Exam2 for Differential Equations II(H)

May 15, 2022

ν always stands for the outward unit normal vector on the boundary.

1 Basic part(19:00-21:30)

1.(Convolution) Suppose $\eta(x) \in C_0^\infty(B_1)$, $\eta(x) = \begin{cases} C_0 e^{\frac{1}{|x|^2-1}} & ,|x|<1 \\ 0 & ,|x| \geq 1 \end{cases}$, such that $\int_{\mathbb{R}^n} \eta(x)\,dx = 1$.
Let $f^\epsilon(x) = (\eta^\epsilon * f)(x)$, where $\eta^\epsilon(x) = \epsilon^{-n}\eta(\frac{x}{\epsilon})$. Prove that:
(a) If $f \in C^0(U)$, then for any subset $V \subset\subset U$, we have: $f^\epsilon(x) \to f$ in V.
(b) If $f \in L^p_{loc}(U)$,$1 < p < +\infty$. Prove that: $f^\epsilon(x) \to f$ in $L^p_{loc}(U)$.
(c) If f is a convex function. Prove that $f^\epsilon(x)$ is also a convex function.
(d) If $f(t) \in C^1_{loc}(\mathbb{R})$, $f'(t) \geq 0$. Prove that $f^\epsilon(t)$ is non-decreasing in $\mathbb{R}$.
(e) If $f(t) \in C^0(U)$, $U \subset \mathbb{R}^n$. Prove that:

$$\left|\frac{\partial}{\partial x_i} f^\epsilon(x)\right| \leq \frac{C}{\epsilon} \sup_{B(x,\epsilon)} |f|, \qquad \left|\frac{\partial}{\partial \epsilon} f^\epsilon(x)\right| \leq \frac{C}{\epsilon} \sup_{B(x,\epsilon)} |f|.$$

2.(Sobolev space)
(a) For which $\alpha \in \mathbb{R}, n \in \mathbb{N}$ does the function $f(x) = \frac{|\ln|x||^\alpha}{|x|^2}$ belong to $H^1(B^n_{\frac{1}{2}}(0))$?
(b) Prove the Trace Theorem: Assume U is bounded and ∂U is C^1. Then there exists a bounded linear operator $T : W^{1,p}(U) \to L^p(\partial U)$, such that:
(i) $Tu = u|_{\partial U}$ if $u \in W^{1,p}(U) \cap C(\overline{U})$
(ii) $\|Tu\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)}$, for each $u \in W^{1,p}(U)$, with $C \sim n, p, U$.
(c) State the Rellich-Kondrachov Compactness Theorem, and apply it to prove the Poincare's inequality: Let U be a bounded, connected, open subset of $\mathbb{R}^n$, ∂U is C^1, $1 \leq p \leq +\infty$. Then there exists $C \sim n, p, U$, such that

$$\|u - (u)_U\|_{L^p(U)} \leq C\|Du\|_{L^p(U)}$$

for each $u \in W^{1,p}(U)$, where $(u)_U = \int_U u(x)\,dx/|U|$.
(d) Assume $1 \leq p < \infty$, U is bounded, $u \in W^{1,p}(U)$. Prove that $u^+ \in W^{1,p}(U)$ and

$$Du^+ = \begin{cases} Du & ,a.e.\, in \{u > 0\} \\ 0 & ,a.e.\, in \{u \leq 0\} \end{cases}$$

3.(Fredholm alternative)
(a) Consider the eigenvalue problem:

$$\begin{cases} \Delta u + \lambda u = 0 & in \quad U = [0,\pi] \times [0,\pi] \\ u = 0 & on \quad \partial U \end{cases}$$

Find the first and the second eigenvalue λ_1, λ_2 and the corresponding eigenfunctions u_1, u_2.
(b) Consider the equation:

$$\begin{cases} \Delta u + 5u = x - a & in \quad U \\ u = 0 & on \quad \partial U \end{cases}$$

For which $a \in \mathbb{R}$ does this equation exist at least one solution ?
(c) Consider the equation:

$$\begin{cases} \Delta u + 3u = x^2 + y^2 & in \quad U \\ u = 0 & on \quad \partial U \end{cases}$$

Show that there exists a unique solution to this equation, and find the appropriate constant C, such that $\|u\|_{L^2(U)} \le C$.
4.Suppose $a_{ij}(x) \in C^1(\overline{U})$, $\partial U \in C^1$, $f \in L^2(U)$, $\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2$, $c(x) \in L^\infty(U)$. Consider the equation:

$$\begin{cases} -\sum\limits_{i,j}(a_{ij}(x)u_i)_j + c(x)u = f & , in \quad U \\ u = 0 & , on \quad \partial U \end{cases}$$

Assume $u \in H_0^1(U)$ is a weak solution of this equation.
(a) For any subset $V \subset\subset U$, prove that:

$$\int_V |Du|^2 dx \le C \int_U (f^2 + u^2) dx$$

where $C \sim V, U$, coefficients of L.
Hint: Consider $v = \xi^2 u$
(b) For any $x_0 \in \partial U$, prove that:

$$\int_{B_r(x_0)\cap U} |Du|^2 dx \le C \int_U (f^2 + u^2) dx$$

where $C \sim r$ and coefficients of L.
Hint: Consider $v = \xi^2 u$
(c) For any subset $V \subset\subset U$, prove that:

$$\int_V |D^2 u|^2 dx \le C \int_U (f^2 + u^2) dx$$

where $C \sim V, U$, coefficients of L.
5.(Maximum principal) Assume U is a bounded, smooth and convex open subset in $\mathbb{R}^2$. Consider the Dirichlet problem:

$$\begin{cases} \Delta u = -2 & in \quad U \\ u = 0 & on \quad \partial U \end{cases}$$

(a) Prove that $u > 0$ in U and $\frac{\partial u}{\partial \nu}|_{\partial U} < 0$
(b) Let $v = -\sqrt{u} < 0$ in U, then we have

$$\begin{cases} v\Delta v = -(1 + |Dv|^2) & in \quad U \\ v = 0 & on \quad \partial U \end{cases}$$

Prove that if $\varphi = v_{11}v_{22} - v_{12}^2 \ge 0$ in U, then $\varphi \equiv 0$ or $\varphi > 0$ in U.
(c) Based on (b), prove that v is strictly convex in U.

2 Additional part(21:30-22:30)

6. Consider a positive harmonic function $\Delta u + u = 0$, $u > 0$ in $B_1(0) \subset \mathbb{R}^n$, prove that there is a positive constant $C = C(n)$, such that: $|\nabla \log u| \leq C$ in $B_{\frac{1}{2}}(0)$.

7. Let $U \subset \mathbb{R}^n$ to be a bounded domain with smooth boundary, $\varphi \in C^\infty(\bar{U})$. $u \in C^\infty(\bar{U})$ is a solution to:

$$\begin{cases} \Delta u = 1 & in \quad U \\ \frac{\partial u}{\partial \nu} = \varphi & on \quad \partial U \end{cases}$$

Prove that:

(a) Prove that there is a positive constant $C \sim U, g, n$, such that:

$$\sup_{\bar{U}} |Du| \leq C(1 + \sup_{\partial U} |Du|)$$

(b) Let $d_0 > 0$ is a small real number such that $U_{d_0} = \{x \in U | \text{dist}(x, \partial U) \geq d_0\}$ is nonempty. Prove that:

$$\sup_{U_{d_0}} |Du| \leq \frac{C_1}{d_0}$$

Where $C_1 \sim n, \sup_{\bar{U}} |u|$.

(c) Prove that:

$$\sup_{\bar{U} \setminus U_{d_0}} |Du| \leq C_2$$

Where $C_2 \sim \varphi, \sup_{\bar{U}} |u|, U$.