School for Mathematical Sciences, USTC

2021 Algebraic Geometry Final Exam

Time: 14:30–16:30. Good luck!

Problem 1

Let X be a projective nonsingular variety over k. For any $n > 0$ we define the nth plurigenus of X to be $P_n = \dim_k \Gamma(X, \omega_X^{\otimes n})$. And for any $q, 0 \leq q \leq \dim X$ we define the Hodge numbers $h^{q,0} = \dim_k \Gamma(X, \Omega^q_{X/k})$. where $\Omega^q_{X/k} = \Lambda^q \Omega_{X/k}$ is the sheaf of regular q-forms on X. Show that P_n and $h^{q,0}$ are birational invariants of X, i.e., if X and X' are birationally equivalent nonsingular projective varieties, then $P_n(X) = P_n(X')$ and $h^{q,0}(X) = h^{q,0}(X')$.

Problem 2

Let $X = \mathbb{P}^n_k$. Show that $H^q(X, \Omega^p_X) = 0$ for $p \neq q$ and $H^q(X, \Omega^p_X) = k$ for $p = q, 0 \leq p, q \leq n$.

Problem 3

Let $f : X \to Y$ be a morphism of ringed spaces, let $\mathcal{F}$ be an $\mathcal{O}_X$-module, and let $\mathcal{E}$ be a locally free O_Y-module of finite rank. Prove the projection formula:

$$R^i f_*(\mathcal{F} \otimes f^*\mathcal{E}) \cong R^i f_*(\mathcal{F}) \otimes \mathcal{E}.$$

Problem 4

Let $f : X \to Y$ be a proper, flat morphism of varieties over k. Suppose for some point $y \in Y$ that the fiber X_y is smooth over $k(y)$. Then show that there is an open neighbourhood U of y in Y such that $f : f^{-1}(U) \to U$ is smooth.

Problem 5

Let X be a normal, projective variety over an algebraically closed field k. Let $|D|$ be a linear system (of effective Cartier divisors) without base points, and assume that $|D|$ is not composite with a pencil, which means that if $f : X \to \mathbb{P}^n_k$ is the morphism determined by $|D|$, then $\dim f(X) \geq 2$. Then show that every divisor in $|D|$ is connected.

Problem 6

Let $\{X_t\}$ be a family of hypersurfaces of the same degree in $\mathbb{P}^n_k$. Show that for each i, the function $h^i(X_t, \mathcal{O}_{X_t})$ is a constant function of t.