# 微分方程II(H) - 期中测验

2020年4月16日，周四，9:45 - 12:00

注意事项：

1. 请完成不少于五个题，成绩为得分最高的五个题的分数加和；
2. 开卷考试，可查阅课本和上课讲义，但请独立完成答题；
3. 请给出详细计算和证明过程。
4. 请在中午12:00 前上传解答到Bb平台作业区- 期中测验(和平时提交作业方式一样)；

试卷正文：

1. [20 分]

   (a) Let $\Omega = B(0, \frac{1}{2}) = \{x \in \mathbb{R}^n : \ |x| < 1/2\}$ be a ball of radius $1/2$ in $\mathbb{R}^n$. Consider the function $u(x) = \log|\log|x||$. Such a function is smooth in the domain $\Omega \setminus \{0\}$ but approaches $\infty$ as $|x|$ goes to zero. Whether $u(x)$ belongs to some Sobolev space $W^{1,p}(\Omega)$? If so, for which values of $p$ does $u(x)$ belong to Sobolev space $W^{1,p}(\Omega)$?

   (b) Let $\Omega = \{0 < x < 1, 0 < y < x^4\}$. Show that the function $u(x,y) = 1/x$ belongs to $H^1(\Omega)$, but does not belong to $L^5(\Omega)$. Is this consistent with the Sobolev embedding theorem?

2. [20 分] Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Prove the following properties of Sobolev spaces.

   (a) Let $u \in W^{1,p}(\Omega)$ and $v \in W_0^{1,q}(\Omega)$, where $1 < p, q < \infty$ satisfies $\frac{1}{p} + \frac{1}{q} = 1$. Then
   $$\int_\Omega D^\alpha u v dx = -\int_\Omega u D^\alpha v dx, \qquad |\alpha| = 1.$$

   (b) Assume $u \in W_0^{1,p}(\Omega)$, $1 \le p < \infty$. Let $\bar{u}$ be the zero extension of $u$ to $\mathbb{R}^n \setminus \Omega$ i.e., we define
   $$\bar{u}(x) := \begin{cases} u(x), & x \in \Omega \\ 0, & x \in \mathbb{R}^n \setminus \Omega \end{cases}$$
   Show that $\bar{u} \in W_0^{1,p}(\tilde{\Omega})$ for any open set $\tilde{\Omega}$ satisfying $\Omega \subset\subset \tilde{\Omega}$.

   (c) Given two nonnegative functions $u, v \in W^{1,p}(\Omega)$, we define $\omega(x) = \min\{u(x), v(x)\}$ for $x \in \Omega$. Then $\omega \in W^{1,p}(\Omega)$.

3. [20 分] Let $B = B(0,1)$ be an open unit ball centered at the origin, and denote $B^+ = B \cap \{x_n > 0\}$ and $B^- = B \cap \{x_n < 0\}$. Given $u \in W^{1,p}(B^+)$, $1 \le p \le \infty$, we consider the reflection of $u$ across the plane $\{x_n = 0\}$, i.e., we set

$$\bar{u}(x) := \begin{cases} u(x), & x \in B^+ \\ u(x_1, \cdots, x_{n-1}, -x_n), & x \in B^- \end{cases}$$

Show that $\bar{u} \in W^{1,p}(B)$ and

$$\|\bar{u}\|_{W^{1,p}(B)} \le 2\|u\|_{W^{1,p}(B^+)}.$$

(Hint: write down the weak derivatives of $\bar{u}$; you may need to use a cut-off function.)

4. [20 分] Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a $C^1$ boundary.

   (a) Show that $H^2(\Omega) \subset\subset H^1(\Omega)$ (Hint: use the fact $H^1(\Omega) \subset\subset L^2(\Omega)$).

   (b) For any $\varepsilon > 0$, prove that there exists a constant $C_\varepsilon > 0$ such that

   $$\|Du\|_{L^2(\Omega)} \le \varepsilon\|u\|_{H^2(\Omega)} + C_\varepsilon\|u\|_{L^2(\Omega)}$$

   holds for all $u \in H^2(\Omega)$, by a contradiction argument based on the compact embedding in item (a).

5. [20 分] Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $C^1$ boundary $\partial\Omega$. Assume that $a^{ij}, b^i, c \in L^\infty(\Omega)$ and

$$a^{ij} = a^{ji} \quad (i,j = 1, \cdots, n), \qquad (a^{ij}(x)) \ge \theta \mathrm{I} > 0 \text{ a.e. } x \in \Omega.$$

Let $L$ denote a second-order elliptic partial differential operator of divergence form

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij}(x)u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u.$$

Suppose that the uniqueness of weak solution to the following Dirichlet boundary value problem

$$\begin{cases} Lu = & f \quad \text{in } \Omega \\ \;\;u = & g \quad \text{on } \partial\Omega \end{cases} \tag{1}$$

holds for each $f \in H^{-1}(\Omega)$ and $g \in H^1(\Omega)$. Show that there exists a constant $C > 0$ such that the unique weak solution $u \in H^1(\Omega)$ of (1) satisfies

$$\|u\|_{H^1(\Omega)} \le C\left(\|f\|_{H^{-1}(\Omega)} + \|g\|_{H^1(\Omega)}\right).$$

6. [20 分] Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Assume that $a^{ij}, b^i, c \in L^\infty(\Omega)$ and

$$a^{ij} = a^{ji} \quad (i, j = 1, \cdots, n), \qquad (a^{ij}(x)) \geq \theta \mathrm{I} > 0 \text{ a.e. } x \in \Omega.$$

Let $L$ denote a second-order elliptic partial differential operator of divergence form

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij}(x) u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x) u_{x_i} + c(x) u.$$

Prove that either ($\alpha$) for each $f \in H^{-1}(\Omega)$ there exists a unique weak solution of

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

or else ($\beta$) there exists a nonzero weak solution of

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$