tag{3.2.12}$$

Since $f(t,\cdot) \in L^2(U)$ is given, by Grönwall's inequality, we can prove that there exists a constant $C_T > 0$ depending on T, U, L such that

$$\begin{aligned}\|\mathbf{u}_m(t)\|^2_{L^2(U)} + \theta\int_0^t \|\nabla\mathbf{u}_m(\tau)\|^2_{L^2(U)} &\le C_T\left(\|\mathbf{u}_m(0)\|^2_{L^2(U)} + \int_0^T \|f(t,\cdot)\|^2_{L^2(U)}\,\mathrm{d}t\right) \\ &\le C_T\left(\|g\|^2_{L^2(U)} + \int_0^T \|f(t,\cdot)\|^2_{L^2(U)}\,\mathrm{d}t\right). \end{aligned} \tag{3.2.13}$$

It now remains to control $\|\mathbf{u}'_m\|^2_{L^2(0,T;H^{-1}(U))}$. Recall that $\mathbf{u}_m(t) = \sum_{k=1}^m d^k_m(t)w_k$ implies $\mathbf{u}'_m(t) = \sum_{k=1}^m (d^k_m)'(t)w_k$ which belongs to $\operatorname{span}\{w_1 \cdots, w_m\}$. We now fix $t \in [0,T]$ and pick a test function $\varphi \in H^1_0(U)$ with $\|\varphi\|_{H^1_0(U)} \le 1$. Then

$$\begin{aligned}\langle\mathbf{u}'_m, \varphi\rangle = \langle\mathbf{u}'_m, \varphi_m\rangle &= B[\mathbf{u}_m, \varphi_m; t] + (\boldsymbol{f}, \varphi_m) \\ &\le C\|\nabla\mathbf{u}_m\|_{L^2(U)}\|\nabla\varphi_m\|_{L^2(U)} + (\|\nabla\mathbf{u}_m\|_{L^2(U)} + \|f\|_{L^2(U)})\|\varphi_m\|_{L^2(U)} \\ &\le C(\|f\|_{L^2} + \|\nabla\mathbf{u}_m\|_{L^2(U)}),\end{aligned}$$

where we use $\|\varphi_m\|_{H^1_0(U)} \le \|\varphi\|_{H^1_0(U)} = 1$ and Poincaré's inequality in the last step. Taking supremum over all choices of φ, by the definition of $H^{-1}(U)$ norm, we get

$$\|\mathbf{u}'_m(t)\|^2_{H^{-1}(U)} \le C\left(\|f\|^2_{L^2} + \|\nabla\mathbf{u}_m\|^2_{L^2(U)}\right), \tag{3.2.14}$$

and thus using the estimate (3.2.13) leads to

$$\|\mathbf{u}'_m(t)\|^2_{L^2(0,T;H^{-1}(U))} \le C_T\left(\|g\|^2_{L^2(U)} + \int_0^T \|f(t,\cdot)\|^2_{L^2(U)}\,\mathrm{d}t\right) \tag{3.2.15}$$

for some $C_T > 0$ depending on T, U, L. $\square$

Step 3: Existence and uniqueness of the weak solution

The last step is to pass to the weak limit, which is expected to be the unique weak solution to (3.0.1), by using the uniform bounds in Theorem 3.2.2.

Theorem 3.2.3. System (3.0.1) admits a unique weak solution.

Proof. The uniform bounds in Theorem 3.2.2 ensure that there exists a subsequence $\{\mathbf{u}_{m_l}\} \subset \{\mathbf{u}_m\}$ such that

$$\mathbf{u}_{m_l} \xrightarrow{\text{weakly in } L^2(0,T;H_0^1(U))} \mathbf{u}, \ \ \mathbf{u}'_{m_l} \xrightarrow{\text{weakly in } L^2(0,T;H^{-1}(U))} \mathbf{u}',$$

where we also use the conclusion of exercise 3.1.1. By Proposition 3.1.5, we also know $\mathbf{u} \in C^1([0,T]; H_0^1(U))$. Next, we verify that this $\mathbf{u}$ is exactly the weak solution that we want. We fix $N \in \mathbb{N}$ and choose a function $\mathbf{v} \in C^1([0,T]; H_0^1(U))$ having the form

$$\mathbf{v}(t) = \sum_{k=1}^{N} d^k(t) w_k.$$

Then we choose $m \geq N$ and multiply (3.2.7) with $d^k(t)$ and integrate in t variable to get

$$\int_0^T \langle \mathbf{u}'_m, \mathbf{v}\rangle + B[\mathbf{u}_m, \mathbf{v}; t]\,\mathrm{d}t = \int_0^T (\boldsymbol{f}, \mathbf{v})\,\mathrm{d}t, \tag{3.2.16}$$

and setting $m = m_l$ and passing to the weak limit to get

$$\int_0^T \langle \mathbf{u}', \mathbf{v}\rangle + B[\mathbf{u}, \mathbf{v}; t]\,\mathrm{d}t = \int_0^T (\boldsymbol{f}, \mathbf{v})\,\mathrm{d}t, \ \ \forall \mathbf{v} \in L^2(0,T; H_0^1(U)), \tag{3.2.17}$$

where we use the fact that functions of the form $\sum d^k(t) w_k$ are dense in this space. This then gives us

$$\langle \mathbf{u}', v\rangle + B[\mathbf{u}, v; t] = (\boldsymbol{f}, v), \ \ \forall v \in H_0^1(U), \text{ a.e. } t \in [0,T]. \tag{3.2.18}$$

Proposition 3.1.3 then implies $\mathbf{u} \in C(0,T]; L^2(U))$.

We next verify the initial data of $\mathbf{u}$ is g. Integrating by parts in t variable, we can see that

$$\int_0^T \langle \mathbf{v}', \mathbf{u}\rangle + B[\mathbf{u}, \mathbf{v}; t]\,\mathrm{d}t = \int_0^T (\boldsymbol{f}, \mathbf{v})\,\mathrm{d}t + (\mathbf{u}(0), \mathbf{v}(0)), \ \ \forall \mathbf{v} \in C^1(0,T; H_0^1(U)) \text{ with } \mathbf{v}(T) = 0, \tag{3.2.19}$$

and

$$\int_0^T \langle \mathbf{v}', \mathbf{u}_m \rangle + B[\mathbf{u}_m, \mathbf{v}; t] \, \mathrm{d}t = \int_0^T (\boldsymbol{f}, \mathbf{v}) \, \mathrm{d}t + (\mathbf{u}_m(0), \mathbf{v}(0)), \quad \forall \mathbf{v} \in L^2(0, T; H_0^1(U)). \tag{3.2.20}$$

Again let $m = m_l$ and pass to the limit. Since $\mathbf{u}_{m_l}(0) \to g$ in $L^2(U)$ and $\mathbf{v}(0)$ is arbitrary, we know $\mathbf{u}(0) = g$ must hold.

To prove the uniqueness, it suffices to check the only weak solution to (3.0.1) with $\boldsymbol{f} = g = 0$ is zero. Setting $\mathbf{v}$ to be $\mathbf{u}$ itself in (3.2.17) and we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \|\mathbf{u}(t)\|_{L^2(U)}^2 + B[\mathbf{u}, \mathbf{u}; t] = \langle \mathbf{u}', \mathbf{u} \rangle + B[\mathbf{u}, \mathbf{u}; t] = 0. \tag{3.2.21}$$

Since there exist $C_1, C_2 > 0$ such that

$$B[\mathbf{u}, \mathbf{u}; t] \geq C_1 \|\mathbf{u}\|_{H_0^1(U)}^2 - C_2 \|\mathbf{u}\|_{L^2(U)}^2 \geq -C_2 \|\mathbf{u}\|_{L^2(U)}^2,$$

using Grönwall's inequality immediately leads to $\|\mathbf{u}\|_{L^2(U)} = 0$ and so $\mathbf{u} = 0$. □

Exercise 3.2

Exercise 3.2.1. Suppose $f \in L^2(U)$ and assume that $u_m = \sum_{k=1}^{m} d_m^k w_k$ solves

$$\int_U \nabla u_m \cdot \nabla w_k \, \mathrm{d}\boldsymbol{x} = \int_U f \, w_k \, \mathrm{d}\boldsymbol{x}, \quad \forall 1 \leq k \leq m$$

where $\{w_k\}$ is an orthonormal basis of $H_0^1(U)$. Show that a subsequence of $\{u_m\}$ converges weakly in $H_0^1(U)$ to the weak solution u to the equation $-\Delta u = f$ in U with $u = 0$ on ∂U.

Exercise 3.2.2. Suppose $g \in L^2(U)$ and assume that u is a smooth solution to

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } U_T, \\ u = 0 & \text{on } [0, T] \times \partial U, \\ u = g & \text{on } \{t = 0\} \times U. \end{cases} \tag{3.2.22}$$

Prove that

$$\|u(t, \cdot)\|_{L^2(U)} \leq e^{-\lambda_1 t} \|g\|_{L^2(U)}, \quad \forall t \geq 0.$$

Here $\lambda_1 > 0$ is the principal eigenvalue of $-\Delta$ in U with zero Dirichlet boundary condition.

3.3 Regularity of parabolic equations

After obtaining the weak solution to the parabolic equation (3.0.1), it is natural to ask further questions: Is this weak solution a classical one? What is the regularity of the solution? To see this, we can first try to compute the *a priori* estimates for the heat equation

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } [0,T] \times \mathbb{R}^d, \\ u(0,\boldsymbol{x}) = g(\boldsymbol{x}) & \text{on } \{t=0\} \times \mathbb{R}^d, \end{cases} \tag{3.3.1}$$

where u is assumed to be a smooth solution that rapidly decays to 0 as $|\boldsymbol{x}| \to \infty$.

In Theorem 3.2.2, we actually get the $L_t^\infty L_{\boldsymbol{x}}^2 \cap L_t^2 H_{\boldsymbol{x}}^1$ a priori estimates

$$\sup_{0\le t\le T} \int_{\mathbb{R}^d} u(t,\boldsymbol{x})^2 \,\mathrm{d}\boldsymbol{x} + \int_0^T \int_{\mathbb{R}^d} |\nabla u(t,\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \le C\left(\int_{\mathbb{R}^d} g(\boldsymbol{x})^2 \,\mathrm{d}\boldsymbol{x} + \int_0^T \int_{\mathbb{R}^d} (f(t,\boldsymbol{x}))^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \right).$$

However, the Laplacian term in the heat equation contains second-order derivative. It is natural to ask if we could establish H^2 regularity results for the solution u. In fact, we can take the square and integrate by parts to get

$$\begin{aligned} \int_{\mathbb{R}^d} f^2 \,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d} (u_t - \Delta u)^2 \,\mathrm{d}\boldsymbol{x} &= \int_{\mathbb{R}^d} (\partial_t u)^2 - 2\partial_t u \Delta u + (\Delta u)^2 \,\mathrm{d}\boldsymbol{x} \\ &= \int_{\mathbb{R}^d} (\partial_t u)^2 + 2\partial_t \nabla u \cdot \nabla u + (\Delta u)^2 \,\mathrm{d}\boldsymbol{x}, \end{aligned}$$

which yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} |\nabla u(t,\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^d} (\partial_t u)^2 + (\Delta u)^2 \,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d} f^2 \,\mathrm{d}\boldsymbol{x}. \tag{3.3.2}$$

Integrating in the time variable t, we get

$$\begin{aligned} &\sup_{0\le t\le T} \int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}\boldsymbol{x} + \int_0^T \left(\int_{\mathbb{R}^d} (\partial_t u)^2 + (\Delta u)^2 \,\mathrm{d}\boldsymbol{x} \right) \mathrm{d}t \\ &\le C\left(\int_{\mathbb{R}^d} |\nabla g|^2 \,\mathrm{d}\boldsymbol{x} + \int_0^T \int_U f(t,\boldsymbol{x})^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \right). \end{aligned} \tag{3.3.3}$$

This shows that the solution $\mathbf{u}$ lies in $L_t^2 H_{\boldsymbol{x}}^2 \cap L_t^\infty H_{\boldsymbol{x}}^1$ and $\mathbf{u}' \in L_t^2 L_{\boldsymbol{x}}^2$ if the initial data $g \in H_0^1$ and the source term $f \in L_t^2 L_{\boldsymbol{x}}^2$. Next, we want to further ask if it is possible to establish pointwise H^2 estimates (instead of L_t^2-type) for the solution. This can be achieved, but it also requires higher regularity of the initial data g and the source term f. In fact, we take ∂_t in the heat equation to get

$$\partial_t^2 u - \Delta \partial_t u = \partial_t f \text{ in } (0,T] \times \mathbb{R}^d, \quad \partial_t u(0,\cdot) = f(0,\cdot) + \Delta g(\cdot).$$

Then testing this equation with $\partial_t u$ in $L^2(\mathbb{R}^d)$ and integrating by parts, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{R}^d} |\partial_t u|^2 \,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^d} |\nabla \partial_t u|^2 \,\mathrm{d}\boldsymbol{x} \leq \|\partial_t f\|_{L^2(\mathbb{R}^d)} \|\partial_t u\|_{L^2(\mathbb{R}^d)},$$

which together with Grönwall's inequality gives

$$\sup_{0\leq t\leq T}\int_{\mathbb{R}^d} |\partial_t u|^2 \,\mathrm{d}\boldsymbol{x} + \int_0^T\int_{\mathbb{R}^d} |\nabla \partial_t u|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \leq C\left(\int_0^T\int_{\mathbb{R}^d} (\partial_t f)^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_{\mathbb{R}^d} |\nabla^2 g|^2 + f(0,\cdot)^2 \,\mathrm{d}\boldsymbol{x}\right). \tag{3.3.4}$$

Then using fundamental theorem of calculus as in Proposition 3.1.4, we have

$$\sup_{0\leq t\leq T} \|f(t,\cdot)\|^2_{L^2(U)} \leq C\left(\|f\|^2_{L^2((0,T)\times\mathbb{R}^d)} + \|\partial_t f\|^2_{L^2((0,T)\times\mathbb{R}^d)}\right). \tag{3.3.5}$$

This gives the regularity of $\partial_t u$. For the $L_t^\infty H_{\boldsymbol{x}}^2$ regularity of u, we use the elliptic regularity theorem (Theorem 2.5.2, essentially testing the equation by Δu) to get

$$\int_{\mathbb{R}^d} |\nabla^2 u|^2 \,\mathrm{d}\boldsymbol{x} \leq C\int_{\mathbb{R}^d} f^2 + (\partial_t u)^2 \,\mathrm{d}\boldsymbol{x}. \tag{3.3.6}$$

Therefore, we conclude that

$$\begin{aligned} &\sup_{0\leq t\leq T}\int_{\mathbb{R}^d} |\partial_t u|^2 + |\nabla^2 u|^2 \,\mathrm{d}\boldsymbol{x} + \int_0^T\int_{\mathbb{R}^d} |\nabla \partial_t u|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &\leq C\left(\int_0^T\int_{\mathbb{R}^d} (\partial_t f)^2 + f^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_{\mathbb{R}^d} |\nabla^2 g|^2 \,\mathrm{d}\boldsymbol{x}\right). \end{aligned} \tag{3.3.7}$$

Based on the above analysis for the standard heat equation, we expect to prove the following regularity result for the parabolic equation (3.0.1). Also, for technical simplicity, we assume $\{w_k\}$ is the complete collection of eigenfunctions of $-\Delta$ in $H_0^1(U)$, U is a bounded domain with a smooth boundary ∂U and also assume the coefficients a^{ij}, b^i, c are smooth in $\overline{U}$ and independent of t.

Theorem 3.3.1 (Parabolic regularity)**.** Let $\mathbf{u} \in L^2(0,T;H_0^1(U))$ with $\mathbf{u}' \in L^2(0,T;H^{-1}(U))$ be the weak solution to (3.0.1) with initial data $g \in H_0^1(U)$ and source term $\boldsymbol{f} \in L^2(0,T;L^2(U))$. Then the solution $\mathbf{u}$ actually satisfies

$$\mathbf{u} \in L^2(0,T;H^2(U)) \cap L^\infty(0,T;H_0^1(U)), \;\; \mathbf{u}' \in L^2(0,T;L^2(U))$$

with the estimate

$$\operatorname*{ess\,sup}_{0\le t\le T}\|\mathbf{u}(t)\|^2_{H^1_0(U)} + \|\mathbf{u}\|^2_{L^2(0,T;H^2(U))} + \|\mathbf{u}'\|^2_{L^2(0,T;L^2(U))} \le C\left(\|\boldsymbol{f}\|^2_{L^2(0,T;H^2(U))} + \|g\|^2_{H^1_0(U)}\right), \tag{3.3.8}$$

where the constant $C > 0$ depends on U, T and the coefficients of L.

Moreover, if we additionally have $g \in H^2(U)$ and $\boldsymbol{f}' \in L^2(0,T;L^2(U))$, then $\mathbf{u}$ satisfies

$$\mathbf{u} \in L^\infty(0,T;H^2(U)),\ \mathbf{u}' \in L^\infty(0,T;L^2(U)) \cap L^2(0,T;H^1_0(U)),\ \mathbf{u}'' \in L^2(0,T;H^{-1}(U))$$

with the estimate

$$\begin{aligned}&\operatorname*{ess\,sup}_{0\le t\le T}\left(\|\mathbf{u}(t)\|^2_{H^2(U)} + \|\mathbf{u}'(t)\|^2_{L^2(U)}\right) + \|\mathbf{u}'\|^2_{L^2(0,T;H^1_0(U))} + \|\mathbf{u}''\|^2_{L^2(0,T;H^{-1}(U))}\\ &\le C\left(\|\boldsymbol{f}\|^2_{H^1(0,T;L^2(U))} + \|g\|^2_{H^2(U)}\right).\end{aligned} \tag{3.3.9}$$

Proof. The first part is essentially obtained by testing the equation by $\mathbf{u}'$. However, currently we do not know $\mathbf{u}' \in L^2(0,T;L^2(U))$, so we shall first do this for the approximate sequence $\{\mathbf{u}_m'\}$ defined in (3.2.5). Testing the equation (3.2.7) with ${d_m^k}'(t)$ and taking the sume over $1 \le k \le m$, we get

$$(\mathbf{u}_m', \mathbf{u}_m') + B[\mathbf{u}_m, \mathbf{u}_m'] = (\boldsymbol{f}, \mathbf{u}_m'), \quad \text{a.e. } t \in [0,T].$$

So, it remains to analyze the term $B[\mathbf{u}_m, \mathbf{u}_m']$. Invoking the concrete form of L, we get

$$B[\mathbf{u}_m, \mathbf{u}_m'] = \int_U a^{ij}\partial_i\mathbf{u}_m\,\partial_j\mathbf{u}_m'\,\mathrm{d}\boldsymbol{x} + \int_U b^i\partial_i\mathbf{u}_m\,\mathbf{u}_m' + c\mathbf{u}_m\mathbf{u}_m'\,\mathrm{d}\boldsymbol{x}.$$

Since $a^{ij} = a^{ji}$ and the coefficients are time-independent, we then find that the second-order term gives an energy structure

$$\int_U a^{ij}\partial_i\mathbf{u}_m\,\partial_j\mathbf{u}_m'\,\mathrm{d}\boldsymbol{x} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_U a^{ij}\partial_i\mathbf{u}_m\,\partial_j\mathbf{u}_m\,\mathrm{d}\boldsymbol{x}.$$

Then, using Young's inequality, we get $\forall\delta > 0$

$$\left|\int_U b^i\partial_i\mathbf{u}_m\,\mathbf{u}_m' + c\mathbf{u}_m\mathbf{u}_m'\,\mathrm{d}\boldsymbol{x}\right| \le \delta\|\mathbf{u}_m'\|^2_{L^2(U)} + \frac{C}{\delta}\|\mathbf{u}_m\|^2_{H^1_0(U)}$$

and

$$|(\boldsymbol{f}, \mathbf{u}_m')| \le \delta\|\mathbf{u}_m'\|^2_{L^2(U)} + \frac{C}{\delta}\|\boldsymbol{f}\|^2_{L^2(U)}.$$

This leads to the inequality

$$\|\mathbf{u}_m'\|^2_{L^2(U)} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_U a^{ij}\partial_i\mathbf{u}_m\,\partial_j\mathbf{u}_m\,\mathrm{d}x \le 2\delta\|\mathbf{u}_m'\|^2_{L^2(U)} + \frac{C}{\delta}\left(\|\boldsymbol{f}\|^2_{L^2(U)} + \|\mathbf{u}_m\|^2_{H^1_0(U)}\right).$$

We choose $\delta \in (0, \frac{1}{4})$ such that the δ-term can be absorbed by the right side, and integrate in time to get

$$\begin{aligned}
&\int_0^T \|\mathbf{u}_m'\|_{L^2(U)}^2 \,\mathrm{d}t + \sup_{0\leq t\leq T} \int_U a^{ij}\partial_i \mathbf{u}_m \,\partial_j \mathbf{u}_m \,\mathrm{d}\boldsymbol{x} \\
&\leq C\left(\int_U a^{ij}\partial_i \mathbf{u}_m \,\partial_j \mathbf{u}_m \Big|_{t=0} + \int_0^T \|\boldsymbol{f}\|_{L^2(U)}^2 + \|\mathbf{u}_m\|_{H_0^1(U)}^2 \,\mathrm{d}t \right) \\
&\leq C\left(\|\boldsymbol{f}\|_{L^2(0,T;H^2(U))}^2 + \|g\|_{H_0^1(U)}^2 \right).
\end{aligned} \tag{3.3.10}$$

Finally, using the uniform ellipticity, we know $\int_U a^{ij}\partial_i \mathbf{u}_m \,\partial_j \mathbf{u}_m \,\mathrm{d}\boldsymbol{x} \geq \theta \|\nabla \mathbf{u}_m\|_{L^2(U)}^2$. Now, we have

$$\int_0^T \|\mathbf{u}_m'\|_{L^2(U)}^2 \,\mathrm{d}t + \sup_{0\leq t\leq T} \|\nabla \mathbf{u}_m\|_{L^2(U)}^2 \leq C\left(\|\boldsymbol{f}\|_{L^2(0,T;H^2(U))}^2 + \|g\|_{H_0^1(U)}^2 \right), \tag{3.3.11}$$

where the right side does not depend on m and so we can pass to limits as $m = m_l \to \infty$ to deduce

$$\mathbf{u} \in L^\infty(0,T;H_0^1(U)), \;\; \mathbf{u}' \in L^2(0,T;L^2(U))$$

together with the estimate

$$\int_0^T \|\mathbf{u}'\|_{L^2(U)}^2 \,\mathrm{d}t + \sup_{0\leq t\leq T} \|\nabla \mathbf{u}\|_{L^2(U)}^2 \leq C\left(\|\boldsymbol{f}\|_{L^2(0,T;H^2(U))}^2 + \|g\|_{H_0^1(U)}^2 \right). \tag{3.3.12}$$

Note that this estimate is obtained with the help of Exercise 3.1.2. Then the $L_t^2 H_{\boldsymbol{x}}^2$ regularity of $\mathbf{u}$ is quite straightforward. In fact, since we already obtain that

$$(\mathbf{u}', \varphi) + B[\mathbf{u}, \varphi] = (\boldsymbol{f}, \varphi) \;\; \forall \varphi \in H_0^1(U), \text{ a.e. } t \in [0,T],$$

then we can rewrite it to be

$$B[\mathbf{u}, \varphi] = (\mathbf{h}, \varphi) \;\; , \mathbf{h} := \boldsymbol{f} - \mathbf{u}' \in L^\infty(0,T;L^2(U)).$$

Using the elliptic regularity theorem (Theorem 2.5.5), we find $\mathbf{u}(t) \in H^2(U)$ for a.e. $t \in [0,T]$ and

$$\|\mathbf{u}\|_{H^2(U)}^2 \leq C(\|\mathbf{h}\|_{L^2(U)}^2 + \|\mathbf{u}\|_{L^2(U)}^2) \leq C(\|\boldsymbol{f}\|_{L^2(U)}^2 + \|\mathbf{u}'\|_{L^2(U)}^2 + \|\mathbf{u}\|_{L^2(U)}^2).$$

This together with the bounds obtained before gives us the desired estimate. The proof of (3.3.8) is completed.

Next, we prove the enhanced parabolic regularity (3.3.9) by assuming $g \in H^2(U) \cap H_0^1(U)$ and $\boldsymbol{f} \in H^1(0,T;L^2(U))$. Again, we shall first differentiate the approximate equation (3.2.7) in the time variable t instead of the original equation, as we currently do not have the differentiability (in time) of the solution.

We now have

$$(\mathbf{u}_m'', w_k) + B[\mathbf{u}_m', w_k] = (\boldsymbol{f}', w_k),$$

and so multiplying ${d_m^k}'(t)$ and taking sum over $1 \le k \le m$ yields

$$(\mathbf{u}_m'', \mathbf{u}_m') + B[\mathbf{u}_m', \mathbf{u}_m'] = (\boldsymbol{f}', \mathbf{u}_m').$$

Invoking the uniform ellipticity and Grönwall's inequality, we deduce

$$\begin{aligned}\sup_{0\le t\le T} \|\mathbf{u}_m'(t)\|_{L^2(U)}^2 + \int_0^T \|\mathbf{u}_m'\|_{H_0^1(U)}^2 \,\mathrm{d}t &\le C(\|\mathbf{u}_m'(0)\|_{L^2(U)}^2 + \|\boldsymbol{f}'\|_{L^2(0,T;L^2(U))}^2) \\ &\le C(\|\mathbf{u}_m(0)\|_{H^2(U)}^2 + \|\boldsymbol{f}\|_{H^1(0,T;L^2(U))}^2).\end{aligned} \tag{3.3.13}$$

To complete the proof, we must seek for the control of $\|\mathbf{u}_m(0)\|_{H^2(U)}^2$. Since L is **not necessarily symmetric**, we may have to control $\|\mathbf{u}_m(0)\|_{H^2(U)}$ by $\|\Delta\mathbf{u}_m(0)\|_{L^2(U)}$. This can be achieved by expanding $\mathbf{u}_m$ into the basis $\{w_k\}$. In fact, by Elliptic Regularity Theorem (and $\mathbf{u}_m|_{\partial U} = 0$), we have

$$\|\mathbf{u}_m(0)\|_{H^2(U)}^2 \le C(\|\mathbf{u}_m(0)\|_{L^2(U)}^2 + \|\Delta\mathbf{u}_m(0)\|_{L^2(U)}^2).$$

Since $\mathbf{u}_m(0) = \sum_{k=1}^m d_m^k(0) w_k(\boldsymbol{x})$ and $-\Delta\mathbf{u}_m(0) = \sum_{k=1}^m d_m^k(0)\lambda_k w_k(\boldsymbol{x})$ (recall w_k is an eigenfunction of $-\Delta$), the orthogonality implies that

$$\|\mathbf{u}_m(0)\|_{L^2(U)}^2 = \sum_{k=1}^m (d_m^k(0))^2 \le \lambda_1^{-2} \sum_{k=1}^m (d_m^k(0)\lambda_k)^2 = \lambda_1^{-2}\|\Delta\mathbf{u}_m(0)\|_{L^2(U)}^2,$$

and so

$$\|\mathbf{u}_m(0)\|_{H^2(U)}^2 \le C(1 + \lambda_1^{-2})\|\Delta\mathbf{u}_m(0)\|_{L^2(U)}^2$$

where $\lambda_1 > 0$ is the principal eigenvalue of $-\Delta$ with the Dirichlet boundary condition. Next, using $\mathbf{u}_m|_{\partial U} = \Delta\mathbf{u}_m|_{\partial U} = 0$, we integrate by parts twice to get:

$$\|\mathbf{u}_m(0)\|_{H^2(U)}^2 \le C(\mathbf{u}_m(0), \Delta^2\mathbf{u}_m(0)).$$

Since $\Delta^2\mathbf{u}_m(0) \in \mathrm{Span}\{w_1, \cdots, w_m\}$ and $(\mathbf{u}_m(0), w_k) = (g, w_k)$, we have

$$\|\mathbf{u}_m(0)\|_{H^2(U)}^2 \le C(\Delta g, \Delta\mathbf{u}_m(0)) \le \frac{1}{2}\|\mathbf{u}_m(0)\|_{H^2(U)}^2 + C\|g\|_{H^2(U)}^2,$$

and so $\|\mathbf{u}_m(0)\|_{H^2(U)} \leq C\|g\|_{H^2(U)}$ for certain $C > 0$. Therefore, we have

$$\begin{aligned}&\sup_{0\leq t\leq T}\|\mathbf{u}_m'(t)\|_{L^2(U)}^2 + \int_0^T \|\mathbf{u}_m'\|_{H_0^1(U)}^2\,\mathrm{d}t \leq C(\|\mathbf{u}_m'(0)\|_{L^2(U)}^2 + \|\boldsymbol{f}'\|_{L^2(0,T;L^2(U))}^2)\\ \leq{}& C(\|g\|_{H^2(U)}^2 + \|\boldsymbol{f}\|_{H^1(0,T;L^2(U))}^2).\end{aligned} \tag{3.3.14}$$

Now, we use the equation to prove the $L_t^\infty H_{\boldsymbol{x}}^2$ regularity. Recall that

$$B[\mathbf{u}_m, w_k] = (\boldsymbol{f} - \mathbf{u}_m', w_k), \ \ 1 \leq k \leq m.$$

Mutiplying this by $\lambda_k d_m^k(t)$ and taking sum over $1 \leq k \leq m$, we get

$$B[\mathbf{u}_m, -\Delta\mathbf{u}_m] = (\boldsymbol{f} - \mathbf{u}_m', -\Delta\mathbf{u}_m), \ \ t \in [0, T].$$

Since $\Delta\mathbf{u}_m|_{\partial U} = 0$, we know $B[\mathbf{u}_m, -\Delta\mathbf{u}_m] = (L\mathbf{u}_m, -\Delta\mathbf{u}_m)$. Invoking the conclusion of Exercise 3.3.1, we see that

$$\|\mathbf{u}_m\|_{H^2(U)}^2 \leq C(\|\boldsymbol{f}\|_{L^2(U)}^2 + \|\mathbf{u}_m'\|_{L^2(U)}^2 + \|\mathbf{u}_m\|_{L^2(U)}^2). \tag{3.3.15}$$

Combining this with the bounds for $\mathbf{u}_m'$, we deduce

$$\begin{aligned}&\sup_{0\leq t\leq T}\left(\|\mathbf{u}_m'(t)\|_{L^2(U)}^2 + \|\mathbf{u}_m(t)\|_{H^2(U)}^2\right) + \int_0^T \|\mathbf{u}_m'\|_{H_0^1(U)}^2\,\mathrm{d}t\\ \leq{}& C(\|\mathbf{u}_m(0)\|_{H^2(U)}^2 + \|\boldsymbol{f}\|_{H^1(0,T;L^2(U))}^2).\end{aligned} \tag{3.3.16}$$

Passing to the limit $m = m_l \to \infty$ leads to the desired estimate for $\mathbf{u}$.

It now remains to prove $\mathbf{u}'' \in L^2(0, T; H^{-1}(U))$. Recall that $H^{-1}(U)$ is the dual space of $H_0^1(U)$. So we pick $\varphi \in H_0^1(U)$ satisfying $\|\varphi\|_{H_0^1(U)}$ and write $\varphi = \varphi_m + \varphi_m^\perp$ satisfying $\varphi_m \in \mathrm{Span}\{w_1, \cdots, w_m\}$ and $\varphi_m^\perp \in \mathrm{Span}\{w_{m+1}, w_{m+2}, \cdots\}$. By definition, we have

$$\begin{aligned}\|\mathbf{u}_m''\|_{H^1(U)} &= \sup_{\|\varphi\|_{H_0^1}\leq 1} |\langle\mathbf{u}_m'', \varphi\rangle| = \sup_{\|\varphi\|_{H_0^1}\leq 1} \left|\langle\boldsymbol{f}', \varphi_m\rangle - B[\mathbf{u}_m', \varphi_m; t]\right|\\ &\leq \sup_{\|\varphi\|_{H_0^1}\leq 1}\left(\|\boldsymbol{f}'\|_{L^2(U)}\|\varphi_m\|_{L^2(U)} + \|\mathbf{u}_m'\|_{H_0^1(U)}\|\varphi_m\|_{H_0^1(U)}\right) \leq \|\boldsymbol{f}'\|_{L^2(U)} + \|\mathbf{u}_m'\|_{H_0^1(U)}.\end{aligned}$$

Since we already prove the uniform bound for $\mathbf{u}_m'$, then letting $m = m_l$ and passing to the limit immediately leads to $\mathbf{u}'' \in L^2(0, T; H^{-1}(U))$ and the stated estimate (3.3.9). □

By induction on the Sobolev index, one can also prove the higher-order parabolic regularity results. We only state the conclusions and skip the proof.

Theorem 3.3.2 (High-order parabolic regularity). Let $m \in \mathbb{N}^*$. For the parabolic equation (3.0.1), assume $g \in H^{2m+1}(U)$ and $\frac{\mathrm{d}^k\boldsymbol{f}}{\mathrm{d}t^k} \in L^2(0, T; H^{2m-2k}(U))$ for $0 \leq k \leq m$. Assume also the initial data g

satisfies the **compatibility condition up to m-th order**, namely

$$g_0 = g \in H_0^1(U),\ g_j := \frac{\mathrm{d}^{j-1}\boldsymbol{f}}{\mathrm{d}t^{j-1}}(0) - Lg_{j-1} \in H_0^1(U),\ \ 1 \le j \le m.$$

Then we have $\frac{\mathrm{d}^k\mathbf{u}}{\mathrm{d}t^k} \in L^2(0,T;H^{2m+2-2k}(U))$ for $0 \le k \le m+1$ with the estimate

$$\sum_{k=0}^{m+1} \left\| \frac{\mathrm{d}^k\mathbf{u}}{\mathrm{d}t^k} \right\|^2_{L^2(0,T;H^{2m+2-2k}(U))} \le C\left(\sum_{k=0}^{m} \left| \frac{\mathrm{d}^k\boldsymbol{f}}{\mathrm{d}t^k} \right|^2_{L^2(0,T;H^{2m-2k}(U))} + \|g\|^2_{H^{2m+1}(U)} \right), \tag{3.3.17}$$

where the constant $C > 0$ depending only on m, U, T and the coefficients of L.

Remark 3.3.1. The compatibility conditions are necessary in order that each of the functions $g_0, g_1, \cdots, g_m$ has zero trace on the boundary ∂U. Otherwise, the boundary conditions for the time-differentiated system no longer hold.

Theorem 3.3.3 (C^∞ parabolic regularity). Let $m \in \mathbb{N}^*$. For the parabolic equation (3.0.1), assume $g \in C^\infty(\overline{U})$ and $f \in C^\infty(\overline{U_T})$ and the initial data g satisfies the **compatibility condition up to infinite order**, namely

$$g_0 = g \in H_0^1(U),\ g_j := \frac{\mathrm{d}^{j-1}\boldsymbol{f}}{\mathrm{d}t^{j-1}}(0) - Lg_{j-1} \in H_0^1(U),\ \ j = 1, 2, \cdots$$

Then (3.0.1) has a unique solution $u \in C^\infty(\overline{U_T})$.

Exercise 3.3

Exercise 3.3.1. Suppose $u \in C^\infty(U)$ also satisfies $u = \Delta u = 0$ on ∂U. Prove that there exist constants $\beta > 0, \gamma \ge 0$ such that

$$\beta\|u\|^2_{H^2(U)} \le (Lu, -\Delta u)_{L^2(U)} + \gamma\|u\|^2_{L^2(U)}.$$

Here $Lu := -\partial_j(a^{ij}\partial_j u) + b_i\partial_i u + cu$ with $a^{ij} = a^{ji}$ is uniformly elliptic.

(This estimate is used in the proof of parabolic regularity. To simplify the proof you may assume $b^i = c = 0$ as well.)

Exercise 3.3.2. Give a simplified proof for the enhanced regularity (3.3.9) when $L = -\partial_j(a^{ij}\partial_i u)$ (that is, L is symmetric).

(Hint: choose $\{w_k\}$ to be the eigenfunctions of L instead of $-\Delta$ in $H_0^1(U)$. Then the inequality in Exercise 3.3.1 is not needed.)

3.4 Parabolic maximum principles

This section is devoted to the maximum principle and Harnack's inequality for second-order parabolic operators. For technical simplicity, it would be more convenient to assume the elliptic operator L has the non-divergence form

$$Lu = -a^{ij}\partial_i\partial_j u + b^i\partial_i u + cu, \quad a^{ij}, b^i, c \in C(\overline{U_T}),\ a^{ij} = a^{ji}.$$

We also recall that the parabolic cylinder $U_T := (0,T] \times U$ and the parabolic boundary $\Gamma_T := \overline{U_T} \backslash U_T$ where $U \subset \mathbb{R}^d$ is a given bounded domain with sufficiently smooth boundary.

In this lecture notes, we only prove the weak maximum principle, while the proof of the strong maximum principle requires parabolic Harnack's inequality and we skip the proof here.

3.4.1 Weak maximum principle

Given an interval $I \subseteq \mathbb{R}$ and a domain $U \subseteq \mathbb{R}^d$, we define $C_1^2(I \times U) = \{u : I \times U \to \mathbb{R} : u, \partial_{x_i}u, \partial_{x_i}\partial_{x_j}u, \partial_t u \in C(I \times U), \forall 1 \le i,j \le d\}$. The variables $t \in I$ and $\boldsymbol{x} \in U$. In this section, we assume U to be bounded.

Theorem 3.4.1 (Weak Maximum Principle). Assume $u \in C_1^2(U_T) \cap C(\overline{U_T})$ and $c = 0$ in U_T. Then $\partial_t u + Lu \le 0$ (≥ 0, resp.) in U_T implies $\max\limits_{\overline{U_T}} u = \max\limits_{\Gamma_T} u$ ($\min\limits_{\overline{U_T}} u = \min\limits_{\Gamma_T} u$, resp.). In such case the function u is called a subsolution (supersolution, resp.).

Proof. The proof is quite similar to that of Theorem 2.6.1. We first consider that case that $\partial_t u + Lu < 0$ in U_T and assume there exists a $(t_0, \boldsymbol{x}_0) \in U_T$ such that $u(t_0, \boldsymbol{x}_0) = \max\limits_{\overline{U_T}} u$. If $0 < t_0 < T$, then we know $(t_0, \boldsymbol{x}_0)$ belongs to the interior of U_T and thus $\partial_t u(t_0, \boldsymbol{x}_0) = 0$. On the other hand, we can mimic the proof of Theorem 2.6.1 to show that $Lu \ge 0$ at $(t_0, \boldsymbol{x}_0)$. Therefore, we deduce $\partial_t u + Lu \ge 0$ at $(t_0, \boldsymbol{x}_0)$, which leads to a contradiction to the assumption $\partial_t u + Lu < 0$. If $t_0 = T$, then we must have $\partial_t u(t_0, \boldsymbol{x}_0) \ge 0$ and again this leads to $\partial_t u + Lu \ge 0$ at $(t_0, \boldsymbol{x}_0)$.

Now, we only assume $\partial_t u + Lu \le 0$ in U_T. In such case, we consider the perturbed function $u^\varepsilon(t, \boldsymbol{x}) = u(t, \boldsymbol{x}) - \varepsilon t$ for $\varepsilon > 0$. Then we can directly compute that $\partial_t u^\varepsilon + Lu^\varepsilon = \partial_t u + Lu - \varepsilon < 0$ in U_T. The above argumet then implies $\max\limits_{\overline{U_T}} u^\varepsilon = \max\limits_{\Gamma_T} u^\varepsilon$. Let $\varepsilon \to 0$ and we obtain that

$$\max_{\overline{U_T}} u = \lim_{\varepsilon \to 0} \max_{\overline{U_T}} (u - \varepsilon t) = \lim_{\varepsilon \to 0} \max_{\Gamma_T} (u - \varepsilon t) \le \max_{\Gamma_T} u.$$

The reverse inequality is trivial as $\Gamma_T \subsetneq \overline{U_T}$. □

When $c \ge 0$ in the operator L, we can prove similar results.

Theorem 3.4.2 (Weak Maximum Principle, $c \ge 0$). Assume $u \in C_1^2(U_T) \cap C(\overline{U_T})$ and $c \ge 0$ in U_T. Then $\partial_t u + Lu \le 0$ (≥ 0, resp.) in U_T implies $\max\limits_{\overline{U_T}} u \le \max\limits_{\Gamma_T} u^+$ ($\min\limits_{\overline{U_T}} u \ge \min\limits_{\Gamma_T} u^-$, resp.). In particular, if $\partial_t u + Lu = 0$ in U_T, then $\max\limits_{\overline{U_T}} |u| = \max\limits_{\Gamma_T} |u|$.

3.4.2 Strong maximum principle

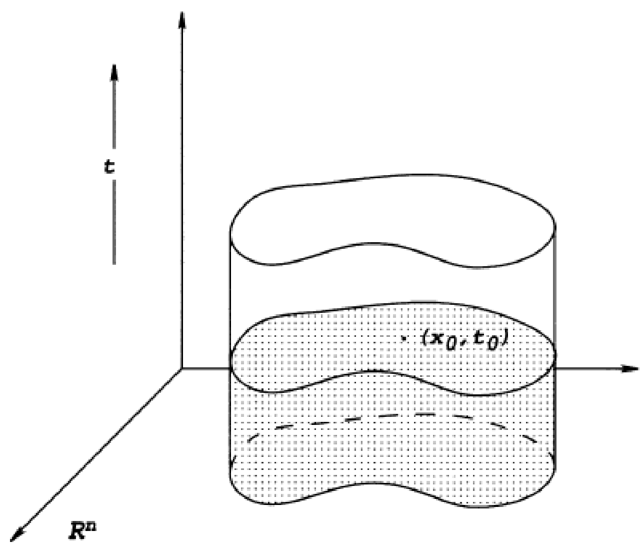
To prove the strong maximum principles for parabolic equations, we shall first introduce the parabolic Harnack's inequality.

Theorem 3.4.3 (Parabolic Harnack's inequality). Assume $u \in C_1^2(U_T)$ solves $\partial_t u + Lu = 0$ in U_T and $u \geq 0$ in U_T. Suppose $V \Subset U$ is connected. Then for each $0 < t_1 < t_2 \leq T$, there exists a constant $C > 0$ depending on V, t_1, t_2 and the coefficients of L, such that

$$\sup_V u(t_1, \cdot) \leq \inf_V u(t_2, \cdot).$$

The proof of Theorem 3.4.3 again relies on the technique logarithemic gradient estimates and we refer to Evans [6, Theorem 7.1.10, pp. 391] for the details. Below, we show how to prove the strong maximum principle with the help of the parabolic Harnack's inequality.

Theorem 3.4.4 (Strong Maximum Principle). Assume $u \in C_1^2(U_T) \cap C(\overline{U_T})$ and $c = 0$ in U_T. Suppose also U is connected. If $\partial_t u + Lu \leq 0$ (≥ 0, resp.) in U_T and u attains its maximum (minimum, resp.) over $\overline{U_T}$ at a point $(t_0, \boldsymbol{x}_0) \in U_T$, then u is a constant on U_{t_0} (not U_T!).



Parabolic strong maximum principle

The conclusion of Theorem 3.4.3 indicates that parabolic PDEs have "infinite propagation speed of disturbances".

Proof. Assume $\partial_t u + Lu \leq 0$ in U_T and u attains its maximum at $(t_0, \boldsymbol{x}_0) \in U_T$. Now we select an open set $W \Subset U$ with a smooth boundary ∂W such that $\boldsymbol{x}_0 \in W$. Then we consider the parabolic equation

$$\partial_t v + Lv = 0 \quad \text{in } W_T, \qquad v = u \quad \text{on } \Delta_T := \overline{W_T} \backslash W_T.$$

By the weak maximum principle, we know $u \le v \le M := \max_{\overline{U_T}} u$ and thus $v = M$ at $(t_0, \boldsymbol{x}_0)$.

Now, it remains to prove $w := M - v \equiv 0$, which immediately leads to our desired result because $W \Subset U$ is arbitrary. Since $c = 0$, we know $\partial_t w + Lw = 0$ in W_T and $w \ge 0$. Then by Harnack's inequality, for any $V \Subset W$ with $\boldsymbol{x}_0 \in V$ and $0 < t < t_0$, we have

$$\sup_V w(t, \cdot) \le C \inf_V w(t_0, \cdot).$$

On the other hand, $\inf_V w(t_0, \cdot) \le w(t_0, \boldsymbol{x}_0) = 0$ forces $w \equiv 0$ on $\{t\} \times V$ for any $0 < t < t_0$. Since $V \Subset W$ and $t \in (0, t_0)$ are both arbitrary, we know $w = 0$ in W_{t_0} and therefore $v = M$ in W_{t_0}. Because of $v = u$ on Δ_T, we conclude that $u = M$ on $[0, t_0] \times \partial W$. □

For $c \ge 0$, we can also prove an analogous result.

Theorem 3.4.5 (Strong Maximum Principle, $c \ge 0$). Assume $u \in C_1^2(U_T) \cap C(\overline{U_T})$ and $c \ge 0$ in U_T. Suppose also U is connected. If $\partial_t u + Lu \le 0$ (≥ 0, resp.) in U_T and u attains its non-negative maximum (non-positive minimum, resp.) over $\overline{U_T}$ at a point $(t_0, \boldsymbol{x}_0) \in U_T$, then u is a constant on U_{t_0} (not U_T!).

Exercise 3.4

Exercise 3.4.1. Let u be a smooth solution to the heat equation $\partial_t u - \Delta u + cu = 0$ in $\mathbb{R}_+ \times U$ with $u = g$ on $\{t = 0\} \times \bar{U}$ and boundary condition $u = 0$ on $[0, \infty) \times \partial U$. Here $g(\boldsymbol{x}), c(t, \boldsymbol{x})$ are given continuous functions.

(1) When $c \ge \gamma > 0$ for some constant γ, prove that there exists a constant $A > 0$ such that $|u(t, \boldsymbol{x})| \le Ae^{-\gamma t}$ for any $(t, \boldsymbol{x}) \in (0, T] \times U$.
(2) When $g \ge 0$ and c is bounded, prove that u is non-negative.

Exercise 3.4.2. In Exercise 3.2.2, if we additionally assume $g \in C(\overline{U})$, then prove that the solution u uniformly converges to 0 as $t \to \infty$, that is,

$$\lim_{t \to \infty} \sup_{x \in U} |u(t, \boldsymbol{x})| = 0$$

and find the decay rate.

(Hint: What is the solution if the initial data is an eigenfunction w_1 for the principal eigenvalue λ_1?)

Exercise 3.4.3. Prove Theorem 3.4.2. You may refer to the proof of Theorem 2.6.2.

Exercise 3.4.4. Prove Theorem 3.4.5.

(Hint: Consider $\partial_t u + L'u$ with $L'u := Lu - cu$.)

3.5 Vanishing viscosity method

In this section, we introduce the method of **vanishing viscosity limit** (also called **inviscid limit**) to prove the local existence of first-order linear symmetric hyperbolic system. Indeed, the vanishing viscosity limit plays a signficantly role in the study of hyperbolic conservation laws and produces numerous challenging problems. The core idea of this method is to adding a Laplacian term with a small coefficient ε to get a "regularized" parabolic system and solve this parabolic system and finally take the limit $\varepsilon \to 0$ in order to obtain the existence of the original hyperbolic system.

In what follows, we assume the domain $U = \mathbb{R}^d$. It should also be remarked that this method may not be applicable if the domain has a boundary due to the possible appearance of "boundary layer". For example, when taking the inviscid limit for Navier-Stokes equations (characterizing the motion of viscoud fluids) in a domain with boundary, the boundary layer may appear and cause the mismatch in the boundary conditions. There are still many unsolved problems in this area.

Consider the following first-order system of PDEs

$$\begin{cases} \partial_t \mathbf{u} + \sum_{j=1}^{d} \mathbf{B}_j \partial_j \mathbf{u} = \boldsymbol{f} & \text{in } (0,\infty) \times \mathbb{R}^d, \\ \mathbf{u} = \mathbf{g} & \text{on } \{t=0\} \times \mathbb{R}^d. \end{cases} \tag{3.5.1}$$

Here $\mathbf{u} = (u^1, \cdots, u^m) : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^m$ is the collection of unknowns. $\mathbf{B}_j : [0,\infty) \times \mathbb{R}^d \to \mathbb{M}^{m\times m}$, $(1 \le j \le d)$ are the coefficient matrices. $\boldsymbol{f} : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^m$ and $\mathbf{g} : \mathbb{R}^d \to \mathbb{R}^m$ are given functions.

Definition 3.5.1 (Hyperbolicity). System (3.5.1) is called **hyperbolic** if the $m \times m$ matrix $\mathbf{B}(t, \boldsymbol{x}; \boldsymbol{y})$ is diagonalizable for each $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ and $t \ge 0$. Here $\mathbf{B}(t, \boldsymbol{x}; \boldsymbol{y})$ is defined by

$$\mathbf{B}(t, \boldsymbol{x}; \boldsymbol{y}) := \sum_{j=1}^{d} y_j \mathbf{B}_j(t, \boldsymbol{x}) \quad (\boldsymbol{x} \in \mathbb{R}^d, t \ge 0).$$

If each $\mathbf{B}_j$ is symmetric, then we say (3.5.1) is **symmetric hyperbolic.**

Equivalently, (3.5.1) is hyperbolic provided that for each $\boldsymbol{x}, \boldsymbol{y}, t$, the matrix $\mathbf{B}(t, \boldsymbol{x}; \boldsymbol{y})$ has m real eigenvalues

$$\lambda_1(t, \boldsymbol{x}; \boldsymbol{y}) \le \lambda_2(t, \boldsymbol{x}; \boldsymbol{y}) \le \cdots \le \lambda_m(t, \boldsymbol{x}; \boldsymbol{y})$$

and corresponding eigenvectors $\{\mathbf{r}_k(t, \boldsymbol{x}; \boldsymbol{y})\}$ form a basis of $\mathbb{R}^m$. In particular, if all $\le$ are replaced by strict inequalities $<$, then we say (3.5.1) is **strictly hyperbolic.**

Remark 3.5.1 (Motivation for the definition of hyperbolicity). For simplicity we assume each $\mathbf{B}_j$ has constant coefficient and $\boldsymbol{f} = 0$. Consider the **plane wave** solution to (3.5.1) having the form $\mathbf{u}(t, \boldsymbol{x}) = \mathbf{v}(\boldsymbol{y} \cdot \boldsymbol{x} - \sigma t)$ with the profile $\mathbf{v} : \mathbb{R} \to \mathbb{R}^m$ sufficiently smooth. Plugging this ansatz into (3.5.1), we obtain that

$$\left(-\sigma I_m + \sum_{j=1}^{d} y_j \mathbf{B}_j\right)\mathbf{v}' = 0.$$

So, $\mathbf{v}'$ is an eigenvector of the matrix $\mathbf{B}(\boldsymbol{y})$ corresponding to the eigenvalue σ. Then the hyperbolic condition requires that there are m **distinct** plane wave solutions for each given $\boldsymbol{y} \in \mathbb{R}^d$, given by

$$(\boldsymbol{y} \cdot \boldsymbol{x} - \lambda_k(\boldsymbol{y})t)\mathbf{r}_k(\boldsymbol{y}) \qquad (1 \le k \le m),\ \ \lambda_1(\boldsymbol{y}) \le \cdots \le \lambda_m(\boldsymbol{y}).$$

The eigenvalues for $\|y\| = 1$ are the wave speeds.

Now we want to prove the local existence of the weak solution to the following symmetric hyperbolic system (3.5.1)

$$\begin{cases} \partial_t \mathbf{u} + \sum_{j=1}^{d} \mathbf{B}_j \partial_j \mathbf{u} = \boldsymbol{f} & \text{in } (0,\infty)\times\mathbb{R}^d, \\ \mathbf{u} = \mathbf{g} & \text{on } \{t=0\}\times\mathbb{R}^d. \end{cases} \tag{3.5.2}$$

The unknown is $\mathbf{u} : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}^m$. The coefficient matrices $\mathbf{B}_j \in C^2([0,T]\times\mathbb{R}^d;\mathbb{M}^{m\times m})$ are symmetric for $1 \le j \le d$ satisfying

$$\sup_{[0,T]\times\mathbb{R}^d} |\mathbf{B}_j| + |\nabla_{t,\boldsymbol{x}}\mathbf{B}_j| + |\nabla^2_{\boldsymbol{x},t}\mathbf{B}_j| < \infty \qquad 1 \le j \le d. \tag{3.5.3}$$

The initial data $\mathbf{g} \in H^1(\mathbb{R}^d \to \mathbb{R}^m)$ and the given source term $\boldsymbol{f} \in H^1((0,T)\times\mathbb{R}^d \to \mathbb{R}^m)$.

Next we define the weak solution to system (3.5.1).

Definition 3.5.2. For given $\mathbf{u}, \mathbf{v} \in H^1(\mathbb{R}^d \to \mathbb{R}^m)$ and $0 \le t \le T$, define the bi-linear form

$$B[\mathbf{u},\mathbf{v};t] := \int_{\mathbb{R}^d} \sum_{j=1}^{d} (\mathbf{B}_j(t,\cdot)\partial_j \mathbf{u}) \cdot \mathbf{v}\,\mathrm{d}\boldsymbol{x}.$$

We say $\mathbf{u} \in L^2(0,T;H^1(\mathbb{R}^d \to \mathbb{R}^m))$ with $\mathbf{u}' \in L^2(0,T;L^2(\mathbb{R}^d \to \mathbb{R}^m))$ is a weak solution to the initial-value problem for the symmetric hyperbolic system if

- $(\mathbf{u}',\mathbf{v}) + B[\mathbf{u},\mathbf{v};t] = (\boldsymbol{f},\mathbf{v})$ for each $\mathbf{v} \in H^1(\mathbb{R}^d \to \mathbb{R}^m)$ and a.e. $t \in [0,T]$.
- $\mathbf{u}(0) = \mathbf{g}$.

Note that we already have $\mathbf{u} \in C([0,T];L^2(\mathbb{R}^d \to \mathbb{R}^m))$ and so $\mathbf{u}(0) = \mathbf{g}$ holds pointwisely.

We aim to prove that

Theorem 3.5.1. There exists a unique weak solution to (3.5.2).

3.5.1 Existence of the regularized parabolic system

Now we introduce the ε-regularized parabolic system

$$\begin{cases} \partial_t \mathbf{u}^\varepsilon - \varepsilon\Delta \mathbf{u}^\varepsilon + \sum_{j=1}^d \mathbf{B}_j \partial_j \mathbf{u}^\varepsilon = \boldsymbol{f} & \text{in } (0,\infty)\times\mathbb{R}^d, \\ \mathbf{u}^\varepsilon = \mathbf{g}^\varepsilon & \text{on } \{t=0\}\times\mathbb{R}^d, \end{cases} \tag{3.5.4}$$

where $0<\varepsilon<1$ and $\mathbf{g}^\varepsilon := \eta_\varepsilon * g$. The idea of the vanishing viscosity method is that one first solves the ε-regularized parabolic system for fixed $\varepsilon>0$, then prove uniform-in-ε estimates, and finally pass to the limit as $\varepsilon\to 0$ to obtain a solution to the hyperbolic system ($\varepsilon=0$) (3.5.2).

The first step is to solve the parabolic system for each fixed $\varepsilon>0$.

Theorem 3.5.2. For each $\varepsilon>0$, there exists a unique solution $\mathbf{u}^\varepsilon$ to (3.5.4) satisfying

$$\mathbf{u}^\varepsilon \in L^2(0,T;H^3(\mathbb{R}^d\to\mathbb{R}^m)),\ \ \mathbf{u}^{\varepsilon\prime}\in L^2(0,T;H^1(\mathbb{R}^d\to\mathbb{R}^m)).$$

The difficulty in the parabolic system is that the ε-regularization term is a higher-order term. Therefore, we may treat the first-order term $\mathbf{B}_j\partial_j\mathbf{u}^\varepsilon$ as a source term. It should also be noted that the H^3 and H^1 regularity for $\mathbf{u}^\varepsilon, \mathbf{u}^{\varepsilon\prime}$ is a consequence of the parabolic regularity theorem (Theorem 3.3.2). In other words, the existence of system (3.5.4) should be proved in suitable lower-order space.

Motivation to find suitable function spaces

To "predict" the function space for the local existence, let us first look at the linear parabolic equation before considering the system (3.5.4).

$$\partial_t u - \varepsilon\Delta u = f \quad \text{in } \mathbb{R}_+\times\mathbb{R}^d, \qquad u(0)=g\in H^1(\mathbb{R}^d) \quad \text{on } \{t=0\}\times\mathbb{R}^d.$$

By Duhamel principle, we know

$$u(t) = e^{\varepsilon t\Delta}g + \int_0^t e^{\varepsilon(t-\tau)\Delta} f(\tau)\,\mathrm{d}\tau,$$

where $e^{\varepsilon t\Delta}g := (e^{-\varepsilon t|\xi|^2}\hat{g}(\boldsymbol{\xi}))^\vee$. Then for $s\ge 0$, we have

$$\|e^{\varepsilon t\Delta}g\|_{H^s(\mathbb{R}^d)} = \|e^{-\varepsilon t|\xi|^2}\hat{g}(\boldsymbol{\xi})\langle\boldsymbol{\xi}\rangle^s\|_{L^2} \le C\|\hat{g}(\boldsymbol{\xi})\langle\boldsymbol{\xi}\rangle^s\|_{L^2} = C\|g\|_{H^s}.$$

Also, let $\boldsymbol{\eta} = \sqrt{\varepsilon t}\boldsymbol{\xi}$ and we find that $e^{-\varepsilon t|\xi|^2}\langle\boldsymbol{\xi}\rangle^s = e^{-\eta^2}\langle(\varepsilon t)^{-\frac12}\boldsymbol{\eta}\rangle^s \le C(\varepsilon t)^{-\frac{s}{2}}$. Therefore, we expect the homogeneous part $e^{\varepsilon t\Delta}g\in L^\infty(0,T;H^1(\mathbb{R}^d))$ when the initial data $g\in H^1(\mathbb{R}^d)$. For the non-homogeneous part, let $\Phi(t,\boldsymbol{x}) := \frac{1}{(4\pi t)^{\frac{d}{2}}}e^{-\frac{|x|^2}{4\varepsilon t}}$ be the fundamental solution to the standard heat equation

($\varepsilon = 1$). Then we can write

$$\int_0^t e^{\varepsilon(t-\tau)\Delta} f(\tau)\,\mathrm{d}\tau = \int_0^t \Phi(\varepsilon(t-\tau),\cdot) * f(\tau,\cdot)\,\mathrm{d}\tau.$$

Using Minkowski's inequality for integrals and Young's inequality, we have

$$\left\|\int_0^t \Phi(\varepsilon(t-\tau),\cdot) * f(\tau,\cdot)\right\|_{L^2} \le \int_0^t \left\|\Phi(\varepsilon(t-\tau),\cdot) * f(\tau,\cdot)\right\|_{L^2}\,\mathrm{d}\tau$$
$$\le \int_0^t \underbrace{\|\Phi(\varepsilon(t-\tau),\cdot)\|_{L^1}}_{=1} \|f(\tau,\cdot)\|_{L^2}\,\mathrm{d}\tau \le \int_0^t \|f(\tau,\cdot)\|_{L^2}\,\mathrm{d}\tau \le \begin{cases} T\|f\|_{L_t^\infty L_x^2} \\ T^{\frac12}\|f\|_{L_t^2 L_x^2} \end{cases}.$$

Similarly, we have

$$\left\|\nabla_x \int_0^t \Phi(\varepsilon(t-\tau),\cdot) * f(\tau,\cdot)\,\mathrm{d}\tau\right\|_{L^2} = \left\|\int_0^t \nabla\Phi(\varepsilon(t-\tau),\cdot) * f(\tau,\cdot)\,\mathrm{d}\tau\right\|_{L^2}$$
$$\le \int_0^t \underbrace{\|\nabla\Phi(\varepsilon(t-\tau),\cdot)\|_{L^1}}_{\le C(\varepsilon(t-\tau))^{-\frac12}} \|f(\tau,\cdot)\|_{L^2}\,\mathrm{d}\tau \le C_\varepsilon T^{1/2}\|f\|_{L_t^\infty L_x^2}.$$

Since the given source term $f \in H^1((0,T)\times\mathbb{R}^d)$, we know that $f \in L^\infty(0,T;L^2(\mathbb{R}^d))$ and also $L^2((0,T)\times\mathbb{R}^d)$ for each given $T < \infty$.

Therefore, we may "predict" that the local existence of the ε-regularized system (3.5.4) can be proved in $L^\infty(0,T;H^1(\mathbb{R}^d))$.

Proof of Theorem 3.5.2

We use the Contraction Mapping Theorem to prove the existence of solution to (3.5.4).

Let $X := L^\infty(0,T;H^1(\mathbb{R}^d \to \mathbb{R}^m))$ and define

$$\mathcal{T} : \mathbf{w} \longmapsto e^{\varepsilon t\Delta}\mathbf{g} + \int_0^t e^{\varepsilon(t-\tau)\Delta}\left(\boldsymbol{f}(\tau,\cdot) - \sum_{j=1}^d \mathbf{B}_j(\tau,\cdot)\partial_j\mathbf{w}(\tau,\cdot)\right)\mathrm{d}\tau. \tag{3.5.5}$$

What we shall prove is that

- $\mathcal{T}$ maps X to X, namely $R(\mathcal{T}) \subseteq X$.
- $\mathcal{T}$ is a contraction on X, namely there exists a constant $C \in (0,1)$ such that $\|\mathcal{T}\mathbf{w} - \mathcal{T}\mathbf{v}\|_X \le C\|\mathbf{w}-\mathbf{v}\|_X$ holds for any $\mathbf{w},\mathbf{v} \in X$.

Once we prove these two facts, then the Contraction Mapping Theorem indicates that $\mathcal{T}$ has a unique fixed point in X, which is exactly our desired solution to the ε-regularized parabolic system (3.5.4).

From the a priori estimates, it is straightforward to see that for any $\mathbf{w} \in X$

$$\|\mathcal{T}\mathbf{w}\|_X \leq C\|g\|_{H^1} + C_\varepsilon T^{\frac{1}{2}}(\|\boldsymbol{f}\|_{L_t^\infty L_x^2} + \|\nabla \mathbf{w}\|_{L_t^\infty L_x^2}) < \infty.$$

This gives $R(\mathcal{T}) \subseteq X$.

Next, we verify that $\mathcal{T}$ is a contraction on X. Given any $\mathbf{w}, \mathbf{v} \in X$, we compute that

$$\mathcal{T}\mathbf{w} - \mathcal{T}\mathbf{v} = \sum_{j=1}^{d} \int_0^t e^{\varepsilon(t-\tau)\Delta} \mathbf{B}_j \partial_j (\mathbf{w} - \mathbf{v})(\tau)\, d\tau.$$

Again, by the a priori estimates, we have

$$\|\mathcal{T}\mathbf{w} - \mathcal{T}\mathbf{v}\|_X \leq C_\varepsilon T^{\frac{1}{2}} \|\mathbf{w} - \mathbf{v}\|_X.$$

Choosing $T_1 > 0$ sufficiently small such that $C_\varepsilon T_1^{\frac{1}{2}} < 0.5$, we find that $\mathcal{T}$ is a contraction on $L^\infty(0, T_1; H^1)$. Therefore, there exists a unique fixed point, say $\mathbf{u}^\varepsilon$, in $X = L^\infty(0, T_1; H^1)$, which is also the solution to (3.5.4). Then we repeat this argument in $[T_1, 2T_1], [2T_1, 3T_1], \cdots$ and obtain the existence of solution in $\mathbb{R}_+ \times \mathbb{R}^d$. (Note that the lifespan T_1 does not depend on $\boldsymbol{f}, \mathbf{g}$ in the contraction argument.)

Finally, we prove the solution $\mathbf{u}^\varepsilon$ satisfies the regularity

$$\mathbf{u}^\varepsilon \in L^2(0, T; H^3(\mathbb{R}^d \to \mathbb{R}^m)), \;\; \mathbf{u}^{\varepsilon\prime} \in L^2(0, T; H^1(\mathbb{R}^d \to \mathbb{R}^m)).$$

In fact, since the source term $\boldsymbol{f} - \sum \mathbf{B}_j \partial_j \mathbf{u}^\varepsilon$ belongs to $L^2(\mathbb{R}^d \to \mathbb{R}^m)$ for a.e. $t \in [0, T]$, the parabolic regularity theorem (Theorem 3.3.1) indicates that

$$\mathbf{u}^\varepsilon \in L^\infty(0, T; H^2(\mathbb{R}^d \to \mathbb{R}^m)), \;\; \mathbf{u}^{\varepsilon\prime} \in L^2(0, T; H^1(\mathbb{R}^d \to \mathbb{R}^m)),$$

and so $\boldsymbol{f} - \sum \mathbf{B}_j \partial_j \mathbf{u}^\varepsilon \in L^\infty(0, T; H^1(\mathbb{R}^d \to \mathbb{R}^m))$. Using Theorem 3.3.2, we obtain $\mathbf{u}^\varepsilon \in L^2(0, T; H^3(\mathbb{R}^d \to \mathbb{R}^m))$ as desired.

3.5.2 Uniform estimates and the vanishing viscosity limit

The a priori estimates derived in the last section depend on ε^{-1}. Therefore, we must seek for another uniform-in-ε estimate for the ε-regularized parabolic system (3.5.4) in order for the vanishing viscosity limit $\varepsilon \to 0$.

Theorem 3.5.3 (Uniform-in-ε energy estimates). Let $\mathbf{u}^\varepsilon$ be the solution to (3.5.4) obtained in Theorem 3.5.2. Then there exists a constant $C > 0$ (independent of ε) such that for any $\varepsilon \in (0, 1)$,

$$\sup_{0 \leq t \leq T} \left(\|\mathbf{u}^\varepsilon(t)\|_{H^1}^2 + \|\mathbf{u}^{\varepsilon\prime}(t)\|_{L^2}^2 \right) \leq C \left(\|\mathbf{g}\|_{H^1}^2 + \|\boldsymbol{f}\|_{L^2(0,T;H^1)}^2 + \|\boldsymbol{f}'\|_{L^2(0,T;L^2)}^2 \right). \tag{3.5.6}$$

Proof. We compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{R}^d}|\mathbf{u}^\varepsilon|^2\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d}\mathbf{u}^\varepsilon\cdot\partial_t\mathbf{u}^\varepsilon\,\mathrm{d}\boldsymbol{x}$$
$$= \int_{\mathbb{R}^d}\mathbf{u}^\varepsilon\cdot(\varepsilon\Delta\mathbf{u}^\varepsilon)\,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^d}\mathbf{u}^\varepsilon\cdot\boldsymbol{f}\,\mathrm{d}\boldsymbol{x} - \sum_{j=1}^d\int_{\mathbb{R}^d}\mathbf{u}^\varepsilon\cdot(\mathbf{B}_j\partial_j\mathbf{u}^\varepsilon)\,\mathrm{d}\boldsymbol{x}.$$

Integrating by parts in the first term, we get

$$\int_{\mathbb{R}^d}\mathbf{u}^\varepsilon\cdot(\varepsilon\Delta\mathbf{u}^\varepsilon)\,\mathrm{d}\boldsymbol{x} = -\varepsilon\int_{\mathbb{R}^d}|\nabla\mathbf{u}^\varepsilon|^2\,\mathrm{d}\boldsymbol{x}.$$

The second term can be directly controlled

$$\left|\int_{\mathbb{R}^d}\mathbf{u}^\varepsilon\cdot\boldsymbol{f}\,\mathrm{d}\boldsymbol{x}\right| \le \|\mathbf{u}^\varepsilon\|_{L^2}\|\boldsymbol{f}\|_{L^2}.$$

For the third term, we can use the symmetry of $\mathbf{B}_j$'s to eliminate the derivative ∂_j on $\mathbf{u}^\varepsilon$. Integrating by parts, we get

$$\int_{\mathbb{R}^d}\mathbf{u}^\varepsilon\cdot(\mathbf{B}_j\partial_j\mathbf{u}^\varepsilon)\,\mathrm{d}\boldsymbol{x}$$
$$= -\int_{\mathbb{R}^d}(\mathbf{B}_j\partial_j\mathbf{u}^\varepsilon)\cdot\mathbf{u}^\varepsilon\,\mathrm{d}\boldsymbol{x} - \int_{\mathbb{R}^d}|\mathbf{u}^\varepsilon|^2(\partial_j\mathbf{B}_j)\,\mathrm{d}\boldsymbol{x},$$

and so

$$\left|\int_{\mathbb{R}^d}\mathbf{u}^\varepsilon\cdot(\mathbf{B}_j\partial_j\mathbf{u}^\varepsilon)\,\mathrm{d}\boldsymbol{x}\right| = \frac{1}{2}\left|\int_{\mathbb{R}^d}|\mathbf{u}^\varepsilon|^2(\partial_j\mathbf{B}_j)\,\mathrm{d}\boldsymbol{x}\right| \le C\|\mathbf{u}^\varepsilon\|_{L^2}^2.$$

Summing up the above three terms, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{R}^d}|\mathbf{u}^\varepsilon|^2\,\mathrm{d}\boldsymbol{x} = -\varepsilon\int_{\mathbb{R}^d}|\nabla\mathbf{u}^\varepsilon|^2\,\mathrm{d}\boldsymbol{x} + C(\|\mathbf{u}^\varepsilon\|_{L^2}^2 + \|\boldsymbol{f}\|_{L^2}^2) \le C(\|\mathbf{u}^\varepsilon\|_{L^2}^2 + \|\boldsymbol{f}\|_{L^2}^2).$$

Using Grönwall's inequality, we get

$$\sup_{0\le t\le T}\|\mathbf{u}^\varepsilon(t)\|_{L^2}^2 \le C(\|\mathbf{g}\|_{L^2}^2 + \int_0^T\|\boldsymbol{f}(t)\|_{L^2}^2\,\mathrm{d}t),$$

where we also use $\|\mathbf{g}^\varepsilon\|_{L^2} \le \|\mathbf{g}\|_{L^2}$.

Similarly, the estimates of $\nabla\mathbf{u}^\varepsilon$ and $\mathbf{u}^{\varepsilon\prime}$ can be proved by differentiating the system by ∂_{x_k} and ∂_t

respectively. We omit the proof and only list the results here.

$$\begin{aligned}
\sup_{0\le t\le T} \|\nabla \mathbf{u}^\varepsilon(t)\|_{L^2}^2 &\le C(\|\nabla \mathbf{g}\|_{L^2}^2 + \|\boldsymbol{f}\|_{L^2(0,T;H^1)}^2),\\
\sup_{0\le t\le T} \|\mathbf{u}^{\varepsilon\prime}(t)\|_{L^2}^2 &\le C(\|\nabla \mathbf{g}\|_{L^2}^2 + \varepsilon^2\|\Delta \mathbf{g}^\varepsilon\|_{L^2}^2 + \|\boldsymbol{f}(0)\|_{L^2}^2 + \|\boldsymbol{f}\|_{L^2(0,T;H^1)}^2 + \|\boldsymbol{f}'\|_{L^2(0,T;L^2)}^2)\\
&\le C(\|\nabla \mathbf{g}\|_{L^2}^2 + \|\mathbf{g}^\varepsilon\|_{L^2}^2 + \|\boldsymbol{f}\|_{L^2(0,T;H^1)}^2 + \|\boldsymbol{f}'\|_{L^2(0,T;L^2)}^2),
\end{aligned}$$

where we use the facts that $\|\Delta \mathbf{g}^\varepsilon\|_{L^2} \le C\varepsilon^{-1}\|\nabla \mathbf{g}\|_{L^2}$ and

$$\|\boldsymbol{f}(0)\|_{L^2}^2 \le C(\|\boldsymbol{f}\|_{L^2(0,T;L^2)}^2 + \|\boldsymbol{f}'\|_{L^2(0,T;L^2)}^2).$$

Note that the appearance of $\Delta \mathbf{g}^\varepsilon$ is necessary because the initial data of $\mathbf{u}^{\varepsilon\prime}$ is $\boldsymbol{f} - \sum \mathbf{B}_j\partial_j \mathbf{g}^\varepsilon + \varepsilon\Delta\mathbf{g}^\varepsilon$. $\square$

The uniform boundedness of $\{\mathbf{u}^\varepsilon\}$ then leads to the weak limit.

Proof of Theorem 3.5.1. The uniform bounds and Exercise 3.1.1 show that there exists a subsequence $\{\varepsilon_k\}$ such that

$$\mathbf{u}^{\varepsilon_k} \rightharpoonup \mathbf{u} \quad \text{in } L^2(0,T;H^1), \quad \mathbf{u}^{\varepsilon_k\prime} \rightharpoonup \mathbf{u}' \quad \text{in } L^2(0,T;L^2).$$

It remains to verify the weak limit $\mathbf{u}$ is exactly the unique weak solution to (3.5.2). Pick a test function $\boldsymbol{\varphi} \in C^1([0,T];H^1)$ and we compute that (by integrating by parts in the term $\varepsilon\Delta\mathbf{u}^\varepsilon$)

$$\int_0^T (\mathbf{u}^{\varepsilon\prime},\boldsymbol{\varphi}) + \varepsilon\nabla\mathbf{u}^\varepsilon : \nabla\boldsymbol{\varphi} + B[\mathbf{u}^\varepsilon,\boldsymbol{\varphi};t]\,\mathrm{d}t = \int_0^T (\boldsymbol{f},\boldsymbol{\varphi})\,\mathrm{d}t.$$

Let $\varepsilon = \varepsilon_k \to 0$ and we get

$$\int_0^T (\mathbf{u}',\boldsymbol{\varphi}) + B[\mathbf{u},\boldsymbol{\varphi};t]\,\mathrm{d}t = \int_0^T (\boldsymbol{f},\boldsymbol{\varphi})\,\mathrm{d}t, \quad \forall \boldsymbol{\varphi} \in C^1([0,T];H^1),$$

and thus we conclude $(\mathbf{u}',\boldsymbol{\varphi}) + B[\mathbf{u},\boldsymbol{\varphi};t] = (\boldsymbol{f},\boldsymbol{\varphi})$ for a.e. $t \in [0,T]$ and all $\boldsymbol{\varphi} \in C^1([0,T];H^1)$.

Assume now $\boldsymbol{\varphi}(T) = 0$ and we integrate ∂_t by parts in $(\mathbf{u}^{\varepsilon\prime},\boldsymbol{\varphi})$ to get

$$\int_0^T -(\mathbf{u}^\varepsilon,\boldsymbol{\varphi}') + \varepsilon\nabla\mathbf{u}^\varepsilon : \nabla\boldsymbol{\varphi} + B[\mathbf{u}^\varepsilon,\boldsymbol{\varphi};t]\,\mathrm{d}t = \int_0^T (\boldsymbol{f},\boldsymbol{\varphi})\,\mathrm{d}t + (\mathbf{g},\mathbf{v}(0)).$$

Let $\varepsilon = \varepsilon_k \to 0$ and again we get

$$\int_0^T (\mathbf{u},\boldsymbol{\varphi}') + B[\mathbf{u},\boldsymbol{\varphi};t]\,\mathrm{d}t = \int_0^T (\boldsymbol{f},\boldsymbol{\varphi})\,\mathrm{d}t + (\mathbf{g},\boldsymbol{\varphi}(0)).$$

On the other hand, integrating ∂_t by parts in $(\mathbf{u}', \boldsymbol{\varphi})$ yields

$$\int_0^T (\mathbf{u}, \boldsymbol{\varphi}') + B[\mathbf{u}, \boldsymbol{\varphi}; t]\, \mathrm{d}t = \int_0^T (\boldsymbol{f}, \boldsymbol{\varphi})\, \mathrm{d}t + (\mathbf{u}(0), \boldsymbol{\varphi}(0)),$$

and thus $\mathbf{u}(0) = \mathbf{g}$ because $\boldsymbol{\varphi}(0)$ is arbitrary.

The uniqueness immediately follows from the linearity and setting $\boldsymbol{\varphi} = \mathbf{u}$, $\boldsymbol{f} = \mathbf{g} = 0$. □

Remark 3.5.2. At the end of this section, we again emphasize that the above method may not be applicable to the initial-boundary-value problems because the ε-regularized problem may have different boundary conditions, or even different numbers of boundary conditions, from the original hyperbolic system. This, in particular, happens in the study of inviscid limits from the Navier-Stokes equations (describing the motion of viscous fluids) to the Euler equations (describing the motion of inviscid fluids) in a domain with boundary. The former one may be imposed with the no-slip condition $\mathbf{u} = 0$ on the boundary, while the latter one can only be imposed with the slip condition $\mathbf{u} \cdot N = 0$ on the boundary. Such mismatch in the boundary condition actually results from the boundary layer which sticks to the boundary when the viscosity is neglected and its thickness converges to 0 as the viscosity converges to 0.

For the classical theory of first-order symmetric hyperbolic system in a domain with boundary, we refer to the following papers or books

- Peter D. Lax, Phillips, R. S. Local boundary conditions for dissipative symmetric linear differential operators. *Commun. Pure. Appl. Math.*, 13(3), 427–455, 1960.
- Rauch, J. Symmetric Positive Systems with Boundary Characteristic of Constant Multiplicity. *Trans. Amer. Math. Soc.*, 291(1), 167-187, 1985.
- Métivier, G. Small viscosity and boundary layer methods: Theory, stability analysis, and applications. Springer Science & Business Media, 2004.

3.6 Semigroup theory (TBA)

36.

Chapter 4 Linear Wave Equations

In undergraduate PDE course, we have learned how to find the classical solutions to the standard linear wave equation $\partial_t^2 u - \Delta u = f$ in $\mathbb{R}^d$ or in 1D intervals under certain boundary conditions. The explicit solutions also show that the propagation speed of the waves must be finite, which is also called the Huygens' principle. However, in many physical models, the wave propagations obey some nonlinear wave equations with *variable coefficients*, including the curved space-time in general relativity (Einstein equations), the sound waves in a perfect compressible fluid (compressible Euler equations), the motion of elastic media (elastodynamics), the magneto-sonic waves in a compressible plasma (compressible MHD equations) and many other widely-applied physical models.

In this chapter, we consider the wave-type equations with *given* variable coefficients, namely we study the following equation in $\mathbb{R}^{1+d}$ with $\varphi : I \times \mathbb{R}^d \to \mathbb{R}$

$$\begin{cases} \partial_\alpha \left(a^{\alpha\beta}\partial_\beta\varphi\right) = F & \text{in } I \times \mathbb{R}^d, \\ (\varphi, \partial_t\varphi) = (\varphi_0, \varphi_1) & \text{on } \{t = 0\} \times \mathbb{R}^d. \end{cases} \tag{4.0.1}$$

Here the indices α, β range from 0 to d and the 0-th component refers to the time variable t. $F : I \times \mathbb{R}^d \to \mathbb{R}$ is a given source term. $I \subseteq \mathbb{R}$ is an open interval for the range of the time variable.

4.1 Existence and regularity of linear wave equations

In (4.0.1), we require $\mathbf{a} = [a^{\alpha\beta}]$ to be a symmetric $(1+d)\times(1+d)$ matrix on $I \times \mathbb{R}^d$ which satisfies

$$\sum_{\alpha,\beta} \left|a^{\alpha\beta} - m^{\alpha\beta}\right| < \frac{1}{10}. \tag{4.1.1}$$

Here the matrix $m = \operatorname{diag}(-1, 1, 1, \cdots, 1)$. We will also need some regularity assumptions on $a^{\alpha\beta} : I \times \mathbb{R}^d \to \mathbb{R}, F : I \times \mathbb{R}^d \to \mathbb{R}, \varphi_0 : \mathbb{R}^d \to \mathbb{R}$ and $\varphi_1 : \mathbb{R}^d \to \mathbb{R}$ later on.

4.1.1 Regularity of linear wave equations

Before proving the local existence of the linear wave equation (4.0.1), let us prove the energy estimates, also considered as regularity results, for the linear variable-coefficient wave equation. It should be noted that the conservation of energy for this class of equations may not hold as in the case of standard wave equations, but we can still show that some appropriately defined "energy" has the property that its growth is controlled.

We first introduce the notation

$$|\partial\varphi|^2 := (\partial_t\varphi)^2 + \sum_{i=1}^{d} (\partial_{x^i}\varphi)^2 \tag{4.1.2}$$

Theorem 4.1.1 (L^2 energy estimates)**.** Let φ be a solution to (4.2), then for some constant $C = C(d,T) > 0$, the following energy estimates hold:

$$\begin{aligned}&\sup_{t\in[0,T]} \|\partial\varphi(t)\|^2_{L^2(\mathbb{R}^d)}\\ &\leq C\left(\|(\varphi_0,\varphi_1)\|^2_{\dot{H}^1(\mathbb{R}^d)\times L^2(\mathbb{R}^d)} + \int_0^T \|F(t)\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}t\right)\exp\left(C\int_0^T \|\partial\mathbf{a}(t)\|^2_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}t\right).\end{aligned} \tag{4.1.3}$$

Here $\|\cdot\|_{\dot{H}^1} = \|\partial(\cdot)\|_{L^2}$ denots the homogeneous Sobolev norm.

Proof. The proof is in fact the similar to that for the constant-coefficient linear wave equation. We test the equation with $\partial_t\varphi$ to get

$$\int_0^t\int_{\mathbb{R}^d} \partial_t\varphi\left(\partial_\alpha\left(a^{\alpha\beta}\partial_\beta\varphi\right) - F\right) = 0$$

and integrate ∂_α by parts. When $\alpha = \beta = 0$, we have

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^d} \partial_t\varphi\partial_t\left(a^{00}\partial_t\varphi\right)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau = \int_0^t\int_{\mathbb{R}^d} \left(\partial_t a^{00}\right)(\partial_t\varphi)^2 + \frac{1}{2}\partial_t(\partial_t\varphi)^2 a^{00}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau\\ &= \frac{1}{2}\int_{\mathbb{R}^d} a^{00}(\partial_t\varphi)^2\,\mathrm{d}\boldsymbol{x}\Big|_0^t + \frac{1}{2}\int_0^T\int_{\mathbb{R}^a} \left(\partial_t a^{00}\right)(\partial_t\varphi)^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau.\end{aligned}$$

When we only have $i, j = 1, 2, \cdots, d$, we have the following identity (here and thereafter we use the convention $i, j = 1, 2, \cdots, d$):

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^d} \partial_t\varphi\partial_i\left(a^{ij}\partial_j\varphi\right)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau = -\frac{1}{2}\int_0^t\int_{\mathbb{R}^d} \left(\partial_t\partial_i\varphi a^{ij}\partial_j\varphi + \partial_t\partial_j\varphi a^{ij}\partial_i\varphi\right)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau\\ &= -\frac{1}{2}\int_0^t\int_{\mathbb{R}^d} \partial_t\left(\partial_i\varphi\partial_j\varphi\right)a^{ij}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau\\ &= -\frac{1}{2}\int_{\mathbb{R}^d} a^{ij}\partial_i\varphi\partial_j\varphi\,\mathrm{d}\boldsymbol{x}\Big|_0^t + \frac{1}{2}\int_0^t\int_{\mathbb{R}^d} \left(\partial_t a^{ij}\right)\left(\partial_i\varphi\partial_j\varphi\right)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau.\end{aligned}$$

Note that in the derivation above, we have used the symmetry of $\mathbf{a}$.

For the terms with $\alpha = 0$ and $i = 1, \cdots, d$, we have

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} \partial_t\varphi \left(\partial_i \left(a^{i0}\partial_t\varphi\right) + \partial_t \left(a^{i0}\partial_i\varphi\right)\right) \mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau \\
=& \int_0^t \int_{\mathbb{R}^d} \left(\left(\partial_i a^{i0}\right)\left(\partial_t\varphi\right)^2 + a^{i0}\partial_i\left(\partial_t\varphi\right)^2 + \left(\partial_t a^{i0}\right)\left(\partial_t\varphi\right)\left(\partial_i\varphi\right)\right) \mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau \\
=& \int_0^t \int_{\mathbb{R}^d} \left(\left(\partial_i a^{i0}\right)\left(\partial_t\varphi\right)^2 - \left(\partial_i a^{i0}\right)\left(\partial_t\varphi\right)^2 + \left(\partial_t a^{i0}\right)\left(\partial_t\varphi\right)\left(\partial_i\varphi\right)\right) \mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau.
\end{aligned}$$

We now combine the above equalities to get

$$\begin{aligned}
&\left|\frac{1}{2}\int_{\mathbb{R}^d} a^{00}\left(\partial_t\varphi\right)^2 - a^{ij}\partial_i\varphi\partial_j\varphi\,\mathrm{d}\boldsymbol{x}\Big|_{\tau=t}\right| \\
\leq& \left|\frac{1}{2}\int_{\mathbb{R}^d} a^{00}\left(\partial_t\varphi\right)^2\,\mathrm{d}\boldsymbol{x} - \frac{1}{2}\int_{\mathbb{R}^d} a^{ij}\partial_i\varphi\partial_j\varphi\,\mathrm{d}\boldsymbol{x}\Big|_{\tau=0}\right| \\
&+ C\int_0^t \|\boldsymbol{\partial}\varphi\|_{L^2(\mathbb{R}^d)}\|F\|_{L^2(\mathbb{R}^d)} + \|\boldsymbol{\partial}\mathbf{a}\|_{L^\infty(\mathbb{R}^d)}\|\boldsymbol{\partial}\varphi\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}\tau
\end{aligned} \tag{4.1.4}$$

By the assumption (4.1.1), we know there exists a constant $C > 0$ such that the left hand side of (4.1.4) dominates $\|\boldsymbol{\partial}\varphi\|^2_{L^2(\mathbb{R}^d)}$

$$\|\boldsymbol{\partial}\varphi\|^2_{L^2(\mathbb{R}^d)} \leq C\left|\frac{1}{2}\int_{\mathbb{R}^d} a^{00}\left(\partial_t\varphi\right)^2\,\mathrm{d}\boldsymbol{x} - a^{ij}\partial_i\varphi\partial_j\varphi\,\mathrm{d}\boldsymbol{x}\right|. \tag{4.1.5}$$

Combining the above two inequalities, we get

$$\|\boldsymbol{\partial}\varphi(t)\|^2_{L^2(\mathbb{R}^d)} \leq C\|\boldsymbol{\partial}\varphi(0)\|^2_{L^2(\mathbb{R}^d)} + C\int_0^t \|\boldsymbol{\partial}\varphi\|_{L^2(\mathbb{R}^d)}\|F\|_{L^2(\mathbb{R}^d)} + \|\boldsymbol{\partial}\mathbf{a}\|_{L^\infty(\mathbb{R}^d)}\|\boldsymbol{\partial}\varphi\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}\tau. \tag{4.1.6}$$

Then, using Young's inequality, we have

$$\int_0^t \|\boldsymbol{\partial}\varphi\|_{L^2(\mathbb{R}^d)}\|F\|_{L^2(\mathbb{R}^d)}\,\mathrm{d}\tau \leq \delta \sup_{t\in[0,T]} \|\boldsymbol{\partial}\varphi\|^2_{L^2(\mathbb{R}^d)}(t) + C\delta^{-1}\left(\int_0^t \|F\|_{L^2(\mathbb{R}^d)}\,\mathrm{d}\tau\right)^2.$$

Choosing $\delta > 0$ sufficiently small such that the term $\delta \sup_{t\in[0,T]} \|\boldsymbol{\partial}\varphi\|_{L^2(\mathbb{R}^d)}(t)$ is absorbed to the left side, we get by using Jensen's inequality that

$$\sup_{t\in[0,T]} \|\boldsymbol{\partial}\varphi(t)\|^2_{L^2(\mathbb{R}^d)} \leq C\|\boldsymbol{\partial}\varphi(0)\|^2_{L^2(\mathbb{R}^d)} + C\int_0^T T\|F(t)\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}t + \|\boldsymbol{\partial}\mathbf{a}\|_{L^\infty(\mathbb{R}^d)}\|\boldsymbol{\partial}\varphi\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}t$$

The conclusion then follows from Grönwall's inequality and Jensen's inequality. □

For the constant coefficient linear wave equation $\partial_t^2 u - \Delta u = 0$, we know that is φ is a solution and then so are $\partial_t \varphi$ and $\partial_{x_i}\varphi$. Therefore, all high-order derivatives of φ are bounded in L^2 if the initial data are sufficiently smooth and vanishes as $|\boldsymbol{x}| \to \infty$, for example, the case $\varphi_0, \varphi_1 \in C_c^\infty(\mathbb{R}^d)$. In general, φ being a solution to (4.0.1) does not necessarily imply that its derivatives are also solutions to (4.0.1). Following the proof of Theorem 4.1.1, we can also control high-order Sobolev norms of φ. We have the following corollary and the proof is left as an exercise.

Corollary 4.1.2. Let φ be a solution to (4.0.1) and $k \in \mathbb{N}^*$. Then for some constant $C = C(d, k, T) > 0$, the following energy estimates hold:

$$\begin{aligned} &\sup_{t\in[0,T]} \|(\varphi(t), \partial_t\varphi(t))\|^2_{H^k\times H^{k-1}} \\ &\leq C\left[\|(\varphi_0,\varphi_1)\|^2_{H^k\times H^{k-1}} + \int_0^T \|F\|^2_{H^{k-1}} + \sum_{|\alpha|+|\beta|\leq k-1} \left\|\partial\partial_{\boldsymbol{x}}^\alpha \mathbf{a}\,\partial\partial_{\boldsymbol{x}}^\beta\varphi\right\|^2_{L^2} + \left\|\partial_{\boldsymbol{x}}^\alpha \mathbf{a}\,\partial\partial_{\boldsymbol{x}}^\beta\varphi\right\|^2_{L^2} \mathrm{d}t\right] \\ &\times \exp\left(C\int_0^T \|\partial\mathbf{a}\|^2_{L^\infty}\,\mathrm{d}t\right). \end{aligned} \tag{4.1.7}$$

4.1.2 Existence of linear wave equations

In this section, we prove the local existence of solutions to (4.0.1) by using the energy estimates together with Hahn-Banach theorem in linear functional analysis.

Below, we assume that $\mathbf{a}$ and all of its derivatives (of all orders) are bounded in $[0,T]\times\mathbb{R}^d$. The source term $F : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ is be assumed to satisfy $F \in L^2(0,T;H^{k-1}(\mathbb{R}^d))$ for a given number $k \in \mathbb{N}$.

Let us recall the Hahn-Banach theorem that will be used to prove the local existence.

Theorem 4.1.3 (Hahn-Banach Theorem). Let X be a normed vector space and $Y \subseteq X$ be a subspace with the norm $\|y\|_Y = \|y\|_X$ for every $y \in Y$. Suppose $f \in Y^*$ is a bounded linear functional on Y, then there exists $\tilde{f} \in X^*$ such that $\tilde{f}|_Y = f$ and $\|\tilde{f}\|_X = \|f\|_{Y^*}$.

We also need the following lemma

Lemma 4.1.4. Let $L^*\psi := \partial_\alpha(a^{\alpha\beta}\partial_\beta\psi)$ be defined as the (formal) adjoint of L defined by $L\varphi := \partial_\alpha(a^{\alpha\beta}\partial_\beta\varphi)$. Suppose $\psi \in C_c^\infty((-\infty,T)\times\mathbb{R}^d)$, then for every $m \in \mathbb{Z}$, there exists $C = C(m,T,\mathbf{a}) > 0$ such that

$$\|\psi(t)\|_{H^m(\mathbb{R}^d)} \leq C\int_t^T \|L^*\psi\|_{H^{m-1}(\mathbb{R}^d)}(\tau)\,\mathrm{d}\tau$$

for every $t \in [0,T]$.

Proof. For $m \geq 1$, this is a consequence of Corollary 4.1.2. We now carry out an induction for the cases $m \leq 0$. Assume that the result holds for $m_0 + 2$ for some negative integer m_0, we wish to prove the same result for m_0.

Define $\Psi = (1-\Delta)^{-1}\psi$ via the Fourier transform, or equivalently $(1-\Delta)\Psi = \psi$. Then there exists a $C > 0$ depending on m_0, T, a and b such that

$$\left|L^*\psi - (1-\Delta)L^*\Psi\right| = \left|L^*(1-\Delta)\Psi - (1-\Delta)L^*\Psi\right| \leq C \sum_{1\leq|\alpha|\leq 3} |\partial_{\boldsymbol{x}}^\alpha \Psi|$$

and therefore

$$\|L^*\Psi\|_{H^{m_0+1}} \leq C\left(\|L^*\psi\|_{H^{m_0-1}} + \|\Psi\|_{H^{m_0+2}}\right)$$

By the induction hypothesis, we get

$$\|\Psi(t)\|_{H^{m_0+2}(\mathbb{R}^d)} \leq C\int_t^T \left(\|L^*\psi(\tau)\|_{H^{m_0-1}} + \|\Psi(\tau)\|_{H^{m_0+2}}\right) \mathrm{d}\tau \leq C\int_t^T \|L^*\psi\|_{H^{m_0-1}}\, \mathrm{d}\tau$$

where in the last line we have used Grönwall's inequality. This then leads to

$$\|\psi(t)\|_{H^{m_0}} \leq C\|\Psi\|_{H^{m_0+2}} \leq C\int_t^T \|L^*\psi(\tau)\|_{H^{m_0-1}}\, \mathrm{d}\tau$$

□

We are now ready for the local existence

Theorem 4.1.5 (Local existence of the linear wave equation)**.** Let $k \in \mathbb{N}$. Given $F \in L^2([0,T]; H^{k-1}(\mathbb{R}^d))$, there exists a unique solution

$$(\varphi, \partial_t\varphi) \in L^\infty([0,T]; H^k(\mathbb{R}^d)) \times L^\infty([0,T]; H^{k-1}(\mathbb{R}^d))$$

solving (4.0.1).

Proof. We start with the case $(\varphi_0, \varphi_1) = (0,0)$, which also gives the uniqueness of (4.0.1). For every element $L^*\psi \in L^*(C_c^\infty((-\infty,T)\times\mathbb{R}^d))$, define a map to $\mathbb{R}$ by

$$L^*\psi \longmapsto \int_0^T \int_{\mathbb{R}^d} \psi F\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t =: \langle F, \psi\rangle,$$

where $L^*(C_c^\infty((-\infty,T)\times\mathbb{R}^d))$ denote the image of $C_c^\infty((-\infty,T)\times\mathbb{R}^d)$ under the map L^*. Note that this map is well-defined because the uniqueness of $L^*\psi = f$ with $\psi(T,\cdot) = 0$ has been proved in the L^2 energy estimates (Theorem 4.1.1). From the assumption on F and Lemma 4.1.4, we have the bound

$$\left|\int_0^T \int_{\mathbb{R}^d} \psi F\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t\right| \leq C\left(\int_0^T \|F\|_{H^{k-1}}\, \mathrm{d}t\right)\left(\sup_{t\in[0,T]} \|\psi\|_{H^{-k+1}}\right) \leq C\int_0^T \|L^*\psi(t)\|_{H^{-k}}\, \mathrm{d}t.$$

Using the Hahn-Banach theorem, we know there exists a function

$$\varphi \in (L^1((-\infty,T); H^{-k}(\mathbb{R}^d)))^* = L^\infty((-\infty,T); H^k(\mathbb{R}^d))$$

with $\varphi = 0$ for $t < 0$ as the extension of the map defined above, i.e.,

$$\langle F, \psi \rangle = \langle \varphi, L^* \psi \rangle$$

for every $\psi \in C_c^\infty((-\infty, T) \times \mathbb{R}^d)$. Therefore, φ is a solution in the sense of distribution. Finally, we use the equation to show that $\varphi \in C^1\left([0,T]; L^2(\mathbb{R}^d)\right)$ and therefore $(\varphi, \partial_t \varphi)|_{\{t=0\}} = (0,0)$.

For the general case $(\varphi_0, \varphi_1) \in H^k(\mathbb{R}^d) \times H^{k-1}(\mathbb{R}^d)$, we introduce $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ such that $(u, \partial_t u)|_{\{t=0\}} = (\varphi_0, \varphi_1)$ solves the equation $L\eta = F - Lu$ with initial data $(\eta, \partial_t \eta)|_{\{t=0\}} = (0,0)$. Then $\varphi := \eta + u$ gives us the desired solution. Using energy estimates in Theorem 4.1.2, we know the solution φ has the corresponding regularity.

We also need to verify $\partial_t \varphi \in L^\infty(0,T; H^{k-1})$. From the equation, we have

$$a^{0\alpha} \partial_t \partial_\alpha \varphi = F - a^{ij} \partial_i \partial_j \varphi - \partial_\alpha a^{\alpha\beta} \partial_\beta \varphi \in H^{k-2},$$

which shows $\partial_t \varphi \in L^\infty(0,T; H^{k-2})$ and $\partial_t^2 \varphi \in L^\infty(0,T; H^{k-3})$. If $F \in C_c^\infty$, then this together with $\varphi_0 = \varphi_1 = 0$ allows us to enhance $\varphi \in L_t^\infty H_{\boldsymbol{x}}^k$ to $\varphi \in L_t^\infty H_{\boldsymbol{x}}^{k+1}$ such that $\partial_t \varphi \in L^\infty(0,T; H^{k-1})$.

For a general $F \in L^1(0,T; H^{k-1})$, we can find $\{F_n\} \subset C_c^\infty((-\infty, T) \times \mathbb{R}^d)$ vanishing in $\{t < 0\}$ such that $\int_0^T \|F(t,\cdot) - F_n(t,\cdot)\|_{H^{k-1}}\, \mathrm{d}t \to 0$. Then if $\varphi_n \in L^\infty(0,T; H^k) \cap W^{1,\infty}(0,T; H^{k-1})$ solve $L\varphi_n = F_n$m with zero initial data, it follows from the regularity theorem that

$$\|\partial(\varphi_l - \varphi_n)\|_{H^{k-1}} \leq C \int_0^T \|F_l(t,\cdot) - F_n(t,\cdot)\|_{H^{k-1}}\, \mathrm{d}t \to 0.$$

□

Remark 4.1.1. The above theorem only give the existence of solutions in the sense of distribution. Even so, using the Sobolev embedding $H^s(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ with $s > d/2$, if we assume that the initial data is sufficiently regular, then the solution also has the corresponding regularity as in Theorem 4.1.2 and so has pointwise definition when k is suitably large.

Exercise 4.1

Exercise 4.1.1. Prove Theorem 4.1.2.

Exercise 4.1.2. Prove that for any constant $D \in \mathbb{R}$, the following equation has at most one smooth solution $u \in C^\infty([0,T] \times U)$. Here $U \subset \mathbb{R}^d$ is a domain with smooth boundary and $\varphi, \psi \in C_c^\infty(U)$.

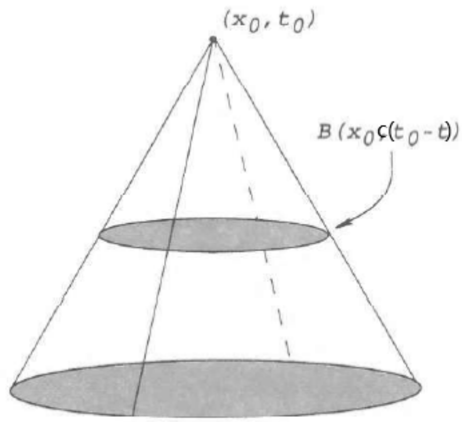
$$\begin{cases} \partial_t^2 u + D \partial_t u - \Delta u = 0 & \text{in } (0,T] \times U, \\ u = \varphi,\ \partial_t u = \psi & \text{on } \{t=0\} \times U, \\ u = 0 & \text{on } [0,T] \times \partial U. \end{cases}$$

4.2 Finite propagation speed

In undergraduate PDE course, we already learned that the standard wave equation $\partial_t^2 u - c^2 \Delta u = 0$ has finite propagation speed.

Theorem 4.2.1 (Finite propagation speed). For the wave equation $\partial_t^2 \varphi - c^2 \Delta \varphi = 0$ in $\mathbb{R}_+ \times \mathbb{R}^d$ with initial data $(\varphi_0, \varphi_1) \in C_c^\infty(\mathbb{R}^d)$, if the initial data $\varphi_0 = \varphi_1 = 0$ in $\{\boldsymbol{x} \in \mathbb{R}^d : |\boldsymbol{x} - \boldsymbol{x}_0| \le ct_0\}$ for some $t_0 > 0$ and $\boldsymbol{x}_0 \in \mathbb{R}^d$, then the solution $\varphi(t, \boldsymbol{x})$ must be zero in the past light cone

$$K(t_0, \boldsymbol{x}_0) := \{(t, \boldsymbol{x}) \in \mathbb{R}_+ \times \mathbb{R}^d : 0 \le t \le t_0,\ |\boldsymbol{x} - \boldsymbol{x}_0| \le c(t_0 - t)\}.$$



Cone of dependence

The finite propagation speed of wave equations addresses an "opposite" phenomenon to the parabolic maximum principle (infinitely propagation speed). This property actually denies the possibility of maximum principle for hyperbolic PDEs. We then ask if there is any similar result about the finite propagation speed for a variable-coefficient wave equation. The answer is yes, and for technical symplicity we assume $a^{00} = -1$ and $a^{i0} = a^{0i} = 0$ in the metric $\mathbf{a}$, that is, we write the wave equation as

$$\partial_t^2 \varphi - a^{ij} \partial_i \partial_j \varphi = 0 \tag{4.2.1}$$

where $[a^{ij}]$ is symmetric and uniformly elliptic.

Fix a point $(t_0, \boldsymbol{x}_0) \in \mathbb{R}_+ \times \mathbb{R}^d$, we want to construct some sort of a "curved cone-like" region C with vertex $(t_0, \boldsymbol{x}_0)$ such that the solution $\varphi = 0$ within C if its initial data vanishes on $C \cap \{t = 0\}$. Recall that the key idea to prove Theorem 4.2.1 is to verify the energy on the slice $\{(t, \boldsymbol{x}) : |\boldsymbol{x} - \boldsymbol{x}_0| \le c(t_0 - t)\}$ is decreasing in time t. Therefore, we may alternatively find suitable function $q(\boldsymbol{x})$ such that analogous argument holds for the variable-coefficient wave equation. Precisely speaking, we want to find $q(\boldsymbol{x})$ such that some energy function $e(t)$ on the slice $K_t := \{\boldsymbol{x} : q(\boldsymbol{x}) < t_0 - t\}$ is decreasing in t. In particular, this $q(\boldsymbol{x})$ is $(\boldsymbol{x} - \boldsymbol{x}_0)/c$ in the case of the standard wave equation.

4.2.1 Motivation: geometric optics

In fact, the boundary of the past cone can be understood as the level set of certain function $p(t,\boldsymbol{x})$ which actually solves a Hamilton-Jacobi PDE, that is, the boundary of the cone consists of the characteristic curves of certain Hamilton-Jacobi PDE. Let us first see this fact from the standard wave equation $\partial_t^2 u - \Delta u = 0$ in $\mathbb{R}_+ \times \mathbb{R}^d$ and consider complex-valued solutions having the form $u^\varepsilon(t,\boldsymbol{x}) := U^\varepsilon(t,\boldsymbol{x}) \cdot \exp(ip^\varepsilon(t,\boldsymbol{x})\varepsilon^{-1})$ for $t \geq 0$, $\boldsymbol{x} \in \mathbb{R}^d$ and $\varepsilon > 0$.

Inserting this ansatz to the wave equation, we obtain that

$$
\begin{aligned}
0 = (\partial_t^2 - \Delta)u^\varepsilon =& e^{\frac{ip^\varepsilon}{\varepsilon}} \left(\partial_t^2 U^\varepsilon + 2i\varepsilon^{-1}\partial_t p^\varepsilon\, \partial_t U^\varepsilon - \varepsilon^{-2}(\partial_t p^\varepsilon)^2 U^\varepsilon + i\varepsilon^{-1}\partial_t^2 p^\varepsilon U^\varepsilon\right) \\
&- e^{\frac{ip^\varepsilon}{\varepsilon}} \left(\Delta U^\varepsilon + 2i\varepsilon^{-1}\nabla p^\varepsilon \cdot \nabla U^\varepsilon - \varepsilon^{-2} U^\varepsilon |\nabla p^\varepsilon|^2 + i\varepsilon^{-1} U^\varepsilon\, \Delta p^\varepsilon\right).
\end{aligned}
$$

Taking the real part of the above equality, we find

$$
U^\varepsilon((\partial_t p^\varepsilon)^2 - |\nabla p^\varepsilon|^2) = \varepsilon^2(\partial_t^2 U^\varepsilon - \Delta U^\varepsilon). \tag{4.2.2}
$$

Assume we have the convergence[1] $p^\varepsilon \to p$ and $U^\varepsilon \to U \neq 0$ as $\varepsilon \to 0$ in some sense, then formally we get

$$
\partial_t p \pm |\nabla p| = 0 \qquad \text{in } \mathbb{R}_+ \times \mathbb{R}^d. \tag{4.2.3}
$$

By separation of variables, we can write $p(t,\boldsymbol{x}) = q(\boldsymbol{x}) + t - t_0$ and q solves $|\nabla q|^2 = 1$ with $q > 0$ in $\mathbb{R}^d \backslash \{\boldsymbol{x}_0\}$ and $q(\boldsymbol{x}_0) = 0$ and then we can solve that $q(\boldsymbol{x}) = |\boldsymbol{x} - \boldsymbol{x}_0|$.

For the variable coefficient case, we can also solve that

$$
\partial_t p \pm \left(a^{ij}\partial_i p \partial_j p\right)^{\frac{1}{2}} = 0 \qquad \text{in } \mathbb{R}_+ \times \mathbb{R}^d. \tag{4.2.4}
$$

Using separation of variable again, we have

$$
p(t,\boldsymbol{x}) = q(\boldsymbol{x}) + t - t_0 \tag{4.2.5}
$$

$$
a^{ij}\partial_i q \partial_j q = 1, \quad q > 0 \text{ in } \mathbb{R}^d \backslash \{\boldsymbol{x}_0\}, \quad q(\boldsymbol{x}_0) = 0. \tag{4.2.6}
$$

In fact, this q is the distance of $\boldsymbol{x}$ to $\boldsymbol{x}_0$ in the Riemann metric determined by $\mathbf{a}$.

4.2.2 Proof of finite propagation speed

We define $K := \{(t,\boldsymbol{x}) : p(t,\boldsymbol{x}) < 0\} = \{(t,\boldsymbol{x}) : q(\boldsymbol{x}) < t_0 - t\}$ and for each $t > 0$ we define

$$
K_t := \{\boldsymbol{x} : q(\boldsymbol{x}) < t_0 - t\}.
$$

[1] As $\varepsilon \to 0$, there is a singular term ε^{-1} that exhibits a highly oscillating feature. The rigorous proof of this convergence is a special case for the oscillatory integrals with stationary phase. We refer to Evans [6, Chapter 4.5.2(b)] for details.

Since $\nabla q \neq \mathbf{0}$ away from $\boldsymbol{x}_0$, we know ∂K_t is a smooth, $(d-1)$-dimensional hypersurface for $0 \le t \le t_0$.

Theorem 4.2.2 (Finite propagation speed). Assume φ is a smooth solution to (4.2.1). If $\varphi \equiv \partial_t \varphi \equiv 0$ on K_0, then $\varphi \equiv 0$ within K.

Remark 4.2.1. We see in particular that if φ is a solution to (4.2.1) with the initial data (φ_0, φ_1), then $\varphi(t_0, \boldsymbol{x}_0)$ depends only upon the values of φ_0 and φ_1 within K_0.

Proof. We define the energy

$$e(t) := \frac{1}{2}\int_{K_t} (\partial_t\varphi)^2 + \sum_{i,j=1}^{n} a^{ij}\partial_i\varphi\partial_j\varphi \,\mathrm{d}\boldsymbol{x} \quad (0 \le t \le t_0)$$

In order to compute $\dot{e}(t)$, we invoke the co-area formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{K_t} f \,\mathrm{d}\boldsymbol{x}\right) = \int_{K_t} \partial_t f \,\mathrm{d}\boldsymbol{x} - \int_{\partial K_t} \frac{f}{|\nabla q|}\,\mathrm{d}S_{\boldsymbol{x}}.$$

Thus

$$e'(t) = \int_{K_t} \partial_t\varphi\,\partial_t^2\varphi + a^{ij}\partial_i\varphi\partial_t\partial_j\varphi \,\mathrm{d}\boldsymbol{x} - \frac{1}{2}\int_{\partial K_t} \left((\partial_t\varphi)^2 + a^{ij}\partial_i\varphi\partial_j\varphi\right)\frac{1}{|\nabla q|}\,\mathrm{d}S_{\boldsymbol{x}}. \tag{4.2.7}$$

For the first term, we integrate by parts and invoke the wave equation to get

$$\begin{aligned}
&\int_{K_t} \partial_t\varphi\,\partial_t^2\varphi + a^{ij}\partial_i\varphi\partial_t\partial_j\varphi \,\mathrm{d}\boldsymbol{x}\\
=&\int_{K_t} \partial_t\varphi\left(\partial_t^2\varphi - \partial_j(a^{ij}\partial_i\varphi)\right)\,\mathrm{d}\boldsymbol{x} + \int_{\partial K_t} a^{ij}\partial_i\varphi N_j\,\partial_t\varphi\,\mathrm{d}S_{\boldsymbol{x}}\\
=&-\int_{K_t} \partial_t\varphi(\partial_i\varphi\,\partial_j a^{ij})\,\mathrm{d}\boldsymbol{x} + \int_{\partial K_t} a^{ij}\partial_i\varphi N_j\,\partial_t\varphi\,\mathrm{d}S_{\boldsymbol{x}}
\end{aligned}$$

where $N = (N_1, \cdots, N_d)$ is the outer unit normal to ∂K_t. We notice that the first term is controlled by $Ce(t)$ by direct calculation

$$\left|-\int_{K_t} \partial_t\varphi(\partial_i\varphi\,\partial_j a^{ij})\,\mathrm{d}\boldsymbol{x}\right| \le Ce(t).$$

For the second term, we want to produce a cancellation with the boundary term appearing in $e'(t)$. We have

$$\left|a^{ij}\partial_i\varphi N_j\right| \le \left(a^{ij}\partial_i\varphi\partial_j\varphi\right)^{\frac{1}{2}}\left(a^{ij}N_iN_j\right)^{\frac{1}{2}}$$

which is a consequence of the generalized Cauchy-Schwartz inequality: For a positive-definite symmetric matrix $\mathcal{M} = \mathcal{P}\mathcal{P}^\top$ and vectors $\boldsymbol{x}, \boldsymbol{y}$, there holds

$$|\boldsymbol{x}^\top\mathcal{M}\boldsymbol{y}| = |(\mathcal{P}\boldsymbol{x})^\top(\mathcal{P}\boldsymbol{y})| \le |\mathcal{P}\boldsymbol{x}||\mathcal{P}\boldsymbol{y}| = \sqrt{(\mathcal{P}\boldsymbol{x}^\top)\mathcal{P}\boldsymbol{x}}\sqrt{(\mathcal{P}\boldsymbol{y})^\top\mathcal{P}\boldsymbol{y}} = \sqrt{\boldsymbol{x}^\top\mathcal{M}\boldsymbol{x}}\sqrt{\boldsymbol{y}^\top\mathcal{M}\boldsymbol{y}}.$$

Then we recall that $q = t_0 - t$ on K_t implies $N = \frac{\nabla q}{|\nabla q|}$ on ∂K_t and thus

$$a^{ij} N_i N_j = \frac{a^{ij}\partial_i q \partial_j q}{|\nabla q|^2} = \frac{1}{|\nabla q|^2}.$$

Then using Young's inequality, we have

$$\begin{aligned}
&\left|\int_{\partial K_t} a^{ij}\partial_i\varphi N_j\, \partial_t\varphi \,\mathrm{d}S_{\boldsymbol{x}}\right| \leq \int_{\partial K_t} \left(a^{ij}\partial_i\varphi\partial_j\varphi\right)^{\frac{1}{2}} \left(a^{ij}N_iN_j\right)^{\frac{1}{2}} |\partial_t\varphi| \,\mathrm{d}S_{\boldsymbol{x}}\\
\leq & \int_{\partial K_t} \left(a^{ij}\partial_i\varphi\partial_j\varphi\right)^{\frac{1}{2}} |\nabla q|^{-1} |\partial_t u| \,\mathrm{d}S_{\boldsymbol{x}}\\
\leq & \frac{1}{2}\int_{\partial K_t} \left(a^{ij}\partial_i\varphi\partial_j\varphi + (\partial_t\varphi)^2\right) |\nabla q|^{-1} \,\mathrm{d}S_{\boldsymbol{x}} = -e'(t) + \int_{K_t} \partial_t\varphi\, \partial_t^2\varphi + a^{ij}\partial_i\varphi\partial_t\partial_j\varphi \,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Summing up the above analysis, we have

$$e'(t) \leq Ce(t).$$

Since $e(0) = 0$, we get $e(t) = 0$ by Grönwall's inequality. □

Exercise 4.2

Exercise 4.2.1. Verify (4.2.4) for the wave equation $\partial_t^2\varphi - a^{ij}\partial_i\partial_j\varphi = 0$.

Exercise 4.2.2. Consider the initial-value problem of the semilinear wave equation

$$\begin{cases} \partial_t^2\varphi - \Delta\varphi + f(\varphi) = 0 & t > 0,\ \boldsymbol{x} \in \mathbb{R}^d; \\ \varphi(0, \boldsymbol{x}) = \varphi_0(\boldsymbol{x}),\ \partial_t\varphi(0, x) = \psi_1(\boldsymbol{x}) & \boldsymbol{x} \in \mathbb{R}^d. \end{cases} \tag{4.2.8}$$

Here f is a continuous function in its arguments. φ is assumed to be vanishing as $|\boldsymbol{x}| \to \infty$.

(1) Show that the following quantity is conserved in time

$$E(t) := \int_{\mathbb{R}^d} \frac{1}{2}(|\partial_t\varphi|^2 + |\nabla\varphi|^2) + F(\varphi)\,\mathrm{d}\boldsymbol{x}.$$

Here $F(u) := \int_0^u f(s)\,\mathrm{d}s$.

(2) Given $t_0 > 0$ and $\boldsymbol{x}_0 \in \mathbb{R}^d$ and define the backwards wave cone with apex $(\boldsymbol{x}_0, t_0)$ by

$$K(\boldsymbol{x}_0, t_0) := \{(t, \boldsymbol{x}) \in [0, \infty) \times \mathbb{R}^d \,:\, 0 \leq t \leq t_0,\ |\boldsymbol{x} - \boldsymbol{x}_0| \leq t_0 - t\},$$

and the curved part of the boundary of $K(\boldsymbol{x}_0, t_0)$ is

$$\Gamma(\boldsymbol{x}_0, t_0) := \{(t, \boldsymbol{x}) \in [0, \infty) \times \mathbb{R}^d : 0 \le t \le t_0, \ |\boldsymbol{x} - \boldsymbol{x}_0| = t_0 - t\}.$$

Define the energy flux on the cone by

$$e(t) := \int_{B(\boldsymbol{x}_0, t_0 - t)} \frac{1}{2}(|\partial_t \varphi|^2 + |\nabla \varphi|^2) + F(\varphi)\,\mathrm{d}\boldsymbol{x} \ \ 0 \le t \le t_0.$$

Prove that

$$\frac{1}{\sqrt{2}} \int_{\Gamma(\boldsymbol{x}_0, t_0)} \frac{1}{2}|(\partial_t \varphi)\nu - \nabla \varphi|^2 + F(\varphi)\,\mathrm{d}S = e(0) \tag{4.2.9}$$

where $\nu := \frac{\boldsymbol{x} - \boldsymbol{x}_0}{|\boldsymbol{x} - \boldsymbol{x}_0|}$.

(3) Assume $F \ge 0$, Use (2) to show that $\varphi \equiv 0$ within the cone $K(\boldsymbol{x}_0, t_0)$ if $\varphi_0 = \varphi_1 \equiv 0$ within $B(\boldsymbol{x}_0, t_0)$.

(4) Prove that (3) also holds for the smooth solution φ to the quasilinear wave equation $\partial_t^2 \varphi - \Delta \varphi + f(\varphi, \partial \varphi) = 0$ where f is continuous in its arguments and $f(0, \mathbf{0}) = 0$.

(Hint: (2) Compute $e'(t)$ and compare it with the quantity on the left side of the desired identity. Think about how the factor $1/\sqrt{2}$ appears. (4) Consider $E(t) = \int_{B(\boldsymbol{x}_0, t_0 - t)} \frac{1}{2}(|\partial_t \varphi|^2 + |\nabla \varphi|^2) + \varphi^2\,\mathrm{d}\boldsymbol{x}$ and use $|f(\varphi, \partial \varphi)| \le C(|\varphi| + |\partial \varphi|)$ for some constant $C > 0$ depending on $\|\varphi, \partial \varphi\|_{L^\infty}$.)

4.3 *Local existence of quasi-linear wave equations (TBA)

In many physical models, wave equations are usually nonlinear, and particularly quasilinear, which means the source term is nonlinear and depends on both φ and $\partial \varphi$. In this section, we consider the following quasilinear wave equation:

$$\begin{cases} \partial_\alpha \left(a^{\alpha\beta} \partial_\beta \varphi\right) = F(\varphi, \partial \varphi) & \text{in } I \times \mathbb{R}^d, \\ (\varphi, \partial_t \varphi) = (\varphi_0, \varphi_1) \in H^k(\mathbb{R}^d) \times H^{k-1}(\mathbb{R}^d) & \text{on } \{t = 0\} \times \mathbb{R}^d, \end{cases} \tag{4.3.1}$$

where $k \in \mathbb{N}^*$ is given. We also require that

$$\sum_{\alpha,\beta} |a^{\alpha\beta} - m^{\alpha\beta}| < \frac{1}{10}, \ \ \mathbf{a}(0) = 0, \ \ F(0,0) = 0, \tag{4.3.2}$$

and $\mathbf{a}$, F are both smooth in their arguments. As a result, for a given number $A > 0$ and $N \in \mathbb{N}^*$, there exists a constant $C_{A,N}$ such that

$$\sum_{|\gamma|\le N}\left(\sum_{\alpha,\beta}\sup_{|\boldsymbol{x}|\le A}|\partial_{\boldsymbol{x}}^{\gamma}(a^{\alpha\beta})(\boldsymbol{x})| + \sup_{|\boldsymbol{x}|,|\boldsymbol{p}|\le A}|\nabla_{\boldsymbol{x},\boldsymbol{p}}^{\gamma}F(\boldsymbol{x},\boldsymbol{p})|\right) < C_{A,N}. \tag{4.3.3}$$

Under these assumption, we aim to prove the local existence of (4.3.1).

Theorem 4.3.1 (Local well-posdness of the quasi-linear wave equation).
Given $\mathbf{a}$, F satisfying the assumptions (4.3.2)-(4.3.3) and $d + 1 \le s \in \mathbb{R}$, then we have

- (Existence and uniqueness of local-in-time solutions) There exists some $T > 0$ depending on $\|\varphi_0\|_{H^{s+2}(\mathbb{R}^d)}$ and $\|\varphi_1\|_{H^{s+1}(\mathbb{R}^d)} > 0$, such that (4.3.1) admits a unique (classical) solution φ sastisfying
$$(\varphi, \partial_t\varphi) \in L^\infty([0,T]; H^{s+2}(\mathbb{R}^d)) \times L^\infty([0,T]; H^{s+1}(\mathbb{R}^d)).$$
- (Continuous dependence on initial data) Let $\varphi_0^{(n)}, \varphi_1^{(n)}$ satisfy $\varphi_0^{(n)} \to \varphi_0$ in $H^{s+2}(\mathbb{R}^d)$ and $\varphi_1^{(n)} \to \varphi_1$ in $H^{s+1}(\mathbb{R}^d)$ as $n \to \infty$. Then for $T > 0$ sufficiently small, we have
$$\|(\varphi^{(n)} - \varphi, \partial_t(\varphi^{(n)} - \varphi))\|_{L^\infty([0,T];H^s(\mathbb{R}^d))\times L^\infty([0,T];H^{s-1}(\mathbb{R}^d))} \to 0$$
as $n \to \infty$ for any $1 \le s < d + 2$. Here, φ is the solution with initial data (φ_0, φ_1) and $\varphi^{(n)}$ is the solution with initial data $(\varphi_0^{(n)}, \varphi_1^{(n)})$.

Remark 4.3.1. An evolution equation is said to be well-posed in the Hadamard sense, if existence, uniqueness of solutions and continuous dependence on initial data hold. Theorem 4.3.1 therefore implies that the equation (4.3.1) is locally well-posed.

Chapter 5 Hyperbolic Conservation Laws

5.1 Systems of quasilinear hyperbolic conservation laws

5.2 Discontinuities

5.3 Nonlinear Waves: Classical solutions

5.4 Nonlinear Waves: Discontinuities

Chapter 6 Sobolev Spaces: Fourier Theory

In Chapter 1, we introduce the Sobolev spaces with integer order. However, many estimates are not sharp in the Sobolev indices if we only use integer-order Sobolev spaces. What's more, in the study of dispersive and wave equations in $\mathbb{R}^d$, Fourier analysis becomes a powerful tool and also allows us to obtain more refined estimates by analyzing different frequencies of the Fourier transform of functions. Therefore, one may ask if there is any generalization to $W^{s,p}(\Omega)$ for $s \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^d$. The answer is yes, but one may have to use different ways to define such generalizations for different domains Ω. When $\Omega = \mathbb{R}^d$, the most efficient tool is the Fourier transform, as taking Fourier transform converts derivatives to polynomial-type multipliers. When the domain has a boundary, one of the generalizations is called "Sobolev-Slobodeckiĭ spaces" defined by using the differential quotients.

In this lecture notes, we only discuss the case $\Omega = \mathbb{R}^d$ and $p = 2$, that is, the most widely-used Sobolev spaces $H^s(\mathbb{R}^d)$ and the homogeneous counterpart $\dot{H}^s(\mathbb{R}^d)$.

6.1 Fractional Sobolev spaces

Given $\xi \in \mathbb{R}^d$, we define $\langle\xi\rangle := \sqrt{1+|\xi|^2}$, which corresponds to the Fourier symbol of $\sqrt{1-\Delta}$. Now, given $s \in \mathbb{R}$, we can define $H^s(\mathbb{R}^d)$ by using Fourier transform.

Definition 6.1.1 (Non-homogeneous Sobolev spaces). Given $s \in \mathbb{R}$, we define

$$H^s(\mathbb{R}^d) := \left\{u \in \mathcal{S}'(\mathbb{R}^d) : \langle\xi\rangle^s \hat{u}(\xi) \in L^2(\mathbb{R}^d)\right\} \tag{6.1.1}$$

to be the s-th order Sobolev space. $H^s(\mathbb{R}^d)$ is a Hilbert space with norm $\|u\|_{H^s(\mathbb{R}^d)} := \|\langle\xi\rangle^s \hat{u}\|_{L^2}$ induced by the inner product $\langle u, v\rangle_{H^s} := \int_{\mathbb{R}^d} \langle\xi\rangle^{2s} \hat{u}(\xi)\overline{\hat{v}(\xi)}\,\mathrm{d}\xi$.

Remark 6.1.1. It should be noted that when s is negative, $\langle\xi\rangle^s \hat{u} \in L^2(\mathbb{R}^d)$ does not necessarily imply u is a function. On the other hand, since Fourier transform maps a tempered distribution to a tempered distribution, so we shall really define fractional Sobolev spaces for tempered distributions instead of L^2 functions.

The above definition coincides with the integer-order Sobolev spaces defined in Chapter 1 when $s \in \mathbb{N}$.

Proposition 6.1.1. The fractional Sobolev space $H^s(\mathbb{R}^d)$ satisfies the following properties

(1) $C_c^\infty(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$ are both dense in $H^s(\mathbb{R}^d)$.
(2) When $s \in \mathbb{N}$, $H^s(\mathbb{R}^d)$ coincides with $W^{s,2}(\mathbb{R}^d)$ defined in Chapter 1.

We also introduce the homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}^d)$, which contains the tempered distributions whose s-th order derivatives (not including any lower-order deriatives) are $L^2(\mathbb{R}^d)$ functions.

Definition 6.1.2 (Homogeneous Sobolev spaces). Given $s \in \mathbb{R}$, we define

$$\dot{H}^s(\mathbb{R}^d) := \left\{u \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^d) : |\boldsymbol{\xi}|^s \hat{u}(\boldsymbol{\xi}) \in L^2(\mathbb{R}^d)\right\} \tag{6.1.2}$$

to be the s-th order homogeneous Sobolev space. Here $\mathcal{P}$ is the collection of polynomials.

Remark 6.1.2. The quotient space $\mathcal{S}'/\mathcal{P}$ actually ignores those tempered distributions whose Fourier transforms contain a tempered distribution supported at $\boldsymbol{\xi} = \mathbf{0}$ (in particular, any nonzero polynomial does not belong to $\mathcal{S}'/\mathcal{P}$). In fact, any tempered distribution supported at a single point must be a finite linear combination of Dirac's delta at that point and its derivatives, and so its inverse Fourier transform is exactly a polynomial. In other words, we actually have

$$\mathcal{S}'/\mathcal{P} \cong \mathcal{S}'_h := \left\{u \in \mathcal{S}'(\mathbb{R}^d) : (P(\boldsymbol{\xi})\hat{u}(\boldsymbol{\xi}))(\mathbf{0}) = 0,\ P \in \mathcal{P}\right\}.$$

Note that $\mathcal{S}'/\mathcal{P}$ is NOT a closed subspace of $\mathcal{S}'$ in the weak-* topology.

For simplicity of notations, we also introduce the fractional-order derivatives as Fourier multipliers.

Definition 6.1.3. Given $f \in \mathcal{S}$ and $s \in \mathbb{R}$, we define $P(\nabla)f$ via Fourier transform

$$\widehat{P(\nabla)f}(\boldsymbol{\xi}) := P(i\boldsymbol{\xi})\hat{f}(\boldsymbol{\xi}), \quad P \text{ is a polynomial.}$$

Similarly. given a locally integrable complex-valued function m, we define the **Fourier multiplier** by

$$\widehat{m(\nabla/i)f}(\boldsymbol{\xi}) := m(\boldsymbol{\xi})\hat{f}(\boldsymbol{\xi}).$$

In particular, we write $\langle\nabla\rangle f$ and $|\nabla| f$ by

$$\widehat{\langle\nabla\rangle f}(\boldsymbol{\xi}) := \langle\boldsymbol{\xi}\rangle\hat{f}(\boldsymbol{\xi}), \qquad \widehat{|\nabla| f}(\boldsymbol{\xi}) := |\boldsymbol{\xi}|\hat{f}(\boldsymbol{\xi}).$$

Under this setting, we know

$$f \in H^s(\mathbb{R}^d) \Leftrightarrow \langle\nabla\rangle^s f \in L^2(\mathbb{R}^d)$$
$$f \in \dot{H}^s(\mathbb{R}^d) \Leftrightarrow |\nabla|^s f \in L^2(\mathbb{R}^d).$$

Proposition 6.1.2. The homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d)$ satisfies the following properties

(1) $\dot{H}^s(\mathbb{R}^d)$ is a Hilbert space if and only if $s < \frac{d}{2}$.

(2) When $s < \frac{d}{2}$, the set

$$\mathcal{S}_0(\mathbb{R}^d) := \{u \in \mathcal{S}(\mathbb{R}^d) : \hat{u}(\boldsymbol{\xi}) \text{ vanishes near } \boldsymbol{\xi} = \mathbf{0}\}$$

is a dense subset of $\dot{H}^s$.

(3) The dual space of $\dot{H}^s(\mathbb{R}^d)$ is $\dot{H}^{-s}(\mathbb{R}^d)$.

Proof. Here we only prove (1) and (2). For (1), when $s < \frac{d}{2}$, we may define the inner product

$$(u, v)_{\dot{H}^s} := \int_{\mathbb{R}^d} |\boldsymbol{\xi}|^{2s} \hat{u}(\boldsymbol{\xi}) \overline{\hat{v}(\boldsymbol{\xi})} \,\mathrm{d}\boldsymbol{\xi}.$$

It suffices to prove the completeness. Let $\{u_n\}$ be a Cauchy sequence in $\dot{H}^s(\mathbb{R}^d)$. Then by definition, $|\boldsymbol{\xi}|^s \hat{u}_n(\boldsymbol{\xi})$ is a Cauchy sequence in $L^2(\mathbb{R}^d)$. By the completeness of L^2, we know there exists $f \in L^2(\mathbb{R}^d)$ such that $|\boldsymbol{\xi}|^s \hat{u}_n(\boldsymbol{\xi}) \xrightarrow{L^2} f$.

Now, writing $f = |\boldsymbol{\xi}|^s g$ and we shall prove that g is a tempered distribution. In fact, this is straightforward by splitting $g(\boldsymbol{\xi})$ into $g(\boldsymbol{\xi})\chi_{|\boldsymbol{\xi}|\le 1}$ and $g(\boldsymbol{\xi})\chi_{|\boldsymbol{\xi}|>1}$. For the low-frequence part, we have

$$\int_{B(\mathbf{0},1)} |g(\boldsymbol{\xi})| \,\mathrm{d}\boldsymbol{\xi} = \int_{B(\mathbf{0},1)} |\boldsymbol{\xi}|^s |g(\boldsymbol{\xi})| |\boldsymbol{\xi}|^{-s} \,\mathrm{d}\boldsymbol{\xi} \le \underbrace{\||\boldsymbol{\xi}|^s g(\boldsymbol{\xi})\|_{L^2}}_{=\|f\|_{L^2}<\infty} \underbrace{\left(\int_{B(\mathbf{0},1)} |\boldsymbol{\xi}|^{-2s} \,\mathrm{d}\boldsymbol{\xi}\right)^{\frac{1}{2}}}_{<\infty \text{ iff } 2s<d} < \infty.$$

Therefore, $(g(\boldsymbol{\xi})\chi_{|\boldsymbol{\xi}|\le 1})^\vee$ is a bounded function. For the high-frequency part, since $|\boldsymbol{\xi}| > 1$ leads to $|\boldsymbol{\xi}| \simeq \langle \boldsymbol{\xi} \rangle$, we have

$$\int_{|\boldsymbol{\xi}|>1} \langle \boldsymbol{\xi} \rangle^{2s} |g(\boldsymbol{\xi})|^2 \,\mathrm{d}\boldsymbol{x} \le C \int_{|\boldsymbol{\xi}|>1} |\boldsymbol{\xi}|^{2s} |g(\boldsymbol{\xi})|^2 \,\mathrm{d}\boldsymbol{\xi} \le C \int_{\mathbb{R}^d} |f(\boldsymbol{\xi})|^2 \,\mathrm{d}\boldsymbol{x} < \infty.$$

Finally, we define the desired limit u by $u := \mathcal{F}^{-1}(g)$. Then the above analysis shows that $u_n \to u$ in $\dot{H}^s$ and $u \in \dot{H}^s$.

When $s \ge \frac{d}{2}$, we can prove that $\dot{H}^s(\mathbb{R}^d)$ endowed with $\|\cdot\|_{\dot{H}^s}$ norm is not complete by contradiction. First, we claim that

Claim (Exercise 6.1.4). When $s \ge \frac{d}{2}$, then $N : u \longmapsto \|\hat{u}\|_{L^1(B(\mathbf{0},1))} + \|u\|_{\dot{H}^s}$ is a norm over $\dot{H}^s(\mathbb{R}^d)$ and $(\dot{H}^s(\mathbb{R}^d), N)$ is indeed a Banach space.

Under this claim, if $\dot{H}^s(\mathbb{R}^d)$ endowed with $\|\cdot\|_{\dot{H}^s}$ norm is also complete, then $\|\cdot\|_{\dot{H}^s}$ must be equivalent to the norm N (because the norm N is always stronger than $\|\cdot\|_{\dot{H}^s}$). This, in particular, leads to

$$\|\hat{u}\|_{L^1(B(\mathbf{0},1))} \le C\|u\|_{\dot{H}^s}.$$

However, we next construct a counterexample that violates this inequality. Let $\mathcal{A} = \{\frac{1}{4} < |\boldsymbol{\xi}| < \frac{1}{3}\}$ be an annulus inside the unit ball such that $\mathcal{A} \cap 2\mathcal{A} = \emptyset$. Then we define v_n by

$$\hat{v_n} := \sum_{k=1}^{n} \frac{2^{(s+\frac{d}{2})k}}{k} \chi_{2^{-k}\mathcal{A}}.$$

We compute that

$$\|\hat{v_n}\|_{L^1(B(\mathbf{0},1))} = C\sum_{k=1}^{n} \frac{2^{(s+\frac{d}{2})k}}{k} 2^{-kd} = C\sum_{k=1}^{n} \frac{2^{(s-\frac{d}{2})k}}{k} \to \infty, \quad \text{as } n \to \infty,$$

but we also compute that

$$\|v_n\|_{\dot{H}^s}^2 = \sum_{k=1}^{n} \int_{2^{-k}\mathcal{A}} |\boldsymbol{\xi}|^{2s} k^{-2} 2^{2k(s+\frac{d}{2})} \, \mathrm{d}\boldsymbol{\xi} \leq C \sum_{k=1}^{n} \frac{1}{k^2} < \infty.$$

(2) When $s < \frac{d}{2}$, $\dot{H}^s$ is a Hilbert space. It suffices to show that: If $u \in \dot{H}^s$ satisfies

$$\text{}\}\}\forall \varphi \in \mathcal{S}_0,\ (u, \varphi)_{\dot{H}^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\boldsymbol{\xi}|^{2s} \hat{u}(\boldsymbol{\xi}) \overline{\hat{\varphi}(\boldsymbol{\xi})} \, \mathrm{d}\boldsymbol{\xi} = 0'', \text{ then } u = 0.$$

But this is quite straightforward. Indeed, if $u \in \dot{H}^s$ satisfies $(u, \varphi)_{\dot{H}^s(\mathbb{R}^d)} = 0$ for any $\varphi \in \mathcal{S}_0$, then by definition of $\mathcal{S}_0$ we have $\hat{u} = 0$ in $\mathbb{R}^d \backslash \{\mathbf{0}\}$. Using Plancherel's identity, we deduce that $u = 0$. □

Exercise 6.1

Exercise 6.1.1. Prove the completeness of $H^s(\mathbb{R}^d)$.

Exercise 6.1.2. Prove Proposition 6.1.2 (3). Precisely speaking, if $|s| < \frac{d}{2}$, prove that

(1) The bilinear functional

$$\begin{aligned} B : \mathcal{S}_0 \times \mathcal{S}_0 &\to \mathbb{C} \\ (\phi, \varphi) &\longmapsto \int_{\mathbb{R}^d} \phi(\boldsymbol{x}) \varphi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \end{aligned}$$

can be extended to a continuous bilinear functional on $\dot{H}^{-s} \times \dot{H}^s$.

(2) If L is a continuous linear functional on $\dot{H}^s$, then there exists a unique tempered distribution $u \in \dot{H}^{-s}$ such that $\langle L, \phi \rangle = B[u, \phi]$ holds for all $\phi \in \dot{H}^s$ and $\|L\|_{(\dot{H}^s)^*} = \|u\|_{\dot{H}^{-s}}$.

Exercise 6.1.3. Assume $s_0 \leq s \leq s_1$. Prove that $\dot{H}^{s_0}(\mathbb{R}^d) \cap \dot{H}^{s_1}(\mathbb{R}^d) \subseteq \dot{H}^s(\mathbb{R}^d)$.

Exercise 6.1.4. When $s > \frac{d}{2}$, prove that $(\dot{H}^s(\mathbb{R}^d), N)$ is complete where the norm N is defined by

$$N : u \longmapsto \|\hat{u}\|_{L^1(B(\mathbf{0},1))} + \|u\|_{\dot{H}^s}.$$

Exercise 6.1.5. Let $0 < s < 1$ and $u \in \dot{H}^s(\mathbb{R}^d)$. Prove that $u \in L^2_{\text{loc}}(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(\boldsymbol{x}+\boldsymbol{y})-u(\boldsymbol{x})|^2}{|\boldsymbol{y}|^{d+2s}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} < \infty.$$

This gives the equivalence between the homogeneous Sobolev norm and the Sobolev-Slobodeckiĭ norm.

(Hint: Split $\hat{u}$ into $\{|\boldsymbol{\xi}| \leq 1\}$ part and $\{|\boldsymbol{\xi}| > 1\}$ part. Then apply Plancherel's identity to the $\boldsymbol{x}$ variable in the Sobolev-Slobodeckiĭ norm.)

6.2 Sobolev embedding theorems

In this section, we aim to prove Sobolev embedding theorems for the fractional Sobolev spaces $H^s(\mathbb{R}^d)$ and $\dot{H}^s(\mathbb{R}^d)$, namely the analogues of Theorem 1.5.1.

6.2.1 Sub-critical and critical Sobolev embedding

First, we prove the "subcritical" Gagliargo-Nirenberg-Sobolev type inequality.

Theorem 6.2.1 (Sobolev embedding)**.** Assume $0 \leq s < \frac{d}{2}$. Then $H^s(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ for $2 \leq q < 2^* := \frac{2d}{d-2s}$ with the inequality $\|f\|_{L^q(\mathbb{R}^d)} \leq C(s,q,d)\|f\|_{H^s(\mathbb{R}^d)}$.

Proof. It suffices to prove this inequality for all $f \in \mathcal{S}(\mathbb{R}^d)$. Given $f \in \mathcal{S}(\mathbb{R}^d)$, we have $f = (\hat{f})^\vee$. Since $2 \leq q < \infty$, we have its conjugate exponent $1 < q' \leq 2$. By Hausdorff-Young inequality, we know

$$\|f\|_{L^q} = \|(\hat{f})^\vee\|_{L^q} \leq C\|\hat{f}\|_{L^{q'}}.$$

We then write $\hat{f}(\boldsymbol{\xi}) = \langle\boldsymbol{\xi}\rangle^{-s}(\langle\boldsymbol{\xi}\rangle^s\hat{f}(\boldsymbol{\xi}))$ and use Hölder's inequality to get

$$\|\hat{f}\|_{L^{q'}} = \|\langle\boldsymbol{\xi}\rangle^{-s}(\langle\boldsymbol{\xi}\rangle^s\hat{f}(\boldsymbol{\xi}))\|_{L^{q'}} \leq \|\langle\boldsymbol{\xi}\rangle^{-s}\|_{L^r}\|\langle\boldsymbol{\xi}\rangle^s\hat{f}\|_{L^2} = \|\langle\boldsymbol{\xi}\rangle^{-s}\|_{L^r}\|f\|_{H^s}.$$

It remains to verify $\|\langle\boldsymbol{\xi}\rangle^{-s}\|_{L^r(\mathbb{R}^d)} < \infty$ which is equivalent to $sr > d$. To do this, we just compute r from Hölder's inequality:

$$\frac{1}{q'} = \frac{1}{r} + \frac{1}{2} \Rightarrow \frac{1}{r} = \frac{1}{q'} - \frac{1}{2} = \frac{1}{2} - \frac{1}{q} < \frac{1}{2} - \frac{1}{2^*} = \frac{s}{d},$$

which leads to our desired inequality $sr > d$. □

Remark 6.2.1. Although the index q is strictly less than the critical one 2^*, the above embedding is still not compact because of the unboundedness of $\mathbb{R}^d$. In fact, we can easily construct a counterexample by setting $f \in H^s(\mathbb{R}^d)$ with $\|f\|_{H^s} = 1$ and define $f_n(\boldsymbol{x}) = f(\boldsymbol{x} + n\mathbf{e}_1)$. Then as $n \to \infty$, f_n weakly converges to 0, but its L^q norm remains the same as f itself. Therefore, $\{f_n\}$ does not have any strongly convergent subsequence.

When $0 < s < \frac{d}{2}$, the critical embedding $H^s(\mathbb{R}^d) \hookrightarrow L^{2^*}(\mathbb{R}^d)$ with $2^* := \frac{2d}{d-2s}$ holds as a corollary of the Hardy-Littlewood-Sobolev inequality (Theorem C.3.4). For simplicity, we only prove this conclusion for the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d)$.

Theorem 6.2.2 (Critical Sobolev embedding)**.** When $0 \le s < \frac{d}{2}$, the space $\dot{H}^s(\mathbb{R}^d)$ is continuously embedded into $L^{2^*}(\mathbb{R}^d)$ with $2^* := \frac{2d}{d-2s}$.

Proof. Again, it suffices to prove this for any $f \in \mathcal{S}$. Define $g = (|\boldsymbol{\xi}|^{d-\gamma}\hat{f})^\vee$. Then $\hat{f} = |\boldsymbol{\xi}|^{-d+\gamma}\hat{g}$ and so $f = (|\boldsymbol{\xi}|^{-d+\gamma})^\vee * g$. By Exercise D.2.4, we know $(|\boldsymbol{\xi}|^{-d+\gamma})^\vee = C_{d,\gamma}|\boldsymbol{x}|^{-\gamma}$. Therefore, the Hardy-Littlewood-Sobolev inequality shows that

$$\|f\|_{L^q} = C_{d,\gamma}\||\cdot|^{-\gamma} * g\|_{L^q(\mathbb{R}^d)} \le C\|g\|_{L^p(\mathbb{R}^d)},\ \ 1 + \frac{1}{q} = \frac{1}{p} + \frac{\gamma}{d}.$$

Picking $p = 2, s = d - \gamma$ and using Plancherel's identity, the right side becomes

$$\|g\|_{L^2} = \|\hat{g}\|_{L^2} = \||\boldsymbol{\xi}|^{d-\gamma}\hat{f}\|_{L^2} = \|f\|_{\dot{H}^{d-\gamma}} = \|f\|_{\dot{H}^s}.$$

Then the exponent q exactly coincides with the critical exponent 2^*:

$$\frac{1}{q} = \frac{\gamma}{d} - \frac{1}{2} = \frac{d-s}{d} - \frac{1}{2} = \frac{1}{2} - \frac{s}{d} = \frac{1}{2^*} \Rightarrow q = 2^*.$$

□

6.2.2 Morrey's embedding and the critical space $\dot{H}^{\frac{d}{2}}$

In this section, we investigate the Sobolev embedding theorems for $H^s(\mathbb{R}^d)$ with $s \ge \frac{d}{2}$. First, there is a rather simple result: $H^s(\mathbb{R}^d)$ is a Banach algebra and embeds into $L^\infty(\mathbb{R}^d)$ for $s > \frac{d}{2}$.

Theorem 6.2.3. Assume $s > \frac{d}{2}$. Then:

(1) $H^s(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ with $\|f\|_{L^\infty(\mathbb{R}^d)} \le C\|f\|_{H^s(\mathbb{R}^d)}$ for all $f \in H^s$.
(2) $\|fg\|_{H^s} \le C\|f\|_{H^s}\|g\|_{H^s}$ for all $f, g \in H^s(\mathbb{R}^d)$.

Proof. Again, it suffices to prove these two inequalities for Schwartz functions.

(1) Since $s > \frac{d}{2}$, we know $\langle\boldsymbol{\xi}\rangle^{-s} \in L^2(\mathbb{R}^d)$ and thus

$$\begin{aligned}|f(\boldsymbol{x})| = |(\hat{f})^\vee(\boldsymbol{x})| &= (2\pi)^{-\frac{d}{2}}\left|\int_{\mathbb{R}^d} e^{i\boldsymbol{x}\cdot\boldsymbol{\xi}}\hat{f}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}\right| \le C\int_{\mathbb{R}^d}|\hat{f}(\boldsymbol{\xi})|\,\mathrm{d}\boldsymbol{\xi}\\ &= C\int_{\mathbb{R}^d}\langle\boldsymbol{\xi}\rangle^{-s}(\langle\boldsymbol{\xi}\rangle^s\hat{f}(\boldsymbol{\xi}))\,\mathrm{d}\boldsymbol{\xi} \le C\|\langle\boldsymbol{\xi}\rangle^{-s}\|_{L^2}\|\langle\boldsymbol{\xi}\rangle^s\hat{f}\|_{L^2} \le C\|f\|_{H^s}.\end{aligned}$$

Then taking supremum over all $\boldsymbol{x} \in \mathbb{R}^d$ yields $f \in L^\infty$ with $\|f\|_{L^\infty(\mathbb{R}^d)} \le C\|f\|_{H^s(\mathbb{R}^d)}$.

(2) For $f, g \in \mathcal{S}$, we have $\widehat{fg} = (2\pi)^{-\frac{d}{2}}(\hat{f} * \hat{g})$ and then

$$\begin{aligned}\|fg\|_{H^s(\mathbb{R}^d)} &= \|\langle \boldsymbol{\xi}\rangle^s \widehat{fg}(\boldsymbol{\xi})\|_{L^2} = (2\pi)^{-\frac{d}{2}}\|\langle \boldsymbol{\xi}\rangle^s(\hat{f} * \hat{g})\|_{L^2}\\ &= (2\pi)^{-\frac{d}{2}}\left\|\int_{\mathbb{R}^d}\langle \boldsymbol{\xi}\rangle^s \hat{f}(\boldsymbol{\xi}-\boldsymbol{\eta})\hat{g}(\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}\right\|_{L^2_{\xi}}.\end{aligned}$$

Next, we claim that

Claim. For all $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^d$ and $s > 0$, there holds

$$\langle \boldsymbol{\xi}\rangle^s \le C_s(\langle \boldsymbol{\xi}-\boldsymbol{\eta}\rangle^s + \langle \boldsymbol{\eta}\rangle^s), \quad C_s = \max\{2^{s/2}, 2^{s-1}\}.$$

With this claim, we have by triangle inequality that

$$\begin{aligned}\left\|\int_{\mathbb{R}^d}\langle \boldsymbol{\xi}\rangle^s \hat{f}(\boldsymbol{\xi}-\boldsymbol{\eta})\hat{g}(\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}\right\|_{L^2_{\xi}} &= C\left(\left\|\int_{\mathbb{R}^d}\langle \boldsymbol{\xi}-\boldsymbol{\eta}\rangle^s \hat{f}(\boldsymbol{\xi}-\boldsymbol{\eta})\hat{g}(\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}\right\|_{L^2_{\xi}} + \left\|\int_{\mathbb{R}^d}\langle \boldsymbol{\eta}\rangle^s \hat{f}(\boldsymbol{\xi}-\boldsymbol{\eta})\hat{g}(\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}\right\|_{L^2_{\xi}}\right)\\ &= C\left(\|(\langle\cdot\rangle^s\hat{f}) * \hat{g}\|_{L^2} + \|(\langle\cdot\rangle^s\hat{g}) * \hat{f}\|_{L^2}\right)\end{aligned}$$

Using Young's inequality for convolution (Theorem C.3.6), we have

$$\|(\langle\cdot\rangle^s\hat{f}) * \hat{g}\|_{L^2} \le \|\langle \boldsymbol{\xi}\rangle^s\hat{f}\|_{L^2}\|\hat{g}\|_{L^1} \le \|f\|_{H^s}\|\langle \boldsymbol{\xi}\rangle^{-s}\|_{L^2}\|\langle \boldsymbol{\xi}\rangle^s\hat{g}\|_{L^2} \le C\|f\|_{H^s}\|g\|_{H^s},$$

thanks to $s > \frac{d}{2}$. The same estimate also applied to $\|(\langle\cdot\rangle^s\hat{g})^*\hat{f}\|_{L^2}$ by interchanging f and g.

It remains to prove the claim above. For $p > 0$, we have

$$\begin{aligned}(1+|\boldsymbol{\xi}|^2)^p &\le (1+2|\boldsymbol{\xi}-\boldsymbol{\eta}|^2+2|\boldsymbol{\eta}|^2)^p \le 2^p(1+|\boldsymbol{\xi}-\boldsymbol{\eta}|^2+1+|\boldsymbol{\eta}|^2)^2\\ &\le \max\{2^p, 2^{2p-1}\}\left((1+|\boldsymbol{\xi}-\boldsymbol{\eta}|^2)^p + (1+|\boldsymbol{\eta}|^2)^p\right).\end{aligned}$$

Choosing $p = s/2$ leads to the claim. □

Next, we prove the analogue of Morrey's embedding theorem.

Theorem 6.2.4 (Morrey's embedding). Assume $s > \frac{d}{2}$ and $s - \frac{d}{2} \notin \mathbb{Z}$. Then $\dot{H}^s(\mathbb{R}^d)$ is included in $C^{k,\rho}(\mathbb{R}^d)$ with $k = [s - \frac{d}{2}]$ and $\rho = \{s - \frac{d}{2}\}$. Also, for all $f \in \dot{H}^s(\mathbb{R}^d)$, we have

$$\sup_{|\alpha|=k}\sup_{\boldsymbol{x}\neq\boldsymbol{y}}\frac{|\partial^\alpha f(\boldsymbol{x}) - \partial^\alpha f(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^\rho} \le C_{d,s}\|f\|_{\dot{H}^s}.$$

Proof. We only prove the theorem only in the case where the integer part of $s - \frac{d}{2}$ is 0 . As $s > d/2$, writing

$$\widehat{f} = \chi_{B(0,1)}\widehat{f} + \left(1 - \chi_{B(0,1)}\right)\widehat{f}$$

we get that $\widehat{f} \in L^1(\mathbb{R}^d)$, and thus f is a bounded continuous function (by Theorem 6.2.3 and Riemann-Lebesgue lemma).

We now decompose f into low and high frequencies. Fix $A > 0$ (to be determined later) and pick a smooth function $\theta \in \mathcal{S}$ such that $\hat{\theta} \in C_c^\infty(\mathbb{R}^d)$, $0 \le \hat{\theta} \le 1$ and $\hat{\theta} = 1$ near $\boldsymbol{\xi} = \mathbf{0}$. Then we define

$$f_{\ell,A} := \left(\hat{\theta}(\frac{\cdot}{A})\hat{f}\right)^\vee, \qquad f_{h,A} := f - f_{\ell,A}.$$

In other words, setting $f_{\ell,A}$ is to localize the frequency variable $\boldsymbol{\xi}$ near $|\boldsymbol{\xi}| = A$.

The low-frequency part $f_{\ell,A}$ is of course smooth. By fundamental theorem of calculus, we have

$$\left|f_{\ell,A}(\boldsymbol{x}) - f_{\ell,A}(\boldsymbol{y})\right| \le \left\|\nabla f_{\ell,A}\right\|_{L^\infty} |\boldsymbol{x} - \boldsymbol{y}|.$$

Using the Fourier inversion formula and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\left\|\nabla f_{\ell,A}\right\|_{L^\infty} &\le C\int_{\mathbb{R}^d} |\boldsymbol{\xi}| \left|\widehat{f}_{\ell,A}(\boldsymbol{\xi})\right| \,\mathrm{d}\boldsymbol{\xi} = C\int_{\mathbb{R}^d} |\boldsymbol{\xi}|^{1-s}|\boldsymbol{\xi}|^s \left|\widehat{f}_{\ell,A}(\boldsymbol{\xi})\right| \,\mathrm{d}\boldsymbol{\xi} \\ &\le C\left(\int_{|\boldsymbol{\xi}|\le CA} |\boldsymbol{\xi}|^{2-2s}\,\mathrm{d}\boldsymbol{\xi}\right)^{\frac{1}{2}} \|f\|_{\dot{H}^s} \le \frac{C}{(1-\rho)^{\frac{1}{2}}} A^{1-\rho}\|f\|_{\dot{H}^s} \quad \text{with } \rho = s - d/2.\end{aligned}$$

For the high-frequency part $f_{h,A}$, we can directly control the pointwise value because of $|\boldsymbol{\xi}|^{-s}$ is L^2-integrable away from the origin.

$$\left\|f_{h,A}\right\|_{L^\infty} \le \int_{\mathbb{R}^d} \left|\widehat{f}_{h,A}(\boldsymbol{\xi})\right| \,\mathrm{d}\boldsymbol{\xi} \le \left(\int_{|\boldsymbol{\xi}|\ge A} |\boldsymbol{\xi}|^{-2s}\,\mathrm{d}\boldsymbol{\xi}\right)^{\frac{1}{2}} \|f\|_{\dot{H}^s} \le \frac{C}{\rho^{\frac{1}{2}}} A^{-\rho}\|f\|_{\dot{H}^s},$$

It then leads to

$$\begin{aligned}|f(\boldsymbol{x}) - f(\boldsymbol{y})| &\le \left\|\nabla f_{\ell,A}\right\|_{L^\infty} |\boldsymbol{x} - \boldsymbol{y}| + 2\left\|f_{h,A}\right\|_{L^\infty} \\ &\le C_s\left(|\boldsymbol{x} - \boldsymbol{y}|A^{1-\rho} + A^{-\rho}\right)\|f\|_{\dot{H}^s}.\end{aligned}$$

Choosing $A = |\boldsymbol{x} - \boldsymbol{y}|^{-1}$ to optimize the above upper bound, we then complete the proof of the theorem. □

Finally, we consider the case $s = \frac{d}{2}$. Exercise 1.4.2 already shows that an $\dot{H}^{d/2}$ function may not belong to L^∞. One way to avoid this counterexample is to slightly enlarge the L^∞ space and the answer is the BMO space.

Definition 6.2.1. The space $BMO(\mathbb{R}^d)$ of **bounded mean oscillations** is the set of locally integrable functions f such that

$$\|f\|_{BMO} := \sup_B \frac{1}{|B|}\int_B |f - f_B| \,\mathrm{d}\boldsymbol{x} < \infty \quad \text{with} \quad f_B := \frac{1}{|B|}\int_B f \,\mathrm{d}\boldsymbol{x}$$

The above supremum is taken over the set of Euclidean balls. We point out that the seminorm $\|\cdot\|_{BMO}$

vanishes on constant functions. Therefore, this is not a norm. We now state the theorem for Sobolev embedding.

Theorem 6.2.5. The space $L^1_{\mathrm{loc}}(\mathbb{R}^d) \cap \dot{H}^{\frac{d}{2}}(\mathbb{R}^d)$ is included in $BMO(\mathbb{R}^d)$.There exists $C > 0$ such that

$$\|f\|_{BMO} \leq C\|f\|_{\dot{H}^{\frac{d}{2}}}$$

for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d) \cap \dot{H}^{\frac{d}{2}}(\mathbb{R}^d)$.

Proof. We again use the decomposition $f = f_{\ell,A} + f_{h,A}$ into low and high frequencies as in Theorem 6.2.4. For any ball B we have (using Cauchy-Schwarz)

$$\int_B |f - f_B| \frac{\mathrm{d}\boldsymbol{x}}{|B|} \leq \left\| f_{\ell,A} - \left(f_{\ell,A}\right)_B \right\|_{L^2(B, \frac{\mathrm{d}x}{|B|})} + \frac{2}{|B|^{\frac{1}{2}}} \left\| f_{h,A} \right\|_{L^2(B)}$$

Let R be the radius of the ball B. We have

$$\begin{aligned} \left\| f_{\ell,A} - \left(f_{\ell,A}\right)_B \right\|_{L^2(B, \frac{\mathrm{d}x}{|B|})} &\leq R \left\| \nabla f_{\ell,A} \right\|_{L^\infty} \leq CR \int_{\mathbb{R}^d} |\boldsymbol{\xi}|^{1-\frac{d}{2}} |\boldsymbol{\xi}|^{\frac{d}{2}} \left| \widehat{f}_{\ell,A}(\boldsymbol{\xi}) \right| \mathrm{d}\boldsymbol{\xi} \\ &\leq CRA\|f\|_{\dot{H}^{\frac{d}{2}}}. \end{aligned}$$

The high-frequency part is directly controlled

$$\left\| f_{h,A} \right\|_{L^2} = \left\| \widehat{f_{h,A}} \right\|_{L^2} \leq \left\| A^{-\frac{d}{2}} |\boldsymbol{\xi}|^{\frac{d}{2}} \widehat{f_{h,A}} \right\|_{L^2} \leq A^{-\frac{d}{2}} \| |\boldsymbol{\xi}|^{\frac{d}{2}} \hat{f} \chi_{|\boldsymbol{\xi}| \geq A} \|_{L^2}.$$

We infer that

$$\int_B |f - f_B| \frac{\mathrm{d}\boldsymbol{x}}{|B|} \leq CRA\|f\|_{\dot{H}^{\frac{d}{2}}} + C(AR)^{-\frac{d}{2}} \left(\int_{|\boldsymbol{\xi}| \geq A} |\boldsymbol{\xi}|^d |\widehat{f}(\boldsymbol{\xi})|^2 \, \mathrm{d}\boldsymbol{\xi} \right)^{\frac{1}{2}}.$$

Choosing $A = R^{-1}$ then completes the proof. □

For the non-homogeneous Sobolev space $H^{d/2}(\mathbb{R}^d)$, we have the so-called Moser-Trudinger inequality, which actually shows the embedding into the Orlicz-type spaces.

Theorem 6.2.6. There exist two constants $c, C > 0$ depending only on d, such that the following inequality holds for any $f \in H^{d/2}(\mathbb{R}^d)$:

$$\int_{\mathbb{R}^d} \exp\left(c \left(\frac{|f(\boldsymbol{x})|}{\|f\|_{H^{d/2}}} \right)^2 \right) - 1 \, \mathrm{d}\boldsymbol{x} \leq C.$$

6.2.3 Compact embedding and the trace theorem

Due to the unboundedness of $\mathbb{R}^d$, the subcritical embedding $H^s(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ for $0 < s < \frac{d}{2}$ and $2 \le q < 2^*$ is not compact. But we still have the compact embedding from H^s to H^t for $t < s$. Note that this compact embedding is not given by the inclusion map unless the functions are compactly supported.

Theorem 6.2.7. For $t < s$, multiplication by a function in $\mathcal{S}(\mathbb{R}^d)$ is a compact operator from $H^s(\mathbb{R}^d)$ in $H^t(\mathbb{R}^d)$.

Proof. Let $\varphi \in \mathcal{S}$. We need to prove that for any sequence $\{f_n\} \subset H^s(\mathbb{R}^d)$ satisfying $\sup_n \|f_n\|_{H^s} \le 1$, there exists a subsequence $\{f_{n_k}\}$ such that u_{n_k} strongly converges in $H^t(\mathbb{R}^d)$.

The Eberlein-Šmulian theorem ensures that the sequence $\{f_n\}$ converges weakly, up to extracting a subsequence, to some $f \in H^s(\mathbb{R}^d)$ with $\|f\|_{H^s} \le 1$. We continue to denote this subsequence by $\{f_n\}$ and set $g_n = f_n - f$. Then direct computation shows that $\sup_n \|\varphi g_n\|_{H^s} \le C$. Our task is thus reduced to proving that $\varphi g_n \to 0$ in $H^t(\mathbb{R}^d)$.

We again truncate the frequency variable ξ at $|\xi| = R$

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \langle \xi \rangle^{2t} \left| \widehat{\varphi g_n}(\xi) \right|^2 \mathrm{d}\xi = \int_{|\xi| \le R} \langle \xi \rangle^{2t} \left| \widehat{\varphi g_n}(\xi) \right|^2 \mathrm{d}\xi + \int_{|\xi| \ge R} \langle \xi \rangle^{2(t-s)} \langle \xi \rangle^{2s} \left| \widehat{\varphi g_n}(\xi) \right|^2 \mathrm{d}\xi \\
\le &\int_{|\xi| \le R} \langle \xi \rangle^{2t} \left| \widehat{\varphi g_n}(\xi) \right|^2 \mathrm{d}\xi + \frac{C \left\| \varphi g_n \right\|_{H^s}^2}{(1+R^2)^{s-t}}
\end{aligned}
$$

As $\{\varphi g_n\}$ is uniformly bounded in $H^s(\mathbb{R}^d)$, for a given $\varepsilon > 0$, we can choose R such that

$$
\frac{C}{(1+R^2)^{s-t}} \left\| \varphi g_n \right\|_{H^s}^2 \le \frac{\varepsilon}{2}.
$$

Now, we shall prove

$$
\int_{|\xi| \le R} \langle \xi \rangle^{2t} \left| \widehat{\varphi g_n}(\xi) \right|^2 \mathrm{d}\xi \to 0.
$$

We have

$$
\begin{aligned}
\widehat{\varphi g_n}(\xi) &= (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \hat{\varphi}(\xi - \eta) \hat{g}_n(\eta) \,\mathrm{d}\eta \\
&= (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \langle \eta \rangle^{2s} \underbrace{\langle \eta \rangle^{-2s} \hat{\varphi}(\xi - \eta)}_{\in \mathcal{S}(\mathbb{R}^d) \text{ for each } \xi \in \mathbb{R}^d} \hat{g}_n(\eta) \,\mathrm{d}\eta = (2\pi)^{-\frac{d}{2}} \left(g_n, (\langle \cdot \rangle^{-2s} \hat{\varphi}(\xi - \cdot))^{\vee} \right)_{H^s}.
\end{aligned}
$$

Since $g_n \rightharpoonup 0$ in H^s, we know the above H^s inner product must converge to 0. That is, we now have the pointwise limit $\widehat{\varphi g_n}(\xi) \to 0$ for each $\xi \in \mathbb{R}^d$.

Next, we claim the uniform boundedness:

Claim.

$$\sup_{\substack{|\xi|\leq R\\ n\in\mathbb{N}}} |\widehat{\varphi g_n}(\xi)| < \infty.$$

Should the claim hold, the by Dominated Convergence Theorem, it is easy to see that

$$\lim_{n\to\infty}\int_{|\xi|\leq R} \langle\xi\rangle^{2t}\left|\widehat{\varphi g_n}(\xi)\right|^2 \mathrm{d}\xi = \int_{|\xi|\leq R}\lim_{n\to\infty}\langle\xi\rangle^{2t}\left|\widehat{\varphi g_n}(\xi)\right|^2 \mathrm{d}\xi = 0,$$

as desired.

To prove the claim, we have

$$|\widehat{\varphi g_n}(\xi)|^2 = (2\pi)^{-d}\left(g_n, (\langle\cdot\rangle^{-2s}\hat{\varphi}(\xi-\cdot))^\vee\right)_{H^s}^2 \leq \|g_n\|_{H^s}^2\|\langle\cdot\rangle^{-s}\hat{\varphi}(\xi-\cdot)\|_{L^2}^2.$$

We then study the last term

$$\int\langle\eta\rangle^{-2s}|\hat{\varphi}(\xi-\eta)|^2\,\mathrm{d}\eta \leq \int_{|\eta|\leq 2R}\langle\eta\rangle^{-2s}|\hat{\varphi}(\xi-\eta)|^2\,\mathrm{d}\eta + \int_{|\eta|\geq 2R}\langle\eta\rangle^{-2s}|\hat{\varphi}(\xi-\eta)|^2\,\mathrm{d}\eta.$$

The first integral is definitely bounded,

$$\int_{|\eta|\leq 2R}\langle\eta\rangle^{-2s}|\hat{\varphi}(\xi-\eta)|^2\,\mathrm{d}\eta \leq C\int_{|\eta|\leq 2R}\langle\eta\rangle^{2|s|}\,\mathrm{d}\eta.$$

While in the second term, we have to use $\varphi \in \mathcal{S}$ to produce more decaying factor to establish the boundedness. As $\hat{\varphi}$ belongs to $\mathcal{S}(\mathbb{R}^d)$, there is a constant $C > 0$ such that

$$|\hat{\varphi}(\xi-\eta)| \leq C\langle\xi-\eta\rangle^{-2N_0} \quad \text{with} \quad N_0 = \frac{d}{2}+|s|+1.$$

We thus use $|\xi-\eta| \geq |\eta|/2$ for $|\xi|\leq R$ and $|\eta|\geq 2R$ to get

$$\begin{aligned}
&\int_{|\eta|\geq 2R}\langle\eta\rangle^{-2s}|\hat{\varphi}(\xi-\eta)|^2\,\mathrm{d}\eta\\
&\leq C_{N_0}\int_{|\eta|\geq 2R}\langle\eta\rangle^{2|s|}\langle\xi-\eta\rangle^{-2N_0}\,\mathrm{d}\eta \leq C\int_{|\eta|\geq 2R}\langle\eta\rangle^{2|s|-2N_0}\,\mathrm{d}\eta < \infty.
\end{aligned}$$

□

Finally, we are concerned with the trace theorem. In Theorem 1.3.1, it is shown that a $W^{1,p}(U)$ function has $L^p(\partial U)$ trace. Let $U = \mathbb{R}^d_+$ be the half-space in $\mathbb{R}^d$ and we will see that the trace of an $H^s(\mathbb{R}^d_+)$ function actually possesses differentiability $H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})$ as long as $s > \frac{1}{2}$.

Theorem 6.2.8 (Trace Theorem)**.** Let $s > 1/2$. The restriction map γ defined by

$$\gamma : \begin{cases} \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{S}(\mathbb{R}^{d-1}) \\ \qquad f \longmapsto \gamma(f) : (x_2, \dots, x_d) \longmapsto \phi(0, x_2, \dots, x_d) \end{cases}$$

can be continuously extended from $H^s(\mathbb{R}^d)$ onto $H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})$.

As a corollary, we conclude a more useful version of the trace theorem. For $s \geq 0$, we define

$$H^s(\mathbb{R}^d_+) := \left\{ u \in L^2(\mathbb{R}^d) : \langle \boldsymbol{\xi}' \rangle^r \partial_d^k \hat{u}(\boldsymbol{\xi}', x_d) \in L^2(\mathbb{R}^d),\ r + k \leq s,\ k \geq 0,\ k \in \mathbb{Z} \right\}.$$

Here the Fourier transform is defined for the tangential variables $\boldsymbol{x}' = (x_1, \cdots, x_{d-1})$.

Corollary 6.2.9 (Trace inequality)**.** Let $s > 1/2$ and $f \in H^s(\mathbb{R}^d_+)$ where $\mathbb{R}^d_+$ is the upper half space $\{x_d > 0\}$. Then $\mathrm{Tr}\, f \in H^{s-\frac{1}{2}}(\partial \mathbb{R}^d_+)$ with estimate $\|\mathrm{Tr}\, f\|_{H^{s-\frac{1}{2}}(\partial \mathbb{R}^d_+)} \leq C \|f\|_{H^s(\mathbb{R}^d_+)}$.

In fact, the proof of Corollary 6.2.9 is very easy, which is essentially a simple application of the Gauss-Green formula.

Proof of Theorem 6.2.9. W.L.O.G assume $f \in C^\infty(\mathbb{R}^d_+)$ for simplicity. Then we compute that

$$\begin{aligned} \|\mathrm{Tr}\, f\|^2_{H^{s-\frac{1}{2}}(\partial \mathbb{R}^d_+)} &= \int_{\partial \mathbb{R}^d_+} \langle \boldsymbol{\xi}' \rangle^{2s-1} |\hat{f}(\boldsymbol{\xi}', 0)|^2 \,\mathrm{d}\boldsymbol{\xi}' \\ = -2\mathbf{Re} \int_0^\infty \int_{\mathbb{R}^{d-1}} &\left(\langle \boldsymbol{\xi}' \rangle^s \overline{\hat{f}(\boldsymbol{\xi}', x_d)} \right) (\langle \boldsymbol{\xi}' \rangle^{s-1} \partial_{x_d} \hat{f}(\boldsymbol{\xi}', x_d)) \,\mathrm{d}\boldsymbol{\xi}' \,\mathrm{d}x_d \\ = -2 \int_0^\infty \int_{\mathbb{R}^{d-1}} &\langle \nabla' \rangle^s f \, \partial_{x_d} \langle \nabla' \rangle^{s-1} f \,\mathrm{d}\boldsymbol{x}' \,\mathrm{d}x_d \\ \leq 2\|f\|^2_{H^s(\mathbb{R}^d_+)}&. \end{aligned}$$

Here we use the fact that $\int_{\mathbb{R}^d} f(\boldsymbol{x}) g(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d} \hat{f}(\boldsymbol{\xi}) \overline{\hat{g}(\boldsymbol{\xi})} \,\mathrm{d}\boldsymbol{\xi}$ in the third line. □

Proof of Theorem 6.2.8. We first prove the existence of γ, that is, we shall find a constant $C > 0$ such that

$$\forall f \in \mathcal{S}, \quad \|\gamma(f)\|_{H^{s-\frac{1}{2}}} \leq C \|f\|_{H^s}.$$

To achieve the above inequality, we may rewrite the trace operator in terms of a Fourier transform:

$$\begin{aligned} f(0, \boldsymbol{x}') &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i \cdot 0 \cdot \xi_1} e^{i \boldsymbol{x}' \cdot \xi'} \hat{f}(\xi_1, \boldsymbol{\xi}') \,\mathrm{d}\xi_1 \,\mathrm{d}\boldsymbol{\xi}' \\ &= (2\pi)^{\frac{-(d-1)}{2}} \int_{\mathbb{R}^{d-1}} e^{i \boldsymbol{x}' \cdot \xi'} \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}\left(\xi_1, \boldsymbol{\xi}'\right) \,\mathrm{d}\xi_1 \right) \mathrm{d}\boldsymbol{\xi}' \end{aligned}$$

We thus have

$$\widehat{\gamma(f)}\left(\xi'\right) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}\left(\xi_1, \xi'\right) \mathrm{d}\xi_1.$$

Using Cauchy-Schwarz inequality, we have

$$\left|\widehat{\gamma(f)}(\xi')\right|^2 \leq \frac{1}{2\pi} \left(\int_{\mathbb{R}} \left(1 + \xi_1^2 + |\xi'|^2\right)^{-s} \mathrm{d}\xi_1 \right) \left(\int_{\mathbb{R}} |\hat{f}(\xi)|^2 \langle \xi \rangle^{2s} \mathrm{d}\xi_1 \right),$$

where the first integral is finite thanks to $s > \frac{1}{2}$. Therefore, we deduce that $\|\gamma(f)\|^2_{H^{s-\frac{1}{2}}} \leq C_s \|f\|^2_{H^s}$ after integrating $\mathrm{d}\xi'$ over $\mathbb{R}^{d-1}$, which completes the proof of the first part of the theorem.

We now prove the existence of the "lifting operator" (indeed, it is the right inverse of the trace operator). Let χ be a function in $C_c^\infty(\mathbb{R})$ such that $\chi(0) = 1$. We define

$$Rv(\boldsymbol{x}) := (2\pi)^{-\frac{d-1}{2}} \int_{\mathbb{R}^{d-1}} e^{i x' \cdot \xi'} \chi(x_1 \langle \xi' \rangle) \hat{v}(\xi') \, \mathrm{d}\xi',$$

which is actually the inverse Fourier transform of $\chi(x_1\langle \xi' \rangle)\hat{v}(\xi')$ in $\mathbb{R}^{d-1}$. Then, taking Fourier transform in $\mathbb{R}^d$ leads to

$$\widehat{Rv}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-it\xi_1} \chi(t\langle \xi' \rangle) \hat{v}\left(\xi'\right) \mathrm{d}t = \langle \xi' \rangle^{-1} \hat{\chi}\left(\frac{\xi_1}{\langle \xi' \rangle}\right) \hat{v}(\xi')$$

and so its $H^s(\mathbb{R}^d)$ norm is given by

$$\begin{aligned}
\|Rv\|^2_{H^s(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} (1 + \xi_1^2 + |\xi'|^2)^s \langle \xi' \rangle^{-2} \left| \hat{\chi}\left(\frac{\xi_1}{\langle \xi' \rangle}\right) \right|^2 |\hat{v}(\xi')|^2 \, \mathrm{d}\xi \\
&= \int_{\mathbb{R}^{d-1}} \left[\int_{\mathbb{R}} \left(1 + \frac{\xi_1^2}{\langle \xi' \rangle^2}\right)^s \langle \xi' \rangle^{-1} \left| \hat{\chi}\left(\frac{\xi_1}{\langle \xi' \rangle}\right) \right|^2 \mathrm{d}\xi_1 \right] \left(\langle \xi' \rangle^{2s-1} |\hat{v}(\xi')|^2 \right) \mathrm{d}\xi'.
\end{aligned}$$

Since $\hat{\chi} \in \mathcal{S}$, we know for any N there exists a constant C_N such that $|\hat{\chi}(t)| \leq C_N t^{-N}$, which then gives

$$\int_{\mathbb{R}} \left(1 + \frac{\xi_1^2}{\langle \xi' \rangle^2}\right)^s \langle \xi' \rangle^{-1} \left| \hat{\chi}\left(\frac{\xi_1}{\langle \xi' \rangle}\right) \right|^2 \mathrm{d}\xi_1 \leq C_N^2 \int_{\mathbb{R}} \left(1 + \frac{\xi_1^2}{\langle \xi' \rangle^2}\right)^{s-2N} \cdot 1 \, \mathrm{d}\xi_1.$$

Here we also use $\langle \xi' \rangle^{-1} \leq 1$. Then pick N sufficiently large such that $s - 2N < -\frac{1}{2}$ and we see that the above integral is finite. Therefore, the right side of $\|\widehat{Rv}\|^2_{H^s(\mathbb{R}^d)}$ is bounded by

$$\|Rv\|^2_{H^s} \leq C \|v\|_{H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})}.$$

Of course, we have $\gamma R v = v$. This completes the proof of the theorem. $\square$

Remark 6.2.2. The construction of the extension map R is not unique. In particular, under the setting of Corollary 6.2.9, given $g \in H^{s-1/2}(\partial\mathbb{R}^d_+)$, its harmonic extension, say G, defined by $-\Delta G = 0$ in $\mathbb{R}^d_+$ with $G = g$ on $\partial\mathbb{R}^d_+$, satisfies the reverse trace inequality: $\|G\|_{H^s(\mathbb{R}^d_+)} \le C\|g\|_{H^{s-\frac{1}{2}}(\partial\mathbb{R}^d_+)}$, which can be proved by analyzing the Poisson integral (the harmonic extension in the half space can be explicity calculated) and even holds for all $s \ge 0$. See, for example, [18, Prop 5.1.8].

Exercise 6.2

Exercise 6.2.1. Prove that for all $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^d$ and $s \in \mathbb{R}$, there holds

$$\left(1 + |\boldsymbol{\xi}|^2\right)^s \left(1 + |\boldsymbol{\eta}|^2\right)^{-s} \le 2^{|s|} \left(1 + |\boldsymbol{\xi} - \boldsymbol{\eta}|^2\right)^{|s|}.$$

Exercise 6.2.2. Let $\varphi \in C(\mathbb{R}^d)$ satisfy $\varphi(\boldsymbol{x}) \to 0$ as $|\boldsymbol{x}| \to \infty$ and $\hat{\varphi}$ satisfies $\int_{\mathbb{R}^d} \langle \boldsymbol{\xi} \rangle^a |\hat{\varphi}(\boldsymbol{\xi})| \, \mathrm{d}\boldsymbol{\xi} < \infty$. prove that the map $M_\varphi(f) := \varphi f$ is a bounded operator on $H^s(\mathbb{R}^d)$ for $|s| \le a$. Moreover, if $\varphi \in \mathcal{S}$, prove that M_φ is a bounded operator on $H^s(\mathbb{R}^d)$ for all $s \in \mathbb{R}$. (Hint: Use Exercise 6.2.1.)

Exercise 6.2.3. Prove that if $f \in H^s(\mathbb{R}^d)$ for all $s \in \mathbb{R}$, then $f \in C^\infty(\mathbb{R}^d)$.

Exercise 6.2.4. If $H^s(\mathbb{R}^d) \subset C_0(\mathbb{R}^d)$ where C_0 represents that continuous function that vanishes at infinity, then prove that $s > d/2$.

(Hint: Use the Closed Graph Theorem to show the inclusion map $H^s \hookrightarrow C_0$ is continuous and hence $\partial^\alpha \delta \in H^{-s}$ for $|\alpha| \le k$.)

Exercise 6.2.5. Suppose that $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $\varphi \not\equiv 0$ and $\{\mathbf{a}_n\}$ be a sequence in $\mathbb{R}^d$ with $|\mathbf{a}_n| \to \infty$. Define $\varphi_n(\boldsymbol{x}) := \varphi(\boldsymbol{x} - \mathbf{a}_n)$. Prove that $\{\varphi_n\}$ is bounded in H^s for all s but has no convergence subsequence in H^t for any t.

Exercise 6.2.6. Fix $m \in \mathbb{N}^*$. Let $P(\partial) = \sum_{|\alpha| \le m} c_\alpha \partial^\alpha$ for $c_\alpha \in \mathbb{R}$ and $c_\alpha \ne 0$ for some $|\alpha| = m$. We define the **principle symbol** P_m by

$$P_m(\boldsymbol{\xi}) := \sum_{|\alpha|=m} c_\alpha \boldsymbol{\xi}^\alpha.$$

We say $P(\partial)$ is **elliptic of order m** if $P_m(\boldsymbol{\xi}) \ne 0$ for all $\boldsymbol{\xi} \ne \mathbf{0}$. Now suppose $P(\partial)$ is of order m.

(1) Prove that $P(\partial)$ is elliptic if and only if there exist some $C, R > 0$ such that $|P(\boldsymbol{\xi})| \ge C|\boldsymbol{\xi}|^m$ for $|\boldsymbol{\xi}| \ge R$.
(2) If $u \in H^s(\mathbb{R}^d)$ and $P(\partial)u \in H^s(\mathbb{R}^d)$, prove that $u \in H^{s+m}(\mathbb{R}^d)$.

Remark 6.2.3. This is the generalization of the elliptic regularity theorem. In fact, when $P(\partial)$ is not elliptic, one may no longer have the regularity theorem but the equation $P(\partial)u = f$ is still solvable in at least $L^2(\Omega)$. However, the proof will be much more difficult as one has to analyze the Fourier support of the solutions. That conclusion is called **Malgrange-Ehrenpreis Theorem** and we refer to Muscalu-Schlag [13, Chap. 10.4].

Exercise 6.2.7. Given a constant-coefficient differential operator L, we say $u \in \mathcal{D}'$ is the **fundamental solution of** L if $Lu = \delta$. Find the fundamental solutions to the Laplacian operator $L_1 = -\Delta$, the heat operator $L_2 = \partial_t - \Delta$ and the wave equator $\square := \partial_t^2 - \Delta$ in $\mathbb{R}^d$. Do note that the fundamental solution to the wave equation is only a distribution which does not coincide with any locally integrable function.

(Reference: Stein [16, Chap. 3.2] and Luk [12, Section 3].)

Exercise 6.2.8. Assume $1 < p \le 2$. Prove that $L^p(\mathbb{R}^d)$ embeds continuously into $\dot{H}^s(\mathbb{R}^d)$ with $\frac{s}{d} = \frac{1}{2} - \frac{1}{p}$.

(Hint: Use duality and critical Sobolev embedding.)

Exercise 6.2.9. Recall that we obtain the inequality $\|f\|_{L^{2^*}(\mathbb{R}^d)} \le C\|f\|_{\dot{H}^s(\mathbb{R}^d)}$ for $0 \le s < \frac{d}{2}$ in Theorem 6.2.2. However, it is NOT invariant under multiplication by a character $e^{i\boldsymbol{x}\cdot\boldsymbol{\eta}}$. In fact, given $\varphi \in \mathcal{S}$ such that $\hat{\varphi} \in C_c^\infty(\mathbb{R}^d)$, we define $\varphi_\varepsilon(\boldsymbol{x}) = e^{i\frac{x_1}{\varepsilon}}\varphi(\boldsymbol{x})$. Verify that $\|\varphi_\varepsilon\|_{L^{2^*}}$ is independent of ε whereas $\|\varphi_\varepsilon\|_{\dot{H}^s}$ has size $O(\varepsilon^{-s})$.

Exercise 6.2.10 (Homogeneous Besov space $\dot{B}^{-\sigma}_{\infty,\infty}$). Let $\theta \in \mathcal{S}$ be a function satisfying $\hat{\theta} \in C_c^\infty$, $0 \le \hat{\theta} \le 1$ and $\hat{\theta} = 1$ near $\boldsymbol{\xi} = \mathbf{0}$. For $f \in \mathcal{S}'$ and $\sigma > 0$, we define

$$\|f\|_{\dot{B}^{-\sigma}_{\infty,\infty}} := \sup_{A>0} A^{d-\sigma}\|\theta(A\cdot) * f\|_{L^\infty}.$$

One can check that $\dot{B}^{-\sigma}_{\infty,\infty}(\mathbb{R}^d)$ equipped with this norm is a Banach space.

Now we assume $s < \frac{d}{2}$. Prove that $\dot{H}^s$ is continuously embedded into $\dot{B}^{s-\frac{d}{2}}_{\infty,\infty}$ with the inequality

$$\|f\|_{\dot{B}^{s-\frac{d}{2}}_{\infty,\infty}} \le \frac{C}{\sqrt{\frac{d}{2} - s}}\|f\|_{\dot{H}^s}, \qquad \forall f \in \dot{H}^s,$$

where C depends only on d and $\operatorname{Spt}\theta$.

Exercise 6.2.11 (Invariance of the homogeneous Besov norm). Under the hypothesis of Exercise 6.2.9 and 6.2.10. Assume $0 < \sigma \le d$. Prove that $\|\varphi_\varepsilon\|_{\dot{B}^{-\sigma}_{\infty,\infty}} \le C\varepsilon^\sigma$ for all $\varepsilon > 0$.

[Hint: When $A\varepsilon > 1$, the conclusion is straightforward. When $A\varepsilon \le 1$, use the fact $(-i\varepsilon\partial_1)^d e^{ix_1/\varepsilon} = e^{ix_1/\varepsilon}$ and integrate by parts d times.]

Exercise 6.2.12 (Refined critical embedding). Let $0 < s < \frac{d}{2}$. Prove that there exists a constant C, depending only on d and $\widehat{\theta}$, such that

$$\|f\|_{L^{2^*}} \le \frac{C}{(2^* - 2)^{\frac{1}{2^*}}}\|f\|^{1-\frac{2}{2^*}}_{\dot{B}^{s-\frac{d}{2}}_{\infty,\infty}}\|f\|^{\frac{2}{2^*}}_{\dot{H}^s}.$$

See more details in Bahouri-Chemin-Danchin [2, Remark 1.44].

(Hint: W.L.O.G assume the homogeneous Besov norm to be 1 and decompose $f = f_{\ell,A} + f_{h,A}$ as in Theorem 6.2.4 and so $\{|f| > \alpha\} \subseteq \{|f_{\ell,A}| > \alpha/2\} \cup \{|f_{h,A}| > \alpha/2\}$. Using the definition of Besov norm, one can see the first set has measure zero if A is suitably chosen. Then use the distribution function to express $\|f\|_{L^{2^*}}^{2^*}$ as in Theorem C.1.8 and use Chebyshev's inequality in Exercise C.1.6 with exponent $p = 2$.)

Exercise 6.2.13 (A refined density argument). If $s \le \frac{d}{2}$ ($< \frac{d}{2}$, resp.), prove that the space $C_c^\infty(\mathbb{R}^d\backslash\{\mathbf{0}\})$ is dense in $H^s(\mathbb{R}^d)$ ($\dot{H}^s(\mathbb{R}^d)$, resp.). If $s > \frac{d}{2}$, then the closure of $C_c^\infty(\mathbb{R}^d\backslash\{\mathbf{0}\})$ in $H^s(\mathbb{R}^d)$ is the set of functions u in $H^s(\mathbb{R}^d)$ such that $\partial^\alpha u(\mathbf{0}) = 0$ for all $|\alpha| < s - \frac{d}{2}$.

(Hint: Let $u_s := (\langle\boldsymbol{\xi}\rangle^{2s}\hat{u})^\vee$. If $(u_s, \varphi)_{L^2} = 0$ for all $\varphi \in C_c^\infty(\mathbb{R}^d\backslash\{\mathbf{0}\})$, then $\operatorname{Spt} u_s = \{\mathbf{0}\}$. Next use Exercise D.2.3(4).)

6.3 The mass-critical nonlinear Schrödinger equation

Let $u : [0,\infty)\times\mathbb{R}^d \to \mathbb{C}$ be a complex-valued function. We introduce the linear Schrödinger equation

$$i\partial_t u + \Delta u = 0 \text{ in } \mathbb{R}_+\times\mathbb{R}^d, \quad u(0,\boldsymbol{x}) = u_0(\boldsymbol{x}),\ \boldsymbol{x}\in\mathbb{R}^d. \tag{6.3.1}$$

The solution is easily solved with the help of Fourier transform

$$u(t,\boldsymbol{x}) = e^{it\Delta}u_0, \qquad e^{it\Delta}f := (e^{-it|\boldsymbol{\xi}|^2}\hat{f}(\boldsymbol{\xi}))^\vee = \frac{1}{(4i\pi t)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4it}} f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}, \tag{6.3.2}$$

which can also be viewed as the convolution of the fundamental solution Φ can the initial data u_0

$$\Phi(t,\boldsymbol{x}) := \frac{1}{(4i\pi t)^{\frac{d}{2}}}e^{-\frac{|\boldsymbol{x}|^2}{4it}}. \tag{6.3.3}$$

The linear Schrödinger equation enjoys the following invariance:

- (Scaling symmetry) If u solves (6.3.1), then for any $\lambda > 0$, $\lambda^{\frac{d}{2}}u(\lambda^2 t, \lambda\boldsymbol{x})$ solves (6.3.1) with initial data $\lambda^{\frac{d}{2}}u_0(\lambda\boldsymbol{x})$.
- (Galilean transformation) If u solves (6.3.1), then for any $\boldsymbol{\xi}_0\in\mathbb{R}^d$, $e^{-it|\boldsymbol{\xi}_0|^2}e^{i\boldsymbol{x}\cdot\boldsymbol{\xi}}u(t,\boldsymbol{x}-2t\boldsymbol{\xi}_0)$ solves (6.3.1) with initial data $e^{i\boldsymbol{x}\cdot\boldsymbol{\xi}_0}u_0(\boldsymbol{x})$.

What people are more interested in are the nonlinear Schrödinger equations (NLS)

$$i\partial_t u + \Delta u = F(u, \partial u, \cdots).$$

In particular, semilinear Schrödinger equations ($F = F(u)$) and quasi-linear Schrödinger equations ($F = F(u,\partial u)$) and their analogues (with Laplacian replaced by fractional-order Laplacian or Laplacian with certain potential terms) have appeared in many physical models. When considering the semilinear case, the nonlinearity is assumed to be

$$F(u) = \mu|u|^{p-1}u, \ \ \mu = \pm 1, \ \ p \ge 1.$$

When $\mu = 1$, we say the NLS is defocusing; when $\mu = -1$, we say the NLS is focusing. The semilinear NLS $i\partial_t u + \Delta u = \pm|u|^{p-1}u$ satisfies the following conservation laws

- (Mass conservation) $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} |u(t,\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x} = 0$.
- (Energy conservation) $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \frac{1}{2}|\nabla u(t,\boldsymbol{x})|^2 \pm \frac{1}{p+1}|u(t,\boldsymbol{x})|^{p+1} \,\mathrm{d}\boldsymbol{x} = 0$.
- (Momentum conservation) $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{Im} \int_{\mathbb{R}^d} u(t,\boldsymbol{x})\overline{\nabla u(t,\boldsymbol{x})} \,\mathrm{d}\boldsymbol{x} = 0$.

When $p - 1 = \frac{4}{d}$, we say the NLS is **mass-critical**. This is due to the fact that the $L^2(\mathbb{R}^d)$ norm of the initial data stays invariant under the scaling symmetry above. When $p-1 = \frac{4}{d-2}$ $(d \geq 3)$, we say the NLS is **energy-critical** because the power of the nonlinear term in the energy $E(t)$ is the critical exponent for the Sobolev embedding.

In this section, we aim to prove the global well-posedness and scattering of the mass-critical NLS

$$i\partial_t u + \Delta u = \pm|u|^{\frac{4}{d}}u \text{ in } \mathbb{R}_+ \times \mathbb{R}^d, \quad u(0,\boldsymbol{x}) = u_0(\boldsymbol{x}),\ \boldsymbol{x} \in \mathbb{R}^d. \tag{6.3.4}$$

6.3.1 Decay estimates and Strichartz estimates of Schrödinger equation

For the non-homogeneous Schrödinger equation

$$i\partial_t u + \Delta u = F \text{ in } \mathbb{R}_+ \times \mathbb{R}^d, \quad u(0,\boldsymbol{x}) = u_0(\boldsymbol{x}),\ \boldsymbol{x} \in \mathbb{R}^d, \tag{6.3.5}$$

Duhamel's principle gives us the solution

$$u(t,\boldsymbol{x}) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-\tau)\Delta}F(\tau)\,\mathrm{d}\tau. \tag{6.3.6}$$

To prove the existence of NLS, we seek for the decay estimate (pointwise in t variable) and the space-time ($L_t^p L_{\boldsymbol{x}}^r$-type) for the Schrödinger semigroup $e^{it\Delta}$.

First, we prove the decay estimate

Proposition 6.3.1 (Decay estimate of $e^{it\Delta}$). Let $1 \leq p \leq 2$ and $f \in \mathcal{S}(\mathbb{R}^d)$. Then

$$\|e^{it\Delta}f\|_{L^{p'}(\mathbb{R}^d)} \leq Ct^{d(\frac{1}{2}-\frac{1}{p})}\|f\|_{L^p(\mathbb{R}^d)}.$$

Proof. Let $Tf = e^{it\Delta}f$. By Plancherel's identity, we know $\|Tf\|_{L^2} = \|f\|_{L^2}$. Then by Young's inequality for convolution, we know

$$\|Tf\|_{L^\infty} = \|\Phi * f\|_{L^\infty} \leq \|\Phi\|_{L^\infty}\|f\|_{L^1} \leq Ct^{-\frac{d}{2}}\|f\|_{L^1}.$$

Therefore, using Riesz-Thorin interpolation theorem (Theorem C.3.5), we know T is bounded from

L^p to L^q for $1 \le p \le 2$ with the estimate

$$\|Tf\|_{L^q} \le C' t^{-\frac{d\theta}{2}} \|f\|_{L^p}, \quad \frac{1}{p} = \frac{\theta}{1} + \frac{1-\theta}{2}, \quad \frac{1}{q} = \frac{\theta}{\infty} + \frac{1-\theta}{2}.$$

This exactly gives us the desired result as $\theta = \frac{2}{p} - 1$. □

Next, we introduce the space-time estimates for the non-homogeneous part. By Christ-Kiselev lemma [4] (see also the argument in [5, Lemma 1.10] by Grillakis and Machedon), it is equivalent to derive the estimates for $\int_{\mathbb{R}} \cdots \,\mathrm{d}\tau$ instead of $\int_0^t \cdots \,\mathrm{d}\tau$. We then want to find suitable requirements for exponents (p, q) and $(\tilde{p}, \tilde{q})$ such that

$$\left\| \int_{\mathbb{R}} e^{i(t-\tau)\Delta} f(\tau) \,\mathrm{d}\tau \right\|_{L_t^p L_x^q} \le C \|f\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}}.$$

Since $p, q \ge 1$, we use the Minkowski's inequality for integrals to see

$$\left\| \int_{\mathbb{R}} e^{i(t-\tau)\Delta} f(\tau) \,\mathrm{d}\tau \right\|_{L_t^p L_x^q} \le \left\| \int_{\mathbb{R}} \|e^{i(t-\tau)\Delta} f(\tau)\|_{L_x^q} \,\mathrm{d}\tau \right\|_{L_t^p}.$$

Then using the decay estimate, we have

$$\left\| \int_{\mathbb{R}} \|e^{i(t-\tau)\Delta} f(\tau)\|_{L_x^q} \,\mathrm{d}\tau \right\|_{L_t^p} \le C \left\| \int_{\mathbb{R}} |t-\tau|^{-d(\frac{1}{2}-\frac{1}{q})} \|f(\tau)\|_{L^{q'}} \,\mathrm{d}\tau \right\|_{L_t^p}.$$

The integrand now is a convolution. By Hardy-Littlewood-Sobolev inequality, we wish to get

$$\left\| |\cdot|^{-d(\frac{1}{2}-\frac{1}{q})} * \|f(\cdot)\|_{L^{q'}} \right\|_{L_t^p} \le C \|f\|_{L_t^{p'} L_x^{q'}}.$$

The exponents should satisfy

$$0 < \gamma := d(\frac{1}{2} - \frac{1}{q}) < 1, \qquad 1 + \frac{1}{p} = \frac{1}{q'} + \frac{\gamma}{d} \Rightarrow \frac{2}{p} = d\left(\frac{1}{2} - \frac{1}{q}\right).$$

This motivates us to define the so-called "admissible pairs".

Definition 6.3.1 (Admissible pair). (p, q) is called an **admissible pair**, if

$$\frac{2}{p} = d\left(\frac{1}{2} - \frac{1}{q}\right)$$

and

- $4 \le p \le \infty$, if $d = 1$;
- $2 < p \le \infty$, if $d = 2$;
- $2 \le p \le \infty$, if $d \ge 3$.

Moreover, if $p > 2$, then we call (p, q) a **non-endpoint admissible pair**; if $p = 2$, then we call $(2, q)$ an **endpoint admissible pair**.

Now, we can introduce the **Strichartz estimates** for the Schrödinger semigroup $e^{it\Delta}$.

Proposition 6.3.2 (Strichartz estimates). Let $(p, q), (\tilde{p}, \tilde{q})$ be two admissible pairs and $t > 0, f \in \mathcal{S}$. Then the following estimates hold

$$\|e^{it\Delta} f\|_{L_t^p L_x^q} \leq C \|f\|_{L^2}. \tag{6.3.7}$$

$$\left\| \int_0^t e^{-i\tau\Delta} f(\tau) \,\mathrm{d}\tau \right\|_{L_x^2} \leq C \|f\|_{L_t^{p'} L_x^{q'}}, \tag{6.3.8}$$

$$\left\| \int_0^t e^{i(t-\tau)\Delta} f(\tau) \,\mathrm{d}\tau \right\|_{L_t^p L_x^q} \leq C \|f\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}}. \tag{6.3.9}$$

Below, we only prove the estimates for non-endpoint admission pairs. The endpoint case is referred to the famous paper:

Markus Keel, Terence Tao: Endpoint Strichartz Estimates. *American Journal of Mathematics*, 120(5), 956-980, 1998.

The proof for the non-endpoint case relies on the so-called TT^*-argument, that is, to estimate the operator norm of T, it suffices to find the operator norm of TT^* which is often easier for NLS, as we already obtain the estimate

$$\left\| \int_{\mathbb{R}} e^{i(t-\tau)\Delta} f(\tau) \,\mathrm{d}\tau \right\|_{L_t^p L_x^q} \leq C \|f\|_{L_t^{p'} L_x^{q'}}. \tag{6.3.10}$$

Proof. Again, by Christ-Kiselev lemma [4], it suffices to show the estimates for $\mathbb{R} \times \mathbb{R}^d$. We first prove (6.3.8). Since L^2 is a Hilbert space, we have

$$\begin{aligned}
&\left\| \int_{\mathbb{R}} e^{-i\tau\Delta} f(\tau) \,\mathrm{d}\tau \right\|_{L_x^2}^2 = \left(\int_{\mathbb{R}} e^{-i\tau\Delta} f(\tau) \,\mathrm{d}\tau, \int_{\mathbb{R}} e^{-it\Delta} f(t) \,\mathrm{d}t \right)_{L_x^2} \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \left(e^{-i\tau\Delta} f(\tau), e^{-it\Delta} f(t) \right)_{L_x^2} \mathrm{d}\tau \,\mathrm{d}t = \int_{\mathbb{R}} \left(f(\tau), \int_{\mathbb{R}} e^{i(\tau-t)\Delta} f(t) \,\mathrm{d}t \right)_{L_x^2} \mathrm{d}\tau \\
&\leq \|f\|_{L_t^{p'} L_x^{q'}} \left\| \int_{\mathbb{R}} e^{i(\tau-t)\Delta} f(t) \,\mathrm{d}t \right\|_{L_t^p L_x^q} \leq C \|f\|_{L_t^{p'} L_x^{q'}}^2.
\end{aligned}$$

Then, (6.3.7) can be proved with the help of (6.3.8) and the duality representation of L^p norm (C.1.1).

$$
\begin{aligned}
\|e^{it\Delta} f\|_{L_t^p L_{\boldsymbol{x}}^q} &= \sup_{\|\varphi\|_{L_t^{p'} L_{\boldsymbol{x}}^{q'}} \le 1} \left| \int_{\mathbb{R}} \int_{\mathbb{R}^d} e^{it\Delta} f(t,\boldsymbol{x}) \overline{\varphi(t,\boldsymbol{x})} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \right| \\
&= \sup_{\|\varphi\|_{L_t^{p'} L_{\boldsymbol{x}}^{q'}} \le 1} \left| \int_{\mathbb{R}} \left(e^{it\Delta} f, \varphi \right)_{L_{\boldsymbol{x}}^2} \mathrm{d}t \right| = \sup_{\|\varphi\|_{L_t^{p'} L_{\boldsymbol{x}}^{q'}} \le 1} \left| \left(f, \int_{\mathbb{R}} e^{it\Delta} \varphi \, \mathrm{d}t \right)_{L_{\boldsymbol{x}}^2} \right| \\
&\le \sup_{\|\varphi\|_{L_t^{p'} L_{\boldsymbol{x}}^{q'}} \le 1} \|f\|_{L^2} \left\| \int_{\mathbb{R}} e^{it\Delta} \varphi \, \mathrm{d}t \right\|_{L^2} \le C \|f\|_{L^2}
\end{aligned}
$$

where we use (6.3.8) in the last but two inequality.

For (6.3.9), we also use the duality representation to find that

$$
\begin{aligned}
&\left\| \int_{\mathbb{R}} e^{i(t-\tau)\Delta} f(\tau) \, \mathrm{d}\tau \right\|_{L_t^p L_{\boldsymbol{x}}^q} = \sup_{\|\varphi\|_{L_t^{p'} L_{\boldsymbol{x}}^{q'}} \le 1} \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \left(e^{i(t-\tau)\Delta} f(\tau), \varphi(t,\cdot) \right)_{L_{\boldsymbol{x}}^2} \mathrm{d}t \, \mathrm{d}\tau \right| \\
&= \sup_{\|\varphi\|_{L_t^{p'} L_{\boldsymbol{x}}^{q'}} \le 1} \left| \left(\int_{\mathbb{R}} e^{-i\tau\Delta} f(\tau) \, \mathrm{d}\tau, \int_{\mathbb{R}} e^{-it\Delta} \varphi(t,\cdot) \, \mathrm{d}t \right)_{L_{\boldsymbol{x}}^2} \right| \le \sup_{\|\varphi\|_{L_t^{p'} L_{\boldsymbol{x}}^{q'}} \le 1} \left\| \int_{\mathbb{R}} e^{i\tau\Delta} f(\tau) \, \mathrm{d}\tau \right\|_{L_{\boldsymbol{x}}^2} \left\| \int_{\mathbb{R}} e^{it\Delta} \varphi(t) \, \mathrm{d}t \right\|_{L_{\boldsymbol{x}}^2} \\
&\le \sup_{\|\varphi\|_{L_t^{p'} L_{\boldsymbol{x}}^{q'}} \le 1} \|f\|_{L_t^{\tilde{p}'} L_{\boldsymbol{x}}^{\tilde{q}'}} \|\varphi\|_{L_t^{p'} L_{\boldsymbol{x}}^{q'}} \le C \|f\|_{L_t^{\tilde{p}'} L_{\boldsymbol{x}}^{\tilde{q}'}}.
\end{aligned}
$$

□

6.3.2 Small-data global well-posedness and scattering for mass-critical NLS

Now, we to turn to prove the small-data global well-posedness and scattering of the mass-critical NLS.

$$
i\partial_t u + \Delta u = \pm |u|^{\frac{4}{d}} u (=: F(u)), \quad u(0,\boldsymbol{x}) = u_0(\boldsymbol{x}). \tag{6.3.11}
$$

Theorem 6.3.3. For any $d \ge 1$, there exists $\varepsilon_0(d) > 0$ such that if $\|u_0\|_{L^2(\mathbb{R}^d)} \le \varepsilon_0$, then (6.3.11) is globally well-posed in $L_{t,\boldsymbol{x}}^{\frac{2(d+2)}{d}}(\mathbb{R} \times \mathbb{R}^d)$ and scatters in $L^2(\mathbb{R}^d)$.

Roughly speaking, global well-posedness and scattering means that (6.3.11) has a solution, and the solution is close to a solution to the linear Schrödinger equation as $t \to \pm\infty$.

Definition 6.3.2 (Well-posedness). An initial value problem is said to be well posed on an interval $I, 0 \in I \subset \mathbb{R}$, if

- there exists a unique solution to the initial value problem,
- the solution is continuous in time,
- the solution depends continuously on the initial data.

Definition 6.3.3 (Scattering). A solution to NLS is said to scatter forward in time if the solution exists on $[0, \infty)$ and there exists u_+such that

$$u(t) - e^{it\Delta}u_+ \to 0$$

as $t \to +\infty$. A solution to NLS is said to scatter backward in time if the solution exists on $(-\infty, 0]$ and there exists u_-such that

$$u(t) - e^{it\Delta}u_- \to 0$$

as $t \to -\infty$. An initial value problem with initial data in some set is said to be scattering if the problem is globally well posed, scatters both forward and backward in time, and u_+ and u_- depend continuously on the initial data.

Proof. We first prove the existence theorem. For any $d \geq 1$ we set $p = q$ for simplicity. In order for (p, q) to be an admissible pair, we have $p = q = \frac{2(d+2)}{d}$, so let X be the set of functions

$$X = \left\{ u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C} : \|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbb{R}\times\mathbb{R}^d)} \leq C\varepsilon_0 \right\}.$$

for some constant C. By Strichartz estimates, there exists some constant $C(d)$ such that

$$\left\| e^{it\Delta}u_0 \right\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbb{R}\times\mathbb{R}^d)} \leq C\varepsilon_0.$$

Now we define the map

$$\Phi(u)(t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-\tau)\Delta}F(u(\tau))\,\mathrm{d}\tau.$$

If $u \in X$ satisfies

$$\Phi(u)(t) = u(t)$$

then u solves (6.3.11). By the contraction mapping principle, to prove the existence of a unique solution to (6.3.11) in X, it suffices to prove $\Phi(X) \subseteq X$ and Φ is a contraction on X.

By Strichartz estimates for $p = q = \frac{2(d+2)}{d}$ and Hölder's inequality, if $u \in X$, then

$$\begin{aligned}
\left\| \int_0^t e^{i(t-\tau)\Delta}F(u(\tau))\,\mathrm{d}\tau \right\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbb{R}\times\mathbb{R}^d)} &\leq C\|F(u)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(\mathbb{R}\times\mathbb{R}^d)} \\
&\leq C\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbb{R}\times\mathbb{R}^d)}^{1+\frac{4}{d}} \\
&\leq (C\varepsilon_0)^{1+\frac{4}{d}}.
\end{aligned}$$

Hence, choosing ε_0 sufficiently small, we get $\Phi(X) \subseteq X$.

To prove Φ is a contraction, for $u, v \in X$, we compute that

$$\begin{aligned}\|\Phi(u) - \Phi(v)\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbb{R}\times\mathbb{R}^d)} &\leq C\|F(u) - F(v)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(\mathbb{R}\times\mathbb{R}^d)} \\ &\leq C\left\||u|^{\frac{4}{d}} + |v|^{\frac{4}{d}}\right\|_{L_{t,x}^{\frac{d+2}{2}}(\mathbb{R}\times\mathbb{R}^d)} \|u - v\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbb{R}\times\mathbb{R}^d)}.\end{aligned}$$

Here we use the simple fact that

$$||u|^\alpha u - |v|^\alpha v| \leq (1 + \alpha)(|u|^\alpha + v^\alpha)|u - v|.$$

For $\varepsilon_0 > 0$ sufficiently small,

$$\|\Phi(u) - \Phi(v)\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbb{R}\times\mathbb{R}^d)} \leq \frac{1}{2}\|u - v\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbb{R}\times\mathbb{R}^d)}$$

proving Φ is a contraction. Therefore, there exists a unique $u \in X$ as the fixed point of Φ, which solves the mass-critical NLS (6.3.11). The uniqueness and continuous dependence on initial data can be proved in the same way, so we omit the proof here.

Finally, we prove the scattering.Set

$$u_+ = u_0 - i\int_0^\infty e^{-it\Delta}F(u(t))\,\mathrm{d}t$$

and

$$u_- = u_0 + i\int_{-\infty}^0 e^{-it\Delta}F(u(t))\,\mathrm{d}t$$

Again by Strichartz estimates and Christ-Kiselev lemma [4], $u_+, u_- \in L_x^2(\mathbb{R}^d)$ are well-defined. Additionally, by the dominated convergence theorem,

$$\lim_{T\to\infty} \|F(u)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}([T,\infty)\times\mathbb{R}^d)} = 0.$$

Therefore,

$$\left\|e^{iT\Delta}u_+ - u(T)\right\|_{L_x^2(\mathbb{R}^d)} = \left\|\int_T^\infty e^{i(t-\tau)\Delta}F(u(\tau))d\tau\right\|_{L_x^2(\mathbb{R}^d)} \to 0$$

This proves that each initial data function $\|u_0\|_{L^2} \leq \varepsilon_0$ has a solution that is global in time and scatters forward in time. The proof that the solution also scatters backward in time is identical. One can also prove that u_+ and u_- depend continuously on the initial data and we skip the proof here. $\square$

6.3.3 Virial identity of NLS

When $\mu = -1$ in NLS

$$i\partial_t u + \Delta u = \mu|u|^{p-1}u, \quad u(0,\boldsymbol{x}) = u_0(\boldsymbol{x}), \quad \mu = \pm 1, \tag{6.3.12}$$

that is, for focusing NLS, the (conserved) energy

$$E(t) := \int_{\mathbb{R}^d} \frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1}\,\mathrm{d}\boldsymbol{x}$$

might be negative. In such case, we can prove the finite-time blow up with the help of **Virial identity**. In this section, we again consider the mass-critical problem, that is, $p-1=\frac{4}{d}$. It should be noted that this result does not violate the small-data global well-posedness.

We define the **Virial potential** by

$$V(t) := \int_{\mathbb{R}^d} |\boldsymbol{x}|^2|u(t,\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}. \tag{6.3.13}$$

It is easy to see that $V(t)$ must be non-negative. On the other hand, we can prove that, for any solution u to (6.3.12) with $\boldsymbol{x}u \in L^2(\mathbb{R}^d)$.

Proposition 6.3.4 (Virial identity of NLS). Assume u to be a smooth solution to (6.3.12) and $u_0 \in H^1(\mathbb{R}^d)$, $|\boldsymbol{x}|u_0 \in L^2(\mathbb{R}^d)$.

$$V''(t) = 8\int_{\mathbb{R}^d} |\nabla u|^2 \pm 4d\int_{\mathbb{R}^d}\left(1-\frac{2}{p+1}\right)|u|^{p+1}\,\mathrm{d}\boldsymbol{x}. \tag{6.3.14}$$

In particular, if we set $p-1=\frac{4}{d}$, then we get $V''(t) = 16E(t)$. Therefore, we conclude the following theorem.

Theorem 6.3.5. Assume $u : [0,T_*)\times\mathbb{R}^d \to \mathbb{C}$ to be a smooth solution to the focusing, mass-critical NLS ((6.3.11) with $\mu=-1$) and $u_0 \in H^1(\mathbb{R}^d)$, $|\boldsymbol{x}|u_0 \in L^2(\mathbb{R}^d)$. Then $T_* < \infty$ if one of the following holds

(a) $E(0) < 0$;
(b) $E(0) = 0$, $V'(0) < 0$;
(c) $E(0) > 0$, $(V'(0))^2 - 32E(0)V(0) > 0$.

It should be noted that this result does not violate the small-data global well-posedness in Theorem 6.3.3, as the latter one does not require the initial data to be H^1 and $|\boldsymbol{x}|u_0 \in L^2$.

Proof of "Prop. 6.3.4 ⇒ Theorem 6.3.5". For the focusing, mass-critical NLS, we already have $V''(t) = 16E(t) = 16E(0)$ thanks to the energy conservation. Then by the fundamental theorem of calculus, we get

$$V(t) = V(0) + V'(0)t + 8E(0)t^2,$$

which is a constant-coefficient quadratic polynomial of t. Thus, if any one of (a)-(c) holds, we get $V(t) \to -\infty$ as $t \to \infty$, which violates the non-negativity of $V(t)$. So, T_* must be finite if any one of (a)-(c) holds. $\square$

Now, it remains to prove the Virial identity (6.3.14).

Proof of Prop. 6.3.4. **Step 1: Compute $V'(t)$.** Notice that u is a complex-valued function, so $|u|^2 = u\bar{u}$ and $|\nabla u|^2 = \nabla u \cdot \overline{\nabla u}$. We will repeatedly use the facts such as $2\mathbf{Re}\,(z\bar{w}) = z\bar{w} + \bar{z}w$, $\mathbf{Re}\,(iz) = -\mathbf{Im}\,(z)$, etc, for $z, w \in \mathbb{C}$. Now, we have

$$
\begin{aligned}
V'(t) &= 2\mathbf{Re} \int_{\mathbb{R}^d} |\boldsymbol{x}|^2 \bar{u} \partial_t u \\
&= 2\mathbf{Re} \int_{\mathbb{R}^d} i|\boldsymbol{x}|^2 \bar{u} \Delta u \, \mathrm{d}\boldsymbol{x} \pm \underbrace{2\mathbf{Re} \int_{\mathbb{R}^d} i|\boldsymbol{x}|^2 |u|^{p+1} \, \mathrm{d}\boldsymbol{x}}_{=0} \\
&= -2\mathbf{Im} \int_{\mathbb{R}^d} |\boldsymbol{x}|^2 (\bar{u}\Delta u + \underbrace{|\nabla u|^2}_{\mathbf{Im}\,|\nabla u|^2=0}) \, \mathrm{d}\boldsymbol{x} = -2\mathbf{Im} \int_{\mathbb{R}^d} |\boldsymbol{x}|^2 \nabla \cdot (\bar{u} \nabla u) \, \mathrm{d}\boldsymbol{x} \\
&= 2\mathbf{Im} \int_{\mathbb{R}^d} \underbrace{\nabla(|\boldsymbol{x}|^2)}_{=2\boldsymbol{x}} \cdot (\bar{u}\nabla u) \, \mathrm{d}\boldsymbol{x} = 4\mathbf{Im} \int_{\mathbb{R}^d} (\boldsymbol{x} \cdot \nabla) u \, \bar{u} \, \mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

Step 2: Compute $V''(t)$. Taking one more time derivative and integrating by parts, we get

$$
\begin{aligned}
V''(t) &= 4\mathbf{Im} \int_{\mathbb{R}^d} (\boldsymbol{x} \cdot \nabla) u \, \overline{\partial_t u} \, \mathrm{d}\boldsymbol{x} + 4\mathbf{Im} \int_{\mathbb{R}^d} (\boldsymbol{x} \cdot \nabla) \partial_t u \, \bar{u} \, \mathrm{d}\boldsymbol{x} \\
&= 4\mathbf{Im} \int_{\mathbb{R}^d} (\boldsymbol{x} \cdot \nabla) u \, \overline{\partial_t u} \, \mathrm{d}\boldsymbol{x} - 4\mathbf{Im} \int_{\mathbb{R}^d} \partial_t u \, (\boldsymbol{x} \cdot \nabla) \bar{u} \, \mathrm{d}\boldsymbol{x} - 4\mathbf{Im} \int_{\mathbb{R}^d} (\nabla \cdot \boldsymbol{x}) \partial_t u \, \bar{u} \, \mathrm{d}\boldsymbol{x} \\
&= -8\mathbf{Im} \int_{\mathbb{R}^d} \partial_t u \, (\boldsymbol{x} \cdot \nabla) \bar{u} \, \mathrm{d}\boldsymbol{x} - 4d\mathbf{Im} \int_{\mathbb{R}^d} \partial_t u \, \bar{u} \, \mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

Now, we write $J_1 := \mathbf{Im} \int_{\mathbb{R}^d} \partial_t u \, (\boldsymbol{x} \cdot \nabla) \bar{u} \, \mathrm{d}\boldsymbol{x}$ and $J_2 := \mathbf{Im} \int_{\mathbb{R}^d} \partial_t u \, \bar{u} \, \mathrm{d}\boldsymbol{x}$. Then $V''(t) = -8J_1 - 4dJ_2$. The term J_2 is easy to compute after inserting the equation and integrating by parts.

$$
J_2 = -\mathbf{Re} \int_{\mathbb{R}^d} \bar{u}(i\partial_t u) \, \mathrm{d}\boldsymbol{x} = \mathbf{Re} \int_{\mathbb{R}^d} \bar{u}\Delta u \mp |u|^{p+1} \, \mathrm{d}\boldsymbol{x} = -\mathbf{Re} \int_{\mathbb{R}^d} |\nabla u|^2 \pm |u|^{p+1} \, \mathrm{d}\boldsymbol{x}.
$$

For J_1, we also insert the NLS equation to get

$$
\begin{aligned}
J_1 =& \mathbf{Im} \int_{\mathbb{R}^d} (i\Delta u \mp i|u|^{p-1} u) \, (\boldsymbol{x} \cdot \nabla) \bar{u} \, \mathrm{d}\boldsymbol{x} \\
=& \mathbf{Re} \int_{\mathbb{R}^d} \Delta u \, (\boldsymbol{x} \cdot \nabla) \bar{u} \, \mathrm{d}\boldsymbol{x} \mp \mathbf{Re} \int_{\mathbb{R}^d} |u|^{p-1} u \, (\boldsymbol{x} \cdot \nabla) \bar{u} \, \mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

Now, we write $K_1 := \mathbf{Re} \int_{\mathbb{R}^d} \Delta u\,(\boldsymbol{x}\cdot\nabla)\bar{u}\,\mathrm{d}\boldsymbol{x}$ and $K_2 := \mathbf{Re} \int_{\mathbb{R}^d} |u|^{p-1}u\,(\boldsymbol{x}\cdot\nabla)\bar{u}\,\mathrm{d}\boldsymbol{x}$. Then $J_1 = K_1 \mp K_2$. In K_2, we introduce $g(x) = x^{\frac{p-1}{2}}$ and its anti-derivative $G(x) := \frac{2}{p+1}x^{\frac{p+1}{2}}$, and so K_2 can be easily computed after integrating by parts.

$$
\begin{aligned}
K_2 &= \frac{1}{2}\mathbf{Re}\int_{\mathbb{R}^d}(\boldsymbol{x}\cdot\nabla)(|u|^2)g(|u|^2)\,\mathrm{d}\boldsymbol{x} = \frac{1}{2}\mathbf{Re}\int_{\mathbb{R}^d}(\boldsymbol{x}\cdot\nabla)G(|u|^2)\,\mathrm{d}\boldsymbol{x}\\
&= -\frac{1}{2}\int_{\mathbb{R}^d}(\nabla\cdot\boldsymbol{x})G(|u|^2)\,\mathrm{d}\boldsymbol{x} = -\frac{d}{p+1}\int_{\mathbb{R}^d}|u|^{p+1}\,\mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

For K_1, we integrate by parts to see

$$
\begin{aligned}
K_1 &= -\sum_{j,k}\mathbf{Re}\int_{\mathbb{R}^d}\partial_j u\,\partial_j(x_k\partial_k\bar{u})\,\mathrm{d}\boldsymbol{x}\\
&= -\sum_{j,k}\mathbf{Re}\int_{\mathbb{R}^d}\partial_j u\,\delta_{jk}\partial_k\bar{u}\,\mathrm{d}\boldsymbol{x} - \mathbf{Re}\int_{\mathbb{R}^d}\partial_j u\,x_k\partial_k\partial_j\bar{u}\,\mathrm{d}\boldsymbol{x}\\
&= -\int_{\mathbb{R}^d}|\nabla u|^2\,\mathrm{d}\boldsymbol{x} - \frac{1}{2}\int_{\mathbb{R}^d}(\boldsymbol{x}\cdot\nabla)(|\nabla u|^2)\,\mathrm{d}\boldsymbol{x}\\
&= -\left(1-\frac{d}{2}\right)\int_{\mathbb{R}^d}|\nabla u|^2\,\mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

Summing up the above quantities, we get

$$
J_1 = -\left(1-\frac{d}{2}\right)\int_{\mathbb{R}^d}|\nabla u|^2\,\mathrm{d}\boldsymbol{x} \pm \frac{d}{p+1}\int_{\mathbb{R}^d}|u|^{p+1}\,\mathrm{d}\boldsymbol{x}.
$$

Therefore,

$$
V''(t) = -8J_1 - 4dJ_2 = 8\int_{\mathbb{R}^d}|\nabla u|^2 \pm 4d\int_{\mathbb{R}^d}\left(1-\frac{2}{p+1}\right)|u|^{p+1}\,\mathrm{d}\boldsymbol{x}.
$$

□

Exercise 6.3

Exercise 6.3.1. Prove the conservation laws of mass, energy and momentum for the nonlinear Schrödinger equation (6.3.12).

- (Mass conservation) $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}|u(t,\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x} = 0$.
- (Energy conservation) $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\frac{1}{2}|\nabla u(t,\boldsymbol{x})|^2 \pm \frac{1}{p+1}|u(t,\boldsymbol{x})|^{p+1}\,\mathrm{d}\boldsymbol{x} = 0$.
- (Momentum conservation) $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{Im}\int_{\mathbb{R}^d}u(t,\boldsymbol{x})\overline{\nabla u(t,\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x} = 0$.

Exercise 6.3.2 (Pseudo-conformal conservation law). If u solves $i\partial_t u + \Delta u = |u|^{\frac{4}{d}}u$ with initial data

$u_0(\boldsymbol{x})$, then prove that the following quantity is conserved in time.

$$\|(x+2it\nabla)u(t)\|^2_{L^2(\mathbb{R}^d)}+\frac{8t^2}{p+2}\int_{\mathbb{R}^d}|u(t,\boldsymbol{x})|^{p+2}\,\mathrm{d}\boldsymbol{x},\quad p=\frac{4}{d}.$$

6.4 *Littlewood-Paley characterization of Sobolev spaces

Many nonlinear evolution equations can be analyzed by viewing them as describing the oscillation and interaction between low, medium, and high frequencies. To make this type of analysis rigorous, we of course need the notation and tools of harmonic analysis, and in particular the Fourier transform. However, merely taking Fourier transform does not distinguish the behaviors of different frequncy bands, which are actually pretty important when treating the derivatives. Therefore, a natural idea is to find a suitable decomposition, actually a partition of unity, in the frequency space. This is the so-called **Littlewood-Paley theory**.

Let $\varphi(\boldsymbol{\xi})$ be a real-valued radial-symmetric C_c^∞ function supported in the ball $\{|\boldsymbol{\xi}|\leq 2\}$ which equals 1 on the ball $\{|\boldsymbol{\xi}|\leq 1\}$. The exact choice of bump function turns out to be not important and here we use $1,2$ instead of any other radius of the ball because 1 and 2 are both dyadic numbers. Now, we define the Littlewood-Paley projections.

Definition 6.4.1. Let φ be defined above, $N\in\mathbb{Z}$ and $f\in\mathcal{S}$. Set $\psi(\boldsymbol{\xi})=\varphi(\boldsymbol{\xi})-\varphi(2\boldsymbol{\xi})$ and $\varphi_j(\boldsymbol{\xi})=\varphi(\boldsymbol{\xi}/2^j),\psi_j(\boldsymbol{\xi})=\psi(\boldsymbol{\xi}/2^j)$ for $j\in\mathbb{Z}$. Then we define the **Littlewood-Paley projections** as follows:

$$P_Nf:=(\psi_N(\boldsymbol{\xi})\hat{f}(\boldsymbol{\xi}))^\vee,\quad P_{\leq N}f:=(\varphi_N(\boldsymbol{\xi})\hat{f}(\boldsymbol{\xi}))^\vee,\quad P_{\geq N}f:=\left((1-\varphi_N(\boldsymbol{\xi}))\hat{f}(\boldsymbol{\xi})\right)^\vee.$$

In other words, P_N localizes f near the frequency band $|\boldsymbol{\xi}|\sim 2^N$. One can easily show that $\sum\limits_{N\in\mathbb{Z}}\psi_N(\boldsymbol{\xi})=1$ for all $\boldsymbol{\xi}\neq\mathbf{0}$. Also, we have $\operatorname{Spt}\psi_N\cap\operatorname{Spt}\psi_M=\emptyset$ for all $N,M\in\mathbb{Z}$ with $|N-M|\geq 2$. Therefore, $\{\psi_N\}$ actually gives a partition of unity of the frequency space.

6.4.1 Bernstein-type inequalities

Now we introduce the Bernstein-type inequalities which present the basic properties of Littlewood-Paley projections.

Proposition 6.4.1 (Bernstein-type inequalities). Let $N\in\mathbb{Z}$, $s\geq 0$ and $1\leq p\leq q\leq\infty$. Then in $\mathbb{R}^d$ we have

$$\|P_{\geq N}f\|_{L^p} \leq C(p,s,d)2^{-Ns}\||\nabla|^s P_{\geq N}f\|_{L^p}, \tag{6.4.1}$$
$$\|P_{\leq N}|\nabla|^s f\|_{L^p} \leq C(p,s,d)2^{Ns}\|P_{\leq N}f\|_{L^p}, \tag{6.4.2}$$
$$\|P_N|\nabla|^{\pm s}f\|_{L^p} \simeq C(p,s,d)2^{\pm Ns}\|P_N f\|_{L^p}, \tag{6.4.3}$$
$$\|P_{\leq N}f\|_{L^q} \leq C(p,q,d)2^{Nd(\frac{1}{p}-\frac{1}{q})}\|P_{\leq N}f\|_{L^p}, \tag{6.4.4}$$
$$\|P_N f\|_{L^q} \leq C(p,q,d)2^{Nd(\frac{1}{p}-\frac{1}{q})}\|P_N f\|_{L^p} \tag{6.4.5}$$

Proof. The proof are quite straightforward consequences of Young's inequality for convolution (Theorem C.3.6). Here we only prove (3) and (5).

For (3), we have $P_N f = \tilde{P}_N(P_N f)$ with $\tilde{P}_N := P_{N-2} + P_{N-1} + \cdots + P_{N+2}$. (Notice that $\psi_{N-2} + \cdots + \psi_{N+2} = 1$ on $\operatorname{Spt}\psi_N$.) So, we know $P_N(|\nabla|^{\pm s}f) = \tilde{P}_N(|\nabla|^{\pm s}P_N f) = (\tilde{\psi}_N(\boldsymbol{\xi})|\boldsymbol{\xi}|^s\widehat{P_N f}(\boldsymbol{\xi}))^\vee = (\psi_N(\boldsymbol{\xi})|\boldsymbol{\xi}|^s)^\vee * f$. Thus, Young's inequality for convolution leads to

$$\|P_N(|\nabla|^{\pm s}f)\|_{L^p} \leq \|(\psi_N(\boldsymbol{\xi})|\boldsymbol{\xi}|^s)^\vee\|_{L^1}\|P_N f\|_{L^p}.$$

Now, we compute

$$\begin{aligned}(\psi_N(\boldsymbol{\xi})|\boldsymbol{\xi}|^s)^\vee(\boldsymbol{x}) =& C\int_{\mathbb{R}^d} e^{i\boldsymbol{x}\cdot\boldsymbol{\xi}}\psi(2^{-N}\boldsymbol{\xi})|\boldsymbol{\xi}|^s \,\mathrm{d}\boldsymbol{\xi} \\ \overset{\boldsymbol{\xi}=2^N\boldsymbol{\eta}}{=\!=\!=}& C2^{Nd}\int_{\mathbb{R}^d} e^{i\boldsymbol{x}\cdot 2^N\boldsymbol{\eta}}\psi(\boldsymbol{\eta})2^{Ns}|\boldsymbol{\eta}|^s \,\mathrm{d}\boldsymbol{\eta} = C2^{N(d+s)}\check{\psi}(2^N\boldsymbol{x}),\end{aligned}$$

and so

$$\|(\psi_N(\boldsymbol{\xi})|\boldsymbol{\xi}|^s)^\vee\|_{L^1} \leq C2^{N(d+s)}\|\check{\psi}(2^N\cdot)\|_{L^1} = C2^{Ns}\|\check{\psi}\|_{L^1} \leq C2^{Ns}.$$

The same proof applies to $|\nabla|^{-s}f$.

For (5), we again use $P_N f = \tilde{P}_N(P_N f)$ to get

$$\|P_N f\|_{L^q} = \|(\psi_{N-2} + \cdots + \psi_{N+2})^\vee * P_N f\|_{L^q} \leq C\|\psi_N^\vee\|_{L^r}\|P_N f\|_{L^p}, \;\; 1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p}.$$

Then

$$\|\psi_N^\vee\|_{L^r} = \|(\psi(\cdot/2^N))^\vee\|_{L^r} = 2^{Nd}\|\check{\psi}(2^N\cdot)\|_{L^r} = 2^{Nd(1-\frac{1}{r})}\|\check{\psi}\|_{L^r}.$$

Since $1 - \frac{1}{r} = \frac{1}{p} - \frac{1}{q}$, we know

$$\|P_N f\|_{L^q} \leq C(p,q,d)2^{Nd(\frac{1}{p}-\frac{1}{q})}\|P_N f\|_{L^p}.$$

□

We also have the Littlewood-Paley characterization of the L^p space. However, the proof relies on Hörmander-Mikhlin multiplier theorem which is purely a conclusion in harmonic analysis, so we skip the proof here.

Theorem 6.4.2 (Littlewood-Paley Square Function Theorem). For any $1 < p < \infty$, there exists a constant $C = C(p, d) > 0$ such that

$$\forall f \in \mathcal{S}, \quad C^{-1}\|f\|_{L^p} \le \|S(f)\|_{L^p} \le C\|f\|_{L^p},$$

where $S(f)$ is the **Littlewood-Paley square function** of f, defined by

$$S(f) := \left(\sum_{N\in\mathbb{Z}} |P_N f|^2\right)^{\frac{1}{2}} = \|P_N f\|_{\ell^2(\mathbb{Z})}.$$

6.4.2 Littlewood-Paley characterization of Sobolev spaces

With the Littlewood-Paley projections, we can establish more general estimates for fractional-order derivatives in certain L^p spaces. We now define the Sobolev spaces $W^{s,p}(\mathbb{R}^d)$ and $\dot{W}^{s,p}(\mathbb{R}^d)$ as follows.

Definition 6.4.2. Given $s \in \mathbb{R}$ and $1 < p < \infty$, we define the non-homogeneous Sobolev space $W^{s,p}(\mathbb{R}^d)$ by

$$W^{s,p}(\mathbb{R}^d) := \left\{f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{W^{s,p}(\mathbb{R}^d)} := \|\langle\nabla\rangle^s f\|_{L^p(\mathbb{R}^d)} < \infty\right\}$$

and define the homogeneous Sobolev space $\dot{W}^{s,p}(\mathbb{R}^d)$ by

$$\dot{W}^{s,p}(\mathbb{R}^d) := \left\{f \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^d) : \|f\|_{\dot{W}^{s,p}(\mathbb{R}^d)} := \||\nabla|^s f\|_{L^p(\mathbb{R}^d)} < \infty\right\}.$$

Based on Bernstein's inequality (Proposition 6.4.1(3)) and Plancherel's identity, we have the Littlewood-Paley characterization of Sobolev norms

$$\|f\|_{W^{s,p}(\mathbb{R}^d)} \simeq \|P_{\le 1} f\|_{L^p(\mathbb{R}^d)} + \left\|\left(\sum_{N=1}^{\infty} 2^{2Ns}|P_N f|^2\right)^{\frac{1}{2}}\right\|_{L^p(\mathbb{R}^d)}. \tag{6.4.6}$$

The first conclusion is the Sobolev embedding theorem for $W^{s,p}(\mathbb{R}^d)$, not only the special case $p = 2$.

Theorem 6.4.3 (Non-endpoint Gagliardo-Nirenberg-Sobolev Inequality). Let $1 < p < q \le \infty$ and $s > 0$ be satisfy $\frac{1}{q} = \frac{1}{p} - \frac{\theta s}{d}$ for some $0 < \theta < 1$. Then there exists a constant $C = C(d, p, q, s) > 0$ such that for any $f \in W^{s,p}(\mathbb{R}^d)$, the following inequality holds

$$\|f\|_{L^q(\mathbb{R}^d)} \le C\|f\|_{L^p}^{1-\theta}\|f\|_{\dot{W}^{s,p}(\mathbb{R}^d)}^{\theta}.$$

Proof. We adopt the Littlewood-Paley decomposition $f = \sum_N P_N f$. From the triangle inequality followed by Bernstein's inequality we have

$$\|f\|_{L^q} \le \sum_N \|P_N f\|_{L^q} \le C(d,p,q) \sum_N 2^{N(\frac{d}{p}-\frac{d}{q})} \|P_N f\|_{L^p} = \sum_N 2^{N\theta s} \|P_N f\|_{L^p}.$$

On the other hand, from Proposition 6.4.1(3) and the boundedness of P_N, we have

$$\|P_N f\|_{L^p} \le C(d,p)\|f\|_{L^p}, \quad \|P_N f\|_{L^p} \le C(d,p,s) 2^{-Ns} \|P_N |\nabla|^s f\|_{L^p} \le C(d,p,s)2^{-Ns} \||\nabla|^s f\|_{L^p}.$$

Inserting this into the previous estimate we obtain for some K that

$$\begin{aligned}\|f\|_{L^q} &\le C(d,p,s) \sum_{N\le K} 2^{N\theta s}\|f\|_{L^p} + \sum_{N>K} 2^{N\theta s}\cdot 2^{-Ns} \||\nabla|^s f\|_{L^p} \\ &\le C(d,p,s)\left(\|f\|_{L^p} \frac{2^{K\theta s}}{1-2^{-\theta s}} + \|f\|_{\dot W^{s,p}} \frac{2^{(K+1)(\theta-1)s}}{1-2^{(\theta-1)s}} \right).\end{aligned}$$

Now, we optimize this upper bound by selecting a suitable K. Let these two terms be equal and this leads to

$$2^{-Ks} = \frac{\|f\|_{L^p}}{\|f\|_{\dot W^{s,p}}} \cdot \frac{1-2^{(\theta-1)s}}{2^{(1-\theta)s}-1}.$$

Substituting this back to the above inequality to eliminate the K-terms, we get the RHS is equal to

$$2C(d,p,s)\|f\|_{L^p} \frac{\|f\|^\theta_{\dot W^{s,p}}}{\|f\|^\theta_{L^p}} \cdot \left(\frac{1-2^{(\theta-1)s}}{2^{(1-\theta)s}-1} \right)^{-\theta} \le C \|f\|^{1-\theta}_{L^p} \|f\|^\theta_{\dot W^{s,p}(\mathbb{R}^d)}.$$

□

The critical embedding is again a corollary of the Hardy-Littlewood-Sobolev inequality

Theorem 6.4.4 (Endpoint Gagliardo-Nirenberg-Sobolev Inequality). Let $1 < p < q < \infty$ and $s > 0$ be satisfy $\frac{1}{q} = \frac{1}{p} - \frac{s}{d}$. Then there exists a constant $C = C(d,p,q,s) > 0$ such that for any $f \in W^{s,p}(\mathbb{R}^d)$, the following inequality holds

$$\|f\|_{L^q(\mathbb{R}^d)} \le C\|f\|_{\dot W^{s,p}(\mathbb{R}^d)}.$$

In particular, when $\frac{1}{q} \ge \frac{1}{p} - \frac{s}{d}$, we have the embedding theorem for non-homogeneous Sobolev spaces

$$\|f\|_{L^q(\mathbb{R}^d)} \le C\|f\|_{W^{s,p}(\mathbb{R}^d)}.$$

We can also prove an analogue of Theorem 6.2.3 (1).

Theorem 6.4.5. Let $1 < p < \infty, s > 0$ satisfy $sp > d$. Then

$$\|f\|_{L^\infty(\mathbb{R}^d)} \le C(p,s,d)\|f\|_{W^{s,p}(\mathbb{R}^d)}$$

6.4.3 Fractional Leibniz rule and chain rule

We also want to establish analogues of the Leibniz's rule and chain rule for fractional-order derivatives. They actually provide us with some "sharp" estimates that are frequently used in recent research.

Example 6.4.1 (Fractional Leibnitz rule). Let f, g be functions on $\mathbb{R}^d$, and let D^α be some sort of differential or pseudodifferential operator of positive order $\alpha > 0$. Heuristically, we know

- (High-Low/Low-High interactions) If f has significantly higher frequency than g (e.g. if $f = P_N F$ and $g = P_{<N-3} G$ for some F, G), or is "rougher" than g (e.g. $f = \nabla u$ and $g = u$ for some u) then fg will have comparable frequency to f, and we expect $D^\alpha(fg) \approx (D^\alpha f)g$ and $P_N(fg) \approx (P_N f)\, g$.
- (High-High/Low-Low interactions) If f and g have comparable frequency (e.g. $f = P_N F$ and $g = P_N G$ for some F, G) then fg should have frequency comparable or lower than f, and we expect $D^\alpha(fg) \lesssim (D^\alpha f)\, g \approx f\,(D^\alpha g)$.
- (Full Leibnitz rule) With no frequency assumptions on f and g, we expect

$$D^\alpha(fg) \approx f\,(D^\alpha g) + (D^\alpha f)\, g$$

Example 6.4.2 (Fractional chain rule). Let u be a function on $\mathbb{R}^d$, and let $F : \mathbb{R} \to \mathbb{R}$ be a "suitably regular" function (e.g. $F(u) = |u|^{p-1}u$). Then we have the fractional chain rule

$$D^\alpha(F(u)) \approx F'(u)D^\alpha u$$

for any differential operator D^α of positive order $\alpha > 0$, as well as the Littlewood-Paley variants

$$P_{<N}(F(u)) \approx F(P_{<N}u)$$

and

$$P_N(F(u)) \approx F'(P_{<N}u)P_N u.$$

Observe that when D^α is a differential operator of order k, then the heuristics are accurate to top order in k (i.e. ignoring any terms which only differentiate f, g, u for $k-1$ or fewer times). Indeed, the above two principles are instances of a more general principle:

Example 6.4.3. (Top order terms dominate). When distributing derivatives, the dominant terms are usually the terms in which all the derivatives fall on a single factor; if the factors have unequal degrees of smoothness, the dominant term will be the one in which all the derivatives fall on the roughest (or highest frequency) factor.

A complete and rigorous treatment of these heuristics (sometimes called paradifferential calculus) is beyond the scope of this lecture notes. We now prove the above heuristics in the case $D = \partial$.

Theorem 6.4.6 (Moser-type inequality). If $s \geq 0$, then we have the estimate

$$\|fg\|_{H^s(\mathbb{R}^d)} \leq C(s,d)\|f\|_{H^s(\mathbb{R}^d)}\|g\|_{L^\infty(\mathbb{R}^d)} + \|f\|_{L^\infty(\mathbb{R}^d)}\|g\|_{H^s(\mathbb{R}^d)}$$

for all $f, g \in H^s(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. In particular, if $s > d/2$, the conclusion degenerates to Theorem 6.2.3(2).

The basic strategy with these multilinear estimates is to decompose using the Littlewood-Paley decomposition, eliminate any terms that are obviously zero (because of impossible frequency interactions), estimate each remaining component using the Bernstein and Hölder inequalities, and then sum. One should always try to apply Bernstein on the lowest frequency possible, as this gives the most efficient estimates. In some dases one needs to apply Cauchy-Schwarz to conclude the summation.

Proof. Assume $s > 0$. From Proposition 6.4.1 we have

$$\|fg\|_{H^s} \lesssim C(s,d)\|P_{\leq 1}(fg)\|_{L^2} + \left(\sum_{N>1} 2^{2Ns}\,\|P_N(fg)\|_{L^2}^2\right)^{1/2}.$$

We shall just bound the latter term because the first term is easily controlled by using the L^2 boundedness of $P_{\leq 1}$ and Cauchy-Schwarz inequality. We split f into two frequency bands

$$\|P_N(fg)\|_{L^2} \leq C\|P_N((P_{<N-3}f)g)\|_{L^2} + \sum_{M>N-3} \|P_N((P_M f)g\|_{L^2}.$$

For the first term, observe from Fourier analysis that we may freely replace g by $P_{N-3<\cdot<N+3}g$, and so by Hölder's inequality

$$\begin{aligned}\|P_N((P_{<N-3}f)g)\|_{L^2} &\leq C\|(P_{<N-3}f)P_{N-3<\cdot<N+3}g\|_{L^2}\\ &\leq C\|f\|_{L^\infty}\|P_{N-3<\cdot<N+3}g\|_{L^2}\end{aligned}$$

and so the total contribution of this term to the square function above is $C(s,d)(\|f\|_{L^\infty}\|g\|_{H^s})$. For the second term, we simply bound

$$\begin{aligned}\sum_{M>N-3} \|P_N((P_M f)g)\|_{L^2} &\leq C \sum_{M>N-3} \|(P_M f)g\|_{L^2}\\ &\leq C\|g\|_{L^\infty} \sum_{M>N-3} 2^{-Ms}\|P_M f\|_{L^2}\end{aligned}$$

and so by Cauchy-Schwarz

$$2^{2Ns} \sum_{M>N-3} \|P_N((P_M f)g)\|_{L^2}^2 \leq C\|g\|_{L^\infty}^2 \sum_{M>N-3} 2^{2(N-M)s}\|P_M f\|_{L^2}^2.$$

Summing this in N (and using the hypothesis $s > 0$) we see that the total contribution of this term is $C(s,d)(\|f\|_{H^s}\|g\|_{L^\infty})$, and we are done. □

Theorem 6.4.7 (Paralinearization/Schauder-type estimate)**.** Let V be a finite-dimensional normed vector space, let $f \in H^s(\mathbb{R}^d \to V) \cap L^\infty(\mathbb{R}^d \to V)$ for some $s \geq 0$. Let k be the first integer greater than s, and let $F \in C^k_{\mathrm{loc}}(V \to V)$ be such that $F(0) = 0$. Then $F(f) \in H^s(\mathbb{R}^d \to V)$ as well, with a bound of the form

$$\|F(f)\|_{H^s(\mathbb{R}^d)} \leq C\left(F, \|f\|_{L^\infty(\mathbb{R}^d)}, V, s, d\right)\|f\|_{H^s(\mathbb{R}^d)}.$$

Proof. The strategy to prove nonlinear is related, though not quite the same as, that used to prove multilinear estimates. Basically, one should try to split $F(f)$ using Taylor expansion into a rough error, which one estimates crudely, and a smooth main term, which one estimates using information about its derivatives. Again, one uses tools such as Hölder, Bernstein, and Cauchy-Schwarz to estimate the terms that appear.

Let us write $A := \|f\|_{L^\infty}$. Since F is C^k_{loc} and $F(0) = 0$, we see that $|F(f)| \leq C(F, A, V)|f|$. This already establishes the claim when $s = 0$. Applying Proposition 6.4.1, it thus suffices to show that

$$\left(\sum_{N>1} 2^{2Ns}\|P_N F(f)\|^2_{L^2}\right)^{1/2} \leq C(F, A, V, s)\|f\|_{H^s}, \qquad \forall s > 0.$$

We first throw away a "rough" portion of $F(f)$ in $P_N F(f)$. Fix N, s,and split $f = P_{<N}f + P_{\geq N}f$. Note that f and $P_{<N}f$ are both bounded by $C(V, d, A)$. Now F is C^k_{loc}, hence Lipschitz on the ball of radius $C(V, d, A)$, hence we have

$$F(f) = F(P_{<N}f) + C(V, d, A, F)(|P_{\geq N}f|)$$

and thus

$$\|P_N F(f)\|_{L^2} \leq C(F, A, V, d)\|P_N F(P_{<N}f)\|_{L_2} + \|P_{\geq N}f\|_{L^2}.$$

To control the latter term, observe from the triangle inequality and Cauchy-Schwarz that

$$2^{2Ns}\|P_{\geq N}f\|^2_{L^2} \leq C(s)\sum_{N' \geq N} 2^{N's}2^{Ns}\|P_{N'}f\|^2_{L^2}$$

and summing this in N and using the Littlewood-Paley characterization of the H^s norm (6.4.6), we see that this term can be bounded by $\|f\|^2_{H^s}$. Thus it remains to show that

$$\left(\sum_{N>1} 2^{2Ns}\|P_N F(P_{<N}f)\|^2_{L^2}\right)^{1/2} \leq C(F, A, V, s)\|f\|_{H^s}.$$

We will exploit the smoothness of $P_{<N}f$ and F by using Bernstein's inequality to estimate

$$\|P_N F(P_{<N}f)\|_{L^2} \lesssim C(d, k)2^{-Nk}\|\nabla^k F(P_{<N}f)\|_{L^2}.$$

Applying the chain rule repeatedly, and noting that all derivatives of F are bounded on the ball of radius $C(V, d, A)$, we obtain the pointwise estimate

$$|\nabla^k F(P_{<N}f)| \leq C(F, A, V, d, k)\sup_{k_1+\ldots+k_r=k}|\nabla^{k_1}(P_{<N}f)| \ldots |\nabla^{k_r}(P_{<N}f)|$$

where r ranges over $1, \dots, k$ and $k_1, \dots, k_r$ range over non-negative integers that add up to k. We split this up further using Littlewood-Paley decomposition as

$$\left|\nabla^k F\left(P_{<N} f\right)\right| \leq C(F, A, V, d, k) \sup_{k_1+\ldots+k_r=k_1} \sum_{1 \leq N_1, \ldots, N_r < N} \left|\nabla^{k_1}(\tilde{P}_{N_1} f)\right| \ldots \left|\nabla^{k_r}(\tilde{P}_{N_r} f)\right|$$

where we adopt the convention that $\tilde{P}_N := P_N$ when $N > 1$ and $\tilde{P}_1 := P_{\leq 1}$.

We may take $N_1 \leq N_2 \ldots \leq N_r$. where $k_1, \dots, k_r$ range over all positive integers that add up to k. Now from Bernstein's inequality, we have

$$\left\|\nabla^{k_i}(\tilde{P}_{N_i} f)\right\|_{L^\infty} \leq C(d, k) 2^{N_i k_i} \|f\|_{L^\infty} \leq C(d, k, A) 2^{N k_i}$$

for $i = 1, \dots, r-1$, and similarly

$$\left\|\nabla^{k_r}(\tilde{P}_{N_r} f)\right\|_{L^2} \leq C(d, k) 2^{N_r k_r} \|\tilde{P}_{N_r} f\|_{L^2}$$

and hence we have

$$\left\|\nabla^k F(P_{<N} f)\right\|_{L^2} \leq C(F, A, V, d, k) \sup_{k_1+\ldots+k_r=k} \sum_{1 \leq N_1 \leq \ldots \leq N_r < N} 2^{N_1 k_1} \ldots 2^{N_r k_r} \left\|\tilde{P}_{N_r} f\right\|_{L^2}.$$

Performing the sum in N_1, then N_2, then finally N_{r-1}, and rewriting $N' := N_r$, we obtain

$$\left\|\nabla^k F(P_{<N} f)\right\|_{L^2} \leq C(F, A, V, d, k) \sum_{1 \leq N' < N} 2^{N' k} \|\tilde{P}_{N'} f\|_{L^2}$$

By Cauchy-Schwarz, we conclude

$$\|P_N F(P_{<N} f)\|_{L^2}^2 \leq C(F, A, V, d, k) \sum_{1 \leq N' < N} 2^{N' k} 2^{-Nk} \|\tilde{P}_{N'} f\|_{L^2}^2.$$

Summing this in N and using the Littlewood-Paley characterization of the H^s norm (6.4.6), we see that this term can be bounded by $\|f\|_{H^s}^2$. □

Exercise 6.4

Exercise 6.4.1. Complete the proof of Proposition 6.4.1.

Exercise 6.4.2. Prove Theorem 6.4.4. (Hint: Use Hardy-Littlewood-Sobolev inequality.)

Exercise 6.4.3. Prove Theorem 6.4.5. (Hint: Use Bernstein's inequality and $s > d/p$.)

Exercise 6.4.4 (Heat semigroup characterization of the Littlewood-Paley projections)**.** Let $\mathcal{A}$ be an annulus and let $f \in \mathcal{S}$ satisfy $\operatorname{Spt} \hat{f} \subseteq \lambda \mathcal{A}$. Prove that there exist $c, C > 0$ such that for any $1 \leq p \leq \infty$, $t, \lambda > 0$, there holds

$$\|e^{t\Delta} f\|_{L^p} \leq C e^{-ct\lambda^2} \|f\|_{L^p}.$$

This is the heat semigroup characterization of the Littlewood-Paley projections (which can be further generalized to manifolds where the Fourier transform cannot be defined).

Exercise 6.4.5 (Bernstein-type inequalities for the heat semigroup characterization)**.** Let $u_0 \in \mathcal{S}$ satisfy the assumption of f in Exercise 6.4.4. Consider the solution u to the equation

$$\partial_t u - \nu \Delta u = 0, \ \ u|_{t=0} = u_0;$$

and the solution v to the equation

$$\partial_t v - \nu \Delta v = f(t, \boldsymbol{x}), \ \ v|_{t=0} = 0$$

with $f(t, \cdot)$ satisfies the assumption of f in Exercise 6.4.4 for all $t > 0$. Prove that

$$\|u\|_{L_t^q L_{\boldsymbol{x}}^b} \le C(\nu\lambda^2)^{-\frac{1}{q}} \lambda^{d(\frac{1}{a}-\frac{1}{b})} \|u_0\|_{L^a}, \ \ \|u\|_{L_t^q L_{\boldsymbol{x}}^b} \le C(\nu\lambda^2)^{-1+\frac{1}{p}-\frac{1}{q}} \lambda^{d(\frac{1}{a}-\frac{1}{b})} \|f\|_{L_t^p L_{\boldsymbol{x}}^a}$$

for all $1 \le a \le b \le \infty$, $1 \le p \le q \le \infty$.

Chapter 7 Calculus of Variations

In the proof of variation principle for the eigenvalue problem of $-\Delta$, we apply the method of calculus of variation to the potential energy and find that the minimizer (with certain constraints) exactly gives the solution to the eigenvalue problem. Now we would like to apply this method to derive other important nonlinear PDEs. Also, we briefly introduce the general theory, i.e., the so-called "Euler-Lagrange equation". For a systematic study, please refer to Evans PDE chapter 8.

7.1 Euler-Lagrange equation: First variation

As discussed before, suppose we wish to solve some certain nonlinear PDE $A[u] = 0$ where $A[\cdot]$ denotes a given (nonlinear) differential operator and u is the unknown. Of course, there is no general theory, but the method of calculus of variation identifies an important class of such nonlinear problems that can be solved using simple techniques in nonlinear functional analysis. Such problems are called "variational problems", where the nonlinear operator $A[\cdot]$ is the "derivative" of an appropriate "energy functional" $I[\cdot]$. Then the PDE is rewritten as $I'[u] = 0$, that is, we turn to solve the "critical point(s)" of $I[\cdot]$, which is usually easier than directly solving the nonlinear PDE itself.

We now briefly introduce a general theory. Suppose now $\Omega \subset \mathbb{R}^d$ is a bounded domain with smooth boundary. We introduce a smooth function $L : \mathbb{R}^d \times \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$, called "Lagrangian". We will write

$$L = L(\boldsymbol{p}, z, \boldsymbol{x}) = L(p_1, \cdots, p_d, z, x_1 \cdots, x_d)$$

for $\boldsymbol{p} \in \mathbb{R}^d,\ z \in \mathbb{R},\ \boldsymbol{x} \in \mathbb{R}^d$. This $\boldsymbol{p}$ is the name of the variable for which we will substitute $\nabla w(\boldsymbol{x})$ and z is the variable that will be replaced by $w(\boldsymbol{x})$. We also denote

$$\nabla_{\boldsymbol{p}} L = (\partial_{p_1} L, \cdots, \partial_{p_d} L),\ \nabla_z L = \partial_z L,\ \nabla_{\boldsymbol{x}} L = (L_{x_1}, \cdots, L_{x_d}).$$

Now, we introduce the energy functional

$$I[w] := \int_\Omega L(\nabla w(\boldsymbol{x}), w(\boldsymbol{x}), \boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$$

where $w : \overline{\Omega} \to \mathbb{R}$ is chosen from some admissible set, for example, $\mathcal{A} := \{w \in C^\infty(\overline{\Omega}) : w|_{\partial\Omega} = g\}$ for a given (smooth) function g.

Claim. If $u \in \mathcal{A}$ is the minimizer of $I[\cdot]$, then u satisfies the so-called "**Euler-Lagrange equation**"

$$-\sum_{i=1}^{d} \partial_{x_i}(\partial_{p_i} L(\nabla u, u, \boldsymbol{x})) + \partial_z L(\nabla u, u, \boldsymbol{x}) = 0 \quad \text{in } \Omega. \tag{7.1.1}$$

Let us temporarily assume the claim is correct and try to see how this applies to PDEs. Indeed, if we let the Lagrangian be

$$L(\boldsymbol{p}, z, \boldsymbol{x}) = \frac{1}{2}|\boldsymbol{p}|^2 = \frac{1}{2}(p_1^2 + \cdots + p_d^2),$$

then

$$L_z = 0,\ \partial_{p_i} L = p_i.$$

Replacing $(\boldsymbol{p}, z, \boldsymbol{x})$ by $(\nabla u(\boldsymbol{x}), u(\boldsymbol{x}), \boldsymbol{x})$, we get the Euler-Lagrange equation

$$-\sum_{i=1}^{d} \partial_{x_i}(\partial_{x_i} u) + 0 = 0 \Rightarrow -\Delta u = 0,$$

which is exactly the Dirichlet's principle.

Proof of Claim. To prove the claim, we pick any $v \in C_c^\infty(\Omega)$ and consider the real-valued function

$$j(\varepsilon) := I[u + \varepsilon v],\ \ \varepsilon \in \mathbb{R}.$$

Since u is the minimizer of $I[\cdot]$, we know $j(\varepsilon)$ reaches a minimum at $\varepsilon = 0$ and thus $j'(0) = 0$.

Next we compute $j'(0)$. First we have

$$j(\varepsilon) = \int_\Omega L(\nabla u + \varepsilon \nabla v, u + \varepsilon v, \boldsymbol{x})\, \mathrm{d}\boldsymbol{x},$$

and then

$$j'(\varepsilon) = \int_\Omega \sum_{i=1}^{d} \partial_{p_i} L(\nabla u + \varepsilon \nabla v, u + \varepsilon v, \boldsymbol{x}) \partial_{x_i} v + \partial_z L(\nabla u + \varepsilon \nabla v, u + \varepsilon v, \boldsymbol{x}) v\, \mathrm{d}\boldsymbol{x}.$$

Let $\varepsilon = 0$, we have

$$0 = j'(0) = \int_\Omega \sum_{i=1}^{d} \partial_{p_i} L(\nabla u, u, \boldsymbol{x}) \partial_{x_i} v + \partial_z L(\nabla u, u, \boldsymbol{x}) v\, \mathrm{d}\boldsymbol{x}.$$

Finally, integrating ∂_{x_i} by parts in the first term yields the result. $\square$

7.1.1 Examples

Example 7.1.1 (Nonlinear Poisson equation). Assume we are given a smooth function $f : \mathbb{R} \to \mathbb{R}$ and define its antiderivative $F(z) = \int_0^z f(y)\,\mathrm{d}y$. We introduce the energy functional

$$I[w] = \int_\Omega \frac{1}{2}|\nabla w|^2 - F(w)\,\mathrm{d}\boldsymbol{x}.$$

Following the above claim, we have $L(\boldsymbol{p}, z, \boldsymbol{x}) = \frac{1}{2}|\boldsymbol{p}|^2 - F(z)$ and thus compute

$$\partial_{p_i} L = p_i, \quad \partial_z L = -f(z)$$

and thus the Euler-Lagrange (E-L) equation is given by

$$-\Delta u = f(u) \text{ in } \Omega.$$

In practice, the function $F(z)$ is usually a polynomial such as $F(z) = \pm\frac{1}{p+1}x^{p+1}$ which gives a semi-linear elliptic equation $-\Delta u = \pm u^p$.

Remark 7.1.1. We can also introduce the time variable t into the functional (i.e., replace $\boldsymbol{x}$ by $(t, \boldsymbol{x})$ and ∇w by $(\partial_t w,\ \nabla w)$.) to derive wave equation. For example, the minimizer of

$$I[w] = \int_0^T \int_\Omega \frac{1}{2}w_t^2 - \left(\frac{1}{2}|\nabla w|^2 + F(w)\right)\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

solves the nonlinear wave equation

$$u_{tt} - \Delta u + f(u) = 0.$$

Example 7.1.2 (Minimal surfaces). Let us consider a classical problem (Plateau's problem): Find the surface with minimal area enclosed by a given curve (in 3D).

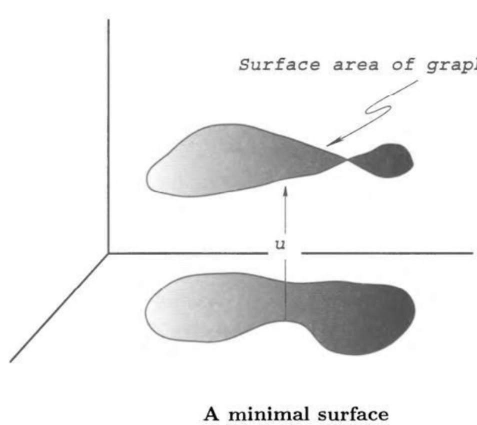


Figure 7.1: An example of minimal surfaces

Mathematically, this is the so-called "minimal surface" whose equation is expected to be the minimizer of the area functional

$$I[w] = \int_\Omega \sqrt{1 + |\nabla w|^2}\, \mathrm{d}\boldsymbol{x}.$$

The corresponding Lagrangian is $L(\boldsymbol{p}, z, \boldsymbol{x}) = \sqrt{1 + |\boldsymbol{p}|^2}$, and thus $\partial_{p_i} L(\boldsymbol{p}, z, \boldsymbol{x}) = \frac{p_i}{\sqrt{1+|\boldsymbol{p}|^2}}$ and $\partial_z L = 0$. Replacing $(\boldsymbol{p}, z)$ by $(\nabla u, u)$, we get the equation of the minimizer u

$$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) := \sum_{i=1}^{d} \partial_{x_i} \left(\frac{\partial_{x_i} u}{\sqrt{1 + |\nabla u|^2}} \right) = 0.$$

Remark 7.1.2. The quantity $\nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right)$ is d times the mean curvature of the graph of u. Thus, a minimal surface has zero mean curvature. This quantity also has physical meaning: it is proportional to the surface tension of a liquid drop.

Example 7.1.3 (Harmonic maps). We now replace the scalar function w by a vector $\mathbf{w} : \Omega \subset \mathbb{R}^d \to \mathbb{R}^m$. Consider the minimizer of the energy

$$I[\mathbf{w}] := \int_\Omega \frac{1}{2} |\nabla \mathbf{w}|^2\, \mathrm{d}\boldsymbol{x}$$

over the admissible set $\mathcal{A} := \{\mathbf{w} \in C^2(\Omega \to \mathbb{R}^m) : w|_{\partial\Omega} = \mathbf{g},\ |\mathbf{w}| = 1\}$. It seems like a geometric problem, but indeed its variants are used to characterize the steady state for the motion of liquid crystals. We prove that the minimizer $\mathbf{u}$ satisfies the following quasilinear elliptic equations

$$\begin{cases} -\Delta \mathbf{u} = |\nabla \mathbf{u}|^2 \mathbf{u} & \text{in } \Omega \\ \mathbf{u} = \mathbf{g} & \text{on } \partial\Omega. \end{cases}$$

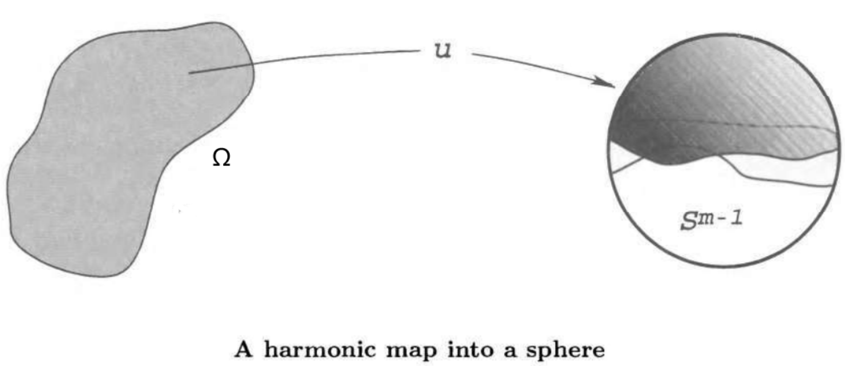


Figure 7.2: A harmonic map to a sphere

Proof. Fix any $\mathbf{v} \in C_c^\infty(\Omega \to \mathbb{R}^m)$. Then since $|\mathbf{u}| = 1$, we have $|\mathbf{u} + \varepsilon\mathbf{v}| \neq 0$ for small ε, and thus $\mathbf{v}_\varepsilon := \dfrac{\mathbf{u}+\varepsilon\mathbf{v}}{|\mathbf{u}+\varepsilon\mathbf{v}|} \in \mathcal{A}$.

We define $j(\varepsilon) := I[\mathbf{v}_\varepsilon]$ and then j has a minimum at $\varepsilon = 0$, and so $j'(0) = 0$. Now we compute

$$j'(0) = \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}'(0)\,\mathrm{d}\boldsymbol{x},$$

where $A : B$ is the Hadamard product of two matrix, i.e., $A : B := \sum_{i,j} A_{ij}B_{ij}$; and $' := \frac{d}{d\varepsilon}$. Then we compute

$$\mathbf{v}'(\varepsilon) = \frac{\mathbf{v}}{|\mathbf{u}+\varepsilon\mathbf{v}|} - \frac{((\mathbf{u}+\varepsilon\mathbf{v})\cdot\mathbf{v})(\mathbf{u}+\varepsilon\mathbf{v})}{|\mathbf{u}+\varepsilon\mathbf{v}|^3} \Rightarrow \mathbf{v}'(0) = \mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{u}.$$

Inserting this into the first identity, we find that

$$0 = \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} - \nabla\mathbf{u} : \nabla((\mathbf{u}\cdot\mathbf{v})\mathbf{u})\,\mathrm{d}\boldsymbol{x}.$$

Since $|\mathbf{u}|^2 = 1$, we have $(\nabla\mathbf{u})^\top\mathbf{u} = \mathbf{0}$ (i.e., $\sum_{j=1}^m (\partial_i u_j)u_j = 0\ \ \forall 1 \le i \le d$). Using this fact, we have

$$\begin{aligned}\nabla\mathbf{u} : \nabla((\mathbf{u}\cdot\mathbf{v})\mathbf{u}) &= \sum_i\sum_j \partial_i\mathbf{u}_j\partial_i(\sum_k \mathbf{u}_k\mathbf{v}_k\mathbf{u}_j)\\ &= |\nabla\mathbf{u}|^2(\mathbf{u}\cdot\mathbf{v}) + \sum_i \underbrace{\sum_j \partial_i\mathbf{u}_j\mathbf{u}_j}_{=0}\,\partial_i(\sum_k \mathbf{u}_k\mathbf{v}_k) = |\nabla\mathbf{u}|^2(\mathbf{u}\cdot\mathbf{v}).\end{aligned}$$

Thus, we get

$$\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\,\mathrm{d}\boldsymbol{x} = \int_\Omega |\nabla\mathbf{u}|^2(\mathbf{u}\cdot\mathbf{v})\,\mathrm{d}\boldsymbol{x}\ \ \forall v \in C_c^\infty(\Omega\to\mathbb{R}^m).$$

Integrating by parts on the left side, we get

$$\int_\Omega (-\Delta\mathbf{u})\cdot\mathbf{v}\,\mathrm{d}\boldsymbol{x} = \int_\Omega |\nabla\mathbf{u}|^2(\mathbf{u}\cdot\mathbf{v})\,\mathrm{d}\boldsymbol{x}\ \ \forall v \in C_c^\infty(\Omega\to\mathbb{R}^m),$$

and thus we get $-\Delta\mathbf{u} = |\nabla\mathbf{u}|^2\mathbf{u}$ in Ω. □

7.1.2 *Second variation: Convexity assumption

Let us discuss a bit more about the Euler-Lagrange equation. As we can see, $I[w]$ attains minimum at u which forces $j'(0) = 0$. It also indicates that $j''(0) \ge 0$. Recall that

$$j(\varepsilon) = \int_\Omega L(\nabla u + \varepsilon\nabla v, u + \varepsilon\nabla v, \boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

Then

$$j''(\varepsilon) = \int_\Omega \sum_{i,j=1}^{d} \partial_{p_i}\partial_{p_j} L(\nabla u + \varepsilon \nabla v, u + \varepsilon v, \boldsymbol{x})\partial_{x_i} v \partial_{x_j} v$$
$$+ 2\sum_{i=1}^{d} \partial_{p_i}\partial_z L(\nabla u + \varepsilon \nabla v, u + \varepsilon v, \boldsymbol{x})\partial_{x_i} v v$$
$$+ \partial_z^2 L(\nabla u + \varepsilon \nabla v, u + \varepsilon v, \boldsymbol{x}) v^2 \,\mathrm{d}\boldsymbol{x}.$$

Letting $\varepsilon = 0$, we find

$$0 \leq \int_\Omega \sum_{i,j=1}^{d} \partial_{p_i}\partial_{p_j} L(\nabla u, u, \boldsymbol{x})\partial_{x_i} v \partial_{x_j} v + 2\sum_{i=1}^{d} \partial_{p_i}\partial_z L(\nabla u, u, \boldsymbol{x})\partial_{x_i} v v + \partial_z^2 L(\nabla u, u, \boldsymbol{x}) v^2 \,\mathrm{d}\boldsymbol{x}$$

holds for all $v \in C_c^\infty(\Omega)$. This indeed implies

$$\sum_{i,j=1}^{d} \partial_{p_i}\partial_{p_j} L(\nabla u, u, \boldsymbol{x})\xi_i\xi_j \geq 0 \quad \xi \in \mathbb{R}^d,\ \boldsymbol{x} \in \Omega,$$

which is the so-called "convexity assumption" on the Lagrangian L. The proof is to pick the test function v to be

$$v(\boldsymbol{x}) = \varepsilon\rho\left(\frac{\boldsymbol{x}\cdot\xi}{\varepsilon}\right)\eta(\boldsymbol{x}),\quad \boldsymbol{x} \in \Omega,$$

where $\eta \in C_c^\infty(\Omega)$ and $\rho : \mathbb{R} \to \mathbb{R}$ is the periodic "zig-zag" function defined to be $\begin{cases} x & 0 \leq x \leq \frac{1}{2} \\ 1 - x & \frac{1}{2} \leq x \leq 1 \end{cases}$ and $\rho(x+1) = \rho(x)\ \ x \in \mathbb{R}$.

Therefore, the convexity assumption should be a necessary condition for the existence of minimizer. For more details, please refer to Evans PDE Chapter 8.2 to 8.4.

Exercise 7.1

Exercise 7.1.1. Find the Euler-Lagrangian equation for the energy functional

$$I[w] := \int_0^T \int_\Omega \frac{1}{2} w_t^2 - \frac{1}{2}|\nabla w|^2 - \frac{m^2}{2} w^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,$$

where $w(t, \boldsymbol{x}) \in C^\infty([0, T] \times \mathbb{R}^d)$ and $m > 0$ is a given constant.

Exercise 7.1.2. Define the energy functional

$$I[w] = \int_0^T \int_{\mathbb{R}^3} \frac{1}{2}w_t^2 - \left(\frac{1}{2}|\nabla w|^2 + \frac{1}{6}w^6\right) \mathrm{d}\boldsymbol{x}\,\mathrm{d}t,$$

where $w(t,\boldsymbol{x})$ belongs to $\mathcal{A} = \{w \in C^\infty([0,T]\times\mathbb{R}^3) : w(t,\cdot) \in \mathcal{S}(\mathbb{R}^3)\ \ \forall t \in [0,T]\}$. Find the equation of the minimizer u of $I[w]$ over $\mathcal{A}$, i.e., $I[u] = \min\limits_{w\in\mathcal{A}} I[w]$ and show that

$$\int_{\mathbb{R}^3} \frac{1}{2}(u_t)^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{6}u^6 \,\mathrm{d}\boldsymbol{x}$$

is independent of time.

Exercise 7.1.3. Find $L = L(\boldsymbol{p}, z, \boldsymbol{x})$ so that the PDE $-\Delta u + \nabla\varphi\cdot\nabla u = f$ in Ω is the Euler-Lagrange equation corresponding to the functional $I[w] := \int_\Omega L(\nabla w, w, \boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$. (Hint: Find a Lagrangian with an exponential term.)

Exercise 7.1.4. Let $\varepsilon > 0$ and $\Omega_T := (0,T]\times\Omega$. Show that the elliptic regularization of the heat equation, namely the PDE $\partial_t u - \Delta u - \varepsilon\partial_t^2 u = 0$, is the Euler-Lagrange equation orresponding to an functional $I_\varepsilon[w] := \iint_{\Omega_T} L(\partial_t w, \nabla w, w, t, \boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$. (Hint: Find a Lagrangian with an exponential term involving t.)

Exercise 7.1.5. Assume $L(\boldsymbol{p},\boldsymbol{x})$ is a Lagrangian that is smooth in its components and is uniformly convex in $\boldsymbol{p}$-components, that is, there exists $\theta > 0$ such that

$$\sum_{i,j=1}^d \partial_{p_i}\partial_{p_j} L(\nabla u, u, \boldsymbol{x})\xi_i\xi_j \geq \theta|\xi|^2 \quad \xi \in \mathbb{R}^d,\ \boldsymbol{x}\in\Omega.$$

Show that the minimizer of $I[w] := \int_\Omega L(\nabla u, \boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ over the admissible set $\mathcal{A} := \{w \in C^\infty(\overline{\Omega}) : w|_{\partial\Omega} = g\}$ is unique. Here $g : \partial\Omega \to \mathbb{R}$ is a given smooth function.

(Hint: Consider $I[\frac{u_1+u_2}{2}]$ if u_1, u_2 are two different minimizers.)

7.2 *Variational inequality: elliptic free-boundary problem

Now let us revisit the Dirichlet principle. If we replace the constraint $w|_{\partial\Omega} = g$ by a one-sided inequality $w \geq g$ in Ω ($g \in C^\infty(\overline{\Omega})$ is called an obstacle), i.e., we define the admissible set to be

$$\mathcal{A} = \{w \in C^2(\overline{\Omega}) : w \geq g \text{ in } \Omega,\ \ w|_{\partial\Omega} = 0\},$$

and consider the same functional $I[w] := \int_\Omega \frac{1}{2}|\nabla w|^2 - wf\,\mathrm{d}\boldsymbol{x}$. The existence and uniqueness of the minimizer of $I[w]$ over $\mathcal{A}$ can be proven by using linear functional analysis, so we assume these are already known. Also we assume we already know $\mathcal{A}$ is convex. However, the minimizer is no longer a solution to Poisson equation. Instead, it satisfies a so-called "variational inequality" which asserts that

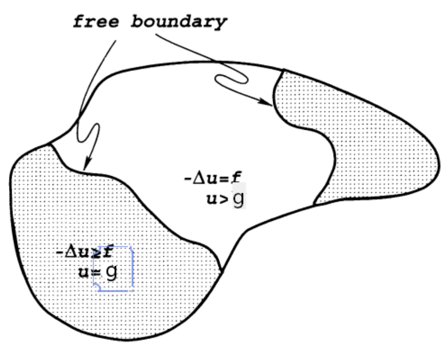
- $-\Delta u = f$ still holds as long as u is "beyond the obstacle"(i.e. when $u(x) > g(x)$).
- $u \geq g$ and $-\Delta u \geq f$ in Ω.

Theorem 7.2.1 (Variational inequality)**.** Let $u \in \mathcal{A}$ be the unique minimizer of $I[w]$ over $\mathcal{A}$. Then

$$\forall w \in \mathcal{A}, \quad \int_\Omega (\nabla u) \cdot (\nabla(w-u))\,\mathrm{d}\boldsymbol{x} \geq \int_\Omega f(w-u)\,\mathrm{d}\boldsymbol{x}.$$

The proof of this inequality is not hard, so we postpone the proof later. The more important thing is how to interprete this inequality. Let us denote

$$\mathcal{O} := \{\boldsymbol{x} \in \Omega : u(\boldsymbol{x}) > g(\boldsymbol{x})\}, \quad \mathcal{C} := \{\boldsymbol{x} \in \Omega : u(\boldsymbol{x}) = g(\boldsymbol{x})\}.$$



The free boundary for the obstacle problem

Figure 7.3: The free boundary in the obstacle problem

Since u, g are continuous, we know $\mathcal{O}$ is an open set and $\mathcal{C}$ is (relatively) closed. (Recall that the pre-image of an open/closed set under a continuous mapping must be open/closed.)

Claim 1: $-\Delta u = f$ in $\mathcal{O}$.

Proof of Claim 1. Fix any test function $v \in C_c^\infty(\mathcal{O})$. Since $u(\boldsymbol{x}) > g(\boldsymbol{x})$ in $\mathcal{O}$, we know for sufficiently small $|\varepsilon|$, we still have $w(\boldsymbol{x}) := u(\boldsymbol{x}) + \varepsilon v(\boldsymbol{x}) \geq g(\boldsymbol{x})$ and thus $w \in \mathcal{A}$. Using the above variational inequality, we have

$$\varepsilon \int_{\mathcal{O}} (\nabla u) \cdot (\nabla v) - f v \,\mathrm{d}\boldsymbol{x} \geq 0.$$

Note that this is true for both small $\varepsilon > 0$ and $\varepsilon < 0$, which forces the left side to be zero! Thus, u has to satisfy $-\Delta u = f$ in Ω. □

Claim 2: $u \geq g$ and $-\Delta u \geq f$ in Ω.

Proof of Claim 2. In general, we just pick the above v to be non-negative and $\varepsilon \in (0,1]$ sufficiently small. Then we integrate by parts in the variational inequality to get

$$\int_\Omega (-\Delta u - f)v \,\mathrm{d}\boldsymbol{x} \geq 0 \;\; \forall v \in C_c^\infty(\Omega),\; v \geq 0,$$

and thus $-\Delta u \geq f$ in Ω. □

Remark 7.2.1. The set $F := \partial\mathcal{O} \cap \Omega$ is called a "free boundary" as we do not know the specific position of this "virtual interface". Such free-boundary problems can be studied in an easier way with the help of variational inequality. These problems arise in the study of Brownian motion, plasticity theory, etc...

Finally let us prove this variational inequality

Proof of the variational inequality. Fix any element $w \in \mathcal{A}$. Then for any $0 \leq \varepsilon \leq 1$, by the convexity of $\mathcal{A}$, we have

$$u + \varepsilon(w-u) = (1-\varepsilon)u + \varepsilon w \in \mathcal{A}.$$

Thus, if we set $j(\varepsilon) = I[u + \varepsilon(w-u)]$, we see that $j(\varepsilon) \geq j(0)$ for any $\varepsilon \in [0,1]$. Hence $j'(0) \geq 0$.

Now we compute $j'(0)$ by definition. For $\varepsilon \in (0,1]$, we have

$$\begin{aligned}\frac{j(\varepsilon)-j(0)}{\varepsilon} &= \frac{1}{\varepsilon}\int_\Omega \frac{|\nabla u + \varepsilon \nabla(w-u)|^2 - |\nabla u|^2}{2} - f\,(u + \varepsilon(w-u) - u)\,\mathrm{d}\boldsymbol{x} \\ &= \int_\Omega \nabla u \cdot \nabla(w-u) + \frac{\varepsilon|\nabla(w-u)|^2}{2} - f(w-u)\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Letting $\varepsilon \to 0_+$ and using $j'(0) \geq 0$, we obtain

$$\int_\Omega \nabla u \cdot \nabla(w-u) - f(w-u)\,\mathrm{d}\boldsymbol{x} \geq 0.$$

□

Exercise 7.2

Exercise 7.2.1. Prove the uniqueness of the minimizer of the functional $I[w] := \int_\Omega \frac{1}{2}|\nabla w|^2 - wf\,\mathrm{d}\boldsymbol{x}$ over the admissible set $\mathcal{A} = \{w \in C^2(\overline{\Omega}) : w \geq g \text{ in } \Omega,\;\; w|_{\partial\Omega} = 0\}$. Here $g \in C^\infty(\overline{\Omega})$ is given.

Exercise 7.2.2. Show that the variational inequality in Theorem 7.2.1 can be rewritten as $-\Delta u + \beta(u - g) \ni f$ for the multi-valued function

$$\beta(z) := \begin{cases} 0 & z > 0 \\ (-\infty, 0] & z = 0\,. \\ \emptyset & z < 0 \end{cases}$$

Exercise 7.2.3. Let $f \in L^2(\Omega)$ be given and u be the minimizer of the functional $I[w] := \int_\Omega \frac{1}{2}|\nabla w|^2 - wf \,\mathrm{d}\boldsymbol{x}$ over the admissible set $\mathcal{A} = \{w \in C^2(\overline{\Omega}) : |\nabla w| \le 1 \text{ in } \Omega,\ w|_{\partial\Omega} = 0\}$. Show that u satisfies the inequality

$$\int_\Omega \nabla u \cdot \nabla(w-u)\,\mathrm{d}\boldsymbol{x} \ge \int_\Omega (w-u)f\,\mathrm{d}\boldsymbol{x} \quad \forall w \in \mathcal{A}.$$

7.3 Nöether's theorem

We now study variational integrands that are invariant under appropriate domain/function variations and show that the solutions to the corresponding Euler-Lagrange equations automatically solve certain divergence-type conservation laws. In particular, we will see how to find some "good multipliers" that induce helpful identities/monotonic quantities. We consider the functional

$$I[w] = \int_\Omega L(\nabla w, w, \boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$

where $\Omega \subset \mathbb{R}^d$ and $w : \Omega \to \mathbb{R}$. We also write $L = L(\boldsymbol{p}, z, \boldsymbol{x})$.

7.3.1 Statement and proof

Before going to the rigorous statement, we must introduce some notations about the invariant variational problems.

Notation 7.3.1 (Domain variation). Let $\mathfrak{X} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$, $\mathfrak{X} = \mathfrak{X}(\boldsymbol{x}, \tau)$ be a smooth family of vector fields satisfying $\mathfrak{X}(\boldsymbol{x}, 0) = \boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{R}^d$. Then for small τ, the mapping $\boldsymbol{x} \longmapsto \mathfrak{X}(\boldsymbol{x}, \tau)$ is called a **domain variation**. We also define $\mathbf{v}(\boldsymbol{x}) := \mathfrak{X}_\tau(\boldsymbol{x}, 0)$ and $\Omega(\tau) := \mathfrak{X}(\Omega, \tau)$.

Notation 7.3.2 (Function variation). Given a smooth $u : \mathbb{R}^d \to \mathbb{R}$, we consider a smooth family of **function variations** $w : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$, $w = w(\boldsymbol{x}, \tau)$ such that $w(\boldsymbol{x}, 0) = u(\boldsymbol{x})$. Also we write $m(\boldsymbol{x}) := w_\tau(\boldsymbol{x}, 0)$ and call it a **multiplier**.

Definition 7.3.1. We say the functional $I[\cdot]$ is **invariant** under the domain variation $\mathfrak{X}$ and the function variation w if

$$\int_\Omega L(\nabla w(\boldsymbol{x}, \tau), w(\boldsymbol{x}, \tau), \boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{\Omega(\tau)} L(\nabla w, w, \boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \tag{7.3.1}$$

holds for all small τ and all open sets $\Omega \subset \mathbb{R}^d$.

Now we can state Nöether's theorem, which shows that the invariance of the functional implies that the corresponding Euler-Lagrange equation can be transformed into divergence form.

Theorem 7.3.1. [Nöether's theorem] Suppose $I[\cdot]$ is **invariant** under the domain variation $\mathfrak{X}$ and the function variation w corresponding to a smooth function u. Then

1. We have the identity

$$\begin{aligned}&\nabla_{\boldsymbol{x}} \cdot \left(m \nabla_{\boldsymbol{p}} L(\nabla u, u, \boldsymbol{x}) - L(\nabla u, u, \boldsymbol{x}) \mathbf{v}\right) \\ &= m \left(\nabla_{\boldsymbol{x}} \cdot \nabla_{\boldsymbol{p}} L(\nabla u, u, \boldsymbol{x}) - \partial_z L(\nabla u, u, \boldsymbol{x})\right).\end{aligned}$$

2. In particular, if u is a critical point of $I[\cdot]$ and so solves the E-L eq $-\nabla_{\boldsymbol{x}} \cdot (\nabla_{\boldsymbol{p}} L) + \partial_z L = 0$, we have the divergence identity.

$$\nabla_{\boldsymbol{x}} \cdot \left(m \nabla_{\boldsymbol{p}} L(\nabla u, u, \boldsymbol{x}) - L(\nabla u, u, \boldsymbol{x}) \mathbf{v}\right) = 0.$$

Proof. It suffices to differentiate with respect to τ in the invariance identity (7.3.1) and set $\tau = 0$. We have the identity

$$\int_\Omega \nabla_{\boldsymbol{p}} L \cdot \nabla_{\boldsymbol{x}} m + \partial_z L m \, \mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} L(\mathbf{v} \cdot N) \, \mathrm{d}S$$

Integrating by parts and using divergence theorem, we get

$$\begin{aligned}&\int_\Omega m \left(\nabla_{\boldsymbol{x}} \cdot \nabla_{\boldsymbol{p}} L(\nabla u, u, \boldsymbol{x}) - \partial_z L(\nabla u, u, \boldsymbol{x})\right) \\ =&\int_\Omega \nabla_{\boldsymbol{x}} \cdot \left(m \nabla_{\boldsymbol{p}} L(\nabla u, u, \boldsymbol{x}) - L(\nabla u, u, \boldsymbol{x}) \mathbf{v}\right).\end{aligned}$$

This is true for any domain $\Omega \subset \mathbb{R}^d$, so the integrands must coincide everywhere. □

7.3.2 Examples

As stated before the theorem, we can sometimes first "predict" a domain variation $\mathfrak{X}$ and then look for a corresponding function variation w as some formula involving $u(\mathfrak{X}(\boldsymbol{x}, \tau))$. Then we can compute the multiplier m in terms of u and its derivatives. In this section, we introduce some examples to illustrate this procedure.

Example 7.3.1 (Translation invariance). Assume $L = L(\boldsymbol{p}, z)$ does not depend on $\boldsymbol{x}$. Then $I[w] := \int_\Omega L(\nabla w, w) \, \mathrm{d}\boldsymbol{x}$ is invariant under shifts. Given $k \in \{1, \cdots, d\}$ and define $\mathfrak{X}(\boldsymbol{x}, \tau) := \boldsymbol{x} + \tau e_k$ and $w(\boldsymbol{x}, \tau) := u(\boldsymbol{x} + \tau e_k)$. Then one can compute that $\mathbf{v} = e_k$ and $m = \partial_{x_k} u$. Consequently, if u is a critical point, then Theorem 7.3.1 leads to the following identity

$$\sum_{i=1}^d \partial_{x_i} (\partial_{p_i} L \, \partial_{x_k} u - L \delta_{ik}) = 0, \quad k = 1, \cdots, d. \tag{7.3.2}$$

For example, we may apply this conclusion to the wave equation. That is, consider

$$I[w] = \int_0^T \int_{\mathbb{R}^d} \frac{1}{2}(\partial_t w)^2 - \frac{1}{2}|\nabla w|^2 + F(w)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

and let u be the minimizer. We already know that u satisfies a semi-linear wave equation

$$\partial_t^2 u - \Delta u + f(u) = 0, \quad f = F'.$$

For this functional, we write $\boldsymbol{p} = (\partial_t w, \partial_1 w, \cdots, \partial_d w)$ and $\boldsymbol{x} = (t, x_1, \cdots, x_d)$ and choose $k = 0$ (time variable). Then $L(\boldsymbol{p}, z) = \frac{1}{2}p_0^2 - \frac{1}{2}(p_1^2 + \cdots + p_d^2) + F(z)$ and $\mathbf{v} = e_0, m = \partial_t u$. So, Nöether's theorem implies that

$$\nabla_{\boldsymbol{x}} \cdot (\nabla u\, u_t) + \partial_t \left(u_t^2 - \frac{1}{2}(u_t^2 - |\nabla u|^2) + F(u) \right) = 0.$$

This yields that $e := \frac{1}{2}(u_t^2 + |\nabla u|^2) + F(u)$ satisfies $e_t - \nabla_{\boldsymbol{x}} \cdot (u_t \nabla u) = 0$. If u is sufficiently smooth, then this implies the energy conservation $\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega e(t, \boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0$.

Example 7.3.2 (Scaling invariance of the wave equation). Recall that the linear wave equation $\partial_t^2 u - \Delta u = 0$ corresponds to the action functional

$$I[w] = \frac{1}{2}\int_0^T \int_{\mathbb{R}^d} (\partial_t w)^2 - |\nabla w|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

It is straightforward to verify that this functional is invariant under the following scaling transformation

$$(\boldsymbol{x}, t) \longmapsto (\lambda \boldsymbol{x}, \lambda t), \quad u \longmapsto \lambda^{\frac{d-1}{2}} u(\lambda \boldsymbol{x}, \lambda t), \quad \lambda > 0.$$

We set $\lambda = e^\tau$ and define

$$\boldsymbol{\mathfrak{X}}(t, \boldsymbol{x}, \tau) = (e^\tau t, e^\tau \boldsymbol{x}),\ w(t, \boldsymbol{x}, \tau) := e^{\frac{(d-1)\tau}{2}} u(e^\tau t, e^\tau \boldsymbol{x}).$$

Then $\mathbf{v} = (t, \boldsymbol{x})$ and $m = t u_t + \boldsymbol{x} \cdot \nabla u + \frac{d-1}{2}u$. After some tedious computations, we can find that the scaling invariance leads to the conservation law $\partial_t p - \operatorname{div} \mathbf{q} = 0$ where

$$\begin{aligned} p :=& \frac{t}{2}((\partial_t u)^2 + |\nabla u|^2) + (\boldsymbol{x} \cdot \nabla u)\partial_t u + \frac{d-1}{2} u u_t, \\ \mathbf{q} :=& \left(t u_t + \boldsymbol{x} \cdot \nabla u + \frac{d-1}{2} u \right) \nabla u + \frac{1}{2}(u_t^2 - |\nabla u|^2)\boldsymbol{x}. \end{aligned}$$

It should be noted that this identity is useful when establishing the global existence of the 3D energy-critical wave equation and we refer to Evans [6, Chap. 12.4] for details.

Exercise 7.3

Exercise 7.3.1 (Scaling invariance of p-Laplacian). Given $p > 0$, consider $I[w] = \int_\Omega |\nabla w|^p \,\mathrm{d}\boldsymbol{x}$. Prove that

(1) The minimizer (assume it exists) u satisfies the p-Laplacian equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$.

(2) $I[w]$ is invariant under the scaling transformation $x \longmapsto \lambda \boldsymbol{x}$, $u \longmapsto \lambda^{\frac{d-p}{p}} u(\lambda \boldsymbol{x})$ for any $\lambda > 0$. Then put $\lambda = e^\tau$ and use Noether's theorem to derive

$$\nabla \cdot \left(\left(\boldsymbol{x} \cdot \nabla u + \frac{d-p}{p} u \right) p |\nabla u|^{p-2} \nabla u - |\nabla u|^p \boldsymbol{x} \right) = 0. \tag{7.3.3}$$

Exercise 7.3.2 (The monotonicity formula). Let $B(\mathbf{0}, r)$ lies within Ω.

(1) Integrate the identity (7.3.3) over $B(\mathbf{0}, r)$ and use Gauss-Green theorem to derive that

$$(d-p) \int_{B(\mathbf{0},r)} |\nabla u|^p \,\mathrm{d}\boldsymbol{x} = r \int_{\partial B(\mathbf{0},r)} |\nabla u|^p - p |\nabla u|^{p-2} (\partial_r u)^2 \,\mathrm{d}S_{\boldsymbol{x}}$$

where $\partial_r u := \frac{x}{|x|} \cdot \nabla u$ represents the radial derivative.

(2) Show that

$$r \to \frac{1}{r^{d-p}} \int_{B(\mathbf{0},r)} |\nabla u|^p \,\mathrm{d}\boldsymbol{x} \text{ is non-decreasing in } r. \tag{7.3.4}$$

Exercise 7.3.3 (*Derrick-Pohozaev's identity). Let u be the smooth solution to $-\Delta u = |u|^{p-1} u$ in Ω with $u|_{\partial\Omega} = 0$. Here $\Omega := B(\mathbf{0}, 1) \subset \mathbb{R}^d$ ($d \geq 3$) and $p > \frac{d+2}{d-2}$. Prove that

(1) $\int_\Omega |\nabla u|^2 \,\mathrm{d}\boldsymbol{x} = \int_\Omega |u|^{p+1} \,\mathrm{d}\boldsymbol{x}$.

(2) $\int_\Omega ((\boldsymbol{x} \cdot \nabla) u) |u|^{p-1} u \,\mathrm{d}\boldsymbol{x} = -\frac{d}{p+1} \int_\Omega |u|^{p+1} \,\mathrm{d}\boldsymbol{x}$. Here $(\boldsymbol{x} \cdot \nabla) = x_1 \partial_{x_1} + \cdots + x_d \partial_{x_d}$.

(3) $\frac{d-2}{2} \int_\Omega |\nabla u|^2 \,\mathrm{d}\boldsymbol{x} \leq \frac{d}{p+1} \int_\Omega |u|^{p+1} \,\mathrm{d}\boldsymbol{x}$, which together with (1) forces $u \equiv 0$ in Ω.

(Hint: $u = 0$ on $\partial\Omega$ implies that $\nabla u(\boldsymbol{x}) \parallel N(\boldsymbol{x})$ on $\partial\Omega$.)

Question 7.3

Question 7.3.1 (Almgren monotocity formula). Let u be a harmonic function in some region $\Omega \subseteq \mathbb{R}^d$ and assume $B(\mathbf{0}, R) \subset \Omega$, $u(\mathbf{0}) = 0$ and $u \not\equiv 0$. For $0 < r < R$, define the functions

$$a(r) := \frac{1}{r^{d-1}} \int_{\partial B(\mathbf{0},r)} u^2 \,\mathrm{d}S_{\boldsymbol{x}}, \quad a(r) := \frac{1}{r^{d-2}} \int_{B(\mathbf{0},r)} u^2 \,\mathrm{d}\boldsymbol{x}.$$

Inserting $p = 2$ in the monotonicity formula obtained in the previous exercise, we find that $b'(r) = \frac{2}{r^{d-2}} \int_{\partial B(\mathbf{0},r)} (\partial_r u)^2 \,\mathrm{d}S_{\boldsymbol{x}}$.

(1) Prove that

$$a'(r) = \frac{2}{r^{d-1}} \int_{\partial B(\mathbf{0},r)} u\, \partial_r u \,\mathrm{d}S_{\boldsymbol{x}} = \frac{2}{r} b.$$

(2) Prove that $b(r)^2 \leq \frac{r}{2} a(r) b'(r)$.

(3) Define the frequency function $f := \frac{b}{a}$ and derive **Almgren's monotonicity formula** $f'(r) \geq 0$. (Hint: Use the conclusion of (1).)

(4) Prove that $\frac{a'(r)}{a(r)} \leq \frac{\beta}{r}$ and thus $a(r) \geq \gamma r^{\beta}$ for all $0 < r < R$. Here $\beta := \frac{2b(R)}{a(R)}$ and $\gamma := \frac{a(R)}{R^{\beta}}$. This is an estimate from below on how fast a non-constant harmonic function must grow near a point where it vanishes. (Hint: Use the conclusion of (3) and the use (1).)

Question 7.3.2 (Conformal energy for the wave equation). The mapping

$$(t, \boldsymbol{x}) \longmapsto (\bar{t}, \bar{\boldsymbol{x}}) := \left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|^2 - t^2}, \frac{t}{|\boldsymbol{x}|^2 - t^2} \right), \quad \forall |\boldsymbol{x}| \neq t \tag{7.3.5}$$

is called hyperbolic inversion. The hyperbolic Kelvin transform $\mathcal{K}u = \bar{u}$ is defined by

$$\bar{u}(t, \boldsymbol{x}) := u(\bar{t}, \bar{\boldsymbol{x}}) \left| |\bar{\boldsymbol{x}}|^2 - \bar{t}^2 \right|^{\frac{d-1}{2}}. \tag{7.3.6}$$

(1) Show that if $\partial_t^2 u - \Delta u = 0$, then $\partial_t^2 \bar{u} - \Delta \bar{u} = 0$.

(2) Consider the variations

$$\boldsymbol{\mathfrak{X}}(t, \boldsymbol{x}, \tau) := \gamma(t + \tau(|\boldsymbol{x}|^2 - t^2), \boldsymbol{x}), \quad w(t, \boldsymbol{x}, \tau) := \gamma^{\frac{d-1}{2}} u(\boldsymbol{x}(t, \boldsymbol{x}, \tau)),$$

where $\gamma := \frac{|\boldsymbol{x}|^2 - t^2}{|\boldsymbol{x}|^2 - (t + \tau(|\boldsymbol{x}|^2 - t^2))^2}$. This is equivalent to apply the hyperbolic inversion (hyperbolic transform, resp.) to the variables $(t, \boldsymbol{x})$ (the function u, resp.), and then add τe_0, and lastly re-apply the hyperbolic inversion (hyperbolic transform, resp.) again. Show that the corresponding speed and multiplier are respectively given by

$$\mathbf{v} = (|\boldsymbol{x}|^2 + t^2, 2t\boldsymbol{x}), \quad m = (|\boldsymbol{x}|^2 + t^2)\partial_t u + 2t\boldsymbol{x} \cdot \nabla u + (d-1)tu.$$

(3) Prove the **Morawetz's identity** $c_t - \operatorname{div} \mathbf{r} = 0$ where

$$\mathbf{r} := \left((|\boldsymbol{x}|^2 + t^2)\partial_t u + 2t\boldsymbol{x} \cdot \nabla u + (d-1)tu \right) \nabla u + t((\partial_t u)^2 - |\nabla u|^2)\boldsymbol{x}, \tag{7.3.7}$$

$$\begin{aligned} c :=& \frac{(t + |\boldsymbol{x}|)^2}{4} \left(\partial_t u + \partial_r u + \frac{d-1}{2|\boldsymbol{x}|} u \right)^2 + \frac{(t - |\boldsymbol{x}|)^2}{4} \left(\partial_t u - \partial_r u - \frac{d-1}{2|\boldsymbol{x}|} u \right)^2 \\ &+ \frac{t^2 + |\boldsymbol{x}|^2}{2} \left(|\nabla u|^2 - (\partial_r u)^2 + \frac{(d-3)(d-1)}{4|\boldsymbol{x}|^2} u^2 \right) - \frac{d-1}{\operatorname{div}} \left(\frac{|\boldsymbol{x}|^2 + t^2}{|\boldsymbol{x}|^2} u \boldsymbol{x} \right). \end{aligned} \tag{7.3.8}$$

In Evans' book [6, Chapter 8.6], the author says "After a longish calculation, we derive Morawetz's identity."

Question 7.3.3 (Local energy decay for the wave equation). Assume u to be a smooth solution to the wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } (0,\infty)\times\Omega, \\ u(t,\boldsymbol{x}) = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ (u,\partial_t u)|_{t=0} = (u_0,u_1) \in C_c^\infty(\Omega). \end{cases} \tag{7.3.9}$$

Here $\Omega := \mathbb{R}^d \backslash \overline{O}$ with $O \subset \mathbb{R}^d$ a bounded, smooth open set that is star-shaped with respect to the origin, that is, the segment between any $\boldsymbol{x} \in O$ and $\mathbf{0}$ totally lies in O. Recall that when $d = 3, \Omega = \mathbb{R}^3, O = \emptyset$, the solution has $O(t^{-1})$ decay rate. In this example we aim to establish $O(t^{-2})$ decay within any bounded region. Assume also $O \subset B(\mathbf{0},R)$ and $d = 3$.

(1) Prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega c\,\mathrm{d}\boldsymbol{x} = \int_{\partial O} \mathbf{r}\cdot\nu\,\mathrm{d}S_{\boldsymbol{x}} \leq 0$$

where $c, \mathbf{r}$ are given by the Morawetz's identity and ν is the unit inward normal vector of ∂O.

(2) Use (1) and the concrete form of c to show that for any $t > 0$

$$\int_{B(\mathbf{0},R)\backslash O} \frac{(t+|\boldsymbol{x}|)^2}{4}\left(\partial_t u + \partial_r u + \frac{d-1}{2|\boldsymbol{x}|}u\right)^2 + \frac{(t-|\boldsymbol{x}|)^2}{4}\left(\partial_t u - \partial_r u - \frac{d-1}{2|\boldsymbol{x}|}u\right)^2 + \frac{|\boldsymbol{x}|^2+t^2}{2}(|\nabla u|^2 - (\partial_r u)^2)\,\mathrm{d}\boldsymbol{x} \lesssim 1.$$

(3) Taking $t \geq 2R$ to derive

$$\int_{B(\mathbf{0},R)\backslash O} |\nabla u|^2 - (\partial_r u)^2\,\mathrm{d}\boldsymbol{x} \lesssim \frac{1}{t^2} \tag{7.3.10}$$

$$\int_{B(\mathbf{0},R)\backslash O} (\partial_t u)^2 + (\partial_r u)^2 + \frac{d-1}{|\boldsymbol{x}|}u\,\partial_r u + \frac{(d-1)^2}{4|\boldsymbol{x}|^2}u^2\,\mathrm{d}\boldsymbol{x} \lesssim \frac{1}{t^2}. \tag{7.3.11}$$

(4) Use $\frac{u}{|\boldsymbol{x}|}\partial_r u = \operatorname{div}\left(\frac{u^2}{2|\boldsymbol{x}|^2}\boldsymbol{x}\right) - \frac{d-2}{2}\frac{u^2}{|\boldsymbol{x}|^2}$ and the second inequality in (3) to show that

$$\int_{B(\mathbf{0},R)\backslash O} (\partial_t u)^2 + (\partial_r u)^2\,\mathrm{d}\boldsymbol{x} \lesssim \frac{1}{t^2}, \quad t \geq 2R.$$

7.4 Existence and regularity of minimizers

4

7.5 Mountain-Pass theorem

Appendix A Notations

The first part of the appendix records the notations used throughout the lecture notes.

A.1 Geometric notations

- $\mathbb{R}^d = d$-dimensional real Euclidean space, $\mathbb{R} = \mathbb{R}^1$.
- A typical point in $\mathbb{R}^d$ is $\boldsymbol{x} = (x_1, \cdots, x_d)$.
- Given a set $U \subset \mathbb{R}^d$, we write $\partial U =$ boundary of U, $\overline{U} = U \cup \partial U =$ closure of U.
- $B(\boldsymbol{x}, r) \subset \mathbb{R}^d$: the open ball with center located at $\boldsymbol{x} \in \mathbb{R}^d$ and radius $r > 0$. $\bar{B}(\boldsymbol{x}, r) =$ closure of $B(\boldsymbol{x}, r)$, $\check{B}(\boldsymbol{x}, r) = B(\boldsymbol{x}, r) \backslash \{\boldsymbol{x}\} =$ punctured open ball with center located at $\boldsymbol{x} \in \mathbb{R}^d$ and radius $r > 0$.
- $\mathbb{S}^{d-1} = \partial B(\mathbf{0}, 1) = (d-1)$-dimensional unit sphere in $\mathbb{R}^d$.
- $\alpha(d)$= volume of unit ball in $\mathbb{R}^d = \frac{\pi^{\frac{d}{2}}}{\Gamma(1+\frac{d}{2})}$. $d\alpha(d) =$ surface area of $\mathbb{S}^{d-1}$.
- $\mathbb{R}^d_+ = \{\boldsymbol{x} \in \mathbb{R}^d : x_d > 0\} =$ open upper half-space.
- $e_i = (0, \cdots, 0, 1, \cdots, 0) = i^{\text{th}}$ standard coordinate vector.
- Let U, V be open subsets of $\mathbb{R}^d$. We write $V \Subset U$ if $V \subset \overline{V} \subset U$ and $\overline{V}$ is compact. We also say V is compactly contained in U.
- Given $T > 0$ and an open set $U \subset \mathbb{R}^d$, we define the parabolic cylinder by $UT := U \times (0, T]$ and its parabolic boundary $\Gamma_T := \overline{U}_T \backslash U_T$.

A.2 Notations for functions

- If $u : U \to \mathbb{R}$, we write $u(\boldsymbol{x}) = u(x_1, \cdots, x_d)$ ($\boldsymbol{x} \in U$). We say u is smooth if it is infinitely differentiable.
- If u, v are two functions, we write $u \equiv v$ to mean that u is identically equal to v. We write $u := v$ to define u as equaling v.
- The support of a function $u : U \to \mathbb{R}$ is denoted by Spt $u := \{\boldsymbol{x} \in U : u(\boldsymbol{x}) \neq 0\}$.
- The positive (negative, resp.) part of a function u is defined by $u^+ := \max\{u, 0\}$ ($u^- :=$

$-\min\{u, 0\}$, resp.). Then $u = u^+ - u^-$ and $|u| = u^+ + u^-$. The sign function is

$$\operatorname{sgn}(x) := \begin{cases} 1 & x > 0, \\ 0 & x = 0 \\ -1 & x < 0. \end{cases}$$

- If $\mathbf{u} : U \to \mathbb{R}^m$, we write $\mathbf{u}(\boldsymbol{x}) = (u_1(\boldsymbol{x}), \cdots, u_m(\boldsymbol{x}))$ $(\boldsymbol{x} \in U)$.
- If Σ is a smooth $(d-1)$-dimensional hypersurface in $\mathbb{R}^d$, we write $\int_\Sigma f(\boldsymbol{x})\,\mathrm{d}S_{\boldsymbol{x}}$ for the integral of f over Σ, with respect to $(d-1)$-dimensional surface measure. If C is a curve in $\mathbb{R}^d$, we write $\int_C f\,\mathrm{d}\ell$ for the integral of f over C with respect to arc-length.
- Averages:

$$\fint_U f\,\mathrm{d}\boldsymbol{x} = \frac{1}{\operatorname{Vol}(U)} \int_\Omega f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},$$

$$\fint_{\partial U} f\,\mathrm{d}S_{\boldsymbol{x}} = \frac{1}{\operatorname{Area}(\partial U)} \int_{\partial U} f(\boldsymbol{x})\,\mathrm{d}S_{\boldsymbol{x}}.$$

- indicator function $E \subset \mathbb{R}^d$. $\chi_E(\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} \in E \\ 0 & \boldsymbol{x} \notin E \end{cases}.$
- The convolution of two functions f, g in $\mathbb{R}^d$ is denoted by $f * g$, satisfying

$$(f * g)(\boldsymbol{x}) = \int_{\mathbb{R}^d} f(\boldsymbol{x}-\boldsymbol{y})g(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = \int_{\mathbb{R}^d} f(\boldsymbol{y})g(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}.$$

- We write $f = O(g)$ as $\boldsymbol{x} \to \boldsymbol{x}_0$ provided that there exists a constant C such that $|f(\boldsymbol{x})| \le C|g(\boldsymbol{x})|$ for all $\boldsymbol{x}$ sufficiently close to $\boldsymbol{x}_0$.
- We write $f = o(g)$ as $\boldsymbol{x} \to \boldsymbol{x}_0$ provided that $\lim\limits_{\boldsymbol{x}\to\boldsymbol{x}_0} \frac{|f(\boldsymbol{x})|}{|g(\boldsymbol{x})|} = 0$.

A.3 Notations for derivatives

Assume $u : U \to \mathbb{R}$, $\quad \boldsymbol{x} \in U$.

- $\frac{\partial u}{\partial x_i}(\boldsymbol{x}) = \lim\limits_{h\to 0} \frac{u(\boldsymbol{x}+he_i)-u(\boldsymbol{x})}{h}$, provided the limit exists. We usually write $\partial_{x_i} u$, $\partial_i u$, u_{x_i} for simplicity of notations. Similarly, we can define high-order derivatives.
- Multi-index notations:
  1. We write

$$\partial^\alpha u(\boldsymbol{x}) := \frac{\partial^{|\alpha|} u(\boldsymbol{x})}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} u,$$

  where $\alpha = (\alpha_1, \cdots, \alpha_d)$ is the multi-index with length $|\alpha| = \alpha_1 + \cdots + \alpha_k$.

2. Given a nonnegative integer k, we write $\partial^k u(\boldsymbol{x}) := \{\partial^\alpha u(\boldsymbol{x}) : |\alpha| = k\}$ to be the set of all partial derivatives of order k. We also reagrd $\partial^k u(\boldsymbol{x})$ as a point in $\mathbb{R}^{d^k}$ with

$$|\partial^k u| = \left(\sum_{|\alpha|=k} |\partial^\alpha u|^2 \right)^{\frac{1}{2}}.$$

3. If $k = 1$, we regard ∂u as being arranged in a vector and denote it by

$$\nabla u := (\partial_{x_1} u, \cdots, \partial_{x_d} u) = \text{ gradient vector.}$$

4. $u_r := \frac{\boldsymbol{x}}{|\boldsymbol{x}|} \cdot \nabla u$ represents the radial derivative of u.
5. If $k = 2$, then

$$\nabla^2 u := \begin{bmatrix} u_{x_1 x_1} & \cdots & u_{x_1 x_d} \\ & \cdots & \\ u_{x_d x_1} & \cdots & u_{x_d x_d} \end{bmatrix}$$

represents the Hessian matrix. $\Delta u = \sum_{i=1}^{d} u_{x_i x_i} = \mathrm{Tr}(\nabla^2 u)$ is the Laplacian of u.

- Let $\mathbf{u} : \mathbb{R}^d \to \mathbb{R}^m$ be a vector-valued function. We define

1. $\partial^\alpha \mathbf{u} = (\partial^\alpha u_1, \cdots, \partial^\alpha u_m)$ for each multi-index α. We can similarly define $\partial^k \mathbf{u}$ and $|\partial^k \mathbf{u}|$.
2. When $k = 1$, we write $\nabla \mathbf{u} := \begin{bmatrix} \partial_{x_1} u_1 & \cdots & \partial_{x_d} u_1 \\ & \cdots & \\ \partial_{x_1} u_m & \cdots & \partial_{x_d} u_m \end{bmatrix}$ to be the gradient matrix.
3. When $m = d$, we define the divergence of $\mathbf{u}$ to be

$$\mathrm{div}\, \mathbf{u} := \nabla \cdot \mathbf{u} = \mathrm{Tr}\, \nabla \mathbf{u} = \sum_{i=1}^{d} \partial_{x_i} u_i.$$

4. When $m = d = 3$, we define the curl of $\mathbf{u}$ to be curl $\mathbf{u} := \nabla \times \mathbf{u} = (\partial_{x_2} u_3 - \partial_{x_3} u_2, \partial_{x_3} u_1 - \partial_{x_1} u_3, \partial_{x_1} u_2 - \partial_{x_2} u_1)$. When $m = d = 2$, the curl of $\mathbf{u}$ becomes a scalar, defined by $\nabla^\perp \cdot \mathbf{u} = -\partial_{x_2} u_1 + \partial_{x_1} u_2$.

A.4 Notations for function spaces

Let U be an open subset of $\mathbb{R}^d$.

- $C(U) = \{u : U \to \mathbb{R} | u \text{ is continuous.}\}$
- $C(\overline{U}) = \{u \in C(U) : u \text{ is uniformly continuous on bounded subsets of } U.\}$.
- $C^k(U) = \{u : U \to \mathbb{R} : \partial^\alpha u \text{ exists and is continuous in } U,\ \forall 0 \leq |\alpha| \leq k\}$.
- $C^k(\overline{U}) = \{u \in C^k(U) : \partial^\alpha u \text{ is uniformly continuous on bounded subsets of } U,\ \forall 0 \leq |\alpha| \leq k\}$.

- $C^\infty(U) = \{u : U \to \mathbb{R} | u \text{ is infinitely differentiable in } U.\} = \bigcap_{k=0}^{\infty} C^k(U)$. $C^\infty(\overline{U}) = \bigcap_{k=0}^{\infty} C^k(\overline{U})$.
- $C_c(U), C_c^k(U), C_c^\infty(U)$ denote the functions in $C(U), C^k(U), C^\infty(U)$ (respectively) that have compact support.
- $C_1^2(I \times U) = \{u : I \times U \to \mathbb{R} : u, \partial_{x_i} u, \partial_{x_i}\partial_{x_j} u, \partial_t u \in C(I \times U), \forall 1 \leq i, j \leq d\}$. Here $I \subset \mathbb{R}$ is an interval and $U \subset \mathbb{R}^d$ is a domain. The variables $t \in I$ and $\boldsymbol{x} \in U$.
- $L^p(U) = \{u : U \to \mathbb{R} | u \text{ is Lebesgue measurable in } U,\ \|u\|_{L^p(U)} < \infty\}$ with

$$\|u\|_{L^p(U)} := \left(\int_\Omega |u|^p \, \mathrm{d}\boldsymbol{x} \right)^{1/p}, \ (1 \leq p < \infty).$$

- $L^\infty(U) = \{u : U \to \mathbb{R} | u \text{ is Lebesgue measurable in } U,\ \|u\|_{L^\infty(U)} < \infty\}$ with

$$\|u\|_{L^\infty(U)} = \operatorname{ess\,sup}_U u := \inf\{M \in \mathbb{R} | \text{The set } \{\boldsymbol{x} | u(\boldsymbol{x}) > M\} \text{ has zero Lebesgue measure.}\}.$$

- $L^p_{\text{loc}}(U) = \{u : U \to \mathbb{R} | u \in L^p(V) \text{ for each } V \Subset U\}$.
- $\|\partial^k u\|_{L^p(U)} = \||\partial^k u|\|_{L^p(U)}$.
- The spaces $C(U \to \mathbb{R}^m), L^p(U \to \mathbb{R}^m)$, etc. consist of those functions $\mathbf{u} : U \to \mathbb{R}^m$ with each component belonging to the correspondiong spaces.
- Schwartz class

$$\mathcal{S}(\mathbb{R}^d) := \{u \in C^\infty(\mathbb{R}^d) : \|u\|_{(N,\alpha)} < \infty \ \forall N \in \mathbb{N} \text{ and multi-indices } \alpha\}.$$

  Here the semi-norm is defined by

$$\|u\|_{(N,\alpha)} := \sup_{\boldsymbol{x} \in \mathbb{R}^d} (1 + |\boldsymbol{x}|)^N |\partial^\alpha u(\boldsymbol{x})|.$$

  Then $(\mathcal{S}(\mathbb{R}^d), \|\cdot\|_{(N,\alpha)})$ is a Fréchet space.

Appendix B Tools in Multi-variable Calculus

Notations: Let $\Omega \subset \mathbb{R}^d$ be a bounded open subset of $\mathbb{R}^d$ and its boundary $\partial\Omega$ is C^1. Denote $\overline{\Omega}$ to be the closure of Ω. Let $N = (N_1, \cdots, N_d)$ be the unit outer (pointing outward) normal vector of $\partial\Omega$. Let $\nabla := (\partial_1, \cdots, \partial_d)$ be the gradient (nabla) operator and $\Delta := \partial_{x_1}^2 + \cdots + \partial_{x_d}^2 = \nabla \cdot \nabla$ be the Laplacian operator. Let $u, v, w : \overline{\Omega} \to \mathbb{R}$ be functions and $\mathbf{u}, \mathbf{v}, \mathbf{w} : \overline{\Omega} \to \mathbb{R}^d$ be vector fields in $\mathbb{R}^d$. Let $B(\boldsymbol{x}, r) \subset \Omega$ be an open ball centered at $\boldsymbol{x} \in \mathbb{R}^d$ with radius $r > 0$.

B.1 Formulas of integration by parts

Let us assume the following lemmas are true.

Lemma B.1.1. Suppose $u \in C^1(\Omega)$. Then

$$\int_\Omega \partial_{x_i} u \,\mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} u\, N_i \,\mathrm{d}S, \quad 1 \le i \le d.$$

Based on this lemma, we conclude the following propositions

Proposition B.1.2. The following identities hold.

1. (divergence theorem) For each vector field $\mathbf{u} \in C^1(\overline{\Omega} \to \mathbb{R}^d)$, we have

$$\int_\Omega \nabla \cdot \mathbf{u} \,\mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} \mathbf{u} \cdot N \,\mathrm{d}S.$$

2. (integration by parts) Suppose $u, v \in C^1(\overline{\Omega})$. Then

$$\int_\Omega \partial_{x_i} u\, v \,\mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} uv\, N_i \,\mathrm{d}S - \int_\Omega u\, \partial_{x_i} v \,\mathrm{d}\boldsymbol{x}, \quad 1 \le i \le d.$$

Recall that $\Delta u := \mathrm{div}\,(\nabla u) = \nabla \cdot (\nabla u) = \sum_{i=1}^{d} \partial_{x_i}^2 u$. Then we have the following properties

Proposition B.1.3 (Green's formulas). Assume $u, v \in C^2(\overline{\Omega})$. Prove the following variants of Gauss-Green formulas.

1. $\int_\Omega \Delta u \,\mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} \frac{\partial u}{\partial N} \,\mathrm{d}S$.
2. $\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}\boldsymbol{x} = -\int_\Omega u\Delta v \,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} u\frac{\partial v}{\partial N} \,\mathrm{d}S$.
3. $\int_\Omega u\Delta v - v\Delta u \,\mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} u\frac{\partial v}{\partial N} - v\frac{\partial u}{\partial N} \,\mathrm{d}S$.

When $\Delta u = 0$ in Ω, we say u is a harmonic function in Ω. A straightforward corollary of Proposition B.1.3 is concluded as follows.

Corollary B.1.4. Assume $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a harmonic function in Ω. Show that

1. $\int_{\partial\Omega} \frac{\partial u}{\partial N} \,\mathrm{d}S = 0$.
2. $\int_\Omega |\nabla u|^2 \,\mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} u\frac{\partial u}{\partial N} \,\mathrm{d}S$.

When the spatial dimensions $d = 3$, we define curl $\mathbf{u} := \nabla\times\mathbf{u} = (\partial_{x_2}u_3 - \partial_{x_3}u_2, \partial_{x_3}u_1 - \partial_{x_1}u_3, \partial_{x_1}u_2 - \partial_{x_2}u_1)$ to be the curl operator. This is important in the analysis of lots of PDEs arising from physics, especially fluids mechanics and electro-magnetics.

Proposition B.1.5 (Calculus of the vector product). Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in C^2(\overline{\Omega} \to \mathbb{R}^3)$, $f \in C^1(\overline{\Omega})$. Then

1. $\nabla \times (\nabla f) = \mathbf{0}$, $\nabla \cdot (\nabla \times \mathbf{u}) = 0$.
2. $\nabla \times (f\mathbf{u}) = f(\nabla \times \mathbf{u}) + (\nabla f) \times \mathbf{u}$. In particular, $\nabla \times (f(|\boldsymbol{x}|)\boldsymbol{x}) = \mathbf{0}$ for $f \in C^1(\mathbb{R} \to \mathbb{R})$. (Thus, the static electric field has no vorticity.)
3. Assume Ω is simply-connected. If $\nabla\times\mathbf{u} = 0$ in Ω, then there exists a function φ such that $\mathbf{u} = \nabla\varphi$.
4. $\nabla \cdot (\mathbf{u} \times \mathbf{v}) = (\nabla \times \mathbf{u}) \cdot \mathbf{v} - (\nabla \times \mathbf{v}) \cdot \mathbf{u}$.
5. $\nabla \times (\mathbf{u} \times \mathbf{v}) = \mathbf{u}(\nabla \cdot \mathbf{v}) - \mathbf{v}(\nabla \cdot \mathbf{u}) + (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v}$
6. $\mathbf{u} \times (\nabla \times \mathbf{v}) = (\nabla\mathbf{v}) \cdot \mathbf{u} - \mathbf{u} \cdot (\nabla\mathbf{v})$.
7. $\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \Delta\mathbf{u}$.
8. $\int_\Omega \nabla \times \mathbf{u} \,\mathrm{d}\boldsymbol{x} = -\int_{\partial\Omega}(\mathbf{u} \times N) \,\mathrm{d}S_{\boldsymbol{x}}$.
9. $\int_\Omega \mathbf{u} \cdot (\nabla \times \mathbf{v}) \,\mathrm{d}\boldsymbol{x} = -\int_{\partial\Omega}(\mathbf{u} \times \mathbf{v}) \cdot N \,\mathrm{d}S_{\boldsymbol{x}} + \int_\Omega(\nabla \times \mathbf{u}) \cdot \mathbf{v} \,\mathrm{d}\boldsymbol{x}$.

B.2 Polar coordinates and moving regions

The following lemma allows us to convert d-dimensional integrals into integrals over spheres.

Lemma B.2.1 (Polar coordinates). Suppose $u : \mathbb{R}^d \to \mathbb{R}$ is continuous and integrable. Then

1. For each $\boldsymbol{x}_0 \in \mathbb{R}^d$,
$$\int_{\mathbb{R}^d} u \,\mathrm{d}\boldsymbol{x} = \int_0^\infty \left(\int_{\partial B(\boldsymbol{x}_0,\rho)} u(\boldsymbol{y}) \,\mathrm{d}S_{\boldsymbol{y}} \right) \mathrm{d}\rho.$$
2. For each $R > 0$ and $\boldsymbol{x}_0 \in \mathbb{R}^d$,
$$\int_{B(\boldsymbol{x}_0,R)} u \,\mathrm{d}\boldsymbol{x} = \int_0^R \left(\int_{\partial B(\boldsymbol{x}_0,\rho)} u(\boldsymbol{y}) \,\mathrm{d}S_{\boldsymbol{y}} \right) \mathrm{d}\rho.$$

The above lemma is a special case of the so-called "co-area formula".

Theorem B.2.2 (Coarea formula)**.** Let $u : \mathbb{R}^d \to \mathbb{R}$ be Lipschitz continuous and assume that, for a.e. $r \in \mathbb{R}$, the level set $\{\boldsymbol{x} \in \mathbb{R}^d | u(\boldsymbol{x}) = r\}$ is a smooth, $(d-1)$-dimensional hypersurface in $\mathbb{R}^d$. Suppose also $f : \mathbb{R}^d \to \mathbb{R}$ is continuous and Lebesgue integrable. Then

$$\int_{\mathbb{R}^d} f(\boldsymbol{x}) |\nabla u(\boldsymbol{x})| \, \mathrm{d}\boldsymbol{x} = \int_{-\infty}^{+\infty} \left(\int_{\{u=r\}} f(\boldsymbol{x}) \, \mathrm{d}S_{\boldsymbol{x}} \right) \mathrm{d}r.$$

Lemma B.2.1 follows from Theorem B.2.2 by taking $u(\boldsymbol{x}) = |\boldsymbol{x} - \boldsymbol{x}_0|$. The proof of Theorem B.2.2 is referred to Evans-Gariepy [7, Chapter 3].

Next, we introduce the differentiaion formula for integrals over moving regions. Consider a family of smooth bounded regions $\Omega(t) \subset \mathbb{R}^d$ that smoothly depends on the parameter $t \in \mathbb{R}$. Let $\mathbf{v}$ be the velocity of the moving boundary $\partial\Omega(t)$ and N be the unit outward normal vector of $\partial\Omega(t)$.

Theorem B.2.3. Let $f = f(\boldsymbol{x}, t)$ be a smooth function. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega(t)} f \, \mathrm{d}\boldsymbol{x} = \int_{\partial\Omega(t)} f(\mathbf{v} \cdot N) \, \mathrm{d}S_{\boldsymbol{x}} + \int_{\Omega(t)} \partial_t f \, \mathrm{d}\boldsymbol{x}.$$

B.3 Grönwall-type inequalities

This section records Grönwall-type inequalities which play an important role in the energy estimates of various evolutionary PDEs.

Theorem B.3.1 (Differential form of Grönwall-type inequalities)**.** Let $\eta(\cdot)$ be a nonnegative, absolutely continuous function on $[0, T]$, which satisfies for a.e. t the differential inequality

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t)$$

where $\phi(t)$ and $\psi(t)$ are nonnegative, summable functions on $[0, T]$. Then

$$\eta(t) \leq e^{\int_0^t \phi(s) ds} \left[\eta(0) + \int_0^t \psi(s) ds \right]$$

for all $0 \leq t \leq T$. In particular, if $\eta' \leq \phi\eta$ on $[0, T]$ and $\eta(0) = 0$, then $\eta \equiv 0$ on $[0, T]$.

Proof. From the differential inequality, we see

$$\frac{\mathrm{d}}{\mathrm{d}s} \left(\eta(s) e^{-\int_0^s \phi(r) \, \mathrm{d}r} \right) = e^{-\int_0^s \phi(r) \, \mathrm{d}r} \left(\eta'(s) - \phi(s)\eta(s) \right) \leq e^{-\int_0^s \phi(r) \, \mathrm{d}r} \psi(s)$$

holds for a.e. $0 \leq s \leq T$. Consequently for each $0 \leq t \leq T$, we have

$$\eta(t) e^{-\int_0^t \phi(r) \, \mathrm{d}r} \leq \eta(0) + \int_0^t e^{-\int_0^s \phi(r) dr} \psi(s) \, \mathrm{d}s \leq \eta(0) + \int_0^t \psi(s) \, \mathrm{d}s.$$

□

Equivalently, we have the following Grönwall-type inequalities in the integral form.

Theorem B.3.2 (Integral form of Grönwall-type inequalities). Let $\xi(t)$ be a nonnegative, summable function on $[0,T]$ which satisfies for a.e. t the integral inequality

$$\xi(t) \leq C_1 \int_0^t \xi(s)\,\mathrm{d}s + C_2$$

for constants $C_1, C_2 \geq 0$. Then

$$\xi(t) \leq C_2 \left(1 + C_1 t e^{C_1 t}\right), \qquad \text{a.e. } t \in [0,T].$$

In particular, if $\xi(t) \leq C_1 \int_0^t \xi(s)\,\mathrm{d}s$ for a.e. $0 \leq t \leq T$, then $\xi(t) = 0$ for a.e. $t \in [0,T]$.

Proof. Let $\eta(t) := \int_0^t \xi(s)\,\mathrm{d}s$ and then $\eta' \leq C_1\eta + C_2$ a.e. in $[0,T]$. According to the differential form of Gronwall's inequality above

$$\eta(t) \leq e^{C_1 t}\left(\eta(0) + C_2 t\right) = C_2 t e^{C_1 t}$$

Then we get

$$\xi(t) \leq C_1\eta(t) + C_2 \leq C_2\left(1 + C_1 t e^{C_1 t}\right).$$

□

Remark B.3.1. The above two inequalities, especially the second one, are often used to establish continuation criteria for various linear or nonlinear PDEs, that is, the solutions blow up at certain time T_* if and only if some quantities blow up at this time T_*. Also, if $\phi(t)$ is a decaying factor, we may also use the above inequalities to compute the lifespan of solutions to various PDEs. However, for some quasilinear or fully nonlinear equations, we may use a more general version of Grönwall-type inequality to close the energy estimates, that is,

$$E(t) \leq P(E(0)) + P(E(t)) \int_0^t P(E(s))\,\mathrm{d}s \Rightarrow \exists T > 0, \text{ such that } \sup_{t \in [0,T]} E(t) \leq P(E(0)),$$

where $P(\cdots)$ represents a generic polynomial (with non-negative coefficients) in its arguments. See Tao [17, Chapter 2] for details.

Appendix C L^p spaces

This section records basic properties of L^p spaces, smooth approximations and frequently-used inequalities related to L^p spaces. They can be found in Evans [6, Appendix B] and Folland [8, Chapter 6].

C.1 L^p inequalities and dual spaces

Let $(X, \mathcal{M}, \mu)$ be a measure space. For $1 \le p \le \infty$, we define $L^p(X)$ by

$$L^p(X, \mathcal{M}, \mu) := \left\{ f : X \to \mathbb{C} : f \text{ is measurable and } \|f\|_{L^p} < \infty \right\},$$

where

$$\|f\|_{L^p} := \begin{cases} \left(\int_X |f|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}} & 1 \le p < \infty \\ \operatorname*{ess\,sup}_X f = \inf \{M : \mu\{\boldsymbol{x} : |f(\boldsymbol{x})| > M\} = 0\} & p = \infty \end{cases}.$$

In many cases, we abbreviate $L^p(X, \mathcal{M}, \mu)$ by $L^p(\mu)$ or $L^p(X)$ or simply L^p. $L^p(X, \mathcal{M}, \mu)$ equipped with $\|\cdot\|_X$ norm is a Banach space when $1 \le p \le \infty$ and we refer to Folland [8, Theorem 6.6, 6.8] for the proof.

C.1.1 Basic L^p inequalities

There are also two important inequalities that are repeatedly used

- Hölder's inequality: $\|fg\|_{L^1} \le \|f\|_{L^p}\|g\|_{L^{p'}}$ with $p^{-1} + (p')^{-1} = 1,\ 1 \le p, p' \le \infty$.
- Minkowski's inequality (triangle inequality): $\|f + g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p},\ 1 \le p \le \infty$.

L^p spaces satisfy the following inclusions

Proposition C.1.1. Let $1 \le p \le q \le r \le \infty$ be three indices. Then

(1) $L^p \cap L^r \subset L^q \subset L^p + L^r$;

(2) If $\mu(X) < \infty$, then $L^q \subset L^p$ and $\|f\|_{L^p} \le \|f\|_{L^q}\mu(X)^{\frac{1}{p}-\frac{1}{q}}$;

(3) If $\inf\{\mu(F) : F \in \mathcal{M}, F \subset X, \mu(F) > 0\} \ge c_0 > 0$ for some positive constant c_0, then $L^p \subset L^q$. In particular, $\ell^p(\mathbb{Z}) \subset \ell^q(\mathbb{Z})$.

Proof. (1) Using Hölder's inequality, we have

$$\|f\|_{L^q} \leq \|f\|_{L^p}^{\theta} \|f\|_{L^r}^{1-\theta}, \quad \frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r}.$$

This proves $L^p \cap L^r \subset L^q$. Next, we verify $L^q \subset L^p + L^r$. Given any $f \in L^q$, we split it into two parts $f = f\chi_E + f\chi_{E^c}$ with $E := \{\boldsymbol{x} : |f(\boldsymbol{x})| > 1\}$. Then we can directly verify that

$$|f\chi_E|^p = |f|^p \chi_E \leq |f|^q \chi_E \Rightarrow f\chi_E \in L^p$$

and

$$|f|^q \chi_{E^c} \geq |f\chi_{E^c}|^r \Rightarrow f\chi_{E^c} \in L^r.$$

(2) If $q = \infty$, then the conclusion is trivial. When $q < \infty$, we use Hölder's inequality to get

$$\|f\|_{L^p}^p = \int_X |f|^p \cdot 1 \mathrm{d}\mu \leq \||f|^p\|_{L^{q/p}} \|1\|_{L^{q/(q-p)}} = \|f\|_{L^q}^p \mu(X)^{1-\frac{p}{q}}.$$

(3) For sake of simplicity, we assume $\|f\|_{L^p} = 1$ and $c_0 = 1$. Then for any $\varepsilon > 0$, we see that

$$\mu\{\boldsymbol{x} \in X : |f(\boldsymbol{x})| > 1+\varepsilon\} \leq (1+\varepsilon)^{-p} \int_X |f|^p \, \mathrm{d}\mu < 1 \Rightarrow \mu\{\boldsymbol{x} \in X : |f(\boldsymbol{x})| > 1+\varepsilon\} = 0,$$

and thus $\mu\{\boldsymbol{x} \in X : |f(\boldsymbol{x})| > 1 = 0$. Then it is easy to see that $\int_X |f|^q \mathrm{d}\mu \leq \int_X |f|^p \mathrm{d}\mu = 1$. □

When $\mu(X) < \infty$, then $L^p(X)$ norm converges to $L^\infty(X)$ norm as $p \to \infty$.

Proposition C.1.2. Suppose $\mu(X) < \infty$ and $f \in L^\infty(X)$. Then $f \in L^p(X)$ for any $p < \infty$ and $\lim\limits_{p\to\infty} \|f\|_{L^p} = \|f\|_{L^\infty}$.

Proof. The $\leq$ part is straightforward thanks to Proposition C.1.1(2), that is, $\limsup\limits_{p\to\infty} \|f\|_{L^p} \leq \|f\|_{L^\infty}$. For the $\geq$ part, given $\varepsilon > 0$, by definition of L^∞-norm, we know there exists some $\delta > 0$ such that

$$\mu\{\boldsymbol{x} : |f(\boldsymbol{x})| \geq \|f\|_{L^\infty} - \varepsilon\} \geq \delta,$$

and hence

$$\int_X |f|^p \, \mathrm{d}\mu \geq \delta(\|f\|_{L^\infty} - \varepsilon)^p.$$

Therefore, we know $\liminf\limits_{p\to\infty} \|f\|_{L^p} \geq \|f\|_{L^\infty} - \varepsilon$ holds for all $\varepsilon > 0$. Letting $\varepsilon \to 0$, we obtain our desired result. □

Remark C.1.1. The condition $\mu(X) < \infty$ can be removed if we additionally assume $f \in L^p \cap L^\infty$ (which implies $f \in L^q$ for all $q > p$ thanks to Proposition C.1.1(1)).

C.1.2 Equivalent definition of L^p norms via duality

Let p, p' be conjugate exponents. Hölder's inequality shows that each $g \in L^q$ defines a bounded linear functional ϕ_g on L^p by $\phi_g(f) := \int_X fg$ and its operator norm does not exceed $\|g\|_{p'}$. In fact, the map $g \longmapsto \phi_g$ is almost always an isometry from $L^{p'}$ into $(L^p)^*$.

Proposition C.1.3. Suppose p, p' are conjugate exponents and $1 \le p' < \infty$. If $g \in L^{p'}$, then

$$\|g\|_{L^{p'}} = \|\phi_g\| = \sup\left\{\left|\int_X fg\right| : \|f\|_{L^p} = 1\right\}.$$

If μ is semi-finite[1] , then the result also holds for $p' = \infty$.

Proof. The $\ge$ part is trivial thanks to Hölder's inequality. For the $\le$ part, when $p' < \infty$, we shall pick

$$f = \frac{|g|^{p'-1}\mathrm{sgn} g}{\|g\|_{p'}^{p-1}}.$$

When $p' = \infty$, for $\varepsilon > 0$ we define $E := \{\boldsymbol{x} : |g(\boldsymbol{x})| \ge \|g\|_{L^\infty} - \varepsilon\}$. Then $\mu(E) > 0$. By the semi-finite property, there exists $F \subset A$ with $0 < \mu(F) < \infty$. Then pick $f = \mu(F)^{-1}\chi_F \mathrm{sgn}\, g$. For details, we refer to Folland [8, Prop. 6.13]. □

Conversely, if $f \longmapsto \int fg$ is a bounded linear functional on L^p, then $g \in L^{p'}$ in almost all cases.

Proposition C.1.4 ([8, Theorem 6.14]). Suppose

- μ is semi-finite;
- g is a measurable function on X such that $fg \in L^1$ for all f that are simple functions supported in a finite-measure set;
- $M_{p'}(g) := \sup\{|\int fg| : f \text{ simple }, \|f\|_{L^p} = 1\} < \infty$.

Then $g \in L^{p'}$ and $M_{p'}(g) = \|g\|_{L^{p'}}$.

Then we can conclude the following duality theorem

Theorem C.1.5 ([8, Theorem 6.15], Duality of L^p). When $1 < p < \infty$, for each $\phi \in (L^p)^*$, there exists a $g \in L^{p'}$ such that $\phi(f) = \int fg$ for all $f \in L^p$ and hence $L^{p'}$ is isometrically isomorphic to $(L^p)^*$. When μ is σ-finite, then the same conclusion holds for $p = 1$. In particular, $L^p(X)$ is reflexive when $1 < p < \infty$.

As a result of the above theorem, we conclude that L^p norm has the following equivalent definition

$$\|f\|_{L^p} = \sup_{\substack{g \in L^{p'} \\ \|g\|_{L^{p'}} \le 1}} \int_X fg \, \mathrm{d}\mu. \tag{C.1.1}$$

Based on this definition, we can prove the Minkowski's inequality for integrals.

[1]We say μ is semi-finite if for any $F \in \mathcal{N}$ with $\nu(F) = \infty$, there exists a subset $K \in \mathcal{N}$, $K \subset F$ satisfying $0 < \nu(K) < \infty$.

Theorem C.1.6 (Minkowski's inequality for integrals). Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces and let $f : X \times Y \to \mathbb{R}$ be an $(\mathcal{M} \otimes \mathcal{N})$-measurable function.

- If $f \geq 0$ and $1 \leq p < \infty$, then

$$\left[\int_X \left(\int_Y f(\boldsymbol{x}, \boldsymbol{y})\, \mathrm{d}\nu(\boldsymbol{y})\right)^p \mathrm{d}\mu(\boldsymbol{x})\right]^{1/p} \leq \int_Y \left[\int_X f(\boldsymbol{x}, \boldsymbol{y})^p\, \mathrm{d}\mu(\boldsymbol{x})\right]^{1/p} \mathrm{d}\nu(\boldsymbol{y}).$$

- If $1 \leq p \leq \infty$, $f(\cdot, \boldsymbol{y}) \in L^p(\nu)$ for a.e. $\boldsymbol{y} \in Y$ and $\boldsymbol{y} \longmapsto \|f(\cdot, \boldsymbol{y})\|_{L^p}$ is in $L^1(\nu)$, then $f(\boldsymbol{x}, \cdot) \in L^1(\nu)$ for a.e. $\boldsymbol{x} \in X$, the function $\boldsymbol{x} \to \int_Y f(\boldsymbol{x}, \boldsymbol{y})\, \mathrm{d}\nu(\boldsymbol{y})$ is in $L^p(\mu)$ and

$$\left\|\int_Y f(\cdot, \boldsymbol{y})\, \mathrm{d}\nu(\boldsymbol{y})\right\|_{L^p} \leq \int_Y \|f(\cdot, \boldsymbol{y})\|_{L^p}\, \mathrm{d}\nu(\boldsymbol{y}).$$

Proof. We only prove (1). (2) is a direct consequence of (1) (with f replaced by $|f|$) and Fubini's theorem.

When $p = 1$, (1) again becomes Tonelli's theorem. When $1 < p < \infty$, let p' be the conjugate exponent to p and let $g \in L^{p'}(\mu)$ with $\|g\|_{L^{p'}} \leq 1$. Then by Tonelli's theorem and Hölder's inequality, we have

$$\begin{aligned}\int_X \left(\int_Y f(\boldsymbol{x}, \boldsymbol{y})\, \mathrm{d}\nu(\boldsymbol{y})\right) |g(\boldsymbol{x})|\, \mathrm{d}\mu(\boldsymbol{x}) &= \iint_{X\times Y} f(\boldsymbol{x}, \boldsymbol{y}) |g(\boldsymbol{x})|\, \mathrm{d}\mu(\boldsymbol{x})\, \mathrm{d}\nu(\boldsymbol{y}) \\ &\leq \|g\|_{L^{p'}} \int_Y \left[\int_X f(\boldsymbol{x}, \boldsymbol{y})^p\, \mathrm{d}\mu(\boldsymbol{x})\right]^{1/p} \mathrm{d}\nu(\boldsymbol{y}).\end{aligned}$$

Taking supremum of the left side over all $g \in L^{p'}(\mu)$ with $\|g\|_{L^{p'}} \leq 1$ leads to our desired inequality thanks to (C.1.1). □

C.1.3 Equivalent definition of L^p norms via distribution function

The L^p norm can also be equivalently written as the weighted integral of the measure of the level sets. This is actually the generalization of the equivalent definition of Lebesgue measure, that is, partitioning the range of a function f instead of the domain of f. To be precise, let f be a measurable function on $(X, \mathcal{M}, \mu)$, we define its **distribution function** $\lambda_f : \mathbb{R}_+ \to [0, \infty]$ by

$$\lambda_f(\alpha) := \mu\{\boldsymbol{x} \in X : |f(\boldsymbol{x})| > \alpha\}.$$

The distribution function satisfies the following basic properties

Proposition C.1.7.

(1) λ_f is decreasing and right continuous.
(2) If $f \leq g$, then $\lambda_f \leq \lambda_g$.

(3) If $|f_n|$ increases to $|f|$, then so does λ_{f_n} to λ_f.
(4) If $f = g + h$, then $\lambda_f(\alpha) \le \lambda_g(\alpha/2) + \lambda_h(\alpha/2)$.

The equivalent definition of L^p norm is concluded as follows

Theorem C.1.8. Let $0 < p < \infty$. Then

$$\int_X |f|^p \, d\mu = p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha) \, d\alpha.$$

Proof. Write $\lambda_f(\alpha) = \int_X \chi_{\{\boldsymbol{x} : |f(\boldsymbol{x})| > \alpha\}} \, d\mu$ and commute the integrals on the right side. One can always commutes these two integrals because of the non-negativity of the integrands and Tonelli's theorem. □

Exercise C.1

Exercise C.1.1. Let $1 \le p \le r \le \infty$. Prove that $(L^p \cap L^r, \|\cdot\|_{L^p} + \|\cdot\|_{L^r})$ is a Banach space.

Exercise C.1.2. Let $1 \le p \le r \le \infty$. Prove that $(L^p + L^r, \|\cdot\|_{L^p+L^r})$ is a Banach space. Here

$$\|f\|_{L^p+L^r} := \inf\{\|f_0\|_{L^p} + \|f_1\|_{L^r} : f = f_0 + f_1, f_0 \in L^p, f_1 \in L^r\}.$$

Exercise C.1.3. Suppose $1 \le p < \infty$. If $\|f_n - f\|_{L^p} \to 0$, then $f_n \to f$ in measure and hence has a subsequence converging to f a.e.. Conversely, if $f_n \to f$ in measure and $|f_n| \le g \in L^p$ for all n and a.e. $\boldsymbol{x}$, then $\|f_n - f\|_{L^p} \to 0$.

Exercise C.1.4. Suppose $1 \le p < \infty$. If $f_n, f \in L^p$ and $f_n \to f$ a.e., then $\|f_n - f\|_{L^p}$ if and only if $\|f_n\|_{L^p} \to \|f\|_{L^p}$.

Exercise C.1.5. Prove that $L^p(\mathbb{R}^d)$ (with Lebesgue measure) is separable when $1 \le p < \infty$ but not separable when $p = \infty$.

(Hint: When $p < \infty$, consider rational linear combination of characteristic functions of finite unions of rectangles with sides that are intervals with rational endpoints. When $p = \infty$, consider $f_r := \chi_{B(\mathbf{0},r)}$.)

Exercise C.1.6 (Chebyshev's inequality). Prove that if $f \in L^p(0 < p < \infty)$, then for any $\alpha > 0$,

$$\mu(\{x : |f(x)| > \alpha\}) \le \left[\frac{\|f\|_p}{\alpha}\right]^p.$$

Exercise C.1.7. Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces and let $f : X \times Y \to \mathbb{R}$ be an $(\mathcal{M} \otimes \mathcal{N})$-measurable function. Suppose there exists $C > 0$ such that $\int_X |K(\boldsymbol{x}, \boldsymbol{y})| \, d\mu(\boldsymbol{x}) \le C$ for a.e. $\boldsymbol{y} \in Y$ and $\int_Y |K(\boldsymbol{x}, \boldsymbol{y})| \, d\nu(\boldsymbol{y}) \le C$ for a.e. $\boldsymbol{x} \in X$. If $f \in L^p(\nu)$, $1 \le p \le \infty$, then the integral

$$Tf(\boldsymbol{x}) := \int_Y K(\boldsymbol{x}, \boldsymbol{y}) f(\boldsymbol{y}) \, d\nu(\boldsymbol{y})$$

converges absolutely for a.e. $\boldsymbol{x} \in X$, and the function Tf is defined in $L^p(\mu)$ with $\|Tf\|_{L^p} \le C\|f\|_{L^p}$.

Exercise C.1.8. Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces and $K \in L^2(\mu \times \nu)$. If $f \in L^2(\nu)$, then prove that the integral

$$Tf(\boldsymbol{x}) := \int_Y K(\boldsymbol{x}, \boldsymbol{y}) f(\boldsymbol{y}) \, \mathrm{d}\nu(\boldsymbol{y})$$

converges absolutely for a.e. $\boldsymbol{x} \in X$, and the function Tf is defined in $L^2(\mu)$ with $\|Tf\|_{L^2} \leq \|K\|_{L^2} \|f\|_{L^2}$.

Exercise C.1.9. Prove that

(1) $f \in L^p$ iff $\sum\limits_{k=-\infty}^{\infty} 2^{kp} \lambda_f(2^k) < \infty$.
(2) If $f \in L^p$, then $\lim\limits_{\alpha \to 0} \alpha^p \lambda_f(\alpha) = \lim\limits_{\alpha \to \infty} \alpha^p \lambda_f(\alpha) = 0$.

Exercise C.1.10 (Non-decreasing rearrangement)**.** If f is a measurable function on X, its decreasing rearrangement is the function $f^* : (0, \infty) \to [0, \infty]$ defined by

$$f^*(t) = \inf \left\{ \alpha : \lambda_f(\alpha) \leq t \right\} \quad (\text{ where } \inf \varnothing = \infty)$$

(1) f^* is decreasing. If $f^*(t) < \infty$ then $\lambda_f(f^*(t)) \leq t$, and if $\lambda_f(\alpha) < \infty$ then $f^*(\lambda_f(\alpha)) \leq \alpha$.
(2) $\lambda_f = \lambda_f$, where α_{f^*} is defined with respect to Lebesgue measure on $(0, \infty)$.
(3) If $\lambda_f(\alpha) < \infty$ for all $\alpha > 0$ and $\lim\limits_{\alpha \to \infty} \lambda_f(\alpha) = 0$ (so that $f^*(t) < \infty$ for all $t > 0$), and ϕ is a nonnegative measurable function on $(0, \infty)$, then $\int_X \phi \circ |f| \, \mathrm{d}\mu = \int_0^\infty \phi \circ f^*(t) dt$. In particular, $\|f\|_{L^p} = \|f^*\|_{L^p}$ for $0 < p < \infty$.
(4) If $0 < p < \infty$, then the weak L^p norm satisfies $[f]_{L^p} := (\sup\limits_{\alpha > 0} \alpha^p \lambda_f(\alpha))^{1/p} = \sup\limits_{t > 0} t^{1/p} f^*(t)$.

C.2 Convolution and smooth approximation

We next introduce tools that allow us to construct smooth approximations to given functions by means of convolution.

Definition C.2.1. We introduce the following notations

- If $U \subset \mathbb{R}^d$ is open and $\varepsilon > 0$, we write $U_\varepsilon := \{\boldsymbol{x} \in U : \operatorname{dist}(\boldsymbol{x}, \partial U) > \varepsilon\}$.
- Define $\eta \in C^\infty(\mathbb{R}^d)$ to be the bump function

$$\eta(\boldsymbol{x}) = \begin{cases} C \exp\left(\frac{1}{|\boldsymbol{x}|^2 - 1}\right) & |\boldsymbol{x}| < 1 \\ 0 & |\boldsymbol{x}| \geq 1 \end{cases},$$

  where the constant $C > 0$ is selected such that $\int_{\mathbb{R}^d} \eta \, \mathrm{d}\boldsymbol{x} = 1$. We call η the standard mollifier.
- (Mollifier) For each $\varepsilon > 0$, we define

$$\eta_\varepsilon(\boldsymbol{x}) := \frac{1}{\varepsilon^d} \eta\left(\frac{\boldsymbol{x}}{\varepsilon}\right).$$

We call η the mollifier with parameter $\varepsilon > 0$. It is easy to verify that $\int_{\mathbb{R}^d} \eta_\varepsilon \,\mathrm{d}\boldsymbol{x} = 1$ and Spt $\eta_\varepsilon \subset B(\mathbf{0}, \varepsilon)$.

- Given $f \in L^p(U)$ for some $1 \le p \le +\infty$, we define its mollification by $f_\varepsilon(\boldsymbol{x}) := (\eta_\varepsilon * f)(\boldsymbol{x})$.

The following theorem records several frequently-used properties of mollifiers.

Theorem C.2.1 (Property of mollifiers). Let $f : U \to \mathbb{R}$ be a locally integrable function.

1. $f_\varepsilon \in C^\infty(U_\varepsilon)$. (not U! Because the support of f_ε is slightly larger (size ε) than the support of f.)
2. $f_\varepsilon \to f$ a.e. as $\varepsilon \to 0$.
3. If $f \in C(U)$, then $f_\varepsilon \rightrightarrows f$ (uniform convergence) on any compact subset of U.
4. If $1 \le p < \infty$ and $f \in L^p_{\mathrm{loc}}(U)$, then $f_\varepsilon \to f$ in $L^p_{\mathrm{loc}}(U)$.

Proof. Fix $\boldsymbol{x} \in U_\varepsilon$ and $i \in \{1, \cdots, d\}$ and h sufficiently small such that $\boldsymbol{x} + he_i \in U_\varepsilon$. Then we compute the differential quotient

$$\begin{aligned}\frac{f_\varepsilon(\boldsymbol{x}+he_i) - f_\varepsilon(\boldsymbol{x})}{h} &= \frac{1}{\varepsilon^d}\int_U \frac{1}{h}\left(\eta\left(\frac{\boldsymbol{x}+he_i-\boldsymbol{y}}{\varepsilon}\right) - \eta\left(\frac{\boldsymbol{x}-\boldsymbol{y}}{\varepsilon}\right)\right) f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\\ &= \frac{1}{\varepsilon^d}\int_V \frac{1}{h}\left(\eta\left(\frac{\boldsymbol{x}+he_i-\boldsymbol{y}}{\varepsilon}\right) - \eta\left(\frac{\boldsymbol{x}-\boldsymbol{y}}{\varepsilon}\right)\right) f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\end{aligned}$$

for some open set $V \Subset U$. Thanks to the uniform convergence

$$\frac{1}{h}\left(\eta\left(\frac{\boldsymbol{x}+he_i-\boldsymbol{y}}{\varepsilon}\right) - \eta\left(\frac{\boldsymbol{x}-\boldsymbol{y}}{\varepsilon}\right)\right) \rightrightarrows \frac{1}{\varepsilon}\partial_{x_i}\eta\left(\frac{\boldsymbol{x}-\boldsymbol{y}}{\varepsilon}\right) \text{ in } V \quad \text{as } h \to 0,$$

we know the differential quotient converges, whose limit is denoted by $\partial_{x_i} f_\varepsilon(\boldsymbol{x})$ and is equal to

$$\int_\Omega \partial_{x_i}\eta_\varepsilon(\boldsymbol{x}-\boldsymbol{y}) f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}.$$

Similar argument is applicable to the derivative of f.

Next, we prove the pointwise convergence. By definition, we have

$$\begin{aligned}|f_\varepsilon(\boldsymbol{x}) - f(\boldsymbol{x})| &= \left|\int_{B(\boldsymbol{x},\varepsilon)} \eta_\varepsilon(\boldsymbol{x}-\boldsymbol{y})(f(\boldsymbol{y}) - f(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{y}\right| \le \frac{1}{\varepsilon^d}\int_{B(\boldsymbol{x},\varepsilon)} \eta\left(\frac{\boldsymbol{x}-\boldsymbol{y}}{\varepsilon}\right)|f(\boldsymbol{y}) - f(\boldsymbol{x})|\,\mathrm{d}\boldsymbol{y}\\ &\le C \fint_{B(\boldsymbol{x},\varepsilon)} |f(\boldsymbol{y}) - f(\boldsymbol{x})|\,\mathrm{d}\boldsymbol{y} \to 0 \quad \text{a.e. } \boldsymbol{x} \in U.\end{aligned}$$

where the last step is achieved by using Lebesgue's differentiation theorem. Moreover, if f is continuous, then, given any subset $V \Subset U$, we choose an open set W such that $V \Subset W \Subset U$. Then f is uniformly continuous in $\overline{W}$ and the convergence in Lebesgue's differentiation theorem holds for all $\boldsymbol{x} \in \overline{V}$. Thus, we obtained the uniform convergence as desired in (3).

It remains to prove (4). Choose $V \Subset W \Subset U$ as above. We first prove $f_\varepsilon \in L^p_{\mathrm{loc}}(U)$ for $1 \le p < \infty$.

Fix $\boldsymbol{x} \in V$, we have

$$
\begin{aligned}
|f_\varepsilon(\boldsymbol{x})| &= \left| \int_{B(\boldsymbol{x},\varepsilon)} \eta_\varepsilon(\boldsymbol{x}-\boldsymbol{y}) f(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y} \right| \le \int_{B(\boldsymbol{x},\varepsilon)} \eta_\varepsilon^{1-\frac{1}{p}} \eta_\varepsilon^{\frac{1}{p}} |f(\boldsymbol{y})| \,\mathrm{d}\boldsymbol{y} \\
&\le \left(\int_{B(\boldsymbol{x},\varepsilon)} \eta_\varepsilon(\boldsymbol{x}-\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y} \right)^{1-\frac{1}{p}} \left(\int_{B(\boldsymbol{x},\varepsilon)} \eta_\varepsilon(\boldsymbol{x}-\boldsymbol{y}) |f(\boldsymbol{y})|^p \,\mathrm{d}\boldsymbol{y} \right)^{\frac{1}{p}} \\
&= 1 \cdot \left(\int_{B(\boldsymbol{x},\varepsilon)} \eta_\varepsilon(\boldsymbol{x}-\boldsymbol{y}) |f(\boldsymbol{y})|^p \,\mathrm{d}\boldsymbol{y} \right)^{\frac{1}{p}}.
\end{aligned}
$$

Next, we integrate both sides on V and use Minkowski's inequality (for integrals) to get

$$
\begin{aligned}
\int_V |f_\varepsilon(\boldsymbol{x})|^p \,\mathrm{d}\boldsymbol{x} &\le \int_V \int_{B(\boldsymbol{x},\varepsilon)} \eta_\varepsilon(\boldsymbol{x}-\boldsymbol{y}) |f(\boldsymbol{y})|^p \,\mathrm{d}\boldsymbol{y} \,\mathrm{d}\boldsymbol{x} \\
&\le \int_W |f(\boldsymbol{y})|^p \left(\int_{B(\boldsymbol{y},\varepsilon)} \eta_\varepsilon(\boldsymbol{x}-\boldsymbol{y}) \,\mathrm{d}\boldsymbol{x} \right) \mathrm{d}\boldsymbol{y} = \int_W |f|^p < \infty.
\end{aligned}
$$

Finally, the convergence in L^p norm is obtained by approximating f by $g \in C(W)$. That is, fixing V, W as above and $\delta > 0$, we can find $g \in C(W)$ such that $\|f - g\|_{L^p(W)} < \delta$. Then

$$
\begin{aligned}
\|f_\varepsilon - f\|_{L^p(V)} &\le \|f_\varepsilon - g_\varepsilon\|_{L^p(V)} + \|g_\varepsilon - g\|_{L^p(V)} + \|g - f\|_{L^p(V)} \\
&\le 2\|f - g\|_{L^p(W)} + \|g_\varepsilon - g\|_{L^p(V)}.
\end{aligned}
$$

Using (3), we have that $\limsup_{\varepsilon \to 0} \|f_\varepsilon - f\|_{L^p(V)} \le 2\delta$. □

C.3 L^p interpolations

In this section, we record two important interpolation theorem for L^p spaces that have been widely used in Fourier analysis and PDEs. Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces, $1 \le p, q \le \infty$.

Definition C.3.1. Let $T : L^p \to L^q$ be an operator. We say

- T is sublinear, if $|T(f_0 + f_1)(\boldsymbol{x})| \le |Tf_0(\boldsymbol{x})| + |Tf_1(\boldsymbol{x})|$ and $|T(\lambda f)(\boldsymbol{x})| = |\lambda||Tf(\boldsymbol{x})|$ hold for all $\boldsymbol{x} \in X$ and $\lambda \in \mathbb{C}$;
- T is strong (p, q), if there exists a constant $C_{p,q} > 0$ such that $\|Tf\|_{L^q} \le C_{p,q}\|f\|_{L^p}$;
- T is weak (p, q), if there exists a constant $C_{p,q} > 0$ such that $\lambda_\alpha(Tf)^{\frac{1}{q}} \le C_{p,q}\alpha^{-1}\|f\|_{L^p}$.

It should be noted that "strong (p, q)" immediately implies "weak (p, q)"

C.3.1 Marcinkiewicz interpolation theorem

We first introduce the Marcinkiewicz interpolation theorem.

Theorem C.3.1 (Marcinkiewicz Interpolation Theorem). Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are measure spaces; $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ are exponents satisfying $p_0 \leq q_0$, $p_1 \leq q_1$, and $q_0 \neq q_1$; and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \text{ and } \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \text{where } 0 < \theta < 1.$$

If T is a sublinear map from $L^{p_0}(\mu) + L^{p_1}(\mu)$ to the space of measurable functions on Y that is weak types (p_0, q_0) and (p_1, q_1), then T is strong type (p, q).

Proof. We only prove that case $p_0 = q_0$ and $p_1 = q_1$. The general case is just more technically complicated but the core idea is the same, and we refer to Folland [8, Theorem 6.28] for details.

Given $f \in L^p$, for each $\alpha > 0$ decompose f as $f_0 + f_1$ as $f_0 = f\chi_{\{x:|f(x)|>c\alpha\}}$, $f_1 = f\chi_{\{x:|f(x)|\leq c\alpha\}}$, where the constant c will be fixed later. Then $f_0 \in L^{p_0}(\mu)$ and $f_1 \in L^{p_1}(\mu)$. Furthermore,

$$|Tf(x)| \leq |Tf_0(x)| + |Tf_1(x)| \Rightarrow \lambda_{Tf}(\alpha) \leq \lambda_{Tf_0}(\alpha/2) + \lambda_{Tf_1}(\alpha/2)$$

We consider two cases. Case 1: $p_1 = \infty$. Choose $c = 1/(2A_1)$, where A_1 is such that $||Tg||_\infty \leq A_1||g||_\infty$. Then $\lambda_{Tf_1}(\alpha/2) = 0$. By the weak (p_0, p_0) inequality,

$$\lambda_{Tf_0}(\alpha/2) \leq \left(\frac{2A_0}{\alpha}\,||f_0||_{p_0}\right)^{p_0}$$

hence,

$$\begin{aligned} ||Tf||_p^p &\leq p\int_0^\infty \alpha^{p-1-p_0}\,(2A_0)^{p_0}\int_{\{x|f(x)|>c\alpha\}} |f(x)|^{p_0}\,\mathrm{d}\mu\,\mathrm{d}\alpha \\ &= p\,(2A_0)^{p_0}\int_X |f(x)|^{p_0}\int_0^{|f(x)|/c} \alpha^{p-1-p_0}\,\mathrm{d}\alpha\,\mathrm{d}\mu = \frac{p}{p-p_0}\,(2A_0)^{p_0}\,(2A_1)^{p-p_0}\,||f||_p^p. \end{aligned}$$

Case 2: $p_1 < \infty$. We now have the pair of inequalities

$$\lambda_{Tf_i}(\alpha/2) \leq \left(\frac{2A_i}{\alpha}\,||f_i||_{p_i}\right)^{p_i}, \quad i = 0, 1$$

From these we get (arguing as above) that

$$
\begin{aligned}
\|Tf\|_p^p &\le p\int_0^\infty \alpha^{p-1-p_0}\,(2A_0)^{p_0}\int_{\{x\,:\,|f(x)|>c\alpha\}} |f(x)|^{p_0}\,\mathrm{d}\mu\,\mathrm{d}\alpha \\
&\quad + p\int_0^\infty \alpha^{p-1-p_1}\,(2A_1)^{p_1}\int_{\{x\,:\,|f(x)|\le c\alpha\}} |f(x)|^{p_1}\,\mathrm{d}\mu\,\mathrm{d}\alpha \\
&= \left(\frac{p2^{p_0}}{p-p_0}\frac{A_0^{p_0}}{c^{p-p_0}} + \frac{p2^{p_1}}{p_1-p}\frac{A_1^{p_1}}{c^{p-p_1}}\right)\|f\|_p^p.
\end{aligned}
$$

□

An application of Marcinkiewicz interpolation theorem is the L^p boundedness of the Hardy-Littlewood maximal function.

Definition C.3.2 (Hardy-Littlewood maximal function)**.** Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. We define its Hardy-Littlewood maximal function by

$$
Mf(\boldsymbol{x}) := \sup_{r>0} \frac{1}{|B(\boldsymbol{x},r)|}\int_{B(\boldsymbol{x},r)} |f(\boldsymbol{y})|\,\mathrm{d}\boldsymbol{y}.
$$

Then

Proposition C.3.2 ([15, Chapter 3])**.** The Hardy-Littlewood maximal operator M is weak (1,1) and strong (∞,∞).

This together with Marcinkiewicz interpolation theorem gives us

Theorem C.3.3. Let $1 < p < \infty$, then

$$
\|Mf\|_{L^p(\mathbb{R}^d)} \le C\frac{p}{p-1}\|f\|_{L^p(\mathbb{R}^d)}.
$$

We note that the constant above can be computed by following the proof of Marcinkiewicz interpolation theorem.

An important result is the so-called Hardy-Littlewood-Sobolev inequality, which actually gives the critical Sobolev embedding for fractional Sobolev spaces.

Theorem C.3.4 (Hardy-Littlewood-Sobolev inequality)**.** Let $f \in L^p(\mathbb{R}^d)$ and the exponents p, q, γ satisfy $0 < \gamma < d$, $1 < p < q < \infty$ and $1 + \frac{1}{q} = \frac{1}{p} + \frac{\gamma}{d}$. Then there exists a constant $C > 0$ depending on p, q, d such that

$$
\||\cdot|^{-\gamma} * f\|_{L^q(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}.
$$

Proof. We truncate the convolution into two parts

$$
|\cdot|^{-\gamma} * f(\boldsymbol{x}) = \int_{|\boldsymbol{y}|>R} f(\boldsymbol{x}-\boldsymbol{y})|\boldsymbol{y}|^{-\gamma}\,\mathrm{d}\boldsymbol{y} + \int_{|\boldsymbol{y}|\le R} f(\boldsymbol{x}-\boldsymbol{y})|\boldsymbol{y}|^{-\gamma}\,\mathrm{d}\boldsymbol{y} =: I_1 + I_2.
$$

This $R > 0$ may depend on $\boldsymbol{x}$ and will be determined later in order to optimze the upper bound.

For I_1, we notice that the exponents satisfy $\frac{1}{\gamma p'} = \frac{1}{d} - \frac{1}{\gamma q} < \frac{1}{d}$. Then I_1 can be directly controlled by using Hölder's inequality

$$I_1 \leq \|f(\boldsymbol{x} - \cdot)\|_{L^p} \left\| |\cdot|^{-\gamma} \chi_{B(\boldsymbol{0},R)^c} \right\|_{L^{p'}} = C\|f\|_{L^p} R^{-\frac{d}{q}}.$$

For I_2, we eliminate the singularity at the origin through the dyadic decomposition of the ball $B(\boldsymbol{0}, R)$.

$$\begin{aligned}
|I_2| &\leq \sum_{j=0}^{\infty} \int_{2^{-(j+1)R} \leq |\boldsymbol{y}| \leq 2^{-j}R} |\boldsymbol{y}|^{-\gamma} |f(\boldsymbol{x} - \boldsymbol{y})| \, \mathrm{d}\boldsymbol{y} \\
&\leq \sum_{j=0}^{\infty} (2^{-(j+1)}R)^{-\gamma} \int_{2^{-(j+1)R} \leq |\boldsymbol{y}| \leq 2^{-j}R} |f(\boldsymbol{x} - \boldsymbol{y})| \, \mathrm{d}\boldsymbol{y} \\
&\leq \sum_{j=0}^{\infty} 2^{(j+1)\gamma} R^{-\gamma} (2^{-j}R)^d \underbrace{\int_{|\boldsymbol{y}| \leq 2^{-j}R} \frac{|f(\boldsymbol{x} - \boldsymbol{y})}{(2^{-j}R)^d} \, \mathrm{d}\boldsymbol{y}}_{\leq C_d Mf(\boldsymbol{x})} \\
&\leq \sum_{j=0}^{\infty} 2^{\gamma} 2^{-j(d-\gamma)} R^{d-\gamma} Mf(\boldsymbol{x}) = CR^{d-\gamma} Mf(\boldsymbol{x}).
\end{aligned}$$

Therefore, we have

$$I_1 + I_2 \leq C \left(\|f\|_{L^p} R^{-\frac{d}{q}} + R^{d-\gamma} Mf(\boldsymbol{x}) \right).$$

Take

$$R := \frac{\|f\|_{L^p}^{\frac{p}{d}}}{Mf(\boldsymbol{x})^{\frac{p}{d}}}$$

such that the two terms are equal. Then we get

$$I_1 + I_2 \leq C\|f\|_{L^p}^{1-\frac{p}{q}} (Mf)^{\frac{p}{q}}.$$

Finally, using the L^p boundedness of the Hardy-Littlewood maximal operator, we get

$$\| |\cdot|^{-\gamma} * f\|_{L^q(\mathbb{R}^d)} \leq \|I_1 + I_2\|_{L^q} \leq C\|f\|_{L^p}^{1-\frac{p}{q}} \underbrace{\|(Mf)^{\frac{p}{q}}\|_{L^q}}_{=\|Mf\|_{L^p}^{\frac{p}{q}}} \leq C\|f\|_{L^p}.$$

□

C.3.2 Riesz-Thorin interpolation theorem

The drawback of Marcinkiewicz interpolation theorem is that the upper bound for the constant in the L^p-boundedness inequality does not have quantitative expression. However, if we replace the assumption "weak (p_0, q_0)" by "strong (p_0, q_0)", then we really obtain explicit expression for the constant of the inequality.

Theorem C.3.5 (Riesz-Thorin Interpolation Theorem). Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are measure spaces and $p_0, p_1, q_0, q_1 \in [1, \infty]$. For $\theta \in [0, 1]$, we define p_θ, q_θ by

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

If T is a linear map from $L^{p_0}(\mu) + L^{p_1(\mu)} \to L^{q_0}(\nu) + L^{q_1}(\nu)$ such that

$$\|Tf\|_{L^{q_i}} \le M_i \|f\|_{L^{p_i}} \qquad i = 0, 1,$$

then

$$\|Tf\|_{L^{q_\theta}} \le M_\theta \|f\|_{L^{p_\theta}}, \ \ M_\theta := M_0^{1-\theta} M_1^\theta.$$

If $q_0 = q_1 = \infty$, then ν should be semi-finite.

The proof relies on the Three-Line Lemma in complex analysis, so we omit the details here. See Folland [8, Theorem 6.27].

An application of Riesz-Thorin interpolation theorem is the strong (p, p') boundedness of Fourier transform. This is concluded as the Hausdorff-Young inequality (Theorem D.1.9). Another important application is the Young's inequality for convolution. This is frequently used in the study of heat equation and Schrödinger's equation.

Theorem C.3.6 (Young's inequality for convolution). Let $1 \le p, q, r \le \infty$ satisfy $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$ and $f \in L^p, g \in L^r$. Then $f * g \in L^q$ and

$$\|f * g\|_{L^q} \le \|f\|_{L^p} \|g\|_{L^r}.$$

Proof. Fix $g \in L^r$ and define $Tf := f * g$. Then by Minkowski's inequality for integrals (Theorem C.1.6), we have $\|Tf\|_{L^r} \le M\|f\|_{L^1}$. Also, by Hölder's inequality, we have $\|Tf\|_{L^\infty} \le M\|f\|_{L^{r'}}$. Then by Riesz-Thorin interpolation theorem with $(p_0, q_0) = (1, r)$ and $(p_1, q_1) = (r', \infty)$, we obtain the desired inequality. □

Exercise C.3

Exercise C.3.1. Show that Hardy-Littlewood operator is not strong (1,1).

(Hint: If $f \not\equiv 0$, then there exists $R > 0$ such that $\int_{B(\mathbf{0},R)} |f| \ge \varepsilon > 0$. Then prove $Mf(\boldsymbol{x}) \ge C\varepsilon|\boldsymbol{x}|^{-d}$ for $|x| > R$ by noticing that $B(\mathbf{0}, R) \subset B(\boldsymbol{x}, 2|\boldsymbol{x}|)$.)

Exercise C.3.2. If B is a bounded subset of $\mathbb{R}^d$, then

$$\int_B Mf \le 2|B| + C\int_{\mathbb{R}^d} |f| \log^+ |f|,$$

where $\log^+ t = \max(\log t, 0)$.

Exercise C.3.3. Let $\{\eta_\varepsilon\}$ be a family of approximation to identity. Then prove that

$$\sup_{\varepsilon>0} |(\eta_\varepsilon * f)(\boldsymbol{x})| \le \|\eta\|_{L^1} Mf(\boldsymbol{x}).$$

Appendix D Preliminaries on Fourier Analysis

D.1 Fourier transform

Given $f \in L^1(\mathbb{R}^d)$, we define the **Fourier transform** of f to be

$$\hat{f}(\boldsymbol{\xi}) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(\boldsymbol{x}) e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}} \, \mathrm{d}\boldsymbol{x}, \tag{D.1.1}$$

where $i = \sqrt{-1}$, $e^{i\theta} := \cos\theta + i\sin\theta$, $\boldsymbol{x} \cdot \boldsymbol{\xi} = x_1\xi_1 + \cdots + x_d\xi_d$ and $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_d)$ is called the "frequency variable". We also define the **inverse Fourier transform** of $f \in L^1(\mathbb{R}^d)$ by

$$\check{f}(\boldsymbol{x}) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(\boldsymbol{\xi}) e^{i\boldsymbol{x}\cdot\boldsymbol{\xi}} \, \mathrm{d}\boldsymbol{\xi}. \tag{D.1.2}$$

The Fourier transform and its inverse are denoted by $\mathcal{F}$ and $\mathcal{F}^{-1}$ respectively.

The reason why we call (D.1.2) the "inverse" Fourier transform is the following property.

Proposition D.1.1 (Fourier inversion formula). If $f, \hat{f} \in L^1(\mathbb{R}^d)$, then there exists a function $f_0 \in C_0(\mathbb{R}^d)$ (continuous functions that vanish at infinity) such that $f = f_0$ a.e. and $f_0 = (\hat{f})^\vee = (\check{f})^\wedge$.

The proof requires either the Riemann-Lebesgue lemma or the property of the Gaussian kernel and we postpone it to later sections.

From the definition (D.1.1), we know that $f \in L^1$ does not necessarily imply $\hat{f} \in L^1$. People want to find a suitable class of functions, say X, such that the Fourier transform maps X into X and is also invertible in X. Such a function space does exist, which is called **Schwartz function space**

$$\mathcal{S}(\mathbb{R}^d) := \{u \in C^\infty(\mathbb{R}^d) : \|u\|_{(N,\alpha)} < \infty \ \forall N \in \mathbb{N} \text{ and multi-indices } \alpha\}. \tag{D.1.3}$$

Here the semi-norm is defined by

$$\|u\|_{(N,\alpha)} := \sup_{\boldsymbol{x}\in\mathbb{R}^d} (1 + |\boldsymbol{x}|)^N |\partial^\alpha u(\boldsymbol{x})|.$$

Then $(\mathcal{S}(\mathbb{R}^d), \|\cdot\|_{(N,\alpha)})$ is a Fréchet space.

Briefly speaking, a Schwartz function is a smooth function whose derivatives (including the function itself) are all decaying faster than any polynomial order (i.e., decay faster than $O(|\boldsymbol{x}|^{-N})$ for any $N \in \mathbb{N}^*$). For example, $e^{-|\boldsymbol{x}|}, e^{-|\boldsymbol{x}|^2}$ are Schwartz funtions.

Proposition D.1.2 (Folland [8, Prop. 8.3 and 8.17]). For Schwartz functions, we have the following properties.

1. If $f \in C^\infty$, then $f \in \mathcal{S}$ if and only if $\boldsymbol{x}^\beta \partial^\alpha f$ is bounded for all multi-indices α, β, if and only if $\partial^\alpha(\boldsymbol{x}^\beta f)$ is bounded for all multi-indices α, β.
2. C_c^∞ and $\mathcal{S}$ are both dense in L^p $(1 \le p < \infty)$ and in C_0.

From the definitions above, Fourier transform seems to be far away from PDEs. However, Fourier transform converts derivatives to be multipliers, which then reduces a linear PDE to an ODE. Such property makes solving certain linear PDEs much easier. We list the following properties. For simplicity, we assume the functions f, g in the below proposition are Schwartz functions.

Proposition D.1.3. Let $f, g \in L^1(\mathbb{R}^d)$. Then

1. (Derivative $\leftrightarrow$ Multiplier) If $f \in C^k$, $\partial^\alpha f \in L^1$ for $|\alpha| \le k$ and $\partial^\alpha f \in C_0$ for $|\alpha| \le k-1$, then $(\widehat{\partial_{\boldsymbol{x}}^\alpha f})(\boldsymbol{\xi}) = (i\boldsymbol{\xi})^\alpha \hat{f}(\boldsymbol{\xi})$. Similarly, if $\boldsymbol{x}^\alpha f \in L^1$ for $|\alpha| \le k$, then $\hat{f} \in C^k$ and $((-i\boldsymbol{x})^\alpha f(\boldsymbol{x}))^\wedge(\boldsymbol{\xi}) = \partial_{\boldsymbol{\xi}}^\alpha \hat{f}(\boldsymbol{\xi})$.
2. If T is an invertible linear transform of $\mathbb{R}^d$ and $S = (T^*)^{-1}$ is the inverse transpose, then $\widehat{f \circ T} = |\det T|^{-1} \hat{f} \circ S$. In particular, we have
   - (Translation) $(f(\boldsymbol{x} - \mathbf{h}))^\wedge(\xi) = e^{-i\mathbf{h}\cdot\xi} \hat{f}(\boldsymbol{\xi})$ for any $\mathbf{h} \in \mathbb{R}^d$.
   - (Scaling) $(f(\lambda \boldsymbol{x}))^\wedge(\boldsymbol{\xi}) = |\lambda|^{-d} \hat{f}(\boldsymbol{\xi}/\lambda)$ for any $\lambda \in \mathbb{R}$.
   - (Symmetry) If $f, \hat{f} \in L^1$, then $\check{f}(\boldsymbol{\xi}) = \hat{f}(-\boldsymbol{\xi})$. As a corollary, $\mathcal{F}^4 =$Id.
3. (Convolution $\leftrightarrow$ Multiplication) $\widehat{f * g}(\boldsymbol{\xi}) = (\sqrt{2\pi})^d \hat{f}(\boldsymbol{\xi}) \hat{g}(\boldsymbol{\xi})$. Here $(f * g)(\boldsymbol{x}) = \int_{\mathbb{R}^d} f(\boldsymbol{x} - \boldsymbol{y}) g(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y} = \int_{\mathbb{R}^d} f(\boldsymbol{y}) g(\boldsymbol{x} - \boldsymbol{y}) \,\mathrm{d}\boldsymbol{y}$ represents the convolution of f, g. This integral converges if one of f, g is L^1 and the other one is L^∞.
4. (Riemann-Lebesgue lemma) For any $f \in L^1(\mathbb{R}^d)$, the Fourier transform $\hat{f} \in C(\mathbb{R}^d)$ and satisfies $|\hat{f}(\boldsymbol{\xi})| \to 0$ as $|\boldsymbol{\xi}| \to \infty$.

Proof. (1): For simplicity, we only consider the case $f \in \mathcal{S}(\mathbb{R}^d)$ and $\partial^\alpha = \partial_j$ for some $j \in \{1, \cdots, d\}$. Using the definition of Fourier transform and integrating by parts, we have

$$
\begin{aligned}
(\widehat{\partial_{x_j} f})(\xi) =& \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \partial_{x_j} f(\boldsymbol{x}) e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}} \,\mathrm{d}\boldsymbol{x} \\
\overset{\partial_j}{=}& -\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(\boldsymbol{x}) \partial_{x_j}(e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}}) \,\mathrm{d}\boldsymbol{x} = (i\xi_j) \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(\boldsymbol{x}) e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}} \,\mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

The second equality in (1) is also similarly proved.

$$(-ix_j f(\boldsymbol{x}))^\wedge(\boldsymbol{\xi}) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}(-ix_j)f(\boldsymbol{x})e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}}\,\mathrm{d}\boldsymbol{x} = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} f(\boldsymbol{x})\partial_{\xi_j}(e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}})\,\mathrm{d}\boldsymbol{x} = \partial_{\xi_j}\hat{f}(\boldsymbol{\xi}).$$

(2): Using the definition of Fourier transform and change of variable $\boldsymbol{y} = T\boldsymbol{x}$, we get

$$\begin{aligned}\widehat{f\circ T}(\boldsymbol{\xi}) &= \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} f(T\boldsymbol{x})e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x} = |\det T|^{-1}\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} f(\boldsymbol{x})e^{-i\boldsymbol{\xi}\cdot T^{-1}\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x}\\ &= |\det T|^{-1}\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} f(\boldsymbol{x})e^{-i(S\boldsymbol{\xi})\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x} = |\det T|^{-1}\hat{f}(S\boldsymbol{\xi}).\end{aligned}$$

(3): WLOG (Without loss of generality) we assume $f, g \in \mathcal{S}(\mathbb{R}^d)$, we know all the integrals below are convergent and we can commute integrals. The general case can be proved by using the approximation in Proposition D.1.2.

$$\begin{aligned}\widehat{f * g}(\boldsymbol{\xi}) &= \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}\underbrace{\left(\int_{\mathbb{R}^d} f(\boldsymbol{x}-\boldsymbol{y})g(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right)}_{(f*g)(\boldsymbol{x})} e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}}\,\mathrm{d}\boldsymbol{x}\\ &= \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f(\boldsymbol{x}-\boldsymbol{y})g(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right) e^{-i(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{\xi}}e^{-i\boldsymbol{y}\cdot\boldsymbol{\xi}}\,\mathrm{d}\boldsymbol{x}\\ &= (\sqrt{2\pi})^d\cdot\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}\underbrace{\left(\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} f(\boldsymbol{x}-\boldsymbol{y})e^{-i(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{\xi}}\,\mathrm{d}\boldsymbol{x}\right)}_{\hat{f}(\boldsymbol{\xi})} g(\boldsymbol{y})e^{-i\boldsymbol{y}\cdot\boldsymbol{\xi}}\,\mathrm{d}\boldsymbol{y}\\ &= (\sqrt{2\pi})^d\hat{f}(\boldsymbol{\xi})\hat{g}(\boldsymbol{\xi}).\end{aligned}$$

(4): Recall the definition of Fourier transform: $\hat{f}(\boldsymbol{\xi}) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} f(\boldsymbol{x})e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}}\,\mathrm{d}\boldsymbol{x}$. The continuity is rather easy to prove, as we have

$$|\hat{f}(\boldsymbol{\xi}-\mathbf{h}) - \hat{f}(\boldsymbol{\xi})| = \left|\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} f(\boldsymbol{x})e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}}(e^{-i\boldsymbol{x}\cdot\mathbf{h}}-1)\,\mathrm{d}\boldsymbol{x}\right| \le \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}|f(\boldsymbol{x})||e^{-i\boldsymbol{x}\cdot\mathbf{h}}-1|\,\mathrm{d}\boldsymbol{x}.$$

Since $|e^{-i\boldsymbol{x}\cdot\mathbf{h}}-1| \le 2$ and $f \in L^1(\mathbb{R}^d)$, the integrand is bounded by $2|f| \in L^1(\mathbb{R}^d)$ (independent of $\mathbf{h}$) for any $\boldsymbol{x} \in \mathbb{R}^d$. Using the "dominated convergence theorem", we can commute $\lim\limits_{\mathbf{h}\to\mathbf{0}}$ with $\int_{\mathbb{R}^d}$ in order to get the convergence to zero.

Now we do a translation in the phase function by the change of variable $\boldsymbol{x} = \boldsymbol{y} + \frac{\pi\xi}{|\xi|^2}$. So, the Fourier transform $\hat{f}(\boldsymbol{\xi})$ can be rewritten in $\boldsymbol{y}$-variables

$$\hat{f}(\boldsymbol{\xi}) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f\left(\boldsymbol{y} + \frac{\pi \boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\right) e^{-i(\boldsymbol{y}+\frac{\pi\xi}{|\xi|^2})\cdot\xi} \,\mathrm{d}\boldsymbol{y} \overset{e^{-i\pi}=-1}{=\!=\!=} -\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f\left(\boldsymbol{y} + \frac{\pi \boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\right) e^{-i\boldsymbol{y}\cdot\xi} \,\mathrm{d}\boldsymbol{y}.$$

Then, we add this to the original definition of Fourier transform to get

$$2\hat{f}(\boldsymbol{\xi}) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \left(f(\boldsymbol{x}) - f\left(\boldsymbol{x} + \frac{\pi \boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\right)\right) e^{-i\boldsymbol{x}\cdot\xi} \,\mathrm{d}\boldsymbol{x},$$

and thus

$$|\hat{f}(\boldsymbol{\xi})| \le \frac{1}{2}\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \left|f(\boldsymbol{x}) - f\left(\boldsymbol{x} + \frac{\pi \boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\right)\right| \mathrm{d}\boldsymbol{x} \to 0 \text{ as } \boldsymbol{\xi} \to \infty$$

because $\pi\boldsymbol{\xi}/|\boldsymbol{\xi}|^2 \to 0$ as $\boldsymbol{\xi} \to \infty$. In the last step, we use the "translation continuity of L^1 norm", that is, $\lim_{\mathbf{h}\to\mathbf{0}} \int_{\mathbb{R}^d} |f(\boldsymbol{x}+\mathbf{h}) - f(\boldsymbol{x})| \,\mathrm{d}\boldsymbol{x} = 0$ for any $f \in L^1(\mathbb{R}^d)$. □

An important corollary is that

Corollary D.1.4 ([8, Corollary 8.23]). The Fourier transform, denoted by $\mathcal{F}$, maps $\mathcal{S}$ continuously into $\mathcal{S}$ itself.

We next introduce the following lemma showing that Fourier transform preserves the L^2 inner product.

Lemma D.1.5. If $f, g \in L^1$, then $\int_{\mathbb{R}^d} \hat{f}(\boldsymbol{x}) g(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d} f(\boldsymbol{\xi}) \hat{g}(\boldsymbol{\xi}) \,\mathrm{d}\boldsymbol{\xi}$.

Proof. Using Fubini's theorem, it is easy to get

$$\begin{aligned}\int_{\mathbb{R}^d} \hat{f}(\boldsymbol{x}) g(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} =& \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(\boldsymbol{y}) e^{-i\boldsymbol{x}\cdot\boldsymbol{y}} \,\mathrm{d}\boldsymbol{y}\right) g(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \\ \overset{\text{Fubini's theorem}}{=\!=\!=}& \int_{\mathbb{R}^d} f(\boldsymbol{y}) \left(\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-i\boldsymbol{x}\cdot\boldsymbol{y}} g(\boldsymbol{x})\right) \mathrm{d}\boldsymbol{y} = \int_{\mathbb{R}^d} f(\boldsymbol{y}) \hat{g}(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y}.\end{aligned}$$

□

Now, we turn to prove the Fourier inversion formula, namely Theorem D.1.1.

Proof of Theorem D.1.1. Let $\Phi(\boldsymbol{x}) = e^{-\frac{|\boldsymbol{x}|^2}{2}}$. Given $t > 0$, we consider the following approximation of $(\hat{f})^\vee$

$$\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-t|\xi|^2} e^{i\xi\cdot\boldsymbol{x}} \hat{f}(\boldsymbol{\xi}) \,\mathrm{d}\boldsymbol{\xi} = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \Phi(\sqrt{2t}\boldsymbol{\xi}) e^{i\xi\cdot\boldsymbol{x}} \hat{f}(\boldsymbol{\xi}) \,\mathrm{d}\boldsymbol{\xi}.$$

Using the translation and scaling properties in Proposition D.1.3, we know $\varphi(\boldsymbol{\xi}) := e^{i\xi\cdot x}e^{-t|\xi|^2}$ satisfies

$$\hat{\varphi}(\boldsymbol{y}) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{-i\xi\cdot(x-y)}\Phi(\sqrt{2t}\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} = \frac{1}{(\sqrt{2t})^d}\Phi(\frac{\boldsymbol{x}-\boldsymbol{y}}{\sqrt{2t}}).$$

Thus, invoking Lemma D.1.5, we obtain

$$\begin{aligned}
&\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{-t|\xi|^2}e^{i\xi\cdot x}\hat{f}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}\varphi(\boldsymbol{\xi})\hat{f}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}\\
&= \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}\hat{\varphi}(\boldsymbol{y})f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}\frac{1}{(\sqrt{2t})^d}\Phi(\frac{\boldsymbol{x}-\boldsymbol{y}}{\sqrt{2t}})f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\\
&= (\eta_{\sqrt{2t}} * f)(\boldsymbol{x}),
\end{aligned}$$

where $\eta(\cdot) := \frac{1}{(2\pi)^{\frac{d}{2}}}\Phi(\cdot) \in \mathcal{S}$ and $\eta_\varepsilon(\cdot) := \frac{1}{\varepsilon^d}\eta(\frac{\cdot}{\varepsilon})$. By direct computation, we can prove $\int_{\mathbb{R}^d}\eta = 1$, so $\eta_{\sqrt{2t}}$ is a family of "approximation to identity". By a similar argument of Theorem ?? (see Folland [8, Lemma 8.25]), we have $f * \eta_{\sqrt{2t}} \to f$ in L^1 norm as $t \to 0_+$, and thus leads to a subsequence that has a.e. convergence to f. On the other hand, since $\hat{f} \in L^1$, we can use the dominated convergence theorem to see

$$\lim_{t\to 0}\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{-t|\xi|^2}e^{i\xi\cdot x}\hat{f}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{i\xi\cdot x}\hat{f}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} = (\hat{f})^\vee(\boldsymbol{x}).$$

It then follows that $f = (\hat{f})^\vee$ a.e. Finally, the Riemann-Lebesgue lemma indicates that both functions are continuous and vanish at infinity. The proof is complete. □

Corollary D.1.6 ([8, Corollary 8.27]). If $f \in L^1$ and $\hat{f} = 0$, then $f = 0$ a.e.

Corollary D.1.7 ([8, Corollary 8.28]). $\mathcal{F}$ is an isomorphism of $\mathcal{S}$ onto itself.

Finally, we conclude this section by Plancherel's theorem, which reveals that Fourier transform is an L^2-isometry.

Theorem D.1.8 (Plancherel's theorem). If $f \in L^1 \cap L^2$, then $\hat{f} \in L^2$; and $\mathcal{F}|_{L^1\cap L^2}$ extends uniquely to a unitary isomorphism on L^2.

Proof. Let $\mathfrak{X} := \{f \in L^1 | \hat{f} \in L^1\}$. Since $\hat{f} \in L^1$ implies $f \in L^\infty$, then we have $\mathfrak{X} \subset L^2$. Also note that $\mathcal{S} \subset \mathfrak{X}$, so $\mathfrak{X}$ is dense in L^2. Given $f, g \in \mathfrak{X}$, let $h = \bar{\hat{g}}$. Using Fourier inversion formula, we have $\hat{h}(\boldsymbol{\xi}) = \overline{g(\boldsymbol{\xi})}$. Hence, by lemma D.1.5, we get

$$\int_{\mathbb{R}^d} f\bar{g} = \int_{\mathbb{R}^d} f\hat{h} = \int_{\mathbb{R}^d}\hat{f}h = \int \hat{f}\bar{\hat{g}}.$$

Thus, $\mathcal{F}|_{\mathfrak{X}}$ preserves the L^2 inner product. In particular, taking $g = f$ yields the L^2 isometry $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$ (Plancherel's identity). The inversion formula indicates the $\mathcal{F}(\mathfrak{X}) = \mathfrak{X}$, by B.L.T. theorem, $\mathcal{F}|_{\mathfrak{X}}$

extends continuously to a unitary isomorphism on L^2. Finally, it remains to show that this extension agrees with $\mathcal{F}$ on $\mathfrak{X}$. The proof follows in the same way as that of Theorem D.1.1 and we omit the proof. $\square$

Next, we prove the L^p boundedness of the Fourier transform $\mathcal{F}$ for $1 \le p \le 2$.

Theorem D.1.9 (Hausdorff-Young Inequality). For $1 \le p \le 2$, there exists a constant C, only depending on p, d, such that $\|\hat{f}\|_{L^{p'}(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}$.

This theorem is a direct consequence of Riesz-Thorin Interpolation Theorem (Theorem C.3.5) in the case $p_0 = q_0 = 2,\ M_0 = 1$ and $p_1 = 1,\ q_1 = \infty, M_1 = (2\pi)^{-d/2}$.

Exercise D.1

Exercise D.1.1 (*Heisenberg's uncertainty principle). Given $\boldsymbol{x}_0, \boldsymbol{\xi}_0 \in \mathbb{R}^d$ and $f \in \mathcal{S}(\mathbb{R}^d)$, prove the Heisenberg's uncertainty principle

$$\left(\int_{\mathbb{R}^d} |(\boldsymbol{x}-\boldsymbol{x}_0)f(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}\right)\left(\int_{\mathbb{R}^d} |(\boldsymbol{\xi}-\boldsymbol{\xi}_0)\hat{f}(\boldsymbol{\xi})|^2\,\mathrm{d}\boldsymbol{\xi}\right) \ge \frac{d^2}{4}\left(\int_{\mathbb{R}^d} |f(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}\right)^2. \tag{D.1.4}$$

This inequality shows that the momentum and position cannot be simultaneously localised around a given position $\boldsymbol{x}_0$ and a given momentum $\boldsymbol{\xi}_0$.

(Hint: It suffices to prove the case $\boldsymbol{\xi}_0 = \boldsymbol{x}_0 = 0$, otherwise consider $g(\boldsymbol{x}) = f(\boldsymbol{x}+\boldsymbol{x}_0)e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}_0}$. First notice that $|\boldsymbol{\xi}\hat{f}(\boldsymbol{\xi})|^2 = |\widehat{\nabla f}(\boldsymbol{\xi})|^2$. Then use the Plancherel's identity and Cauchy-Schwarz inequality to show the left side $\ge (\int_{\mathbb{R}^d} |(\boldsymbol{x}\cdot\nabla f)f|\,\mathrm{d}\boldsymbol{x})^2$. Finally use $(\nabla f)f = \frac{1}{2}\nabla(f^2)$ and integrate by parts.)

Exercise D.1.2. Show that an inequality

$$\|\{a_n\}\|_{L^q} \le A\|f\|_{L^p}, \quad \text{for all } f \in L^p$$

with $a_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}d\theta$, is possible only if $1/p + 1/q \le 1$. (Hint: Let $D_N(\theta) = \sum_{|n|\le N} e^{in\theta}$ be the Dirichlet kernel. Then $\|D_N\|_{L^p} \approx N^{1-1/p}$ as $N \to \infty$, if $p > 1$ and $\|D_N\|_{L^1} \approx \log N$.)

Exercise D.1.3. The following are simple generalizations of the Hausdorff-Young inequalities.

(1) Suppose $\{\varphi_n\}$ is an orthonormal sequence on $L^2(X,\mu)$. Assume also that $|\varphi_n(x)| \le M$ for all n. If $a_n = \int f\overline{\varphi_n}\,\mathrm{d}\mu$, then $\|a_n\|_{L^q} \le M^{(2/p)-1}\|f\|_{L^p(X)}$, $1 \le p \le 2, 1/p + 1/q = 1$.

(2) Suppose $f \in L^p$ on the torus $\mathbb{T}^d$, and $a_n = (2\pi)^{-\frac{d}{2}}\int_{\mathbb{T}^d} f(\boldsymbol{x})e^{-i\mathbf{n}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x}, \mathbf{n} \in \mathbb{Z}^d$. Then $\|\{a_n\}\|_{L^q(\mathbb{Z}^d)} \le \|f\|_{L^p(\mathbb{T}^d)}$, where $1/q \le 1 - 1/p$.

Exercise D.1.4. Check that an inequality of the form $\|\hat{f}\|_{L^q(\mathbb{R}^d)} \le A\|f\|_{L^p(\mathbb{R}^d)}$ (holding for all simple functions f) is possible if and only if $1/p + 1/q = 1$.

(Hint: Let $f_r(\boldsymbol{x}) = f(r\boldsymbol{x}), r > 0$. Then $\hat{f}_r(\boldsymbol{\xi}) = \hat{f}(\boldsymbol{\xi}/r)r^{-d}$.)

Exercise D.1.5. Prove that another necessary condition for the inequality in the previous exercise is that $p \le 2$. In fact the estimate

$$\int_{|\xi|\le 1} |\hat{f}(\boldsymbol{\xi})| \, \mathrm{d}\boldsymbol{\xi} \le A\|f\|_{L^p}$$

can hold only if $p \le 2$.

(Hint: Let $f^s(\boldsymbol{x}) = s^{-d/2}e^{-|\boldsymbol{x}|^2/s}, s = \sigma + it, \sigma > 0$. Then let $s = 1/2,\ t \to \infty$.)

D.2 Tempered distribution

The fundamental idea of the theory of distributions is that it is generally easier to work with linear functionals acting on spaces of "nice" functions than to work with "bad" functions directly. The set of "nice" functions we consider is closed under the basic operations in analysis, and these operations are extended to distributions by duality. This wonderful interpretation has proved to be an indispensable tool that has clarified many situations in analysis.

In the appendix, we only introduce the distributions in $\mathbb{R}^d$ as we only use it to define the Fourier transform of certain distributions. Let us recall that $C_c^\infty(\mathbb{R}^d) \subsetneq \mathcal{S}(\mathbb{R}^d) \subset C^\infty(\mathbb{R}^d)$ and

- $f_n \to f$ in C^∞ if and only if $f_n, f \in C^\infty$ and $\sup\limits_{|\boldsymbol{x}|\le N} |\partial^\alpha(f_n - f)| \to 0$ as $n \to \infty$ for all multi-indices α and $N \in \mathbb{N}^*$.
- $f_n \to f$ in $\mathcal{S}$ if and only if $f_n, f \in \mathcal{S}$ and $\sup\limits_{\boldsymbol{x}\in\mathbb{R}^d}(1 + |\boldsymbol{x}|)^N |\partial^\alpha(f_n - f)| \to 0$ as $n \to \infty$ for all multi-indices α and $N \in \mathbb{N}^*$.
- $f_n \to f$ in C_c^∞ if and only if $f_n, f \in C_c^\infty$ have a common compact support K and $\|\partial^\alpha(f_n - f)\|_{L^\infty} \to 0$ as $n \to \infty$ for all multi-indices α.

We now define their dual spaces as

$$\mathcal{D}'(\mathbb{R}^d) := (C_c^\infty(\mathbb{R}^d))^*, \quad \mathcal{S}'(\mathbb{R}^d) := (\mathcal{S}(\mathbb{R}^d))^*, \quad \mathcal{E}'(\mathbb{R}^d) := (C^\infty(\mathbb{R}^d))^*.$$

The dual spaces are nested as follows

$$\mathcal{E}' \subset \mathcal{S}' \subset \mathcal{D}'.$$

They are eqiupped with weak-* topology as they are the dual spaces of $\mathcal{D} := C_c^\infty$, $\mathcal{S}$ and C^∞, that is,

- $T_n \to T$ in $\mathcal{D}'$ if and only if $T_n, T \in \mathcal{D}'$ and $\langle T_n, f\rangle \to \langle T, f\rangle$ as $n \to \infty$ for all $f \in \mathcal{D}$.
- $T_n \to T$ in $\mathcal{S}'$ if and only if $T_n, T \in \mathcal{S}'$ and $\langle T_n, f\rangle \to \langle T, f\rangle$ as $n \to \infty$ for all $f \in \mathcal{S}$.
- $T_n \to T$ in $\mathcal{E}'$ if and only if $T_n, T \in \mathcal{E}'$ and $\langle T_n, f\rangle \to \langle T, f\rangle$ as $n \to \infty$ for all $f \in C^\infty$.

Definition D.2.1. Elements in $\mathcal{D}'$ ($\mathcal{S}', \mathcal{E}'$, resp.) are called **distributions** (**tempered distributions, distributions with compact support**, resp.).

There are several examples of distributions

Example D.2.1.

- L^1_{loc} functions.

- (signed) Borel measures on $U \subset \mathbb{R}^d$ which are finite on each compact subset of Ω (also called Radon measure). We can define $F \in \mathcal{D}'$ via $\langle F, \varphi\rangle = \int_\Omega \varphi(\boldsymbol{x})\,\mathrm{d}\mu(\boldsymbol{x})$. In particular, if μ is the point-mass which assigns total mass of 1 to the origin, it gives the **Dirac delta function** δ_0, that is, $\langle \delta, \varphi\rangle = \varphi(\mathbf{0})$. Note that δ is not a function, it only belongs to $\mathcal{E}'$!
- Every function g that satisfies $|g(\boldsymbol{x})| \leq C(1 + |\boldsymbol{x}|)^k$ for some $k \in \mathbb{R}$ (called **slowly increasing function**) is a tempered distribution.
- $\log |\boldsymbol{x}| \in \mathcal{S}'$.
- The truncation of $P.V.(\frac{1}{x})$ belongs to $\mathcal{E}'$, defined by

$$\langle u, \varphi\rangle := \lim_{\varepsilon\to 0} \int\limits_{\varepsilon\leq|x|\leq 1} \frac{\varphi(x)}{x}\,\mathrm{d}x.$$

D.2.1 Basic operations for distributions

We now define basic operations for distributions.

Definition D.2.2. The multiplication, differentiation, linear transformation and convolution of distributons are defined as the following.

- (Differentiation) Let $Tf = \partial^\alpha f$, defined on $C^{|\alpha|}(\mathbb{R}^d)$. If $\varphi \in C_c^\infty(\mathbb{R}^d)$, integration by parts gives $\int(\partial^\alpha f)\varphi = (-1)^{|\alpha|}\int f(\partial^\alpha\varphi)$; there are no boundary terms since φ has compact support. Hence, we can define the derivative $\partial^\alpha F \in \mathcal{D}'(\mathbb{R}^d)$ of any $F \in \mathcal{D}'(\mathbb{R}^d)$ by

$$\langle \partial^\alpha F, \varphi\rangle = (-1)^{|\alpha|}\langle F, \partial^\alpha \varphi\rangle.$$

  Notice, in particular, that by this procedure we can define derivatives of arbitrary locally integrable functions even when they are not differentiable in the classical sense; this is one of the main reasons for the power of distribution theory. We shall discuss this matter in more detail below.
- (Multiplication by Smooth Functions) Given $\psi \in C^\infty(\mathbb{R}^d)$, define $Tf = \psi f$. Then $T^* = T|_{C_c^\infty(\mathbb{R}^d)}$, so we can define the product $\psi F \in \mathcal{D}'(\mathbb{R}^d)$ for $F \in \mathcal{D}'(\mathbb{R}^d)$ by

$$\langle \psi F, \varphi\rangle = \langle F, \psi\varphi\rangle.$$

  Moreover, if $\psi \in C_c^\infty(\mathbb{R}^d)$, this formula makes sense for any $\varphi \in C_c^\infty\left(\mathbb{R}^d\right)$ and defines ψF as a distribution on $\mathbb{R}^d$.
- (Translation) Given $y \in \mathbb{R}^d$, let $T = \tau_{\boldsymbol{y}}$. (Recall that we have defined $\tau_{\boldsymbol{y}} f(\boldsymbol{x}) = f(\boldsymbol{x} - \boldsymbol{y})$.) Since $\int f(\boldsymbol{x} - \boldsymbol{y})\varphi(\boldsymbol{x})\,\mathrm{d}x = \int f(\boldsymbol{x})\varphi(\boldsymbol{x} + \boldsymbol{y})\,\mathrm{d}\boldsymbol{x}$, we have $T^* = \tau_{-\boldsymbol{y}}|_{C_c^\infty(\mathbb{R}^d)}$. For $F \in \mathcal{D}'(\mathbb{R}^d)$, then, we define the translated distribution $\tau_{\boldsymbol{y}} F \in \mathcal{D}'$ by

$$\langle \tau_{\boldsymbol{y}} F, \varphi\rangle = \langle F, \tau_{-\boldsymbol{y}}\varphi\rangle.$$

  For example, the point mass at $\boldsymbol{y}$ is $\tau_{\boldsymbol{y}}\delta$.

- (Composition with Linear Maps) Given an invertible linear transformation S of $\mathbb{R}^d$, let $V = S^{-1}(\mathbb{R}^d)$ and let $Tf = f \circ S$. Then $T^*\varphi = |\det S|^{-1}\varphi \circ S^{-1}$, so for $F \in \mathcal{D}'(\mathbb{R}^d)$ we define $F \circ S \in \mathcal{D}'(S^{-1}(\mathbb{R}^d))$ by
$$\langle F \circ S, \varphi \rangle = |\det S|^{-1} \langle F, \varphi \circ S^{-1} \rangle$$
In particular, for $S\boldsymbol{x} = -\boldsymbol{x}$ we have $f \circ S = \tilde{f}, S^{-1} = S$, and $|\det S| = 1$, so we define the reflection of a distribution in the origin by
$$\langle \tilde{F}, \varphi \rangle = \langle F, \tilde{\varphi} \rangle,$$
where $\tilde{\varphi}(\boldsymbol{x}) := \varphi(-\boldsymbol{x})$.
- (Convolution, First Method) Given $\psi \in C_c^\infty$ and $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, the integral
$$f * \psi(\boldsymbol{x}) = \int f(\boldsymbol{x} - \boldsymbol{y})\psi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = \int f(\boldsymbol{y})\psi(\boldsymbol{x} - \boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = \int f\left(\tau_{\boldsymbol{x}}\widetilde{\psi}\right)$$
is well defined for all $\boldsymbol{x} \in \mathbb{R}^d$. The same definition works for $F \in \mathcal{D}'(\mathbb{R}^d)$: the convolution $F * \psi$ is the function defined by
$$F * \psi(\boldsymbol{x}) = \langle F, \tau_{\boldsymbol{x}}\widetilde{\psi} \rangle.$$
Since $\tau_{\boldsymbol{x}}\tilde{\psi} \to \tau_{\boldsymbol{x}_0}\tilde{\psi}$ in C_c^∞ as $\boldsymbol{x} \to \boldsymbol{x}_0, F * \psi$ is a continuous function (actually C^∞). As an example, for any $\psi \in C_c^\infty$ we have
$$\delta * \psi(\boldsymbol{x}) = \langle \delta, \tau_{\boldsymbol{x}}\widetilde{\psi} \rangle = \tau_{\boldsymbol{x}}\widetilde{\psi}(\boldsymbol{0}) = \psi(\boldsymbol{x}).$$
so δ **is the multiplicative identity for convolution.**
- (Convolution, Second Method) Let $\psi, \widetilde{\psi}$ be defined as above. If $f \in L^1_{\text{loc}}$ and $\varphi \in C_c^\infty$, we have
$$\int (f * \psi)\varphi = \iint f(\boldsymbol{y})\psi(\boldsymbol{x} - \boldsymbol{y})\varphi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} = \int f(\varphi * \widetilde{\psi}).$$
That is, if $Tf = f * \psi$, then T maps L^1_{loc} into L^1_{loc}. For $F \in \mathcal{D}'(\mathbb{R}^d)$, we can therefore define $F * \psi$ as a distribution on V by
$$\langle F * \psi, \varphi \rangle = \langle F, \varphi * \widetilde{\psi} \rangle.$$
Again, we have $\delta * \psi = \psi$, for
$$\langle \delta * \psi, \varphi \rangle = \langle \delta, \varphi * \widetilde{\psi} \rangle = \varphi * \widetilde{\psi}(\boldsymbol{0}) = \int \varphi(\boldsymbol{x})\psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \langle \psi, \varphi \rangle.$$

These two definitions are equivalent and we refer to Stein [16, Prop. 3.1.1] for the proof.

As we see in Appendix C.2, the approximation to identity $\{\eta_\varepsilon\}$ converges to the dirac delta at the origin in some sense. In fact, the convergence exactly holds in $\mathcal{D}'$ with weak-* topology.

Proposition D.2.1 ([8, Prop. 9.5]). We have

- $\mathcal{D}$ is dense in $\mathcal{D}'$ in weak-* topology.

- Given $F \in \mathcal{D}'$ and the approximation to identity $\{\eta_\varepsilon\}$ as in Appendix C.2, then $\eta_\varepsilon * F \to F$ in $\mathcal{D}'$.

We come next to the notion of the support of a distribution. If f is a continuous function, then its support is defined as the closure of the set where $f(\boldsymbol{x}) \neq 0$. Or put another way, it is the complement of the largest open set on which f vanishes.

Definition D.2.3. For a distribution F we say that F vanishes in an open set if $\langle F, \varphi \rangle = 0$, for all test functions $\varphi \in \mathcal{D}$ which have their supports in that open set. Thus we define the support of a distribution F as the complement of the largest open set on which F vanishes.

Remark D.2.1. This definition is unambiguous because if F vanishes on any collection of open sets $\{\mathcal{O}_i\}_{i\in\mathcal{I}}$, then F vanishes on the union $\mathcal{O} = \bigcup\limits_{i\in\mathcal{I}} \mathcal{O}_i$. Indeed suppose φ is a test function supported in the compact set $K \subset \mathcal{O}$. Since $\mathcal{O}$ covers the compact set K, we may select a sub-cover which (after possibly relabeling the sets $\mathcal{O}_i$) we can write as $K \subset \bigcup\limits_{k=1}^{N} \mathcal{O}_k$. A regularization applied to the partition of unity yields *smooth* functions η_k for $1 \le k \le N$ so that $0 \le \eta_k \le 1$, $\operatorname{spt}(\eta_k) \subset \mathcal{O}_k$, and $\sum\limits_{k=1}^{N} \eta_k(\boldsymbol{x}) = 1$ whenever $\boldsymbol{x} \in K$. Then $F(\varphi) = F(\sum\limits_{k=1}^{N} \varphi\eta_k) = \sum\limits_{k=1}^{N} F(\varphi\eta_k) = 0$, since F vanishes on each $\mathcal{O}_k$. Thus F vanishes on $\mathcal{O}$ as claimed.

Note the following simple facts about the supports of distributions. The supports of $\partial_x^\alpha F$ and $\psi \cdot F$ (with $\psi \in C^\infty$) are contained in the support of F. The support of the Dirac delta function (as well as its derivatives) is the origin. Finally, $F(\varphi) = 0$ whenever the supports of F and φ are disjoint.

Proposition D.2.2 ([8, Prop. 9.3]). The following facts hold true.

- Suppose F is a distribution whose support is C_1, and ψ is in $\mathcal{D}$ and has support C_2. Then the support of $F * \psi$ is contained in $C_1 + C_2 := \{\boldsymbol{x} + \boldsymbol{y} : \boldsymbol{x} \in C_1, \boldsymbol{y} \in C_2\}$.
- If F_1 and F_2 have compact support, then $F_1 * F_2 = F_2 * F_1$. (For this reason we shall sometimes also write $F_1 * F$ for $F * F_1$, when only F_1 has compact support.) With δ the Dirac delta function $F * \delta = \delta * F = F$.
- If F_1 has compact support, then for every multi-index α

$$\partial_{\boldsymbol{x}}^\alpha (F * F_1) = (\partial_{\boldsymbol{x}}^\alpha F) * F_1 = F * (\partial_{\boldsymbol{x}}^\alpha F_1).$$

- If F and F_1 have supports C and C_1 respectively, and C is compact, then the support of $F * F_1$ is contained in $C + C_1$.

D.2.2 Tempered distributions and Fourier transform

In view of the Hausdorff-Young inequality, we cannot even define the Fourier transform of $f \in L^p(\mathbb{R}^d)$ ($p > 2$) as an ordinary function. To extend the definition of Fourier transform, we must seek for a suitable class of distributions such that $\mathcal{F}$ is still an automorphism on this class. In fact, we shall pick $\mathcal{S}'$ instead of $\mathcal{D}'$ to define Fourier transform, because the set of test functions $\mathcal{D}$ does not satisfy $\mathcal{F}(\mathcal{D}) \subseteq \mathcal{D}$.

Proposition D.2.3. Given $\varphi \in C_c^\infty(\mathbb{R}^d)$ that is not identically zero, then $\hat{\varphi}$ cannot be zero in any non-empty open set.

For tempered distribution, one can verify

Proposition D.2.4. Let $F \in \mathcal{S}'$. Then there exist an $N \in \mathbb{N}^*$, a multi-index α and a constant $C > 0$ such that $\langle F, \varphi\rangle \leq C\|\varphi\|_{(N,\alpha)}$ for all $\varphi \in \mathcal{S}$.

Therefore, given any $F \in \mathcal{S}'$, its distributional derivative also belongs to $\mathcal{S}'$, and $\boldsymbol{x}^\alpha F$ also belongs to $\mathcal{S}'$ for all multi-indices α.

We next define the Fourier transform of tempered distributions

Definition D.2.4. Let $F \in \mathcal{S}'$. Define its Fourier transform $\hat{F}$ by

$$\langle \hat{F}, \varphi\rangle = \langle F, \hat{\varphi}\rangle, \qquad \forall \varphi \in \mathcal{S}.$$

Similarly, we can define the inverse Fourier transform $\check{F}$ by

$$\langle \check{F}, \varphi\rangle = \langle F, \check{\varphi}\rangle, \qquad \forall \varphi \in \mathcal{S}.$$

The Fourier inversion theorem formula $\varphi = (\widehat{\varphi})^\vee = (\varphi^\vee)^\wedge$ then extends to $\mathcal{S}'$:

$$\langle (\widehat{F})^\vee, \varphi\rangle = \langle \widehat{F}, \varphi^\vee\rangle = \langle F, (\varphi^\vee)^\wedge\rangle = \langle F, \varphi\rangle,$$

so that $(\widehat{F})^\vee = F$, and likewise $(F^\vee)^\wedge = F$. Thus the Fourier transform is an isomorphism on $\mathcal{S}'$.

Proposition D.2.5. The properties of Fourier transform applied to Schwartz functions are inherited to tempered distributions. Let $F \in \mathcal{S}'$ and then

- $\widehat{(\tau_{\boldsymbol{y}} F)} = e^{-i\boldsymbol{\xi}\cdot\boldsymbol{y}}\widehat{F}, \quad \tau_{\boldsymbol{\eta}}\widehat{F} = \widehat{e^{i\boldsymbol{\eta}\cdot\boldsymbol{x}}F}.$
- $\partial^\alpha \widehat{F} = ((-i\boldsymbol{x})^\alpha F)^\wedge, \quad (\partial^\alpha F)^\wedge = (i\boldsymbol{\xi})^\alpha \widehat{F}.$
- $(F \circ T)^\wedge = |\det T|^{-1}\widehat{F} \circ (T^*)^{-1} \quad (T \in GL(d, \mathbb{R})),$
- $(F * \psi)^\wedge = (2\pi)^{\frac{d}{2}}\widehat{\psi}\widehat{F} \quad (\psi \in \mathcal{S}).$

In particular, any compactly supported distribution is tempered. Moreover, if $F \in \mathcal{E}'$, there is an alternative way to define $\widehat{F}$. Indeed, $\langle F, \varphi\rangle$ makes sense for any $\varphi \in C^\infty$, and if we take $\varphi(\boldsymbol{x}) = e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}$, we obtain a function of $\boldsymbol{\xi}$ that has a strong claim to be called $\widehat{F}(\boldsymbol{\xi})$. In fact, the two definitions are equivalent:

Proposition D.2.6. If $F \in \mathcal{E}'$, then $\widehat{F}$ is a slowly increasing C^∞ function, and it is given by $\widehat{F}(\boldsymbol{\xi}) = (2\pi)^{-\frac{d}{2}}\langle F, E_{-\xi}\rangle$ where $E_\xi(\boldsymbol{x}) = e^{i\xi\cdot\boldsymbol{x}}$.

Based on this, we can assert that any distribution supported at a single point must be a finite linear combination of the dirac delta function and its distributional derivatives.

Theorem D.2.7. Suppose F is a distribution supported at the origin. Then F is a finite sum

$$F = \sum_{|\alpha|\leq N} a_\alpha \partial_x^\alpha \delta.$$

That is,

$$\langle F, \varphi\rangle = \sum_{|\alpha|\leq N} (-1)^{|\alpha|} a_\alpha \left(\partial_x^\alpha \varphi\right)(0), \qquad \text{for } \varphi \in \mathcal{D}.$$

The proof relies on the following claim

Claim. Suppose F_1 is a distribution supported at the origin that satisfies for some M the following two conditions:

(a) $|\langle F_1, \varphi\rangle| \leq c\|\varphi\|_{(N,\alpha)}$, for all $\varphi \in \mathcal{D}$, $N + |\alpha| \leq M$.
(b) $\langle F_1, \boldsymbol{x}^\alpha\rangle = 0$, for all $|\alpha| \leq M$.

Then $F_1 = 0$.

To prove the claim, let $\eta \in \mathcal{D}$, with $\eta(\boldsymbol{x}) = 0$ for $|\boldsymbol{x}| \geq 1$, and $\eta(\boldsymbol{x}) = 1$ when $|\boldsymbol{x}| \leq 1/2$, and write $\eta_\varepsilon(\boldsymbol{x}) = \eta(\boldsymbol{x}/\varepsilon)$. Then since F_1 is supported at the origin, $\langle F_1, \eta_\varepsilon \varphi\rangle = \langle F_1, \varphi\rangle$. Moreover, by the same token $\langle F_1, \eta_\varepsilon \boldsymbol{x}^\alpha\rangle = \langle F_1, \boldsymbol{x}^\alpha\rangle = 0$ for all $|\alpha| \leq M$, and hence

$$\langle F_1, \varphi\rangle = \left\langle F_1, \eta_\varepsilon \left(\varphi(\boldsymbol{x}) - \sum_{|\alpha|\leq M} \frac{\varphi^{(\alpha)}(0)}{\alpha!} \boldsymbol{x}^\alpha \right)\right\rangle$$

with $\varphi^{(\alpha)} = \partial_x^\alpha \varphi(0)$. If $R(\boldsymbol{x}) = \varphi(\boldsymbol{x}) - \sum\limits_{|\alpha|\leq M} \frac{\varphi^{(\alpha)}(0)}{\alpha!} \boldsymbol{x}^\alpha$ is the remainder, then $|R(\boldsymbol{x})| \leq c|\boldsymbol{x}|^{M+1}$ and $|\partial_{\boldsymbol{x}}^\beta R(\boldsymbol{x})| \leq c_\beta |\boldsymbol{x}|^{M+1-|\beta|}$, when $|\beta| \leq M$.

However $|\partial_{\boldsymbol{x}}^\beta \eta_\varepsilon(\boldsymbol{x})| \leq c_\beta \varepsilon^{-|\beta|}$ and $\partial_{\boldsymbol{x}}^\beta \eta_\varepsilon(\boldsymbol{x}) = 0$ if $|\boldsymbol{x}| \geq \varepsilon$. Thus by Leibnitz's rule, $\|\eta_\varepsilon R\|_{(N,\alpha)} \leq c\varepsilon$ for all $|\alpha| + N \leq M$, and our assumption (a) gives $|\langle F_1, \varphi\rangle| \leq c'\epsilon$, which yields the desired conclusion upon letting $\varepsilon \to 0$.

Proof of Theorem D.2.7. Proceeding with the proof of the theorem, we now apply the above lemma to $F_1 = F - \sum_{|\alpha|\leq M} a_\alpha \partial_x^\alpha \delta$ where $M = N + |\alpha|$ is sum of N and $|\alpha|$ appearing in Proposition D.2.4, while the a_α are chosen so that $a_\alpha = \frac{(-1)^{|\alpha|}}{\alpha!}\langle F, \boldsymbol{x}^\alpha\rangle$. Then since $\langle \partial_{\boldsymbol{x}}^\alpha \delta, \boldsymbol{x}^\beta)\rangle = (-1)^{|\alpha|}\alpha!$, if $\alpha = \beta$, and zero otherwise, we see that $F_1 = 0$, which proves the theorem. □

It is an important fact that every distibution is, at least locally, a linear combination of derivatives of continuous functions. The Fourier transform yields an easy proof of this:

Proposition D.2.8 ([8, Prop. 9.14]).

- If $F \in \mathcal{E}'$, there exist $N \in \mathbb{N}$, constants $c_\alpha(|\alpha| \leq N)$, and $f \in C(\mathbb{R}^d)$ vanishing as $|\boldsymbol{x}| \to \infty$, such that $F = \sum\limits_{|\alpha|\leq N} c_\alpha \partial^\alpha f$.

- If $F \in \mathcal{D}'$ and V is a precompact open set, there exist N, c_α, f as above such that $F = \sum_{|\alpha| \le N} c_\alpha \partial^\alpha f$ on V.

Exercise D.2

Exercise D.2.1. Let $H(x) = \chi_{(0,\infty)}$ be the Heaviside function. Prove that its distributional derivative is the dirac delta at the origin.

Exercise D.2.2. If $F \in \mathcal{D}'$ and $\partial_j F = 0$ for $1 \le j \le d$. Prove that F is a constant function over $\mathbb{R}^d$.
(Hint: Consider $F * \eta_\varepsilon$.)

Exercise D.2.3. Prove that, when $0 < \alpha < d$, the Fourier transform of $|\boldsymbol{x}|^{-\alpha}$ in $\mathbb{R}^d$ is $C_{\alpha,d}|\boldsymbol{\xi}|^{\alpha-d}$.
(Hint: First prove that the Fourier transform equals a smooth function away from $\boldsymbol{\xi} = \mathbf{0}$. Then use dilation and rotational symmetry to determine this function.)

Exercise D.2.4. A distribution F on $\mathbb{R}^d$ is called homogeneous of degree λ if $F \circ S_r = r^\lambda F$ for all $r > 0$, where $S_r(\boldsymbol{x}) = r\boldsymbol{x}$.

(1) δ is homogeneous of degree $-d$.
(2) If F is homogeneous of degree λ, then $\partial^\alpha F$ is homogeneous of degree $\lambda - |\alpha|$.
(3) The distribution $\frac{\mathrm{d}}{\mathrm{d}x}\left(\chi_{(0,\infty)}(x) \log x\right)$ discussed in the text is not homogeneous, although it agrees on $\mathbb{R}\backslash\{0\}$ with a function that is homogeneous of degree -1.

Exercise D.2.5. Let f be a continuous function on $\mathbb{R}^d\backslash\{\mathbf{0}\}$ that is homogeneous of degree $-d$ (i.e., $f(r\boldsymbol{x}) = r^{-d} f(\boldsymbol{x})$) and has mean zero on the unit sphere (i.e., $\int_{\mathbb{S}^{d-1}} f \, \mathrm{d}S = 0$). Then f is not locally integrable near the origin (unless $f = 0$), but the formula

$$\langle P.V.(f), \varphi \rangle = \lim_{\epsilon \to 0} \int_{|\boldsymbol{x}| > \epsilon} f(\boldsymbol{x}) \varphi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \quad (\varphi \in C_c^\infty)$$

defines a distribution $P.V.(f)$ - "$P.V.$ " stands for "principal value" - that agrees with f on $\mathbb{R}^d\backslash\{\mathbf{0}\}$ and is homogeneous of degree $-d$ in the sense of Exercise D.2.4.
(Hint: For any $a > 0$, the indicated limit equals $\int_{|\boldsymbol{x}| \le a} f(\boldsymbol{x})[\varphi(\boldsymbol{x}) - \varphi(\mathbf{0})] \, \mathrm{d}\boldsymbol{x} + \int_{|\boldsymbol{x}| > a} f(\boldsymbol{x}) \varphi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$ and these integrals converge absolutely.)

Exercise D.2.6 ([16, Theorem 3.2.5]). Suppose $\lambda > d$; then the function $\boldsymbol{x} \longmapsto |\boldsymbol{x}|^{-\lambda}$ on $\mathbb{R}^d$ is not locally integrable near the origin. Here are some ways to make it into a distribution:

(1) If $\varphi \in C_c^\infty$, let P_φ^k be the Taylor polynomial of φ about $x = 0$ of degree k. Given $k > \lambda - d - 1$ and $a > 0$, define

$$\left\langle F_a^k, \varphi \right\rangle = \int_{|\boldsymbol{x}| \le a} \left[\varphi(\boldsymbol{x}) - P_\varphi^k(\boldsymbol{x})\right] |\boldsymbol{x}|^{-\lambda} \, \mathrm{d}\boldsymbol{x} + \int_{|\boldsymbol{x}| > a} \varphi(\boldsymbol{x}) |\boldsymbol{x}|^{-\lambda} \, \mathrm{d}\boldsymbol{x}.$$

Then F_a^k is a distribution on $\mathbb{R}^d$ that agrees with $|\boldsymbol{x}|^{-\lambda}$ on $\mathbb{R}^d\backslash\{\mathbf{0}\}$.

(2) If $\lambda \notin \mathbb{Z}$ and we take k to be the greatest integer $\leq \lambda - d$, we can let $a \to \infty$ in (1) to obtain another distribution F that agrees with $|\boldsymbol{x}|^{-\lambda}$ on $\mathbb{R}^d \backslash \{\boldsymbol{0}\}$:

$$\langle F, \varphi \rangle = \int \left[\varphi(\boldsymbol{x}) - P_\varphi^k(\boldsymbol{x}) \right] |\boldsymbol{x}|^{-\lambda} \, \mathrm{d}\boldsymbol{x}$$

(3) Let $d = 1$ and let k be the greatest integer $\leq \lambda$. Let

$$f(\boldsymbol{x}) = \begin{cases} [(k-\lambda)\cdots(1-\lambda)]^{-1} (\operatorname{sgn} x)^k |x|^{k-\lambda} & \text{if } \lambda > k \\ (-1)^{k-1} [(k-1)!]^{-1} (\operatorname{sgn} x)^k \log |x| & \text{if } \lambda = k \end{cases}.$$

Then $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, and the distribution derivative $f^{(k)}$ agrees with $|x|^{-\lambda}$ on $\mathbb{R} \backslash \{0\}$.

(4) According to Theorem D.2.7, the difference between any two of the distributions constructed in (a)-(c) is a linear combination of δ and its derivatives. Which one?

Exercise D.2.7 ([16, Theorem 3.2.1]). Prove that the distribution $P.V.(1/x)$ equals

$$\frac{\mathrm{d}}{\mathrm{d}x} \log |x| \text{ and } \frac{1}{2} \left(\frac{1}{x - i0} + \frac{1}{x + i0} \right),$$

and its Fourier transform is $-i\sqrt{\frac{\pi}{2}} \operatorname{sgn}(\xi)$.

Appendix E Functional Analysis

The last section of the appendix records some basic concepts and frequently-used theorem arising in functional analysis. The proofs can be found in either Bühler-Salamon [3, Chapter 2-4] or Evans [6, Appendix D].

E.1 Banach spaces

Let X denote a real vector space.

Definition E.1.1. A mapping $\|\quad\| : X \to [0, \infty)$ is called a **norm** if

- (triangle inequality) $\|u + v\| \le \|u\| + \|v\|$ for all $u, v \in X$;
- $\|\lambda u\| = |\lambda|\|u\|$ for all $u \in X, \lambda \in \mathbb{R}$;
- $\|u\| = 0$ if and only if $u = 0$.

Hereafter we assume X is a normed vector space.

Definition E.1.2. We say a normed vector space X equipped with the norm $\|\cdot\|$ is a **Banach space** if $(X, \|\cdot\|)$ is complete, that is, every Cauchy sequence in $(X, \|\cdot\|)$ is convergent and the limit still belongs to X.

Definition E.1.3. We say X is separable if X contains a countable dense subset.

Let X and Y be real Banach spaces.

Definition E.1.4. A mapping $A : X \to Y$ is a **linear operator** provided

$$A[\lambda u + \mu v] = \lambda Au + \mu Av$$

for all $u, v \in X, \lambda, \mu \in \mathbb{R}$. The range of A is $R(A) := \{v \in Y \mid v = Au \text{ for some } u \in X\}$ and the null space of A is $N(A) := \ker(A) = \{u \in X \mid Au = 0\}$. The domain of A is denoted by $D(A)$.

A linear operator $A : X \to Y$ is **bounded** if

$$\|A\| := \sup\left\{\|Au\|_Y \mid \|u\|_X \le 1\right\} < \infty.$$

It is easy to check that a bounded linear operator $A : X \to Y$ is continuous.

Definition E.1.5. A linear operator $A : X \to Y$ is called **closed** if whenever $u_k \to u$ in X and $Au_k \to v$ in Y, then

$$Au = v.$$

Equivalently, the operator A is closed if its graph $G(A) := \{(u, v) \in X \times Y \mid x \in D(A), v = Au\}$ is a closed linear subspace of $X \times Y$. The graph norm of A on the linear subspace $D(A) \subset X$ is the norm $D(A) \to [0, \infty) : u \longmapsto \|u\|_A$ defined by

$$\|u\|_A := \|u\|_X + \|Au\|_Y$$

for $u \in D(A)$.

A linear operator $A : X \to Y$ is called **open** if the image of every open subset of X under f is an open subset of Y.

E.1.1 Geometry of Banach spaces

First, let us note that a Banach space is finite dimensional if and only if every bounded closed set is compact. This is given by the following Lemma by F. Riesz.

Lemma E.1.1 (Riesz's lemma). Let $(X, \|\cdot\|)$ be a normed vector space and let $Y \subset X$ be a closed linear subspace that is not equal to X. Fix a constant $0 < \delta < 1$. Then there exists a vector $x \in X$ such that

$$\|x\| = 1, \quad \inf_{y \in Y} \|x - y\| \geq 1 - \delta$$

Proof. Let $x_0 \in X \backslash Y$. Then $d := \inf_{y \in Y} \|x_0 - y\| > 0$ because Y is closed. Choose $y_0 \in Y$ such that

$$\|x_0 - y_0\| \leq \frac{d}{1 - \delta}$$

and define $x := \|x_0 - y_0\|^{-1} (x_0 - y_0)$. Then $\|x\| = 1$ and

$$\|x - y\| = \frac{\|x_0 - y_0 - \| x_0 - y_0 \| y\|}{\|x_0 - y_0\|} \geq \frac{d}{\|x_0 - y_0\|} \geq 1 - \delta$$

for all $y \in Y$. □

There are several theorems that characterize the geometry of Banach spaces from different aspects.

Theorem E.1.2 (Open Mapping Theorem). Let X, Y be Banach spaces and $A : X \to Y$ be a surjective bounded operator. Then A is open.

A consequence of Open Mapping Theorem is the special case where A is bijective.

Theorem E.1.3 (Inverse Mapping Theorem). Let X, Y be Banach spaces and $A : X \to Y$ be a bijective bounded operator. Then $A^{-1} : Y \to X$ is bounded.

It is often interesting to consider linear operators on a Banach space X whose domains are not the entire Banach space but instead are linear subspaces of X. In most of the interesting cases the domains are dense linear subspaces.

Example E.1.1. Let $X := C([0,1])$ equipped with the supremum norm. Let $D(A) := C^1([0,1]) = \{f : [0,1] \to \mathbb{R} : f$ is continuously differentiable $\}$ and define the linear operator $A : D(A) \to X$ by

$$Af := f' \qquad f \in C^1([0,1])$$

The subspace $D(A) = C^1([0,1])$ is dense in $X = C([0,1])$ thanks to the Weierstrass Approximation Theorem. Moreover, the graph of A, defined by

$$G(A) := \{(f,g) \in X \times X \mid f \in D(A), g = Af\}$$

is a closed linear subspace of $X \times X$. Namely, if $f_n \in C^1([0,1])$ satisfies that (f_n, Af_n) converges to (f,g) in $X \times X$, then f_n uniformly converges to f and f_n' converges uniformly to g, and hence f is continuously differentiable with $f' = g$.

Theorem E.1.4 (Closed Graph Theorem)**.** Let X and Y be Banach spaces and let $A : X \to Y$ be a linear operator. Then A is bounded if and only if its graph is a closed linear subspace of $X \times Y$.

The Closed Graph Theorem asserts that a linear operator $A : X \to Y$ is continuous if and only if A has a closed graph.

Next, we introduce the Uniform Boundedness Principle (一致有界原理/共鸣定理).

Definition E.1.6. Let X be a set. A family $\{f_i\}_{i \in I}$ of mappings $f_i : X \to Y_i$, indexed by the elements of a set I and each taking values in a normed vector space Y_i, is called **pointwise bounded** if

$$\sup_{i \in I} \|f_i(x)\|_{Y_i} < \infty \quad \forall x \in X.$$

Definition E.1.7. Let X and Y be normed vector spaces. A sequence of bounded linear operators $A_n : X \to Y, n \in \mathbb{N}$, is said to **converge strongly** to a bounded linear operator $A : X \to Y$ if $Au = \lim_{t \to \infty} A_n u$ for all $u \in X$.

The Uniform Boundedness Theorem is stated as follows.

Theorem E.1.5 (Banach-Steinhaus Theorem)**.** Let X and Y be Banach spaces and let $A_n : X \to Y, n \in \mathbb{N}$, be a sequence of bounded linear operators. Then the following are equivalent.

(1) The sequence $(A_n u)_{n \in \mathbb{N}}$ converges in Y for every $u \in X$.
(2) $\sup\limits_{n \in \mathbb{N}} \|A_n\| < \infty$ and there is a dense subset $D \subset X$ such that $\{A_n u\}_{n \in \mathbb{N}}$ is a Cauchy sequence in Y for every $u \in D$.
(3) $\sup\limits_{n \in \mathbb{N}} \|A_n\| < \infty$ and there is a bounded linear operator $A : X \to Y$ such that A_n converges strongly to A and $\|A\| \le \liminf\limits_{n \to \infty} \|A_n\|$.

The equivalence (1) $\Leftrightarrow$ (3) still holds when Y is not complete. The equivalence of (2) $\Leftrightarrow$ (3) still holds when X is not complete.

Next, we introduce the Hahn-Banach Theorem. It deals with bounded linear functionals on a subspace of a Banach space X and asserts that every such functional extends to a bounded linear functional on all of X. This is even true when X is a real vector space and boundedness is replaced by a bound relative to a given quasi-seminorm.

Definition E.1.8 (Quasi-Seminorm). Let X be a real vector space. A function $p : X \to \mathbb{R}$ is called a **quasi-seminorm** if it satisfies

$$p(x+y) \leq p(x) + p(y), \quad p(\lambda x) = \lambda p(x)$$

for all $x, y \in X$ and all $\lambda \geq 0$. It is called a **seminorm** if it is a quasi-seminorm and $p(\lambda x) = |\lambda| p(x)$ for all $x \in X$ and all $\lambda \in \mathbb{R}$. A seminorm has nonnegative values, because $2p(x) = p(x) + p(-x) \geq p(0) = 0$ for all $x \in X$. Thus a seminorm satisfies all the axioms of a norm except nondegeneracy (there may be nonzero elements $x \in X$ satisfying $p(x) = 0$).

Theorem E.1.6 (Hahn-Banach Theorem). Let X be a normed vector space and let $p : X \to \mathbb{R}$ be a quasi-seminorm. Let $Y \subset X$ be a linear subspace and let $\phi : Y \to \mathbb{R}$ be a linear functional such that $\phi(x) \leq p(x)$ for all $x \in Y$. Then there exists a linear functional $\Phi : X \to \mathbb{R}$ such that

$$\Phi|_Y = \phi, \quad \Phi(x) \leq p(x) \qquad \forall x \in X.$$

What we will use to prove the local existence of linear wave equations is that

Corollary E.1.7. Let X be a normed vector space over $\mathbb{R}$, let $Y \subset X$ be a linear subspace, let $\phi : Y \to \mathbb{R}$ be a linear functional, and let $c \geq 0$ such that $|\phi(x)| \leq c\|x\|$ for all $x \in Y$. Then there exists a bounded linear functional $\Phi : X \to \mathbb{R}$ such that

$$\Phi|_Y = \phi, \quad |\Phi(x)| \leq c\|x\| \quad \text{for all } x \in X.$$

Proof. Let $p(x) := c\|x\|$ in Hahn-Banach Theorem and then there exists a linear functional $\Phi : X \to \mathbb{R}$ such that $\Phi|_Y = \phi$ and $\Phi(x) \leq c\|x\|$ for all $x \in X$. Since $\Phi(-x) = -\Phi(x)$ it follows that $|\Phi(x)| \leq c\|x\|$ for all $x \in X$. $\square$

For complex vector spaces, analogous results also hold.

Corollary E.1.8. Let X be a normed vector space over $\mathbb{C}$, let $Y \subset X$ be a linear subspace, let $\psi : Y \to \mathbb{C}$ be a complex linear functional, and let $c \geq 0$ such that $|\psi(x)| \leq c\|x\|$ for all $x \in Y$. Then there exists a bounded complex linear functional $\Psi : X \to \mathbb{C}$ such that

$$\Psi|_Y = \psi, \quad |\Psi(x)| \leq c\|x\| \quad \text{for all } x \in X.$$

E.1.2 Weak convergence

Due to Riesz's lemma, the boundedness no longer gives compactness as in the finite-dimensional case (Bolzzano-Weierstrass Theorem). However, we can introduce the concepts of **weak convergence** and **weak-* convergence** such that the compactness in such "weak" sense automatically holds true as long as we have the boundedness. **In our PDE course, there is no need to introduce the weak topology. Instead, we only need sequential weak convergence.** Let X denote a real Banach space.

Definition E.1.9 (Weak Convergence). We say a sequence $\{u_k\}_{k=1}^\infty \subset X$ converges weakly to $u \in X$, written $u_k \rightharpoonup u$, if $\langle f, u_k\rangle \to \langle f, u\rangle$ for each bounded linear functional $f \in X^*$.

Definition E.1.10 (Weak-* Convergence). We say a sequence $\{f_k\}_{k=1}^\infty \subset X^*$ converges weakly-* to $f \in X^*$, written $f_k \overset{*}{\rightharpoonup} f$, if $\langle f_k, u\rangle \to \langle f, u\rangle$ for each $u \in X$.

It is easy to check that if $u_k \to u$, then $u_k \rightharpoonup u$. It is also true that any weakly convergent sequence is bounded. In addition, if $u_k \rightharpoonup u$, then

$$\|u\| \le \liminf_{k\to\infty} \|u_k\|$$

Theorem E.1.9 (Eberlein-Šmulian). Let X be a reflexive Banach space and suppose the sequence $\{u_k\}_{k=1}^\infty \subset X$ is bounded. Then there exists a subsequence $\{u_{k_j}\}_{j=1}^\infty \subset \{u_k\}_{k=1}^\infty$ and $u \in X$ such that $u_{k_j} \rightharpoonup u$ in X.

In other words, bounded sequences in a reflexive Banach space are weakly precompact. In particular, a bounded sequence in a Hilbert space contains a weakly convergent subsequence. If we drop the reflexive assuption, then we have the weak-* convergence.

Theorem E.1.10 (Banach-Alaoglu). Let X be a normed space. Then the closed unit ball in the dual space X^* (endowed with its usual operator norm) is compact with respect to the weak-* topology.

Mazur's Theorem asserts that a convex, closed subset of X is weakly closed. Moreover, we have

Theorem E.1.11 (Mazur). Let $(X, \|\cdot\|)$ be a normed vector space and let $\{x_j\}_{j\in\mathbb{N}} \subset X$ be a sequence converges weakly to some $x \in X$. Then there exists a sequence $\{y_k\}_{k\in\mathbb{N}} \subset X$ made up of finite convex combination of the x_j 's of the form $y_k = \sum\limits_{j\le k} \lambda_j^{(k)} x_j$ such that $y_k \to x$ strongly that is $\|y_k - x\| \to 0$.

E.2 Hilbert spaces

In many situations in this lecture notes, we only use L^2-based Sobolev spaces $H^k(\Omega)$. In such cases, the Banach space also enjoys an inner product structure. Let H be a real linear space.

Definition E.2.1. A mapping $(\cdot,\cdot) : H \times H \to \mathbb{R}$ is called an inner product if

- $(u, v) = (v, u)$ for all $u, v \in H$,
- the mapping $u \longmapsto (u, v)$ is linear for each $v \in H$,

- $(u, u) \geq 0$ for all $u \in H$,
- $(u, u) = 0$ if and only if $u = 0$.

The associated norm is $\|u\| := (u, u)^{1/2}$. A Hilbert space H is a Banach space endowed with an inner product which generates the norm. If H is a complex linear space, then $(u, v) = \overline{(v, u)}$.

Definition E.2.2. If S is a subspace of $H, S^{\perp} = \{u \in H \mid (u, v) = 0 \text{ for all } v \in S\}$ is the subspace orthogonal to S.

For Hilbert spaces H, one can identify its dual space H^* with H through the following theorem.

Theorem E.2.1 (Riesz Representation Theorem). H^* can be canonically identified with H; more precisely, for each $u^* \in H^*$ there exists a unique element $u \in H$ such that $\langle u^*, v \rangle = (u, v)$, $\forall v \in H$. The mapping $u^* \longmapsto u$ is a linear isomorphism of H^* onto H.

It should be noted that $H^* \ncong H$ via the identity map!

E.3 Spectrum theory of compact operators

One of the most important concepts in the study of bounded linear operators is that of a compact operator which **produces compactness from boundedness.** The notion of a compact operator can be defined in several equivalent ways. The equivalence of these conditions is the content of the following lemma.

Lemma E.3.1. Let X and Y be Banach spaces and let $K : X \to Y$ be a bounded linear operator. Then the following are equivalent.

- If $(x_n)_{n \in \mathbb{N}}$ is a bounded sequence in X, then the sequence $(Kx_n)_{n \in \mathbb{N}}$ has a Cauchy subsequence.
- If $S \subset X$ is a bounded set, then the set $K(S) := \{Kx \mid x \in S\}$ has a compact closure.
- The set $\overline{\{Kx \mid x \in X, \|x\|_X \leq 1\}}$ is a compact subset of Y.

Definition E.3.1 (Compact Operator). Let X and Y be Banach spaces. A bounded linear operator $K : X \to Y$ is said to be

- **compact** if it satisfies the equivalent conditions of the above lemma;
- of **finite rank** if its image is a finite-dimensional subspace of Y;
- **completely** continuous if the image of every weakly convergent sequence in X under K converges in the norm topology on Y.

Proposition E.3.2 (Compact≈completely continuous). Let X and Y be Banach spaces. Then every compact operator $K : X \to Y$ is completely continuous. If additionally X is reflexive, then a bounded linear operator $K : X \to Y$ is compact if and only if it is completely continuous.

Proposition E.3.3 (Composition and duality). Let X, Y, and Z be Banach spaces. Then the following hold.

(1) Let $A : X \to Y$ and $B : Y \to Z$ be bounded linear operators and assume that A is compact or B is compact. Then $BA : X \to Z$ is compact.
(2) Let $K_i : X \to Y$ be a sequence of compact operators that converges to a bounded linear operator $K : X \to Y$ in the norm topology. Then K is compact.
(3) Let $K : X \to Y$ be a bounded linear operator and let $K^* : Y^* \to X^*$ be its dual operator. Then K is compact if and only if K^* is compact.

E.3.1 Riesz-Fredholm Theory: Ker and Im of compact operators

Let X be a Banach space and $\mathfrak{C}(X)$ denotes the set of all compact operators on X. Then we have

Theorem E.3.4 (Fredholm alternative). Let X be a Banach space, $K \in \mathfrak{C}(X)$. Then

(1) $\dim N(I-K) < \infty$, where $N(I-K) = \{x \in X | (I-K)x = 0\}$.
(2) $R(I-K)$ is closed.
(3) $R(I-K) = N(I-K^*)^\perp$ and $R(I-K^*) = {}^\perp N(I-K)$.
(4) $N(I-K) = \{0\}$ if and only if $R(I-K) = X$.
(5) $\dim N(I-K) = \dim(N(I-K^*))$.

Here, for $M \subset X,\ F \subset X'$, we denote

$$^\perp M := \{f \in X' | \langle f, x\rangle = 0,\ \forall x \in M\}, \quad F^\perp := \{x \in X | \langle f, x\rangle = 0,\ \forall f \in X'\}.$$

Remark E.3.1. The Fredholm alternative deals with the solvability of the equation $u - Ku = f$. It says that either for every $f \in X$ the equation $u - Ku = f$ has a unique solution, or the homogeneous equation $u - Ku = 0$ admits n linearly independent solutions. In the latter case, the non-homogeneous equation $u - Ku = f$ is solvable if and only if f satisfies n orthogonality conditions $f \in N(I-K^*)^\perp$.

E.3.2 Riesz-Schauder Theory: Spectrum of compact operators

We finally record the spectrum theorem of compact operators.

Definition E.3.2. Let X be a Banach space and $A : X \to X$ is a bounded linear operator.

- The **resolvent** set of A is defined by $\rho(A) := \{\eta \in \mathbb{R} | A - \eta I \text{ is 1-1 and onto}\}$.
- The **spectrum** of A is defined by $\sigma(A) := \mathbb{R} \backslash \rho(A)$.

Given $\eta \in \rho(A)$, by the Closed Graph Theorem, we know $(A - \eta I)^{-1}$ is a bounded linear operator on X.

- We say $\lambda \in \sigma(A)$ is an **eigenvalue** of A if $N(A - \eta I) \neq \{0\}$. The set of all eigenvalues is denoted by $\sigma_p(A)$, called "point spectrum".
- If λ is an eigenvalue with $Aw = \lambda w$ for some $w \neq 0$, then we say w is an **eigenvector** of A associated with λ.

We now have

Theorem E.3.5 (Riesz-Schauder). Let X be a Banach space and $K \in \mathfrak{C}(X)$. Then

(1) $0 \in \sigma(K)$ unless $\dim X < \infty$.
(2) $\sigma(K)\backslash\{0\} = \sigma_p(K)\backslash\{0\}$.
(3) The accumulation point of $\sigma_p(K)$, if exists, must be 0.

E.3.3 Symmetric operators on Hilbert spaces

Let H be a real Hilbert space.

Definition E.3.3. We say a bounded linear operator $A : H \to H$ is **symmetric** if $(Ax, y) = (x, Ay)$ holds for all $x, y \in H$. Here $(\cdot, \cdot)$ is the inner product of H. It is easy to see that A is symmetric if and only if $A = A^*$.

Proposition E.3.6. Let $A : H \to H$ be a bounded, linear operator. Then A is symmetric if and only if $(Ax, x) \in \mathbb{R}$ for any $x \in H$. In this case, we further have

(1) $\sigma(A) \subset \mathbb{R}$ and $\|(\lambda I - A)^{-1}x\| \leq \frac{\|x\|}{|\mathbf{Im}\,\lambda|}$ for any $x \in H,\ \lambda \in \mathbb{C}$ with $\mathbf{Im}\,\lambda \neq 0$.
(2) Let $H_1 \subset H$ be an A-invariant closed subspace of H, then A_{H_1} is also symmetric on H_1.
(3) For any $\lambda, \lambda' \in \sigma_p(A)$ with $\lambda \neq \lambda'$, we have $N(\lambda I - A) \perp N(\lambda' I - A)$.
(4) $\|A\| = \sup_{\|x\|=1} |(Ax, x)|$.

Now let $S : H \to H$ be linear, bounded, symmetric, and write

$$m := \inf_{\substack{u\in H\\ \|u\|=1}} (Su, u), M := \sup_{\substack{u\in H\\ \|u\|=1}} (Su, u)$$

Proposition E.3.7 (Bounds on spectrum). We have $\sigma(S) \subset [m, M]$ and $m, M \in \sigma(S)$.

Proof. Let $\eta > M$. Then $(\eta u - Su, u) \geq (\eta - M)\|u\|^2 \quad (u \in H)$. Hence the Lax-Milgram Theorem asserts $\eta I - S$ is one-to-one and onto, and thus $\eta \in \rho(S)$. Similarly $\eta \in \rho(S)$ if $\eta < m$. This proves $\sigma(S) \subset [m, M]$.

We next prove $M \in \sigma(S)$. Since the pairing $[u, v] := (Mu - Su, v)$ is symmetric, with $[u, u] \geq 0$ for all $u \in H$, the Cauchy-Schwarz inequality implies $|(Mu - Su, v)| \leq (Mu - Su, u)^{1/2}(Mv - Sv, v)^{1/2}$ for all $u, v \in H$. In particular, $\|Mu - Su\| \leq C(Mu - Su, u)^{1/2} \quad (u \in H)$ for some constant C.

Now let $\{u_k\}_{k=1}^{\infty} \subset H$ satisfy $\|u_k\| = 1 (k = 1, \ldots)$ and $(Su_k, u_k) \to M$. Then we have $\|Mu_k - Su_k\| \to 0$. Now if $M \in \rho(S)$, then

$$u_k = (MI - S)^{-1} (Mu_k - Su_k) \to 0$$

a contradiction. Thus $M \in \sigma(S)$, and likewise $m \in \sigma(S)$. □

Theorem E.3.8 (Eigenvectors of a compact, symmetric operator). Let H be a separable Hilbert space, and suppose $S : H \to H$ is a compact and symmetric operator. Then there exists a countable orthonormal basis of H consisting of eigenvectors of S.

Proof. Let $\{\eta_k\}$ comprise the sequence of distinct eigenvalues of S, excepting 0. Set $\eta_0 = 0$. Write $H_0 = N(S), H_k = N(S - \eta_k I)\,(k = 1, \dots)$. Then $0 \le \dim H_0 \le \infty$, and $0 < \dim H_k < \infty$, according to the Fredholm alternative.

Let $u \in H_k, v \in H_l$ for $k \neq l$. Then $Su = \eta_k u, Sv = \eta_l v$ and so $\eta_k(u,v) = (Su,v) = (u,Sv) = \eta_l(u,v)$. As $\eta_k \neq \eta_l$, we deduce $(u,v) = 0$. Consequently we see the subspaces H_k and H_l are orthogonal.

Now let $\tilde{H}$ be the smallest subspace of H containing $H_0, H_1, \dots$ Thus

$$\tilde{H} = \left\{ \sum_{k=0}^{m} a_k u_k \,:\, m \in \{0, \dots\}, u_k \in H_k, a_k \in \mathbb{R} \right\}.$$

We next demonstrate $\tilde{H}$ is dense in H. Clearly $S(\tilde{H}) \subseteq \tilde{H}$. Furthermore $S\left(\tilde{H}^\perp\right) \subseteq \tilde{H}^\perp$: indeed if $u \in \tilde{H}^\perp$ and $v \in \tilde{H}$, then $(Su, v) = (u, Sv) = 0$.

Now the operator $\tilde{S} \equiv S\big|_{\tilde{H}^\perp}$ is compact and symmetric. In addition $\sigma(\tilde{S}) = \{0\}$, since any nonzero eigenvalue of $\tilde{S}$ would be an eigenvalue of S as well. According to the lemma then, $(\tilde{S}u, u) = 0$ for all $u \in \tilde{H}^\perp$. But if $u, v \in \tilde{H}^\perp$,

$$2(\tilde{S}u, v) = (\tilde{S}(u+v), u+v) - (\tilde{S}u, u) - (\tilde{S}v, v) = 0$$

Hence $\tilde{S} = 0$. Consequently $\tilde{H}^\perp \subset N(S) \subset \tilde{H}$, and so $\tilde{H}^\perp = \{0\}$. Thus $\tilde{H}$ is dense in H.

Choose an orthonormal basis for each subspace $H_k (k = 0, \dots)$, noting that since H is separable, H_0 has a countable orthonormal basis. We obtain thereby an orthonormal basis of eigenvectors. □

On Hilbert spaces, the spectrum and structure of symmetric compact operators are quit similar to those of real symmetric matrices in Euclidean spaces. In particular, we recall that any real symmetric matrix is diagonalizable and the elements on the diagonal are exactly the eigenvalues, which also implies that the eigenvectors of a real symmetric matrix gives an orthogonal (actually orthonormal after normalization) basis of the Euclidean spaces. Also, the critical value of quadratic form is also an eigenvalue. These properties also hold for symmetric compact operators on Hilbert spaces.

Proposition E.3.9. Let $A \in \mathfrak{C}(H)$ be symmetric. Then there exists an $x_0 \in H$, $\|x_0\| = 1$, such that

$$\lambda := |(Ax_0, x_0)| = \sup_{\|x\|=1} |(Ax, x)|, \quad Ax_0 = \lambda x_0.$$

Proposition E.3.10. Let $A \in \mathfrak{C}(H)$ be symmetric. Then there is a at most countable sequence of real numbers $\{\lambda_k\}_{k\in\mathbb{N}^*}$ whose only possible accumulation point (if exists) is 0, such that $\{\lambda_k\}$ are exactly the eigenvalues of A. Also, there exists an orthonormal basis $\{e_k\}$ of H such that

$$x = \sum_{k\ge1} (x, e_k) e_k, \quad Ax = \sum_k \lambda_k (x, e_k) e_k.$$

Proposition E.3.11 (Courant minimax characterization). Let $A \in \mathfrak{C}(H)$ be symmetric and have eigenvalues $\lambda_1^+ \geq \lambda_2^+ \geq \cdots \geq 0 > \cdots \geq \lambda_2^- \geq \lambda_1^-$. Then

$$\lambda_n^+ = \inf_{E_{n-1}} \sup_{\substack{x \in E_{n-1}^\perp \\ x \neq 0}} \frac{(Ax, x)}{(x, x)}, \qquad \lambda_n^- = \sup_{E_{n-1}} \max_{\substack{x \in E_{n-1}^\perp \\ x \neq 0}} \frac{(Ax, x)}{(x, x)}.$$

Here E_{n-1} can be any $(n-1)$-dimensional closed subspace of H.

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