USTC Midterm Exam | Riemannian Geometry | 2016 Spring

Your NAME: ____________________

Your STUDENT ID: ____________________

About this exam:

- It is a closed-book exam.
- 6 problems. 120 minutes.
- If you cheat, you will fail.

For some of you who feel nervous, here are some randomly chosen formulae:

[Warning: It is very possible that most of them are **NOT** needed in this exam.]

$\frac{dE(\gamma_s)}{ds}(0) = -\int_a^b \langle V(t), \nabla_{\dot\gamma}\dot\gamma\rangle\, dt + \langle V, \dot\gamma\rangle\,|_a^b$.

$\frac{d^2E(\gamma_s)}{ds^2}(0) = -\int_a^b \langle V(t), \nabla_{\dot\gamma}\nabla_{\dot\gamma}V(t) + R(\dot\gamma, V)\dot\gamma(t)\rangle\, dt + \langle V, \nabla_{\dot\gamma}V\rangle|_a^b + \langle \nabla_V f_s, \dot\gamma\rangle|_a^b$.

$(\nabla_X T)(\omega_1, \cdot, \omega_r, Y_1, \cdot, Y_s) = X(T(\omega_1, \cdot, \omega_r, Y_1, \cdot, Y_s)) - \sum_i T(\cdot, \nabla_X\omega_i, \cdots) - \sum_j T(\cdots, \nabla_X Y_j, \cdot)$.

$2\Gamma_{ij}^l = g^{lk}(\partial_j g_{ki} + \partial_i g_{jk} - \partial_k g_{ij})$.

$R_{ijk}{}^l = -\Gamma_{jk}^s\Gamma_{is}^l + \Gamma_{ik}^s\Gamma_{js}^l - \partial_i\Gamma_{jk}^l + \partial_j\Gamma_{ik}^l$

$(\nabla_X R)(Y, Z, W) + (\nabla_Y R)(Z, X, W) + (\nabla_Z R)(X, Y, W) = 0$.

$Rm = W + \frac{1}{m-2}E \owedge g + \frac{S}{2m(m-1)}g \owedge g$

$W = Rm - \frac{1}{m-2}\left(\mathrm{Ric} - \frac{S}{2(m-1)}g\right) \owedge g$

$T_1 \owedge T_2(X, Y, Z, W) = T_1(X, Z)T_2(Y, W) - T_1(Y, Z)T_2(X, W) - T_1(X, W)T_2(Y, Z) + T_1(Y, W)T_2(X, Z)$.

$\ddot x^k(t) + \dot x^i(t)\dot x^j(t)\Gamma_{ij}^k = 0, \quad 1 \le k \le m$.

$\nabla_{\dot\gamma}\nabla_{\dot\gamma}X + R(\dot\gamma, X)\dot\gamma = 0$

☺ Good Luck! ☺

1.($20'$) **Write down the definitions of**

(a) *Isometry* and *local isometry*:

- An *isometry* $f : (M, g) \to (N, h)$ is ______________________________

  ______________________________

- A *local isometry* $f : (M, g) \to (N, h)$ is ______________________________

  ______________________________

(b) **<u>Three</u>** definitions of the *Laplace-Beltrami operator* $\Delta$:

- $\Delta =$ ______________________________
- $\Delta =$ ______________________________
- $\Delta =$ ______________________________

(c) The *linear connection* and the *Levi-Civita connection*

- A *linear connection* on $M$ is ______________________________

  ______________________________

- A *Levi-Civita connection* on $(M, g)$ is ______________________________

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(d) The *conjugate points* and *cut points* of $p = \gamma(0)$ along a geodesic $\gamma : [0, +\infty] \to M$:

- We say $q = \gamma(t_0)$ is a *conjugate point* of $p$ along $\gamma$ if ______________________________

  ______________________________

- We say $q = \gamma(t_0)$ is a *cut point* of $p$ along $\gamma$ if ______________________________

  ______________________________

(e) The *length* and *energy* of a curve $\gamma : [a, b] \to M$:

- $L(\gamma) =$ ______________________________
- $E(\gamma) =$ ______________________________

2.(20$'$) Which of the following statements are correct? Write a "T" before the correct ones and write an "F" before the wrong ones.

( ) Any two points on a Riemannian manifold can be connected by a geodesic.

( ) The distance function $f(x) := d(p,x)$ is absolutely continuous.

( ) If $\nabla^1$ and $\nabla^2$ are linear connections on $M$, and $a, b > 0$, then $a\nabla^1 + b\nabla^2$ is also a linear connection on $M$.

( ) If $X_1(p) = Y_1(p)$ and $X_2(p) = Y_2(p)$, then $\left(\nabla_{X_1} Y_1\right)(p) = \left(\nabla_{X_2} Y_2\right)(p)$.

( ) If $p = \gamma(a)$ has no conjugate point along $\gamma : [a, b] \to M$, then $\gamma$ is a minimizing geodesic connecting $p$ and $q = \gamma(b)$.

( ) The Riemann curvature tensor satisfies $Rm(U, S, T, C) = Rm(C, T, S, U)$.

( ) Any pseudo-Riemannian manifold admits a unique linear connection that is torsion free and compatible with the pseudo-Riemannian metric.

( ) $S^2$ admits no Riemannian metric of non-positive sectional curvature.

( ) $\mathbb{T}^2$ admits a Riemannian metric of positive Ricci curvature.

( ) $\mathbb{RP}^2 \times \mathbb{RP}^2$ admits neither a metric of positive sectional curvature, nor a metric of negative sectional curvature.

3.(20′) Let $M$ be a smooth manifold and let $\nabla, \nabla'$ be two linear connections on $M$. For any vector fields $X, Y \in \Gamma(TM)$, define
$$A(X,Y) = \nabla_X Y - \nabla'_X Y.$$

(a) Prove: $A$ is a 2-tensor.

(b) Suppose $g$ is a Riemannian metric on $M$, and suppose $\nabla$ is a $g$-compatible linear connection. Prove: $\nabla'$ is $g$-compatible if and only if
$$g(A(X,Y),Z) = -g(Y, A(X,Z)), \qquad \forall X, Y, Z \in \Gamma(TM).$$

(c) As in the Levi-Civita case, a curve $\gamma$ is called a geodesic of $\nabla$ if $\nabla_{\dot\gamma}\dot\gamma = 0$. [The geodesic equations, exponential maps etc can be extended to this setting without any difficulty.] Prove: $\nabla$ and $\nabla'$ have the same geodesics if and only if
$$A(X,Y) = -A(Y,X), \quad \forall X, Y \in \Gamma(TM).$$

4.(20′) Endow $S^2 \times S^2$ with the canonical product metric $g = \pi_1^* g_1 + \pi_2^* g_2$, where $g_1 = g_2 = g_{round}$. Prove: It is an Einstein manifold, but it is not locally conformally flat.

5.(20′) Let $M, N$ be connected Riemannian manifolds of the same dimension and let $f_1, f_2 : M \to N$ be local isometries. Moreover, suppose there exists a point $p \in M$ so that

$$f_1(p) = f_2(p) \quad \text{and} \quad (df_1)_p = (df_2)_p.$$

Prove: $f_1 = f_2$. [Hint: First prove this near $p$.]

6.(20′) Let $M = \{(x, y) \mid y > 0\}$ be the upper half plane. Define a Riemannian metric on $M$ by

$$g = y^{2k}(dx \otimes dx + dy \otimes dy),$$

where $k \in \mathbb{R}$ is a constant.

(a) Calculate the Christoffel symbols of $(M, g)$.

(b) Prove: the vertical lines are geodesics, and $\frac{\partial}{\partial x}$ is a Jacobi field along any such geodesics.

(c) Find the (sectional) curvature of $(M, g)$.
[Hint: You can use the results from (a) **AND** (b).]

(d) Find all $k$ so that $(M, g)$ is complete.

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