高等实分析第三次作业

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Description:

For any $A \in \mathbb{R}^d$, we define s-dimensioanl Hausdorff measure as follows:

$$\mathcal{H}^s(A) := \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(A),\ \ \mathcal{H}^s_\delta(A) := \inf\{\sum_{j=1}^{\infty} \alpha(s) \cdot \left(\frac{diam(C_j)}{2}\right)^s : A \subseteq \bigcup_{j=1}^{\infty} C_j, C_j \in \mathbb{R}^d, diam(C_j) < \delta\}.$$

If we restrict $\{C_j\}$ to be a family of closed balls, then we actually construct the **spherical Hausdorff measure** $\mathcal{S}^s$ as follows:

$$\mathcal{S}^s(A) := \lim_{\delta \to 0^+} \mathcal{S}^s_\delta(A),\ \ \mathcal{S}^s_\delta(A) := \inf\{\sum_{j=1}^{\infty} \alpha(s) \cdot \left(\frac{diam(C_j)}{2}\right)^s : A \subseteq \bigcup_{j=1}^{\infty} C_j, C_j = B(x_j, r_j) \in \mathbb{R}^d, 2r_j < \delta\}.$$

Now, we want to discuss on the density defined by measures. Suppose also $0 < s \le d$.

1. For any $A \subseteq \mathbb{R}^d$ which is $\mathcal{L}^d$-measurable with finite d-dim Lebesgue measure, the upper density defined as follows satisfies

$$\limsup_{r \to 0+} \frac{\mathcal{L}^d(A \cap B(x,r))}{\mathcal{L}^d(B(x,r))} = 1,\ \ a.e.\ x \in A.$$

2. For any $A \subseteq \mathbb{R}^d$ which is $\mathcal{H}^s$-measurable with finite s-dim Hausdorff measure, the upper density defined as follows satisfies

$$\limsup_{r \to 0+} \frac{\mathcal{H}^s(A \cap B(x,r))}{\mathcal{H}^s(B(x,r))} \in [2^{-s}, 1],\ \ a.e.\ x \in A.$$

Problem:

Prove or disprove: For any $A \subseteq \mathbb{R}^d$ which is $\mathcal{S}^s$-measurable with finite s-dim spherical Hausdorff measure, the upper density defined as follows satisfies

$$\limsup_{r \to 0+} \frac{\mathcal{S}^s(A \cap B(x,r))}{\mathcal{S}^s(B(x,r))} = 1,\ \ a.e.\ x \in A.$$

Remark: If the last formula holds, then we may draw a conclusion that the reason why the upper density of Huasdorff measure may not attain 1 a.e. is that there is no restriction for the covering family $\{C_j\}$.