Linear Elliptic PDE 2020Fall final

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Exercise 1. *Given* $\varphi \in C(\partial B_1)$*, let*

$$u(x) := \begin{cases} \frac{1-|x|^2}{n|B_1|}\int_{\partial B_1} \frac{\varphi(y)}{|x-y|^2} dS_y, & x \in B_1; \\ \varphi(x), & x \in \partial B_1. \end{cases} \tag{1}$$

(1) *Prove that* $\Delta u(x) = 0$ *in* B_1*.*

(2) *Prove that* $u \in C(\bar{B}_1)$*.*

Exercise 2. *Solve the following problems.*

(1) *Prove one of the interpolation inequalities in Holder space, that for any* $u \in C^1(B_R)$ *and* $\epsilon > 0$*, then*

$$R^\alpha [u]_{C^{0,\alpha}(B_R)} \le \epsilon R |u|_{L^\infty(B_R)} + C_\epsilon |u|_{L^\infty}. \tag{2}$$

(2) *Assume that* $osc_{B_r(x)} u \le C_0 r^\alpha$ *for any* $B_r(x) \subset \bar{B}_1$*, then show that*

$$[u]_{C^{0,\alpha}(\bar{B}_1)} \le C C_0, \tag{3}$$

where C *depends on* n *and* α*.*

(3) *Suppose that for any nonnegative* $u \in H^1(\Omega)$ *solving* $\partial_j(a_{ij}\partial_i u) = 0$ *weakly, we have*

$$\sup_{B_{r/2}(x)} u \le C \inf_{B_r(x)} u \tag{4}$$

for any $B_r(x) \subset \bar{B}_1$*, where* C *is a constant. Try to prove that*

$$[u]_{C^{0,\alpha}(B_{1/2})} \le C |u|_{L^\infty(B_1)}. \tag{5}$$

Exercise 3. *If* $u \in C^3(\Omega) \cap C^1(\bar{\Omega})$ *satifies the following PDE*

$$a_{ij} D_{ij} u + b_i D_i u = f(x, u) \text{ in } \Omega, \tag{6}$$

where $(a_{ij})_{n\times n} \ge \lambda I$*,* $a_{ij}, b_i \in C^1$ *and* $f \in C^1(\bar{\Omega} \times \mathbb{R})$*.*

(1) *prove that there exists a constant depending on* $\lambda, |a_{ij}|, |b_i|, |f|_{C^1(\bar{\Omega}\times[-|u|_{L^\infty}, |u|_{L^\infty}])}$ *s.t.*

$$L(|Du|^2) \geq \lambda |D^2u|^2 - C|Du|^2 - C. \tag{7}$$

(2) *Prove that*

$$\sup_{\Omega} |Du| \leq \sup_{|Du|} + c, \tag{8}$$

where C *depends on* $|u|^{L^\infty}$ *also.*

Exercise 4. $L = \partial_j(a_{ij}\partial_i)$, $Lu = 0$, $u \in H^1(B_1)$ *weak solution.*

(1) $k \geq 0$, $v = (u-k)^+$ *prove for any cutoff function* $\eta \in C_0^1(B_1)$ *that*

$$\int |D(\eta v)|^2 \leq C \int v^2 |D\eta|^2. \tag{9}$$

(2) $A(k,r) := \{x | u(x) \geq k\}$, *prove that*

$$\int_{A(k,r)} (u-k)^2 \leq \frac{1}{(R-r)^2} |A(k,R)|^{\frac{2}{n}} \int_{A(k,R)} (u-k)^2, \tag{10}$$

where $0 < r < R \leq 1$.

(3) *prove that*

$$\sup_{B_{\frac{1}{2}}} u \leq C|u^+|_{L^2}. \tag{11}$$

默题工具人写的一些 hint：题目都是从书上摘的，我漏写了很多条件（比如 Ω 的边界正则性、算子的椭圆形等、常数 C 的依赖性等），但老师试卷上都注明了，不必担心，做不出来可以看看对比下教材看是否漏条件了。

Ex1. 可以看 Evans 第 2 章，学过微分方程 1 应该就能写了。

Ex2. 第 1 问参考 [Han Qing]Lemma4.1.2。第 2、3 问需要参考 [Regularity Theory for Elliptic PDE, Xavier Fern´andez-Real, Xavier Ros-Oton] 一书，第 2 问看 Appendix A (H1), 第 3 问看 2.1 节。

Ex3. [Han-Lin] Proposition 2.18。

Ex4. 这是 De Giorgi 迭代，算子比书上简单些，而且从 subsolution 改成了 solution，参考 [Han-Lin] Theorem 4.1。