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中国科学技术大学数学科学学院

2021～2022 学年第一学期期中试卷(A卷)

课程名称：微分流形　课程编号：MATH5003P

开课院系：数学科学学院　考试形式：闭卷

姓名：＿＿＿＿＿　学号：＿＿＿＿＿＿　班级：＿＿＿＿＿＿

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General facts about this exam:

- This is a closed-book examination.
- There are 9 problems (10 pages including this cover) in total.
- Read all the questions carefully before starting to work.
- Attempt to answer all questions for partial credit.
- You have two hours to complete this exam.

Unless otherwise stated, smoothness about this exam:

- All manifolds in this exam are smooth and are of dimensions at least 1.
- All maps in this exam are smooth.
- All vector fields in this exam are smooth.
- All distributions in this exam are smooth.
- All group actions in this exam are smooth.

Problem 1 (4 points each, 20 points in total)

Linearization is one of the major themes in this course. [Write down as much details as you can.]

(1) Let M be a smooth manifold and $p \in M$. The tangent space T_pM can be viewed as the linearization of M at p. Write down the definition of T_pM.

(2) Let $f : M \to N$ be a smooth map. The differential $df_p : T_pM \to T_{f(p)}N$ is the linearization of f at p. Write down the definition of $df_p : T_pM \to T_{f(p)}N$.

(3) Let G be a Lie group. The Lie algebra $\mathfrak{g}$ of G is the linearization of G. Write down the definition of the Lie algebra $\mathfrak{g}$ of a Lie group G.

(4) Any Lie group homomorphism $\phi : G \to H$ can be linearized to a Lie algebra homomorphism $d\phi : \mathfrak{g} \to \mathfrak{h}$. Write down the definition of $d\phi : \mathfrak{g} \to \mathfrak{h}$.

(5) The canonical submersion theorem tells us that locally, any submersion can be "linearized" to a simple linear one. Write down the canonical submersion theorem.

(6) The local Frobenius theorem tells us that if $\mathcal{V}$ is a k-dimensional involutive distribution, then locally, $\mathcal{V}$ can be "linearized" to a simple linear one. Write down the local Frobenius theorem.

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Problem 2 (2 points each, 20 points in total)

Which of the following statements are correct? Put a "T" before correct ones, and an "F" before wrong ones.

( ) Any connected component of a smooth manifold M is an open subset in M.

( ) Let K be a compact subset in M. Then there exists a smooth function f on M so that $K = f^{-1}(0)$.

( ) The complex projective plane $\mathbb{CP}^2$ is non-orientable.

( ) There exists no submersion from $\mathrm{O}(n)$ to $\mathrm{SL}(n)$.

( ) The composition of two constant rank maps is still a constant rank map.

( ) Any proper injective immersion $f : M \to N$ between two smooth manifolds is an embedding.

( ) For any smooth map $f : S^2 \to \mathbb{R}^4$ and any $\varepsilon > 0$, there is an immersion $f_\varepsilon : S^2 \to \mathbb{R}^4$ so that $|f - f_\varepsilon| < \varepsilon$.

( ) For any smooth manifolds M, N, the property "$f \in C^\infty(M, N)$ is an immersion" is a stable property.

( ) There exists no smooth vector field on S^2 so that $X_p = 0$ for exactly one point.

( ) If X_1, X_2 are linearly independent smooth vector fields on M, then near each $p \in M$, there is a local chart(φ, U, V) so that $X_1 = \partial_{x_1}$ and $X_2 = \partial_{x_2}$ on U.

( ) On any smooth manifold, there exists at most one Lie group structure.

Problem 3 (4 points each, 20 points in total)

For each of the following statements, write down an example.(No detail is needed.)

(1) A smooth map $f : \mathbb{R}^2 \setminus \{(1,0)\} \to \mathbb{R}^2 \setminus \{(0,1)\}$ that is a local diffeomorphism everywhere but not a diffeomorphism.

(2) A smooth function $f : S^2 \to \mathbb{R}$ and a regular point $p \in S^2$ of f, so that $f(p)$ is not a regular value of f.

(3) A continuous function $f : \mathbb{R} \to \mathbb{R}$ so that the graph Γ_f of f is a smooth submanifold of $\mathbb{R}^2$, but f itself is not a smooth function.

(4) A smooth vector field on $\mathbb{R}^2$ that is not complete.

(5) A smooth distribution on $\mathbb{R}^4$ that is not integrable.

(6) A smooth function $f \in C^\infty(\mathbb{R})$ whose critical values form an uncountable set.

Problem 4 (10 points)

Define a map $f : \mathbb{RP}^1 \to \mathbb{RP}^n$ $(n \geq 2)$ by the formula

$$f([t:s]) := [t^n : t^{n-1}s : \cdots : ts^{n-1} : s^n].$$

Prove: f is an embedding of $\mathbb{RP}^1$ to $\mathbb{RP}^n$.

Problem 5 (15 points)

(a) State the theorem of the existence of partition of unity (Yes, you need to define P.O.U.).

(b) Let $U \subset \mathbb{R}^n$ be open. Define a bilinear map $R_0 : \Gamma^\infty(TU) \times \Gamma^\infty(TU) \to \Gamma^\infty(TU)$ by

$$R_0(X,Y) = \sum_{i,j} X^i(\partial_i Y^j)\partial_j, \qquad \text{if } X = \sum_i X^i \partial_i,\, Y = \sum_j Y^j \partial_j.$$

Prove: for any $f \in C^\infty(U)$ and any $X, Y \in \Gamma^\infty(TU)$, we have

$$R_0(fX,Y) = fR_0(X,Y) \quad \text{and} \quad R_0(X,fY) = fR_0(X,Y) + (Xf)Y.$$

c) Prove: for any smooth manifold M, there exists (not unique) a bilinear map $R : \Gamma^\infty(TM) \times \Gamma^\infty(TM) \to \Gamma^\infty(TM)$ so that for any $f \in C^\infty(M)$ and any $X, Y \in \Gamma^\infty(TM)$,

$$R(fX,Y) = fR(X,Y) \quad \text{and} \quad R(X,fY) = fR(X,Y) + (Xf)Y.$$

Problem 6 (10 points)

Let M, N be smooth manifolds. Prove: A smooth map $f : M \to N$ is a diffeomorphism if and only if it is an invertible immersion.

Problem 7 (20 points)

Consider the set

$$M = \left\{(x_1, x_2, y_1, y_2) \in \mathbb{R}^4 \ : \ (x_1 - y_1)^2 + (x_2 - y_2)^2 = 2\right\}.$$

(a) Prove: M is a smooth submanifold of $\mathbb{R}^4$.

(b) Consider a constant vector field on $\mathbb{R}^4$,

$$X = a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + b_1 \frac{\partial}{\partial y_1} + b_2 \frac{\partial}{\partial y_2}.$$

Find out all (a_1, a_2, b_1, b_2) so that X is tangent to M at the point $(2, 2, 1, 1) \in M$.

(c) Prove: M is diffeomorphic to $\mathbb{R}^2 \times S^1$.

(d) Does the plane $P := \{(c, c, d, d) : c, d \in \mathbb{R}\}$ intersect M transversally in $\mathbb{R}^4$? Prove you claim.

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Problem 8 (15 points)

(a) Consider the smooth manifold $G = \mathrm{GL}(n, \mathbb{R}) \times \mathbb{R}^n$. Define a multiplication operation on G via

$$(X, x) \cdot (Y, y) := (XY, Xy + x).$$

Prove: G is a Lie group with respect to this multiplication.(The affine group of $\mathbb{R}^n$)

(b) Consider the vector space $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R}) \times \mathbb{R}^n$. Define a bracket on $\mathfrak{g}$ via

$$[(A, a), (B, b)] := (AB - BA, Ab - Ba).$$

Prove: $(\mathfrak{g}, [\cdot, \cdot])$ is a Lie algebra.

(c) Define a map $e : \mathfrak{g} \to G$ by

$$e(A, a) := \left(e^A, \sum_{n=1}^{\infty} \frac{1}{n!} A^{n-1} a \right).$$

For any $(A, a) \in \mathfrak{g}$, let $h(t) := e(tA, ta)$. Prove: $\{h(t) \mid t \in \mathbb{R}\}$ is a one-parameter subgroup of G.

Problem 9 (10 points)

In class we defined the exponential map for Lie groups. Now for any smooth manifold M and any $p \in M$, I want to define a local "exponent map" at p. Here is how I define it:

> Take a coordinate chart (φ, U, V) around p. For any $X_p \in T_pM$, we can write $X_p = \sum c_i \partial_i|_p$. Define a local smooth vector field X on U by letting $X_q = \sum c_i \partial_i|_q$ for any $q \in U$. By the fundamental theorem of differential equations, there is an $\varepsilon_p > 0$ so that the integral curve of X starting at p exists for $|t| < \varepsilon_p$. It follows that for any $\varepsilon < \varepsilon_p$, the integral curve γ_ε of the vector field εX starting at p exists for $|t| < \varepsilon_p/\varepsilon$. I define $\exp_p(\varepsilon X_p) := \gamma_\varepsilon(1)$. By this way I can define $\exp_p$ for all vectors in T_pM that are small enough.

Find out the mistake(s).

<u>Your answer:</u> [write down the mistake(s) with brief explanation/counterexample]