6.2 习题课

Ch7 习题

1. $$\begin{cases} u_t-\Delta u=f & \text{in } U_T \\ \frac{\partial u}{\partial \nu}=0 & \text{on } \partial U\times[0,T] \\ u=g & \text{on } U\times\{t=0\} \end{cases}$$

至多一个光滑解

Proof: ~~令 w =~~ 设 u_1. u_2 为原方程 2 个光滑解. $v=u_1-u_2$ 要证 $v=0$

v 满足 $$\begin{cases} v_t-\Delta v=\cancel{f}0 & \text{in } U_T \\ \frac{\partial v}{\partial \nu}=0 & \text{on } \partial U\times[0,T] \\ v=0 & \text{on } U\times\{t=0\} \end{cases}$$

两边乘以 v. 积分得.

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{L^2}^2-\int_U v\Delta v=0$$

分部积分

$$\Longrightarrow \frac{1}{2}\frac{d}{dt}\|v(t)\|_{L^2}^2=-\int_U|\nabla v|^2dx\le 0.$$

又: $\|v(0)\|_{L^2}^2=0$ 故 $\forall t\in(0,T]$. $\|v(t)\|_{L^2}^2\le 0$

$$\Rightarrow \|v(t)\|_{L^2}=0 \quad \underset{v\in C^\infty}{\Rightarrow} \quad v=0 \quad \text{in } [0,T]$$

□

2. 设 u 是如下方程的光滑解.

$$\begin{cases} u_t-\Delta u=\cancel{f}0 & \text{in } U\times(0,\infty) \\ u=0 & \text{on } \partial U\times[0,\infty) \\ u=g & \text{on } \cancel{\partial}U\times\{t=0\} \end{cases}$$

证明: $\|u(\cdot,t)\|_{L^2(U)}\le e^{-\lambda_1 t}\|g\|_{L^2(U)}$

$\lambda_1>0$ 是 $-\Delta$ 的主特征值

Proof: 方程两边乘以 u. 积分得

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2=-\int_U|\nabla u|^2dx$$

又: $$\lambda_1=\inf_{\substack{u\in H_0^1(U)\\ u\neq 0}}\frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^2}^2}$$

故 $\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2\le-\lambda_1\|u(t)\|_{L^2}^2.$

由 Gronwall 不等式

$$\|u(t)\|_{L^2}^2\le e^{-2\lambda_1 t}\|u(0)\|_{L^2}^2 = e^{-2\lambda_1 t}\|g\|_2^2$$

□

7. 设u是光滑解：
$$\begin{cases} u_t - \Delta u + cu = 0 & \text{in } U\times(0,\infty). \\ u = 0 & \text{on } \partial U\times[0,\infty). \\ u = g & \text{on } U\times\{t=0\}. \end{cases}$$

且函数c 满足 $c \geq \gamma > 0$.

证明：$|u(x,t)| \leq Ce^{-\gamma t}$.

Proof：设 $v = e^{\gamma t}u$.

则 $\partial_t v - \Delta v + cv = \gamma e^{\gamma t}u + e^{\gamma t}u_t - e^{\gamma t}\Delta u + ce^{\gamma t}u$.

$$= \gamma v + \big(\underbrace{(\partial_t - \Delta + c)u}_{0}\big)e^{\gamma t}$$

$$= \gamma v$$

$$\Rightarrow \begin{cases} \partial_t v - \Delta v + \underbrace{(c-\gamma)}_{\geq 0} v = 0 & \text{in } U\times(0,\infty) \\ v = 0 & \text{on } \partial U\times[0,\infty). \\ v = g & \text{on } U\times\{t=0\} \end{cases}$$

由弱极大值原理：

$\forall (x,t) \in U_T$.

$$|v(x,t)| = e^{\gamma t}|u(x,t)| \leq \sup_{\Gamma_T}|v(x,t)|.$$

$$= \sup_{x\in U}|g(x)|.$$

$$\Rightarrow |u(x,t)| \leq e^{-\gamma t}\|g\|_{L^\infty}$$

8. 若u是7中方程的光滑解：$g \geq 0$，但c 有界但不一定非负. 证明 $u \geq 0$

Proof：令 $v = e^{-(\|c\|_{L^\infty}+1)t}u$.

同7.
$$\Rightarrow \begin{cases} \partial_t v - \Delta v + (c + \|c\|_{L^\infty} + 1)v = 0 & \text{in } U\times(0,\infty) \\ v = 0 & \text{on } \partial U\times[0,\infty) \\ v = g & \text{on } U\times\{t=0\}. \end{cases}$$

由弱极大值原理. $\min\limits_{U_T} v \geq -\max\limits_{\Gamma_T} u^- = -\max\limits_{U} g^-$.

$g \geq 0 \Rightarrow g^- \equiv 0 \rightarrow \min\limits_{\overline{U}_T} v \geq 0 \rightarrow \min\limits_{\overline{U}_T} u \geq 0$. □

4. (Galerkin Method for Poisson).

$f\in L^2(U)$. $\quad u_m=\sum_{k=1}^{m} d_m^k w_k$ solves $\int_U Du_m\cdot Dw_k\,dx=\int_U f\cdot w_k\,dx$, $\quad 1\le k\le m$.

证明: $\{u_m\}$存在子列, 在 $H_0^1(U)$ 中弱收敛于 $\begin{cases}-\Delta u=f & \text{in } U\\ u=0 & \text{on } \partial U\end{cases}$ 的弱解.

Pf. Step 1: $\{u_m\}$ 在 H_0^1 中一致有界

$\int Du_m\cdot Dw_k\,dx=\int_U f\cdot w_k\,dx$. 两边乘以 d_m^k, 对 k 求和有:

$$\|Du_m\|_{L^2}^2=\int_U f u_m\,dx$$

$$C\|u_m\|_{H_0^1}^2\le\ \le \|f\|_2\|u_m\|_{H_0^1}\le \varepsilon\|u_m\|_{H_0^1}^2+C(\varepsilon)\|f\|_{L^2}^2.$$

ε 充分小 $\Rightarrow$ $\|u_m\|_{H_0^1}^2\le C\|f\|_{L^2}^2$ ✓

Step 2: $\exists$ 子列 $u_{m_k}\rightharpoonup u$ in $H_0^1(U)$.

$Du_{m_k}\rightharpoonup v$ in $L^2(U)$

$v\overset{a.e.}{=}Du$?

$\forall \varphi\in C_c^\infty$

$$\int Du_{m_k}\cdot D\varphi \to \int v\cdot D\varphi$$

$$\int Du_{m_k}\cdot D\varphi=\int f\varphi$$

$$\| $$

$$-\int u_{m_k}\cdot D^2\varphi$$

$$\Downarrow k\to\infty$$

$$-\int u\cdot D^2\varphi=\int Du\cdot D\varphi$$

~~同理可得 左 = $\int v\,D\varphi$~~

□.

Ch7 复习:

1. 抛物方程弱解理论.

$$\begin{cases} \partial_t u + Lu = f & \text{in } U_T \\ u = g & \text{on } \{t=0\}\times U \\ u = 0 & \text{on } [0,T]\times\partial U \end{cases}$$

$f\in L^2(U_T)$. $g\in L^2(U)$

(1) Def: $u\in L^2(0,T;H_0^1)$, $u'\in L^2(0,T;H^{-1})$. 为弱解.

iff: ① $\underbrace{\langle u',v\rangle + B[u,v;t] = (f,v)}_{\text{相当于 Fix } t}$. $\forall v\in H_0^1(U)$. a.e. $t\in[0,T]$

② $u(0)=g$. (因 $u\in C([0,T];L^2)$. 故可以谈论逐点值)

(2) 弱解存在性: Galerkin 逼近

按 $-\Delta$ (零边值) 的特征函数系 (作为 L^2 的标正基, H_0^1 的正交基) 展开

- 目标: 找一列 $\{u_m\}:[0,T]\to H_0^1(U)$. s.t.

$$u_m(t)=\sum_{k=1}^m d_m^k(t)w_k. \leftarrow \text{(有限维截断)}$$

$$\begin{cases} d_m^k(0)=(g,w_k) \\ (u_m',w_k)+B[u_m,w_k;t]=(f,w_k) & 0\le t\le T. \quad 1\le k\le m \end{cases}$$

- 为何存在这样的 $\{u_m\}$?

设 $u_m(t)$ 有形式 $\sum_{k=1}^m d_m^k(t)w_k$.

则 $(u_m'(t),w_k)=d_m^{k\,\prime}(t)$.

$$B[u_m,w_k;t]=\sum_{l=1}^m \underbrace{B[w_l,w_k;t]}_{=e^{kl}(t)} d_m^l(t)$$

令 $f^k(t)=(f(t),w_k)$.

若要满足 $(u_m',w_k)+B[u_m,w_k;t]=(f,w_k)$, 就要满足.

如下ODE: $d_m^{k\,\prime}(t)+\sum_{l=1}^m e^{kl}(t)d_m^l(t)=f^k(t)$ $1\le k\le m$

这可由 ODE 存在唯一性理论得出

☑.

• 如何造出方程本身的弱解：Banach-Alaoglu 定理：

①：$\{u_m\}$，$\{u_m'\}$ 在某些空间中一致有界

(能量估计)．$\underbrace{\sup_{0\le t\le T}\|u_m(t)\|_{L^2}}_{\text{取 sup 后设法用 Gronwall 不等式}} + \underbrace{\|u_m\|_{L^2(0,T;H_0^1)}}_{\text{用抛物方程中的方法}} + \underbrace{\|u_m'\|_{L^2(0,T;H^{-1})}}_{\text{用 } H^{-1} \text{ norm 的对偶表示}}$

$$\le C\left(\|f\|_{L^2(0,T;L^2)} + \|g\|_{L^2}\right)$$

②：Banach-Alaoglu Thm.

存在弱收敛子列．$u_{m_l} \rightharpoonup u$ in $L^2(0,T;H_0^1)$

$u_{m_l}' \rightharpoonup u'$ in $L^2(0,T;H^{-1})$

希望借此证明 u 是该方程弱解

(取 $v=\sum_{k=1}^{N} d^k(t)w_k$)

□

• 唯一性：Gronwall 不等式（能量估计）

(3) 正则性：先用热方程预估得出的结果．(P380-381)．

结论：

(1) $g\in H_0^1$．$f\in L_t^2L_x^2$．$u\in L^2(0,T;H_0^1)$，$u'\in L^2(0,T;H^{-1})$．

$$\Rightarrow u\in L^2(0,T;H^2)\cap L^\infty(0,T;H_0^1),\quad u'\in L^2(0,T;L^2).$$

(2) $g\in H^2$．$f'\in L^2(0,T;L^2)$

$$\Rightarrow u\in L^\infty(0,T;H^2).\quad u'\in L^\infty(0,T;L^2)\cap L^2(0,T;H^1)$$

$$u''\in L^2(0,T;H^{-1}).$$

□

2. 经典解理论:

弱极大值原理:

$u \in C_1^2(U_T) \cap C(\bar{U}_T)$

$c = 0$ in U_T $\Rightarrow$ $u_t + Lu \begin{cases} \le 0 \\ \ge 0 \end{cases}$ in U_T $\Rightarrow$ $\max\limits_{\bar{U}_T} u = \max\limits_{\Gamma_T} u$. / $\min\limits_{\bar{U}_T} u = \min\limits_{\Gamma_T} u$.

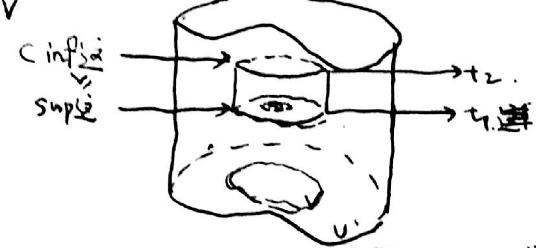
$c \ge 0$ in U_T $\Rightarrow$ $u_t + Lu \begin{cases} \le 0 \\ \ge 0 \end{cases}$ in U_T. $\Rightarrow$ $\max\limits_{\bar{U}_T} u \le \max\limits_{\Gamma_T} u^+$. / $\min\limits_{\bar{U}_T} u \ge -\max\limits_{\Gamma_T} u^-$.

$\Rightarrow$ $= 0$ $\quad \max\limits_{\bar{U}_T} |u| = \max\limits_{\Gamma_T} |u|$.

Harnack 不等式:

$u \in C_1^2(U_T)$. $\begin{cases} u_t + Lu = 0 & \text{in } U_T \\ u \ge 0 & \text{in } U_T. \end{cases}$

$V \subset\subset U$ 连通. 则 $\forall\, 0 < t_1 < t_2 \le T$. $\exists C$, $\sup\limits_V u(\cdot, t_1) \le C \inf\limits_V u(\cdot, t_2)$.



强极大值原理:

$u \in C_1^2(U_T) \cap C(\bar{U}_T)$ U 连通.

~~$c=0$ in U_T.~~ $\quad c = 0 \; (c \ge 0)$ in U_T $\Rightarrow$

~~U 连通.~~

若 $\begin{cases} u_t + Lu \le 0 \text{ 且 } u \text{ 在 } (x_0, t_0) \in U_T \text{ 达 } \bar{U}_T \text{ 中(非负)最大值} \\ \quad \cdots \ge 0 \; \cdots\cdots \text{(非正)极小值} \end{cases}$

~~$u_t + Lu \le 0$ in U_T 且.~~

~~则 $c=0$~~

则 u 在 U_{t_0} 中 const.

□.

抛物方程经典解 $\begin{cases} \text{极大值原理} \\ \text{分离变量法.} \\ \text{特征函数展开法} \end{cases}$

唯一性：$\begin{cases} \text{特征函数展开法} \\ \text{能量法} + \text{Gronwall 不等式.} \end{cases}$

eg：$U=(0,\pi)$. $\quad u_t = u_{xx}$ in $(0,\pi)\times(0,\infty)$.

$u(0)=\varphi(x)$. $\quad \varphi(0)=\varphi(\pi)=0$

何时

则 $\lim\limits_{t\to\infty} e^t u(x,t)=0$?

Pf：设 $u(t)= c(t)\,w(x)$.

$\Rightarrow \dfrac{c'}{c}=\dfrac{w''}{w} =: -\lambda$ 要么只与 t 有关，要么只与 x 有关 ⇒ 必须都无关

⇒ λ const

$\Rightarrow \begin{cases} w''+\lambda w=0 \\ w(0)=w(\pi)=0. \end{cases} \Rightarrow \lambda=k^2 \quad k\in\mathbb{N}$

$c'+\lambda c=0 \Rightarrow c(t)=e^{-\lambda t}.$

$\Rightarrow u(x,t)=\sum\limits_{k=1}^{\infty} C_k e^{-k^2 t}\sin kx.$

$u(x,0)=\sum\limits_{k=1}^{\infty} C_k \sin kx = \varphi(x)$. 其中. $C_k=\dfrac{2}{\pi}\displaystyle\int_0^\pi \varphi(x)\sin kx\,dx$

则 $\lim\limits_{t\to\infty} e^t u(x,t)=0 \iff \sum\limits_{k=1}^{\infty} C_k e^{(1-k^2)t}\sin kx \longrightarrow 0.$

$\iff C_1=0.$

$\iff \displaystyle\int_0^\pi \varphi(x)\sin x\,dx=0.$

除此之外，抛物方程衰减估计也可以对解的L^p范数操作.（补充内容）

eg: $\begin{cases} \partial_t u - \Delta u = 0 & \text{in } (0,\infty)\times\mathbb{R}^d \\ u(0) = f. & \end{cases}$

则直接计算可得 $u(t,x) = (\Phi * f)(t,x).$

其中 $\Phi(t,x) = \dfrac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}$

$\Rightarrow \|u(t,x)\|_{L^q_x(\mathbb{R}^d)} \le \|\Phi\|_{L^p(\mathbb{R}^d)} \|f\|_{L^r(\mathbb{R}^d)}.\qquad 1+\frac{1}{q} = \frac{1}{p}+\frac{1}{r}.$

$$\|\Phi\|_p = \left(\int \frac{1}{(4\pi t)^{d/2}}\left(\int e^{-\frac{p|x|^2}{4t}} dx\right)\right)^{\frac{1}{p}} \underset{dx=\sqrt{\frac{2t}{p}}^d dy}{\overset{y=\sqrt{\frac{p}{2t}}|x|}{=\!=}} C t^{-\frac{d}{2}} t^{\frac{d}{2}\cdot\frac{1}{p}} \left(\int e^{-y^2} dy\right)$$

$$\cancel{=\left(\int \frac{1}{(4\pi t)^{d/2}} e^{-}\right)} \qquad = C t^{-\frac{d}{2}(1-\frac{1}{p})}$$

$$\Rightarrow \|u\|_{L^q} \le C t^{-\frac{d}{2}(1-\frac{1}{p})} \|f\|_{L^r} = C t^{-\frac{d}{2}(\frac{1}{r}-\frac{1}{q})} \|f\|_{L^r}.$$

$q>r$时 即有衰减性. □

~~抛物方程的先验估计~~ 时空

时空范数估计.（补充，不考）

$\begin{cases} \partial_t u - \Delta u = f & \text{in } (0,\infty)\times\mathbb{R}^d. \\ u(0) = g & \text{on } \{0\}\times\mathbb{R}^d. \end{cases}$

Du'Hamel原理 $\cancel{\Rightarrow u(t,x) = e^{t\Delta} g + \int_0^t e^{(t-s)\Delta} f(s)\, ds}$ $\quad u(t,x) = e^{t\Delta} g + \int_0^t e^{(t-s)\Delta} f(\cdot,s)\, ds$

其中 $e^{t\Delta} f := \Phi * f := \int \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{4t}} f(y)\, dy$

$$\|u\|_{L^q_t L^r_x} \lesssim \|e^{t\Delta} g\|_{L^q_t L^r_x} + \left\|\int_0^t e^{(t-s)\Delta} f(s)\, ds\right\|_{L^q_t L^r_x}$$

Step 1: $\cancel{\left\|\int_0^t e^{(t-s)\Delta} f(s)\, ds\right\|_{L^q_t L^r_x} \overset{\text{decay estimate}}{\lesssim} \left\|\int_0^t (t-s)^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{r})} \|f(s)\|_r\, ds\right\|_{L^q_t}}.$

~~Minkowski~~

$\cancel{\Leftarrow \int \|f\|_r}\ \|f\|_r \sim t-s$

Step 1: Christ-Kiselev lemma implies it suffices to prove the estimates for $\int_{\mathbb{R}}$ instead of $\int_0^t$.

非齐次项化为.

$$\Big\| \Big\| \| e^{(t-s)\Delta} f(s) \|_{L_s^1} \Big\|_{L_x^r} \Big\|_{L_t^q} \overset{\text{反Minkowski}}{\leq} \Big\| \Big\| \| e^{(t-s)\Delta} f(s) \|_{L_x^r} \Big\|_{L_s^1} \Big\|_{L_t^q}$$

decay estimates

$$\lesssim \Big\| \int_{\mathbb{R}} |t-s|^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{r})} \| f(s) \|_{L^{\tilde{r}}} \, ds \Big\|_{L_t^q}$$

$$= \Big\| |\cdot|^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{r})} * \| f(\cdot) \|_{L^{\tilde{r}}} \Big\|_{L_t^q}$$

Hardy-Littlewood-Sobolev Ineq.

$$\lesssim \| f \|_{L_t^{\tilde{q}} L_x^{\tilde{r}}}$$

H-L-S:

$$\| |\cdot|^{-\sigma} * f \|_{L^q} \lesssim \| f \|_{L^{\tilde{q}}}$$

$$1 + \frac{1}{q} = \frac{1}{\tilde{q}} + \cancel{d}\frac{\sigma}{d}.$$

利用重排不等式 (Lieb)

or 极大函数的有界性 (GTM250. Ch6).

Step 2:

$$\| e^{t\Delta} g \|_{L_t^q L_x^r} \lesssim \| u_0 \|_{L^2} \quad ?$$

套路:

$$\text{左} = \sup_{\|\varphi\|_{L_t^{q'} L_x^{r'}} \leq 1} \Big| \iint e^{t\Delta} g(x) \varphi(t,x) \, dx \, dt \Big|$$

$$= \sup_{\varphi} \Big| \Big\langle g(x), \int_{\mathbb{R}} e^{t\Delta} \varphi \, dt \Big\rangle_{L_x^2} \Big|.$$

$$\leq \Big\| \int_{\mathbb{R}} e^{t\Delta} \varphi \, dt \Big\|_{L_x^2} \| g \|_{L^2}$$

而

$$\Big\| \int_{\mathbb{R}} e^{(t-s)\Delta} \varphi(s) \, ds \Big\|_{L_x^2}^2$$

$$= \iint_{\mathbb{R}\mathbb{R}} \langle e^{t\Delta} \varphi(t), e^{s\Delta} \varphi(s) \rangle_{L_x^2} \, ds \, dt$$

$$\leq \| \varphi \|_{L_t^{q'} L_x^{r'}} \Big\| \int e^{(s-t)\Delta} f(t) \, dt \Big\|_{L_s^q L_x^r}$$

化为 Step 1.

以上过程 → Strichartz 估计.

(TT* Method. 适用于热、Schrödinger、波、KdV方程等 发展方程). □

Ref: PDE的调和分析方法. 苗长兴.

双曲方程与波方程：

· 双曲方程没有极大值原理

· 双曲方程正则性估不动提高两阶

例：$\partial_t^2 u + Lu = f$

$u(0) = g,\ \partial_t u(0) = h$

$u|_{\partial U} = 0$

$g \in H_0^1 \quad h \in L^2 \quad f \in L_t^2 L_x^2$

~~$u \in L^2[0,T]$~~ $\quad u \in L_t^2 H_0^1 \quad u' \in L_t^2 L_x^2 \quad u'' \in L_t^2 H_x^{-1}$

$\Rightarrow \quad u \in L_t^\infty H_0^1,\ u' \in L_t^\infty L_x^2$

$\not\Rightarrow \quad u \in L_t^2 H_x^2$

· 波动方程的基本解是个分布，不是函数

关于波方程，守恒律、有限传播速度需注意，← 乘上恰当的函数，造出守恒量/单调量

由于波方程基本解是广义函数，且没有热方程一样的解的半群表示，

故对波方程进行衰减估计是困难的：
$\begin{cases} \text{基本解(硬算)} \\ \text{Strichartz估计(插值空间不是Sobolev空间)} \\ \text{Klainerman-Sobolev嵌入} \end{cases}$

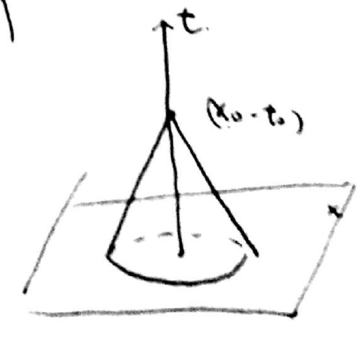
需要流形与黎曼几何

例：
$$\begin{cases} \partial_{tt} u - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = u_t = 0 & \text{in } B(x_0, t_0) \times \{t=0\} \end{cases}$$

$(x_0, t_0) \in \mathbb{R}^n \times (0,\infty)$ fixed.

$K(x_0, t_0) = \{(x,t) \mid 0 \le t \le t_0,\ |x - x_0| \le t_0 - t\}$

则 $u \equiv 0$ in $K(x_0, t_0)$.



Pf：令 $E(t) = \frac{1}{2}\int_{B(x_0, t_0 - t)} u_t^2 + |\nabla u|^2 \, dx$

$$E'(t) = \int_{B(x_0,\, t_0-t)} u_t u_{tt} + \nabla u \cdot \nabla u_t \, dx$$

Coarea formula

$$- \frac{1}{2} \int_{\partial B(x_0,\, t_0-t)} u_t^2 + |\nabla u|^2 \, ds$$

$$= \int_{B(x_0,\, t_0-t)} u_t \underbrace{(u_{tt} - \Delta u)}_{=0} \, dx + \int_{\partial B(x_0,\, t_0-t)} \frac{\partial u}{\partial \nu} \cdot u_t \, ds$$

分部积分得

$$- \frac{1}{2} \int_{\partial B(x_0,\, t_0-t)} u_t^2 + |\nabla u|^2 \, ds$$

$$= \int_{\partial B(x_0,\, t_0-t)} \frac{\partial u}{\partial \nu} u_t - \frac{1}{2} u_t^2 - \frac{1}{2} |\nabla u|^2 \, ds.$$

$$\le \int_{\partial B(x_0,\, t_0-t)} \underbrace{\frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2}_{} - \frac{1}{2} u_t^2 - \frac{1}{2} |\nabla u|^2 \, ds = 0.$$

$\Rightarrow E'(t) \le E(0) = 0, \quad \forall\, 0 \le t \le t_0. \quad \Rightarrow u_t = Du = 0$

$\Rightarrow u = 0 \quad \text{in } K(x_0, t_0)$

$\square$.

Here we use:

$$\frac{d}{dt}\left(\int_{B(x_0,\, t)} f \, dx\right) = \int_{\partial B(x_0,\, t)} f \, ds.$$

this can be derived by $\displaystyle\int_{B(x_0,\, r)} f \, dx = \int_0^r \int_{\partial B(x_0,\, \rho)} f \, ds \, d\rho.$

#

Ch6 复习:

1. Lax-Milgram 定理

回答了 $L+\mu I$ 的 H_0^1 弱解存在性 ($\mu \geq$ 能量估计中的 ν).

L-M Thm: H Hilbert:

$B: H\times H \to \mathbb{R}$. bilinear. satisfies.

Boundedness: $B[u,v] \leq \alpha \|u\|_H \|v\|_H$

Coercivity: $B[u,u] \geq \beta \|u\|_H^2$. for some $\alpha, \beta > 0$.

$\Rightarrow \exists ! f \in H^*$. s.t. $B[u,v] = \langle f, v \rangle$.

~~步骤~~ 用在方程上:

① Find H

How to Find? Determine "what is the weak sol?"

$\Uparrow$

形式推导.

② Construct. $B[u,v]$

③ Check $\begin{cases} \text{Boundedness} & (\text{easier}) \\ \text{Coercivity.} & (\text{hard}) \end{cases}$

Example: 习题 2. 3. 4. 5, 6.

2. Fredholm 二择一.

问: $\begin{cases} Lu = f & \text{in } U \\ u=0 & \text{on } \partial U \end{cases}$ 的 H_0^1 弱解. 是否一定存在唯一?

Answer: $\boxed{\text{并不一定}}$.

2种情况 二择一：

①: $\forall f \in L^2$ ~~时~~, $\exists!$ weak sol. to $\begin{cases} Lu = f & \text{in } U \\ u=0 & \text{on } \partial U \end{cases}$

Recall 线代！

②: $\exists\, u \neq 0$. as the weak sol to $\begin{cases} Lu = 0 & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$

① holds iff. $\forall v \in N^*$. $(f, v) = 0$.

② holds $\Rightarrow$. $N \subset H_0^1(U)$ 零空间 finite-dimensional.

$\dim N = \dim N^*$. null space of L^* with zero boundary data.

Steps: ① Find N and N^*.

② ~~Determ~~ check $(f, v) = \begin{cases} = 0 \ ? \ \forall v. & \Rightarrow \text{① holds} \\ \neq 0 \ \exists v & \Rightarrow \text{② holds}. \end{cases}$

Proof of Fredholm Alternative: $B_\gamma = B[u,v] + \gamma(u,v)$.

L_γ: invertible. $L_\gamma^{-1}: L^2 \to L^2$ compact.

$Lu = f \iff u = L_\gamma^{-1} g. \ (g = f + \gamma u)$.

$\iff u = L_\gamma^{-1}(\gamma u + f)$

$= Ku + h.$ $\quad K = \gamma L_\gamma^{-1}$. $h = L_\gamma^{-1} f$.

$\iff (Id - K) u = h.$

K compact: $L^2 \to L^2$.

then use the fredholm alternative for compact operator.

corollary: Σ: L的谱集

(1). Σ 至多可数. $\Sigma \subset \mathbb{R}$.

$\begin{cases} Lu = \lambda u + f & \text{in } U \\ u = 0. & \text{on } \partial U. \end{cases}$

$\exists!$ 弱解 $\iff$. $\lambda \notin \Sigma$.. 且此时 $\|u\|_2 \leq C\|f\|_2$.

(2) 若 $|\Sigma| = \infty$. 则 $\Sigma = \{\lambda_k\}_1^\infty$. $\lambda_k \nearrow +\infty$

具体计算：

eg：
$$\begin{cases} \Delta u + 2u = x - a & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \qquad U = (0,\pi)\times(0,\pi).$$

对哪些 $a \in \mathbb{R}$，该方程有唯一弱解？

$$Lu = -\Delta u - 2u.$$
$$\Rightarrow L^* u = -\Delta u - 2u.$$

Step 1：考虑齐次方程解空间：

$$\begin{cases} -\Delta u - 2u = 0 & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

分离变量：$u(x,y) = \cancel{u(x)}\, f(x)\, g(y)$

$$\Rightarrow -f''g - g''f - 2fg = 0.$$

$$\Rightarrow \underbrace{\frac{f''}{f}}_{\text{只与}x\text{有关}} + \underbrace{\frac{g''}{g}}_{\text{只与}y\text{有关}} + 2 = 0$$

$$\Rightarrow \frac{f''}{f} = \text{const}, \quad \frac{g''}{g} = \text{const}.$$

设 $\frac{f''}{f} = -\lambda$. $\Rightarrow \begin{cases} f'' + \lambda f = 0 \\ f(0) = f(\pi) = 0 \end{cases} \Rightarrow \lambda = k^2$. $f_k(x) = \sin kx$. $k \in \mathbb{Z}_+$

$\frac{g''}{g} = -\beta \Rightarrow \beta = l^2$

$g_l(x) = \sin lx \quad l \in \mathbb{Z}_+$.

又：$\lambda + \beta = 2 \Rightarrow k^2 + l^2 = 2.$

$$\Rightarrow k = l = 1.$$

$$\Rightarrow u = C \sin x \sin y.$$

Step 2：
由二择一，原方程有解 $\Longleftrightarrow (a - x, \sin x \sin y) = 0.$

$$\Longleftrightarrow \int_0^\pi \int_0^\pi (a - x)\sin x \sin y = 0.$$

$$\Longleftrightarrow a = \frac{\pi}{2}.$$

$\square$

eg (Lax-Milgram)

$$\begin{cases} \Delta u + \frac{1}{4} u = f & \text{in } U. \\ u = 0 & \text{on } \partial U \end{cases} \qquad U = (0, 2\pi) \times (0, 2\pi)$$

求 ① L-M 证明弱解 ∃!

② $\|u\|_{L^2} \le 4 \|f\|_{L^2}$.

Pf: Step 1: Define weak sol:

$Lu := -\Delta u - \frac{1}{4} u$.

$(Lu, v) = (f, v) \Rightarrow \int -(\Delta u) v - \frac{1}{4} uv = \int fv \qquad u, v \in C_c^\infty$

$\overset{u|_{\partial U} = 0}{\Longrightarrow} \int Du \cdot Dv - \frac{1}{4} uv \, dx = \int fv \, dx.$

$u \in H_0^1(U)$ 是弱解 $\Leftrightarrow \forall v \in H_0^1(U)$.

$\int Du \cdot Dv - \frac{1}{4} uv \, dx = \int fv \, dx.$

Step 2: Check L-M

$B[u, v] := \int Du \cdot Dv - \frac{1}{4} uv.$

① $B[u, v] \le \alpha \|u\|_{H_0^1} \|v\|_{H_0^1}$. trivial.

② $B[u, u] = \int |Du|^2 - \frac{1}{4} u^2 \ \overset{?}{\ge} \ \beta \|u\|_{H_0^1}^2$.

Poincaré? 不知最佳常数!

Recall, Principal Eigenvalue of $-\Delta$ $\lambda_1 := \inf_{\substack{\|u\|_{L^2} = 1 \\ u \in H_0^1}} \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^2}^2}$

Similarly as the example above we can get $\lambda_1 = \frac{1}{2}$.

(Consider $-\Delta u = \lambda u$.)

$$\Rightarrow \int |Du|^2 dx \geq \frac{1}{2} \int |u|^2 dx.$$

$$\Rightarrow B[u,u] = \frac{1}{2} \int_\Omega |Du|^2 + \frac{1}{2} \int_\Omega \left(|Du|^2 - \frac{1}{2} u^2 \right) dx$$

$$\geq \frac{1}{2} \int_\Omega |Du|^2 dx$$

$$\cancel{\geq C \|Du\|^2_{H_0^1}}$$

$$\geq \frac{1}{6} \int_\Omega |Du|^2 dx + \frac{1}{3} \times \frac{1}{2} \int_\Omega |u|^2 dx$$

$$= \frac{1}{6} \|u\|^2_{H_0^1}.$$

By L-M. done.

Step 3: Take $v = u$.

$$\cancel{\int u \Delta u \, dx + \frac{1}{4} \int u^2 dx = \int f u \, dx}$$

$$\Rightarrow \frac{1}{4} \int_U u^2 dx - \int |Du|^2 dx = \int f u \, dx$$

$$\Rightarrow \int |Du|^2 dx = \frac{1}{4} \int_U u^2 dx - \int f u \, dx.$$

$$\frac{1}{2} \int u^2 dx \leq$$

$$\Rightarrow \int u^2 dx \leq 4 \|u\|_{L^2} \|f\|_{L^2}. \Rightarrow \|u\|_{L^2} \leq 4 \|f\|_{L^2}$$

$\square$

4. 极大值原理： 习题 8-12（除去 11）

关键：构造出符合极大值原理的函数。

eg（Ex 6.8）. $u\in C^\infty$ solves $Lu=-\sum_{ij}a^{ij}u_{x_ix_j}=0$ in U.

$a^{ij}\in C^\infty$ 证明：$\|Du\|_{L^\infty(U)}\le C\left(\|Du\|_{L^\infty(\partial U)}+\|u\|_{L^\infty(\partial U)}\right)$

Pf. $v=|\nabla u|^2+\lambda u^2$.

$\partial_i|\nabla u|^2=2u_ju_{ij}$.

$\partial_{ij}|\nabla u|^2=2u_ku_{kij}+2u_{kj}u_{ki}$.

$(u^2)_{ij}=2u_iu_j+2uu_{ij}$

$$\Rightarrow\ Lv=-a^{ij}(|\nabla u|^2)_{ij}-\lambda a^{ij}(u^2)_{ij}$$

$$=-a^{ij}(2u_ku_{ijk}+2u_{kj}u_{ki})\underbrace{-\lambda a^{ij}(2u_iu_j+2uu_{ij})}_{\text{the } \lambda a^{ij}\cdot 2uu_{ij}\text{ part } =0}$$

$$=-2\sum_{ij}a^{ij}\Big(\sum_k u_{ki}u_{kj}+\lambda u_iu_j\Big)$$

$$-2\sum_{k=1}^n u_k\sum_{i,j=1}^n a^{ij}u_{ijk}.$$

$$=-2\sum_{k=1}^n\sum_{i,j=1}^n a^{ij}u_{ki}u_{kj}+\lambda\sum_{ij}a^{ij}u_iu_j.$$

$$-2\sum_{k=1}^n u_k\Big(\underbrace{\Big(\sum_{ij}^n a^{ij}u_{ij}\Big)}_{0}{}_k-\sum_{ij}a^{ij}_k u_{ij}\Big).$$

$$\le -\theta\sum_{k=1}^n|\nabla u_k|^2-\lambda\theta|\nabla u|^2+2\sum_k\sum_{ij}u_ka^{ij}_ku_{ij}.$$

$$\le -\theta|D^2u|^2-\lambda\theta|Du|^2+\frac{C}{\varepsilon}|D^2u|^2+C(\varepsilon)|\nabla u|^2.$$

$C(\varepsilon)=C_\varepsilon$.

$$\overset{\varepsilon=\frac{C}{\theta}}{\le}\ \Big(-\lambda\theta+\frac{C^2}{\theta}\Big)|\nabla u|^2\le 0.\quad \lambda\text{充分大}.$$

从而由极大值原理。

$\sup_U v\le\sup_{\partial U}v$

$\Rightarrow \|\nabla u\|^2_{L^\infty(U)}+\lambda\|u\|^2_{L^\infty(U)}$

$\le\|\nabla u\|^2_{L^\infty(\partial U)}+\lambda\|u\|^2_{L^\infty(\partial U)}$

$\Rightarrow\|\nabla u\|_{L^\infty(U)}$

$\le\|\nabla u\|_{L^\infty(\partial U)}+\|u\|_{L^\infty(\partial U)}$

极大值原理

应用于 { 经典解存在性.
梯度估计. Consider. $\log|u|$. $\log|\log|u||$
$-|\nabla u|^2 + \beta\frac{|x|^2}{2n}$ 之类的函数

~~eg: $u\in C^2(\Omega)\cap C(\bar{\Omega})$.~~ $\begin{cases}\Delta u + Xu\end{cases}$

eg: $\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \gamma} + u = \varphi & \text{on } \partial\Omega. \end{cases}$ $\quad f\in C(\Omega)$. $\varphi\in C(\partial\Omega)$.

解的唯一性?

Consider $\begin{cases} -\Delta u + u = 0. & \text{in } \Omega. \\ \frac{\partial u}{\partial \gamma} + u = 0. & \text{on } \partial\Omega \end{cases}$

Pf1: $-\Delta u + u = 0$ in Ω

由弱极大值原理. u 的最大值在 $x_0\in\partial\Omega$ 达到.
非负

$u(x_0)\geq 0$.

Hopf lem $\Rightarrow$. $\left.\frac{\partial u}{\partial n}\right|_{x_0} > 0$.

$\Rightarrow \left.\frac{\partial u}{\partial n} + u\right|_{x_0} > 0$. 与边值矛盾

Pf2. 能量法: $\int_\Omega -u\Delta u + \int_\Omega u^2\,dx = 0$.

$\Rightarrow. -\int_\Omega u\Delta u\,dx = \int_\Omega |\nabla u|^2 - \int_{\partial\Omega} u\cdot\frac{\partial u}{\partial n}\,dS$

$= \int_\Omega |\nabla u|^2 + \int_{\partial\Omega} u^2\,dS$

$\Rightarrow \int_\Omega u^2\,dx + \int_\Omega |\nabla u|^2\,dx + \int_{\partial\Omega} u^2\,dS = 0$

$\Rightarrow u=0 \Rightarrow$ unique! $\Box$