中国科学技术大学数学科学学院

2021年春季学期《微分方程II(II)》期末考试

2021年7月3日，　8:30 - 10:30,　二教2104

姓名:　　　　学号: __

注意事项：

1. 请将解答写在答题纸上，试卷和答题纸一并上交；
2. 闭卷考试，总分110分. 得分超过100分时，成绩取整为100分；
3. 在试卷正文中，我们始终假定$\Omega \subset \mathbb{R}^n$ 为有界集且具有C^∞边界$\partial\Omega$.

试卷正文

1. [10 分] Assume $u \in L^1_{loc}(\Omega)$ and $V \subset\subset \Omega$. For $1 < p < \infty$, if $u \in L^p(V)$ and the differential quotient of u satisfies

$$\|D^h u\|_{L^p(V)} \le C$$

for some constant C and for all $0 < h < \frac{1}{2}\mathrm{dist}(V, \partial\Omega)$, show that

$$u \in W^{1,p}(V), \quad \text{with} \quad \|Du\|_{L^p(V)} \le C.$$

2. [25 分] Assume that $a^{ij}(x) \in C^1(\bar{\Omega})$, $a^{ij}(x) = a^{ji}(x)$ and $(a^{ij}(x)) \ge \theta I > 0$ for all $x \in \Omega$, $f \in L^2(\Omega)$. Suppose that $u \in H^1(\Omega)$ is a weak solution of

$$-\sum_{i,j=1}^{n} \left(a^{ij}(x) u_{x_i}\right)_{x_j} = f(x) \qquad \text{in } \Omega.$$

(a) Show that for each $V \subset\subset \Omega$, there exists a constant $C = C(V, \Omega, a^{ij})$ such that

$$\|u\|_{H^1(V)} \le C\left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right).$$

(b) Show that for each $V \subset\subset \Omega$, there exists a constant $C = C(V, \Omega, a^{ij})$ such that

$$\|u\|_{H^2(V)} \le C\left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right).$$

3. [15 分] Assume that $b^i(x) \in C^1(\bar{\Omega})$. Let $u \in C^3(\Omega) \cap C^1(\bar{\Omega})$ be a solution of

$$\Delta u(x) = \sum_{i=1}^{n} b^i(x) u_{x_i}(x) \qquad \text{in } \Omega.$$

Show that there exists a constant $C = C(\Omega, b^i)$ such that

$$\max_{x \in \bar{\Omega}} |Du|(x) \le C\left(\max_{\partial\Omega} |Du| + \max_{\partial\Omega} |u|\right).$$

4. [20 分] Let $0 < T < \infty$ and $\Omega_T = \Omega \times (0, T]$, $\Gamma_T = \bar{\Omega}_T \setminus \Omega_T$. Denote

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x,t)u_{x_i x_j}(x,t) + \sum_{i=1}^{n} b^i(x,t)u_{x_i}(x,t) + c(x,t)u(x,t)$$

where $a^{ij}, b^i, c \in C(\bar{\Omega}_T)$, $a^{ij} = a^{ji}$ and $(a^{ij}(x,t)) \geq \theta I > 0$. Given $f, \varphi \in C(\bar{\Omega}_T)$ and $g \in C(\bar{\Omega})$, suppose $u \in C_1^2(\Omega_T) \cap C(\bar{\Omega}_T)$ is a solution to the initial/boundary-value problem

$$\begin{cases} u_t + Lu = f & \text{in } \Omega_T \\ u = \varphi & \text{on } \partial\Omega \times [0, T] \\ u = g & \text{on } \Omega \times \{t = 0\}. \end{cases}$$

(a) Assume that $f \geq 0$ in Ω_T, $\varphi \geq 0$ on $\partial\Omega \times [0, T]$ and $g \geq 0$ in Ω. Show that $u \geq 0$ in Ω_T.

(b) Show that there exists a constant C depending on Ω, L, T such that

$$\max_{\bar{\Omega}_T} |u(x,t)| \leq C \left(\max_{\partial\Omega \times [0,T]} |\varphi| + \max_{\bar{\Omega}} |g| + \max_{\bar{\Omega}_T} |f| \right).$$

5. [20 分] Assume u is a smooth solution of the hyperbolic PDE

$$u_{tt} - \sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + c(x)u = 0 \qquad \text{in} \quad \mathbb{R}^n \times (0, \infty),$$

where the coefficients $a^{ij}(x)$ and $c(x) \geq 0$ are smooth and independent of time, $a^{ij} = a^{ji}$ and $(a^{ij}(x)) \geq \theta I > 0$. Fix a space-time point $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$, and let $q(x) = \mathrm{dist}_{a^{ij}}(x, x_0)$ denote the distance of x to x_0 with respect to the Riemannian metric (a^{ij}), which means that $q(x)$ satisfies

$$q(x_0) = 0, \qquad \sum_{i,j=1}^{n} a^{ij}(x)q_{x_i}(x)q_{x_j}(x) \equiv 1, \;\; q > 0 \quad \text{in } \mathbb{R}^n \setminus \{x_0\}.$$

Define the curved backwards wave cone

$$K = \{(x,t) \mid q(x) < \overset{t_0 - t}{\cancel{t_0}}, \quad 0 \leq t < t_0\}.$$

Show that if $u \equiv u_t \equiv 0$ in $K_0 = K \cap \{t = 0\}$, then $u \equiv 0$ within K.

6. [20 分] Apply the method of *Calculus of Variation* to prove that for every $f \in L^{\frac{2n}{n+2}}(\Omega)$ and $g \in H^1(\Omega)$ with $n > 2$, there exists a unique weak solution $u \in H^1(\Omega)$ of

$$\begin{cases} -\Delta u = f(x) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$