Riemannian Geometry (Spring, 2023)
Final Exam

Name: No.: Department:

All Riemannian manifolds are assumed to be connected.

1. (15 marks) Let M_k be a complete simply-connected Riemannian manifold with constant sectional curvature $k > 0$. Let

$$\gamma : \left[0, \frac{2\pi}{\sqrt{k}}\right] \to M$$

be a normal geodesic. Answer the following questions directly without proofs.

(i) Which point is the cut point of $\gamma(0)$ along γ?

(ii) What is the index of γ?

(iii) What is the injectivity radius of M_k?

2. (15 marks) Let (M, g) be a Riemannian manifold with nonnegative Ricci curvature and $f : M \to \mathbb{R}$ be a harmonic function. Suppose that the gradient vector field $\mathrm{grad} f$ has constant norm. Show that $\mathrm{grad} f$ is parallel.

3. (15 marks) Suppose that $(\widetilde{M}, \widetilde{g})$ and (M, g) are two Riemannian manifolds and $\pi : \widetilde{M} \to M$ is a Riemannian covering map. Show that (M, g) is complete if and only if $(\widetilde{M}, \widetilde{g})$ is complete.

4. (20 marks) Let (M, g) be a complete Riemannian manifold, and let $N \subset M$ be a compact submanifold of M without boundary. Let $p_0 \in M \setminus N$, and let

$$d(p_0, N) := \inf_{q \in N} d(p_0, q)$$

be the distance from p_0 to N. Show that there exists a point $q_0 \in N$ such that

$$d(p_0, q_0) = d(p_0, N).$$

Moreover, a minimizing geodesic $\gamma : [a, b] \to M$ which joins p_0 to q_0 is orthogonal to N at q_0, that is, $g(\dot{\gamma}(b), V) = 0$, for any $V \in T_{q_0}N \subset T_{q_0}M$.

5. (25 marks) Let (M, g) be a Riemannian manifold, and $\gamma : [0, \ell] \to M$ be a normal geodesic. We say that a vector filed $U(t)$ along γ is **almost parallel** if there exists a parallel vector field $V(t)$ along γ such that

$$U(t) = f(t)V(t), \text{ for some } f \in C^\infty(M).$$

(i) Prove that if M has constant sectional curvature k, then any normal Jacobi field $U(t)$ along γ with $U(0) = 0$ is almost parallel.

(ii) Let (M, g) be a complete simply-connected Riemannian manifold with constant sectional curvature -1. Calculate the volume of the geodesic ball with radius R.

6. (30 marks)

Let (M, g) be a complete noncompact Riemannian manifold with non-negative sectional curvature.

(i) For any given $q \in M$, show that there exists a ray emanating from q, i.e. a normal geodesic $\gamma : [0, \infty) \to M$ with $\gamma(0) = q$ and $d(q, \gamma(t)) = t$ for any $t \geq 0$.

(ii) Let $b^\gamma : M \to \mathbb{R}$ be the Busemann function with respect to the ray γ. Let $p \in M$ be any given point and $\xi : (-a, a) \to M$ be a given normal geodesic with $\xi(0) = p$. For any $\epsilon > 0$, show that there exists a lower barrier g_ϵ for b^γ at p such that

$$\frac{d^2}{dt^2}_{|t=0} g_\epsilon(\xi(t)) \geq -\epsilon.$$

Reference Answer for RG Final Exam (2023). Junhao Tian

1. (i) $\gamma(\frac{\pi}{\sqrt{k}})$ (ii) $\dim M_k - 1$ (iii) $\frac{\pi}{\sqrt{k}}$

2. By Bochner formula: $\frac{1}{2}\Delta|\nabla f|^2 = |\nabla^2 f|^2 + \langle \nabla \Delta f, \nabla f\rangle + Ric(\nabla f)$

given $Ric \geq 0$, $\Delta f = 0$ and $|\nabla f| \equiv$ Constant

we know $0 = |\nabla^2 f|^2 + Ric(\nabla f) \geq |\nabla^2 f|^2 \Rightarrow \nabla^2 f = 0$

that is: "∇f is parallel".

3. By Hopf-Rinow: (M,g) complete $\Leftrightarrow \forall p \in M, v \in T_pM$, $\exp_p(tv)$ well-defined on $[0,+\infty)$;

Let $\pi: \tilde{M} \to M$ be the covering map. we know π is local isometric. which means that π preserve geodesic. Thus "$\gamma: [0,+\infty) \to M$" $\Leftrightarrow$ "$\tilde{\gamma}: [0,+\infty) \to \tilde{M}$"

So. (M,g) complete $\Leftrightarrow (\tilde{M}, \tilde{g})$ complete. (Detail elision)

4. Choose a sequence $\{q_i\}$ $q_i \in N$. st: $\lim_{i\to\infty} d(q_i, p_0) = d(p_0, N)$

By compactness of N. $\exists\ q_0 \in N$ st $d(q_0, q_i) \to 0$. as $i \to +\infty$

By continuity of distant function $d(q_i, p_0) \to d(q_0, p_0)$ as $i \to +\infty$

thus $d(q_0, p_0) = d(p_0, N)$

Moreover, if $\exists\ V \in T_{q_0}N$ st $g(\dot\gamma(b), V) < 0$ ~~(Assume~~ consider following ODE:

$$\begin{cases} \nabla_T\nabla_T U + R(U,T)T = 0 \\ U(a) = 0, \quad U(b) = V \end{cases}$$

. the unique solution "U" is Jacobi vector field.

Let $\gamma_s(t) = \exp_{p_0}((t-a)(\dot\gamma(a) + sV))$. by the First Variation of Length

$$\frac{d}{ds}L(\gamma_s)\Big|_{s=0} = -\int_a^b \langle V(t), \nabla_{\dot\gamma}\dot\gamma\rangle dt + \langle V(b), \dot\gamma(b)\rangle - \langle V(a), \dot\gamma(a)\rangle$$

$$= \langle V(b), \dot\gamma(b)\rangle < 0.$$

That means, $\exists\ s > 0$ st $L(\gamma_s) < L(\gamma_0)$, then $d(p_0, N) \leq L(\gamma_s) < L(\gamma_0) = d(p_0, N)$

Contradiction!

5(i). Let $(E_1(t)=\dot{r}(t), E_2, \cdots E_n)$ be a parallel orthonormal frame along r.

then $U(t)=f^i E_i(t)$; the Jacobi equation tell us:

$$\nabla_{E_1}\nabla_{E_1} U + R(U, E_1)E_1 = 0$$

that is $\ddot{f}^i + k f^i = 0$ and $f^i(0)=0$, the solution is:

$$f^i(t)=\begin{cases} c^i \cdot \sin\sqrt{k}\, t & k>0 \\ c^i \cdot t & k=0 \\ c^i \cdot \sinh\sqrt{-k}\, t & k<0 \end{cases} \qquad (\text{assume } c^2 \neq 0)$$

then: $\frac{1}{f^2(t)} \cdot U(t) = \frac{1}{c^2}(c^i \cdot E_i(t))$, $\nabla\left(\frac{1}{f^2(t)} \cdot U(t)\right) = \frac{c^i}{c^2} \nabla E_i(t) = 0$

$U(t)$ is almost parallel.

(ii) In polar coordinates. $(s, \theta_1, \cdots \theta_{n-1})$. ~~wh~~ we know $dV = dy_1 \cdots dy_n = \xi \cdot ds\, d\theta_1 \cdots d\theta_{n-1}$ → Remark: 第一次答案中步骤3一丁's

where ~~$g(y_1 \cdots y_j)$~~ $(y_1, \cdots y_n)$ is the standard metric with $g_{ij} = \frac{4}{(1-(\Sigma y_i^2))^2} \cdot \delta_{ij}$

the normal geodesic is given by $r(s) = \frac{e^s - 1}{e^s + 1}$, $r: [0, +\infty) \to H^n \cong B_n$

then. $Vol(S^{n-1}) = Vol_0(S^{n-1}) \cdot \sqrt{\det(g_{ij}} \cdot \frac{1}{r}) = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \cdot \left(\frac{2}{1-|r|^2} \cdot \frac{1}{r}\right)^{n-1} = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \cdot \sinh(s)^{n-1}$

then. $B(0,R) = \int_0^R \frac{2 \cdot \pi^{n/2}}{\Gamma(\frac{n}{2})} \cdot \sinh(s)^{n-1}\, ds$

$$= \frac{2 \cdot \pi^{n/2}}{n \cdot \Gamma(\frac{n}{2})} \cdot \cosh(R) \cdot \mathrm{sech}(R) \cdot \sinh(R)^n \cdot F_1\left(\tfrac{1}{2}, \tfrac{n}{2}, \tfrac{n+2}{2}, -\sinh(R)^2\right)$$

where F_1 is hypergeometric Function.

6. (i) By noncompactness of M. we can find $\{p_i\}$ st: $d(p_i, q) \to +\infty$.

let r_i be the shortest curve from q to p_i. then $\{\dot{r}_i(0)\}$ has a subsequence converge in T_qM to $v \in T_qM$. let $r(t) = \exp_q(tv)$

the subsequence converge on every bound set $r([0,T])$.

which means $d(r(0), r(T)) = T$ for each $T \geq 0$.

6. (ii) Claim: $b^r(x) = \lim_{t\to+\infty} (t - d(x, r(t)))$ is convex.

i.e. $\lambda_1 b^r(\xi(t_1)) + \lambda_2 b^r(\xi(t_2)) \geq b^r(\xi(\lambda_1 t_1 + \lambda_2 t_2))$

we will prove this claim later

By convexity. there exist a line below $b^r(x)$ at $\xi((-a, a))$:

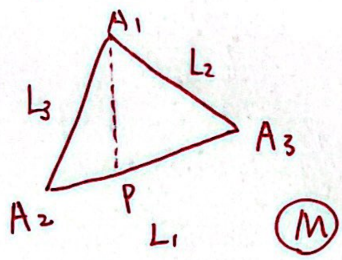
that is $\quad b^r(p) + k\cdot t \leq b^r(\xi(t))$

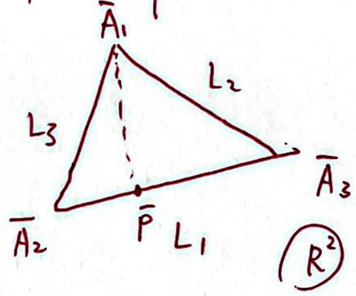
~~let $g_\varepsilon(t) = -\varepsilon t^2 + b^r(p) + k(t)$, then $\frac{d}{dt^2} g_\varepsilon(t)$~~

Let $g_\varepsilon(\xi(t)) = -\varepsilon t^2 + b^r(p) + kt$ then $\frac{d}{dt} g_\varepsilon(\xi(t)) = -\varepsilon \geq -\varepsilon$

We prove the Claim by the following Toponogov comparision Theorem.

Let (M, g) be Riemann mfd with $\sec \geq 0$, then for any three point A_1, A_2, A_3, $d(A_i, A_{i+1}) = L_{i+2}$. (index mod 3). P is a point on line $\overline{A_2 A_3}$.





with $d(A_2, P) = \lambda_2 L_1$, $d(A_3, P) = \lambda_3 \cdot L_1$. We also choose the conterpart in R^2

then. $d(A_1, P) \geq d(\bar{A}_1, \bar{P})$

By this comparision: we have:

$$\lambda_1 b^r(\xi(t_1)) + \lambda_2 b^r(\xi(t_2)) - b^r(\xi(\lambda_1 t_1 + \lambda_2 t_2))$$

$$= \lim_{t\to\infty} \left(d(\xi(\lambda_1 t_1 + \lambda_2 t_2), r(t)) - \lambda_1 d(\xi(t_1), r(t)) - \lambda_2 d(\xi(t_2), r(t)) \right)$$

$$\geq \lim_{t\to\infty} \left(\bar{d}(\xi(\lambda_1 t_1 + \lambda_2 t_2), r(t)) - \lambda_1 d(\xi(t_1), r(t)) - \lambda_2 d(\xi(t_2), r(t)) \right)$$

$$= 0.$$