Homework 3 in Advanced Real Analysis

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Description:

For any $A \in \mathbb{R}^d$, we define s-dimensional Hausdorff measure as follows:

$$\mathcal{H}^s(A) := \lim_{\delta\to 0^+} \mathcal{H}^s_\delta(A),\ \ \mathcal{H}^s_\delta(A) := \inf\{\sum_{j=1}^{\infty} \alpha(s)\cdot\left(\frac{diam(C_j)}{2}\right)^s : A \subseteq \bigcup_{j=1}^{\infty} C_j, C_j \subseteq \mathbb{R}^d, diam(C_j) < \delta\}.$$

If we restrict $\{C_j\}$ to be a family of closed balls, then we actually construct the **spherical Hausdorff measure** $\mathcal{S}^s$ as follows:

$$\mathcal{S}^s(A) := \lim_{\delta\to 0^+} \mathcal{S}^s_\delta(A),\ \ \mathcal{S}^s_\delta(A) := \inf\{\sum_{j=1}^{\infty} \alpha(s)\cdot\left(\frac{diam(C_j)}{2}\right)^s : A \subseteq \bigcup_{j=1}^{\infty} C_j, C_j = B(x_j, r_j) \subseteq \mathbb{R}^d, 2r_j < \delta\}.$$

Problem:

Prove or Disprove: For any Borel set $A \subseteq \mathbb{R}^d$ with finite s-dim spherical Hausdorff measure, the upper density defined as follows satisfies

$$\limsup_{r\to 0+} \frac{\mathcal{S}^s(A\cap B(x,r))}{\alpha(s)r^s} = 1,\ \ \mathcal{S}^s - a.e.\ x \in A. \qquad \cdots(*)$$

Conclusion: The contemporary results are listed as follows:

(1). If $s > d$, then $\limsup_{r\to 0+} \frac{\mathcal{S}^s(A\cap B(x,r))}{\alpha(s)r^s} = 0$;

(2). If $s = d$, then $(*)$ holds;

(3). If $0 < s \le 1$, then $(*)$ does not hold. Plus, the middle-third Cantor set is a counterexample.

(4). If $1 < s < d$, we do not know whether it is true or not. But we can draw a weaker conclusion:

For any $\mathcal{S}^s$-measurable set $A \subseteq \mathbb{R}^d$ with finite s-dim spherical Hausdorff measure,we have

$$\sigma(A,x) = \lim_{\delta\to 0+} \sup\left\{\frac{\mathcal{S}^s(A\cap B)}{\alpha(s)r^s} : x \in B, diam(B) < \delta, B \text{ is a closed ball}\right\} = 1,\ \ \mathcal{S}^s - a.e.\ x \in A.$$

Proof: (1) Trivial Case.

(2) It suffices to prove $\mathcal{L}^d = \mathcal{S}^d$ on $\mathbb{R}^d$. Since $\mathcal{L}^d$ coincides with $\mathcal{H}^d$ on $\mathbb{R}^d$ and $\mathcal{S}^d \ge \mathcal{H}^d$ (by their definitions), we have $\mathcal{S}^d \ge \mathcal{L}^d$. To prove $\mathcal{S}^d \le \mathcal{L}^d$, one may use the fact proved in (4) that the upper density of $\mathcal{S}^s$ is no greater than 1. Detailed proof can be found in [1] (Theorem 2.10.18, 2.10.19, 2.10.35).

(3) In [2] we know that the upper density of Hausdorff measure satisfies

$$\frac{1}{2^s} \le \limsup_{r\to 0+} \frac{\mathcal{H}^s(A\cap B(x,r))}{\alpha(s)r^s} \le 1,\ \ \mathcal{H}^s - a.e.\ x \in A.$$

In [3], Remark 6.4 implies that the lower bound is always sharp when $0 < s \leq 1$. Suppose a Borel set C (C can always be Borel by the Borel regularity of $\mathcal{S}^s$) satisfies

$$\frac{1}{2^s} = \limsup_{r\to 0+} \frac{\mathcal{H}^s(C\cap B(x,r))}{\alpha(s)r^s}, \quad \mathcal{H}^s - a.e.\ x \in C.$$

Claim: $\forall 0 < s < d, A \subseteq \mathbb{R}^d$, $\mathcal{S}^s(A) \leq (\sqrt{3})^s \mathcal{H}^s(A)$.

Should the claim hold, then

$$\frac{1}{2^s} = \limsup_{r\to 0+} \frac{\mathcal{H}^s(C\cap B(x,r))}{\alpha(s)r^s} \geq \limsup_{r\to 0+} \frac{\mathcal{S}^s(C\cap B(x,r))}{(\sqrt{3})^s\alpha(s)r^s}.$$

Thus

$$\limsup_{r\to 0+} \frac{\mathcal{S}^s(C\cap B(x,r))}{\alpha(s)r^s} \leq \left(\frac{\sqrt{3}}{2}\right)^s < 1,$$

which implies $(*)$ does not hold in this case.

The proof of the claim is not very hard. Suppose $\{C_j\}$ is a countable covering of C with $\text{diam}(C_j) = 2r < \delta$. Now we choose $x, y \in C_j$ such that $|x-y| = 2r$ (or $=2r - 0_+$). WLOG $x = (-r, 0, \cdots, 0), y = (r, 0, \cdots, 0)$. If $z = (z_1, \cdots, z_d) \in C_j$, then $|z-x| \leq 2r, |z-y| \leq 2r$. Thus

$$(z_1 - r)^2 + \cdots + z_d^2 \leq 4r^2, \ \ (z_1 + r)^2 + \cdots + z_d^2 \leq 4r^2.$$

Summing up the last 2 formulae we get $z_1^2 + \cdots + z_d^2 \leq 3r^2$. Therefore the closed ball $B_j := \bar{B}(0, \sqrt{3}r)$ covers C_j and $C \subseteq \bigcup_1^\infty B_j$ and $\text{diam}(B_j) < \sqrt{3}\delta$.

Hence by the definitions of Hausdorff measure and spherical Hausdorff measure,

$$\mathcal{S}^s_{\sqrt{3}\delta}(C) \leq \sum_{j=1}^\infty \alpha(s)\left(\frac{diam(B_j)}{2}\right)^s = \sum_{j=1}^\infty \alpha(s)\left(\frac{\sqrt{3}diam(C_j)}{2}\right)^s.$$

Take the infimum over all δ-coverings of C on RHS of the last formula and then set $\delta \to 0+$ so that we finish the proof of the claim.

A counterexample is the middle-third Cantor set $\mathcal{C}$. In [4] (page 330, Chapter 7), Theorem 2.1 implies the Hausdorff dimension of $\mathcal{C}$ is $s = \log 2/\log 3$. [5] proved the Hausdorff upper density of $\mathcal{C}$ is $2/4^s$ a.e. Combining with the fact $\mathcal{H}^s = \mathcal{S}^s$ on $\mathbb{R}$ (This is quite easy because each C_j can be covered by a closed interval I_j with the same diameter on $\mathbb{R}$), we know that the upper density of spherical Hausdorff measure on $\mathcal{C}$ is also $2/4^s < 1$.

(4) The proof of (4) is similar to the proof of the upper density estimate of Hausdorff measure in [2]. The detailed proof can be referred to Theorem 6.6 in [3].

□

Bibliography

[1] Herbert Federer: *Geometry Measure Theory*, Springer Verlag, 1969.

[2] Lawrence C. Evans, Ronald F. Gariepy: *Measure Theory and Fine Properties of Functions*, Revised Version, CRC Press, 2015.

[3] Pertti Mattila: *Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability*, Cambridge University Press, 1995.

[4] Elias M. Stein, Rami Shakarchi: *Real Analysis*, Princeton Lectures in Analysis, 2007.

[5] De-jun Feng, Su Hua, Zhi-ying Wen: *The Pointwise Densities of the Cantoe Measure*, Journal of Mathematical Analysis and Applications Vol. 250, 692-705, 2000.