Spring 2021, Midterm

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1. (15 points) Let $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ be nonnegative, integrable, and adapted to the filtration $(\mathcal{F}_n)_{n\geq 0}$. Suppose that

$$E(X_{n+1}|\mathcal{F}_n) \leq X_n + Y_n, \quad n \geq 0,$$

with $\sum_{n\geq 0} Y_n < \infty$ a.s. Prove that X_n converges a.s. to a finite limit.

2. (15 points) Define $S_0 = 0$ and $S_n = X_1 + \ldots + X_n$ for any $n \geq 1$, where $X_1, X_2, \ldots$ are independent with $EX_n = 0$ and $\mathrm{var}(X_n) = \sigma^2$ for any $n \geq 1$. Use the martingale $(S_n^2 - n\sigma^2)_{n\geq 0}$ to prove that if T is a stopping time with $ET < \infty$, then

$$ES_T^2 = \sigma^2 ET.$$

3. (15 points) Let $X = (X_n, n \geq 0)$ be a Markov chain on the countable state space S. Define

$$T_x = \inf\{n \geq 1 : X_n = x\}, \quad x \in S,$$

and

$$N_{y,x} = \sum_{n=0}^{T_x - 1} \mathbf{1}\{X_n = y\}, \quad x, y \in S.$$

Also define

$$w_{x,y} = P_x(T_y < T_x), \quad x, y \in S.$$

Suppose that $x, y \in S$, $x \neq y$, and $\rho_{y,x} = P_y(T_x < \infty) = 1$. Show that

$$P_x(N_{y,x} \geq k) = w_{x,y}(1 - w_{y,x})^{k-1}, \quad k \geq 1.$$

4. (15 points) Let p be a transition probability on the countable state space S, and for the Markov chain $X = (X_n)_{n\geq 0}$ with the transition probability p, define

$$T_y = \inf\{n \geq 1 : X_n = y\}, \quad y \in S.$$

Prove that if the transition probability p is irreducible and positive recurrent, then

$$E_x T_y < \infty, \quad x, y \in S.$$

5. (20 points) Given a nonnegative supermartingale $X = (X_n)_{n\geq 0}$ and some stopping times $\tau_0 \leq \tau_1 \leq \cdots$, show that the sequence $(X_{\tau_n})_{n\geq 0}$ is again a supermartingale. (Hint: Truncate the times τ_n, and use the conditional Fatou's lemma.)

6) (20 points) Let S be a countable state space. Prove that for every irreducible, positive recurrent subset $S_k \subset S$, there exists a unique invariant distribution ν_k restricted to S_k, and every invariant distribution is a convex combination $\sum_k c_k \nu_k$.