2020年秋季学期 高等概率论期末考试（管理学院统计系）
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# Advanced Probability Theory, STAT5101, Autumn 2020, Final

14:00-16:00, January 18, 2021

1. (20 points) Let $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \ldots$ be a sequence of $\sigma$-algebras.

(a) Prove that $\bigcup_i \mathscr{F}_i$ is an algebra.

(b) Give an example to show that $\bigcup_i \mathscr{F}_i$ may not be a $\sigma$-algebra.

2. (20 points) Let $X_1, X_2, \ldots$ be i.i.d random variables with finite expectation, prove that

$$\frac{\max\limits_{1\le k\le n} |X_k|}{n} \to 0 \;\; a.s. \quad \text{and} \quad \frac{1}{n}E(\max_{1\le k\le n} |X_k|) \to 0$$

3. (20 points) Let F be the distribution function of a random variable X.

(a) Prove that as $c \ge 0$, $\int_{-\infty}^{\infty} (F(x+c) - F(x))\, dx = c$.

(b) Calculate $\int_0^{\infty} x(1-F(x))\, dx$. (Express the result with the moment related to X)

4. (15 points) Let $X_1, X_2, \ldots$ be random variables. Suppose $E(X_n) = 0$ and $E(X_m X_n) \le r(n-m)$ for $m \le n$ with $r(k) \to 0$ as $k \to \infty$, show that:

$$\frac{X_1 + X_2 + \ldots + X_n}{n} \to 0 \;\text{ in probability}$$

5. (25 points) Let $X, Y$ be random variables.

(a) Let $f_X$, $f_Y$ be density functions of $X, Y$. Prove that if $\forall x \in \mathbb{R}$, $f_X(x) \le f_Y(x)$, then X and Y are identically distributed.

(b) Let $X \sim N(0,1)$, find the value of

$$\lim_{x\to\infty} P(X > x + c/x | X > x)$$

(c) If X and Y are independent and X is continuous, prove that $P(X = Y) = 0$.

(d) Let $X \sim N(0,1)$, if $g : \mathbb{R} \to \mathbb{R}$ is continuous and bounded with $|g'(x)| \le 1$, prove that

$$E(g'(X)) = E(Xg(X))$$

(e) Let $X, X_1, X_2, \ldots \in \{1, 2, 3, \cdots\}$, $X_n \to X$ in distribution, prove that

$$\sum_{i=1}^{\infty} |P(X_n = i) - P(X = i)| \to 0 \;\; as\; n \to \infty$$