几何测度论基础

- BV in $\mathbb{R}^d$? → 某种意义下最弱的可导性，导数是 Radon 测度.
- AC　　书本 4.1-4.6 节

教材：
L.C. Evans, R.F. Gariepy: Measure Theory and Fine Properties of Functions, ch 1-4

Ch 1 General Measure Theory

Def: (1) A map $\mu: 2^X \to [0, \infty]$ is called a measure iff.

① $\mu(\phi) = 0$.

② Sub-Additivity.

(2) μ is a measure on X. $C \subseteq X$ is any subset.

$(\mu \llcorner C)(A) := \mu(A \cap C)$. $\forall A \subseteq X$. is the measure ~~defined by~~ called the restriction of μ to C.

(3). $A \subseteq X$ is μ-measurable iff $\forall B \subseteq X$. $\mu(B) = \mu(B \cap A) + \mu(B \cap A^c)$

(4) (Borel measure). All Borel sets are ~~Borel~~ μ-measurable.

(5) (π-system). A nonempty collection of subsets $\mathcal{P} \subseteq 2^X$ is a π-system provided $A, B \in \mathcal{P} \Rightarrow A \cap B \in \mathcal{P}$

(6) (λ-system). A collection of subsets $\mathcal{L} \subseteq 2^X$ is a λ-system provided ① $X \in \mathcal{L}$

② $A, B \in \mathcal{L}$. $B \subseteq A \Rightarrow A - B \in \mathcal{L}$.

③ $A_k \in \mathcal{L} \nearrow \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{L}$.

- Thm (π-λ). $\mathcal{P}$ is a π-system.
  $\mathcal{L}$ is a λ-system
  $\mathcal{P} \subseteq \mathcal{L}$. $\Rightarrow \sigma(\mathcal{P}) \subseteq \mathcal{L}$.

Rmk:

- Used To prove that the ~~sigm~~ σ-Alg generated by $\mathcal{P}$ has property p.

  Steps: ① $\mathcal{P}$ is a π-system.

  ② $\mathcal{L} = \{$ all satisfying $p\}$ is a λ-system.

  ③ $\mathcal{P} \subseteq \mathcal{L}$.

  $\xRightarrow{\pi\text{-}\lambda \text{ System}}$ $\sigma(\mathcal{P}) \subseteq \mathcal{L}$. $\Rightarrow$ done. $\square$.

- λ-system is similar to the monotone class.

Example:

Thm (Borel Measures and Rectangles)

Let μ, ν be 2 finite Borel measures on $\mathbb{R}^n$ such that

$\mu(R) = \nu(R)$, $\forall$ rectangles (closed)

$R = \{x \in \mathbb{R}^n \mid -\infty \le a_i \le x_i \le b_i \le \infty,\ 1 \le i \le n\}$.

then $\mu(B) = \nu(B)$ $\forall$ Borel sets $B \subseteq \mathbb{R}^n$.

Proof: $\mathcal{P} = \{R \in \mathbb{R}^n \mid R$ is a rectangle$\}$ is a π-system.

$\mathcal{L} = \{B \subseteq \mathbb{R}^n \mid B$ is Borel. $\mu(B) = \nu(B)\}$. then $\mathcal{P} \subseteq \mathcal{L}$

If $\mathcal{L}$ is a λ-system, then by π-λ Thm, we have $\underset{\mathcal{B}_{\mathbb{R}^n}}{\underset{\|}{\sigma(\mathcal{P})}} \subseteq \mathcal{L}$. done

* Check: $\mathcal{L}$ is a λ-system

① $\mu(\mathbb{R}^n) = \nu(\mathbb{R}^n)$. Trivial

② (注意~~半~~开半闭 $= \bigcup\limits_{\text{可数}}$ 闭).

③ $A_i \in \mathcal{L}$. $A_i \subseteq A_{i+1}$. $\Rightarrow \bigcup\limits_{i=1}^{\infty} A_i \in \mathcal{L}$.

Proof of π-λ Thm:

Define $S := \bigcap\limits_{\substack{\mathcal{L}' \supseteq P \\ \mathcal{L}'\ \lambda\text{-system}}} \mathcal{L}'$. $\Rightarrow P \subseteq S \subseteq \mathcal{L}$.

Check: $\begin{cases} ①\ S \text{ is a } \lambda\text{-system}. \\ ②\ S \text{ is a } \pi\text{-system} \\ ③\ S \text{ is a } \sigma\text{-Algebra} \end{cases}$ $\xRightarrow{\pi\text{-}\lambda\ \text{thm}}$ $\sigma(P) \subseteq S \subseteq \mathcal{L}$. $\Rightarrow$ done.

① : Trivial

Check ②: $\forall A, B \in S$. To prove $A \cap B \in S$,

↑

好集技术

we define $\mathcal{A} := \{C \subseteq X \mid A \cap C \in S\}$

S is a λ-system. $\Rightarrow \mathcal{A}$ is a λ-system.

$\Rightarrow S \subseteq \mathcal{A}$

$\xRightarrow{B \in S} A \cap B \in S$ ~~$\Rightarrow S \subseteq \mathcal{A}$~~. $\Rightarrow$ ...

③ S is both π-system and λ-system.

- $A \in S \Rightarrow X - A = A^c \in S$ ($X \in S$)

- Countable union:

$A_1, A_2, \ldots \in S$. $\left.\begin{array}{l} B_n = \bigcup\limits_{k=1}^{n} A_k \overset{\pi\text{-system}}{\in} S \\ S \text{ is } \lambda\text{-system} \end{array}\right\} \xRightarrow{(3)} \bigcup\limits_{n=1}^{\infty} B_n \in S$, $\bigcup\limits_{k=1}^{\infty} A_k = \bigcup\limits_{n=1}^{\infty} B_n$ $\Rightarrow S$ is a σ-Alg. D

Def: A measure μ on X.

(1) (regular). if $\forall A \subseteq X$. $\exists$ μ-measurable set B s.t. $A \subseteq B$

$\mu(A) = \mu(B) \not\Rightarrow \mu(B-A)=0$ (not μ-measurable)

(2). (Borel regular). if μ is Borel and ~~for each~~ $\forall A \subseteq \mathbb{R}^n$.

$\exists$ a Borel set $B \supseteq A$ s.t. $\mu(A) = \mu(B)$

(3)*. (Radon Measure) A measure μ on $\mathbb{R}^n$ is a Radon measure,

if $\begin{cases} \mu \text{ is Borel regular} \\ \mu(K) < \infty \quad \forall K \subset\subset \mathbb{R}^n. \end{cases}$

□

eg: ① $\mathbb{R}^n$ 上的 Lebesgue测度 $\mathcal{L}$, $\forall f \in L^1(\mathbb{R}^n)$, $\mu(A) = \int_A f \, d\mathcal{L}$.

满足 (1) (2) (3).

② $\mathbb{R}^2$ 上. 给定 smooth curve Γ

$\forall$ open set A, $\mu(A) = \Gamma$在A中的长度 (Hausdorff Measure)

Radon Measure

质量都聚集在线上 ↓ 以及一些奇异集合

质量聚集在点上 → ③ Dirac (Radon)

Thm (Increasing Sets) μ regular on X. $A_k \uparrow$ then

$$\lim_{k\to\infty} \mu(A_k) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right).$$

Proof: (可测包技术). $\exists C_k$. with $A_k \subseteq C_k$. $\mu(A_k) = \mu(C_k)$ $\forall k$.

$B_k = \bigcap_{j \geq k} C_j \uparrow . \supseteq A_k$. $\Rightarrow \lim_{k\to\infty} \mu(A_k) = \lim_{k\to\infty} \mu(B_k)$

B_k: μ-measurable.

$$= \mu\left(\bigcup_{k=1}^{\infty} B_k\right) \geq \mu\left(\bigcup_{k=1}^{\infty} A_k\right) \geq \mu(A_k) \quad \forall k$$

$\Rightarrow$ Equality holds. □

Thm. (Restriction and Radon Measures) (重要).

Let μ be a Borel regular measure on $\mathbb{R}^n$. $A \subseteq \mathbb{R}^n$ μ-measurable

$\mu(A) < \infty \Rightarrow \underset{\nu =}{\mu \llcorner A}$ is a Radon measure.

Rmk: Hausdorff 测度 Borel 正则 但不 Radon $\xrightarrow{\text{限制在小集合上}}$ 成为 Radon 测度.

Proof: ① $\nu = \mu \llcorner A$ is a measure. ✓

② Radon = measure + Borel + Borel regular.

• Borel ?

$\forall$ Borel set B, why is B ~~μ-m~~ ν-measurable?

By Thm 1.1 (4), every μ-measurable set is ν-measurable.

$\Rightarrow \nu$ is a Borel measure.

• Borel regular ?

(i). μ Borel regular $\Rightarrow \exists$ Borel set B. Such that $A \subseteq B$. $\mu(A) = \mu(B) < \infty$

$\Rightarrow \mu(B - A) = 0$

Consider $\tilde{\nu} = \mu \llcorner B$. $\Rightarrow$. ν coincides with $\tilde{\nu}$.

$\Rightarrow$ WLOG A is a Borel set.

(ii) $\forall C \subseteq \mathbb{R}^n$. $\exists$ Borel set $D \supseteq C$. $\nu(C) = \nu(D)$.

μ Borel regular $\Rightarrow$ $\exists$ Borel set E, $A \cap C \subseteq E$. $\mu(E) = \mu(A \cap C)$.

Let $D = E \cup (\mathbb{R}^n \setminus A)$. ~~State~~ $\Rightarrow D$ Borel.

$*$: $C \subseteq (A \cap C) \cup (\mathbb{R}^n \setminus A) \subseteq D$ $\Rightarrow$ $\nu(D) = \mu(D \cap A)$

$D \cap A = E \cap A$ $\quad = \mu(E \cap A)$

$\quad \leq \mu(E) = \mu(A \cap C) = \nu(C)$.

• $\forall K \subset\subset \mathbb{R}^n$.

$\nu(K) < \infty$ is trivial.

$\square$

$*$. μ is a Radon measure.

B Borel, $\mu(E)$.

How E can be $\underbrace{\text{approximated}}_{\text{measured by } \mu}$ by $\begin{matrix}\text{open sets} \\ \text{compact sets}\end{matrix}$ from $\begin{matrix}\text{outside} \\ \text{inside}\end{matrix}$?

Lem 1.1: μ Borel on $\mathbb{R}^n$.

B Borel set

(1) If $\mu(B)<\infty$. then $\forall \varepsilon>0$. $\exists C$ closed s.t. $C \subseteq B$. $\mu(B \setminus C)<\varepsilon$.

(2) If μ is a Radon measure, then $\forall \varepsilon>0$. there exists an open set U

s.t. $B \subseteq U$ $\mu(U \setminus B)<\varepsilon$.

Proof: $\nu := \mu \llcorner B$. μ is Borel. $\mu(B)<\infty$. ν is a finite Borel measure.

Let $\mathcal{F} := \{A \subseteq \mathbb{R}^n \mid A$ is μ-measurable and $\forall \varepsilon>0$, $\exists$ closed set $C \subseteq A$ with $\nu(A-C)<\varepsilon\}$

① $\{\text{closed set}\} \subseteq \mathcal{F}$.

② $\{A_i\}_1^\infty \subseteq \mathcal{F} \Rightarrow A = \bigcap_{i=1}^\infty A_i \in \mathcal{F}$

check: $\forall \varepsilon>0$ since $A_i \in \mathcal{F}$ then $\exists$ closed $C_i \subseteq A_i$. $\nu(A_i - C_i) < \frac{\varepsilon}{2^i}$

let $C := \bigcap_{i=1}^\infty C_i$. then C closed.

$\nu(A-C) = \nu(\bigcap_{i=1}^\infty A_i - \bigcap_{i=1}^\infty C_i) \le \cancel{\nu(\bigcup_{i=1}^\infty (A_i \setminus C_i))}$

$\nu(\bigcup_{i=1}^\infty (A_i - C_i)) \le \sum_{i=1}^\infty \nu(A_i - C_i) < \varepsilon$.

$\Rightarrow A \in \mathcal{F}$

③ If $\{A_i\}_1^\infty \subseteq \mathcal{F}$. then $A := \bigcup_{i=1}^\infty A_i \in \mathcal{F}$.

check: $\forall \varepsilon>0$. Take C_i as before, $C := \underbrace{\bigcup_{i=1}^\infty C_i}_{\text{not closed}}$

$\nu(A)<\infty$

$\Rightarrow \nu(A-C) \le \sum_{i=1}^\infty \nu(A_i - C_i) < \varepsilon$.

* Consider $A - \bigcup_{i=1}^{m} C_i$

$$\lim_{m\to\infty} \nu\left(A - \bigcup_{i=1}^{m} C_i\right) = \nu\left(\bigcup_{i=1}^{\infty} A_i - \bigcup_{i=1}^{\infty} C_i\right) < \varepsilon.$$

$\Rightarrow$. $\exists m \in \mathbb{Z}_+$. $\nu(A - \bigcup_{i=1}^{m} C_i) < \varepsilon$ $\quad \bigcup_{i=1}^{m} C_i$ closed $\Rightarrow A \in \mathcal{F}$.

③ $\mathcal{F}$ contains all open sets.

$$\mathcal{G} = \{A \in \mathcal{F} \mid \mathbb{R}^n \setminus A \in \mathcal{F}\}.$$

$\begin{cases} A \in \mathcal{G} \Rightarrow \mathbb{R}^n \setminus A \in \mathcal{G} \\ \mathcal{G} \text{ contains all open sets.} \end{cases}$

• If $\{A_i\}_1^{\infty} \subseteq \mathcal{G} \Rightarrow A = \bigcup_1^{\infty} A_i \in \mathcal{G}$

~~Proof of the claim.~~

check: $A \in \mathcal{F}$. $\{\mathbb{R}^n \setminus A_i\}_1^{\infty} \in \mathcal{F}$

$\Rightarrow \mathbb{R}^n \setminus A = \bigcap_1^{\infty} (\mathbb{R}^n \setminus A_i) \in \mathcal{F}$.

$\Rightarrow \mathcal{G}$ is a σ-Alg containing all open sets $\Rightarrow$ contains all Borel sets

In particular. $B \in \mathcal{G} \Rightarrow \forall \varepsilon > 0$. $\exists$ closed set $C \subseteq B$

$\mu(B - C) = \nu(B - C) < \varepsilon$. $\Rightarrow$ (1) holds.

Write $U_m = B^\circ(0, m)$.

$U_m - B$ Borel. $\mu(U_m - B) < \infty$.

By (1), $\exists C_m \subseteq U_m \setminus B$ s.t. $\mu((U_m - C_m) - B) = \mu((U_m - B) - C_m) < \frac{\varepsilon}{2^m}$

$U = \bigcup_{m\geq 1}^{\infty} (U_m - C_m)$. open. Now $B \subseteq \mathbb{R}^n \setminus C_m$

$\Rightarrow U_m \cap B \subseteq U_m \setminus C_m$.

$\Rightarrow$. $B = \bigcup_{m\geq 1}^{\infty} (U_m \cap B) \subseteq \bigcup_{m\geq 1} (U_m - C_m) = U$.

$\mu(U - B) = \mu\left(\bigcup_{m\geq 1}^{\infty} (U_m - C_m) - B\right) \leq \sum_{m\geq 1}^{\infty} \mu((U_m - C_m) - B) < \varepsilon$. □

Thm 1.8 (Approximation by open & compact sets).

Let μ be a Radon measure on $\mathbb{R}^n$

(1) $\forall A \subseteq \mathbb{R}^n$. $\mu(A) = \inf\{\mu(U) \mid A \subseteq U,\ U \text{ open}\}$

(2). $\forall$ μ-measurable $A \subseteq \mathbb{R}^n$. $\mu(A) = \sup\{\mu(K) \mid K \subseteq A\ \ K \text{ compact}\}$.

$\Rightarrow \begin{cases} \mu \text{ is Borel regular.} \quad , \exists B \text{ Borel } \mu(B)=\mu(A). \\ \text{Lemma 1.1.} \end{cases}$

Proof. $\mu(A)=\infty$ ~~$\Rightarrow$ trivial. for (1).~~
$\Rightarrow$ trivial.

(1) WLOG $\mu(A)<\infty$.

① If A Borel. Fix $\varepsilon>0$. by Lemma 1.1. $\exists$ open $U \supseteq A$ with $\mu(U\setminus A)<\varepsilon$. since $\mu(U)=\mu(A)+\mu(U-A)<\infty$ $\Rightarrow$ (1) holds

② If A is an arbitrary set. since μ is Borel regular, then there exists a Borel set $B \supseteq A$ with $\mu(A)=\mu(B)$.

Thus $\mu(A)=\mu(B) = \inf\{\mu(U) \mid B \subseteq U.\ U \text{ open}\}$

$\geq \inf\{\mu(U) \mid A \subseteq U.\ U \text{ open}\}$.

$\geq \mu(A)$. $\Rightarrow$ (1) holds $\forall A \subseteq \mathbb{R}^n$.

(2). A μ-measurable. $\mu(A)<\infty$

Set $\nu = \mu \llcorner A$, then ν is a Radon measure.

Consider $\mathbb{R}^n \setminus A$ By (1). $\forall \varepsilon>0$. $\exists$ open set $U \supseteq \mathbb{R}^n - A$

s.t. $\nu(U) \leq \varepsilon$.

$C := \mathbb{R}^n \setminus U$. $\Rightarrow$ C is closed. $C \subseteq A$. $\Rightarrow$ $\mu(A-C) = \nu(\mathbb{R}^n - C) \leq \varepsilon$.

$\Rightarrow 0 \leq \mu(A)-\mu(C) \leq \varepsilon$.

$\Rightarrow \mu(A) = \sup\{\mu(C) \mid C \subseteq A\ \ C \text{ closed}\}$ ... (*)

Suppose $\mu(A)=\infty$ $\quad D_k=\{x \mid k-1\le |x|<k\}$ Then $A=\bigcup_{k=1}^{\infty}(D_k\cap A)$.

$$\Rightarrow \sum_{k=1}^{\infty}\mu(A\cap D_k)=\infty$$

μ Radon. $\mu(A\cap D_k)<\infty$

$$\Rightarrow \exists\, C_k \text{ closed} \subseteq A\cap D_k.\quad \mu(C_k)\ge \mu(D_k\cap A)-\frac{1}{2^k}$$

$\bigcup_{k=1}^{\infty} C_k \subseteq A.$

$$\lim_{n\to\infty} \mu\left(\bigcup_{k=1}^{n} C_k\right)=\mu\left(\bigcup_{k=1}^{\infty} C_k\right)$$

$$\underset{\text{disjoint}}{=}\sum_{k=1}^{\infty}\mu(C_k)\ge\sum_{k=1}^{\infty}\left(\mu(D_k\cap A)-\frac{1}{2^k}\right)=\infty$$

But $\bigcup_{k=1}^{n} C_k$ is closed for each n. $\Rightarrow$ (*) also holds

~~Finally, Let $B(m)$ denote the closed ball with~~

Finally. C closed. $C_m=\overline{B(0,m)}\cap C$. compact.

$$\mu(C)=\lim_{m\to\infty}\mu(C_m)$$

$\Rightarrow$ $\forall$ μ-measurable set A.

$$\sup\{\mu(K)\mid K\subseteq A.\ K \text{ compact}\}=\sup\{\mu(C)\mid C\subseteq A\ \ C \text{ closed}\}$$

$\square$

Thm 1.9 Caratheodory's criterion.

Let μ be a measure on $\mathbb{R}^n$. If for all sets $A, B\subseteq \mathbb{R}^n$, we have $\mu(A\cup B)=\mu(A)+\mu(B)$, whenever $\mathrm{dist}(A,B)>0$

then μ is a Borel measure

$\square$

Thm 1.12 (Decomposition of nonnegative measurable functions).

Assume that $f: X\to[0,\infty]$ is μ-measurable. Then there exist μ-measurable sets $\{A_k\}_1^{\infty}$ in X such that $f=\sum_{k=1}^{\infty}\frac{1}{k}\chi_{A_k}$

$\square$

Thm 1.13 Extending continuous functions:

Suppose ~~$K \subseteq \mathbb{R}^n$~~ $K \subset\subset \mathbb{R}^n$. $f: K \to \mathbb{R}^n$ continuous, then $\exists$ a continuous mapping

$\tilde{f}: \mathbb{R}^n \to \mathbb{R}^n$ s.t. $\bar{f} = f$ on K. (Tietze 扩张定理)

— 一定要是紧集. 否则考虑 Heaviside 函数反例. $\square$

Thm 1.14. Luzin's Thm

μ Borel regular on $\mathbb{R}^n$. $f: \mathbb{R}^n \to \mathbb{R}^m$ μ-measurable.

$A \subseteq \mathbb{R}^n$ is μ-measurable. $\mu(A) < \infty$

Fix $\varepsilon > 0$, then $\exists$ ~~$K \subset\subset A$~~ $K \subset A$ cpt s.t.

(1) $\mu(A - K) < \varepsilon$

(2) $f\big|_K$ continuous

希望用"在不同紧集上取不同 const"的连续函数来逼近

Proof: (1) $\forall i \in \mathbb{Z}_+$. $\{B_{ij}\}_{j=1}^{\infty} \subset \mathbb{R}^m$ disjoint Borel sets such that

$\mathbb{R}^m = \bigcup_{j=1}^{\infty} B_{ij}$. $\operatorname{diam} B_{ij} < \frac{1}{i}$

Define $A_{ij} := A \cap f^{-1}(B_{ij}) \Rightarrow \mu$-measurable. $A = \bigcup_{j=1}^{\infty} A_{ij}$

Set $f_i = \sum_{j=1}^{+\infty} b_j \chi_{A_{ij}}(x) : A \to \mathbb{R}^n$ $b_j \in B_j$. 任取定.

则 $\Rightarrow f_i \rightrightarrows f$ on A

Write $\nu = \mu \llcorner A$ is a Radon measure

By Thm 1.8, $\Rightarrow \exists$ cpt $K_{ij} \subseteq A_{ij}$ with $\nu(A_{ij} - K_{ij}) < \frac{\varepsilon}{2^{i+j}}$

Then $\mu(A - \bigcup_{j=1}^{\infty} K_{ij}) = \nu(A - \bigcup_{j=1}^{\infty} K_{ij})$

$f_i\big|_{K_{ij}} = \text{const}$.

~~$A_i = \bigcup_{j=1}$~~ $K = \bigcup_{j=1}^{\infty} K_{ij} \subset \bigcup_{j=1}^{\infty} A_{ij} = A$.

$K_{ij} \cap K_{ij'} = \phi$.

$\nu(A - K_i) \leq \sum_{j=1}^{\infty} \nu(A_{ij} - K_{ij}) < \frac{\varepsilon}{2^i}$

取 j 足够大 N. $\nu(A - \bigcup_{j=1}^{N(i)} K_{ij}) \leq \frac{2\varepsilon}{2^i}$

$\nu(A - \bigcup_{j=1}^{N} K_{ij}) \to \nu(A - \bigcup_{j=1}^{\infty} K_{ij}) \leq \frac{\varepsilon}{2^i}$

3. $\tilde{K}_i = \bigcup_{j=1}^{N(i)} K_{ij}$ $\Rightarrow$ cpt. f_i 在 cont. on each K_{ij}

$\Rightarrow$ f_i cont on $\tilde{K}_i$

$K := \bigcap_{i=1}^{\infty} \tilde{K}_i$ cpt. $K \subset A$.

check:

$\nu(A-K) \le \sum_{i=1}^{\infty} \nu(A-\tilde{K}_i) < 2\varepsilon$. □

Def: μ-integrable $\int f^+$, $\int f^-$ 有一个有限.

μ-summable $\int |f| d\mu < \infty$

§1.5 Covering Theorems

Notations: $B(x,r)$. closed balls

$\hat{B}(x,r) = B(x,5r)$.

Def: $\mathcal{F}$ is a fine cover (细), if, additionally $\inf\{\operatorname{diam} B \mid x \in B,\ B \in \mathcal{F}\} = 0$ $\forall x \in A$.

Thm (Vitali Covering Thm) $\mathcal{F}$ is a collection of closed balls

$\sup\{\operatorname{diam} B \mid B \in \mathcal{F}\} < +\infty$. then $\exists$ a countable

sub-covering $\mathcal{G} \subseteq \mathcal{F}$. s.t. ① $\bigcup_{B\in\mathcal{F}} B \subseteq \bigcup_{B\in\mathcal{G}} \hat{B}$

② balls in $\mathcal{G}$ are disjoint

Proof: $D = \sup\{\operatorname{diam} B \mid B \in \mathcal{F}\}$

$\mathcal{F}_j = \{B \in \mathcal{F} \mid \frac{D}{2^j} < \operatorname{diam} B \le \frac{D}{2^{j-1}}\}$ $\quad j \ge 1$

Define $\mathcal{G}_j \subseteq \mathcal{F}_j$ as follows

$\mathcal{G}_1$ be any maximal disjoint collection of balls in ~~A~~, $\mathcal{F}$.

Assume $\mathcal{G}_1 \dots \mathcal{G}_{k-1}$ have been selected. we choose $\mathcal{G}_k$ to be any maximal disjoint subcollection of $\{B \in \mathcal{F}_k \mid B \cap B' = \emptyset \ \forall B' \in \bigcup_{j=1}^{k-1} \mathcal{G}_j\}$

Finally $\mathcal{G} := \bigcup_{j=1}^{\infty} \mathcal{G}_j \subseteq \mathcal{F}$. disjoint collection.

check: $\forall B \in \mathcal{F}$. $\exists B' \in \mathcal{G}$ s.t. $B \cap B' \neq \emptyset$
$B \subseteq \hat{B}'$.

$\exists k \in \mathbb{N}$. $B \in \mathcal{F}_k$. if $B \in \mathcal{G}_k$ then done ✓.

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if $B \notin \mathcal{G}_k \Rightarrow \begin{cases} \exists B' \in \bigcup_{j=1}^{k-1} \mathcal{G}_j \quad B \cap B' \neq \emptyset \\ \text{or} \\ \exists B' \in \mathcal{G}_k \text{ s.t. } B \cap B' \neq \emptyset \end{cases}$

$\Rightarrow \exists B' \in \bigcup_{j=1}^{k} \mathcal{G}_j$ s.t. $B \cap B' \neq \emptyset$.

Claim: $B \subseteq \hat{B}'$.

$B \in \mathcal{F}_k \Rightarrow \operatorname{diam} B \le \frac{D}{2^{k-1}}$

$B' \in \bigcup_{j=1}^{k} \mathcal{G}_j \subseteq \bigcup_{j=1}^{k} \mathcal{F}_j \Rightarrow \operatorname{diam} B' > \frac{D}{2^k} \ge \frac{1}{2} \operatorname{diam} B$

$\Rightarrow B \subseteq \hat{B}'$.

#

Thm 1.25 (Variant of Vitali Covering Thm)

$\mathcal{F}$ is a fine cover of A by closed balls.

$\sup\{\operatorname{diam} B \mid B \in \mathcal{F}\} < \infty$

Then $\exists$ a countable $\mathcal{G}$ of disjoint balls in $\mathcal{F}$. such that for each finite subset $\{B_1, \dots, B_m\} \subseteq \mathcal{F}$. we have

$$A - \bigcup_{k=1}^{m} B_k \subseteq \bigcup_{B \in \mathcal{G} - \{B_1, \dots, B_m\}} \hat{B}$$

<u>Proof</u>: $x \in A - \underbrace{\bigcup_{k=1}^{m} B_k}_{\text{closed}}$. $\quad \delta := \operatorname{dist}(x, \bigcup_{k=1}^{m} B_k) > 0$.

By "fine", $\exists$ $B \in \mathcal{F}$, $x \in B$, $B \cap B_k = \phi$. $\forall 1 \le k \le m$.

$\Rightarrow$. $B' \in \mathcal{G}$ as above. with $B \cap B' \neq \phi$. $B \subseteq \hat{B'}$ #.

Thm 1.25. $\mathcal{F}$:

$\forall U \subset \mathbb{R}^n$ open $\exists$ a countable collection $\mathcal{G}$ of disjoint closed balls in U s.t. $\operatorname{diam} B < \delta$ $\forall B \in \mathcal{G}$.

& $\mathcal{L}^n(U - \bigcup_{B \in \mathcal{G}} B) = 0$.

#.

Counterexample:

An open set $\subset \mathbb{R}^n$ may not be written as the ~~union~~ union of countably many disjoint balls.

$(0,1) = \bigsqcup_{j=1}^{\infty} (a_j, b_j)$? Impossible.

Thm 1-26. (Filling open sets with balls).

$U \subseteq \mathbb{R}^n$ open. $\forall \delta > 0$. $\exists$ countable collection $\mathcal{G}$ of disjoint closed balls in U s.t. $\operatorname{diam} B < \delta$ $\forall B \in \mathcal{G}$

$$\left\{ \mathcal{L}^n \Big(U - \bigcup_{B \in \mathcal{G}} B\Big) = 0. \right.$$

Proof: Fix $1 - \frac{1}{5^n} < \theta < 1$. $\mathcal{L}^n(U) < \infty$

* Claim: $\exists$ finite collection $\{B_i\}_1^{M_1}$ disjoint. closed balls $\subseteq U$.

s.t. $\operatorname{diam} B_i < \delta$. $1 \le i \le M_1$

$$\left\{ \mathcal{L}^n \Big(U - \bigcup_{i=1}^{M_1} B_i\Big) \le \theta \mathcal{L}^n(U) \right. \cdots (*)$$

Let $\mathcal{F}_1 = \{ B \in U \mid \operatorname{diam} B < \delta \}$

By Vitali's Covering Thm. $\exists$ a countable disjoint collection $\mathcal{G}_1 \subseteq \mathcal{F}_1$ s.t. $U \subseteq \bigcup_{B \in \mathcal{G}_1} \hat{B}$.

$$\Rightarrow \mathcal{L}^n(U) \le \sum_{B \in \mathcal{G}_1} \mathcal{L}^n(\hat{B}) \underset{\substack{\uparrow \\ B \text{ disjoint}}}{=} 5^n \mathcal{L}^n\Big(\bigcup_{B \in \mathcal{G}_1} B\Big)$$

$$\Rightarrow \mathcal{L}^n\Big(\bigcup_{B \in \mathcal{G}_1} B\Big) \ge \frac{1}{5^n} \mathcal{L}^n(U) \Rightarrow \mathcal{L}^n\Big(U - \bigcup_{B \in \mathcal{G}_1} B\Big) \le (1 - 5^{-n}) \mathcal{L}^n(U).$$

Since $\mathcal{G}_1$ is countable and $1 - \frac{1}{5^n} < \theta < 1$. then $\exists M_1 \in \mathbb{Z}_+$.

$B_1, \dots, B_{M_1} \in \mathcal{G}_1$ satisfying $(*)$

Next, set $U_2 = U - \bigcup_{i=1}^{M_1} B_1$. $\mathcal{F}_2 = \{ B \mid B \subseteq U_2, \operatorname{diam} B < \delta \}$

$\Rightarrow \exists B_{M_1+1}, \dots, B_{M_2} \in \mathcal{F}_2$ s.t. $\mathcal{L}^n\Big(U - \bigcup_{i=1}^{M_2} B_i\Big) = \mathcal{L}^n\Big(U_2 - \bigcup_{i=M_1+1}^{M_2} B_i\Big) \le \theta \mathcal{L}^n(U_2) \le \theta^2 \mathcal{L}^n(U)$.

$\Rightarrow \cdots$ $\forall k \in \mathbb{Z}_+$. $\exists M_k$ balls (disjoint). $\mathcal{L}^n\Big(U - \bigcup_{i=1}^{M_k} B_i\Big) \le \theta^k \mathcal{L}^n(U)$ $k \to \infty$. done

If $\mathcal{L}^n(U) = \infty$. Set $U_m = \mathcal{L}^n(U) \cap (B(0, m+1) \setminus B(0, m))$

$\square$

Rmk: The proof above depends on a fact that $\mathcal{L}^n(\hat{B}) = 5^n \mathcal{L}^n(B)$.
but for a Radon measure, the similar fact does not hold.

\#

1.5.2. Besicovitch's Covering Thm.

优点：对 Radon 测度. 且仍然可数覆盖

缺点：覆盖多次. 有重叠.

*：但覆盖次数有个控制.

Thm 1.27 (Besicovitch).

$\exists$ a universal constant N_n with the following property:

- If $\mathcal{F}$ is any collection of non-degenerate closed balls in $\mathbb{R}^n$ with $\sup\{\operatorname{diam} B \mid B \in \mathcal{F}\} < \infty$
- 若 A is the set of centers of balls in $\mathcal{F}$, then there exist N_n countable collections $\mathcal{G}_1, \dots, \mathcal{G}_{N_n}$ of disjoint balls in $\mathcal{F}$, such that $A \subseteq \bigcup_{i=1}^{N_n} \bigcup_{B \in \mathcal{G}_i} B$.

Two steps: Find countable balls → 尽大的挑 较易

How to categorise them into N_n groups 较难

Proof: Step 1: First suppose A bdd.

Take $B_1 = B(a_1, r_1)$. s.t. $r_1 \geq \frac{3}{4} \cdot \frac{D}{2}$. → 第一个球要尽量大.

~~归纳地~~, Inductively, suppose $B_2, \dots, B_{j-1}$ are chosen. 半径上不同阶

· Set $A_j = A - \bigcup_{i=1}^{j-1} B_i$

若 $A_j=\phi$. then stop. and set $J=j-1$. (仅是球的选取结束)

若 $A_j\neq\phi$. take $B_j = B(a_j, r_j)\in\mathcal{F}_1$

s.t. $\begin{cases} a_j\in A_j. & \to \text{中心还未被盖住}. \\ r_j \geq \frac{3}{4}\sup\{r \mid B(a,r)\in\mathcal{F}_1,\ a\in A_j\}. \end{cases}$

$\downarrow$

挑余下的半较大的

若 $A_j\neq\phi$ $\forall j$. set $J=+\infty$

Step 2: Basic Props.

(1). $\forall j>i$. $r_j \leq \frac{4}{3}r_i$

(2) $\{B(a_j, \frac{r_j}{3})\}_{j=1}^{J}$ disjoint.

check: $\forall j>i$. $a_j\notin B_i \Rightarrow |a_i-a_j|>r_i = \frac{r_i}{3}+\frac{2r_i}{3}$

$$\geq \frac{r_i}{3}+\frac{2}{3}\cdot\frac{3}{4}r_j > \frac{r_i+r_j}{3}$$ ※.

(3). $r_j\to\infty$ as $j\to\infty$ 若 ~~$J=+\infty$~~ $J=+\infty$

(4) $A\subseteq\bigcup_{j=1}^{J}B_j$.

check: $J<\infty \Rightarrow$ trivial.

$J=\infty \Rightarrow$. 若 $a\in A$. then $\exists r>0$. $B(a,r)\in\mathcal{F}_1$.

By (3). $\exists r_j<\frac{3}{4}r$. , which contradicts with the choice of r_j. 若 $a\notin\bigcup_{i=1}^{j}B_i$,

Step 3: How to categorize?

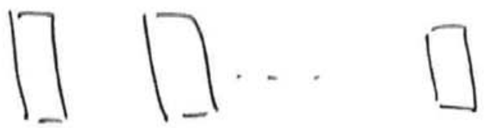
(disjoint in each group? # group is a universal number?)

$\forall k \in \mathbb{Z}_+$, let $I = \{ j < k \mid B_j \cap B_k \neq \phi \}$

need to estimate $|I|$.

<u>Claim</u>: $|I| \overset{(\exists)}{\leq}$ universal number N.

Should claim holds, then our proof would finish.



N个.

归纳地. 若前j前球放进去了.

第j个球: ~~不能放~~ 必有一组, 其中的球与它全不相交.

Step 4: Prove the claim

Set $K_1 = I \cap \{ j \mid r_j \leq 100 r_k \}$.

Estimate $|K_1|$: $j \in K_1$, $B_j(a_j, \frac{r_j}{3}) \supseteq \underbrace{B_j(a_j, \frac{1}{4} r_k)}_{\text{all of which stay inside } B(a_k, 201 r_k)}$ disjoint.

$\Rightarrow \# K_1 \leq \frac{(201)^n}{(\frac{1}{4})^n}$.

$K_2 := I \cap \{ j \mid 100 r_k < r_j \leq 10000 r_k \}$

$\# K_2$? $\leq \frac{(20001 r_k)^n}{(\frac{100}{3} r_k)^n}$.

$K_3 := I \cap \{ j \mid 10000 r_k < r_j \leq 100^3 r_k \}$.

$\# K_3 \leq (\cdots\cdots)^n$.

$\Rightarrow \forall i \in \mathbb{Z}_+$, $\# K_i \leq$ Universal Number. Independent of i.

5. # {非退化 K_i} < universal number.

if K_i 非退化, K_j 非退化 $i, j \ge 2$. $B(\tilde a_i, \tilde r_i) \in K_i$

$B(\tilde a_j, \tilde r_j) \in K_j$.

$\angle \tilde a_i a_k \tilde a_j >$ 非退化的. $= \arccos\frac{1}{2}$.

每个点代表一个非退化组 → 任2个点夹角不能太小. (否则离得太近)

$$\cos\angle \tilde a_i a_k \tilde a_j = \frac{|\tilde a_i a_k|^2 + |\tilde a_j a_k|^2 - |\tilde a_i \tilde a_j|^2}{2|\tilde a_i a_k||\tilde a_j a_k|}.$$

① $r_k < \frac{1}{100} r_i$ ~~$< \frac{1}{100}$~~

② $\tilde r_i < \frac{1}{100} r_j$ …(✱) (有误).

③ ⓒ. $\tilde B_i \cap B_k \ne \phi \Rightarrow |\tilde a_i a_k| \le \tilde r_i + r_k$.

ⓓ 同理. $|\tilde a_j a_k| \le \tilde r_j + r_k$

$\Rightarrow$. $B(\tilde a_j, \tilde r_j)$ 大于. $B(\tilde a_i, \tilde r_i)$. 大于. $B(a_k, r_k)$.

$\tilde a_i \notin B(\tilde a_j, \tilde r_j)$

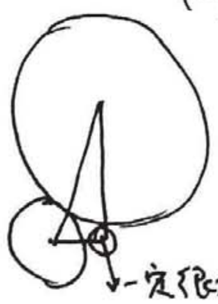
$|\tilde a_i \tilde a_j| > \tilde r_j$.

$|\tilde a_i a_k| > \tilde r_i$.

$|\tilde a_j a_k| > \tilde r_j$.

$$\le \frac{(\tilde r_i + r_k)^2 + (\tilde r_j + r_k)^2 - \tilde r_j^2}{2\tilde r_i \tilde r_j} < \frac{1}{2}.$$

Eg:



一定很大.

□

Besicovitch 覆盖定理:

存在一个万有常数 N_n 满足以下性质:

· 若 $\mathcal{F}$ 是 $\mathbb{R}^n$ 中任一族非退化闭球, A 是 $\mathcal{F}$ 中这些球的球心. 则存在 N_n ~~族~~个

由 $\mathcal{F}$ 中可数个彼此不交的球构成的球族 $\mathcal{G}_1,\dots,\mathcal{G}_{N_n}$. s.t. $A\subseteq \bigcup_{i=1}^{N_n}\bigcup_{B\in\mathcal{G}_i} B$.

* 2 steps: $\begin{cases}\text{找出可数多个球,}\\ \text{将其分为 } N_n \text{ 组}\end{cases}$ 不妨先设 A 是有界集合

Step 1: 找球:

设 $D=\sup\{\operatorname{diam} B \mid B\in\mathcal{F}\}<\infty$

取 $B_1=B(a_1,r_1)$. s.t. $r_1\geq\frac{3}{4}\cdot\frac{D}{2}$

归纳地. 设 $B_2,\dots,B_{j-1}$ 已经取好了, 令 $A_j=A-\bigcup_{i=1}^{j-1}B_i$ (未被覆盖部分).

i) 若 $A_j=\phi$. 则选球结束. 令 $J=j-1$.

ii) 若 $A_j\neq\phi$. 则取 $B_j=B(a_j,r_j)\in\mathcal{F}$. 使得 $\begin{cases} a_j\in A_j & \to B_j \text{的中心还未被前 } j-1 \text{ 个球盖住.}\\ r_j\geq\frac{3}{4}\sup\{r\mid B(a,r)\in\mathcal{F},\ a\in A_j\} & \downarrow \text{挑大球.}\end{cases}$

若 $\forall j\in\mathbb{Z}_+$. $A_j\neq\phi$. 则令 $J=+\infty$.

Step 2: 某些基本性质:

(1) $\forall j>i$. $r_j\leq\frac{4}{3}r_i$.

(2) $\{B(a_j,\frac{r_j}{3})\}_{j=1}^{J}$ 彼此不交.

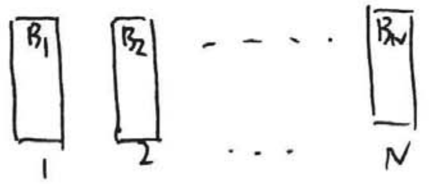
(3) $r_j\to0$ as $j\to\infty$

(4) $A\subseteq\bigcup_{j=1}^{J}B_j$.

Step 3: 分组: $\forall k\in\mathbb{Z}_+$. 令 $I=\{j<k\mid B_j\cap B_k\neq\phi\}$ 即 $B_1\sim B_{k-1}$ 中与 B_k 有交的.

claim: $|I|\leq$ a universal number N.

若 claim 成立. 则我们考虑如下分组: 先将 $B_1\sim B_N$ 分在不同的 N 组.



对 B_{N+1}. 由于 $I=\{j<N+1\mid B_j\cap B_{N+1}\}<N$.

故 $B_1\sim B_N$ 中与 B_{N+1} 有交的少于 N 个.

故必存在 $l\in\{1,2,\dots,N\}$ s.t. $B_l\cap B_{N+1}=\phi$

将 B_{N+1} 放进第 l 组. (l 不唯一).

$\forall k\geq N$. 对 B_k 的分组与上面是一样的.

设这样得到 N 组可数球 $\mathcal{G}_1\dots\mathcal{G}_N$. $\mathcal{G}_i$ 中各球不交(根据构造). $A\subseteq\bigcup_{i=1}^{N}\bigcup_{B\in\mathcal{G}_i}B$.

Step 4：证明 claim：

Idea，将那些与 B_k 有交的球分成小的和大的来考虑。

令 $K = I \cap \{j \mid r_j \le 100\, r_k\}$. 则 $\#I = \#K + \#(I \setminus K)$

$\forall j \in K$. 有 $B_j(a_j, \frac{r_j}{3}) \supseteq B(a_j, \frac{r_k}{4})$ disjoint 且全体 $B(a_j, \frac{r_k}{4}) \subseteq B(a_k, 201 r_k)$.

by step 2 (1).

$\therefore \#K \le \dfrac{|B(a_k, 201 r_k)|}{|B(a_j, \frac{r_k}{4})|} = \dfrac{(201)^n}{(\frac{1}{4})^n}$ is a universal number.

再估计 $I-K$.

$\forall i, j \in I-K$. $i \ne j$. 则 $1 \le i, j < k$. $B_i \cap B_k \ne \phi$. $B_j \cap B_k \ne \phi$.

$B_i = B(a_i, r_i)$, $B_k = B(a_k, r_k)$, $B_j = B(a_j, r_j)$

$r_i, r_j > 100 r_k$.

下面我们证明：$\angle a_j a_i a_k \ge 60^\circ$ ……(*).

如果(*)对的话，那么，先注意到以下事实。 例如 59°

固定某个 $r_0 > 0$. s.t. $\forall x \in \partial B(0,1)$, $y, z \in B(x, r_0)$. $\angle y0z \le$ 某个固定的 θ_0

画图看得出

则 $\exists$ universal number L_n 使得 $\partial B(0,1)$ 可以被 L_n 个球心在 $\partial B(0,1)$ 上，半径为 r_0 的球盖到

但，不可以被 $L_n - 1$ 个这样的球盖住。

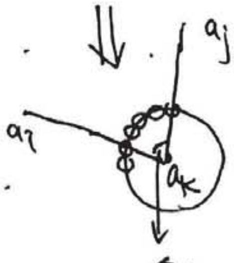
从而 对 $B_k = B(a_k, r_k)$ 它可以被 L_n 个半径为 $r_0 r_k$ 的 球心在 ∂B_k 上，半径为某固定 $r_0 > 0$ 的球盖住（记作 $\tilde{B}_1, \dots, \tilde{B}_{L_n}$）.

~~若 claim~~ 若(*)对了，则 $\forall i \ne j$，向量 $a_i - a_k$. $a_j - a_k$ 不可能穿过同一个 $\tilde{B}_i$.

所以 $\#(I-K) \le L_n$ is a universal number

$\Rightarrow$ 这样 $\#I \le 804^n + L_n =: N_n$ 为所求的 universal number.

余下只需证(*)。这用余弦定理即可、和上课讲的一样。



角太大了，$\tilde{B}_i$ 太小了。

#

eg: $f \uparrow$ on $\mathbb{R}$

derivative: $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \Rightarrow$ Lebesgue-Stieltjes measure

※

Lemma 1.2: Fix $0<\alpha<\infty$. Then.

(i) $A \subseteq \{x \in \mathbb{R}^n \mid D_\mu \nu(x) \le \alpha\} \Rightarrow \nu(A) \le \alpha \mu(A)$.

(ii) $A \subseteq \{x \in \mathbb{R}^n \mid D_\mu \nu(x) \ge \alpha\} \Rightarrow \nu(A) \ge \alpha \mu(A)$.

※

Rmk: A need not be μ-. ν- measurable

<u>Proof</u>: $\cancel{\forall \varepsilon>0}$. WloG. $\mu(\mathbb{R}^n)$. $\nu(\mathbb{R}^n) < \infty$

$\forall \varepsilon>0$. U open. $A \subseteq U$. satisfying (i).

set $\mathcal{F} = \{B \mid B = B(a,r)\ a\in A.\ B\subseteq U.\ \nu(B) \le (\alpha+\varepsilon)\mu(B)\}$

~~Then~~ $\cancel{\nu(A) \le \sum_{B\in\mathcal{G}} \nu(B) \le (\alpha+\varepsilon)\sum_{B\in\mathcal{G}} \mu(B) \le (\alpha+}$

then: $\inf\{r \mid B(a,r) \in \mathcal{F}\} = 0 \quad \forall a \in A$.

so Thm 1.28 implies that. $\exists$ a countable collection $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that $\nu(A - \bigcup_{B\in\mathcal{G}} B) = 0$.

$\Rightarrow \nu(A) \le \sum_{B\in\mathcal{G}} \nu(B) \le (\alpha+\varepsilon)\sum_{B\in\mathcal{G}} \mu(B) \le (\alpha+\varepsilon)\mu(U)$.

$\forall U \supseteq A$.

$\Rightarrow \forall \varepsilon>0$ $\cancel{\mu}$ $\nu(A) \le (\alpha+\varepsilon)\mu(A)$.

※

Exe: 单调函数 a.e. 可微.

证明 用 日升引理.

或用 Radon 测度导数. 可以考虑证明.

check why it is the corollary.

Recall: Besikovitch Covering Thm:

$\exists N_n \in \mathbb{Z}_+$. $\mathcal{F}$: any collection of closed balls $\subseteq \mathbb{R}^n$.

$D := \sup\{\operatorname{diam} B \mid B \in \mathcal{F}\}$. A is the set of the centers of the balls $\in \mathcal{F}$.

then $\exists \mathcal{G}_1 \dots \mathcal{G}_{N_n}$ countable collections of disjoint balls in $\mathcal{F}$

s.t. $A \subseteq \bigcup_{k=1}^{N_n} \bigcup_{B \in \mathcal{G}_k} B$.

□

Thm 1.28 (More on filling open sets with balls).

μ Borel measure on $\mathbb{R}^n$. $\mathcal{F}$ is the Vitali covering of A.

$\mu(A) < \infty$.

i.e. $\forall x \in A$. $\forall \delta > 0$, $\exists B(x, r) \in \mathcal{F}$ s.t. $r < \delta$

then $\forall U \subseteq \mathbb{R}^n$ open, $\exists$ a countable collection $\mathcal{G}$ of disjoint balls in $\mathcal{F}$

Such that $\bigcup_{B \in \mathcal{G}} B \subseteq U$ and $\mu\left((A \cap U) - \bigcup_{B \in \mathcal{G}} B\right) = 0$.

or say: $\exists$ a countable collection $\mathcal{G}$ of disjoint. balls $\in \mathcal{F}$

s.t. $\mu\left(A \setminus \bigcup_{B \in \mathcal{G}} B\right) = 0$.

Proof: ① Fix $1 - \frac{1}{N_n} < \theta < 1$, then $\exists$ a finite collection $\{B_1 \dots B_{M_1}\}$ of disjoint closed balls in U s.t.

$\mu\left((A \cap U) - \bigcup_{i=1}^{M_1} B_i\right) \leq \theta \mu(A \cap U)$.

check: $\mathcal{F}_1 := \{B \in \mathcal{F} \mid \operatorname{diam} B \leq 1 \quad B \subset U\}$ By Besikovitch covering thm, $\exists \mathcal{G}_1 \dots \mathcal{G}_{N_n}$ of disjoint balls in $\mathcal{F}_1$ s.t. $A \cap U \subseteq \bigcup_{i=1}^{N_n} \bigcup_{B \in \mathcal{G}_i} B$

$\Rightarrow$

$$\mu(A\cap U) \le \sum_{i=1}^{N_n} \mu\Big(A\cap U\cap \bigcup_{B\in\mathcal{G}_i} B\Big).$$

$\Rightarrow \exists\, j \in \{1,2,\dots,N_n\}$

$$\mu\Big(A\cap U\cap \bigcup_{B\in\mathcal{G}_j} B\Big) \ge \frac{1}{N_n}\mu(A\cap U).$$

~~B~~ $\Rightarrow \exists\, B_1 \dots B_{M_1} \in \mathcal{G}_j$

$$\mu\Big(A\cap U\cap \bigcup_{i=1}^{M_1} B_i\Big) \ge (1-\theta)\underbrace{\mu(A\cap U)}_{\shortparallel}.$$

$$\mu\Big(A\cap U\cap \bigcup_{i=1}^{M_1} B_i\Big) + \mu\Big((A\cap U) - \bigcup_{i=1}^{M_1} B_i\Big)$$

$\Rightarrow \checkmark$.

Repeat the processes above. Done.

#.

§1.6: Differentiation of Radon measures:

μ, ν Radon measures on $\mathbb{R}^n$.

Def: (1) $\overline{D}_\mu \nu(x) = \begin{cases} \limsup\limits_{r\to 0} \dfrac{\nu(B(x,r))}{\mu(B(x,r))} & \text{if } \mu(B(x,r))>0,\ \forall r>0 \\ +\infty & \text{if } \mu(B(x,r)) = 0,\ \exists r>0 \end{cases}$

(2) $\underline{D}_\mu \nu(x) = \begin{cases} \liminf\limits_{r\to 0} \dfrac{\nu(B(x,r))}{\mu(B(x,r))} & \text{if } \mu(B(x,r))>0,\ \forall r>0 \\ +\infty & \text{if } \mu(B(x,r)) = 0,\ \exists r>0 \end{cases}$

(3). If $\overline{D}_\mu \nu(x) = \underline{D}_\mu \nu(x) < +\infty$, then we say ν is differentiable w.r.t μ at x, and write $D_\mu \nu(x) := \overline{D}_\mu \nu(x) = \underline{D}_\mu \nu(x)$, or the density of ν w.r.t μ.

#.

Thm 1.29 (Differentiating measures)

μ, ν : Radon on $\mathbb{R}^n$.

Then, (1) $D_\mu \nu$ $\exists$ finite μ-a.e.

(2) $D_\mu \nu$ μ-measurable.

Proof. Assume $\nu(\mathbb{R}^n) < \infty$. $\mu(\mathbb{R}^n) < \infty$.

① $I := \{x \mid \overline{D}_\mu \nu(x) = +\infty\} \subseteq \{x \mid \overline{D}_\mu \nu(x) \geq \alpha\}$ $\forall \alpha$

By Lem 1.2. $\mu(I) \leq \frac{1}{\alpha}\nu(I)$. $\xRightarrow{\alpha \to \infty}$ $\mu(I) = 0$.

$\therefore$ $\overline{D_\mu \nu}$. $\underline{D_\mu} \nu$ finite a.e.

② $\forall\, 0 < a < b$.

$$\{x \mid \underline{D}_\mu \nu(x) < \overline{D}_\mu \nu(x) < \infty\} = \bigcup_{\substack{0<a<b \\ a,b \in \mathbb{Q}^+}} \underbrace{\{x \mid \underline{D}_\mu \nu(x) < a < b < \overline{D}_\mu \nu(x) < \infty\}}_{R(a,b)}$$

but by lem 1.2.

$b\,\mu(a,b) \leq \nu(R(a,b)) \leq a\,\mu(R(a,b))$.

$\Rightarrow$ $\mu(R(a,b)) = 0$. $\nu(R(a,b)) = 0$

$\Rightarrow$ $D_\mu \nu$ $\exists$ finite. μ-a.e.

(1) holds.

(2):

* Federer 的可测性论 . Besikovitch. 能否改成开球?

claim:

$\forall x \in \mathbb{R}^n$, $r > 0$. $\limsup\limits_{y \to x} \mu(B(y,r)) \leq \mu(B(x,r))$.

choose $\mathbb{R}^n \ni y_k \to x$

check: $\limsup\limits_{k \to \infty} \chi_{B(y_k,r)} \leq \chi_{\overline{B}(x,r)}$

Fatou's lemma:

$\Longrightarrow$ $\displaystyle\int_{B(x,2r)} \chi_{B(x,r)^c}\, d\mu \leq \liminf_{k \to \infty} \int_{B(x,2r)} \chi_{B(y_k,r)^c}\, d\mu$.

$\Rightarrow$ $\mu(B(x,2r)) - \mu(B(x,r)) \leq \liminf\limits_{k \to \infty} \mu(B(x,2r)) - \mu(B(y_k,r))$

$\Rightarrow$. $\limsup_{y\to x} \mu(B(y,r)) \le \mu(B(x,r)) \le \liminf_{y\to x} \mu(B(y,r))$

③. $D_\mu \nu$ μ-可测？

$$f_r(x) := \begin{cases} \dfrac{\nu(B(x,r))}{\mu(B(x,r))} & \mu(B(x,r))>0 \\ +\infty \end{cases}$$

μ-可测. $\rightarrow$ 因为 $x \mapsto \nu(B(x,r))$ 可测. $=0$.

$x \mapsto \mu(B(x,r))$ 可测.

$D_\mu \nu = \lim_{k\to\infty} f_{\frac{1}{k}}$

$\uparrow$ 分 $\mu(B(x,r))=0$ or $\neq 0$ 简单讨论.

1.6.2. Integration of derivatives & Lebesgue decomposition.

Def: Assume μ, ν Borel

(1). $\nu \ll \mu$, iff $\mu(A)=0 \Rightarrow \nu(A)=0$. $\forall A \subseteq \mathbb{R}^n$.

(2). $\nu \perp \mu$ iff $\exists$ Borel $B \subseteq \mathbb{R}^n$. $\mu(\mathbb{R}^n \setminus B)=\nu(B)=0$

Thm (Radon-Nikodym). ν, μ Radon. on $\mathbb{R}^n$. $\nu \ll \mu$.

then $\nu(A) = \int_A D_\mu \nu \, d\mu$. $\forall A \subseteq \mathbb{R}^n$ & μ可测

Pf: A μ-可测 $\Rightarrow \exists$ Borel $B \supseteq A$ $\mu(B\setminus A)=0 \Rightarrow \nu(B\setminus A)=0$

$\Rightarrow A$ ν-可测.

$\Rightarrow$任一μ-可测 都是 ν-可测.

Set $Z=\{x\in\mathbb{R}^n \mid D_\mu\nu(x)=0\}$ $I:=\{x\in\mathbb{R}^n \mid D_\mu\nu=+\infty\}$. Z, I μ可测.

By 1.29. $\mu(I)=0 \Rightarrow \nu(I)=0$.

By lem 1.2. $\forall \alpha>0$, $\nu(Z) \le \alpha\mu(Z) \Rightarrow \nu(Z)=0$

$\Rightarrow \nu(Z)=0=\int_Z D_\mu\nu \, d\mu$.

Let A μ-measurable. Fix $1<t<\infty$. Define $\forall m\in\mathbb{Z}$. $A_m = A \cap \{x \mid D_\mu\nu \in [t^m, t^{m+1})\}$.

μ-可测 $\Rightarrow \nu$-可测.

$A - \bigcup_{m\in\mathbb{Z}} A_m \subseteq Z \cup I \cup \{x \mid \overline{D}_\mu\nu \neq \underline{D}_\mu\nu\}$

$\Rightarrow \nu(A - \bigcup A_m) = 0$

$\therefore \nu(A) = \sum \nu(A_m) \le \sum_m t^{m+1}\mu(A_m)$

$$= t \sum_m t^m \mu(A_m) \le t \int_A D_\mu \nu \, d\mu$$

By lem 1.? $\nu(A) = \sum_m \nu(A_m) \ge \sum t^m \mu(A_m)$

$$\ge t^{-1}\sum_m \int_{A_m} D_\mu \nu \, d\mu = t^{-1} \int_{A} D_\mu \nu \, d\mu$$

$t \to 1^+$ done

$\square$

Thm (Lebesgue Decomposition)

μ, ν Radon

(1) $\nu = \nu_{ac} + \nu_s$, $\nu_{ac} \ll \mu$ Radon, $\nu_s \perp \mu$ Radon

(2) $D_\mu \nu = D_\mu \nu_{ac}$, $D_\mu \nu_s = 0$ μ-a.e

$\nu(A) = \int_A D_\mu \nu \, d\mu + \nu_s(A)$ $\forall$ Borel $A \subseteq \mathbb{R}^n$

Proof: WLOG $\mu(\mathbb{R}^n) < \infty$, $\nu(\mathbb{R}^n) < \infty$

$\mathcal{E} := \{ A \subseteq \mathbb{R}^n \mid A \text{ Borel}, \ \mu(\mathbb{R}^n \setminus A) = 0 \}$

Proof: choose $B_k \in \mathcal{E}$

$\nu(B_k) \le \inf_{A \in \mathcal{E}} \nu(A) + \frac{1}{k}$.

Write $B := \bigcap_{k=1}^{\infty} B_k$. Since $\mu(\mathbb{R}^n \setminus B) \le \sum_{k=1}^{\infty} \mu(\mathbb{R}^n \setminus B_k) = 0$. $\Rightarrow B \in \mathcal{E}$.

$\therefore \nu(B) = \inf_{A \in \mathcal{E}} \nu(A)$ (*)

Define $\nu_{ac} = \nu \llcorner B$, $\nu_s = \nu \llcorner (\mathbb{R}^n \setminus B)$.

By 1.? ν_{ac}, ν_s Radon.

Now suppose $A \subseteq B$ A Borel $\mu(A) = 0$, $\nu(A) > 0$

$\Rightarrow B - A \in \mathcal{E}$.

$\nu(B - A) < \nu(B)$ 与 with (*).

$\Rightarrow \nu_{ac} \ll \mu$. 另一方面. $\mu(\mathbb{R}^n \setminus B)$

$\Rightarrow \nu_s \perp \mu$.

断言: Fix $\alpha > 0$. set $C := \{ x \in B \mid D_\mu \nu_s(x) \ge \alpha \}$

By lem 1.? $\alpha \mu(C) \le \nu_s(C) = 0$

$\Rightarrow D_\mu \nu_s = 0$ μ a.e. $D_\mu \nu_{ac} = D_\mu \nu$ a.e

§1.7 Lebesgue pts & ap continuity

Thm 1.32 (Lebesgue–Besikovitch Differentiation Thm).

μ Radon. $f \in L^1_{loc}(\mathbb{R}^n, \mu)$. $\Rightarrow \fint_{B(x,r)} f\,d\mu \to f(x)$ as $r \to 0$. μ-a.e. $x \in \mathbb{R}^n$

Proof: ① $\forall$ Borel B. $\nu^{\pm}(B) = \int_B f^{\pm}\,d\mu$

② $\forall A \subseteq \mathbb{R}^n$. $\nu^{\pm}(A) := \inf\{\nu^{\pm}(B) \mid A \subseteq B,\ B \text{ Borel}\}$.

| check: $\nu^{\pm}$ Radon.

$\Rightarrow$ ⊛ $D_\mu \nu^{\pm} = f^{\pm}$ μ-a.e. $\Rightarrow$. $\nu^+(A) = \int_A D_\mu \nu^+\,d\mu$. $\forall \mu$-可测集 $A \subseteq \mathbb{R}^n$.

Claim: ① holds for all μ-measurable sets A.

ⓞ: $\nu^{\pm} \ll \mu$ by ① ~~$\nu^+(A) = \int_A f^{\pm}\,d\mu$~~.

② $\exists$ Borel $\tilde{B}$. $\mu(\tilde{B} \setminus B) = 0$.

$$\begin{array}{ccc} \nu^{\pm}(\tilde{B}) & = & \int_{\tilde{B}} f^{\pm}\,d\mu \\ \vee\!| & & \| \\ \nu^{\pm}(B) & \overset{?}{\underset{\checkmark}{=}} & \int_B f^{\pm}\,d\mu \end{array} \Rightarrow D_\mu \nu^{\pm} = f^{\pm} \quad \mu\text{-a.e.}$$

$$\lim_{r\to 0} \fint_{B(x,r)} f\,d\mu = \lim_{r \to 0} \frac{1}{\mu(B(x,r))}\left(\nu^+(B(x,r)) - \nu^-(B(x,r))\right)$$

$$= D_\mu \nu^+ - D_\mu \nu^-$$

$$= f^+ - f^- = f \quad . \ \forall_\mu\text{a.e. } x$$

□

<u>Recall</u>: μ Radon. $f \in L^1_{loc}$

then $\displaystyle ⨍_{B(x,r)} f \, d\mu \xrightarrow[r \to 0^+]{} f(x)$ μ-a.e.

Thm 1.33

$f \in L^p_{loc}$ $1 \le p < \infty$ $\Rightarrow$ $\displaystyle ⨍_{B(x,r)} |f - f(x)|^p \, d\mu \to 0$.

<u>Pf</u>: $\mathbb{Q} = \{r_i\}_1^\infty$

$\forall i$. By Lebesgue–Besikovitch Thm.

$$⨍_{B(x,r)} |f - r_i|^p \, d\mu \to |f(x) - r_i|^p.$$

$\Rightarrow \exists A$ $\mu(A) = 0$. $\forall x \in A^c$ $\forall i$. $\displaystyle ⨍_{B(x,r)} |f - r_i|^p \to |f(x) - r_i|^p$ a.e.

$\forall \varepsilon > 0$. choose r_i s.t. $|f(x) - r_i|^p < \frac{\varepsilon}{2^p}$

$$\therefore \limsup_{r \to 0} ⨍_{B(x,r)} |f(x) - f|^p \le 2^{p-1} \Big(\limsup_{r \to 0} ⨍ |f(x) - r_i|^p \, d\mu + ⨍ |f - r_i|^p \, d\mu \Big)$$

$$\le 2^{p-1} \cdot 2 \cdot \frac{\varepsilon}{2^p} = \varepsilon$$

□

<u>Def</u>: $\forall f \in L^1_{loc}(\mathbb{R}^n)$

$$f^*(x) := \begin{cases} \lim_{r \to 0^+} ⨍_{B(x,r)} f \, dy & 若 \exists \\ 0 & 否则 \end{cases}$$

$\downarrow$

precise representation of f

$\uparrow$

为了避免L^p函数在零测集上可以任意改变取值.

<u>Rmk</u>: $$\frac{\mathcal{L}^n(B(x,r) \cap E)}{\mathcal{L}^n(B(x,r))} \to \begin{cases} 1 & \text{a.e. } x \in E \\ 0 & \text{a.e. } x \notin E \end{cases}$$

$\to$ 测度论意义下的内点

$\to$ — — — 外点

如何定义测度论意义下的极限？连续？可微？

连续 $\Updownarrow$ (可测)

1.7.2 Approximate limits & continuity

Def: $f: \mathbb{R}^n \to \mathbb{R}^m$. say $l \in \mathbb{R}^m$ is the approximate limit of f as $y \to x$, i.e. $\operatorname{ap}\lim\limits_{y\to x} = l$. iff.

$$\forall \varepsilon > 0. \quad \frac{\mathcal{L}^n\left(B(x,r) \cap \{\,|f - l| \geq \varepsilon\}\right)}{\mathcal{L}^n(B(x,r))} \to 0 \quad \text{as } r \to 0^+.$$

(If exists. then unique).

f在x附近"测度意义上的"上确界 ... 下...

* ap limsup: $\inf\limits_{t} \frac{\mathcal{L}^n(B(x,r) \cap \{f > t\})}{\mathcal{L}^n(B(x,r))} \to 0$ as $r \to 0^+$

ap liminf $\sup\limits_{t} \frac{\mathcal{L}^n(B(x,r) \cap \{f < t\})}{\mathcal{L}^n(B(x,r))} \to 0$. as $r \to 0^+$

* ap continuous.

$f: \mathbb{R}^n \to \mathbb{R}^m$ ap cont. at $x \in \mathbb{R}^n$. if $\operatorname{ap}\lim\limits_{y \to x} f(y) = f(x)$.

Then:

- thm 1.37. (Measurability & ap. cont.).

$f: \mathbb{R}^n \to \mathbb{R}^m$ $\mathcal{L}^n$-可测 $\Longleftrightarrow$ f $\mathcal{L}^n$-a.e. 逼近连续

证明: $\Rightarrow$: * $\exists$ disjoint cpt sets $\{K_i\} \subseteq \mathbb{R}^n$

$\mathcal{L}^n(\mathbb{R}^n \setminus \bigcup_{i=1}^{\infty} K_i) = 0$ $\quad f|_{K_i}$ cont. ✓.

~~f. B. Lusin ⇒~~ (by induction).

$\forall$ a.e. $x \in K_i$. $\frac{\mathcal{L}^n(B(x,r) \setminus K_i)}{\mathcal{L}^n(B(x,r))} = 0$.

For K_i: $\exists \tilde{K}_i \subset K_i$. $\mathcal{L}^n(K_i \setminus \tilde{K}_i)=0$.

$\tilde{K}_i$: density pts ~~of~~ K_i inside.

$A := \bigcup_{i=1}^{+\infty} \tilde{K}_i$ $\quad \mathcal{L}^n(\mathbb{R}^n \setminus A)=0$.

$\forall x\in A$. $\exists i$. $x \in \tilde{K}_i$

$\forall \varepsilon>0$. $\exists s>0$ s.t. $\forall\, r<s$.

$\forall y \in B(x,r)\cap K_i$

$|f(y)-f(x)|<\varepsilon$.

then. if $0<r<s$, $B(x,r) \cap \{ y \mid |f(y)-f(x)| \geq \varepsilon \}$

$\subseteq B(x,r) - K_i$.

$$\therefore \frac{B(x,r) \cap \{ |f - f(x)| \geq \varepsilon \}}{B(x,r)} \to 0. \quad r \to 0^+ \quad \forall \varepsilon>0$$

$\square$

§ 1.8. Riesz Representation Thm.

Thm 1.39: (非负线性泛函即是 Radon 测度).

$L: C_c(\mathbb{R}^n;\mathbb{R}) \longrightarrow \mathbb{R}$. nonnegative & linear function

i.e. $\forall f \geq 0$ $f \in C_c(\mathbb{R}^n)$ $Lf \geq 0$.

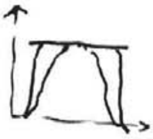
Then $\exists \mu$ Radon s.t. $L(f) = \int_{\mathbb{R}^n} f \, d\mu$.

Pf: say $f \prec \underset{\text{open}}{U}$. if $\operatorname{spt} f \subseteq U$. $0 \leq f(x) \leq 1$

① $\forall$ ~~开~~ U. $\mu(U) := \sup \{ L(f) \mid f \in C_c(\mathbb{R}^n;\mathbb{R}).\ f \prec U \}$

(若 $L(f)=\int f\,d\mu$. How to get $\mu(U)$?

$\mu(U) = \int \chi_U \, d\mu \neq L(\chi_U)$. 但可用 $C_c(\mathbb{R}^n)$ 逼近)



$\forall E \subseteq \mathbb{R}^n$ $\quad \mu(E) := \inf \{ \mu(U) \mid U \overset{\text{open}}{\supseteq} E \}$

Check: μ: ① Measure
② Borel
③ Regular.
④ Radon

Finally ⑤ $L(f) = \int f \, d\mu$.

①: $E \subseteq \bigcup_{i=1}^{\infty} E_i$. check: $\mu(E) \leq \sum_{i=1}^{\infty} \mu(E_i)$.

WLOG $\mu(E_i) < +\infty$ . . ~~otherwise $\exists U_i$ open~~

$\Rightarrow \exists U_i$ open $\supseteq E_i$ $\quad \mu(U_i) \leq \mu(E_i) + \frac{\varepsilon}{2^i}$

$U = \bigcup_{i=1}^{\infty} U_i$. $\quad \mu(U) \overset{?}{\leq} \sum_{i=1}^{\infty} \mu(U_i)$
$\quad \mu(E)$ $\quad \left(\leq \sum_{i=1}^{\infty} \mu(E_i) + \varepsilon \right)$

$\forall f \in C(\mathbb{R}^n, \mathbb{R})$. $f \prec U$.

$Lf \overset{?}{\leq} \sum_{i=1}^{\infty} \mu(U_i)$.

Suppose $K = \operatorname{Spt} f$. $\quad K \subseteq U = \bigcup_{i=1}^{\infty} U_i$.

Since K is compact. then. $\exists N \in \mathbb{Z}_+$. $K \subseteq \bigcup_{i=1}^{N} U_i$.

By P.o.U. $\exists g_i \prec U_i$. s.t. $\sum_{i=1}^{N} g_i = 1$. in K.

$f = \sum_{i=1}^{N} f \cdot g_i \quad \Rightarrow L(f) = \sum_{i=1}^{N} L(fg_i) \leq \sum_{i=1}^{N} \mu(U_i)$.

$g_i \prec U_i$
$\operatorname{Spt} g_i$ 紧.

Take the supremum of Lf over all f.

$\leq \sum_{i=1}^{\infty} \mu(U_i)$

① done ✓

② ~~Borel~~

Take E_1, E_2. $\mathrm{dist}(E_1, E_2) > 0$.

check: $\underset{E_1 \cup E_2}{\mu(E)} = \mu(E_1) + \mu(E_2)$.

$\forall \varepsilon > 0$. $\exists$ open $U \supseteq E$. $\mu(U) \le \mu(E) + \varepsilon$.

Set $U_1 = \{x \in U \mid \mathrm{dist}(x, E_1) < \frac{\mathrm{dist}(E_1, E_2)}{3}\}$

$U_2 = \{x \in U \mid \mathrm{dist}(x, E_2) < \frac{\mathrm{dist}(E_2, E_1)}{3}\}$

$U_1 \cap U_2 = \phi$. $U_1 \cup U_2 \subseteq U$.

$\underset{(\mu(E)+\varepsilon)}{\mu(U)} \overset{?}{\ge} \mu(U_1) + \mu(U_2)$ $(\ge \mu(E_1) + \mu(E_2))$.

Take $f_1, f_2 \in C_c(\mathbb{R}^n)$. $f_i \prec U_i$.

$L(\underset{f}{f_1 + f_2}) = L(f_1) + L(f_2)$. Take supremum on RHS:

$(f \prec U)$

$\mu(U) \ge L(f) = L(f_1) + L(f_2)$

$\Rightarrow \mu(U) \ge L(f) = \mu(U_1) + \mu(U_2)$.

Take sup over all f.

③. Omit.

④: $\forall K \subset\subset \mathbb{R}^n$. why $\mu(K) < \infty$?

$\exists U$ open $\supseteq K$. $\exists \tilde{K}$ cpt. $\tilde{K} \supseteq U$.

Take $\tilde{f} \in C_c(\mathbb{R}^n, \mathbb{R}) = \begin{cases} 1 & \text{on } \tilde{K} \\ \ge 0 & \text{on } \tilde{K}^c \end{cases}$

It suffices to prove $\mu(U) < \infty$

i.e. $\sup\{L(f) \mid f \in C_c(\mathbb{R}^n; \mathbb{R}),\ f \prec U\} < \infty$?

$\forall f < U \quad f \in C_c(\mathbb{R}^n;\mathbb{R})$.

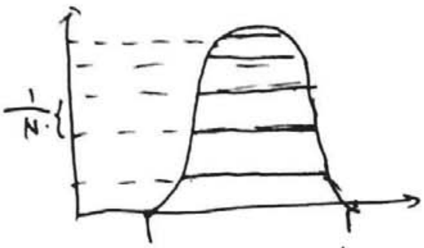
$0 \le f \le \tilde{f}$

$\Downarrow$ L non-negative

$Lf \le L\tilde{f}$

$\therefore \sup_{f \dots} L\tilde{f} < \infty \Rightarrow \mu(U) < \infty$.

⑤ check: $L(f) = \int f \, d\mu$. where $\mu(U) = \sup\left\{ Lf \,\middle|\, \begin{array}{l} f \in C_c(\mathbb{R}^n;\mathbb{R}) \\ f < U \end{array} \right\}$. $\forall f \in C_c(\mathbb{R}^n;\mathbb{R})$.



$\forall N \in \mathbb{Z}_+$. $1 \le j \le N$. $K_j := \{x \mid f(x) > \frac{j}{N}\}$

$K_0 = \operatorname{spt} f$

set $f_j(x) = \begin{cases} 0 & x \notin K_{j-1} \\ f(x) - \frac{j}{N} & x \in K_{j-1} \setminus K_j \\ \frac{1}{N} & x \in K_j \end{cases}$. $\Rightarrow f = \sum_{j=1}^{\infty} f_j$

$\frac{1}{N} \chi_{K_j} \le f_j \le \frac{1}{N} \chi_{K_{j-1}}$

$\therefore \frac{1}{N}\mu(K_j) \le \int f_j \, d\mu \le \frac{1}{N}\mu(K_{j-1})$.

Claim: $\frac{1}{N}\mu(K_j) \le Lf_j \le \frac{1}{N}\mu(K_{j-1})$.

由 Nf_j spt in K_{j-1}. then. $\forall U$ open $\supseteq K_{j-1}$. $\therefore Nf_j < U$.

$\Rightarrow \mu(U) \ge L(Nf_j) \Rightarrow \mu(K_{j-1}) \ge N \cdot L(f_j) \Rightarrow \frac{1}{N}\mu(K_{j-1}) \ge L(f_j)$.

又: $Nf_j \equiv 1$ on $K_j \Rightarrow \forall \varepsilon > 0$. $U = \{x \mid Nf_j(x) > 1-\varepsilon\} \supseteq K_j$.

$\uparrow$ open.

$\forall g < U$. $g \le \frac{Nf_j}{1-\varepsilon}$. $\Rightarrow Lg \le \frac{N}{1-\varepsilon} Lf_j$

$\Rightarrow Lf_j \ge \frac{1}{N} Lg \cdot (1-\varepsilon) \Rightarrow Lf_j \ge \frac{1-\varepsilon}{N}\mu(K_j)$

$\exists M \in \mathbb{Z}_+$

$$\therefore \frac{1}{N}\sum_{j=1}^{M}\mu(K_j) \le \underset{\int f\,d\mu}{Lf} \le \frac{1}{N}\sum_{j=0}^{M-1}\mu(K_j)$$

$$\therefore \left| L(f) - \int f\,d\mu \right| \le \frac{1}{N}\mu(K_0). \qquad K_0 = \operatorname{supp} f.$$

$N \to +\infty \quad \Rightarrow Lf = \int f\,d\mu.$ $\square$.

Now consider:

$L: C_c(\mathbb{R}^n;\mathbb{R}^m) \to \mathbb{R}.$ linear.

$$\sup\{Lf \mid f \in C_c(\mathbb{R}^n;\mathbb{R}^m),\ |f| \le 1,\ \operatorname{spt} f \subseteq K\} < \infty \quad (*) \quad \forall K \subset\subset \mathbb{R}^n.$$

$\Rightarrow$. $\exists$ Radon μ on $\mathbb{R}^n$.

$\exists$ a μ-measurable $\sigma: \mathbb{R}^n \to \mathbb{R}^m$: $|\sigma(x)| = 1$ μ-a.e.

$$L(f) = \int_{\mathbb{R}^n} f \cdot \sigma\,d\mu.$$

避免出现"符号测度"

Pf: ① Decompose L: $\forall\, C_c^+(\mathbb{R}^n;\mathbb{R})$.

$$L^+ f := \sup\{Lg : g \in C_c^+(\mathbb{R}^n;\mathbb{R}),\ 0 \le g \le f\}$$

L^+: 非负线性泛函.

(1.1): $L^+ cf = c L^+ f$. $\checkmark$ $\quad c \ge 0 \quad f \ge 0$

(1.2): $\forall f_1, f_2 \in C_c(\mathbb{R}^n;\mathbb{R})$ $\quad f_1, f_2 \ge 0$

claim: $L^+ f_1 + L^+ f_2 = L^+(f_1 + f_2)$

"$\le$": $\forall \varepsilon > 0$, $\exists g_1, g_2$. $0 \le g_i \le f_i$.

$L^+ f_i \le L g_i + \varepsilon$ $\quad$ 令 $g = g_1 + g_2$. $\quad 0 \le g \le f := f_1 + f_2$

$$L^+ f_1 + L^+ f_2 - 2\varepsilon < \underset{L^+(f_1+f_2)}{Lg} \le Lf. \quad \checkmark$$

"$\geq$": $\forall g \in C_c^+(\mathbb{R}^n; \mathbb{R})$. $0 \le g \le f_1 + f_2$

取 $g_1 = \min\{g, f_1\}$; $g_2 = g - g_1 \geq 0$.

$|g_i| \le f_i$.

$L g_i \le L^+ f_i$

$\overset{L(g)}{\overset{\shortparallel}{L(g_1+g_2)}} \le L g_1 + L g_2 \le L^+ f_1 + L^+ f_2$

取 sup.

$L^+(f_1+f_2)$.

扩充定义 L^+ 到 $C_c(\mathbb{R}^n, \mathbb{R})$.

$\forall f \in C_c(\mathbb{R}^n; \mathbb{R})$. $f = f^+ - f^-$.

Set $L^+ f = L^+ f^+ - L^+ f^-$. check: L^+ 线性 泛定 (omit)

L^+ 显见非负.

令 $L^- f := L^+ - L$.

check: L^- 非负 omit.

We've just proved $\exists \mu^+$ μ^- Radon s.t. $\forall f \in C_c(\mathbb{R}^n; \mathbb{R})$

$L^\pm f = \int f \, d\mu^\pm$. $\Rightarrow Lf \,?\, \int f \, \dfrac{d(\mu^+ - \mu^-)}{\sigma d(\mu)}$? 如何操作?

$\sigma d\mu$ 与 $d\mu^+ - d\mu^-$ 什么关系?

是否 $\sigma > 0$ 时 $\sigma d\mu = d\mu^+$, $\sigma < 0$ 时 $\sigma d\mu = d\mu^-$? ($\mu^+ \perp \mu^-$?) . 股浩猜想.

Introduce a new def: Total Variation measure μ:

$\forall$ open V. $\mu(V) = \sup\{L(f) : f \in C_c(\mathbb{R}^n; \mathbb{R}),\ |f| \prec V\}$

$\forall E \subseteq \mathbb{R}^n$. $\mu(E) = \inf\{\mu(V) \mid \underset{\text{open}}{V} \supseteq E\}$.

claim: μ is a radon measure.

$\forall K \subset\subset \underset{\text{开}}{U} \subset \tilde{K}$ cpt. $\mu(K) < +\infty$? $\Leftarrow \mu(U) < +\infty$

$\Leftarrow \forall f \in \overset{\text{IA}}{\le} \sup\{Lf \ f \in C_c(\mathbb{R}^n; \mathbb{R})\}$

$\underset{\sim\sim\sim}{\mu^+ \ll \mu}$? $\mu^+(U) = \sup\{L^+ f \mid f \in C_c(\mathbb{R}^n; \mathbb{R}),\ f \prec U\}$ $f \prec \tilde{K}$.

$$\mu^+(U) = \sup\{Lg : f, g \in C_c(\mathbb{R}^n; \mathbb{R})\ f \prec U.\ 0 \le g \le f\}$$

$$\le \sup\{Lg : \text{- - - - - - -}\ |g| \prec U\}$$

$$= \mu(U).$$

$\Rightarrow \forall E \subseteq \mathbb{R}^n$.

$\mu^+(E) \le \mu(E) \Rightarrow \mu^+ \ll \mu$.

同理: $\mu^- \ll \mu$.

由 Radon-Nikodym thm. 存在 μ-可测函数. σ^+. σ^-.

$\underline{\sigma^\pm \ge 0 \text{ 局部可积}}$ $\mu^\pm(A) = \int_A \sigma^\pm \, d\mu$. $\sigma := \sigma^+ - \sigma^-$. $\forall A$ 可测

$\Rightarrow \forall f \in C_c(\mathbb{R}^n; \mathbb{R})$. $\int f \, d\mu^\pm = \int f \cdot \sigma^\pm \, d\mu$.

$$Lf = (L^+ - L^-)f = \int f \, d\mu^+ - \int f \, d\mu^- = \int f \sigma^+ \, d\mu - \int f \sigma^- \, d\mu$$

Set $\sigma = \sigma^+ - \sigma^-$.

只剩下: $|\sigma| = 1$ a.e-μ. 稍后证明.

卅.

$\mathbb{R}^n \to \mathbb{R}^m$ 版本:

$f \in C_c(\mathbb{R}^n, \mathbb{R}^m)$

$\quad \| $

$(f^1, \dots, f^m)$

$Lf = L^1(f^1) + \dots + L^m(f^m)$.

先对分量定义全变差测度 μ. 如前

Claim: $\mu^{\pm} \ll \mu$. (omit)

$\mu_i^{\pm} \ll \mu_i \ll \mu$.

By R-N: $\exists\, \sigma^{i\pm}$. μ-可测

~~$L(f) =$~~ $L^{i,\pm}(f) = \int f \sigma^{i\pm} \, d\mu$.

$$Lf = \sum_i L^i f^i = \int (f^1 \cdots f^m) \underbrace{\begin{pmatrix} \sigma_1^+ - \sigma_1^- \\ \vdots \\ \sigma_m^+ - \sigma_m^- \end{pmatrix}}_{= \vec{\sigma}} d\mu.$$

□.

Recall: General Riesz Representation thm.

$L: C_c(\mathbb{R}^n;\mathbb{R}^m) \to \mathbb{R}$ linear functional with $\sup\left\{ Lf \,\middle|\, f\in C_c(\mathbb{R}^n;\mathbb{R}^m),\ |f|\le 1,\ \operatorname{spt} f\subseteq K \right\} < \infty$ ... (*). $\forall K\subset\subset \mathbb{R}^n$.

Then $\exists$ Radon μ, μ-measurable function $\sigma: \mathbb{R}^n\to\mathbb{R}^m$, with $|\sigma(x)|=1$ μ-a.e. and $L(f) = \int_{\mathbb{R}^n} f\cdot\sigma\, d\mu$. $\forall f\in C_c(\mathbb{R}^n;\mathbb{R}^m)$.

Till now, It remains to show $|\sigma|=1$ μ-a.e.

$\lambda(U) = \mu^+(U)+\mu^-(U)$.
$=\sup\{Lg \mid 0\le g\le f,\ f\in U\}$
$+\sup\{L^- g \mid 0\le g\le f,\ f\in U\}$.
$\ge \sup_U\{Lg \mid \dots\}$
$|\sup$ 沿线 $\to \int g\cdot\sigma\, d\mu\, d\lambda$.

Claim: $\forall$ open U. $\int_U |\sigma|\,d\mu = \mu(U)$.

Take $f_k\in C_c(\mathbb{R}^n;\mathbb{R}^m)$ such that $|f_k|\le 1$ $\operatorname{spt} f_k\subseteq U$ $f_k\cdot\sigma\to|\sigma|$ μ-a.e.

~~such functions exist according to~~ $\Rightarrow \int_U |\sigma|\,d\mu = \lim_{k\to\infty}\int f_k\cdot\sigma\,d\mu \le \mu(U)$. ✓

$g_k\sigma \to |\sigma|$

On the other hand. if $f\in C_c(\mathbb{R}^n;\mathbb{R}^m)$, $|f|\le 1$, $\operatorname{spt} f\subseteq U$.

$\Rightarrow \int f\cdot\sigma\,d\mu = \int_U |\sigma|\,d\mu \ge \mu(U)$ by the def of $\mu(U)$. $f_k\to|\sigma|$ ✓ done $\square$.

$Lg + Lg := \int g\,\sigma^+ d\lambda + \sigma^-\lambda$ $\le$ ✓ $g_k = \frac{f_k}{\sigma}$ $\sigma\ne 0$ ✓

Existence?

WLOG $\mu\subseteq B(0,R)$. $\frac{\sigma}{|\sigma|}$ μ-measurable on $\mathbb{R}^n$.

By Thm 1.15, $\forall k, \exists\, \tilde f_k$ continuous: $\mathbb{R}^n\to\mathbb{R}^m$, s.t. $\mu\left(U\cap\left\{\tilde f_k \ne \frac{\sigma}{|\sigma|}\right\}\right) < \frac{1}{k}$.

WLOG $|\tilde f_k|\le 1$. (else we consider $\frac{f_k}{|f_k|}$ if $|f_k|\ge 1$).

$\forall k$. Take $K_k\subset\subset U$. $k\to\infty$. $K_k\uparrow U$.

$\Rightarrow \exists\, \zeta_k\in C_c(\mathbb{R}^n;\mathbb{R}^m)$. $\zeta_k < U$. $\operatorname{spt}\zeta_k\subseteq U$.
$\zeta_k = 1$. on K_1

check: $\int |f_k\cdot\sigma - |\sigma||\,d\mu \le \cancel{\int_{U\setminus K_k} 2|\sigma|\,d\mu + \int_{K_k} \left|f_k - \frac{\sigma}{|\sigma|}\right| |\sigma|}$ 如上. $f_k \xrightarrow{\mu} f$.

$|\sigma| \le \sum_{i=1}^m |\sigma_i|$

$\cancel{\int_{U\setminus K_k} d\mu_i^{\pm} < \infty \Rightarrow \int_{U\setminus K_k} |\sigma|\,d\mu < \infty}$ Radon. $\exists f_{k_j} \to \frac{\sigma}{|\sigma|}$ μ-a.e.

再用DCT 即可 $\square$

§1.9 Weak Convergence

~~Thm 1.40~~: $\mu \cdot \mu_k$ Radon on $\mathbb{R}^n$.

How to define $\mu_k \xrightarrow{?} \mu$.

Say $\mu_k \to \mu$. iff. $\forall f \in C_c(\mathbb{R}^n; \mathbb{R})$. $\int f \, d\mu_k \longrightarrow \int f \, d\mu$.

(Equivalent with weak-* convergence)

eg: $\mu_k = \delta_{\frac{1}{k}}$. $E = (0 \cdot 1)$.

$\forall f \in C_c(\mathbb{R}^n; \mathbb{R})$ $\quad \int f \, d\mu_k = f(\frac{1}{k})$.
$\quad \quad \quad \int f \, d\mu = f(0)$

but . $\forall A \subset (0 \cdot 1)$
$\mu_k(A) = 1_{\{\frac{1}{k} \in A\}} \not\to \mu(A) = 1_{\{0 \in A\}}$

Thm 1.40: $\mu \cdot \mu_k$ Radon on $\mathbb{R}^n$. t.f.a.e

(1) $\int_{\mathbb{R}^n} f \, d\mu_k = \int_{\mathbb{R}^n} f \, d\mu$. $\forall f \in C_c(\mathbb{R}^n)$

(2) $\limsup_{k \to \infty} \mu_k(K) \leq \mu(K)$. $\forall K \subset\subset \mathbb{R}^n$. and $\mu(U) \leq \liminf_{k \to \infty} \mu_k(U)$. $\forall U \subseteq \mathbb{R}^n$.

(3) $\lim_{k \to \infty} \mu_k(B) = \mu(B)$. $\forall B \overset{\text{Borel}}{\subseteq} \mathbb{R}^n$, with $\mu(\partial B) = 0$

<u>Proof</u>: (1) $\Rightarrow$ (2) Fix $\varepsilon > 0$. U open. $K \subset\subset U$.

choose $f \in C_c(\mathbb{R}^n)$ with $\operatorname{spt} f \subseteq U$. $0 \leq f \leq 1$. $f \equiv 1$ on K.

$$\mu(K) \leq \int_{\mathbb{R}^n} f \, d\mu = \lim_{k \to \infty} \int_{\mathbb{R}^n} f \, d\mu_k \leq \liminf_{k \to \infty} \mu_k(U)$$

$$\therefore \mu(U) = \sup_{K \subset\subset U} \mu(K) \leq \liminf_{k \to \infty} \mu_k(U)$$

(2) $\Rightarrow$ (3).
$$\mu(B) = \mu(\mathring{B}) \leq \liminf_{k \to \infty} \mu_k(\mathring{B}) \leq \limsup_{k \to \infty} \mu_k(\bar{B})$$
$$\leq \mu(\bar{B}) = \mu(\mathring{B}) + \mu(\partial B) = \mu(B).$$

(3) $\Rightarrow$ (1). $\forall \varepsilon > 0$. $f \in C_c^+(\mathbb{R}^n; \mathbb{R})$.

let $R > 0$ with $\operatorname{spt} f \subseteq B(0, R)$. $\mu(\partial B_R) = 0$.

WLOG $f \geq 0$.

Choose $0 = t_0 < t_1 < \cdots < t_N$ with $t_N = 2\|f\|_\infty$, $0 < t_i - t_{i-1} < \varepsilon$, $\underline{\mu(f^{-1}\{t_i\}) = 0}$ & $\mu^k(f^{-1}\{t_i\}) = 0$.

如何做到?

$B_i = f^{-1}(t_{i-1}, t_i)$.

$\Rightarrow \mu(\partial B_i) = 0$.

$\int_{\bigcup_{i=2}^N B_i \cup \{f^{-1}(0,\varepsilon)\} \cup \bigcup_{i=2}^{N} \partial B_i}$

$\forall \eta > 0$ $\{x \in (t_i - \eta, t_i + \eta) \mid \mu(f^{-1}\{t_i\}) > 0\}$ 可数.

[illegible] $\Downarrow$ 有限集 $\downarrow$ 否则会与"Radon"矛盾

Now: $\sum_{i=2}^N t_{i-1}\mu_k(B_i) \leq \int_{\mathbb{R}^n} f \, d\mu_k \leq \sum_{i=2}^N t_i \mu_k(B_i) + t_1 \mu_k(B(R))$

$\sum_{i=2}^N t_{i-1}\mu(B_i) \leq \int_{\mathbb{R}^n} f \, d\mu \leq \sum_{i=2}^N t_i \mu(B_i) + t_1 \mu(B(R))$.

$k \to +\infty \Rightarrow \int f \, d\mu_k \to \int f \, d\mu \quad \forall f \in C_c(\mathbb{R}^n, \mathbb{R})$.

$\square$.

Thm 1.41. Weak compactness for measure

$\{\mu_k\}$ Radon. with $\sup_{k \in \mathbb{Z}_+} \mu_k(K) < \infty \quad \forall K \subset\subset \mathbb{R}^n$

$\Rightarrow \exists \mu_{k_j} \to \exists \mu$.

Fact: • 紧集上的连续函数可由多项式一致逼近 (有理系数). (cf. Stone-Weierstrass 定理).

• $C_c(\mathbb{R}^n; \mathbb{R})$ (with "sup" norm) ~~is a Polish spa~~ has a countable dense subset.

一致收敛拓扑.

~~[illegible]~~

Idea. Find countable dense subset.

Proof:

1. Assume first $\mu_k(\mathbb{R}^n) \leq C$ & $\{f_j\}$ is a countable dense subset of $C_c(\mathbb{R}^n)$.

$\Rightarrow \int f_1 \, d\mu_j$ bdd. $\exists$ subsequence $\{\mu_j^1\}_1^\infty$ $a_1 \in \mathbb{R}$.

$\int f_1 \, d\mu_j^1 \to a_1$.

Repeat. $\forall k$. $\exists$ subsequence $\{\mu_j^k\}_{j=1}^\infty$ of $\{\mu_j^{k-1}\}_1^\infty$ & $a_k \in \mathbb{R}$ s.t. $\int f_k \, d\mu_j^k \to a_k$

Set $\nu_j = \mu_j^j$ then $\int f_k \, d\nu_j \to a_k \quad \forall k \geq 1$.

Define $L(f_k) = a_k$. $\quad L$ $\begin{cases} \text{linear} \\ |L(f_k)| \le \|f_k\|_\infty M \end{cases}$ $\xRightarrow{\text{B.L.T.}}$ $\tilde{L}$ on $C_c(\mathbb{R}^n;\mathbb{R}^m)$

Exists? How to construct?

$\forall f \in C_c(\mathbb{R}^n, \mathbb{R})$. $L(f) = ?$

$$\exists\ f_j \rightrightarrows f \quad (\text{by Stone-Weierstrass Thm})$$

存在性：$\{a_j'\}$ 柯西列.(易证)

$$Lf := \lim_{j\to\infty} a_j' = \lim_{j\to\infty} \lim_{k\to\infty} \int f_j'\, d\mu_k .$$

① Well-defined? $\quad f_j' \rightrightarrows f. \quad f_j'' \rightrightarrows f.$

$$\lim_{j\to\infty} \lim_{i\to\infty} \int |f_j' - f_j''|\, d\mu_{k_i} \to 0 \quad (\text{同上证柯西列})$$

② linear $\quad f, \tilde{f} \in C_c(\mathbb{R}^n, \mathbb{R}).\ \lambda \in \mathbb{R}.$

$$L(f+\lambda\tilde{f}) \overset{?}{=} L(f) + \lambda L(\tilde{f}).$$

$f_j' \rightrightarrows f. \quad \tilde{f}_j \rightrightarrows \tilde{f}.$

$$L(f+\lambda\tilde{f}) \ne \lim_{j\to\infty}\lim_{i\to\infty} \int (f_j'+\lambda\tilde{f}_j)\, d\mu_{k_i}$$

不一定在可数[illegible]

$\exists\ f_j'' \rightrightarrows f+\lambda\tilde{f}.$

$$\lim_{j\to\infty}\lim_{i\to\infty} \int f_j''\, d\mu_{k_i}$$
$$\| $$
$$\lim_{j\to\infty}\lim_{i\to\infty} \int f_j' + \lambda \tilde{f}_j\, d\mu_{k_i}.$$
$$\| $$
$$\lim_{j\to\infty}\lim_{i\to\infty} \int f_j'\, d\mu_{k_i} + \lambda \lim_{j\to\infty}\lim_{i\to\infty} \int \tilde{f}_j\, d\mu_{k_i}$$
$$= L(f) + \lambda L(\tilde{f}).$$

③ bddness $\quad \forall f.\ \exists f_j \rightrightarrows f$

$$|Lf| \le \lim_{j\to\infty}\lim_{i\to\infty} \int |f_j|\, d\mu_{k_i} \le \|f\|_\infty$$

④ Non-negative

$\forall f \in C_c(\mathbb{R}^n, \mathbb{R}) \ge 0.$

$\exists f_j' \rightrightarrows f. \quad \lim_{j\to\infty} \min_{\mathbb{R}^n} f_j' = 0.$

$$Lf = \lim_j \lim_i \int f_j'\, d\mu_i .$$

又 $\lim_{j\to\infty} \min_{\mathbb{R}^n} f_j' = 0$.

By Riesz Representation Thm, $\exists$ Radon measure μ.

s.t. $Lf = \int f\, d\mu$. $\quad \forall f \in C_c(\mathbb{R}^n;\mathbb{R})$

Next, it remains to prove μ is what we want

i.e. $\nu_j \longrightarrow \mu$.

$\forall f \in C_c(\mathbb{R}^n)$, $\exists f_i \rightrightarrows f$

Then $\forall \varepsilon > 0$, $\exists N\ \forall i > N$ $\|f - f_i\|_\infty < \frac{\varepsilon}{4M}$, $M = \sup_k \mu_k(\mathbb{R}^n)$.

Choose J s.t. $\forall j > J$.

$$\left|\int f_i\, d\nu_j - \int f_i\, d\mu\right| < \frac{\varepsilon}{2}$$

$\Rightarrow \forall j > J$,

$$\begin{aligned}
&\left|\int f\, d\nu_j - \int f\, d\mu\right| \\
&\le \left|\int f - f_i\, d\nu_j\right| + \left|\int f - f_i\, d\mu\right| \\
&\quad + \left|\int f_i\, d\nu_j - \int f_i\, d\mu\right| \\
&\le 2M\|f - f_i\|_\infty + \frac{\varepsilon}{2} < \varepsilon \longrightarrow 0^+ \text{ as } \varepsilon \to 0^+
\end{aligned}$$

Generally, if we only have $\sup_{k \in \mathbb{Z}_+} \mu_k(K) < \infty$ $\forall K \subset\subset \mathbb{R}^n$

then set $\mu_k^l = \mu_k \llcorner B(0,l)$ and use a diagonal argument. $\square$

- Weak convergence of L^p functions.

$U \subseteq \mathbb{R}^n$ open, $1 \le p < \infty$.

Fact: ~~SW~~ S-W Thm 的结论在 L^p 相容.

~~Thm~~ Def. $\{f_k\} \subseteq L^p(U) \rightharpoonup f \in L^p(U)$ iff $\forall g \in L^{p'}$, $\int_U f_k g \to \int_U fg$.

Proof:

Applications:

L^p $p \geq 1$ $L: L^p \to [-\infty, +\infty]$ linear

$L(f) \leq C\|f\|_p$ $\Big\} \Rightarrow L \in (L^p)^*$

Fact: $(L^p)^* = L^q$ $1 \leq p < \infty$

有一边易知 $\forall g \in L^q$ 令 $Lf = \int fg \leq \|f\|_p \|g\|_q$ V.

为什么 L 由积分给出?

即 $\forall L \in L^{p*}$. $\overset{?}{\exists} g \in L^q$. $Lf = \int f \cdot g$. $\leq \|f\|_p$. $\forall f \in L^p$

$L|_{C_c(\mathbb{R}^n;\mathbb{R})}$ 线性正泛. $\Rightarrow \exists \mu$. s.t. $Lf = \int f d\mu$

只证 "$\mu << m$". 因 $Lf \lesssim \|f\|_p \longrightarrow \exists g$

$\int f \cdot g dm$.

$\square$

Thm: $1<p<\infty$. $\{f_k\} \subseteq L^p$ with $\sup_k \|f_k\|_p < \infty$.

then $\exists f_{k_j} \in L^p$, $f \in L^p$ $\quad f_{k_j} \rightharpoonup f$ in L^p.

(即为 Banach-Alaoglu 定理的特例).

Proof: WLOG $U=\mathbb{R}^n$. $f_k \geq 0$ $\mathcal{L}^n$-a.e.

Define the Radon measures $\mu_k := \mathcal{L}^n \llcorner f_k$. $k=1,2,\dots$

Then $\forall K \subset\subset \mathbb{R}^n$.

$\mu_k(K) = \int_K f_k \, dx \leq \|f_k\|_p \left(\mathcal{L}^n(K)\right)^{1-\frac{1}{p}}$.

$\sup_k \mu_k(K) < \infty$

$\phi \quad \therefore \exists \mu$ Radon. $\exists \mu_{k_j} \rightharpoonup \mu$.

① $\mu \ll \mathcal{L}^n$.

$\forall A \subset \mathbb{R}^n$ $\cancel{\text{bad}}$. $\mathcal{L}^n(A)=0$.

$\forall \varepsilon>0$, $\exists$ open $V \supseteq A$. $\mathcal{L}^n(V)<\varepsilon$.

Then $\mu(V) \leq \liminf_{j\to\infty} \mu_{k_j}(V) = \liminf_{j\to\infty} \int_V f_{k_j} \, dx$

$$\leq \liminf_{j\to\infty} \left(\int_V f_{k_j}^p \, dx\right)^{\frac{1}{p}} \mathcal{L}^n(V)^{1-\frac{1}{p}}$$

$$\lesssim \varepsilon^{1-\frac{1}{p}} \to 0$$

$\cancel{②}$ $\therefore \exists f \in L^1_{loc}$ s.t. $\forall A \subseteq \mathbb{R}^n$ Borel. $\mu(A) = \int_A f \, d\mathcal{L}^n$.

② $f \in L^p(\mathbb{R}^n)$.

$$\|f\|_p = \sup_{\substack{\phi \in C_c(\mathbb{R}^n) \\ \|\phi\|_q \leq 1}} \int_{\mathbb{R}^n} \phi f \, dx \overset{?}{<} \infty.$$

$\forall \phi \in C_c(\mathbb{R}^n)$. $\quad \int_{\mathbb{R}^n} \phi f = \int_{\mathbb{R}^n} \phi \, d\mu = \lim_{j\to\infty} \int_{\mathbb{R}^n} \phi \, d\mu_{k_j}$.

$$= \lim_{j\to\infty} \int_{\mathbb{R}^n} f_{k_j} \phi$$

$$\leq \sup_k \|f_{k_j}\|_p \|\phi\|_q$$

$$\lesssim \|\phi\|_q < \infty.$$

③ $f_{k_j} \to f$ in L^p

$\forall \phi \in C_c^0(\mathbb{R}^n)$ $\int f_{k_j}\phi \to \int f\phi$

$\forall g \in L^q$. $\forall \varepsilon>0$. $\exists \phi \in C_c(\mathbb{R}^n)$ $\|\phi - g\|_q < \varepsilon$

$$\int (f_{k_j}-f)g = \underline{\int (f_{k_j}-f)\phi} \to 0$$

$$+ \underbrace{\int (f_{k_j}-f)(g-\phi)}$$

$$\le \|f_{k_j}-f\|_p \|g-\phi\|_q$$

$$\le (\|f_{k_j}\|_p + \|f\|_p)\|g-\phi\|_q \lesssim \varepsilon \to 0$$

□.

p=1. False!

缺"一致可积".

$$f_n = \begin{cases} n & x\in[0,\frac{1}{n}] \\ 0 & 否则 \end{cases} \quad \sup_n \|f_n\|_1 < \infty$$

无弱收敛子列

因 ~~$f_n(0) \to \infty$~~ ~~$\delta(0) \to$~~ $\delta \notin L^1$.

Thm 1.43 Uniform integrability & weak convergence

U bdd ~~open~~. $f_k \in L^1(U)$. $\sup_k \|f_k\|_{L^1(U)} < \infty$

$\lim_{l\to\infty} \sup_k \int_{\{|f_k|\ge l\}} |f_k|\,dx = 0$ ← 一致可积

Then $\exists f_{k_j}$. $f \in L^1$ ⇒ $f_{k_j} \rightharpoonup f$ in $L^1(U)$.

Proof: WLOG $f_k \ge 0$. $\mu_k = \mathcal{L}^n \llcorner f_k$.

Define, $\mu_k(K) = \int_K f_k\,dx$ $\forall K \subset\subset \mathbb{R}^n$.

$\sup_k \mu_k(U) < \infty$ By 1.41. $\exists$ Radon μ. & $\exists \mu_{k_j} \rightharpoonup \mu$.

claim $\mu \ll \mathcal{L}^n$. $\forall \varepsilon>0$. choose $\delta = \frac{\varepsilon}{2l}$ then. $\forall E \subset U$ $\mathcal{L}^n$-可测.

~~$\forall \varepsilon>0$. $\exists \delta$~~ with $\mathcal{L}^n(E) < \delta$, we have

$$\int_E |f_k|\,dx = \underset{\downarrow 0}{\int_{E\cap\{|f_k|\ge l\}} |f_k|\,dx} + \underbrace{\int_{E\cap\{|f_k|<l\}} |f_k|\,dx}_{< l\,\mathcal{L}^n(E)} \to 0 \text{ as } \varepsilon \to 0, \quad \delta = \frac{\varepsilon}{2l}.$$

$\therefore \mu \ll \mathcal{L}^n$

Claim:

$\forall g \in L^\infty(U) \quad \int_U f_{k_j} g \, dx \to \int_U fg \, dx$

U bdd $\Rightarrow g \in L^1_{loc}(U)$ or $L^1(U)$. then

$\exists\, g_i \in C_c^\infty(U)$ s.t. $g_i \to g$ $\mathcal{L}^n$ a.e

For fixed $\varepsilon > 0$, we can also select a proper δ. ~~for~~

By Egorov's Thm. $\exists E \subset U$

$g_i \rightrightarrows g$ on $U \setminus E$. $\mathcal{L}^n(E) \le \delta$.

$$\left| \int_U (f_{k_j} - f) g \, dx \right| = \left| \int_U (f_{k_j} - f)(g - g_i)\, dx + \int (f_{k_j} - f) g_i \, dx \right|$$

$$\le \int_U |f_{k_j} - f| \cdot |g - g_i| \, dx + \left| \int_U (f_{k_j} - f) g_i \, dx \right|$$

$$= \int_E + \int_{U \setminus E} + \left| \int_U (f_{k_j} - f) g_i \, dx \right|$$

$$\lesssim \underbrace{\int_E |f_{k_j}| + |f| \, dx}_{\lesssim \varepsilon} + \underbrace{\sup_{U \setminus E} |g - g_i|}_{\to 0} + \left| \int_U (f_{k_j} - f) g_i \, dx \right|$$

$$\lesssim \int_U (f_{k_j} - f) g_i \, dx . \quad \text{由 } \mu_{k_j} \rightharpoonup \mu \quad \to 0 .$$

$k_j \to \infty \quad i \to \infty$

$\square$

Ch 2 Hausdorff Measures

$\mathbb{R}^n$中 低维长度/面积 的推广

§2.1 Defs & elementary properties.

$\mathbb{R}^n$的 s维子流形/可求面积集

nice sets $\mathcal{H}^s = S^s$

$\mathcal{H}^s_\delta(A) \le S^s_\delta(A) \le 2^s \mathcal{H}^s_\delta(A)$.

Def 2.1 (1) $\forall A \in \mathbb{R}^n$. $s \in [0, \infty)$. $0 < \delta \le +\infty$.
$\sim$ dimension

改成球，称作球测度 S^s_δ, S^s

define $$\mathcal{H}^s_\delta(A) = \inf\left\{ \sum_{j=1}^{\infty} \alpha(s) \left(\frac{\operatorname{diam} C_j}{2}\right)^s \;\middle|\; A \subseteq \bigcup_{j=1}^{\infty} C_j,\ \operatorname{diam} C_j \le \delta \right\}$$

Any set. 不能改成球

where $\alpha(s) := \dfrac{\pi^{s/2}}{\Gamma(\frac{s}{2}+1)}$.

(2) $\mathcal{H}^s(A) := \lim\limits_{\delta\to 0} \mathcal{H}^s_\delta(A) = \sup\limits_{\delta>0} \mathcal{H}^s_\delta(A)$. s-dimensional Hausdorff measure.

Remark: $\delta \to 0^+$ forces the coverings to follow ~~the~~ local geometry of A.

Thm 2.1 $\forall 0 \le s < \infty$. $\mathcal{H}^s$ is a Borel regular measure in $\mathbb{R}^n$

Rmk: $\mathcal{H}^s$ is not Radon if $0 \le s < n$ since $\mathbb{R}^n$ is not σ-finite with respect to $\mathcal{H}^s$

Proof: ~~Step 1~~ ① $\mathcal{H}^s_\delta$ is a measure. trivial.

② $\mathcal{H}^s$ is a measure

$\{A_i\}_{i=1}^{\infty} \subseteq \mathbb{R}^n$ ~~then~~ $\mathcal{H}^s_\delta\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} \mathcal{H}^s_\delta(A_i) \le \sum_{i=1}^{\infty} \mathcal{H}^s(A_i)$.

$\delta \to 0^+$ $\Rightarrow$ $\mathcal{H}^s\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} \mathcal{H}^s(A_i)$.

③ $\mathcal{H}^s$ is a Borel measure.

Use Carathéodory's criteria: Choose $A, B \subseteq \mathbb{R}^n$. $\operatorname{dist}(A, B) > 0$.

$$\mathcal{H}^s(A) + \mathcal{H}^s(B) \overset{?}{\le} \mathcal{H}^s(A \cup B).$$

Choose $\delta \in (0, \frac{1}{4}\operatorname{dist}(A, B))$. $A \cup B \subseteq \bigcup_{k=1}^{\infty} C_k$. $\operatorname{diam} C_k \le \delta$.

Write $\mathcal{A} = \{C_j \mid C_j \cap A \ne \emptyset\}$. $\mathcal{B} = \{C_j \mid C_j \cap B \ne \emptyset\}$ $\Rightarrow A \subseteq \bigcup_{C_j \in \mathcal{A}} C_j$

$B \subseteq \bigcup_{C_j \in \mathcal{B}} C_j$. $C_i \cap C_j = \emptyset$ if $C_i \in \mathcal{A}$, $C_j \in \mathcal{B}$.

Hence $\sum_{j=1}^{\infty} \alpha(s)\left(\frac{\operatorname{diam} C_j}{2}\right)^s \ge \sum_{C_j \in \mathcal{A}} \alpha(s)\left(\frac{\operatorname{diam} C_j}{2}\right)^s$

$+ \sum_{C_j \in \mathcal{B}} \alpha(s)\left(\frac{\operatorname{diam} C_j}{2}\right)^s \ge \mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B)$.

Take the infimum over all $\{C_j\}_1^\infty \rightarrow \underline{\mathcal{H}_\delta^s(A\cup B) \geq \mathcal{H}_\delta^s(A)+\mathcal{H}_\delta^s(B)}$.

$\forall\, 0 < \boxed{\delta} < \frac{1}{4}\operatorname{dist}(A,B)$.

与A,B选取有关.

从而不能对 $\mathcal{H}_\delta^s$ 用 Carathéodory's criterion.

Set $\delta \to 0^+$ and we obtain $\mathcal{H}^s(A\cup B) \geq \mathcal{H}^s(A)+\mathcal{H}^s(B)$

$$\Downarrow ②$$

$\mathcal{H}^s(A\cup B) = \mathcal{H}^s(A)+\mathcal{H}^s(B)$. $\forall\, A, B \subseteq \mathbb{R}^n$, $\operatorname{dist}(A,B) > 0$.

$$\Downarrow \text{Carathéodory's criterion}$$

$\mathcal{H}^s$ is a Borel measure.

④ $\mathcal{H}^s$ is Borel regular.

Since diam C = diam $\bar{C}$ then

$$\mathcal{H}_\delta^s(A) = \inf\left\{\sum_{j=1}^\infty \alpha(s)\left(\frac{\operatorname{diam} C_j}{2}\right)^s \;\middle|\; A \subseteq \bigcup_{j=1}^\infty C_j,\ \operatorname{diam} C_j \leq \delta,\ C_j \text{ closed}\right\}$$

Choose $A \subset \mathbb{R}^n$ s.t. $\mathcal{H}^s(A) < \infty$, then $\mathcal{H}_\delta^s(A) < \infty$ $\forall \delta > 0$.

$\forall k \geq 1$ choose closed sets $\{C_j^k\}_1^\infty$ s.t. diam $C_j^k \leq \frac{1}{k}$.

$$A \subseteq \bigcup_{j=1}^\infty C_j^k.$$

$$\sum_{j=1}^\infty \alpha(s)\left(\frac{\operatorname{diam} C_j^k}{2}\right)^s \leq \mathcal{H}_{\frac{1}{k}}^s(A) + \frac{1}{k}.$$

Set $A_k = \bigcup_{j=1}^\infty C_j^k$. $B := \bigcap_{k=1}^\infty A_k \Rightarrow B$ Borel. ~~i.e. $B = \limsup_{k\to\infty}$~~

Also $A \subseteq A_k\ \forall k. \Rightarrow A \subseteq B$

$$\Rightarrow \mathcal{H}_{\frac{1}{k}}^s(B) \leq \sum_{j=1}^\infty \alpha(s)\left(\frac{\operatorname{diam} C_j^k}{2}\right)^s \leq \mathcal{H}_{\frac{1}{k}}^s(A) + \frac{1}{k}$$

$$k\to\infty \Rightarrow \mathcal{H}^s(B) \leq \mathcal{H}^s(A) \overset{A\subseteq B}{\leq} \mathcal{H}^s(B)$$

$\square$

Prop 2.2 (Properties of Hausdorff Measure).

(1) $\mathcal{H}^0$ is a counting measure.

(2) $\mathcal{H}^1 = \mathcal{L}^1$ on $\mathbb{R}^1$

(3) $\mathcal{H}^s = 0$ on $\mathbb{R}^n$ $\forall s > n$

(4) $\mathcal{H}^s(\lambda A) = \lambda^s \mathcal{H}^s(A)$. $\forall \lambda > 0$. $A \subseteq \mathbb{R}^n$

(5). $\mathcal{H}^s(L(A)) = \mathcal{H}^s(A)$ $\forall$ isometry $L: \mathbb{R}^n \to \mathbb{R}^n$, $A \subseteq \mathbb{R}^n$

Proof: (4). (5) is trivial.

(1). $\alpha(0) = 1$. $\Rightarrow \forall a \in \mathbb{R}^n$. $\mathcal{H}^0\{a\} = 1$. ✓

(2) $\mathcal{L}^1(A) = \inf\left\{ \sum_{i=1}^\infty (b_i - a_i) \;\middle|\; A \subseteq \bigcup_{i=1}^\infty [a_i, b_i] \right\}$

$= \inf\left\{ \sum_{i=1}^\infty \operatorname{diam} C_i \;\middle|\; A \subseteq \bigcup_{i=1}^\infty C_i,\ C_i \subseteq \mathbb{R} \right\}$

$\overset{\forall \delta > 0}{=} \inf\{ \cdots \quad \operatorname{diam} \leq \delta \}$

$= \cdots \{ \cdots \quad \text{any set} \cdots \}$

$= \mathcal{H}^1_\delta(A)$.

(3) $\forall m \geq 1$. $[0,1]^n$ can be decomposed into m^n cubes with side length $\frac{1}{m}$.

and diam $\frac{\sqrt{n}}{m}$

$\Rightarrow \mathcal{H}^s_{\frac{\sqrt{n}}{m}}(Q) \leq \alpha(s) n^{\frac{s}{2}} m^{n-s} \to 0$ as $m \to \infty$ if $s > n$

$\therefore \mathcal{H}^s(Q) = 0$.

(4)(5) ✓ □.

Lemma 2.1 $A \subset \mathbb{R}^n$. $\mathcal{H}^s_\delta(A) = 0$ for SOME $0 < \delta \leq \infty$, then $\mathcal{H}^s(A) = 0$

Proof: Fix $\varepsilon > 0$ $\exists \{C_j\}_1^\infty$. $A \subseteq \bigcup_{j=1}^\infty C_j$ and $\sum_{j=1}^\infty \alpha(s) \left(\frac{\operatorname{diam} C_j}{2}\right)^s \leq \varepsilon$.

$\therefore \forall i$. $\operatorname{diam} C_i \leq 2\left(\frac{\varepsilon}{\alpha(s)}\right)^{\frac{1}{s}} =: \delta(\varepsilon)$.

$\Rightarrow \mathcal{H}^s_{\delta(\varepsilon)}(A) \leq \varepsilon$. $\varepsilon \to 0^+$. $\therefore \mathcal{H}^s(A) = 0$ □.

Lemma 2.2 $A \subseteq \mathbb{R}^n$. $0 \le s < t < \infty$

(1) If $\mathcal{H}^s(A) < \infty$ then $\mathcal{H}^t(A) = 0$

(2) If $\mathcal{H}^t(A) > 0$, then $\mathcal{H}^s(A) = +\infty$

Proof: (1) $\Longleftrightarrow$ (2) ✓

(1). $\mathcal{H}^s(A) < \infty$. $\delta > 0$.

then $\exists \{C_j\}_1^\infty$ s.t. $\operatorname{diam} C_j \le \delta$ and $\sum_{j=1}^\infty \alpha(s) \left(\frac{\operatorname{diam} C_j}{2}\right)^s \le 1 + \mathcal{H}^s_\delta(A) \le 1 + \mathcal{H}^s(A)$

$A \subseteq \bigcup_{j=1}^\infty C_j$

$$\therefore \mathcal{H}^t_\delta(A) \le \sum_{j=1}^\infty \alpha(t) \left(\frac{\operatorname{diam} C_j}{2}\right)^t$$

$$= \sum_{j=1}^\infty \frac{\alpha(t)}{\alpha(s)} \alpha(s) \left(\frac{\operatorname{diam} C_j}{2}\right)^s \left(\frac{\operatorname{diam} C_j}{2}\right)^{t-s}.$$

$$\le \frac{\alpha(t)}{\alpha(s)} 2^{s-t} \delta^{t-s} \left(\mathcal{H}^s_\delta(A) + 1\right).$$

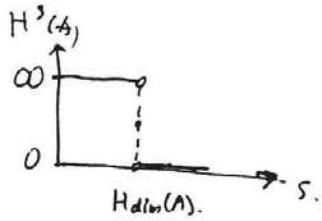
$\delta \to 0. \Rightarrow \mathcal{H}^t(A) = 0. \Rightarrow (1)$ holds. $\square$.

Def 2.2. (Hausdorff dimension). $A \subseteq \mathbb{R}^n$. $H_{\dim}(A) := \inf\{0 < s < \infty \mid \mathcal{H}^s(A) = 0\}$. $\square$.

Rmk: $H\dim(A) \in \mathbb{R}$. may not be an integer

Example: $H\dim(\mathcal{C}) = \frac{\log 2}{\log 3}$

Cantor set



$\square$

§2.2. Isodiametric inequality, $\mathcal{H}^n = \mathcal{L}^n$.

Thm 2.5. $\mathcal{H}^n = \mathcal{L}^n$ on $\mathbb{R}^n$.

<u>Prof</u>: ① $\mathcal{L}^n(A) \overset{\le}{} \mathcal{H}^n(A)$, by def.

② $\mathcal{H}^n(A) \overset{\ge}{} \mathcal{L}^n(A)$.

Isometric Ineq: $\mathcal{L}^n(A) \le \alpha(n) \left(\frac{\operatorname{diam} A}{2}\right)^n$.

~~If this ineq holds: then~~: Goal: $\mathcal{H}^n_\delta(A) \ge \mathcal{L}^n(A)$. $\forall \delta > 0$.

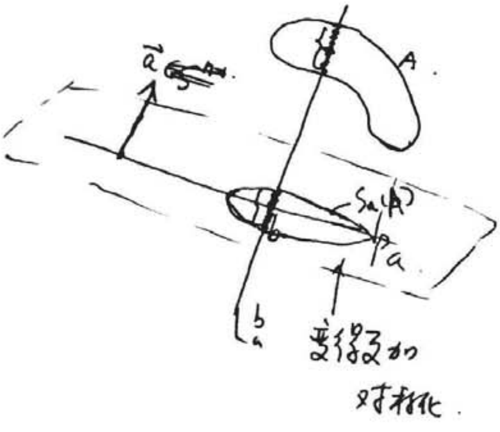
$\forall \delta > 0$ $A \subseteq \bigcup_{i=1}^{\infty} C_i$. C_i any set. $\operatorname{diam} C_i \le \delta$.

$\mathcal{L}^n(A) \le \sum_{i=1}^{\infty} \mathcal{L}^n(C_i) \le \sum_{i=1}^{\infty} \alpha(n) \left(\frac{\operatorname{diam} C_i}{2}\right)^n$. Take inf ✓.

Thus it remains to prove the ~~Isomet~~ isodiametric ineq. □.

Thm 2.4 Isometric Ineq:

$$\mathcal{L}^n(A) \le \alpha(n) \left(\frac{\operatorname{diam} A}{2}\right)^n.$$



· <u>Def</u>: Steiner's symmetrization.

$A \subseteq \mathbb{R}^n$. $L_b^a := \{b + ta \mid t \in \mathbb{R}\}$. $P_a = \{x \in \mathbb{R}^n \mid x \cdot a = 0\}$.

$$S_a(A) := \bigcup_{\substack{b \in P_a \\ A \cap L_b^a \ne \phi}} \left\{ b + ta \;\middle|\; |t| \le \tfrac{1}{2} \mathcal{H}^1(A \cap L_b^a) \right\}, \text{ for } A \in \mathbb{R}^n, \ \underset{\sim}{|a| = 1}.$$

Lemma 2.3. (Steiner Symmetrization).

(1) $\operatorname{diam} S_a(A) \le \operatorname{diam} A$.

(2) A $\mathcal{L}^n$-measurable $\Rightarrow$ $S_a(A)$ $\mathcal{L}^n$-measurable.

$\mathcal{L}^n(S_a(A)) = \mathcal{L}^n(A)$.

Proof of lem:

(a): By the rotation-invariance of $\mathcal{L}^n$, we assume $a = \vec{e_n} = (0, \dots, 0, 1)$,

then $P_{\vec{a}} = P_{\vec{e_n}} = \mathbb{R}^{n-1}$

$\mathcal{L}^1 = \mathcal{H}^1$ on $\mathbb{R}^1$.

By ~~fact~~ Fubini's thm.

$$\mathcal{L}^n(A) = \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \chi_A(b+ta)\, dt \right) db$$

$$= \int_{\mathbb{R}^{n-1}} \mathcal{L}^1(A \cap L_b^a)\, db = \int_{\mathbb{R}^{n-1}} \underbrace{\mathcal{H}^1(A \cap L_b^a)}_{=f(b)}\, db$$

By Fubini's thm. $\mathcal{H}^1(A \cap L_b^a)$ is $\mathcal{L}^{n-1}$-measurable as a function of b.

$$\text{Now } \underset{\text{by def}}{S_a(A) :=} \underbrace{\left\{ (b, y) \,\middle|\, -\frac{f(b)}{2} \le y \le +\frac{f(b)}{2} \right\}}_{\text{是可测函数的图像}} \setminus \underbrace{\{(b, 0) \mid L_b^a \cap A = \phi\}}_{\mathcal{L}^n\text{-zero measured}}$$

is $\mathcal{L}^n$-measurable.

$$\Rightarrow \mathcal{L}^n(S_a(A)) \overset{\text{Fubini}}{=} \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}^1} \chi_{S_a(A)}(t)\, dt \right) db.$$

$$= \int_{\mathbb{R}^{n-1}} \left(\int_{-\frac{f(b)}{2}}^{\frac{f(b)}{2}} 1\, dt \right) db$$

$$= \int_{\mathbb{R}^{n-1}} f(b)\, db = \mathcal{L}^n(A) \quad \Rightarrow \text{(a) holds.}$$

(b): ~~Set~~ $\operatorname{diam} A = \sup_{x, y \in A} \operatorname{dist}(x, y)$.

Set $\boxed{\begin{array}{l} A_b := A \cap L_b^a \\ S_b := S_a(A) \cap L_b^a \end{array}}$. $\operatorname{diam} A \overset{\text{易见}}{=} \sup_{b_1, b_2 \in P_a} \operatorname{diam}(A_{b_1} \cup A_{b_2})$.

Similarly,

$$\operatorname{diam} S_a(A) = \sup_{b_1, b_2 \in P_a} \operatorname{diam}(S_{b_1} \cup S_{b_2})$$

Thus it suffices to compare $\operatorname{diam}(A_{b_1} \cup A_{b_2})$ with $\operatorname{diam}(S_{b_1} \cup S_{b_2})$.

① ~~3~~:

直线的子集,有洞,试着填充

set $\tilde{A}_{b_i} = [\inf A_{b_i}, \sup A_{b_i}]$. (视作 $\mathbb{R}^1$ 的子集). $\Rightarrow \mathcal{H}^1(\tilde{A}_{b_i}) \ge \mathcal{H}^1(A_{b_i}) = \text{length of } S_{b_i}$.

$$\mathcal{H}^1(\tilde{A}_{b_i}) = \mathcal{L}^1(\tilde{A}_{b_i})$$

Thus, to prove "$\geq$", we may as well suppose A_{b_i} is connected.

(这样做不改变 diam), 但减小了 $\tilde{A}_{b_i}$ 长度. 即是没洞的.

再移动位置 $\Rightarrow$ ~~恰好 diam~~ 恰好 $= \operatorname{diam}(S_{b_1} \cup S_{b_2})$.

$\Rightarrow \operatorname{diam}(A_{b_1} \cup A_{b_2}) \geq \operatorname{diam}(S_{b_1} \cup S_{b_2})$.

~~因 $\leq$: 易见~~.

Pf of thm 2.4 WLOG $A \subseteq \mathbb{R}^n$ closed, $\mathcal{L}^n(A) \leq \alpha(n)\left(\frac{\operatorname{diam} A}{2}\right)^2$ is trivial.

To prove $\geq$:

Take Steiner's symmetrization for A w.r.t. P_1. $\{P_1 \dots P_n\}$ 为 $\mathbb{R}^n$ 中 $n-1$ 维超平面

$A_1 := S_a(A)$.

Iterate the steps w.r.t. $P_2, \dots, P_n$ to get $A_2, \dots, A_n =: A^*$.

For A^*, we have

Claim ① $\mathcal{L}^n(A^*) = \alpha(n)\left(\frac{\operatorname{diam} A^*}{2}\right)^n$

② 因 A^* 关于 $P_1 \dots P_n$ 对称 $\Rightarrow$ A^* 中心对称.

$\therefore \forall x \in A^* \Rightarrow -x \in A^*$.

$|x| = \frac{1}{2}\operatorname{dist}(x, -x) \leq \frac{1}{2}\operatorname{diam}(A^*)$.

$\Rightarrow A^* \subseteq B(0, \frac{1}{2}\operatorname{diam} A^*)$ ✓

$\Rightarrow \mathcal{L}^n(A^*) \leq \alpha(n)\left(\frac{\operatorname{diam} A^*}{2}\right)^2$.

$\geq$ trivial.

$\Rightarrow$ claim holds.

$\xRightarrow{S_a \text{保} \mathcal{L}^n}$ $\mathcal{L}^n(A) = \alpha(n)\left(\frac{\operatorname{diam} A}{2}\right)^n$.

~~Recall~~:

§ 2.3. Densities.

Recall: $E \subset \mathbb{R}^n$. $\lim_{r\to 0^+} \frac{\mathcal{L}^n(E\cap B(x,r))}{\mathcal{L}^n(B(x,r))} = 1$. a.e. $x \in E$.

For Hausdorff measure?

- $E \subseteq \mathbb{R}^n$.

Consider $\lim_{r\to 0} \frac{\mathcal{H}^s(B(x,r)\cap E)}{\alpha(s) r^s}$ ~~=0~~. $\exists$? $=$?

$\limsup_{r\to 0}$ - - - $=: \mathcal{H}^{s*}_{E}$

$\liminf_{r\to 0}$ - - - - - $=: \mathcal{H}^{s}_{E,*}$

$0 < s < n$.

Thm 2.6 $E \subseteq \mathbb{R}^n$. E is $\mathcal{H}^s$-measurable. $\mathcal{H}^s(E) < \infty$ then.

$$\lim_{r\to 0} \frac{\mathcal{H}^s(B(x,r)\cap E)}{\alpha(s) r^s} = 0 \quad \mathcal{H}^s\text{-a.e. } x \in \mathbb{R}^n \setminus E.$$

Proof: Fix $t>0$. define:

$$A_t = \left\{ x \in \mathbb{R}^n \setminus E \;\middle|\; \limsup_{r\to 0} \frac{\mathcal{H}^s(B(x,r)\cap E)}{\alpha(s) r^s} > t \right\}$$

Want: $\mathcal{H}^s(A_t) = 0$.

$\mathcal{H}^s \llcorner E$ Radon. $\Rightarrow E$ is $\mathcal{H}^s \llcorner E$ Radon Measurable.

$\forall \varepsilon > 0$. $\exists K \subset\subset E$. $\mathcal{H}^s(E \setminus K) \le \varepsilon$.

Set $U = \mathbb{R}^n \setminus K$. open. $A_t \subseteq U$. $\mathcal{H}^s(E\cap U) < \varepsilon$.

Next we construct the covering:

$$\mathcal{F} = \left\{ B(x,r) \;\middle|\; B(x,r) \subseteq U,\ 0 < r < \delta,\ \frac{\mathcal{H}^s(B(x,r)\cap E)}{\alpha(s) r^s} > t \right\}$$

By Vitali Covering thm, there exists a countable disjoint family of balls $\{B_i\}_1^\infty$ in $\mathcal{F}$, such that $A_t \subseteq \bigcup_{i=1}^\infty \hat{B}_i$

$\mathcal{F}$ is a fine covering

$B_i := B(x_i, r_i)$. $\hat{B}_i = 5B_i$, $\bigcup_{i=1} 5B_i$.

Then

$$\mathcal{H}^s_{10\delta}(A_t) \le \sum_{i=1}^\infty \alpha(s)(5r_i)^s \le \frac{5^s}{t}\sum_{i=1}^\infty \mathcal{H}^s(B_i\cap E).$$

diam $\hat{B}_i < 10\delta$.

$$\le \frac{5^s}{t}\mathcal{H}^s(U\cap E) = \frac{5^s}{t}\mathcal{H}^s(E\setminus K) \le \frac{5^s}{t}\varepsilon.$$

$\delta \to 0^+$

$$\Rightarrow \mathcal{H}^s(A_t) \le \frac{5^s}{t}\varepsilon \Rightarrow \mathcal{H}^s(A_t)=0 \quad \forall t>0.$$

□

Rmk: Density $\Leftarrow$ Covering thm $\Leftarrow$ Fine covering.

Thm 2.7. $E\subseteq\mathbb{R}^n$. E $\mathcal{H}^s$-measurable, $\mathcal{H}^s(E)<\infty$. then

$$\frac{1}{2^s} \le \limsup_{r\to 0}\frac{\mathcal{H}^s(B(x,r)\cap E)}{\alpha(s)r^s} \le 1 \qquad \mathcal{H}^s \text{ a.e. } x\in E.$$

It is possible that $0=\liminf \le \limsup < 1$.

eg: Federer §3.3.19

Mattila. 1995.

thm: E $\mathcal{H}^s$ 可测. $\forall\ \mathcal{H}^s$-a.e. $x\in E$. $\limsup = \liminf \Rightarrow E$ 可求长.

Proof:

Step 1: "≤ 1".

Fix $t>1$. $\varepsilon>0$. define

$$B_t = \left\{x\in E \,\middle|\, \limsup_{r\to 0}\frac{\mathcal{H}^s(B(x,r)\cap E)}{\alpha(s)r^s} > t\right\}$$

define

$$\mathcal{F} = \left\{B(x,r) \,\middle|\, B(x,r)\subseteq U,\ 0<r<\delta,\ \frac{\mathcal{H}^s(E\cap B(x,r))}{\alpha(s)r^s} > t\right\}$$

$\mathcal{H}^s \llcorner E$ is a Radon measure $\Rightarrow \exists\ U$ open $\supseteq B_t$, $\mathcal{H}^s(U\cap E) \le \mathcal{H}^s(B_t)+\varepsilon$.

By Vitali covering, $\exists$ countable disjoint $\{B_i\}\subseteq\mathcal{F}$ s.t.

$$B_t \subseteq \bigcup_{i=1}^m B_i \cup \bigcup_{i=m+1}^\infty \hat{B}_i \qquad \forall m\in\mathbb{Z}_+.$$

write $B_i = B(x_i, r_i)$

$$\mathcal{H}^s_{10\delta}(B_t) \le \sum_{i=1}^m \alpha(s) r_i^s + \sum_{i=m+1}^\infty \alpha(s)(5r_i)^s \le \frac{1}{t}\sum_{i=1}^m \mathcal{H}^s(B_i\cap E) + \frac{5^s}{t}\sum_{i=m+1}^\infty \mathcal{H}^s(B_i\cap E)$$

$$\le \frac{1}{t}\mathcal{H}^s(U\cap E) + \frac{5^s}{t}\mathcal{H}^s\Big(\bigcup_{i=m+1}^\infty B_i\cap E\Big) \quad ✓$$

股洪猜想:

将 $\mathcal{H}^s_\delta$ 中 in any set 改成(闭)球. 得一个外测度 S^s_δ.

$\delta \to 0^+$ 得 s-dim 球测度 S^s.

对 S^s 定义 density. 问: $E \subseteq \mathbb{R}^n$. S^s-可测. 是否有对 S^s-a.e. $x \in E$.

$S^s(E) < \infty$. $\lim\limits_{r\to 0} \dfrac{S^s(E \cap B(x,r))}{\alpha(s) r^s} = 1$. ?

这可以导出. 上一定理中取不到 1 的问题是否是 "$\mathcal{H}^s$ 中对 any set 不还是球~~的问题 是A~~" . □

§ 2.4 Functions and Hausdorff measure.

Thm 2.8. Hausdorff measure under Lipschitz maps

(1). $f \in \mathrm{Lip}(\mathbb{R}^n, \mathbb{R}^m)$. $A \subset \mathbb{R}^n$ $0 \le s < \infty$ then

$$\mathcal{H}^s f(A) \le (\mathrm{Lip} f)^s \mathcal{H}^s(A).$$

(2) $n > k$. $P: \mathbb{R}^n \to \mathbb{R}^k$ projection. $A \subseteq \mathbb{R}^n$. $0 \le s < \infty$.

then $\mathcal{H}^s(P(A)) \le \mathcal{H}^s(A)$

□

Graph of Lipschitz functions.

$G(f; A) := \{(x, f(x)) \mid x \in A\} \subseteq \mathbb{R}^n \times \mathbb{R}^m$.

Thm 2.9. $\mathcal{L}^n(A) > 0$. $f: \mathbb{R}^n \to \mathbb{R}^m$.

(1) $\mathrm{H.dim}(G(f;A)) \ge n$

(2) $\mathrm{H.dim}(G(f;A)) = n$ if f lip.

Pf: (1) ✓.

(2). ~~∃~~ $\mathcal{H}^n(G(f;A)) < +\infty$? i.e. $\forall \delta > 0$. $H^s_\delta(G(f;A)) < C < +\infty$.

$\forall k \in \mathbb{N}$. Q: unit cube. subdivide Q into k^n.

$Q_1 \dots Q_{k^n}$. $\mathrm{diam}\, Q_i = \frac{\sqrt{n}}{k}$ $\Rightarrow H^n_\delta(G(f, Q)) \le \alpha(n)\left(\frac{c}{2}\right)^n$ ~~k~~ δ-

$\mathrm{diam}\, f(Q_i) \le \mathrm{lip} f \,\mathrm{diam}\, Q_i < \frac{c}{k}$

$G(f, R)$. covered by $Q_i \times f(Q_i)$. $\mathrm{diam}\, C_i < \frac{c}{k} = \delta$

$m\to\infty$

$\Rightarrow \mathcal{H}^s_{10\delta}(B_t) \le \frac{1}{t}\mathcal{H}^s(U\cap E) \le \frac{1}{t}(\varepsilon + \mathcal{H}^s(B_t))$.

$\delta\to 0$. $\varepsilon\to 0$ $\forall \Rightarrow \mathcal{H}^s(B_t) \le \frac{1}{t}\mathcal{H}^s(B_t) \Rightarrow \mathcal{H}^s(B_t) = 0$ $\forall t>1$ ✓.

Step2: $\limsup_{r\to 0} \dfrac{\mathcal{H}^s_\infty(E\cap B(x,r))}{\alpha(s)\, r^s} \geqslant \dfrac{1}{2^s}$. $\mathcal{H}^s$ a.e. $x\in E$.

↳ 先对 any set 操作，再考虑球：

$\forall \delta>0$. $0<\tau<1$

$$E(\delta,\tau) := \left\{ x\in E \,\middle|\, H^s_\delta(C\cap E) \le \tau\alpha(s)\left(\frac{\operatorname{diam} C}{2}\right)^s. \right\} \quad \forall C\subseteq \mathbb{R}^n,\ x\in C,\ \operatorname{diam} C\le\delta.$$

• Calculate. $H^s_\delta(E(\delta,\tau))$

Suppose. $E(\delta,\tau)$. covered by any set C. $\operatorname{diam} C_j\le\delta$

$E(\delta)\subseteq \bigcup_{i=1}^\infty C_i$.

$C_i\cap E(\delta,\tau)\ne\phi$.

$$\Rightarrow \mathcal{H}^s_\delta(E(\delta,\tau)) \le \sum_{i=1}^\infty \mathcal{H}^s_\delta(C_i\cap E(\delta,\tau))$$
$$\le \sum_{i=1}^\infty \mathcal{H}^s_\delta(C_i\cap E)$$
$$\le \tau\sum_{i=1}^\infty \alpha(s)\left(\frac{\operatorname{diam} C_i}{2}\right)^s.$$

$\Rightarrow \mathcal{H}^s_\delta(E(\delta,\tau)) \le \tau\,\mathcal{H}^s_\delta(E(\delta,\tau))$. $\Rightarrow \mathcal{H}^s_\delta(E(\delta,\tau)) = 0$.

$0<\tau<1 \Rightarrow$. $\mathcal{H}^s_\delta(E(\delta,\tau)) \underset{\vee}{\le} \mathcal{H}^s_\delta(E) \underset{\delta\to 0^+}{\le} \mathcal{H}^s(E) < \infty$

In particular, $\mathcal{H}^s(E(\delta,1-\delta)) = 0$ ...(*).

Now. if $x\in E$ & $\limsup_{r\to 0} \dfrac{\mathcal{H}^s_\infty(B(x,r)\cap E)}{\alpha(s)\, r^s} < \dfrac{1}{2^s}$ then

$$\exists\delta>0 \text{ s.t. } \frac{\mathcal{H}^s_\infty(B(x,r)\cap E)}{\alpha(s)\, r^s} \le \frac{1-\delta}{2^s} \quad \forall 0<r\le\delta.$$

$\therefore \forall x\in C$. $\operatorname{diam} C\le\delta$.

$$\Rightarrow \mathcal{H}^s_\infty(C\cap E) = \mathcal{H}^s_\infty(C\cap E) \le \mathcal{H}^s_\infty(B(x,\operatorname{diam} C)\cap E)$$
$$\le (1-\delta)\,\alpha(s)\left(\frac{\operatorname{diam} C}{2}\right)^s$$

$\Rightarrow x\in E(\delta,1-\delta)$.

$\therefore \{x\in E \mid \limsup(\quad) < \frac{1}{2^s}\} \subseteq \bigcup_{k=1}^\infty E(\frac{1}{k}, 1-\frac{1}{k})$. 即 $\mathcal{H}^s_\infty \le \mathcal{H}^s$ ...... □

Thm. 10. $f \in L^1_{loc}(\mathbb{R}^n)$, $0 \le s < n$. define

$$\Lambda_s = \left\{ x \in \mathbb{R}^n \,\middle|\, \limsup_{r \to 0} \frac{1}{r^s} \underbrace{\int_{B(x,r)} |f|\, dy}_{O(r^{n-s})} > 0 \right\} \quad \text{then } \mathcal{H}^s(\Lambda_s) = 0.$$

依赖于s. 坏点的刻画.

不再只刻画坏点的Lebesgue测度.

<u>Proof</u>:

Step 1: $\frac{1}{r^n}\int_{B(x,r)} |f|\, dy \to |f(x)|$ as $r \to 0^+$.

$\Rightarrow$ $\forall\, 0 \le s < n$.

$$\frac{1}{r^s}\int_{B(x,r)} |f|\, dy = 0.$$

$\Rightarrow \Lambda_s \subseteq \left\{ x \in \mathbb{R}^n \,\middle|\, \limsup_{r\to 0} ⨍_{B(x,r)} |f| > 0 \right\}$

$\Rightarrow \mathcal{L}^n(\Lambda_s) = 0 \quad \forall\, 0 \le s < n.$

Step 2: $\forall \varepsilon > 0$. set.

$$\Lambda_s^\varepsilon = \left\{ x \in \mathbb{R}^n \,\middle|\, \limsup_{r \to 0} \frac{1}{r^s} \int_{B(x,r)} |f|\, dy > \varepsilon \right\}$$

It suffices to prove $\mathcal{H}^s(\Lambda_s^\varepsilon) = 0$. $\Downarrow$ $\mathcal{L}^n(\Lambda_s^\varepsilon) = 0$.

~~Fix~~

By the absolute continuity of $\int$,

$\forall \varepsilon > 0$ $\forall \sigma > 0$. $\exists \eta > 0$. s.t. $\int_U |f| dx < \sigma \Leftarrow \mathcal{L}^n(U) \le \eta$

$\Rightarrow \exists\, U \supseteq \Lambda_s^\varepsilon$. $\mathcal{L}^n(U) < \eta$.

Define $\mathcal{F} = \left\{ B(x,r) \,\middle|\, x \in \Lambda_s^\varepsilon,\ 0 < r < \delta,\ B(x,r) \subseteq U,\ \int_{B(x,r)} |f| dy > \varepsilon r^s \right\}$,

which is a fine cover of Λ_s^ε.

By Vitali Covering thm, $\exists$ disjoint balls $\{B_i\}_1^\infty$ in $\mathcal{F}$ such that $\Lambda_s^\varepsilon \subseteq \bigcup_{i=1}^\infty \hat{B}_i$ radius$(\hat{B}_i) =: r_i$

$\Rightarrow \mathcal{H}^s_{10\delta}(\Lambda_s^\varepsilon) \le \sum_{i=1}^\infty \alpha(s)(5r_i)^s \le \alpha(s) \frac{5^s}{\varepsilon} \sum_{i=1}^\infty \int_{B_i} |f| dy$

$\le \frac{\alpha(s) 5^s}{\varepsilon} \int_U |f| dy \le \frac{\alpha(s) 5^s}{\varepsilon} \cdot \sigma$

$\delta \to 0$

$\sigma \to 0$.

$\varepsilon \to 0$ $\square$.

Ch 3 Area and Co-area formulae

§3.1. Lipschitz functions, Rademacher's thm

~~Ref~~

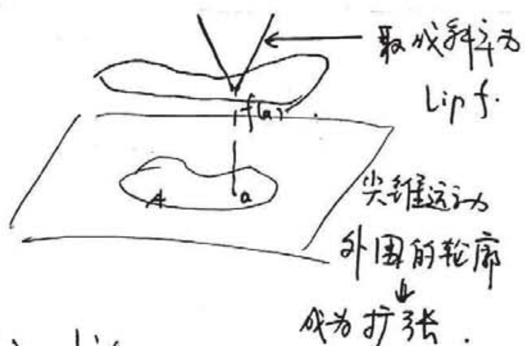
Thm 3.1 (Extension of Lipschitz mappings).

$f: A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$. Lipschitz. then $\exists\ \bar{f}: \mathbb{R}^n \longrightarrow \mathbb{R}^m$. Lipschitz.

s.t. (1) $\bar{f} = f$ on A

(2) $\operatorname{lip}(\bar{f}) \le \sqrt{m}\, \operatorname{lip} f$.

Proof: $\bar{f}(x) := \inf\limits_{a \in A} \{ f(a) + \operatorname{lip}(f)\, |x-a| \}$ $\square$.



Rademacher's Thm

Def: $f: \mathbb{R}^n \to \mathbb{R}^m$ differentiable at $x \in \mathbb{R}^n$.

if $\exists$ linear mapping $L: \mathbb{R}^n \to \mathbb{R}^n$.

s.t. $$\lim_{y \to x} \frac{|f(y) - f(x) - L(y-x)|}{|x-y|} = 0.$$

Thm 3.2 (Rademacher). $f: \mathbb{R}^n \to \mathbb{R}^n$ locally lipschitz.

$\Rightarrow f$ is differentiable $\mathcal{L}^n$ - a.e.

Proof: $n=1$. Trivial.

$n \ge 2$: 方向导数 $\Rightarrow \partial_i \exists \Rightarrow$ 余项一致小 $\Rightarrow$ 可微.

$\partial_v = \nabla \cdot v$

$\forall v \in S^{n-1}$. $D_v f(x) := \lim\limits_{t \to 0} \dfrac{f(x+tv) - f(x)}{t}$ $\quad x \in \mathbb{R}^n$.

Step 1: $D_v f(x)$ $\exists$ $\mathcal{L}^n$-a.e. $\Rightarrow A_v$ Borel

$A_v := \{x \in \mathbb{R}^n \mid D_v f(x) \not\exists\}$. $\quad = \lim\limits_{k \to 0} \sup\limits_{\substack{0<|t|<\frac{1}{k} \\ \cancel{t \in \mathbb{Q}}}} \dfrac{f(x+tv) - f(x)}{t}$.

$= \{x \in \mathbb{R}^n \mid \underline{D}_v f(x) < \overline{D}_v f(x)\}$

Denote:

$P_v := \{ x \mid x \cdot v = 0 \}$

$L_b^v = \{ b + tv \mid t \in \mathbb{R} \}$.

$$\mathcal{L}^n(A_v) = \int_{\mathbb{R}^n} \chi_{A_v} \, dx$$

$$\underset{\text{Tonelli}}{=} \int_{P_v} \left(\int_{\mathbb{R}} \chi_{A_v}(b+tv) \, dt \right) db$$

$$= \int_{P_v} \underbrace{\mathcal{H}^1(A_v \cap L_b^v)}_{0} \, db = 0.$$

v方向导数不存在之点 ⇒ $\mathcal{H}^1$-零测

直线L_b^v上

$\Rightarrow \nabla f \ \exists \ \mathcal{L}^n$-a.e.

Step 2: $D_v f(x) = \nabla f \cdot v$. $\mathcal{L}^n$-a.e.

check: It sufficies to prove. $\forall \zeta \in C_c^\infty(\mathbb{R}^n)$.

$$\int_{\mathbb{R}^n} \underbrace{\left(\frac{f(x+vt) - f(x)}{t} \right)}_{|\cdot| \le \mathrm{Lip}\, f} \zeta(x) \, dx = \int_{\mathbb{R}^n} (\nabla f \cdot v) \cdot \zeta \, dx.$$

$$\int_{\mathbb{R}^n} D_v f(x) \zeta(x) \, dx = -\int_{\mathbb{R}^n} f(x) D_v \zeta(x) \, dx$$

By DCT

$$= -\sum_{i=1}^n v_i \int f(x) \partial_{x_i} \zeta(x) \, dx.$$

$$= \sum_{i=1}^n v_i \int \partial_{x_i} f \, \zeta(x) \, dx.$$

$$= \int (\nabla f \cdot v) \cdot \zeta(x) \, dx.$$

Step 3:

choose $\{v_k\}$ countable dense $\subseteq \partial B(0,1)$.

$\forall k$. $A_k = \{ x \in \mathbb{R}^n \mid D_{v_k} f(x), \ \nabla f(x) \ \exists. \ D_{v_k} f = \nabla f \cdot v_k \}$

$A = \bigcap_{k=1}^\infty A_k$ $\qquad \mathcal{L}^n(\mathbb{R}^n \setminus A) = 0$

Step 3: f is differentiable at $\forall x \in A$

Fix any $x \in A$. choose $v \in \partial B(1)$. $t \in \mathbb{R}\setminus\{0\}$

$$Q(x,v,t) = \frac{f(x+tv)-f(x)}{t} - \nabla f \cdot v.$$

$\forall v, v' \in \partial B(1)$.

$$|Q(x,v,t) - Q(x,v',t)|$$

$$\leq \left|\nabla f \cdot (v-v')\right| + \left|\frac{f(x+tv)-f(x+tv')}{t}\right|$$

$$\leq \operatorname{lip} f\, |v-v'| + |\nabla f|\,|v-v'|$$

$$\leq (\sqrt{n}+1) \operatorname{lip} f\, |v-v'|.$$

Fix $\varepsilon > 0$. chose N. s.t. $\forall \overset{k>N}{\cancel{v\in\partial B(1)}}$. $|v-v_k| < \dfrac{\varepsilon}{2 \operatorname{lip} f(\sqrt{n}+1)}$.

$\exists k \in \{1,2,\dots,n\}$

$Q(x,v_k,t) \to 0$ as $t \to 0$.

$\Rightarrow \exists \delta > 0$.

$$|Q(x,v_k,t)| < \frac{\varepsilon}{2}. \quad \forall\, 0<|t|<\delta.$$

$\Rightarrow \forall v \in \partial B(1)$ $\exists k \in [N]$ s.t. $|Q(x,v,t)| \leq |Q(x,v_k,t)|$

$$+ |Q(x,v,t) - Q(x,v_k,t)| < \varepsilon.$$

~~Now the~~ Note that $\delta > 0$ is uniform for $v \in \partial B(0,1)$ $\forall\, 0<|t|<\delta$.

□

Thm 3.3

(1). $f: \mathbb{R}^n \to \mathbb{R}^m$ locally lipschitz and $Z = \{x \in \mathbb{R}^n \mid f(x) = 0\}$.

then $Df(x) = 0$ for $\mathcal{L}^n$ a.e. $x \in Z$. 不再是开集

Proof: WLOG $m = 1$.

$x \in Z$. so that $Df(x)\ \exists$ & $\lim_{r \to 0} \dfrac{\mathcal{L}^n(Z \cap B(x,r))}{\mathcal{L}^n(B(x,r))} = 1$ ~~$\mathcal{L}^n$-a.~~ $(*)$

$\Rightarrow f(y) = \underbrace{\nabla f(x)}_{a} \cdot (y-x) + o(|y-x|)$ as $y \to x$. $(**)$.

$a := Df(x) \neq 0$.

$S = \{v \in \partial B(1) \mid a \cdot v \geq \frac{1}{2}|a|\}$

$\forall v \in S$. $t > 0$. $y = x + tv$. in $(**)$

$f(x+tv) = a \cdot tv + o(|tv|) \geq \frac{t|a|}{2} + o(t)$ $t \to 0$

$\Rightarrow \exists t_0 > 0$. s.t. $f(x+tv) > 0$ $0 < t < t_0$. $v \in S$.

Contradiction to $(*)$.

(2). $f, g: \mathbb{R}^n \to \mathbb{R}^n$ ~~locally~~ locally lip. and $Y \subset \{x \in \mathbb{R}^n \mid g(f(x)) = x\}$

Then $Dg(f(x))\, Df(x) = I$ for $\mathcal{L}^n$-a.e. $x \in Y$.

([illegible])

$A = \{x \mid Df(x)\ \exists\}$. $B = \{x \mid Dg(x)\ \exists\}$.

$X = Y \cap A \cap f^{-1}(B)$.

$\Rightarrow Y - X \subseteq (\mathbb{R}^n \setminus A) \cup g(\mathbb{R}^n \setminus B)$ $\cdots (***)$

$\uparrow$ $x \in Y - f^{-1}(B) \Longleftrightarrow f(x) \in \mathbb{R}^n \setminus B \Rightarrow x = g(f(x)) \in g(\mathbb{R}^n - B)$.

By Rademacher's thm $\mathcal{L}^n(Y - X) = 0$

Now if $x \in X$. $Dg(f(x))$. $\nabla f(x)\ \exists$

$\Downarrow$

✓ □

§3.3 Area ~~Formula~~ formula.

$n \le m$.

Question 1: Q: unit cube in $\mathbb{R}^n$.

$L: \mathbb{R}^n \to \mathbb{R}^m$ linear mapping.

What is the volume/area of $L(Q)$?

$\mathcal{H}^n(L(Q)) = ?$ 有理由称之为$L(Q)$的体积

$$\sqrt{\det(L^* L)}$$

Question 2: $f: \mathbb{R}^n \to \mathbb{R}^m$ smooth (C^1).

$f(Q)$ volume? 微元法.

Question 3: Lipschitz. a.e. 都对?

极分解. $L: \mathbb{R}^n \to \mathbb{R}^m$ 线性

(1) $n \le m$ $\exists$ symmetric $S: \mathbb{R}^n \to \mathbb{R}^n$. 正交 $O: \mathbb{R}^n \to \mathbb{R}^m$

$L = O \circ S$

(2) $n \ge m$. $\exists$ symmetric $S: \mathbb{R}^m \to \mathbb{R}^m$. $O: \mathbb{R}^m \to \mathbb{R}^n$.

$L = S \circ O^*$. □

O: 旋转 S: 伸缩

Jacobian of L. $[\![L]\!] = |\det S|$.

$[\![L]\!]^2 = \det(L^* \circ L)$ $n \le m$

$[\![L]\!]^2 = \det(L \circ L^*)$. $n \ge m$

. $f: \mathbb{R}^n \to \mathbb{R}^m$. $f = (f^1 \dots f^m)$ Lipschitz.

$Df = (\partial_{x_i} f^j)_{m \times n}$. (切映射).

then. for $\mathcal{L}^n$-a.e. x. $Jf(x) := [\![Df(x)]\!]$

lemma 3.1 : $L: \mathbb{R}^n \to \mathbb{R}^m$ linear. $n \le m$.

Then $\mathcal{H}^n(L(A)) = [\![L]\!]\mathcal{L}^n(A)$. $\forall A \subseteq \mathbb{R}^n$

<u>Proof</u> : $L = O \circ S$. $[\![L]\!] = |S|$.

$[\![L]\!] = 0$. ✓

$[\![L]\!] > 0$.

$$\frac{\mathcal{H}^n(L(B(x,r)))}{\mathcal{L}^n(B(x,r))} = \frac{\mathcal{L}^n(O^* \circ L(B(x,r)))}{\mathcal{L}^n(B(x,r))}$$

$$= \frac{\mathcal{L}^n(O^* \circ O \circ S(B(x,r)))}{\mathcal{L}^n(B(x,r))}$$

$$= \frac{\mathcal{L}^n(S(B(1)))}{\alpha(n)}$$

$$= |S| = [\![L]\!].$$

$\nu(A) := \cancel{\mathcal{H}^n(A)}\ \mathcal{H}^n(L(A))$. $\forall A \subseteq \mathbb{R}^n$.

ν Radon $\ll \mathcal{L}^n$

$$D_{\mathcal{L}^n}\nu(x) = \lim_{r \to 0} \frac{\nu(B(x,r))}{\mathcal{L}^n(B(x,r))} = [\![L]\!]$$

$\Rightarrow \forall B \subseteq \mathbb{R}^n$ ✓ $\overset{\text{Radon}}{\Longrightarrow}$ done.

$\square$.

面积公式:

$f: \mathbb{R}^n \to \mathbb{R}^m$ Lipschitz $n \le m$

then $\forall \mathcal{L}^n$-measurable subset $A \subseteq \mathbb{R}^n$. $\int_A Jf\,dx = \int_{\mathbb{R}^m} \mathcal{H}^0(\overbrace{A \cap f^{-1}\{y\}}^{\text{[illegible]}})\, d\mathcal{H}^n(y)$.

<u>Lemma 3.1</u> $L: \mathbb{R}^n \to \mathbb{R}^m$ linear. $n \le m$. Then $\mathcal{H}^n(L(A)) = [\![L]\!]\mathcal{L}^n(A)$ $\forall A \subseteq \mathbb{R}^n$. □

We've proved $\forall$ ~~Borel sets~~ ball $B \subseteq \mathbb{R}^n$. $\mathcal{H}^n(L(B)) = [\![L]\!]\mathcal{L}^n(B)$.

claim

~~要证~~ $\nu(A) = \mathcal{H}^n(L(A))$ 是Radon测度.

measure ν

Borel: $\operatorname{dist}(A_1, A_2) > 0 \iff \operatorname{dist}(L(A_1), L(A_2)) > 0$

Borel regular: $\mathcal{H}^s$ Borel regular

$\exists$ Borel $B \supseteq L(A)$. $\mathcal{H}^s(B) = \mathcal{H}^s(A)$.

$\tilde{B} = B \cap L(\mathbb{R}^n)$ Borel.

$B \supseteq \tilde{B} \supseteq L(A)$. $\mathcal{H}^s(\tilde{B}) = \mathcal{H}^s(L(A))$

$\tilde{A} = L^{-1}(\tilde{B})$ also <u>Borel in $\mathbb{R}^n$</u>. check...

$\nu(\tilde{A}) = \mathcal{H}^s(\tilde{B}) = \nu(A)$. ☒

Borel B. $\nu \ll \mathcal{L}^n$

$\Rightarrow D_{\mathcal{L}^n}\nu(x) = \lim_{r\to 0} \dfrac{\nu(B(x,r))}{\mathcal{L}^n(B(x,r))} = [\![L]\!]$

$\Rightarrow \forall$ Borel B. $\mathcal{H}^n L(B) = [\![L]\!]\mathcal{L}^n(B)$

□.

lem 3.1 is a special case in Area Formula:

since Jf can be locally considered as a linear mapping

~~f~~ ~~$f = \chi$~~$_{f(A)}$.

<u>lem 3.2</u> $A \subseteq \mathbb{R}^n$ $\mathcal{L}^n$-measurable. Then.

(1) $f(A)$ $\mathcal{H}^n$-measurable.

(2) $y \mapsto \mathcal{H}^0(A \cap f^{-1}\{y\})$ $\mathcal{H}^n$-measurable on $\mathbb{R}^m$.

(3) $\int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}\{y\})\, d\mathcal{H}^n \le (\operatorname{Lip} f)^n \cdot \mathcal{L}^n(A)$

Proof :

Observation : The conclusion of lemma 3.2 (4) holds for finite disjoint sets

(1) If $A \subset\subset \mathbb{R}^n$ then ~~fA~~ $f(A) \subset\subset \mathbb{R}^n \Rightarrow$ Borel $\Rightarrow \mathcal{H}^n$-measurable

(2) If A is a ~~bdd~~ bdd $\mathcal{L}^n$-measurable set, then $\exists K_1 \subset \cdots \subset K_n \subset \cdots$ compact, s.t. $\mathcal{L}^n(A - \bigcup_{i=1}^{\infty} K_i) = 0$.

$$\mathcal{H}^n\left(f(A) - f\left(\bigcup_{i=1}^{\infty} K_i\right)\right) \le \mathcal{H}^n\left(f\left(A - \bigcup_{i=1}^{\infty} K_i\right)\right)$$

$$\le (\mathrm{Lip} f)^n \cdot \mathcal{L}^n\left(A - \bigcup_{i=1}^{\infty} K_i\right) = 0$$

~~(3) If A is unbdd. th~~

(3) If A is ~~an incre~~ an $\mathcal{L}^n$-measurable. choose a increasing sequence done.

(2) $\forall A_1 \cap A_2 = \emptyset$. $A = A_1 \cup A_2$.

$$\mathcal{H}^n(A \cap f^{-1}\{y\}) = \mathcal{H}^n(A_1 \cap f^{-1}\{y\}) + \mathcal{H}^n(A_2 \cap f^{-1}\{y\}).$$

let $B_k := \left\{ Q \;\middle|\; Q = (a_1, b_1] \times \cdots \times (a_n, b_n] \;\middle|\; a_i = \frac{c_i}{2^k},\ b_i = \frac{c_i + 1}{2^k} \quad c_i \in \mathbb{Z} \right\}$

$\mathbb{R}^n = \bigcup_{Q \in B_k} Q$. $\quad g_k = \sum_{Q \in B_k} \chi_{f(A \cap Q)} \quad \leftarrow \mathcal{H}^0$ is counting

$k \to +\infty \quad g_k \uparrow \mathcal{H}^0(A \cap f^{-1}\{y\})$.

By MCT

$$\int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}\{y\})\, d\mathcal{H}^n = \lim_{k \to \infty} \int_{\mathbb{R}^m} g_k \, d\mathcal{H}^n = \lim_{k \to \infty} \sum_{Q \in B_k} \mathcal{H}^n(f(A \cap Q))$$

$$\le \limsup_{k \to \infty} \sum_{Q \in B_k} (\mathrm{Lip}(f))^n \cdot \mathcal{L}^n(A \cap Q)$$

$$= (\mathrm{Lip}(f))^2 \, \mathcal{L}^n(A).$$

Lemma 3 (~~Distortion Estimates~~)
(面积)

$t>1$ $\quad B=\{x \mid Df(x)\ \exists,\ Jf(x)>0\}$
表示 f 在 x 可微

then $\exists \{E_k\}_1^\infty$ of Borel subsets of $\mathbb{R}^n$ s.t.

(1) $B=\bigcup_{k=1}^\infty E_k$

(2) $f|_{E_k}$ 1-1

(3) $\forall k$. $\exists$ symmetric automorphism $T_k: \mathbb{R}^n \to \mathbb{R}^n$ s.t.

$$\mathrm{Lip}\left((f|_{E_k})\circ T_k^{-1}\right) \le t$$

Df 很像 T_k

$$\mathrm{Lip}\left(T_k\circ(f|_{E_k})^{-1}\right) \le t$$

$Df = O\circ S$ $\quad S$ 与 T_k 很像.

$$t^{-n} \le |\det T_k| \le Jf|_{E_k} \le t^n |\det T_k|$$

<u>Proof</u>: choose $\varepsilon>0$ s.t. $\frac{1}{t}+\varepsilon<1<t-\varepsilon$

let C be a countable dense subset of B

S be a countable dense subset of symmetric automorphisms of $\mathbb{R}^n$.

$\forall c\in C$. $T\in S$. $i\in\mathbb{Z}_+$.

$$E(c,T,i)=\left\{b\in B\cap B(c,\tfrac{1}{i}) \;\middle|\; \left(\tfrac{1}{t}+\varepsilon\right)|Tv| \le |Df(b)v| \le (t-\varepsilon)|Tv| \quad (*)\right\}.$$

Borel

即构造集合满足(3)中前2个条件. $\quad |f(a)-f(b)-Df(b)\cdot(a-b)| \le \varepsilon|T(a-b)| \cdots (**)$

因Df Borel.

$\forall a\in B(b,\frac{2}{i})$

$$(*)(**) \Rightarrow \frac{1}{t}|T(a-b)| \le |f(a)-f(b)| \le t|T(a-b)| \quad \cdots (***)$$

$\forall b\in E(c,T,i)$. $a\in B(b,\frac{2}{i})$.

<u>Claim</u>: $b\in E(c,T,i)$. then

$$\left(\tfrac{1}{t}+\varepsilon\right)^n \cancel{\ } |\det T| \le Jf(b) \le (t-\varepsilon)^n|\det T|.$$

If claim holds. then ~~Beta~~ Relabel $\{E(c,T,i) \mid \cdots\}$ as $\{E_k\}_1^\infty$.

$\forall b\in B$ $\quad Df(b)=O\circ S$ $\quad$ choose $T\in S$ s.t. $\mathrm{Lip}(T\circ S^{-1}) \le \dfrac{1}{\frac{1}{t}+\varepsilon}$

Select $i\in\mathbb{Z}_+$. $c\in C$. $|b-c|<\frac{1}{i}$ $\qquad \mathrm{lip}(S\circ T^{-1}) \le t-\varepsilon$

$$|f(a)-f(b)-Df(b)(a-b)| \le \frac{\varepsilon}{\mathrm{Lip}\,T^{-1}}|a-b| \le \varepsilon|T(a-b)| \quad \forall a\in B(b,\tfrac{2}{i}) \Rightarrow b\in E(c,T,i)$$

$\Rightarrow$ (1) holds.

choose any set $E_k = E(e, T, i)$

$T_k = T$.

By $(***)$,

$$\frac{1}{t}|T_k(a-b)| \le |f(a) - f(b)| \le t|T_k(a-b)|. \quad \forall b \in E_k, \ a \in B(b, \tfrac{2}{i}).$$

As $E_k \subseteq B(c, \frac{1}{i}) \subseteq B(b, \frac{2}{i})$

$$\Rightarrow \frac{1}{t}|T_k(a-b)| \le |f(a) - f(b)| \le t|T_k(a-b)| \quad \forall a, b \in E_k.$$

$f|_{E_k}$ 1-1

$\Rightarrow \mathrm{Lip}(f|_{E_k} \circ T_k^{-1}) \le t.$

$\mathrm{Lip}(T_k \circ (f|_{E_k})^{-1}) \le t$ $\Rightarrow \frac{1}{t^n}|\det T_k| \le Jf|_{E_k} \le t^n|\det T_k|.$

Pf of claim:

$Df(b) = L = O \circ S$

$Jf(b) = [\![Df(b)]\!] = |\det S|$.

By $(*)$ $(\frac{1}{t} + \varepsilon)|Tv| \le |(O \circ S)(v)| \le Sv \le (t - \varepsilon)|Tv|$ $\forall v \in \mathbb{R}^n$.

$\Rightarrow (\frac{1}{t} + \varepsilon)|v| \le |(S \circ T^{-1}) v| \le (t-\varepsilon) v$ $\quad v \in \mathbb{R}^n$.

$\Rightarrow (S \circ T^{-1})(B(1)) \subseteq B(t - \varepsilon)$.

$\Rightarrow |\det (S \circ T^{-1})| \alpha(n) \le \mathcal{L}^n B(t-\varepsilon) = \alpha(n)(t-\varepsilon)^n$.

$\Rightarrow [\![S]\!] \le (t-\varepsilon)^n [\![T]\!]$ #

Area Formula:

$f: \mathbb{R}^n \to \mathbb{R}^m$ Lipschitz, $n \le m$. then $\forall\ \mathcal{L}^n$-measurable $A \subseteq \mathbb{R}^n$.

$$\int_A Jf\, dx = \int_{\mathbb{R}^m} \underbrace{\mathcal{H}^0(A \cap f^{-1}\{y\})}_{\text{重数}}\, d\mathcal{H}^n(y).$$

lem 1: $L: \mathbb{R}^n \to \mathbb{R}^m$ linear. then $\mathcal{H}^0(L(A)) = [\![L]\!]\, \mathcal{L}^n(A)$.

lem 2: $A \subseteq \mathbb{R}^n$. $\mathcal{L}^n$-measurable. then.

(1) $f(A)$ $\mathcal{H}^n$-measurable.

(2). $y \mapsto \mathcal{H}^0(A \cap f^{-1}\{y\})$ $\mathcal{H}^n$-measurable on $\mathbb{R}^m$, ~~and~~

(3) $\int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}\{y\})\, d\mathcal{H}^n \le (\mathrm{Lip}\, f)^n \mathcal{L}^n(A)$.

lem 3: $t > 1$. $B = \{x \mid Df(x)\ \exists.\ Jf(x) > 0\}$.

then $\exists\ \{E_k\}_1^\infty$ Borel $\subseteq \mathbb{R}^n$. s.t.

(1) $B = \bigcup_{k=1}^\infty E_k$

(2) $f|_{E_k}$ 1-1

(3) $\forall k$. $\exists$ a symmetric automorphism $T_k: \mathbb{R}^n \to \mathbb{R}^n$ s.t.

$\mathrm{Lip}((f|_{E_k}) \circ T_k^{-1}) \le t$.

$\mathrm{Lip}(T_k \circ (f|_{E_k})^{-1}) \le t$.

$t^{-n} |\det T_k| \le Jf|_{E_k} \le t^n |\det T_k|$.

Date

Proof:

Observation: 定理的形式是可加的: If thm holds for A_i $(A_i \cap A_j = \phi)$ then thm also holds for $\bigcup_{i=1}^{\infty} A_i$.

$$A = \{Df \nexists\} \cup \{Jf(x) > 0\} \cup \{Jf(x) = 0\}$$

① ② ③

①: By Rademacher's thm. $\mathcal{L}^n\{Df \nexists\} = 0$

By lem 2 $\Rightarrow$ $\int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}\{y\}) d\mathcal{H}^n = 0$.

②: $\{Jf(x) > 0\}$.

$\forall t > 0$ WLOG $\{E_k\}_1^\infty$ in lem 3 are disjoint

$A = \bigcup_{k=1}^{\infty} E_k \xRightarrow{\text{WLOG}} A = E_k$ for some k.

~~$\forall l \in \mathbb{Z}_+$ B_l denotes the cube decomposition of $\mathbb{R}^n$.~~

$$\int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}\{y\}) \, d\mathcal{H}^n(y) = \mathcal{H}^n(f(A)) \overset{?}{=} \int_A Jf \, dx.$$

$$\mathcal{H}^n(f(A)) = \mathcal{H}^n(\underbrace{f \circ T^{-1}}_{\text{Lip} \le t} \circ T(A))$$

$$\overset{\text{lem 2}}{\le} t^n \mathcal{H}^n(T(A)) = t^n \mathcal{L}^n(T(A))$$

$$= t^n \mathcal{L}^n(T \circ f^{-1} \circ f(A))$$

$$\le t^{2n} \cdot \mathcal{H}^n(f(A))$$

$$t^{-n} \underbrace{|\det T| \mathcal{L}^n(A)}_{= \mathcal{L}^n(T(A))} \le \int_A Jf \, dx \le t^n \underbrace{|\det T| \mathcal{L}^n(A)}_{= \mathcal{L}^n(T(A))}$$

$$t^{-2n} \cdot \mathcal{H}^n(f(A)) \le \int_A Jf \, dx \le t^{2n} \mathcal{H}^n(f(A)).$$

$t \to 1$ 不能如此操作 因 $A = \bigcup_{k=1} E_k$ 分解依赖于 t.

$$\int_A Jf\,dx \quad \textcircled{1} = \mathcal{H}^n(f(A)).$$

$$= \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}\{y\})\,d\mathcal{H}^n(y).$$

修改：将上述 A 换成 E_k. 再求和. 令 $t \to 1$.

尽管 E_k 依赖于 t, 但 A 并不依赖于 t.

③ $A = \{Jf = 0\}$. It suffices to check $\int_{\mathbb{R}^m} \underbrace{\mathcal{H}^0(A \cap f^{-1}\{y\})}_{\text{Spted in } f(A)} d\mathcal{H}^n(y) = 0$.

$f: \mathbb{R}^n \to \mathbb{R}^m$. $n \le m$. ? $\Uparrow$

$\mathcal{H}^n(f(A)) = 0$.

$\Uparrow$

Define. $g: \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$.

$x \mapsto (f(x), \varepsilon x)$ $\forall \varepsilon > 0$ $\mathcal{H}^n(f(A)) < \varepsilon$.

额外撑出一块 但又任意小.

claim: $0 < Jg(x) \le C_\varepsilon$.

Should the claim holds. $\boxed{\text{WLOG } A \text{ bdd}}$.

By ②, $\int_A Jg\,dx = \int_{\mathbb{R}^m} \underbrace{\mathcal{H}^0(A \cap g^{-1}\{y\})}_{\text{Spted in } g(A)}\,d\mathcal{H}^n(y)$

$\mathbin{\rotatebox[origin=c]{90}{$\geq$}}$ $\ge \int_{\mathbb{R}^m} \chi_{g(A)}\,d\mathcal{H}^n(y)$

$C_\varepsilon \cdot \mathcal{L}^n(A)$.

$\varepsilon \to 0^+$ $= \mathcal{H}^n(g(A))$

check: $f = p \circ g$ $\quad \mathbin{\rotatebox[origin=c]{90}{$\geq$}}$

p: projection. $\mathcal{H}^n(f(A))$

It only remains to prove the claim.

因 $\mathrm{Lip}(p) = 1$ 故有此不等号.

与 ε 无关

$Dg(x) = \begin{pmatrix} Df(x) \\ \varepsilon I \end{pmatrix}_{(n+m)\times n}$. $\quad Dg^*(x) = (Df^*, \varepsilon I)$

$\Rightarrow \underbrace{(Jg(x))^2}_{Dg\cdot Dg^*} = \underbrace{Df^*\cdot Df}_{\text{非负定}} + \underbrace{\varepsilon^2 I}_{\text{正定}}$ ⇒ 严格正定

~~⇒ det $Jg(x) > 0$~~

Jg → 利用Binet-Cauchy公式. $\Rightarrow \det(Dg\cdot Dg^*) > 0$.

→ $\le C\varepsilon^2$.

事实上已用 $Jg(x) = o(1)$.

$\varepsilon \to 0$. $\det(Dg^*\cdot Dg) \to \det(Df^*\cdot Df)$. □

Applications:

· Change of Variables formula:

$f: \mathbb{R}^n \to \mathbb{R}^m$. Lipschitz. $n \le m$. then $\forall$ $\mathcal{L}^n$-summable function $g: \mathbb{R}^n \to \mathbb{R}$.

$$\int_{\mathbb{R}^n} g(x)\, Jf(x)\, dx = \int_{\mathbb{R}^m} \Big(\sum_{x\in f^{-1}\{y\}} g(x) \Big)\, d\mathcal{H}^n(dy)$$

esp: ~~m=n~~. f injective ⇒ change of variable ~~formula~~ formula.

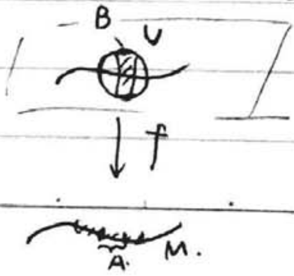
~~Proof~~ □

· Embedded Submanifold:

$M \subseteq \mathbb{R}^m$. Lipschitz continuous & n-dim embedded submanifold.

$U \subseteq \mathbb{R}^n$. $f: U \to M$ a chart of M. $A \subseteq f(U)$ Borel. $B = f^{-1}(A)$

Define $g_{ij} := f_{x_i}\cdot f_{x_j}$ $\quad 1 \le i, j \le n$.



$\Rightarrow (Df)^*\cdot Df = (g_{ij})$

$\Rightarrow Jf = \sqrt{g}$ $\quad g = \det(g_{ij})$

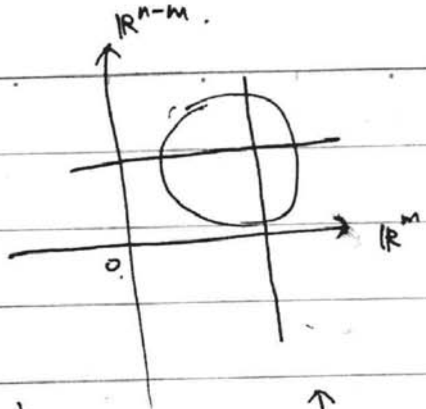
$\Rightarrow \mathcal{H}^n(A) =$ Volume of A in M $= \int_B g^{\frac{1}{2}}\, dx$. □

§3.4 Co-area formula:

Riemann Geo: C^1 + Submersion

Real Analysis: Lipschitz.

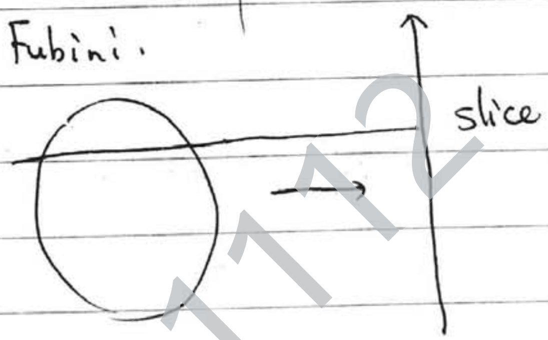
Linear Algebra:



$f:\mathbb{R}^n \to \mathbb{R}^m$. $n \geq m$.

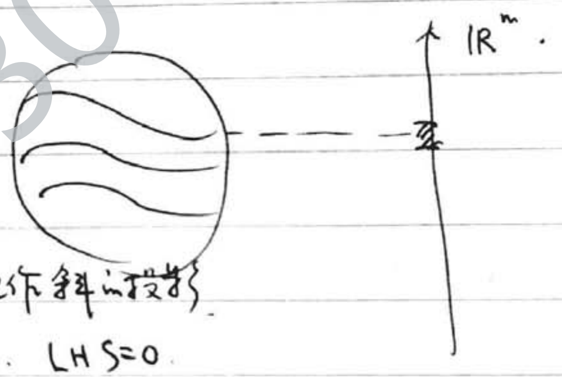
弯曲的 Fubini thm:

Fubini.



* Preliminaries:

Co-area.



lem 3.4 $L:\mathbb{R}^n \to \mathbb{R}^m$ linear. $A \subseteq \mathbb{R}^n$ is $\mathcal{L}^n$-measurable. Then

(1) $y \mapsto \mathcal{H}^{n-m}(A \cap L^{-1}\{y\})$. is $\mathcal{L}^m$-measurable

*(2) $\displaystyle\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap L^{-1}\{y\})\,dy = [\![L]\!]\mathcal{L}^n(A)$.

线性映射视作斜的投影

Case 1: $\dim L(\mathbb{R}^n) < m$. $\Rightarrow [\![L]\!] = 0$. LHS $= 0$

Proof: Case 2: $L = P$ = orthogonal projection of $\mathbb{R}^n$ onto $\mathbb{R}^m$.

$\forall y \in \mathbb{R}^m$. $P^{-1}\{y\}$. $(n-m)$ dim affine subspace of $\mathbb{R}^n$.

By Fubini, $y \mapsto \mathcal{H}^{n-m}(A \cap P^{-1}\{y\})$. $\mathcal{L}^m$ 可测.

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap P^{-1}\{y\})\,dy = \mathcal{L}^n(A).$$

Case 3: $L:\mathbb{R}^n \to \mathbb{R}^m$ $\dim L(\mathbb{R}^n) = m$. $[\![L]\!] > 0$

$L = S \circ O^*$. S 对称 O 正交.

Claim: $O^* = P \circ Q$. $P: \mathbb{R}^n \to \mathbb{R}^m$ 正交投影

$Q: \mathbb{R}^n \to \mathbb{R}^n$ 正交. ✓

$L^{-1}\{0\}$ $n-m$ 维子空间

$L^{-1}\{y\}$ is a translate of $L^{-1}\{0\}$

Recall: Co-area Formula.

$f: \mathbb{R}^n \to \mathbb{R}^m$ Lipschitz. $n \geq m$. $A \subseteq \mathbb{R}^n$.

$$\int_A Jf\,dx = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}\{y\})\, d\mathcal{H}^m(y).$$

lem1 $L: \mathbb{R}^n \to \mathbb{R}^m$ 线性. $A \subseteq \mathbb{R}^n$. $\mathcal{L}^n$ 可测.

(1) $y \mapsto \mathcal{H}^{n-m}(A \cap L^{-1}\{y\})$ $\mathcal{L}^m$-可测.

(2) $\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap L^{-1}\{y\})\, dy = [\![L]\!]\,\mathcal{L}^n(A)$. □

lem2: $f: \mathbb{R}^n \to \mathbb{R}^m$ lipschitz, $A \subseteq \mathbb{R}^n$ $\mathcal{L}^n$-可测. then

(1) $\forall\ \mathcal{L}^m$-a.e. y. $A \cap f^{-1}\{y\}$ $\mathcal{H}^{n-m}$-可测.

(2) $y \mapsto \mathcal{H}^{n-m}(A \cap f^{-1}\{y\})$. $\mathcal{L}^m$-可测

(3). $\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}\{y\})\, d\mathcal{H}^m(y) \leq C\,(\mathrm{Lip} f)^m \mathcal{L}^n(A)$.

observe: lem 2(1) is additive.

(2)(3) 对升链并/降链交封闭.

Proof: (1°) A is compact: (1) trivial.

(2) $y \mapsto \mathcal{H}^{n-m}(A \cap f^{-1}\{y\})$ $\mathcal{L}^m$-measurable.

$\Leftrightarrow \forall t \in \mathbb{R}$. $\{y \mid \mathcal{H}^{n-m}(A \cap f^{-1}\{y\}) \leq t\}$ $\mathcal{L}^m$-measurable

$\Leftarrow$ - - - - - - - - - - - - Borel set ...(*) (要证)

To prove (*). we define $\forall i \in \mathbb{N}$

$U_i = \{y \mid \exists$ a family of open set S_j. $(j = 1, 2, \ldots, l)$. s.t.

$$\begin{cases} A \cap f^{-1}\{y\} \subseteq \bigcup_{j=1}^{l} S_j \\ \mathrm{diam}\, S_j \leq \frac{1}{i} \\ \sum_{j=1}^{l} \alpha(n-m)\left(\frac{\mathrm{diam}\, S_j}{2}\right)^{n-m} < t + \frac{1}{i} \end{cases} \quad \ldots (\#)$$

then $\begin{cases} U_i \text{ open}. \\ \{y \mid \mathcal{H}^{n-m}(A \cap f^{-1}\{y\}) \leq t\} = \bigcap_{i=1}^{\infty} U_i \end{cases} \Longrightarrow$ (*) holds.

① U_i open ?

Suppose not. then. $\exists y \in U$ $\exists z_k \to y$ & $z_k \notin U$.

$\Rightarrow \exists S_1, \dots, S_\ell$ s.t. (#) holds for y.

$z_k \notin U_i \Rightarrow$ (#) does not hold for z_k

especially, $A \cap f^{-1}\{z_k\} \not\subset \bigcup_{i=1}^{\ell} S_i$.

$\Rightarrow \exists x_k \in A$. $x_k \notin \bigcup_{j=1}^{\ell} S_j$.

A compact. WLOG $x_k \to x_\infty \in A$, $f(x_k) = z_k$ $k \to \infty \Rightarrow f(x_\infty) = y$.

$\Rightarrow x_\infty \in A \cap f^{-1}\{y\}$.

But $x_k \notin \bigcup_{j=1}^{\ell} S_j$ open. $\Rightarrow x_\infty \notin \bigcup_{j=1}^{\ell} S_j$. contradicts with $A \cap f^{-1}\{y\} \subset \bigcup_{j=1}^{\ell} S_j$.

Thus U_i open. ✓

② $\{y \mid \mathcal{H}^{n-m}(A \cap f^{-1}\{y\}) \le t\} \stackrel{?}{=} \bigcap_{i=1}^{\infty} U_i$.

$\subseteq$: $\forall \delta > 0$. $\mathcal{H}_\delta^{n-m}(A \cap f^{-1}\{y\}) \le t$ ...... (*)

By (*), then $\exists$ a covering $\{S_j\}_1^\infty$ of $A \cap f^{-1}\{y\}$. s.t.

~~注意~~: diam $S_j < \delta$

$\sum_{j=1}^{\infty} \alpha(n-m) \left(\frac{\operatorname{diam} S_j}{2}\right)^{n-m} < t + \frac{1}{i}$.

WLOG S_j open.

By finite covering thm. $\exists S_1 \dots S_\ell$. cover

$\Rightarrow y \in U_i$ $\forall i$.

$\Rightarrow \subseteq$ holds.

$\supseteq$: $y \in \bigcap_{i=1}^{\infty} U_i$. $\Rightarrow \forall i$. $\mathcal{H}_{\frac{1}{i}}^{n-m}(A \cap f^{-1}\{y\}) < t + \frac{1}{i}$.

$\Rightarrow \mathcal{H}^{n-m}(A \cap f^{-1}\{y\}) \le t$ ✓

(2) ✓

(3): Cover A with a $\{B_i^j\}_{i=1}^{\infty}$, diam $B_i^j \le \frac{1}{j}$. $\sum_{i=1}^{\infty} \mathcal{L}^n(B_i^j) \le \mathcal{L}^n(A) + \frac{1}{j}$.

Define. $g_i^j = \alpha(n-m) \left(\frac{\operatorname{diam} B_i^j}{2}\right)^{n-m} \chi_{f(B_i^j)}$.

g_i^j $\mathcal{L}^m$-可测.

"KEY": $\sum_{i=1}^{\infty} g_i^j(y) \ge \mathcal{H}_{\frac{1}{j}}^{n-m}(A \cap f^{-1}\{y\})$.

Calculate

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A\cap f^{-1}\{y\})\,dy = \int_{\mathbb{R}^m} \lim_{j\to\infty} \mathcal{H}^{n-m}_{\frac{1}{j}}(A\cap f^{-1}\{y\})\,dy$$

$$\le \int_{\mathbb{R}^m} \liminf_{j\to\infty} \sum_{i=1}^{\infty} g_i^j \, dy \qquad \cdots (*2)$$

$$\overset{\text{Fatou}}{\le} \liminf_{j\to\infty} \int_{\mathbb{R}^m} \sum_{i=1}^{\infty} g_i^j \, dy$$

$$= \liminf_{j\to\infty} \int_{\mathbb{R}^m} \sum_{i=1}^{\infty} \alpha(n-m)\left(\frac{\operatorname{diam} B_i^j}{2}\right)^{n-m} \chi_{f(B_i^j)}$$

$$\overset{\text{MCT}}{\le} \liminf_{j\to\infty} \int_{\mathbb{R}^m} \alpha(n-m)\left(\frac{\operatorname{diam} B_i^j}{2}\right)^{n-m} \alpha(m)\left(\frac{\operatorname{diam} f(B_i^j)}{2}\right)^{m}$$

~~$\mathcal{H}$~~ Isometric Inq

$$\le \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} (\operatorname{Lip} f)^m \liminf_{j\to\infty} \sum_{i=1}^{\infty} \mathcal{L}^n(B_i^j)$$

$$\le \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} (\operatorname{Lip} f)^m \mathcal{L}^n(A)$$

(3) holds.

(2°) . compact set $\xrightarrow{\delta\text{-扩}}$ open set $\xrightarrow{\cap}$ G_δ set . ✓.

· 要证: $\forall\ \mathcal{L}^n$-a.e. y $A\cap f^{-1}\{y\}$ $\mathcal{H}^{n-m}$-可测.

反证若 ↑

~~Sup~~ $\exists \eta>0,\ \delta>0$

$$\mathcal{L}^n\{y \mid \mathcal{H}^{n-m}(A\cap f^{-1}\{y\}) \ge \eta\} > \delta$$

$\mathcal{L}^n(A)=0 \Rightarrow \exists$ open $U \supseteq A$ $\mathcal{L}^n(U) \le ((C\operatorname{Lip}(f))^m)^{-1} \cdot \frac{\cancel{4}\delta\eta}{2}$.

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(U\cap f^{-1}\{y\})\,d\mathcal{H}^n \le \frac{\delta\eta}{2}$$

$$\mathcal{H}^{n-m}(A\cap f^{-1}(y)) \ge \eta\cdot\delta$$

□

Lem 3. $t>1$. $h:\mathbb{R}^n\to\mathbb{R}^n$ Lipschitz

$B=\{x \mid Dh(x)\ \exists.\ Jh(x)>0\}$

Then. $\exists \{D_k\}_1^\infty$ Borel s.t.

(1) $\mathcal{L}^n(B-\bigcup_{k=1}^\infty D_k)=0$.

(2) $h|_{D_k}$ 1-1. $\forall k$

(3). $\forall k$. $\exists$ 对称自同~~构~~态 $S_k:\mathbb{R}^n\to\mathbb{R}^n$ s.t.

$\mathrm{Lip}(S_k^{-1}\circ(h|_{D_k}))\le t$.

$\mathrm{Lip}((h|_{D_k})^{-1}\circ S_k)\le t$.

$\frac{1}{t^n}|\det S_k|\le Jh|_{D_k}\le t^n|\det S_k|$.

$\square$.

Sketch of the Proof of Co-area formula:

$A=\{Df\ 非\}\cup\{Df=0\}\cup\{Df>0\}$

· $A=\{Df>0\}$. $A=\bigcup_{i=1}^\infty D_i$. $\int_A Jf\,dx \sim \int_{\mathbb{R}^m}\mathcal{H}^{n-m}(A\cap f^{-1}\{y\})\,d\mathcal{H}^m$.

$\updownarrow$ 近似

$L^{-1}(y)$.

· $f:\mathbb{R}^n\to\mathbb{R}^m$. 分解 $f=g\circ h$.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "f"] \arrow[d, "h"] & \mathbb{R}^m \\ \mathbb{R}^n \arrow[ur, "g\ \text{projection}"'] & \end{tikzcd}$$

$\int \mathcal{H}^{n-m}(D\cap f^{-1}\{y\})$

$=\int \mathcal{H}^{n-m}(D\cap h^{-1}\circ g^{-1}\{y\})$

作线性逼近

说明 该引理是为了将 h 用 L 线性逼近.

$=\int \mathcal{H}^{n-m}(h^{-1}(h(D)\cap g^{-1}(y)))$

$=\int \mathcal{H}^{n-m}(h^{-1}\circ S\circ S^{-1}(h(D)\cap q^{-1}(y)))$

$\sim\int \mathcal{H}^{n-m}(S^{-1}h(D)\cap S^{-1}\circ q^{-1}(y))$

$\sim [\![q\circ S]\!]\,\mathcal{L}^n(D)$.

$\boxtimes$

Date

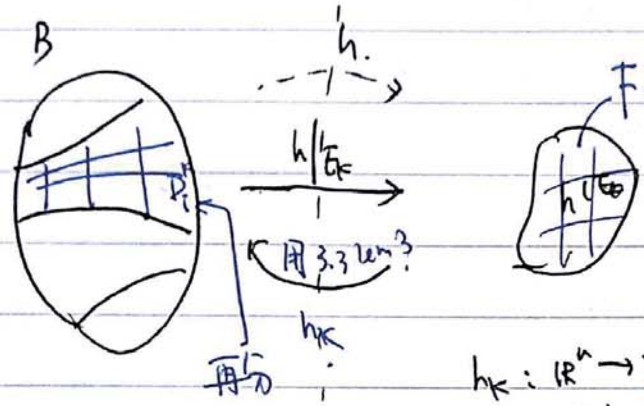
pf of lem3: 由3.3节 lem3 on h, to find Borel sets $\{E_k\}$.

(1) $\mathbb{R} = \bigcup_{k=1}^{\infty} E_k$ (2) $h|_{E_k}$ 1-1. (3) ~~Lipschitz~~ ...

$\mathrm{Lip}(h|_{E_k} \circ T_k^{-1}) \le t$

$\mathrm{Lip}(T_k \circ (h|_{E_k})^{-1}) \le t.$

$|\det T_k| \overset{t}{\approx} Jh|_{E_k}.$



和之前2次用lem3 in §3.3.

$h_k : \mathbb{R}^n \to \mathbb{R}^n$

$h_k = (h|_{E_k})^{-1}$ on $h(E_k)$.

Claim: h_k lipschitz.

$\mathrm{Lip}(T_k \circ h_k) \le t \Rightarrow \mathrm{Lip}(h_k) \le \mathrm{Lip}(T_k^{-1})\,\mathrm{Lip}(T_k \circ h_k)$

再延拓到 $\mathbb{R}^n$ 上. 取 "B" 为 $h(E_k)$ "f" 为 h_k.

用 §3.3 lem 3. $h(E_k) = \bigcup_{i=1}^{\infty} F_i^k$. Borel. ∃ F_i^k Borel. $\{R_j^k\}$ symmetric 同构

s.t. $\mathcal{L}^n\left(h(E_k) - \bigcup_{j=1}^{\infty} F_j^k\right) = 0$

$h_k|_{F_j^k}$ 1-1.

$\mathrm{Lip}((h_k|F_j^k) \circ (R_j^k)^{-1}) \le t.$

$\mathrm{Lip}(R_j^k \circ (h_k|F_j^k)^{-1}) \le t.$

$\frac{1}{t^n}|\det R_j^k| \le Jh_k|_{F_j^k} \le t^n |\det R_j^k|.$

$D_j^k = E_k \cap h^{-1}(F_j^k)$. $S_j^k := R_j^{k-1}$.

Direct calculation yields $\mathcal{L}^n(B - \bigcup_{j,k} D_j^k) = 0$.

{ ...
...

□

Recall:

Thm 3.10 Co-area Formula:

$f: \mathbb{R}^n \to \mathbb{R}^m$. Lipschitz continuous. $n \geq m$.

$\forall A \subseteq \mathbb{R}^n$. $\mathcal{L}^n$-measurable. $\displaystyle\int_A Jf\, dx = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}\{y\})\, dy$.

lem 1: $L: \mathbb{R}^n \to \mathbb{R}^m$. linear. $A \subseteq \mathbb{R}^n$. $\mathcal{L}^n$-measurable $\Rightarrow$
(1). $y \mapsto \mathcal{H}^{n-m}(A \cap L^{-1}\{y\})$. $\mathcal{L}^m$-measurable.
(2) $\displaystyle\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap L^{-1}\{y\})\, dy = [\![L]\!]\, \mathcal{L}^n(A)$.

lem 2: $A \subseteq \mathbb{R}^n$, $\mathcal{L}^n$-measurable. $n \geq m$. Then

(1) $A \cap f^{-1}\{y\}$ $\mathcal{H}^{n-m}$-measurable. $\mathcal{L}^m$-a.e. y

(2) $y \mapsto \mathcal{H}^{n-m}(A \cap f^{-1}\{y\})$ $\mathcal{L}^m$-measurable.

(3). $\displaystyle\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}\{y\})\, dy \leq \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} (\mathrm{Lip}\, f)^m \mathcal{L}^n(A)$.

lem 3: $t > 1$. $h: \mathbb{R}^n \to \mathbb{R}^n$ Lipschitz. → 近似于矩

$B = \{x \mid Dh(x)\ \exists.\ Jh(x) > 0\}$. h. h^{-1} Lip constant.

$\Rightarrow \exists \{D_k\}_1^\infty$ Borel $\subseteq \mathbb{R}^n$, s.t.

(1) $\mathcal{L}^n(B - \bigcup_{k=1}^\infty D_k) = 0$

(2) $h|_{D_k}$ 1-1. $\forall k \in \mathbb{Z}_+$.

(3) $\forall k$. $\exists$ symmetric automorphism $S_k: \mathbb{R}^n \to \mathbb{R}^n$.

$\mathrm{Lip}(S_k^{-1} \circ (h|_{D_k})) \leq t$.

$\mathrm{Lip}((h|_{D_k})^{-1} \circ S_k) \leq t$.

$t^{-n} |\det S_k| \leq Jh|_{D_k} \leq t^n |\det S_k|$

来自于:
希望将f分解作 $f = g \circ h$.

$f: \mathbb{R}^n \to \mathbb{R}^m$, $g: \mathbb{R}^n \to \mathbb{R}^m$, $\boxed{h}: \mathbb{R}^n \to \mathbb{R}^n$ $\exists$?

Df $m \times n$矩阵. $Jf > 0$ 满秩
⇓
必有m个列向量线性无关.
n取m. 有 C_n^m 种方法.
其中一种记作 λ.

若 $x \in A$. 使"λ列"线性无关.

记为 $x \in A_\lambda$ (不一定不交).

$A \subseteq \bigcup_\lambda A_\lambda = \bigsqcup_\lambda \tilde{A}_\lambda$ 两两不交 (shrink if necessary).

Proof of 3.10:

Observation: additivity:

不妨对某个λ证明

$A = \{f\text{不可微}\} \cup \{Jf > 0\} \cup \{Jf = 0\}$

↓ Rademacher. 主要. 扰动

$\mathcal{L}^n$-zero measure.

不用考虑.

λ取成前m列

$Df = \Big(\underbrace{\boxed{m \times m}\ \ \cdots}_{n} \Big)\ m$ 线性无关.

证 (1)

WLOG $A \subseteq \{Jf > 0\}$

取入如前所述（前 m 列）.

WLOG $A = A_\lambda$

再令 $h: \mathbb{R}^n \to \mathbb{R}^n$ Lipschitz.

$x \longmapsto (f(x), x_{m+1}, \dots, x_n)$

$$Dh = \begin{pmatrix} \boxed{Df} \\ 0_{(n-m)\times m} \quad I_{m\times m} \end{pmatrix}_{m\times n} \text{ invertible. (计算行列式).}$$

$\underset{\sim}{>0}$ 为了让 h 有 lemma 中的 "$Dh>0$" 性质. 用 lemma

再令 natural projection:

$g: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ $\qquad f = g \circ h$. √

$(x_1 \dots x_n) \longmapsto (x_1 \dots x_m)$.

Apply lemma to A & h, $\forall t > 0$ we have $\mathcal{L}^n(A - \bigcup_{k=1}^{\infty} D_k) = 0$

→ 在每个 D_k 上运用 co-area formula 再对 k 求和. 再 $t \to 1^+$.

$$\int_{\mathbb{R}^n} \mathcal{H}^{n-m}(D_k \cap f^{-1}\{y\})\,dy.$$

$$= \int_{\mathbb{R}^n} \mathcal{H}^{n-m}(D_k \cap (h^{-1} \circ g^{-1})\{y\})\,dy$$

$$= \int_{\mathbb{R}^n} \mathcal{H}^{n-m}\Big(\underbrace{h^{-1}\big(h(D_k) \cap g^{-1}\{y\}\big)}_{\text{利用线性运算}}\Big)\,dy$$

$$= \int_{\mathbb{R}^n} \mathcal{H}^{n-m}\Big(\underbrace{h^{-1} \circ S_k}_{\text{Lip} \le t} \circ \big(\underbrace{S_k^{-1}(h(D_k))}_{\text{Lip} \le t} \cap S_k^{-1} \circ g^{-1}\{y\}\big)\Big)\,dy$$

$$\overset{t}{\approx} \int_{\mathbb{R}^n} \mathcal{H}^{n-m}\big(S_k^{-1}(h(D_k)) \cap \underbrace{S_k^{-1} \circ g^{-1}}_{\text{线性. 用 lem 1.}}\{y\}\big)\,dy$$

$\in (t^{-1}, t)$.

$$= [\![g \circ S_k]\!] \cdot \mathcal{L}^n(\underbrace{S_k^{-1}}_{\text{Lip}\le t} \circ h(D_k)).$$

$$\overset{t}{\approx} \underline{[\![g \circ S_k]\!]\, \mathcal{L}^n(D_k)}.$$

Claim: $[\![g\circ S_k]\!] \overset{t}{\sim} Jf\big|_{D_k}$.

If claim holds, then

$[\![g\circ S_k]\!]\mathcal{L}^n(D_k) \overset{t}{\sim} \int_{D_k} Jf\,dx$. ~~$t\to 1^+$~~ done.

Sum over $k\in\mathbb{Z}^+$.

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A\cap f^{-1}\{y\})\,dy \overset{t}{\sim} \int_A Jf\,dx.$$

A 与 $\bigcup_{k=1}^{\infty} D_k$ 仅差零测集.

左、右不依赖 t.

令 $t\to 1^+$. done.

$Df = g\cdot Dh$

$= g\circ S_k\circ \underbrace{S_k^{-1}\circ Dh}_{w}$. 希望 w 像一个正交变换

$\mathrm{Lip}\, w \overset{t}{\sim} 1$.

任取 $\Omega\subseteq\mathbb{R}^n$, 由 lem1.

$$[\![g\circ S_k\circ w]\!]\mathcal{L}^n(\Omega) = \int_{\mathbb{R}^m}\mathcal{H}^{n-m}(\Omega\cap w^{-1}\circ S_k^{-1}\circ g^{-1}(y))\,dy$$

$$= \int_{\mathbb{R}^m}\mathcal{H}^{n-m}(w^{-1}(w(\Omega)\cap S_k^{-1}\circ g^{-1}(y)))\,dy$$

$$\overset{t}{\sim} \int_{\mathbb{R}^n}\mathcal{H}^{n-m}(w(\Omega)\cap S_k^{-1}\circ g^{-1}(y))\,dy$$

$$\overset{\text{lem1}}{=} [\![g\circ S_k]\!]\mathcal{L}^n(w(\Omega)).$$

$$\overset{t}{\sim} [\![g\circ S_k]\!]\mathcal{L}^n(\Omega).$$

Pf of claim:

$f = g\circ h \Rightarrow Df = g\circ Dh$

$= g\circ S_k\circ S_k^{-1}\circ Dh$

$= g\circ S_k\circ \underbrace{D(S_k^{-1}\circ h)}_{=C}$

$\forall x\in D_k$, $\forall v\in\mathbb{R}^n$.

$S_k^{-1}h(x+v) - S_k^{-1}h(x) = S_k^{-1}Dh(x)\cdot v + o(|v|)$.

$|S_k^{-1}\cdot Dh(x)v|$

$\le |S_k^{-1}(h(x+v)-h(x))| + o(|v|)$

$\le 2t|v|$. $|v|\ll 1$.

$|S_k^{-1}\cdot Dh(v)|$

$\ge |S_k^{-1}(h(x+v)-h(x))| - o(|v|)$

$\ge \frac{1+\frac{1}{t}}{2}|v|$ $|v|\ll 1$.

By lem3

$\frac{1}{t}\le \mathrm{lip}(S_k^{-1}\circ h) = \mathrm{lip}\, C \le t$. on D_k. $\forall k\in\mathbb{Z}_+$.

~~It suffices to show $\det(S_k^{-1}\circ Dh)\overset{t}{\sim} 1$.~~ 作极分解

$Df = S\circ O^*$. $g\circ S_k = T\circ P^*$.

$S\circ O^* = T\circ P^*\circ C \Rightarrow S = T\circ P^*\circ C\circ O \Rightarrow \det S\neq 0$.

$\forall v\in\mathbb{R}^m$ $|T^{-1}\circ Sv| = |P^*\circ C\circ Ov|$

$\le |C\circ Ov| \le t|Ov| = t|v|$.

$\Rightarrow (T^{-1}\circ S)(B(1)) \subseteq B(t)$

$\Rightarrow \underset{\overset{\shortparallel}{Jf}}{|\det S|} \le t^n \underset{\overset{\shortparallel}{\det(T\circ P^*)}}{|\det T|} = t^n[\![g\circ S_k]\!]$.

Similarly $\det S \ge t^{-n}[\![g\circ S_k]\!]$

$\Rightarrow [\![g\circ S_k]\!] \overset{t}{\sim} Jf\big|_{D_k}$ ✓

(2°) $A=\{Jf=0\}$ $\forall\ 0<\varepsilon\le 1$

$$g(x,y)=f(x)+\varepsilon y$$
$$p(x,y)=y.$$

$\forall x\in\mathbb{R}^n,\ y\in\mathbb{R}^m.$ $Dg=(Df,\ \varepsilon I).$

$$\varepsilon^m\le Jg=[\![Dg]\!]=[\![Dg^*]\!]\le C\varepsilon.$$

observe.

$$\int_B Jg\,dx\,dy\,dw=\int_{\mathbb{R}^m}\mathcal{H}^n(B\cap g^{-1}\{y\})\,dy.$$

claim. $\forall\,\Omega\subseteq\mathbb{R}^{n-m}$, lebesgue 可测 ? $c\,\mathcal{H}^n(\Omega)\ge\int_{\mathbb{R}^n}\mathcal{H}^{n-m}(\Omega\cap p^{-1}(w))\,dw$ (用在 $\Omega=B\cap g^{-1}\{y\}$).

If claim holds,

$$B\cap g^{-1}(y)\cap p^{-1}(w)=\begin{cases}\phi & w\notin B_1.\\ A\cap f^{-1}(y-\varepsilon w). & w\in B_1.\end{cases}$$

$$\int_{\mathbb{R}^m}\mathcal{H}^n(B\cap g^{-1}(y))\,dy\ge c\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\mathcal{H}^{n-m}(B\cap g^{-1}(y)\cap p^{-1}\{w\})\,dw\,dy$$

$\|$

$$\int_B Jg\,dx\,dw\ \le C\varepsilon.$$

$$\overset{\text{Fubini}}{=}c\int_{\mathbb{R}^n}\int_{\mathbb{R}^m}\mathcal{H}^{n-m}(A\cap f^{-1}(\underbrace{y-\varepsilon w}_{平移}))\,dw\,dy$$

$$=c\int_{\mathbb{R}^m}\mathcal{H}^{n-m}(A\cap f^{-1}\{y\})\,dy$$

$$=\boxed{\int_{\mathbb{R}^m}\mathcal{H}^{n-m}(A\cap f^{-1}\{y\})\,dy}\ \ /\!/\,?$$

check 条件.

可测性. 先设B紧 ⇒ ✓. 一般可测集 ⇒ 逼近

done { claim ?

结论. $\mathcal{H}^n(\Omega)=\int_{\mathbb{R}^n}\mathcal{H}^{n-m}(\Omega\cap p^{-1}\{w\})\,dw$ 相当类似 Fubini. Ω Borel $\subseteq\mathbb{R}^{n-m}$

Fact: $\forall\ V\subseteq\mathbb{R}^m$. $\exists\ c$. s.t. $\mathcal{H}^n(U\times V)=c\,\mathcal{H}^{n-m}(U)\,\mathcal{L}^m(V)$, $U\subseteq\mathbb{R}^{n-m}$, c只与V有关.

V: 可求长时 c=1. 但 $\exists V\subset\subset\mathbb{R}^m$. c>1 (存在反例). c<1 未知.

希望: $\mathcal{H}^n=\mathcal{H}^{n-m}\times\mathcal{L}^n$, $\mathbb{R}^n=\mathbb{R}^{n-m}\times\mathbb{R}^m$. 但并不对. (Federer) §2.10.45~46

更进一步 $\mathcal{H}^{s+t}=\mathcal{H}^s\times\mathcal{H}^t$. $\mathbb{R}^n=\mathbb{R}^{n-m}\times\mathbb{R}^m$ 更不对.

~~Proof of~~

Proof of claim is similar to lem2.

lem2 (取了to projection). $\forall \mathcal{L}^n$ - measurable set $A \subseteq \mathbb{R}^n$; $\exists C$.

$$C(\underbrace{\mathrm{Lip}\, f}_{1})^m \mathcal{L}^n(A) \geq \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap 第\, p^{-1}\{w\})\, dw.$$

Steps: $A \underset{?}{\xrightarrow{\text{紧来}}} \xRightarrow{MCT}$ 升链并 $\Rightarrow \ldots$ ⑧

降链 哎.

$\forall j \in \mathbb{Z}_+$. Find. ~~A by choice~~ a covering of A $\{B_i^j\}$ (closed ball), $\mathrm{diam}\, B_i^j < \frac{1}{j}$.

$$\sum_{i=1}^{+\infty} \mathcal{L}^n(B_i^j) \leq \mathcal{L}^n(A) + \frac{1}{j}.$$

只有不交球有界时 lem2 $\longrightarrow \exists B_i^j$

$$\sum_{i=1}^{\infty} \alpha(n)\left(\frac{\mathrm{diam}\, B_i^j}{2}\right)^n \leq \mathcal{H}^n_{\frac{1}{j}}(\Omega) + \frac{1}{j}.$$

$$g_i^j(y) = \sum_{i=1}^{\infty} \alpha(n-m)\left(\frac{\mathrm{diam}\, B_i^j}{2}\right)^{n-m} \chi_{P(B_i^j)}.$$

$$\mathcal{H}^{n-m}_{\frac{1}{j}}(A \cap f^{-1}\{y\}) \leq \sum_{i=1}^{\infty} g_i^j(y).$$

$$\Rightarrow \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}\{y\})\, dy \leq \int_{\mathbb{R}^m} \liminf_{j\to\infty} \sum_{i=1}^{\infty} g_i^j(y)$$

$$\overset{\text{Fatou}}{\leq} \liminf_{j\to\infty} \sum_{i=1}^{\infty} \alpha(n-m)\cdot\left(\frac{\mathrm{diam}\, B_i^j}{2}\right)^{n-m} \mathcal{L}^m(P(B_i^j)).$$

MCT

而 $\mathcal{H}^n(B_i) = \alpha(m)\left(\frac{\mathrm{diam}\, B_i^j}{2}\right)^m$.

$\leq \mathcal{H}^m(B_i^j)$ ~~Lip P≤1~~ 因 $\mathrm{Lip}\, P \leq 1$

~~P限2 下紧区域~~

$\leq_{m.n.}$ $\mathcal{H}^n(A)$.

紧 $\Rightarrow$ Borel $\Rightarrow \mathcal{H}^n$-可测. □

以下 co-area formula 证明.

Thm 3.11

Pf: $g = \chi_A$

$$\int g Jf\, dx = \int_{\mathbb{R}^m} \left(\int_{f^{-1}\{y\}} g \, d\mathcal{H}^{n-m} \right) dy.$$

$$\chi_A \Rightarrow \text{Simple functions} \overset{\text{MCT}}{\Rightarrow} \text{positive } L^1 \Rightarrow f = f^+ - f^-. \qquad \square$$

注记:

~~Recall~~

Thm 3.11. Change of Variables formula:

$f: \mathbb{R}^n \to \mathbb{R}^m$ Lipschitz. $n \geq m$.

then for $\mathcal{L}^n$-summable function $g: \mathbb{R}^n \to \mathbb{R}$.

(1) $g|_{f^{-1}\{y\}}$ $\mathcal{H}^{n-m}$ summable. $\mathcal{L}^m$-a.e. y

(2) $$\int_{\mathbb{R}^n} g Jf \, dx = \int_{\mathbb{R}^m} \int_{f^{-1}\{y\}} g \, d\mathcal{H}^{n-m} \, dy \qquad \square$$

3.12. Polar coordinates

$g: \mathbb{R}^n \to \mathbb{R}$. $\mathcal{L}^n$-summable.

then $$\int_{\mathbb{R}^n} g \, dx = \int_0^\infty \int_{\partial B(r)} g \, d\mathcal{H}^{n-1} \, dr.$$

与流形课中曲面积分上的积分一样. (面积公式).

check

在 co-area 公式中. $f(x) = |x|$ $\square$

3.13. Integration over level sets.

$f: \mathbb{R}^n \to \mathbb{R}$. Lipschitz.

(1). $$\int_{\mathbb{R}^n} |Df| \, dx = \int_{-\infty}^{+\infty} \mathcal{H}^{n-1}\{f = t\} \, dt \qquad (\text{令 } g \equiv 1). \text{ In } 3.11$$

(2) If additionally. $\operatorname{ess\,inf} |Df| > 0$. $g: \mathbb{R}^n \to \mathbb{R}$ $\mathcal{L}^n$-summable. $$\Rightarrow \int_{\{f > t\}} g \, dx = \int_t^{\infty} \left(\int_{\{f = s\}} \frac{g}{|Df|} \, d\mathcal{H}^{n-1} \right) ds$$

(3) In particular:

$$\frac{d}{dt} \int_{\{f > t\}} g \, dx = - \int_{\{f = t\}} \frac{g}{|Df|} \, d\mathcal{H}^{n-1}. \qquad \mathcal{L}^1\text{-a.e. } t$$

(1) $Jf = |Df|$

(2) 记 $E_t := \{f > t\}$.

$$\int_{\{f>t\}} g\,dx = \int_{\mathbb{R}^n} \chi_{\{f>t\}} \frac{g}{|Df|} Jf\,dx$$

$$= \int_{-\infty}^{+\infty} \left(\int_{\partial\{f>t\}} \frac{g}{|Df|} \cancel{Jf}\, \chi_{E_t}\, d\mathcal{H}^{n-1} \right) ds$$

$$= \int_t^{\infty} \left(\int_{\partial\{f>s\}} \frac{g}{|Df|}\, d\mathcal{H}^{n-1} \right) ds$$

(3) 注: $Jf > 0 \Rightarrow \{f = t\} = \partial\{f>s\}$ 参(2).

#.

Thm 3.14 $\emptyset \neq K \subset\subset \mathbb{R}^n$

$d(x) := \mathrm{dist}(x, K) \quad x \in \mathbb{R}^n$.

$\Rightarrow \forall\, 0 < a < b$. $\int_a^b \mathcal{H}^{n-1}\{d = t\}\, dt = \mathcal{L}^n\{a \le d \le b\}$.

类似(1)(2)上 ~~其~~ n维对于距离函数.

$\square$

Pointwise Properties of Sobolev Functions

如何去除人为 modify 零测集的因素?

Lebesgue 微分定理 $f^*(x) := \lim_{r\to 0} ⨍_{B(x,r)} f(z)\,dz$

改变零测集,不改变 $f^*(x)$ $= f(x)$ a.e.

对 $W^{1,p}$ 函数而言,如何刻画更精细的差异(比 Lebesgue 零测集小很多).

如何刻画?
(利用 Capacity).
某种 Capacity 是 0.

将要证明 capacity 是一个外测度

§4.7 Capacity

Def: $K^p := \{ f:\mathbb{R}^n\to\mathbb{R} \mid f\geq 0,\ f\in L^{p^*}(\mathbb{R}^n),\ \nabla f\in L^p(\mathbb{R}^n;\mathbb{R}^n)\}$

$$Cap_p(A) = \inf\left\{\int_{\mathbb{R}^n}|\nabla f|^p\,dx \;\middle|\; f\in K^p,\ A\subseteq \{f\geq 1\}^\circ\right\}$$

(1) $p=2$.

A上有多少电? 使电容=1

$$\begin{cases} \Delta u = 0 & \text{in } A^c \\ u|_\infty = 0 \\ u|_{\partial A} = 1. \end{cases}$$

u. 极小化所在问题. $\inf\left\{\int|\nabla w|^2 \;\middle|\; \begin{array}{l} w\geq 1 \text{ on } A. \\ v=0 \text{ at } \infty \end{array}\right\}$

$\int|\nabla u|^2 = Cap_2(A)$.

$$-\int_{A^c} u\Delta u + \int_{\partial A}\frac{\partial u}{\partial n}\, u\, ds = \text{total charge on } A.$$

(2). 不加内点(改成 $A \subseteq \{f\geq 1\}$)的后果.

A 是 $\mathcal{L}^n$-零测集? 条件没用了

(3). $Cap_p(K) = \inf\left\{\int_{\mathbb{R}^n}|\nabla f|^p\,dx \;\middle|\; f\in C_c^\infty(\mathbb{R}^n),\ f\geq \chi_K\right\}$ $\forall K\subset\subset\mathbb{R}^n$.

Thm 4.12. Approximation in K^p

(1) If $f \in K^p$, $1 \le p < n$ $\Rightarrow$ $\exists \{f_k\} \subseteq W^{1,p}(\mathbb{R}^n)$. s.t.

$$\|f - f_k\|_{p^*} \to 0$$

$$\|\nabla f - \nabla f_k\|_p \to 0$$

(2) $f \in K^p \Rightarrow \|f\|_{p^*} \lesssim \|\nabla f\|_p$.

常数由GNS不等式给出.

<u>Proof</u>: (2) is ~~the~~ an ~~direct~~ immediate result of (1).

(1). Set $\zeta \in C_c^\infty(\mathbb{R}^n)$. 使 $\zeta = 1$ in $B(1)$,

$= 0$ in $B(2)^c$.

$\operatorname{spt} \zeta \subseteq B(2)$. $|\nabla \zeta| \le 2$

$$\zeta_k(x) = \zeta\left(\frac{x}{k}\right)$$

Given $f \in K^p$. $f_k = f\zeta_k$. $f_k \in W^{1,p}(\mathbb{R}^n)$.

$$\int_{\mathbb{R}^n} |f - f_k|^{p^*} dy \le \int_{\mathbb{R}^n - B(k)} |f|^{p^*} dy \to 0 \text{ as } k \to +\infty$$

$$\int_{\mathbb{R}^n} |\nabla f - \underbrace{\nabla f_k}_{= \nabla(f\zeta_k) = \nabla f \zeta_k + \nabla\zeta_k f}|^p$$

$$\le \int_{\mathbb{R}^n} \left(|\nabla f - \nabla f \zeta_k| + |\nabla\zeta_k \cdot f| \right)^p dx$$

$$\le 2^p \int_{\mathbb{R}^n} |\cancel{\nabla f - \nabla f \zeta_k}\ \nabla f(1-\zeta_k)|^p + |(\nabla\zeta_k) f|^p dx$$

$$\le 2^p \int_{\mathbb{R}^n - B(k)} |\nabla f|^p + \frac{2^p}{k^p} \int_{B(2k) - B(k)} |f|^p dy$$

先Hölder 再放缩区域

$$\lesssim \int_{\mathbb{R}^n - B(k)} |\nabla f|^p + \left(\int_{\mathbb{R}^n - B(k)} |f|^{p^*} dy \right)^{1 - \frac{p}{n}}.$$

$\to 0$ as $k \to +\infty$. $\square$

Thm 4.13 (Properties of K^p)

(1). $f, g \in K^p \Rightarrow h := f \vee g \in K^p$

$$\nabla h = \begin{cases} \nabla f & \mathcal{L}^n\text{-a.e. on } \{f \geq g\} \\ \nabla g & \mathcal{L}^n\text{-a.e. } \{f \leq g\} \end{cases}$$

极小值同理.

只用注意 $h = f + (g-f)^+$ 即可. ※

(2). $\forall f \in K^p$. $t \geq 0$. $h := \min\{f, t\} \in K^p$. ※

(3). $\forall \{f_k\}_1^\infty \subseteq K^p$. define $g := \sup\limits_{1 \leq k < \infty} f_k$.

If $h \in L^p(\mathbb{R}^n) \Rightarrow \begin{cases} g \in K^p \\ |Dg| \leq h. \ \mathcal{L}^n\text{-a.e.} \end{cases}$ $h := \sup\limits_{1 \leq k < \infty} |\nabla f_k|$

Proof:

$g_l = \sup\limits_{1 \leq k \leq l} f_k$.

由 (1). $|\nabla g_l| \leq \sup\limits_{1 \leq k \leq l} f_k$.

$g_l \nearrow g \Rightarrow \|g\|_{p^*} = \lim\limits_{l \to \infty} \|g_l\|_{p^*}$

$\lesssim \liminf\limits_{l \to \infty} \|\nabla g_l\|_p$

$\leq \|h\|_p < +\infty \Rightarrow g \in L^{p^*}$.

再证 $|Dg| \leq h$ $\mathcal{L}^n$-a.e.

$\forall \phi \in C_c^\infty(\mathbb{R}^n)$.

$$\int_{\mathbb{R}^n} g \cdot \nabla\phi = \lim_{l \to \infty} \int_{\mathbb{R}^n} g_l \cdot \nabla\phi = -\lim_{l \to \infty} \int_{\mathbb{R}^n} \nabla g_l \cdot \phi$$

$$\leq \int_{\mathbb{R}^n} |\phi| \, h$$

令 $L(\phi) = \int_{\mathbb{R}^n} g \cdot \nabla\phi \, dy$ $\forall \phi \in C_c^\infty(\mathbb{R}^n)$.

由 Riesz 表示定理.

$\exists!$ random μ.

$\mu(A) \leq \int_A h \, dy$. $\forall$ Lebesgue 可测集 A.

$L\phi = \int_{\mathbb{R}^n} \phi \cdot \psi \, dy$ $\forall \psi \in L^p(\mathbb{R}^n, \mathbb{R}^n)$

$|\psi| \leq h$ $\mathcal{L}^n$-a.e.

$\rightarrow g \in K^p$. $|\nabla g| = |\psi| \leq h$ $\mathcal{L}^n$-a.e.

Capacity

~~Def.~~ Recall:

- $K^p = \{ f: \mathbb{R}^n \to \mathbb{R} \mid f \geq 0,\ f \in L^{p^*}(\mathbb{R}^n),\ \nabla f \in L^p(\mathbb{R}^n; \mathbb{R}^n) \}$

- $\mathrm{Cap}_p(A) := \inf \left\{ \int_{\mathbb{R}^n} |\nabla f|^p dx \,\middle|\, f \in K^p,\ A \subseteq \{f \geq 1\}^\circ \right\}$

- C_c^∞, $W^{1,p}$ is dense in K^p.

- GNS不等式的证明：(1) $f \in K^p$, $1 \leq p < n$ $\Rightarrow \exists \{f_k\} \subseteq W^{1,p}(\mathbb{R}^n)$, 使 $\|f - f_k\|_{L^{p^*}} \to 0$

$\|\nabla f - \nabla f_k\|_{L^p} \to 0$.

(2). $f \in K^p$. $\|f\|_{p^*} \lesssim \|\nabla f\|_p$.

Thm 4.13

(1) $f, g \in K^p$ then $\max\{f, g\}$, $\min\{f, g\} \in K^p$.

(2). ~~$\{f_k\}$~~ $\forall t \geq 0$. $h := \min\{f, t\} \in K^p$

(3). $\{f_k\} \subseteq K^p$. $g := \sup_{1 \leq k < \infty} f_k$, $h := \sup_{1 \leq k < \infty} |\nabla f_k|$. If $h \in L^p(\mathbb{R}^n)$, then $g \in K^p$

$|\nabla g| \leq h$ $\mathcal{L}^n$-a.e.

$\square$.

Thm 4.14 Capacity is a measure on $\mathbb{R}^n$.

Proof: It suffices to check the sub-additivity

$\bigcup_{k=1}^\infty A_k$. $\sum_{k=1}^\infty \mathrm{Cap}_p(A_k) < +\infty$

$\forall k$. $\exists f_k \in K^p$ s.t. $A_k \subseteq \{f_k \geq 1\}^\circ$

$$\int_{\mathbb{R}^n} |\nabla f_k|^p dx \leq \mathrm{Cap}_p(A_k) + \frac{\varepsilon}{2^k}.$$

Set $f = \sup_k f_k$. $A \in \{f \geq 1\}^\circ$ $f \in K^p$

~~$\int_{\mathbb{R}^n} |\nabla f|^p$~~

$\int_{\mathbb{R}^n} |\nabla f|^p dx$

$\leq \int_{\mathbb{R}^n} \sup_{k \in \mathbb{Z}_+} |\nabla f_k|^p dx$

$\leq \sum_{k=1}^\infty \int_{\mathbb{R}^n} |\nabla f_k|^p dx$

$\leq \sum_{k=1}^\infty \mathrm{Cap}_p(A_k) + \varepsilon$

$\Rightarrow f \in K^p$

$\Rightarrow \cdots$

$\square$

Rmk: Cap_p is not a Borel measure.

$A = \mathring{B}(0,1)$.

~~设~~ 用调和函数 on $A=1$. ∞

无穷远 $=0$ …

$\square$

Thm 4.15. $A, B \subseteq \mathbb{R}^n$.

(1) $\mathrm{Cap}_p(A) = \inf\{\mathrm{Cap}_p(U) \mid U \text{ open}, A \subseteq U\}$

(2) $\mathrm{Cap}_p(\lambda A) = \lambda^{n-p}\,\mathrm{Cap}_p(A)$

(3) $\mathrm{Cap}_p(L(A)) = \mathrm{Cap}_p(A)$ $\forall$ Affine Isometry $L: \mathbb{R}^n \to \mathbb{R}^n$.

(4). $\mathrm{Cap}_p(B(x,r)) = r^{n-p}\,\mathrm{Cap}_p(B(1))$.

(5). $\mathrm{Cap}_p(A) \leq C_{n,p}\,\mathcal{H}^{n-p}(A)$.

(6). $\mathcal{L}^n(A) \leq C_{p,n}\,\mathrm{Cap}_p(A)^{\frac{n}{n-p}}$.

(7). $\mathrm{Cap}_p(A\cup B) + \mathrm{Cap}_p(A\cap B) \leq \mathrm{Cap}_p(A) + \mathrm{Cap}_p(B)$.

(8). $A_1 \subseteq \cdots \subseteq A_k \subseteq \cdots$ then $\lim\limits_{k\to+\infty} \mathrm{Cap}_p(A_k) = \mathrm{Cap}_p\left(\bigcup_{k=1}^{\infty} A_k\right)$.

(9). $A_1 \supseteq \cdots \supseteq A_k \supseteq \cdots$ compact, then $\lim\limits_{k\to+\infty} \mathrm{Cap}_p(A_k) = \mathrm{Cap}_p\left(\bigcap_{k=1}^{\infty} A_k\right)$.

紧性不可去掉

Proof (5) $\forall\,\delta > 0$. $A \subseteq \bigcup_{k=1}^{+\infty} B(x_k, r_k)$, $2r_k < \delta$. $\leq \mathcal{S}^{n-p}(A) \leq 2^{n-p}\mathcal{H}^{n-p}(A)$.

(6)

$$\mathrm{Cap}(A) \leq \sum_{k=1}^{\infty} \mathrm{Cap}_p(B(x_k, r_k)) \underset{\text{利用(4)}}{=} \mathrm{Cap}_p(B(1))\,\frac{\boxed{\sum_{k=1}^{\infty} r_k^{n-p}\,\alpha(n-p)}}{\alpha(n-p)}.$$

$$\Rightarrow \mathrm{Cap}(A) \leq \cancel{\mathrm{Cap}_p(B(1))}\,\mathcal{H}^{n-p}(A).$$

证: $\left(\int |\nabla f|^p\,dx\right)^{\frac{1}{p}} \gtrsim \left(\int_{\mathbb{R}^n} f^{p^*}\,dx\right)^{\frac{1}{p^*}} \geq \mathcal{L}^n(A)^{\frac{1}{p^*}}$

$\uparrow$ GNS. $\quad \geq \cancel{\chi_A}$.

$\leq (\mathrm{Cap}_p(A)+\varepsilon)$.

Pf:

(7) Fix $\varepsilon>0$. $f\in K^p$ s.t.
$g\in K^p$ $\quad B\subseteq\{g\geq 1\}^\circ$. $\int|\nabla g|^p\leq Cap_p(B)+\varepsilon$

then $\max\{f,g\}$, $\min\{f,g\}\in K^p$

$$A\cup B\subseteq\{\max\{f,g\}\geq 1\}^\circ$$

$$A\cap B\subseteq\{\min\{f,g\}\geq 1\}^\circ$$

$$Cap_p(A\cup B)+Cap_p(A\cap B)\leq\int|\nabla f|^p+|\nabla g|^p$$

$$\leq Cap_p(A)+Cap_p(B)+2\varepsilon.$$

(8). Only prove $1<p<n$.

Suppose $\lim\limits_{k\to\infty}Cap_p(A_k)<\infty$. $\quad\varepsilon>0$.

then $\forall k$, $\exists f_k\in K^p$

$$A_k\subseteq\{x\mid f_k(x)\geq 1\}^\circ$$

$$\int|\nabla f_k|^p<Cap_p(A_k)+\frac{\varepsilon}{2^k}$$

Set $h_m=\sup\limits_{1\leq k\leq m}f_k$.

$h_0=0$

$h_m=h_{m-1}\vee f_m$

$A_{m-1}\subseteq\{h_{m-1}\wedge f_m\geq 1\}^\circ$

$$\Rightarrow\int|\nabla h_m|^p+Cap_p(A_{m-1})\leq\int_{\mathbb{R}^n}|\nabla\max(h_{m-1},f_m)|^p\,dx$$

$$+\int_{\mathbb{R}^n}|\nabla\min(h_{m-1},f_m)|^p\,dx$$

$$=\int_{\mathbb{R}^n}|\nabla h_{m-1}|^p+|\nabla f_m|^p\,dx$$

$$\leq\int_{\mathbb{R}^n}|Dh_{m-1}|^p\,dx+Cap_p(A_m)+\frac{\varepsilon}{2^m}$$

$\longrightarrow$

$$\int_{\mathbb{R}^n}|\nabla h_m|^p-|\nabla h_{m-1}|^p\leq Cap_p(A_m)-Cap_p(A_{m-1})+\frac{\varepsilon}{2^m}.$$

求和

$$\int_{\mathbb{R}^n}|\nabla h_m|^p\leq Cap(A_m)+\varepsilon.$$

$f=\lim\limits_{m\to\infty}h_m\Rightarrow\bigcup\limits_{k=1}^{\infty}A_k\subseteq\{x\mid f(x)\geq 1\}^\circ$

$$\|f\|_{p^*}\lesssim\liminf_{m\to\infty}\|\nabla h_m\|_p\lesssim\left(\lim_{m\to\infty}Cap_p(A_m)+\varepsilon\right)^{\frac{1}{p}}$$

$p>1$

$\Rightarrow\exists\, Dh_m\xrightarrow{\text{弱}}Df$ in L^p

$\therefore f\in K^p$

$$Cap_p\Big(\bigcup_{k=1}^{\infty}A_k\Big)\leq\|Df\|_{L^p}^p\leq\lim_{m\to\infty}Cap_p(A_m)+\varepsilon$$

(9) $\mathrm{Cap}_p\left(\bigcap_{k=1}^{\infty} A_k\right) \le \lim_{k\to\infty} \mathrm{Cap}_p(A_k)$.

choose U 开 $\supseteq \underbrace{\bigcap_{k=1}^{\infty} A_k}_{\text{紧}}$ $\Rightarrow$ $\exists m \in \mathbb{Z}_+$ s.t. $\forall k \ge m$, $A_k \subseteq U$.

$\lim_{k\to\infty} \mathrm{Cap}_p(A_k) \le \mathrm{Cap}_p(U)$.

$\square$.

Recall: $\mathrm{Cap}_p(\cdot)$ 不是Borel测度.

- $\mathrm{Cap}_p(A) \le C_{p,n}\mathcal{H}^{n-p}(A)$
- $\mathcal{L}^n(A) \le C_{p,n}\,\mathrm{Cap}_p(A)^{\frac{n}{n-p}}$

thm 4.16 (Capacity and Hausdorff measure).

(1) $1<p<n$. $\mathcal{H}^{n-p}(A)<\infty \Rightarrow \mathrm{Cap}_p(A)=0$

(2). $A\subseteq\mathbb{R}^n$. $1\le p<\infty$ $\mathrm{Cap}_p(A)=0 \Rightarrow \mathcal{H}^s(A)=0$. $\forall s>n-p$.

Proof: WLOG A is compact.

Check: $\mathcal{H}^{n-p}$. Borel regular.

$\forall A\subseteq\mathbb{R}^n$. $\exists B\subseteq\mathbb{R}^n$ Borel $\mathcal{H}^{n-p}(B)=\mathcal{H}^{n-p}(A)$.
$B\supseteq A$

$\mathrm{Cap}_p(B)=0\Rightarrow\mathrm{Cap}_p(A)=0$.

$\Rightarrow$ 不妨 ~~A~~ Borel.

进一步地, 若 $\mathcal{H}^{n-p}(A)<+\infty$ A Borel

$\mu:=\mathcal{H}^{n-p}\llcorner A$. Radon. A μ-可测.

$\mu(A)=\sup\{\mu(K) \mid K\subset A,\ K紧\}$. $\Rightarrow$

$\Rightarrow\exists K_i$ $\mu(A-\bigcup_{i=1}^\infty K_i)=0$

$\|$

$\mathcal{H}^{n-p}(A-\bigcup_{i=1}^\infty K_i)$.

$\mathrm{Cap}_p(A-\bigcup_{i=1}^\infty K_i)=0 \Leftarrow \mathrm{Cap}_p(\bigcup_{i=1}^\infty K_i) \Leftarrow \mathrm{Cap}_p(K_i)<+\infty\ \forall i$

从而才有不妨设A紧.

Now: A is compact.

claim: $\forall$ open set. $V\supseteq A$. $\exists$ open set W. $A\subset W\subset\subset V$

$\exists C=c(n,A)>0$ $\forall f\in K^p$. $f\ge1$. on W. $\mathrm{spt}\, f\subseteq V$

$\int_{\mathbb{R}^n}|\nabla f|^p<C$

Proof of the claim:

Choose $0<\delta<\frac{1}{2}\operatorname{dist}(A,\partial V)$. $\mathcal{H}_\delta^{n-p}(A)<+\infty$

then. $\exists$ covering $\{B(x_i,r_i)\}_{i}^{m}\supseteq A$. s.t.

$$\sum_{i=1}^{m}\alpha(n-p)\,r_i^{n-p}\le C\left(\mathcal{H}^{n-p}(A)+1\right)$$

Define.
$$f_i(x)=\begin{cases}1 & \mathring{B}(x_i,r_i).\\ \text{Linear} & \\ 0 & \left(\mathring{B}(x_i,2r_i)\right)^c.\end{cases}\qquad \int_{\mathbb{R}^n}|\nabla f_i|^p\,dx\lesssim r_i^{n-p}.$$

(从而 $f_i\in W^{1,\infty}$ ~~$f\in K^p$~~ $\subseteq W^{1,p}\subseteq K^p$)

Set $W=\bigcup_{i=1}^{m}B(x_i,r_i)$.

~~$f(x)=\sum_{i=1}^{m}f_i(x)$~~ $f(x)=\max\limits_{1\le i\le m}f_i(x)\in K^p$. $f|_W\equiv 1$.

$\forall A\subseteq\{f\ge 1\}^\circ$. $\operatorname{Cap}_p(A)\le\int_{\mathbb{R}^n}|\nabla f|^p \lesssim_p \sum_{i=1}^{m}\int_{\mathbb{R}^n}|\nabla f_i|^p$.

$|\nabla f|=\max\limits_{1\le i\le m}|\nabla f_i|$.

$$\lesssim_{n,p}\sum_{i=1}^{m} r_i^{n-p}$$

$$\lesssim_{n,p}\left(\mathcal{H}^{n-p}(A)+1\right)$$

claim 证毕! ※

~~Technical~~

Small Techniques:

命题: $f\in K^p$. $A\subseteq\{f\ge 1\}^\circ$. $\int_{\mathbb{R}^n}|\nabla f|^p\,dx$ 任意小 $\Rightarrow\operatorname{Cap}_p(A)=0$.

技巧: 利用 $s_j=\sum_{n=1}^{j}\frac{1}{n}\longrightarrow+\infty$ as $j\to+\infty$

将 s_j 作为"权", 造出加权平均.

$$g_j=\cancel{\frac{1}{s_j}\sum_{i=1}^{k}\cdots\frac{1}{s_j}\sum_{k=1}^{j}\frac{1}{k}}\quad \frac{1}{s_j}\sum_{k=1}^{j}\frac{1}{k}f_k$$

$g_j\in K^p$. $A\subseteq\{g_j\ge 1\}^\circ$

Notice:

p次方直接拿到里面 因为支集不交.

$$\int|\nabla g_j|^p\,dx=\frac{1}{s_j^p}\int_{\mathbb{R}^n}\sum_{k=1}^{j}\frac{|\nabla f_k|^p}{k^p}$$

$$\le\frac{1}{s_j^p}\sum_{k=1}^{j}\frac{1}{k^p}$$

$$\longrightarrow 0.\ \text{as } j\to\infty$$

ID 证毕. ※

(2) ~~Cap~~ $A \subseteq \mathbb{R}^n$ $1 \le p < \infty$

$\mathrm{Cap}_p(A)=0$ then $\forall s > n-p$ $\mathcal{H}^s(A)=0$

Proof: Indirect

Recall: $f \in L^1_{loc}(\mathbb{R}^n)$.

$$\Lambda_s := \left\{ x \in \mathbb{R}^n \,\middle|\, \limsup_{r \to 0} r^{-s} \int_{B(x,r)} |f(y)|\,dy > 0 \right\}$$

$\Rightarrow \mathcal{H}^s(\Lambda_s)=0$.

Find $g \in K^p$ $\forall s > n-p$.

$$A \in \left\{ x \in \mathbb{R}^n \,\middle|\, \limsup_{r \to 0} r^{-s} \int_{B(x,r)} |\nabla g|^p = +\infty \right\}$$

$\Uparrow$

$\forall x \in A$ $\lim_{r \to 0^+} \langle g \rangle_{x,r} = +\infty$

lemma: $\lim_{r \to 0^+} \langle g \rangle_{x,r} + \infty \Rightarrow \lim_{r \to 0^+} r^{-s} \int_{B(x,r)} |\nabla g|^p = +\infty$

non-trivial part.

Find g?

$\mathrm{Cap}_p(A)=0 \Rightarrow \forall i \ \exists f_i \in K^p$. $A \in \{f_i \ge 1\}^\circ$. $\int_{\mathbb{R}^n} |\nabla f_i|^p dx < \frac{1}{2^i}$.

$\Downarrow$

AGNS

Set $g := \sum_{i=1}^{\infty} f_i$.

check: $g = \sum_{i=1}^{\infty} f_i$ $\quad g \in K^p$.

$g \in L^{p^*}$ ✓

$\nabla g \in L^p$ ✓

$\forall x \in A$. $x \in A \subseteq \{f_i \ge 1\}^\circ$

$\forall m \in \mathbb{Z}_+$. $\exists r_i$ $B(x, r_i) \in \bigcap_{i=1}^{m} \{f_i \ge 1\}^\circ \Rightarrow \forall r < r_x$. $\langle g \rangle_{x,r} \ge m$.

$\Rightarrow \lim_{r \to 0} \langle g \rangle_{x,r} = +\infty$

此时仍欠证引理.

回忆: $\exists M,\ \forall\, 0<r<1$ $\displaystyle\int_{B(x,r)} |\nabla g|^p \le M\cdot r^s.$

By Poincaré's Ineq:

$$⨍_{B(x,r)} |g(x)-\langle g\rangle_{x,r}|^p\,dy \le C r^p ⨍_{B(x,r)} |\nabla g|^p \le C r^{\overset{p+s-n=:\theta}{>0}}.$$

$\overset{\text{Hölder}}{\Longrightarrow}$ $$⨍_{B(x,r)} |g(y)-\langle g\rangle_{x,r}|\,dy \lesssim r^{\frac{\theta}{p}}$$

分平积,因子由Scaling不变性

$$\Rightarrow\ |\langle g\rangle_{x,\frac{r}{2}}-\langle g\rangle_{x,r}| \le 1 ⨍_{B(x,\frac{r}{2})} |g(y)-\langle g\rangle_{x,r}|\,dy$$
$$\lesssim ⨍_{B(x,r)} |g(y)-\langle g\rangle_{x,r}|\,dy$$
$$\lesssim C\cdot r^{\theta/p} \longrightarrow 0 \text{ as } r\to 0^+.$$

令 $a_j=\langle g\rangle_{x,2^{-j}}$

$|a_{j+1}-a_j| \lesssim 2^{-\frac{j\theta}{p}}$. $\Rightarrow \{a_j\}$ converges $\Rightarrow$ 矛盾! ※.

□.

§4.8: Quasi-continuity. Precise Representatives of Sobolev Functions

Thm 4.18 (Capacity Estimate)

Assume $f\in K^p$, $\varepsilon>0$. let $A:=\{x\in\mathbb{R}^n \mid \langle f\rangle_{x,r}>\varepsilon \text{ for some } r>0\}$.

then $\mathrm{Cap}_p(A) \lesssim_{n,p} \frac{1}{\varepsilon^p}\int_{\mathbb{R}^n}|\nabla f|^p\,dx$ (不妨取 $\varepsilon=1$ 即可).

Proof:

Idea: $\mathrm{Cap}_p(A)\le xxx$

Find $g\in K^p$, $A\subseteq\{g\ge 1\}^\circ$. $\mathrm{Cap}_p(A)\le \underbrace{\int_{\mathbb{R}^n}|\nabla g|^p}_{化成\|\nabla g\|_p的估计} \overset{?}{\le} xxx$.

坏处: $\langle f\rangle_{x,r}>1 \not\Rightarrow f(x)>1$.

Modify f to get g

想法:

$\langle f\rangle_{x,r}>1$.

$h:=(\langle f\rangle_{x,r}-f)^+$. $\underbrace{f+h}_{想要 \text{ in } g}\ge\langle f\rangle_{x,r}$.

$\mathcal{F}=\{B(x,r)\mid x\in A,\ \langle f\rangle_{x,r}>1\}$

$A\subseteq$ centers of $\mathcal{F}$. ← 希望用 Besikovitch 覆盖

(但需要验证 $\sup \mathrm{diam}<\infty$) Omit! (直接 $\alpha(n)r^n\le\int_{B(x,r)} f\,dy \overset{\text{Hölder}}{\le} \|1\|_{(p^*)'}\|f\|_{p^*}$.)

Besikovitch Covering

$\exists N_n\in\mathbb{Z}_+$. $\mathcal{F}_1\dots\mathcal{F}_{N_n}$ 可数族, 不交, 闭球.
每族中的球不交.

$A\subseteq\bigcup_{i=1}^{N_n}\bigcup_{B\in\mathcal{F}_i}B$. $\langle f\rangle_B>1$. $\forall B\in\bigcup_{i=1}^{N_n}\mathcal{F}_i$.

~~On each~~ $h_{ij}:=(\langle f\rangle_{B_j^i}-f)^+$ on B_j^i.

延拓: $|\nabla h_j|\le|\nabla f|$ on B_j^i → $\nabla h_j\in L^p(B_j^i)$

Finally $g=f+\sum_{i,j}h_{ij}$

$$\int_{B_j^i} |h_{ij}|^p \le \int_{B_j^i} |f - \langle f \rangle_{B_j^i}|^p dx \le C \int_{B_j^i} |\nabla f|^p dx$$

由延拓定理: $\|h_j\|_{W^{1,p}(\mathbb{R}^n)} \le C \|\nabla f\|_{L^p(B_j)} \Rightarrow \int_{\mathbb{R}^n} |\nabla h_j|^p \le C \int_{B_j^i} |\nabla f|^p$

如何确定C与什么有关?

Scaling 到单位球上

看上述过程

$$\int_{\mathbb{R}^n} |\nabla \tilde{h}_{ij}|^p \lesssim_n \int_{B(0,1)} |\nabla \tilde{f}|^p$$

再 Scaling 回去

两边的体积刚好消去!

~~Set h~~

$h := \sup_{i,j} h_{ij}$ claim $\boxed{h \in K^p}$

只用 $\sup_{i,j} |\nabla h_{ij}| \in L^p$. $\int_{\mathbb{R}^n} \sup_{i,j} |\nabla h_{ij}|^p \le \sum_{i=1}^{N_n} \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} |\nabla h_{ij}|^p dx$

$$\lesssim \int_{\mathbb{R}^n} |\nabla f|^p dx.$$

$f+h \in K^p$

$\forall x \in A \Rightarrow \exists i, j.\ x \in B_j^i$. $(f+h)(x) \ge f(x) + h_{ij}(x)$

$\ge \langle f \rangle_{B_{ij}} \underset{\text{def}}{\ge} 1$

A 开

$\Downarrow$

$A \subset \{f+h \ge 1\}^{\circ}$.

$\Rightarrow \mathrm{Cap}_p(A) \le \int_{\mathbb{R}^n} |\nabla(f+h)|^p dx \lesssim \int_{\mathbb{R}^n} |\nabla f|^p dx.$

对于 Σ_m case, 作 Scaling 即可.

$\square$

Fine Properties of Sobolev Functions:

Def 4.11: f is p-quasicontinuous, if $\forall \varepsilon > 0$ $\exists$ 开 V. s.t. $\begin{cases} \mathrm{Cap}_p(V) < \varepsilon \\ f\big|_{\mathbb{R}^n \setminus V} \text{ continuous} \end{cases}$.

Thm 4.19 (Fine Properties).

$\forall f \in W^{1,p}(\mathbb{R}^n)$ $\quad 1 \le p < n$.

(1) $\exists$ Borel $E \subseteq \mathbb{R}^n$. $\mathrm{Cap}_p(E) = 0$. s.t. $\lim_{r \to 0} \langle f \rangle_{x,r} =: \underline{f^*(x)}$. a.e. $x \in \mathbb{R}^n - E$.

($\Leftarrow$ $\mathcal{H}^s(E) = 0$. $\forall s > n-p$)

Called the precise representation of f.

(2). $\displaystyle \lim_{r \to 0} \fint_{B(x,r)} |f - f^*(x)|^{p^*} dy = 0$. $\quad \forall x \in \mathbb{R}^n - E$.

(3). f^* is p-quasicontinuous.

Proof. $\quad A := \left\{ x \in \mathbb{R}^n \,\middle|\, \underbrace{\limsup_{r \to 0^+} \frac{1}{r^{n-p}} \int_{B(x,r)} |\nabla f|^p dy}_{\text{Scaling invariance}} > 0 \right\}$.

By Thm 2.10. 4.16.

$$|\nabla f|^p \in L^1 \overset{2.10}{\Rightarrow} \mathcal{H}^{n-p}(A) = 0 \overset{4.16}{\Rightarrow} \mathrm{Cap}_p(A) = 0.$$

A Borel 因 $\displaystyle \frac{1}{r^{n-p}} \int_{B(x,r)} |\nabla f|^p dy$ 连续. limsup $= \cap\cup$ Borel $\Rightarrow$ Borel.

$\forall x \notin A$. By Poincaré's Ineq:

$$\lim_{r \to 0} \fint_{B(x,r)} |f - \langle f \rangle_{x,r}|^{p^*} dy = 0 \qquad (*).$$

只有 $=0$，<u>没有速度控制</u>.

否则用柯西列 in argument 即可

作逼近如下:

$\exists f_i \in W^{1,p}(\mathbb{R}^n) \cap C_0^\infty(\mathbb{R}^n)$.

$$\int_{\mathbb{R}^n} |\nabla f - \nabla f_i|^p dy \le \frac{C}{2^{i(p+1)}}$$

为证 (*) 式中 $\langle f \rangle_{x,r}$ 有极限. 则需要将 $\langle f \rangle_{x,r}$ 与 f_i 来建立联系

$$B_i := \left\{ x \in \mathbb{R}^n \,\middle|\, \fint_{B(x,r)} |f - f_i|\, dy > \frac{1}{2^i},\ \text{for some } r > 0 \right\}.$$

By 4.18: $\dfrac{Cap_p(B_i)}{2^{pi}} \lesssim \int_{\mathbb{R}^n} |\nabla f - \nabla f_i|^p\, dy \lesssim \dfrac{1}{2^{i(p+1)}}$.

$$|\langle f\rangle_{x,r} - f_i(x)| = \fint_{B(x,r)} |\langle f\rangle_{x,r} - f_i(x)|\, dy.$$

$$\le \fint_{B(x,r)} \underbrace{|f(y) - \langle f\rangle_{x,r}|}_{\text{Hölder}} + \underbrace{|f - f_i|}_{?} + \underbrace{|f_i - f_i(x)|\, dy}_{\text{由 Lebesgue 微分定理}}$$

Fix i. $\forall x \notin A \cup B_i$.

$$\limsup_{r\to 0} |\langle f\rangle_{x,r} - f_i(x)| \le \frac{1}{2^i}.$$

Set $E := \bigcap_{k=1}^{\infty} \left(\bigcup_{j=k}^{\infty} (A \cup B_j) \right)$. 目标. $Cap_p(E) = 0$.

A. B_j 坏集合 上限集 $=0$? Cap

先考虑 $E_k = A \cup \bigcup_{j=k}^{\infty} (B_j) = A \cup$.

$$\Rightarrow Cap_p(E_k) \le Cap_p(A) + \sum_{j=1}^{\infty} Cap_p(B_j)$$
$$\lesssim \sum_{j=k}^{\infty} \frac{1}{2^j} \lesssim \frac{1}{2^k}.$$

$\Rightarrow Cap_p(E) = 0$.

此时论证已结束. 因为.

$\forall x \in \mathbb{R}^n - E_k$. $i, j \ge k$

$$|f_i(x) - f_j(x)| \le \limsup_{r\to 0} |\langle f\rangle_{x,r} - f_i(x)| + \limsup_{r\to 0} |\langle f\rangle_{x,r} - f_j(x)|$$
$$\le \frac{1}{2^i} + \frac{1}{2^j} \to 0 \quad \text{as } i, j \to \infty$$

$\Rightarrow f_j \rightrightarrows \exists g$ continuous on $\mathbb{R}^n - E_k$

Also. $\limsup_{r\to 0} |g(x) - \langle f\rangle_{x,r}| \le \underbrace{|g(x) - f_i(x)|}_{\to 0} + \limsup_{r\to 0} |f_i(x) - \langle f\rangle_{x,r}|$ as $i \to \infty$.

$\Rightarrow g(x) = f^*(x)$. $\forall x \in \mathbb{R}^n - E_k \Rightarrow \forall x \in \mathbb{R}^n - E$. $f^*(x)$ $\exists$

(2). 显然.

(3). 仅需复述 Fix $\varepsilon > 0$. choose k. $Cap_p(E_k) < \frac{\varepsilon}{2}$. By 4.15 $\exists$ 开 $U \supseteq E_k$. $Cap_p(U) < \varepsilon$.
s.t. $f_i \rightrightarrows f^*$ on $\mathbb{R}^n - U$.

因 p-quasicontinuous 要求这个开集.

□

§4.9 Differentiability on Lines.

Conclusion: $f\in W^{1,p}_{loc}(\mathbb{R}^n)$

⇒ 强导数 $\partial_{x_i} f$ a.e. 存在. 且 L^p_{loc}.

~~$1\le p<\infty$~~

Thm 4.20. $1\le p<\infty$.

(1). $f\in W^{1,p}_{loc}(\mathbb{R})$. then $f^*\in AC_{loc}(\mathbb{R})$. $(f^*)'\in L^p_{loc}(\mathbb{R})$.

(2) Conversely. $f\in L^p_{loc}(\mathbb{R})$. $f=g$ $\mathcal{L}^1$-a.e. $g\in AC_{loc}(\mathbb{R})$. $g'\in L^p_{loc}(\mathbb{R})$.

⇑ then $f\in W^{1,p}_{loc}(\mathbb{R})$

(2'). $g\in AC_{loc}(\mathbb{R})$. $g'\in L^p_{loc}$ ⇒ $g\in W^{1,p}_{loc}(\mathbb{R})$.

Proof: (2'): claim: 弱导数 = 强导数:

$$\forall\varphi\in C_c^\infty(\mathbb{R}),\ \int g\varphi' = -\int g'\varphi \quad ※$$

(1) Consider $f^\varepsilon=\eta_\varepsilon * f$. $0<\varepsilon\le 1$.

$$f^\varepsilon(y)=f^\varepsilon(x)+\int_x^y (f^\varepsilon)'(t)\,dt.$$

希望让 $\varepsilon\to0^+$

Take x_0 as a Lebesgue point of f. (只对 $p=1$ 有用 $p>1$ 直接 Morrey 嵌入)

$\forall \varepsilon,\delta\in(0,1)$

$$|f^\varepsilon(x)-f^\delta(x)| \le \int_{x_0}^x |(f^\varepsilon)'(t)-(f^\delta)'(t)|\,dt + |f^\varepsilon(x_0)-f^\delta(x_0)|.$$

→ $f\in W^{1,p}$, 磨光子收敛. L^p 收敛于弱导数 ⇓ L^1 也对. ⇓ 依 L^1 收敛 ↓ 0. as $\varepsilon\to0^+$. $\delta\to0^+$

↓ 0. as $\varepsilon.\delta\to0^+$.

$$\Rightarrow \lim_{\varepsilon\to0^+} f^\varepsilon(x) \triangleq g.$$

$\varepsilon\to0^+$ 有 $g(y)=g(x)+\int_x^y \underbrace{\cancel{g'(t)}}_{f\text{弱导数}(t)}\,dt.$

⇒ $\begin{cases} g\in AC_{loc}(\mathbb{R}) \\ g' = f \text{ weak derivative} \in L^1 \end{cases}$

又 $\langle f\rangle_{x,r}=\langle g\rangle_{x,r}\to g(x)\quad \forall x\in\mathbb{R}$ ⇒ $g=\cancel{F}\, f^*$

□

$\boxed{\text{Rmk:}}$ $f\in W^{1,1}(\mathbb{R})$ ← critical case

⇒ $f^*\in AC_{loc}(\mathbb{R})$.

$H^1(\mathbb{R}^2)$ 可能不有界.
取极坐标 (r,θ).
$f=\log|\log r|$. 0附近

$$\int_{B(0,\varepsilon)}|\nabla f|^2=\int_{B(0,\varepsilon)}|\partial_r f|^2 r\,dr\,d\theta = \int_{B(0,\varepsilon)}\frac{1}{r^2}\frac{r}{|\log r|^2}\,dr\,d\theta$$

瑕积分收敛.

但 $W^{1,1}$ 是 有界 ?

H^1 不有界.

Thm 4.21 (Sobolev Functions Restricted to line)

(1) ~~$f \in W^{1,p}_{loc}(\mathbb{R}^n)$. then $\forall k = 1,2,\dots,n$~~ ~~$f_k^*(x',t) = f^*(\dots, x_{k-1}, t, x_{k+1}, \dots)$.~~

~~Also $(f_k^*)' \in L^p_{loc}(\mathbb{R}^n)$.~~ ~~$\in AC_{loc}(\mathbb{R})$ $\mathcal{L}^{n-1}$-a.e. x'~~

关于 t. ~~$x' = (x_1 \dots x_{k-1}, x_{k+1} \dots x_n) \in \mathbb{R}^{n-1}$.~~

$x = (x_1 \dots x_n) = (t, x')$, $t \in \mathbb{R}$, $x' \in \mathbb{R}^{n-1}$.

Then. (1) $f \in W^{1,p}_{loc}(\mathbb{R})$. then $\forall$ $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$.

$f^*_{x_1}(t) := f^*(t, x') \in AC_{loc}(\mathbb{R})$ $\quad$ $f^{*\prime}_{x_1}(t) := \underbrace{\partial_{x_1}}_{\text{经典偏导数}} f^*(t, x') \in L^p_{loc}(\mathbb{R}^n)$.

~~$(f_k^*)' \in L^p_{loc}(\mathbb{R}^n)$~~

(2) ~~Conversely suppose $f \in L^p_{loc}(\mathbb{R}^n)$. $f = g$ $\mathcal{L}^n$-a.e. $1 \le k \le n$.~~

(2) $g \in L^p_{loc}(\mathbb{R}^n)$. $\forall$ $\mathcal{L}^{n-1}$ a.e. $x' \in \mathbb{R}^{n-1}$.

$g_{x_1}(t) = g(t, x') \in AC_{loc}(\mathbb{R})$

$g'_{x_1}(t) = \partial_{x_1} g(t, x') \in L^p_{loc}(\mathbb{R}^n)$

$\Rightarrow$ g 的弱偏导 a.e. 存在 且 $= \partial_{x_1} g \in L^p_{loc}(\mathbb{R}^n)$

Fact if $f^*(x) = \lim_{r \to 0^+} ⨍_{B(x,r)} f$ for some x.

~~Choose η.~~

For Mollified f: $f^\varepsilon(x) \overset{?}{\longrightarrow} f^*(x)$ as $\varepsilon \to 0^+$.

Recall: (1) $f \in W^{1,p}(\mathbb{R}^n)$. $1 \le p < n$.

$\exists E$ Borel $Cap_p(E) = 0$ $\langle f \rangle_{x,r} \to f^*(x)$ as $r \to 0^+$

(2) $f \in W^{1,p}_{loc}(\mathbb{R}) \Rightarrow f^*$ A.C.

$(f^*)' \overset{a.e.}{=}$ ~~f~~ f 弱导数 $\in L^{p}$

Lemma: f for some x. $\langle f \rangle_{x,r} \to f^*(x)$ as $r \to 0^+$

$\Rightarrow$ at this x.

$$(\eta^{\varepsilon} * f)(x) \to f^*(x) \text{ as } \varepsilon \to 0^+.$$

Pf: Fix η.

Find a sequence of $\eta^m = \sum a_i \chi_{B_{r_i}}(x)$. s.t. $\eta^m \rightrightarrows \eta$.

$\int \eta^m = 1$

~~Case 1~~
① $\forall \varepsilon$ fixed. $\lim_{m \to +\infty} \eta^{m\varepsilon} * f(x) = \eta^{\varepsilon} * f(x)$.

$$\left| \int \eta^{m\varepsilon}(x-y) f(y) - \int \eta^{\varepsilon}(x-y) f(y) \, dy \right|$$

$$\le \int_{\mathbb{R}^n} |\eta^{m\varepsilon} - \eta^{\varepsilon}|(x-y) \, |f|(y) \, dy \to 0.$$

~~Case 2~~:

② $\eta^{m\varepsilon} * f(x) \to f^*(x)$. as $\varepsilon \to 0$.

$$\eta^{m\varepsilon} * f(x) = \sum a_i w_n r_i^n \fint_{B_{\varepsilon r_i}(x)} f(y) \, dy$$

$$\int_{\mathbb{R}^n} \eta^m dx = 1 \Rightarrow \sum a_i w_n r_i^n = 1.$$

$$\fint_{B_{\varepsilon r_i}(x)} f(y)\,dy \rightrightarrows f^*(x). \quad \text{when } \varepsilon \to 0.$$

$$\eta^{m\varepsilon} * f(x) \longrightarrow f^*(x). \quad \text{uniform in } m.$$

lem 2 : $f_k : \mathbb{R} \to \mathbb{R} \in C^\infty(\mathbb{R})$ #.

且设 $\int_{-L}^{L} |f_k - f|^p - |f_k' - g|^p \to 0$

则 $f^k \to \tilde{f} \in AC$ pointwise.

$\tilde{f} = g$ a.e.

Proof : $f_k(x_0) \to \tilde{f}(x_0)$.

$$f^k(x) = f^k(x_0) + \int_{x_0}^{x} f_k'(\tau)\,d\tau.$$

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Thm (Pointwise derivative of $W^{1,p}$) $f \in W^{1,p}_{loc}$

① $\forall\ i$ a.e. $x' \in \mathbb{R}^{n-1}$

② $f(t, x')$ as a function of t is AC. $f'(t, x')$ pointwise derivative.

③. $f_i(t, x) \overset{a.e.}{\underset{L^n}{=}}$ weak derivative of $\partial_i f \in L^p_{loc}(\mathbb{R}^n)$.

Pf: $\exists E$ $\mathrm{Cap}_p(E)=0$

$\forall x\in\mathbb{R}^n\setminus E$. $\langle f\rangle_{x,r}\longrightarrow f^*$.

$$\begin{cases} p>1 & \text{Thm } 4.17 \quad H^{n-1}(E)=0 \\ p=1 & 5.12. \quad \mathrm{Cap}_1(E)=0 \Longleftrightarrow H^{n-1}(E)=0 \end{cases}$$

$H^{n-1}(E)=0$ $E\subseteq\mathbb{R}^n$. $\pi:\mathbb{R}^n\to\mathbb{R}^{n-1}$ $\Rightarrow H^{n-1}(\pi(E))=0$

$\Rightarrow \mathcal{L}^{n-1}(\pi(E))=0$

$\forall x\notin\pi(E)$. $\mathrm{Line}_{(t,x)}\cap E=\phi$.

f^* 在line上有定义 $\mathcal{L}^n$-a.e.

$\forall\ \mathcal{L}^{n-1}$ a.e. x': 在 (t,x') line 上. $f^\varepsilon(x)\to f^*(x)$ pointwise.

$f\in W^{1,p}_{loc}(\mathbb{R}^n)$. 因此

$$\int|f^\varepsilon-f|^p+|\partial_i f^\varepsilon-\partial_i f|^p\to 0.$$

$$\int_{\mathbb{R}^{n-1}}\int_{-L}^{L}|f^\varepsilon-f|^p+|\partial_i f^\varepsilon-g|^p\to 0.$$

$$\exists\varepsilon_j\quad \int_{-L}^{L}|f^{\varepsilon_j}-f|^p+|\partial_i f^{\varepsilon_j}-g|^p\to 0$$

由lem. 再取. ε_{ij} $f^{\varepsilon_{ij}}\to\tilde f$ pointwise. $\tilde f$ A.C. $\tilde f'\overset{a.e.}{=\!=}g$

f^* ---- f^* A.C. $(f^*)'\overset{a.e.}{=\!=}\partial_{x_i}f$.

$\forall \mathcal{L}^{n-1}$-a.e. x'. $f^*(t, x')$ as a function of t A.C.

$$(f^*)' \overset{\mathcal{L}^n\text{ a.e.}}{=\!=} \partial_{x_i} f$$

Thm: 若 $g(x)=g(t,x')$. $\forall \mathcal{L}^{n-1}$-a.e. $x\in\mathbb{R}^{n-1}$. $g(x',t)$ 关于 $\in$ A.C.

且. $\frac{d}{dt} g(t,x') \in L^p_{loc}(\mathbb{R}^n)$.

then g is 弱可偏导. $\partial_{x_n} g \overset{\mathcal{L}^n\text{-a.e.}}{=\!=} \frac{d}{dt} g(t,x)$

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§5 $BV(\mathbb{R}^n) \Longleftrightarrow$ 一阶弱偏导是 Radon 测度.

$W^{1,p}(\mathbb{R}^n) \Longleftrightarrow L^p + \cdots\cdots L^p$. 比 L^p 更广泛.

$f\in L^p \Rightarrow f\,dx$ Radon.

Def: (1) $BV(U) = \left\{ f\in L^1(U) \,\middle|\, \sup\limits_{\substack{\phi\in C_c^1(U;\mathbb{R}^n)\\ |\phi|\le 1}} \int_U f \,\mathrm{div}\,\phi \,dx < \infty \right\}$

(2). $\mathcal{L}^n$-可测集 $E\subset\mathbb{R}^n$ 有限周长 (finite perimeter) in U if $\chi_E \in BV(U)$.

Thm: $f\in BV_{loc}(U;\mathbb{R}^n)$. Then $\exists$ Radon μ on U, μ 可测 $\sigma: U\to\mathbb{R}^n$, $|\sigma|=1$ μ-a.e.

s.t. $\forall\phi\in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$, $\int_U f\,\mathrm{div}\,\phi = -\int \phi\cdot\sigma\, d\mu$.

Recall: $L: C_c(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$. 线性

$\forall\, \mathrm{Spt}\, K\subset U$. $\sup\{ Lf \mid f\in C_c(\mathbb{R}^n,\mathbb{R}), |f|\le 1, \mathrm{Spt}\, f\subseteq K\} < \infty$.

Pf: $\forall\phi\in C_c(U;\mathbb{R})$. $\mathrm{Spt}\,\phi = K \subset V \subset\subset U$.

Take $\phi_k\in C_c^1(U,\mathbb{R}) \rightrightarrows \phi$.

$L(\phi_k) := \int_U f\,\mathrm{div}\,\phi_k$.

$f\in BV \Rightarrow |L(\phi_k)| \le C\|\phi_k\|_\infty \Rightarrow$ 钦定 $L\phi = \lim\limits_{k\to\infty} L\phi_k$

$\Rightarrow L\phi \le C\|\phi\|_\infty$

☆

Def: (1) $f \in BV(U)$. 全变差测度 $\mu = \|Df\|$.

$[Df] := \|Df\| \llcorner \sigma$.

从而 5.1 表明: $\int_U f(\nabla \cdot \phi)\, dx = -\int_U \phi \cdot \sigma\, d\|Df\| = -\int_U \phi \cdot d[Df]$

② $f = \chi_E$. E 局部周长有限.

$\|\partial E\|$ 记为对应的测度 μ.

$\nu_E := -\sigma$.

$$\int_E \operatorname{div} \phi \, dx = \int_U \phi \cdot \nu_E \, d\|\partial E\|.$$

例: $E = B$. (球)

$\forall \phi \in C_c^1$.

$$\left|\int \chi_B \operatorname{div}\phi\right| = \left|\int_B \operatorname{div}\phi\right|$$

$$= \left|\int_{\partial B} \phi \cdot \vec{n} \, d(\mathcal{H}^{n-1} \llcorner \partial B)\right|.$$