中国科学技术大学

2018-2019 学年第二学期期中试卷

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课程名称：复分析 (H) 课程编号：001701

开课院系：数学科学学院 考试形式：闭卷

姓名：______________ 学号：_______________

题号	1	2	3	4	5	6	7	8	总分
得分									

1. (10 pionts) Recall by definition we have

$$dz = dx + idy$$

$$d\bar{z} = dx - idy$$

Given a smooth function $f(x, y)$ in two variables, we now define

$$\partial f = \frac{df}{dz}dz \qquad \partial(fdz + gd\bar{z}) = \frac{dg}{dz}dz \wedge d\bar{z}$$

$$\bar{\partial} f = \frac{df}{d\bar{z}}d\bar{z} \qquad \bar{\partial}(fdz + gd\bar{z}) = \frac{dg}{d\bar{z}}d\bar{z} \wedge dz$$

(a) Prove:

$$df = \partial f + \bar{\partial} f$$

$$\Delta f dz \wedge d\bar{z} = 4\partial\bar{\partial} f = -4\bar{\partial}\partial f$$

(b) Write $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial \bar{z}}$ and the Laplace operator Δ in polar coordinates.

2. (20 pionts) Compute the following integrals
   (a) $\int_{|z|=1} \frac{\cos^3(z)}{z^3}\, dz$
   (b) $\int_0^\infty \frac{x^{\frac{1}{3}}}{1+x^2}\, dx$

3. (15 pionts) Count the number of zeros of the function $z^7 + z^5 + 9z^4 + 8z^3 + 7z + 8$ in the right half plane.

4. (10 pionts) Prove the Liouville's theorem for harmonic functions: If u is a bounded harmonic function on $\mathbb{C}$, then u is a constant.

5. (15 pionts) Prove the following statements:
   (a) If u is a harmonic function, f is a holomorphic, then $u \circ f$ is still harmonic.
   (b) If u is a positive harmonic function on $\mathbb{R}^2 - \{0\}$, then u is a constant.

6. (20 pionts) Let f be an entire function and let $a, b > 0$ be positive constants.
(a) If $|f(z)| \leq a\sqrt{|z|} + b$ for all z, prove that f is a constant.
(b) What can we say about f if

$$|f(z)| \leq a|z|^{\frac{n}{2}} + b, \ n \in \mathbb{N}$$

for all z?

7. (10 pionts) Prove that, if a holomorphic function f over the unit disc $\mathbb{D}$ can be continuously extended to an arc $\gamma \subset \{z||z| = 1\}$ of positive length and takes value 0 on this arc γ, then $f \equiv 0$.

8. (20 pionts) Let, for $R > 1$,

$$A = \{z \in \mathbb{C}|1 < |z| < R\}$$

Find the automorphism group $Aut(A)$ of A by the following steps:
(a) Assume $\phi \in Aut(A)$. If a sequence $\{w_j\}$ in A converges to the boundary, then so does the sequence $\{\phi(w_j)\}$.
(b) Moreover, if $|w_j| \to 1$, then either for all such sequences, $|\phi(w_j)| \to 1$ or $|\phi(w_j)| \to R$. And in the second case, if $|w_j| \to R$, then $|\phi(w_j)| \to 1$(Similar for the first case).
(c) Find out the automorphism group finally.
(d) True or false: If we assume,

$$A_1 = \{z \in \mathbb{C}|R_1 < |z| < R_2\}$$

$$A_2 = \{z \in \mathbb{C}|S_1 < |z| < S_2\}$$

then A_1 is conformally equivalent to A_2 iff $\frac{R_2}{R_1} = \frac{S_2}{S_1}$. Tell me your answer and explain why (do not require details).