RIEMANNIAN GEOMETRY (SPRING, 2020)
FINAL EXAM

Name: No.: Department:

1. (20 marks) Let S^n be the unit sphere with the canonical round sphere metric. (You can give the answers without explanation.)

(i) Let $p \in S^n$ be a point. What is the cut locus of p?
(ii) Let $\gamma : [0, \pi + 2.5] \to S^n$ be a normal geodesic. What is the index of γ?
(iii) What is the first non-zero eigenvalue of the Beltrami-Laplace operator on S^n?
(iv) What is the injectivity radius of S^n?

2. (20 marks)

(i) Consider the differential manifold $S^1 \times S^2$. Decide whether there exists a Riemannian metric on it with **Ricci curvature** everywhere positive. Explain the reason of your judgement.
(ii) Consider the differential manifold $RP^n \times RP^n$. Decide whether there exists a Riemannian metric on it with **sectional curvature** everywhere positive. Explain the reason of your judgement.

3.(15 marks) Let (M,g) be a Riemannian manifold. Let ∇ be the Levi-Civita connection of the metric g.

(i) For $X, Y, Z \in \Gamma(TM)$, compute

$$\nabla^2 Z(Y, X) - \nabla^2 Z(X, Y).$$

Here we use $\nabla^2 \cdot$ for the second order covariant differentiation.
(ii) Use Ricci Identity to prove the Bochner formula: For any $f \in C^\infty(M)$, it holds that

$$\frac{1}{2}\Delta|\mathrm{grad} f|^2 - \langle \mathrm{grad}(\Delta f), \mathrm{grad} f\rangle = |\mathrm{Hess} f|^2 + \mathrm{Ric}(\mathrm{grad} f, \mathrm{grad} f).$$

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4 (15 marks)

Let $f : M \to M$ be an isometry of a compact orientable Riemaniann manifold (M, g) with dimension n. Suppose n is odd and f reverses the orientation of M. Assume that there exists $p \in M$ such that

$$d(p, f(p)) = \inf_{q \in M} d(q, f(q)), \text{ and } d(p, f(p)) \neq 0.$$

Let $\gamma : [0, \ell] \to M$ be a normal minimizing geodesic from $\gamma(0) = p$ to $\gamma(\ell) = f(p)$.

(i) Let $\overline{\gamma} : [0, 2\ell] \to M$ be a curve given by

$$\overline{\gamma}(t) = \begin{cases} \gamma(t), & t \in [0, \ell]; \\ f(\gamma(t - \ell)), & t \in [\ell, 2\ell]. \end{cases}$$

Show that $\overline{\gamma}$ is a smooth curve.

(ii) Show that there exists a nontrivial parallel **normal** vector field $V(t)$, $t \in [0, \ell]$ along γ satisfying

$$V(\ell) = df_p(V(0)).$$

5 (30 marks)

Let (M^n, g) be an n dimensional complete Riemannian manifold with $\text{Ric} \geq 0$. Given $p \in M$, let $\gamma : [0, b] \to M$ be a normal geodesic with no cut point of $\gamma(0) = p$. Let $J_1, J_2, \dots, J_{n-1}$ be Jacobi fields along γ satisfying

$$\begin{aligned} & J_i(0) = 0, \ \ i = 1, 2, \dots, n-1; \\ & \langle J_i(b), J_j(b) \rangle = \delta_{ij}, \ \ i, j = 1, 2, \dots, n-1; \\ & \langle J_i(b), T(b) \rangle = 0, \ \ i = 1, 2, \dots, n-1, \text{ where } T(t) := \dot{\gamma}(t). \end{aligned}$$

(i) Show that for any $i = 1, 2, \dots, n-1$ and for any $t \in (0, b]$, we have

$$J_i(t) = \left(d\exp_p\right)_{(tT(0))} (t \nabla_T J_i(0)).$$

(ii) Let $\rho(x) := d(x, p)$. Show that

$$\Delta \rho(\gamma(b)) = \sum_{i=1}^{n-1} \langle \nabla_T J_i(b), J_i(b) \rangle.$$

(You can use the variational formulae for length functional without proof.)

(iii) Define

$$\psi(t) := \frac{|J_1(t) \wedge \cdots \wedge J_{n-1}(t)|}{t^{n-1} |\nabla_T J_1(0) \wedge \cdots \wedge \nabla_T J_{n-1}(0)|}, \ \ t \in (0, b].$$

Show that

$$\frac{d}{dt}_{|t=b} \psi^2(t) \leq 0.$$

(Hint: Use Laplacian comparison theorem.)

(iv) Recall that there exists a function $\varphi : E(p) \subset T_pM \to \mathbb{R}$, such that,

$$\text{Vol}(B_p(r)) = \int_{E(p) \cap B(0,r)} \varphi d\text{vol}_{T_pM}, \forall r > 0,$$

where $d\text{vol}_{T_pM}$ is the Euclidean volume measure on T_pM, $B(0, r)$ is the ball of radius r centered at the origin in T_pM, and $B_p(r)$ is the ball of radius r centered at p in M.

Use (iii) to show that

$$\mathrm{Vol}(B_p(r)) \leq \omega_n r^n,$$

where ω_n is the volume of unit ball in $\mathbb{R}^n$. (You are NOT allowed to apply Bishop-Gromov Theorem directly.)