Exam1 for Differential Equations II(H)

19:00-21:30 April 3, 2022

This exam is of full mark 120. ν always stands for the outward unit normal vector on the boundary.

1.(The monotonicity formula of Green's function) Suppose $u \in C^2(U)$, $U \subset \mathbb{R}^n$ to be a bounded domain.

(a)(15 marks) If $\Delta u \geq 0$, prove that for any ball $B_R(x_0) \subset\subset U$, we have:

$$u(x_0) \leq \frac{1}{\mathrm{Vol}(\partial B_R(x_0))} \int_{\partial B_R(x_0)} u(y)dy$$

(b)(10 marks) If $\Delta u = 0$ in $B_1(0) \subset \mathbb{R}^n$, $0 < r < 1$, $D(r) = \int_{B_r(0)} |\nabla u|^2 dx$. Prove that:

$$D'(r) = \frac{n-2}{r} D(r) + 2 \int_{\partial B_r(0)} (\frac{\partial u}{\partial \nu})^2$$

2. Consider the Dirichlet problem:

$$\begin{cases} \Delta u = 2 & in \quad U \\ u = 0 & on \quad \partial U \end{cases}$$

where U is a bounded domain with smooth boundary in $\mathbb{R}^n$. $u \in C^2(U) \cap C^1(\overline{U})$ is a solution to it. Prove that:

(a)(5 marks) $u < 0$ in U

(b)(5 marks) $\frac{\partial u}{\partial \nu} < 0$ on ∂U

(c)(10 marks) Let $\varphi = |\nabla u|^2 + \alpha u$, find the appropriate real number α, such that φ attains its maximum over $\bar{U}$ on ∂U.

3.(10 marks) Consider a positive harmonic function $\Delta u = 0$, $u > 0$ in $B_1(0) \subset \mathbb{R}^n$, prove that there is a positive constant $C = C(n)$, such that:

$$|\nabla \log u| \leq C$$

in $B_1(0)$.

4.(Integrate by parts)

(a)(5 marks) Consider the Neumann problem:

$$\begin{cases} \Delta u + cu = f & in \quad U \\ \frac{\partial u}{\partial \nu} = \varphi & on \quad \partial U \end{cases}$$

where U is a bounded domain in $\mathbb{R}^n$ with smooth boundary, and $c \leq 0$. Prove that if the solution exists, it is unique.

(b)(10 marks) Suppose U to be a bounded domain in $\mathbb{R}^n$, and $U' \subset\subset U$. u is a solution to $\Delta u = f$ in U.Prove that there is a positive constant $C \sim n, U, U'$, such that:

$$\int_{U'} (|\nabla^2 u|^2 + |\nabla u|^2) dx \leq C(\int_U (f^2 + u^2) dx)$$

(c)(5 marks) Suppose $\Delta u = 0$ in $\mathbb{R}^n$, and $u \in L^2(\mathbb{R}^n)$. Prove that u is constant valued.

5. Consider the Dirichlet problem:

$$\begin{cases} \Delta u = f & in \quad U \\ u = g & on \quad \partial U \end{cases}$$

where $U \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. $u \in C^3(U) \cap C^1(\bar{U})$ is a solution.

(a)(4 marks) Suppose $R = \sup_{x \in \partial U} |x - x_0|$, $V(x) = \frac{|x-x_0|^2 - R^2}{2n}$. Verify that: $\Delta V = 1$, $V|_{\partial U} \leq 0$.

(b)(6 marks) Let $f_+ = \max(f, 0)$, $f_- = \max(-f, 0)$. Prove that:

$$u(x) \geq (\sup f_+) V(x) + \inf g$$

$$u(x) \leq (\inf f_+) V(x) + \sup g$$

(c)(6 marks) Prove that there is a positive constant $C \sim U, f, g, n$, such that:

$$\sup_{\bar{U}} |\nabla u| \leq C(1 + \sup_{\partial U} |\nabla u|)$$

6.(10 marks) Let $\Delta u = 0$ in $\Omega = \{x \in \mathbb{R}^n | |x| > 1\}$. $u \in L^2(\bar{\Omega})$, and $\lim_{|x| \to +\infty} u(x) = 0$. Prove that:

$$\max_{\Omega} |u| = \max_{\partial \Omega} |u|$$

Hint: Consider the set $B_R(0) \setminus B_1(0)$, R is sufficiently large.

7. Let $U \subset \mathbb{R}^n$ to be a bounded domain with smooth boundary, $\varphi \in C^\infty(\bar{U})$. $u \in C^\infty(\bar{U})$ is a solution to:

$$\begin{cases} \Delta u = 1 & in \quad U \\ \frac{\partial u}{\partial \nu} + u = \varphi & on \quad \partial U \end{cases}$$

Prove that:

(a)(6 marks) $\sup_{\bar{U}} |u| \leq \sup_{\partial U} |\varphi| + C$. Where $C \sim n, U$.

(b)(6 marks) Let $d_0 > 0$ is a small real number such that $U_{d_0} = \{x \in U | \text{dist}(x, \partial U) \geq d_0\}$ is nonempty. Prove that:

$$\sup_{U_{d_0}} |\nabla u| \leq \frac{C_1}{d_0}$$

Where $C_1 \sim n, \sup_{\bar{U}} |u|$.

(c)(7 marks) Prove that:

$$\sup_{\bar{U} \setminus U_{d_0}} |\nabla u| \leq C_2$$

Where $C_2 \sim \varphi, u, U$.