2020年秋季学期 黎曼曲面 期末考试
考试时间：2021年元月15日
授课教师：许斌、张磊、赵晨

Open-book Final: Riemann Surfaces (15 Jan 2021)

- **Answer five of the first seven questions before 4:00 pm.**

(1) Let $\Sigma = \mathbb{P}^1_{\mathbb{C}}$ be endowed with homogenous coordinate $[X, Y]$, let T_Σ denote its tangent bundle, and let $P = [0, 1]$.
(i) Show that T_Σ is isomorphic to the line bundle $L_{[2P]}$.
(ii) Construct a basis of the space of holomorphic vector fields vanishing at P.

(2) Let Σ be a compact Riemann surface and let γ be a smooth closed route. We may regard γ as an element of $H_1(\Sigma, \mathbb{Z}) \cong H^1(\Sigma, \mathbb{Z})$ via Poincaré duality. Please construct a smooth closed 1-form α such that $[\alpha] \in H^1(\Sigma, \mathbb{Z})$ coincides with γ.

(3) Consider the Riemann surface $\Sigma_n = \{(w, z) \in \mathbb{C}^2 : w^2 = z^3 + n^2\}$. Find a holomorphic function $f : \Sigma_n \to \mathbb{C}$ whose branched set coincides with $\{0, 1\}$.

(4) Show that the genus of $F_n = \{(z, w) \in \mathbb{C}^2 : z^n + w^n = 1\}$ is $g(F_n) = \frac{(n-1)(n-2)}{2}$.

(5) Consider the Riemann surface X defined by $\{(x, y) \in \mathbb{C}^2 : y^2 = x(x-1)(x-2)\}$. If $R(x, y) = P(x, y)/Q(x, y)$ is a rational function in two variables where $Q(x, y)$ does not vanish identically on X, then $\omega = R(x, y)dx$ is a meromorphic 1-form on X. Show that if $R(x, y) = p(x)/y$, with $p(x)$ a polynomial, then ω is holomorphic on X.

(6) Prove that every compact Riemann surface of genus 2 admits a double branched cover over the Riemann sphere.

(7) Prove that there exists on the Riemann sphere a meromorphic one-form ω which has a single zero of multiplicity 3 and five simple poles with residues -3, -1, -1, 2 and 3, respectively.

(8) Read the 2-page paper "What is a Dessin d'Enfant" and submit a report before 10:00 am of Jan 17 via Tencent QQ.