Stochastic Processes, MA04243, Spring 2020, Final

Student ID: Name:

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1. (10 points) Let $(X_n)_{n\geq 0}$ be a submartingale with $\sup_{n\geq 0} X_n < \infty$. For any $n \geq 1$, let $\xi_n = X_n - X_{n-1}$ and further suppose that $E(\sup_{n\geq 1} \xi_n^+) < \infty$. Prove that X_n converges a.s.

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2. (10 points) Let $X = (X_n)_{n\geq 0}$ be a Markov chain with values in the countable state space S. Define $T_y = \inf\{n > 0 : X_n = y\}$ for any $y \in S$ and $\rho_{xy} = P_x(T_y < \infty)$ for any $x, y \in S$. Use the strong Markov property to prove that $\rho_{xz} \geq \rho_{xy}\rho_{yz}$ for any $x, y, z \in S$.

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3. (15 points) Let $(B_t)_{t\geq 0}$ be a real Brownian motion started from 0. Using a proper zero-one law, show that, a.s.,

$$\limsup_{t\downarrow 0} \frac{B_t}{\sqrt{t}} = +\infty, \quad \liminf_{t\downarrow 0} \frac{B_t}{\sqrt{t}} = -\infty.$$

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4. (15 points) Let $(B_t)_{t\geq 0}$ be a real Brownian motion started from $x > 0$. Set

$$T_0 = \inf\{t \geq 0 : B_t = 0\}.$$

Give the law of $\sup_{t\leq T_0} B_t$.

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5. (15 points) Let $X = (X_n)_{n\geq 0}$ be a Markov chain with values in the finite state space $S = (s_1, s_2, \ldots, s_m)$, and define

$$\tau = \inf\{n > 0 : X_n = X_0\}.$$

Suppose that X is irreducible, and $\pi = (\pi(s))_{s\in S}$ is a stationary distribution of X. First show that $\pi(s) > 0$ for any $s \in S$, then compute $E_\pi \tau$.

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6. (15 points) Let $B = (B_t)_{t\geq 0}$ be a real Brownian motion started from 0. First show that a.s. local maxima exist for the sample paths of B, then show that for any $t \geq 0$, $P(A_t) = 0$, where

$$A_t = \{\text{the time } t \text{ is a time of a local maximum for the sample paths of } B\}.$$

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7. (20 points) In this problem we interpret $\mathbb{R}^2$ as the complex plane. Hence a 2-dimensional Brownian motion becomes a complex Brownian motion. A complex-valued stochastic process is called a martingale, if its real and imaginary parts are martingales. Let $(B_t)_{t\geq 0}$ be a complex Brownian motion started in i, the imaginary unit.

(a) Show that $(\exp(i\lambda B_t))_{t\geq 0}$ is a martingale, for any $\lambda \in \mathbb{R}$.

(b) Let T be the first time when $(B_t)_{t\geq 0}$ hits the real axis. Using a proper optional stopping theorem, show that

$$E[\exp(i\lambda B_T)] = \exp(-\lambda).$$