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教材：Folland实分析
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中国科学技术大学
2020-2021 秋季学期 高等实分析 期末试卷

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注意. 所有题目的解答要有详细过程，其中使用的定理或者命题需要注明.

I. Assume n is a positive integer. Let $1 \le q_0 < q_1 < \infty$ and $1 \le p < \infty$. For each $t \in (0,1)$, define the index q_t by

$$\frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

1. Show that if $f \in L^{q_0}(\mathbb{R}^n) \cap L^{q_1}(\mathbb{R}^n)$ then $f \in L^{q_t}(\mathbb{R}^n)$.
2. Show that if $f \in$ weak-$L^{q_0}(\mathbb{R}^n) \cap$ weak-$L^{q_1}(\mathbb{R}^n)$ then $f \in L^{q_t}(\mathbb{R}^n)$.
3. Using the result in the second item, show the following simple case of the Marcinkiewicz interpolation inequality. If T is a sublinear map from $L^p(\mathbb{R})$ to the space of measurable functions on $\mathbb{R}^n$ that is weak types (p, q_0) and p, q_1, then T is strong type (p, q_t)

II. Assume $1 \le p, q, r \le \infty$ satisfy the relation $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

1. In the non-endpoint case $p > 1, q > 1$ and $r < \infty$, prove the following assertion. If $f \in L^p(\mathbb{R}^n)$ and $g \in$ weak-L^q, then $f * g \in L^r$ and $\|f * g\|_{L^r(\mathbb{R}^n)} \le C_{pq}\|f\|_{L^p(\mathbb{R}^n)}[g]_q$. Here C_{pq} is independent of f and g.
2. In the endpoint case $p = 1$ and $r = q > 1$, prove the following assertion. If $f \in L^1(\mathbb{R}^n)$ and $g \in$ weak-L^q, then $f * g \in$ weak-$L^q(\mathbb{R}^n)$ and $[f * g]_q \le C_q\|f\|_{L^1(\mathbb{R}^n)}[g]_q$. Here C_q is independent of f and g.
3. If $0 < \alpha < n$, define an operator T_α on functions on $\mathbb{R}^n$ by

$$T_\alpha f(x) := \int_{\mathbb{R}^n} \frac{1}{|x-y|^\alpha} f(y)dy.$$

Use the second item to show that T_α is weak type $(1, n/\alpha)$.

III. Complete the following computations or proofs.

1. Show that $\widehat{\chi}_{[-1,1]}(x) = \frac{\sin 2\pi x}{\pi x}$.
2. For each $k > 0$, let $f_k(x) = \chi_{[-1,1]} * \chi_{[-k,k]}$.

   (a) Show that $f_k \in C_0(\mathbb{R})$ by definition.

(b) Show that $\widehat{f_k}(x) = \frac{\sin 2\pi k x \sin 2\pi x}{(\pi x)^2}$. Prove that $\|\widehat{f_k}\|_{L^1(\mathbb{R})} \to \infty$ as $k \to \infty$.

3. Let f_k be as in the second item, show

$$4k = \sum_{n \in \mathbb{Z}} f_k(n)$$

for any positive integer k.

IV. We know that the Fourier transform maps Schwartz functions to be Schwartz functions. In this paragraph, we consider the analogous properties of Fourier transform on the circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$.

For $f \in L^1(\mathbb{T})$, we define its Fourier transform $\hat{f}$ on $\mathbb{Z}$, by

$$\hat{f}(k) = \int_{\mathbb{T}} f(x) e^{-2\pi i k x} dx.$$

1. If $f \in C^1(\mathbb{T})$, show then

$$\hat{f}(k) = \mathcal{O}\left(\frac{1}{|k|}\right), \text{ as } |k| \to \infty.$$

Conclude from this that if $f \in L^1(\mathbb{T})$, then $\hat{f}(k) \to 0$ as $|k| \to \infty$.

2. If $f \in C^\infty(\mathbb{T})$, show

$$\hat{f}(k) = \mathcal{O}\left(\frac{1}{|k|^n}\right) \text{ as } |k| \to \infty,$$

for any positive integer n.

3. Let $f(z) := \frac{1}{1-z/2}$ be a function in the complex plane. If viewing f as a function on the circle, show that $\hat{f}(k)$ decays exponentially as $|k| \to \infty$.

V. Let $X = [0, 1]$ be equipped with the Lebesgue measure.

1. For $n \in \mathbb{N}$, let $f_n(x) = \cos(2\pi n x)$. Then as n tends to infinity, $f_n \to 0$ weakly in L^2, but it does not converge to 0 in measure.

2. For $n \in \mathbb{N}$, let $g_n(x) = n\chi_{(0,1/n)}(x)$. Then as n tends to infinity, g_n converges to 0 in measure, but it does not converges to 0 weakly in L^p for any $p > 0$.