1. Let $\mathbb{B}^n$ be the ~~r~~unit ball in $\mathbb{C}^n$, $\mathcal{U}^n := \{(z_1,\dots,z_n)\in\mathbb{C}^n \mid \operatorname{Im} z_n > \sum_{j=1}^{n-1}|z_j|^2\}$

Prove: $(z_1,\dots,z_n)\longmapsto \left(\frac{z_1}{1+z_n},\dots,\frac{z_{n-1}}{1+z_n}, i\frac{1-z_n}{1+z_n}\right)$ is a biholomorphism from ~~for~~ $\mathbb{B}^n$ to $\mathcal{U}^n$ and calculate its inverse.

2. ~~Denote~~ Let $\Omega\subset\mathbb{C}^n$ be a domain, $p>1$, $\varphi\in PSH(\Omega)$.

Define $A^p(\Omega,\varphi) := \{f\in\mathcal{O}(\Omega) \mid \int_\Omega |f|^p e^{-\varphi}\, d\lambda < +\infty\}$

$\|\cdot\|_{p,\varphi} := \left(\int_\Omega |f|^p e^{-\varphi}\, d\lambda\right)^{\frac{1}{p}}$.

(1) Prove $(A^p(\Omega,\varphi), \|\cdot\|_{p,\varphi})$ is a complex Banach Space.

(2) Calculate $\dim_{\mathbb{C}} A^1(\cancel{\Omega}, \log(1+|\cdot|^{2021}))$. 此处Ω应改为C

3. For $p=(p_1,\dots,p_n)\in(0,+\infty)^n$, $E_p := \{(z_1,\dots,z_n)\in\mathbb{C}^n \mid \sum_{j=1}^n |z_j|^{p_j} < n\}$

(1) Prove $E_p$ is pseudoconvex;

(2) Prove $E_p$ is strongly pseudoconvex if and only if $p_j = 2$, $\forall 1\le j\le n$.

(3) Let $\{f_j\}_{j=1}^\infty \subset \mathcal{O}(\mathbb{B}^n, E_p)$ and $\{f_j(0)\}_{j=1}^\infty$ converges to $(1,\dots,1)$. Show that $\{f_j\}_{j=1}^\infty$ converges locally uniformly to $(1,\dots,1)$.

4. Prove a domain is pseudoconvex if and only if it is a domain of holomorphy.

5. Let $0\in\Omega\subseteq\mathbb{C}^n$ be a Reinhardt domain, $f\in\mathcal{O}(\Omega)$. prove that $f$ can be written as $\sum_{k=0}^\infty P_k(z)$, where $P_k(z) = \sum_{|\alpha|=k} a_\alpha z^\alpha$ is a (complex) homogeneous polynomial of $k$-th order, and the series converges normally.

(Choose one of the following two questions to answer)

~~6. Let $\Omega\in\mathbb{R}^n$, $u\in sh(\Omega)$, $u\ge 0$ on $\Omega$. Prove~~

6. $\forall v\in C^\infty(\Omega,\mathbb{C})$, show that there exist $u_1,\dots,u_n\in C^\infty(\Omega,\mathbb{C})$, s.t. $\sum_{j=1}^n \frac{\partial u}{\partial \bar z_j} = v$ on $\Omega$.

$\Omega\subset \cancel{\mathbb{R}}\,\mathbb{C}^n$

7. Let $\Omega\subset\mathbb{R}^n$. $u\in sh(\Omega)$, $u\ge 0$ on $\Omega$. Prove: $u\in W^{1,2}_{loc}(\Omega)$ and $\int_\Omega |\operatorname{grad} u|^2 \chi^2\, d\lambda \le 4\int_\Omega u^2 |\operatorname{grad}\chi|^2\, d\lambda$, $\forall \chi\in C_c^1(\Omega,\mathbb{R})$.