Stochastic Processes, MA04243, Spring 2019, Midterm

Student ID: Name:

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1. Suppose that $X^1 = (X_n^1)_{n\geq 0}$ and $X^2 = (X_n^2)_{n\geq 0}$ are two supermartingales with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$, and N is a stopping time with respect to $(\mathcal{F}_n)_{n\geq 0}$ such that $X_N^1 \geq X_N^2$ on $\{N < \infty\}$. Define $Y = (Y_n)_{n\geq 0}$ by

$$Y_n = X_n^1 1_{\{n\leq N\}} + X_n^2 1_{\{n>N\}}.$$

Prove that $Y = (Y_n)_{n\geq 0}$ is a supermartingale with respect to $(\mathcal{F}_n)_{n\geq 0}$.

2. Let $X = (X_n)_{n \geq 0}$ be a Markov chain with values in the countable state space S. Define

$$T_y = \inf\{n > 0 : X_n = y\}$$

for any $y \in S$ and

$$\rho_{xy} = P_x(T_y < \infty)$$

for any $x, y \in S$. Use the strong Markov property to prove that

$$\rho_{xz} \geq \rho_{xy}\rho_{yz}$$

for any $x, y, z \in S$.

3. Define $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for any $n \geq 1$, where $X_1, X_2, \cdots$ are independent with $EX_n = 0$ and $\mathrm{var}(X_n) = \sigma^2$ for any $n \geq 1$. Use the martingale $(S_n^2 - n\sigma^2)_{n\geq 0}$ to prove that if T is a stopping time with $ET < \infty$, then

$$ES_T^2 = \sigma^2 ET.$$

4. Let p be a transition probability on the countable state space S, and for the Markov chain $X = (X_n)_{n \geq 0}$ with the transition probability p, define

$$T_y = \inf\{n > 0 : X_n = y\}$$

for any $y \in S$. Prove that if the transition probability p is irreducible and positive recurrent, then

$$E_x T_y < \infty$$

for any $x, y \in S$.

5. Let $X = (X_n)_{n\geq 0}$ be a martingale with values in $\mathbf{Z}_+ = \{0, 1, 2, \cdots\}$, such that $X_0 = a$, $|X_{n+1} - X_n| \leq 1$ for any $n \geq 0$, and $\lim_{n\to\infty} X_n = b \leq a$. Define $M = \sup_{n\geq 0} X_n$. Compute $p_c = P(M = c)$ for all $c \in \mathbf{Z}_+$.

6. Let $X = (X_n)_{n \geq 0}$ be a Markov chain with values in the countable state space S, and define

$$\tau = \inf\{n > 0 : X_n = X_0\}.$$

Suppose that π is a stationary distribution of X. Compute $E_\pi \tau$.