Exam 2 for Differential Equations

May 25, 2023

ν always stands for the outward unit normal vector on the boundary, and U, Ω is always a bounded domain in $\mathbb{R}^n$.

1.(Sobolev space)

(a)(10 Marks) Suppose $|\log|x||^\alpha \in W^{1,2}(B_{\frac{1}{2}}(0)), B_{\frac{1}{2}}(0) \subseteq R^2$. Show the range of α.

(b)(10 Marks) Suppose $u \in W_0^{1,2}(U), U$ is an open bounded domain in R^n. Prove that $u^+ \in W_0^{1,2}(U)$.

2.(Morrey inequality) Let $u \in C^1(R^n)$, prove the Morrey inequality in the following two steps:

(a)(10 Marks) $\frac{1}{|B(x,r)|}\int_{B(x,r)} |u(x) - u(y)|dy \leq C(n)\int_{B(x,r)} \frac{|\nabla u(y)|}{|y-x|^{n-1}}dy$

(b)(10 Marks) $[u]_{C^\alpha(R^n)} + |u|_{L^\infty(R^n)} \leq C(n)||u||_{W^{1,p}(R^n)}$, where $p > n$, $\alpha = 1 - \frac{n}{p}$.

3.(20 Marks) Consider the equation

$$\begin{cases} -\Delta u = f \ in \ U \subseteq R^n, f \in L^2(U) \\ u \in W_0^{1,2}(U) \end{cases}$$

Using the method of minimizing sequence to prove the existence(15 Marks) and uniqueness(5 Marks) of the weak solution.

4.(Fredholm alternative) Suppose $U = [0, \pi] \times [0, 2\pi]$.

(a)(10 marks) Consider the eigenvalue problem:

$$\begin{cases} \Delta u + \lambda u = 0 & in \quad U \\ u = 0 & on \quad \partial U \end{cases}$$

Find the eigenvalues λ_k and the corresponding eigenfunctions u_k.

(b)(10 marks) Consider the equation:

$$\begin{cases} \Delta u + \frac{5}{4}u = 2x + y + a & in \quad U \\ u = 0 & on \quad \partial U \end{cases}$$

For which $a \in \mathbb{R}$ does this equation exist at least one solution ?

5. (a)(10 marks) Suppose $\eta(x) \in C_0^\infty(\mathbb{R})$, $\eta(x) = \begin{cases} C_0 e^{\frac{1}{|x|^2-1}} & , |x| < 1 \\ 0 & , |x| \geq 1 \end{cases}$, such that $\int_{\mathbb{R}} \eta(x)\, dx = 1$.

Let $f^\epsilon(x) = (\eta^\epsilon * f)(x)$, where $\eta^\epsilon(x) = \epsilon^{-n}\eta(\frac{x}{\epsilon})$, $f \in C^{0,1}(\mathbb{R})$, $f'(t) \geq 0$ and f is a convex function. Prove that:

$$\frac{df^\epsilon(t)}{dt} \geq 0, \frac{d^2 f^\epsilon(t)}{dt^2} \geq 0$$

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(b)(10 marks) Suppose $u \in H_0^1(\Omega)$ is a super-solution in the following sense:

$$\int_\Omega \sum_{i,j} a_{ij} u_i v_j \, dx \geq 0$$

, for any $v \geq 0$ in Ω and $v \in H_0^1(\Omega)$, where $0 < \lambda I \leq (a_{ij}) \leq \Lambda I$. Prove that $u^+ := \max\{u, 0\}$ is also a super-solution.

6. (a)(10 marks)(Moser iteration) Suppose $a_{ij}(x) \in L^\infty(B_R(0))$, $\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$. Consider the equation:

$$\begin{cases} -\sum\limits_{i,j}(a_{ij}(x)u_i)_j = 0 & , in \quad B_R(0) \\ u = 0 & , on \quad \partial B_R(0) \end{cases}$$

Assume $u > 0$, $u \in C^\infty(B_R(0))$ is a weak solution of this equation. Prove that for any $p > 1$, $\theta \in (0,1)$, we have:

$$\left(\int_{B_{\theta R}} u^{\frac{np}{n-2}} dx\right)^{\frac{n-2}{np}} \leq C\left(\int_{B_R} u^p dx\right)^{\frac{1}{p}}$$

and give the expression of C.

(b)(10 marks) Suppose $a_{ij}(x) \in C^1(\overline{\Omega})$, $\partial\Omega \in C^1$, $f \in L^2(\Omega)$, $\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$. Consider the equation:

$$\begin{cases} -\sum\limits_{i,j}(a_{ij}(x)u_i)_j = f & , in \quad \Omega \\ u = 0 & , on \quad \partial\Omega \end{cases}$$

Assume $u \in H_0^1(\Omega)$ is a weak solution of this equation. For any subset $V \subset\subset \Omega$, prove that: $u \in H^2(V)$ and

$$\int_V |D^2u|^2 dx \leq C\int_\Omega (f^2 + u^2) dx$$

where $C \sim V, \Omega$, coefficients of L.