Advanced Probability, MATH5007P
Autumn 2020, Midterm

Student ID: Name:

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1. (10 points) Let f be a mapping between two measurable spaces $(\Omega, \mathcal{F})$ and $(\mathbb{R}, \mathcal{B})$. Show that if $\mathcal{A}$ generates the σ-field $\mathcal{B}$, then $f^{-1}\mathcal{A} = \{f^{-1}A; A \in \mathcal{A}\}$ generates the σ-field $f^{-1}\mathcal{B}$.

2. (10 points) Let μ be a finite measure on $(\mathbb{R}, \mathcal{R})$ and $F(x) = \mu((-\infty, x])$. Show that

$$\int (F(x+c) - F(x))dx = c\mu(\mathbb{R}).$$

3. (10 points) Let $X_1, X_2, \ldots$ be uncorrelated with $EX_n = \mu_n$ and $\text{var}(X_n)/n \to 0$ as $n \to \infty$. Let $S_n = X_1 + \cdots + X_n$ and $\nu_n = ES_n/n$. Show that as $n \to \infty$, $S_n/n - \nu_n \to 0$ in L^2 and in probability.

4. (15 points) Let $X_1, X_2, \ldots$ be independent Poisson random variables with $EX_n = \lambda_n \in (0, \infty)$, and let $S_n = X_1 + \cdots + X_n$. Show that if $\sum_n \lambda_n = \infty$, then as $n \to \infty$, $S_n/ES_n \to 1$ a.s.

5. (15 points) Suppose the nth light bulb burns for an amount of time X_n and then remains burned out for time Y_n before being replaced. Suppose the X_n, Y_n are positive and independent with the X's having the common distribution F and the Y's having the common distribution G, both of which have finite mean. Let R_t be the amount of time in $[0, t]$ that we have a working light bulb. Show that as $t \to \infty$, $R_t/t \to EX_1/(EX_1 + EY_1)$ a.s.

6. (20 points) Let $X, X_1, X_2, \dots$ be a sequence of i.i.d. random variables and let B be a Borel set in $\mathbb{R}$ such that $0 < P(X \in B) < 1$. Define N to be the first index n such that $X_n \in B$ if there is at least one such index, and $N = \infty$ if there is no such index. Then for any $\omega \in \Omega$ such that $N(\omega) < \infty$, define that $X_N(\omega) = X_{N(\omega)}(\omega)$. First show that $N < \infty$ a.s. Then show that X_N has the same distribution as X conditioned to belong to B, that is, for any Borel set B' in $\mathbb{R}$ such that $B' \subset B$,

$$P(X_N \in B') = P(X \in B' | X \in B) = \frac{P(X \in B')}{P(X \in B)}.$$

7. (10 points) Let $X_1, X_2, \ldots$ be a sequence of independent random variables in L^2. Recall that the metric space L^2 is complete. Show that $\sum_n X_n$ converges in L^2 if and only if $\sum_n EX_n$ and $\sum_n \mathrm{var}(X_n)$ both converge.

8. (10 points) Construct a sequence of independent random variables $X_1, X_2, \ldots$ in L^2, such that $\sum_n X_n$ converges in L^2 but not a.s.