2019年秋季学期 代数几何引论期末考试

授课教师：张磊

(10) 1. Prove that

(1) $\mathbb{P}^n \setminus V(F)$ is an affine variety where $F(T_0, \cdots, T_n)$ is a homogeneous polynomial;

(2) for finitely many points $x_1, \cdots, x_m$ on a projective variety X, there exists an affine open subset U containing those points.

(15) 2. Let $p = [1, 1] \in \mathbb{P}^1$ be a point and also a prime divisor.

(1) Give a basis to the linear space $\mathcal{L}(\mathbb{P}^1, 2p)$.

(2) Describe the map defined by the linear system $|2p|$.

(15) 3. Assume char $k \neq 2, 3$. Let $X = V(x_0^3 + x_1^3 + x_2x_3^2, x_0^2 + x_1^2 + x_2x_3) \subset \mathbb{P}^3$.

(1) Construct a finite regular map $g : X \to \mathbb{P}^1$.

(2) $\dim X$.

(3) Is X is a normal variety and why?

(15) 4. Let $X \subset \mathbb{A}^2 \times \mathbb{A}^1$ be a variety defined by $x^3 + y^2 + t = 0$ where (x, y) are the coordinate of $\mathbb{A}^2$ and t is the coordinate of $\mathbb{A}^1$. Denote by $f : X \to \mathbb{A}^1$ the projection to the second factor.

(1) Assume char $k = 0$, find a nonempty subset U of $\mathbb{A}^1$ such that for every $t \in U$, the fiber X_t is nonsingular.

(2) Give counterexamples to the second Bertini Theorem in every characteristic $p > 0$?

(20) 5. Let $X = V(F_1, \cdots, F_r) \subset \mathbb{P}^n$ be a projective variety, where the $F_i(T_0, \cdots, T_n)$'s are homogeneous polynomials. The affine cone $C(X) \subseteq V = k^{n+1}$ consists the lines passing through $O = (0, \cdots, 0)$ and $(x_0, \cdots, x_n)$ such that $[x_0, \cdots, x_n] \in X$. We know that there is a projection $\pi : C(X) \setminus \{O\} \to X$.

Let $\sigma : \tilde{V} \to V$ be the blowup at O and $\tilde{C}(X)$ the strict transform of $C(X)$ under the blow up. And denote by $f : \tilde{C}(X) \to C(X)$ the natural map.

(1) The ring of regular functions $k[C(X)]$.

(2) Prove that X is irreducible if and only if $C(X)$ is.

(3) The map $\pi' : \tilde{C}(X) \setminus f^{-1}(O) \to C(X) \setminus \{O\} \to X$ extends to a regular map $\tilde{C}(X) \to X$.

(4) Show that the $f^{-1}(O) \subset \sigma^{-1}(O) \cong \mathbb{P}^n$ is isomorphic to X.

(15) 6. Assume char $k \neq 2, 3$. Let $X = V(x^2y + y^2z + xz^2) \subset \mathbb{P}^2$ and $p = [0, 0, 1] \in X$.

(1) Tangent line of X at p. (2) $\dim \mathcal{L}(X, 2p)$. (3) Describe the map induced by $|2p|$.

(15) 7. (1) Give a natural identification between the set of hyperplanes of $\mathbb{P}^n$ and $\mathbb{P}^n$.

(2) Let X be a projective nonsingular hypersurface of degree $d \geq 2$. With the identification above, prove that the set of hyperplanes tangent to X is a closed subset V.

(3) Compute $\dim V$.