Exam1 for Differential Equations

April 18, 2023

ν always stands for the outward unit normal vector on the boundary, and Ω is always a bounded domain in $\mathbb{R}^n$.

1.(10 marks) Suppose $\Delta u = 0 \;\; in \; B_{2R} \subseteq \mathbb{R}^n, u > 0$ .Prove that:

$$\sup_{B_R} |\nabla \log u| \leq \frac{C_n}{R}$$

2.(15 marks) Suppose $\sum_{ij} a_{ij}(x)u_{ij} = 0 \;\; in \; B_2(0) \subseteq \mathbb{R}^n$, $u > 0$, $0 < \lambda I \leq (a_{ij}) \leq \Lambda I$, $a_{ij} \in C^1(\bar{B}_2)$.Prove that:

$$\sup_{B_1} |\nabla \log u| \leq C(n, |a_{ij}|_{C^1(\bar{B}_2)}, \lambda, \Lambda)$$

3.(10 marks) Consider $Lu = a_{ij}(x)u_{ij} + b_i(x)u_i + c(x)u$, $\lambda I \leq (a_{ij}) \leq \Lambda I, \max_{\bar{\Omega}} |a_{ij}| + \max_{\bar{\Omega}} |b_i| \leq \Lambda$, $f \in C(\bar{\Omega})$, $\varphi \in C(\partial\Omega)$, $c(x) \leq 0$. If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ solves the equation:

$$\begin{cases} Lu = f & in \quad \Omega \\ u = \varphi & on \quad \partial\Omega \end{cases}$$

prove that $\max_{\bar{\Omega}} |u(x)| \leq \max_{\partial\Omega} |\varphi| + C \max_{\bar{\Omega}} |f|$.

(Hint:We can assume $\Omega \subseteq \{0 < x_1 < d\}$, and consider $w = u + (e^{\alpha d} - e^{\alpha x_1})F$.)

4.(5 marks) Consider the equation

$$\begin{cases} \Delta u = f & in \quad \Omega^c \\ u = \varphi & on \quad \partial\Omega \\ u \to 0 & as \; |x| \to \infty. \end{cases}$$

Prove the uniqueness of the solution.

5.(20 marks) Consider the equation

$$\begin{cases} \Delta u = f & in \quad \Omega \\ u = \varphi & on \quad \partial\Omega \end{cases}$$

Assume that $u \in C^2(\Omega) \cap C(\bar{\Omega})$, $\varphi \in C^1(\bar{\Omega})$.

(a)(5 marks) Reduce the gradient estimates to the boundary gradient estimates. That is:

$$\sup_{\Omega} |\nabla u| \leq C(\sup_{\partial\Omega} |\nabla u| + 1)$$

,where C depends on $|f|_{C^0}|, |\varphi|_{C^0}, \Omega$.
(b)(15 marks) Prove the boundary gradient estimates. That is: $\sup_{\partial\Omega} |\nabla u| \leq C$, where C depends on $|f|_{C^1}|, |\varphi|_{C^1}, \Omega$.
6.(Green function)
(a)(10 marks) Find the green function $G(x, y)$ in $B_R(0) \subset \mathbb{R}^2$, and calculate $\frac{\partial G(x,y)}{\partial \nu_y}$, $y \in \partial B_R(0)$.
Hint: $\Gamma(x, y) = \frac{1}{2\pi} \log |x - y|$.
(b)(15 marks) Suppose that $\Omega \subset \mathbb{R}^n$ is a C^1 bounded domain, $G(x, y)$ is the Green function on Ω, $f \in C(\overline{\Omega})$, $\varphi \in C(\partial\Omega)$. Consider the equation:

$$\begin{cases} \Delta u = f & in \quad \Omega \\ u = \varphi & on \quad \partial\Omega \end{cases}$$

Prove that $u(x) = \int_\Omega G(x, y)f(y)dy + \int_{\partial\Omega} \varphi(y)\frac{\partial G(x,y)}{\partial \nu_y} d\sigma_y$.

7.(integration by parts)
(a)(5 marks) Consider the equation:

$$\begin{cases} \Delta u = 2 & in \quad \Omega \\ u = 0 & on \quad \partial\Omega \end{cases}$$

Prove that:$\frac{n-2}{2} \int_\Omega |Du|^2 dx + \frac{1}{2} \int_{\partial\Omega} (x \cdot \nu)|Du|^2 d\sigma = -2n \int_\Omega u dx$
Hint: Multiply $(x \cdot Du)$.
(b)(5 marks) Assume that $\Omega' \subset\subset \Omega$, $\Delta u = 0$ in Ω, $u \in L^2(\Omega)$. Prove that:

$$\int_{\Omega'} |Du|^2 dx \leq C(n, dist(\Omega', \Omega)) \int_\Omega u^2 dx$$

Hint: Multiply $u\xi^2$.
(c)(5 marks) Assume that $\Omega' \subset\subset \Omega$, $\Delta u = f$ in Ω, $u \in L^2(\Omega)$, $f \in L^2(\Omega)$. Prove that:

$$\int_{\Omega'} |D^2u|^2 dx \leq C(n, dist(\Omega', \Omega)) \int_\Omega f^2 + u^2 dx$$

(d)(5 marks) Assume that $\Delta u = 0$ in B_2, $u > 0$. Prove that:

$$\int_{B_1} |D \log u|^2 dx \leq C(n)$$

Hint: Multiply $u^{-1}\xi^2$.
8.(Critical index problem)(15 marks)
Assume that $1 < \alpha < \frac{n-2}{n+2}$, $u > 0$. Consider the equation: $\Delta u + u^\alpha = 0$ in $\mathbb{R}^n$. Multiply this equation by $u^a \Delta u$ and find the appropriate constant a, such that we can prove:

$$u^{a-2}|Du|^4 \leq \partial_i(C_1 u^{a-1}|Du|^2 u_i + C_2 u^{a+\alpha} u_i + C_3 u^a u_{ij} u_j)$$