MIDTERM EXAM — ARA 2018 FALL

TIME: 15:55 — 18:00, NOVEMBER 9TH, 2018 LECTURE ROOM 5306 CLOSED-BOOK

Write your answers by order in the independent answer sheet, which you have to hand back together with this exam sheet in the end. For any questions, you need to state the reason rigorously, except with particular indication.

"i.e." means "id est" or "in another word/that is". "*s.t.*" means "such that". "*a.e.*" means almost everywhere. By "$A \lesssim B$", we mean $A \leq C B$, for some C, which we do not care in the context. All functions are complex-valued, measurable, and are finite, a.e. in a measure space, except with particular indication. $\|\cdot\|_p := (\int_\Omega |\cdot|^p d\mu)^{1/p}$ denotes the standard scale in L^p ($p \neq 0$) space, on a measurable subset Ω of $\mathbb{R}^n$. $f_j \xrightarrow{L^p} f$ means the strong convergence. For any nonzero exponent p, the exponent p' always denote the conjugate exponent, such that $1/p + 1/p' = 1$.

PROBLEMS

Problem 1 (10'). (1) *Let Ω be a set, $\mathcal{A} \subset \mathcal{P}(\Omega)$ be a subalgebra, and $\Sigma = \sigma(\mathcal{A})$, the smallest sigma-algebra, generated by $\mathcal{A}$. If μ is a $\mathcal{A}$-strong sigma-finite measure, and ν is another measure s.t. $\nu(A) = \mu(A)$, $\forall A \in \mathcal{A}$, then prove that $\mu = \nu$.*

(2) *State the definition of* ***product measure*** *and prove the* ***Fubini theorem.***

(3) *Is it necessary for the measures to be sigma-finite in (1) and (2)?*

Problem 2 (12'). (1) *Let $f_j \to f$, a.e. (pointwise a.e. convergent). If there exists a strong convergent dominating sequence F_j, s.t. $|f_j| \leq F_j \xrightarrow{L^1} F \in L^1$, then prove that, $f, f_j \in L^1$, and*

$$\int f = \lim_j \int f_j .$$

(2) *Is there any uniform bounded, integrable, pointwise a.e. convergent sequence, i.e. $f_j \to f, a.e.$, with $\sup_j \|f_j\|_1 + \sup_{j,x} |f_j| < \infty$, which*

(a) *however does NOT satisfy the interchangement of integral and limit?*

(b) *satisfies the interchangement of integral and limit, but has NO dominating sequence?*

(c) *has a dominating sequence, but has NO (nonnegative, integrable) dominating function?*

Problem 3 (14'). (1) *Let $J : \mathbb{C} \to \mathbb{R}$, continuous, convex, and $J(0) = 0$. Prove that, for any small $\epsilon > 0$, there exists two nonnegative, continuous functions ϕ_ϵ and ψ_ϵ, s.t., for any $a, b \in \mathbb{C}$,*

$$|J(a+b) - J(a)| \leq \epsilon \phi_\epsilon(a) + \psi_\epsilon(b) .$$

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(2) *Let* $f_j \to f$, *a.e.. If* $J(cf) \in L^1$ *for any* $c \in \mathbb{R}$, *and there exists some* $k \in \mathbb{N}$, $k \geq 2$, *s.t.,* $\sup_j \|[J(k\cdot) - kJ(\cdot)] \circ (f_j - f)\|_1 < \infty$, *then prove that*

$$\int |J(f_j) - J(f) - J(f_j - f)| \to 0\,.$$

(3) *Can we eliminate the uniform assumption in (2), or weaken it to* $\sup_j \|J(f_j - f)\|_1 < \infty$?

(4) *Let* $\{f_j\} \subset L^p$, $p \in (0,\infty)$, $f_j \to f$, *a.e., and* $\{\|f_j\|_p\}$ *converges, then does this imply* $f_j \xrightarrow{L^p} f$?

Problem 4 (18'). (1) (a) *State the definition of* **convergence by measure**, *denoted by* $f_j \xrightarrow{\mu} f$, *and (pointwise)* **almost uniform convergence**, *denoted by* $f_j \rightrightarrows_{alm} f$.

(b) *Prove the* **Egorov theorem**. *Is there any sequence, s.t.* $f_j \to f$, *a.e., and* $f_j \xrightarrow{\mu} f$, *but* f_j *is not almost uniformly convergent ?*

(c) *Prove the* **Riesz subsquence lemma** *(i.e., if* $f_j \xrightarrow{\mu} f$, *then there exists a subsequence, still denoted by* $\{f_j\}$, *s.t.,* $f_j \to f$, *a.e.). Is there any sequence,* $f_j \xrightarrow{\mu} f$, *but* f_j *is not pointwise convergent at any point?*

(2) *Let* $1 < p < \infty$. *Prove that, if* $f_j \rightharpoonup f$ *(***weak convergent** *in* L^p*), and* $\|f_j\|_p \to \|f\|_p$, *then* $f_j \xrightarrow{L^p} f$. *Is there any weak convergent sequence, which*

(a) *is NOT strong convergent, and also is NOT pointwise convergent at any point?*

(b) *is NOT strong convergent, but is pointwise a.e. convergent, and* $\lim_j \|f_j\|_p$ *exists?*

Problem 5 (12'). *Let* $p \in (0,1)$. *Is* L^p *a* **normed linear space**? *Is it a* **complete linear space**? *Is there any* **logarithmic convexity** *for the scale* $\|\cdot\|_p$?

Problem 6 (12'). (1) *For* $1 \leq p \leq 2$, *prove that*

$$\|f+g\|_p^{p'} + \|f-g\|_p^{p'} \leq 2\,(\|f\|_p^p + \|g\|_p^p)^{p'/p}.$$

Using Hanner inequality directly, you might only get partial points.

(2) *A normed linear space* $(X, \|\cdot\|)$ *is called* **uniformly convex** *if*

$$\delta(\epsilon) := \inf\{1 - \|(x+y)/2\| : \|x\| = \|y\| = 1, \|x-y\| = \epsilon\} > 0.$$

Furthermore, it is called $p-$*order uniformly convex, if* $\delta(\epsilon) \gtrsim \epsilon^p$.

Prove that $L^p\,(1 < p < \infty)$ *is* $(\max\{p,p'\})-$*order uniformly convex, using the inequality above.*

(3) *Show that, for some p at least,* L^p *is not* $(2-\epsilon)$*-order uniformly convex, for any small* $\epsilon > 0$.

Problem 7 (12'). (1) *State the* **projection theorem** *on a closed convex subset in* L^p *for any* $p \in (1,\infty)$ *(without proof). Is the projection unique?*

(2) *Using the projection theorem to prove the* **Ascoli theorem** *in real* L^p: *for any closed convex proper subset* $K \subsetneq L^p$, *and any* $f \notin K$, *there is a (real) bounded linear functional* $L \in (L^p)^*$, *strictly separating* f *and* K, *i.e., there exists a constant* $\alpha \in \mathbb{R}$, *s.t.,* $L(f) > \alpha > L(g)$, $\forall\, g \in K$.

Problem 8 (10'). *Let* $p \in [1,\infty)$. *State and prove the theorem of* **identity approximation** *of* L^p *functions by smooth functions with compact support, using convolution (i.e.,* $\overline{C_c^\infty} = L^p$*).*