2022 年春季学期微分方程 II 期末试卷

2022 年 6 月 17 日

1. Suppose U is open, bounded, and ∂U is smooth. Suppose $u \in L^2(0,T;H^2(U))$, $u' \in L^2(0,T;L^2(U))$. Show that $u \in C([0,T];H^1(U))$ (after possibly being redefined on a set of measure zero).

2. Let $k \in \mathbb{Z}$, $T > 0$.

(a) Give the definition of weak solution to the wave equation

$$(W)\begin{cases} u_{tt} - \sum_{j,k=1}^{n} g^{jk}(t,x)\partial_j\partial_k u = F \quad in \quad [0,T)\times\mathbb{R}^n \\ u(x,0) = f \quad \partial_t u(x,0) = g \quad in \quad \mathbb{R}^n \end{cases}$$

Here g^{jk} is smooth and symmetric on $[0,T]\times\mathbb{R}^n$ and there exists $0 < \lambda < \Lambda < \infty$ such that

$$\lambda|\xi|^2 \leq g^{jk}(x,t)\xi_j\xi_k \leq \Lambda|\xi|^2 \text{ for any } (t,x) \in [0,T]\times\mathbb{R}^n.$$

(b) Assume there holds the energy estimate

$$\sum_{|\alpha|\leq 1} ||\partial^\alpha u(t)||_{H^k} \leq C(\sum_{|\alpha|\leq 1} ||\partial^\alpha u(0)||_{H^k} + \int_0^T ||F(\tau)||_{H^k} d\tau)$$

when $f = g = 0$ and $F \in C_c^\infty([0,T)\times\mathbb{R}^n)$, prove by the Hahn-Banach method that (W) has a unique weak solution $u \in C([0,T];H^{k+1}(\mathbb{R}^n)) \cap C^1([0,T];H^k(\mathbb{R}^n))$.

3. Suppose $U \subset \mathbb{R}^n$ is a bounded open set with smooth boundary. Let $2 < p < \frac{2n}{n-2}, n \geq 3$. Consider the nonlinear elliptic equation

$$\begin{cases} -\Delta u + \lambda u = |u|^{p-2}u \quad in \quad U \\ u > 0 \quad in \quad U \\ u = 0 \quad on \quad \partial U \end{cases}$$

Prove that for any $\lambda > -\lambda_1$, there exists a positive solution $u \in C^2(U) \cap C(\bar{U})$, where λ_1 is the principal eigenvalue of $-\Delta$ in $H_0^1(U)$.

4. State the Hille-Yosida Theorem for the semigroups of operators and use this theorem to prove that there exists a unique solution $u \in X = L^1((0,+\infty),\mathbb{R})$ to the equation

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} + \dfrac{\partial u(x,t)}{\partial x} = 0 \quad t>0, x>0 \\ u(t,0) = 0 \quad t>0 \\ u(0,\cdot) = \varphi \in L^1((0,+\infty),\mathbb{R}). \end{cases}$$

5. Let $U = (0,1) \subset \mathbb{R}$. For any $\varepsilon > 0$, take for granted that there is a smooth solution $u = u^\varepsilon(x,t)$ of the parabolic equation

$$(P)\begin{cases} u^\varepsilon_t - \varepsilon u^\varepsilon_{xx} - a(x,t)u^\varepsilon_x = 0 \quad 0<x<1, 0\le t<T \\ u^\varepsilon(0,t) = u^\varepsilon(1,t) = 0 \quad 0 \le t < T \\ u^\varepsilon(x,0) = g(x) \in C_c^\infty(U). \end{cases}$$

(a) Suppose $\sup_{[0,1]\times[0,T]}(|a(x,t)| + |\partial_{t,x}a(x,t)|) \le M$. Prove that there exists $C>0$ such that

$$\max_{0\le t\le T}(||u^\varepsilon(t)||_{H^1_0(U)} + ||u^{\varepsilon'}(t)||_{L^2(U)}) \le C||g||_{H^1(U)}, \forall 0<\varepsilon\le 1.$$

(b) There exists a weak solution $u \in L^2(0,T;H^1_0(U))$ with $u' \in L^2(0,T;L^2(U))$ of the above equation with $\varepsilon = 0$ in the sense that

$$(u',v) - \int_U a(x,t)u_x v \mathrm{d}x = 0$$

for each $v \in H^1_0(U)$ and $a.e.$ $0 \le t \le T$, and

$$u(0) = g.$$

(Hint: Note that u^ε is a weak solution to the parabolic equation (P) and take some subsequence $\varepsilon_k \downarrow 0$.)