Riemannian Geometry (Spring, 2023)
Mid-term Exam

Name: No.: Department:

1. (15 marks) Let M be a smooth manifold. Let ∇ be an affine connection on M. We define for any $X, Y \in \Gamma(TM)$

$$\overline{\nabla}_X Y := \nabla_X Y - \frac{1}{2} T(X, Y),$$

where T is the torsion tensor of ∇.

(i) Prove that $\overline{\nabla}$ is an affine connection. 5

(ii) Prove that $\overline{\nabla}$ is torsion free. 5

(iii) A parametrized curve $\gamma = \gamma(t)$ on M is called *a geodesic with respect to an affine connection* ∇ if $\nabla_{\dot\gamma(t)} \dot\gamma(t) = 0$ for any t. Prove that ∇ and $\overline{\nabla}$ have the same geodesics. 10

2. (20 marks) Let $c : [0, a] \to M$ be a piecewise smooth curve in a Riemannian manifold (M, g).

(i) Let $F : [0, a] \times (-\epsilon, \epsilon) \to M$ be a piecewise smooth variation of c. Derive the First Variation Formula of the energy functional. 10

(ii) Let $V(t)$ be a piecewise smooth vector filed along the curve c. Show that there exists a piecewise smooth variation $F : [0, a] \times (-\epsilon, \epsilon) \to M$ such that $V(t)$ is the variational field of F; in addition, If $V(0) = V(a) = 0$, it is possible to choose F as a *proper* variation, i.e., $F(0, s) = c(0)$, $F(a, s) = c(a)$ for all $s \in (-\epsilon, \epsilon)$. 10

(iii) Prove that a piecewise smooth curve $c : [0, a] \to M$ is a geodesic if and only if, for every proper piecewise smooth variation F of c, we have 10

$$E'(0) = 0.$$

3. (15 marks) Let (M,g) be a Riemannian manifold. Let ∇ be the Levi-Civita connection of the metric g.

(i) For $X, Y, Z \in \Gamma(TM)$, compute 5

$$\nabla^2 Z(Y, X) - \nabla^2 Z(X, Y).$$

Here we use $\nabla^2 \cdot$ for the second order covariant differentiation.

(ii) Use Ricci Identity to prove the Bochner formula: For any $f \in C^\infty(M)$, it holds that 15

$$\frac{1}{2}\Delta |\mathrm{grad} f|^2 - \langle \mathrm{grad}(\Delta f), \mathrm{grad} f \rangle = |\mathrm{Hess} f|^2 + \mathrm{Ric}(\mathrm{grad} f, \mathrm{grad} f).$$

4. (20 marks)
Let M be the upper half-plane $\{(x,y) \in \mathbb{R}^2 : y > 0\}$, with the Riemannian metric

$$\frac{1}{y^2}(dx \otimes dx + dy \otimes dy).$$

(i) Find a function $t \mapsto y(t), t \in (0, +\infty)$ such that the curve

$$\gamma : (0, +\infty) \to M, \ t \mapsto (x_0, y(t)), \text{ where } x_0 \in \mathbb{R},$$

is a geodesics parametrized by arclength. 10

(ii) Let $0 < a < b < +\infty$. Let γ be as in (i) and $\sigma : [a, b] \to M$ be a smooth curve connecting $\gamma(a)$ and $\gamma(b)$. Prove that

$$\text{Length}(\sigma_{|[a,b]}) \geq \text{Length}(\gamma_{|[a,b]}).$$

Characterize the case when the equality holds. 10

5. (30 marks)
Let (M^n, g) be a compact orientable Riemaniann manifold (M, g) with dimension n, where n is odd. Suppose that there exists an isometry $f : M \to M$ which reverses the orientation of M, and there exists $p \in M$ such that

$$d(p, f(p)) = \inf_{q \in M} d(q, f(q)), \text{ and } d(p, f(p)) \neq 0.$$

Let $\gamma : [0, \ell] \to M$ be a normal minimizing geodesic from $\gamma(0) = p$ to $\gamma(\ell) = f(p)$.

(i) Let $\overline{\gamma} : [0, 2\ell] \to M$ be a curve given by

$$\overline{\gamma}(t) = \begin{cases} \gamma(t), & t \in [0, \ell]; \\ f(\gamma(t - \ell)), & t \in [\ell, 2\ell]. \end{cases}$$

Show that $\overline{\gamma}$ is a smooth curve. 10

(ii) Show that there exists a nontrivial parallel **normal** vector field $V(t)$, $t \in [0, \ell]$ along γ satisfying

$$V(\ell) = df_p(V(0)).$$ 10

(iii) Show that M can not have positive sectional curvature. 10