5月13日 习题课.

内容：Ch 6 : 13, 14, 15.

Ch 7 : 5, 6, 9.

计划进程：6.15 → H-S定理 → 6.13 ~~[illegible]~~

~~[illegible] → [illegible]~~ → 热方程衰减

→ Stein...

⇒ → Ch7简介 (Galerkin (存在))

→ Ex 5. 6. 11.

→ Ex. 9

[6.15] $U(\tau) \subset \mathbb{R}^n$. $\partial U(\tau)$速度为$\vec{v}$

$\forall \tau$，考虑
$$\begin{cases} -\Delta w = \lambda w & \text{in } U(\tau) \\ w = 0 & \text{on } \partial U(\tau) \end{cases} \qquad \|w\|_{L^2} = 1.$$

证明：
$$\dot{\lambda} = -\int_{\partial U(\tau)} \left|\frac{\partial w}{\partial \nu}\right| \vec{v} \cdot \vec{\nu} \, dS \qquad \cdot = \frac{d}{d\tau}.$$

Hint：
$$\frac{d}{d\tau}\int_{U(\tau)} f \, dx = \int_{\partial U(\tau)} f \vec{v} \cdot \nu \, ds + \int_{U(\tau)} \partial_\tau f \, dx$$

Prof：
$$-\Delta w = \lambda w \Rightarrow \langle -\Delta w, w \rangle = \lambda \langle w, w \rangle = \lambda.$$

$$\int_{U(\tau)} -\Delta w \cdot w = \int_{U(\tau)} |\nabla w|^2 dx - \int_{\partial U(\tau)} \underset{\overset{\|}{0}}{w} \cdot \frac{\partial w}{\partial \nu} \, dS$$

$$\Rightarrow \lambda = \int_{U(\tau)} |\nabla w|^2 dx$$

$$\dot{\lambda} = \int_{\partial U(\tau)} |\nabla w|^2 \vec{v} \cdot \vec{\nu} \, ds + \int_{U(\tau)} \partial_\tau |\nabla w|^2 dx.$$

~~而 $\int_{U(\tau)} \partial_\tau |\nabla w|^2 dx = \int_{U(\tau)} \partial_\tau (\nabla w \cdot \nabla w)$~~ dx.

~~$= -\int_{U(\tau)} \partial_\tau (\underbrace{w \Delta w}_{\lambda w^2}) \, dx + \int_{\partial U(\tau)} \partial_\tau \cdot (\underset{\overset{\|}{0}}{w} \cdot \frac{\partial w}{\partial \nu}) \, dS$~~

~~$= \frac{d}{d\tau} \int_{U(\tau)} |\nabla w|^2 dx - \int_{\partial U(\tau)} |\nabla w|^2 \vec{v} \cdot \nu \, dS.$~~

$$= \int_{\partial U(\tau)} \left|\frac{\partial w}{\partial \nu}\right|^2 \vec{v} \cdot \nu \, ds + \int_{U(\tau)} 2 \nabla w : \nabla(\partial_\tau w) \, dx.$$

$$= \int_{\partial U(\tau)} \left|\frac{\partial w}{\partial \nu}\right|^2 \vec{v}\cdot\vec{\nu}\, dS + \int_{U(\tau)} 2w\,\underbrace{(-\Delta \partial_\tau w)}_{\shortparallel}\, dx$$

$$\frac{d}{d\tau}(\lambda w) = \lambda \partial_\tau w + \dot{\lambda} w$$

$$= (\qquad\qquad) + \int_{U(\tau)} \underline{2\lambda w w_\tau} + 2\dot{\lambda}\, \underset{\shortparallel\atop 1}{\|w\|_{L^2}^2}$$

$$2\lambda\int \partial_\tau w^2 = 0 \cdot 2\lambda\partial_\tau \int w^2 = 0$$

$$= (\qquad\qquad) + 2\dot{\lambda}$$

$$\Rightarrow \dot{\lambda} = \int_{\partial U(\tau)} \left|\frac{\partial w}{\partial \nu}\right|^2 \vec{v}\cdot\vec{\nu}\, ds.$$

$\square$.

13: $L = -\sum_{i,j}^{n} \partial_j(a^{ij}\partial_i u) \qquad a^{ij} = a^{ji}$ $\qquad \begin{cases} Lw_k = \lambda_k w_k & \text{in } U \\ w_k = 0 & \text{on } \partial U \end{cases}$

证明: $\lambda_k = \sup\limits_{S\in\Sigma_{k-1} \atop \uparrow H_0^1} \inf\limits_{u\in S^\perp \atop \|u\|_{L^2}=1} B[u,u]$.

Proof: 先证: $\lambda_k = \sup\limits_{S\in E_{k-1} \atop \cap L^2} \inf\limits_{u\in S^\perp \atop \|u\|_{L^2}=1} B[u,u]$.

令 $A = L^{-1}: L^2 \to H_0^1(U) \hookrightarrow\hookrightarrow L^2(U)$

$f \mapsto u \quad \mapsto u.$

$A: L^2 \to L^2$ 紧. $\qquad Lw_k = \lambda_k w_k \Rightarrow Aw_k = \frac{1}{\lambda_k} w_k$

$\Rightarrow$ A的特征值为 $\lambda_1^{-1} \geq \cdots \geq 0$.

由 Hilbert-Schmidt 定理 设 A 关于 λ_k^{-1} 有特征向量 $\{e_k\}$. $\|e_k\|_{L^2}=1$

↳ L^2 的标准正交基

则 $\forall f\in L^2$. 有 $f=\sum_i (f, e_i)e_i$

$$\Rightarrow B[u,u]=\langle Lu, u\rangle=\sum_{i=1}^{\infty}\lambda_i(u,e_i)^2, \quad \forall \|u\|_{L^2}=1$$

① $\forall S\in E_{k-1}$ (L^2 $k-1$ dim 子空间).

$\exists u_k\in \text{Span}\{e_1,\dots,e_k\}$. s.t. $u_k\perp S$. (H-S 定理推论)

$$\Rightarrow \inf_{\substack{\|u\|_{L^2}=1\\ u\in S^\perp}} B[u,u]\le B[u_k,u_k]=\sum_{i=1}^{k}\lambda_i(u,e_i)^2 \le \lambda_k$$

② 取 $S=\text{Span}\{e_1,\dots,e_{k-1}\}$. 则 $\forall u\in S^\perp$.

有 $\lambda_k=B[e_k,e_k]\le\sum_{j=k}^{\infty}\lambda_j(u,e_j)^2=B[u,u]$

$$①② \Rightarrow \lambda_k=\sup_{S\in E_{k-1}}\inf_{\substack{\|u\|_{L^2}=1\\ u\in S^\perp}} B[u,u].$$

从而 $\lambda_k \ge \sup_{S\in\Sigma_{k-1}}\inf_{\substack{\|u\|_{L^2}=1\\ u\in S^\perp}} B[u,u]$.

为证 $\le$. 只用取 $S=\text{Span}\{\frac{e_1}{\sqrt{\lambda_1}},\dots,\frac{e_k}{\sqrt{\lambda_k}}\}$.

($\{\frac{e_j}{\sqrt{\lambda_j}}\}_1^\infty$ 是 H_0^1 标正基. 内积 $B[\cdot,\cdot]$)

从而 $\forall u\in S^\perp$. $\|u\|_{L^2}=1$. 设 $u=\sum_{j\ge k}\mu_j\frac{e_j}{\sqrt{\lambda_j}}$, $\mu_j=a_j\sqrt{\lambda_j}$.

$$\Rightarrow B[u,u]=\sum_{j\ge k}a_j^2\lambda_j \ge \lambda_k.$$

□

14. λ_1是如下椭圆算子主特征值.

$$Lu = -\sum_{i,j} a^{ij}\partial_{ij}u + \sum_{i} b^i \partial_i u + cu.$$

则 $\lambda_1 = \sup_{\substack{u\in C^\infty(\bar{U}),\\ u>0 \text{ in } U\\ u=0 \text{ on } \partial U}} \inf_{x\in U} \frac{Lu(x)}{u(x)}.$

<u>Prf</u>: $X = \{u\in C^\infty(\bar{U}) \mid \underset{\text{in } U}{u>0}.\ \underset{\text{on } \partial U}{u=0}\}$.

① 由 Thm 6.5.3 $\exists w_1>0$ $w_1\in H^1(U)$. 为 L 关于 λ_1 的特征向量.

从而. $\exists \{u_n\}\subset X$. $u_n \to w_1$ in H^1.

$$\Rightarrow \sup_X \inf_x \frac{Lu}{u} \geq \inf_x \frac{Lu_n}{u_n} \underset{n\to\infty}{\uparrow}= \lambda_1.$$

② $\forall u\in X$. $\inf_{x\in U} \frac{Lu}{u} \leq \lambda$.

$$\Longleftrightarrow \inf_{x\in U}(Lu - \lambda_1 u) \leq 0.$$

Consider. $L^* w_1^* = \lambda_1 w_1^*$. $w_1^*>0$ 为 L^* 关于 λ_1 的特征向量.

$$\Longleftrightarrow (L^* w_1^*, u) = (\lambda_1 w_1^*, u).$$

$$\Longleftrightarrow \langle Lu, w_1^*\rangle = \langle \lambda_1 u, w_1^*\rangle$$

$$\Longleftrightarrow \langle Lu - \lambda_1 u, w_1^*\rangle = 0$$

$$\Longrightarrow \inf_x (Lu - \lambda_1 u) \leq 0$$

check: λ_1为L^*主特征值 (設为λ_1^*)

由: $\lambda_1^* (w_1^*, w_1)_{L^2} = \langle L^* w_1^*, w_1\rangle_{L^2}$

$$= \langle w_1^*, Lw_1\rangle_{L^2}$$

$$= \lambda_1 \langle w_1^*, w_1\rangle_{L^2}$$

$\Rightarrow \lambda_1^* = \lambda_1$.

□

Ch7

习题5 · 设 $u_k \rightharpoonup u$ in $L^2(0,T;H_0^1)$　　证明：$v=u'$.

$u_k' \rightharpoonup v$ in $L^2(0,T;H^{-1})$.

Proof：首先, 据弱收敛定义有:

$$\int_0^T \langle w, u_n\rangle \to \int_0^T \langle w, u\rangle \qquad \forall w\in L^2(0,T;H^{-1}(U))$$

$$\int_0^T \langle u_n', w\rangle \to \int_0^T \langle v, w\rangle \qquad \forall w\in L^2(0,T;H_0^1(U))$$

$\forall \varphi\in C_c^\infty(0,T)$. $w\in L^2(0,T;H_0^1(U))$ 有: $\varphi w\in L^2(0,T;H^{-1}(U))$

由 $H_0^1 \hookrightarrow\hookrightarrow L^2 \hookrightarrow H^{-1}$.

从而:

$$\left\langle \int_0^T \varphi'(t)u(t)\,dt, w\right\rangle = \int_0^T\langle\varphi'(t)u(t), w\rangle\, dt$$

$$= \int_0^T \langle u(t), \varphi'(t) w\rangle\, dt$$

$$\overset{u_n\rightharpoonup u \text{ in } L_t^2H_x^{-1}}{=} \lim_{n\to\infty}\int_0^T \langle u_n(t), \varphi'(t)w\rangle\, dt$$

$$= \lim_{n\to\infty} \left\langle \int_0^T u_n(t)\varphi'(t)\,dt, w\right\rangle$$

$$\overset{\text{分部积分}}{=} \lim_{n\to\infty}\int_0^T \langle -u_n', \varphi w\rangle\, dt$$

$$\overset{u_n'\rightharpoonup v \text{ in } L_t^2H_0^1}{=} -\int_0^T\langle v, \varphi w\rangle\, dt$$

$$= -\left\langle\int_0^T \varphi v\, dt, w\right\rangle$$

$\cancel{\Rightarrow u'=v}$

$\Rightarrow \forall w\in H_0^1(U).\ \varphi\in C_c^\infty(0,T).\ \left\langle \int_0^T \varphi'(t)u(t)+\varphi(t)v(t)\,dt, w\right\rangle = 0$

$\Rightarrow \forall \varphi\in C_c^\infty(0,T).\ \int_0^T \varphi'(t)u(t)+\varphi(t)v(t)\,dt = 0.$

$\Rightarrow v=u'$

□

6. 设 H 是 Hilbert 空间，$u_k \rightharpoonup u$ in $L^2(0,T;H)$，$\operatorname*{ess\,sup}_{0\le t\le T} \|u_k(t)\| \le C$ $\forall k \in \mathbb{Z}_+$

证明：$\operatorname*{ess\,sup}_{0\le t\le T} \|u(t)\| \le C$

Proof：先证：$\forall\, 0\le a\le b\le T$. $v\in H$，有：

(hint). $\int_a^b (v, u_k(t))\, dt \le C\|v\|\cdot|b-a|$. （显然）

若 hint 成立，则

$$\frac{1}{b-a}\left|\int_a^b (v, u_k(t))\, dt\right| \le C\|v\|$$

这样 由 Lebesgue 微分定理。

对 a.e. $t\in[0,T]$ 成立：$|\langle u_k^{(t)}, v^{(t)}\rangle| \le C\|v\|$. t是 $[a,b]$ 中点，

$k\to\infty$，由弱收敛知，$|\langle u, v\rangle| \le C\|v\|$ a.e. $t\in[0,T]$

$$\Rightarrow \operatorname*{ess\,sup}_{0\le t\le T} \|u(t)\| \le C.$$

□.

9. 证明 7.1.3 中的 (54)。
7.2.3 中的 (59)。

Proof:
(54) 是什么？ $\forall u\in H^2(U)\cap H_0^1(U)$ 成立：$\beta\|u\|_{H^2}^2 \le (Lu, -\Delta u) + \gamma\|u\|_{L^2(U)}^2$.

$(\exists \beta>0,\ \gamma\ge 0)$.

实际用上的 只是对 Galerkin 逼近序列 用此不等式.

在此，为了方便，我们设 $Lu = -\sum_{i,j=1}^n \partial_j(a^{ij}\partial_i u)$ 在实际构造中 $\Delta u_m = 0$ on ∂U.

$\{u\in C^\infty,\ u|_{\partial U}=0.\ \Delta u|_{\partial U}=0.$

要证：$-\int_U \sum_{ij} a^{ij}\partial_i u\, \partial_j(\Delta u)\, dx \ge \frac{\theta}{2}\int_U |D^2u|^2 dx - C\int_U |u|^2 dx$

①出现三阶导，希望分部积分.

分部积分可得:

$$\sum_{ij}\int_U a^{ij}\partial_i u\,\partial_j(\Delta u) = -\sum_{ij}\int_U a^{ij}\partial_i u\,\nabla\cdot(\nabla\partial_j u)\,dx$$

$$= \sum_{ij}\int_U \nabla(a^{ij}\partial_i u)\cdot\nabla(\partial_j\nabla u)\,dx$$

$$-\sum_{ij}\int_{\partial U} a^{ij}\partial_i u\cdot(\partial_j\nabla u)\cdot\vec{n}\,dS$$

$$= \sum_{ijk}\left(\int a^{ij}\partial_{ik}u\,\partial_{jk}u\,dx + \int_U \partial_k a^{ij}\,\partial_i u\,\partial_{jk}u\,dx\right)$$

$$-I.\qquad I := \sum_{ijk}\int_{\partial U} a^{ij}\partial_i u\,\underbrace{\partial_{jk}u\cdot\cos(\vec{n},e_k)}_{=\frac{\partial(\partial_j u)}{\partial\vec{n}}}\,dS$$

~~方向导数~~

希望对I有何控制?

问题:

- 计算出 u 的所有二阶导数 (along ∂U).
- 如何处理 $\frac{\partial u}{\partial x_n}$ (边界的法向导数)?

手段: 边界拉直 (or 带边流形定义), 写成局部坐标.

- 直接计算. 先求 $\partial_{nn}u$. 再求 $\partial_\alpha\partial_n u$. 再求 $\partial_\alpha\partial_\beta u$.

- 希望的结果. $|I| \lesssim \varepsilon\int_U |D^2u|^2 + C_\varepsilon\int_U |u|^2\,dx.$

~~这是~~ $\xLeftarrow{\text{Young's}}$ $|I| \lesssim \int_U |\nabla u|^2\,dx$

因I本身是 ∂U的积分, 转化为U中的积分只有靠"迹定理" $\xLeftarrow{\text{迹定理}}$ $|I| \lesssim \int_{\partial U}\left|\frac{\partial u}{\partial n}\right|^2 dS.$

$\xLeftarrow{\text{局部坐标下,法向为}e_n}$ $|I| \lesssim \int_\Sigma (\partial_{x_n}u)^2\,dx^{n-1}$ ← 于是这成为了我们的目标.

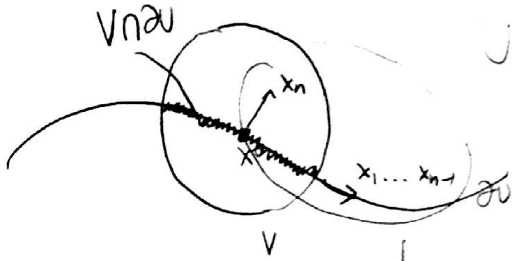
即: 设法用 $\partial_{x_n}u$ 等, 来给出I中各个导数 (尤其是二阶导) 的估计.

$\partial_{x_\alpha}\partial_{x_n}u$.

圈

于是，我们现在要做的是，将∂U上的各阶导数用 $\partial_{x_n}u$、$\partial_\alpha\partial_{x_n}u$ 表示出来。
令中。

Step 1：由单位分解（因∂U紧），可以假设 a^{ij} 的支撑包含于某一点 $x_0\in\partial U$ 的邻域V内。



不妨设：x^0 是 ~~原点~~ 原点，
∂U在x^0处的法向是 x_n轴（$\vec{e}_n$）。

Σ = V∩∂U 在 x_n 轴的投影。

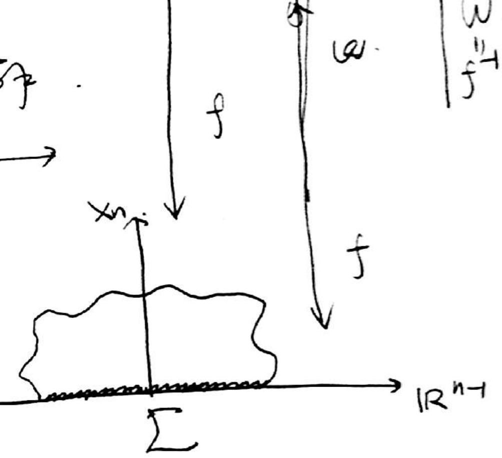
记 $x_n = w(x')$，　$x'=(x_1,\dots,x_{n-1})\in\Sigma$
$w\in C^2(\Sigma)$.

局部微分同胚于上半平面

这一段文字其实可以用带边流形之定义来理解。

如图则存在 ~~坐标~~ 微分同胚 f：

w 即是"该微分同胚之逆 坐标映射之逆"



$\left(\partial_\alpha=\frac{\partial}{\partial x_\alpha}\right)$.

Step 2：计算 $\partial_\alpha\partial_\beta u$.

(1°) 令 $v(x') = \partial_n u(x', w(x'))$.　即 V∩∂U 任一点，用局部坐标可写成 $(x', w(x'))$

~~① 计算~~：求 $\partial_n^2 u$.

对上式求导 $(\alpha=1,2,\dots,n-1)$.

有：$\partial_\alpha v = \partial_\alpha\partial_n u + \partial_n^2 u\cdot\partial_\alpha w$.　…①

又：$u|_{\partial U}=0$.　故 $u(x', w(x'))=0$　$\forall x'\in\Sigma$.

对 x_α 求导：$\partial_\alpha u + \partial_n u\,\partial_\alpha w = \partial_\alpha u + v\cdot\partial_\alpha w = 0$.　on V∩∂U.

对 x_β 求导. $\partial_\alpha\partial_\beta u + \partial_\alpha\partial_n u\,\partial_\beta w + \partial_\beta v\,\partial_\alpha w + v\cdot\partial_\alpha\partial_\beta w = 0$　…②

$1\le\alpha,\beta\le n-1$.

取 $\alpha=\beta$ 有：$\partial_\alpha^2 u + \partial_\alpha\partial_n u\,\partial_\alpha w + \partial_\alpha v\,\partial_\alpha w + v\cdot\partial_\alpha^2 w = 0$　…③

~~对手~~

对α从$1\sim n-1$求和：

注意到 $\Delta u\big|_{\partial u}=0 \Rightarrow -\partial_n^2 u=\sum_{1\le\alpha\le n-1}\partial_\alpha^2 u$.

有：$-\partial_n^2 u+\sum_{1\le\alpha\le n-1}\partial_\alpha\partial_n u\cdot\partial_\alpha w+\partial_\alpha v\,\partial_\alpha w+v\,\partial_\alpha^2 w=0$ $\cdots$ ④

①代入④，有：（重复指标代表求和）. 希腊字母

$$-\partial_n^2 u+(\partial_\alpha v-\partial_n^2 u\,\partial_\alpha w)\partial_\alpha w+\partial_\alpha v\,\partial_\alpha w+v\,\partial_\alpha^2 w=0$$

$$\Rightarrow \partial_n^2 u\left(1+\sum_\alpha(\partial_\alpha w)^2\right)=v\Delta_{n-1}w+2\partial_\alpha v\,\partial_\alpha w$$

$$\Rightarrow \partial_n^2 u=\frac{2\partial_\alpha w}{\sqrt{1+|\nabla_{n-1}w|^2}}\partial_\alpha v+\frac{\Delta w}{\sqrt{1+|\nabla_{n-1}w|^2}}v$$

这样，对$\partial_n^2 u$，我们达到了目的，即用$\partial_\alpha v$. v (i.e. $\partial_\alpha\partial_n u$. $\partial_n u$)表示.

方便起见，令 $\sigma_{nn}^{\alpha}=\dfrac{2\partial_\alpha w}{\sqrt{1+|\nabla_{n-1}w|^2}}\in C^1(\Sigma)$

$$T_{nn}=\frac{\Delta w}{\sqrt{1+|\nabla_{n-1}w|^2}}\in C(\Sigma).$$

$$\Rightarrow \partial_n^2 u=\sigma_{nn}^{\alpha}\partial_\alpha v+T_{nn}v. \quad\cdots⑤$$

$\boxed{\text{上下指标表示求和}}$

(2) 求$\partial_\alpha\partial_n u$.

⑤代入①，即有： α换成β，因避免与①中α重复.

$$\partial_\alpha v=\partial_\alpha\partial_n u+\partial_\alpha w\left(\sigma_{nn}^{\beta}\partial_\beta v+T_{nn}v\right).$$

$$\Rightarrow \partial_\alpha\partial_n u=\partial_\alpha v-\partial_\alpha w\left(\sigma_{nn}^{\beta}\partial_\beta v+T_{nn}v\right).$$

$$=\partial_\beta v\left(\delta_\alpha^\beta-\sigma_{nn}^{\beta}\partial_\alpha w\right)-T_{nn}\partial_\alpha w\cdot v.$$

令，$T_{\alpha n}=T_{nn}\partial_\alpha v\in C(\Sigma)$ $\quad\sigma_{\alpha n}^{\beta}=\delta_\alpha^\beta-\sigma_{nn}^{\beta}\partial_\alpha w\in C^1(\Sigma)$

有：$\partial_\alpha\partial_n u=\sigma_{\alpha n}^{\beta}\partial_\beta v-T_{\alpha n}v.$ $\cdots$ ⑥

(3°). 求 $\partial_\beta \partial_\alpha u$.

①代入②有:

$$\partial_{\alpha\beta} u + (\sigma_{\alpha n}^{\nu} \partial_\nu v + \tau_{\alpha n} v) \partial_\beta w + \partial_\beta v \partial_\alpha w + v \partial_\alpha \partial_\beta w = 0$$

$$\Rightarrow \partial_{\alpha\beta} u = -(\sigma_{\alpha n}^{\nu} \partial_\nu v + \tau_{\alpha n} v) \partial_\beta w - \partial_\beta v \partial_\alpha w - \partial_{\alpha\beta} w \cdot v$$

$$= \partial_\nu v \left(-\sigma_{\alpha n}^{\nu} \partial_\beta w - \delta_\beta^\nu \partial_\alpha w\right)$$

$$+ v \left(-\partial_{\alpha\beta} w - \tau_{\alpha n} \partial_\beta w\right)$$

令 $\sigma_{\alpha\beta}^{\nu} = -\sigma_{\alpha n}^{\nu} \partial_\beta w - \delta_\beta^\nu \partial_\alpha w$

$$\tau_{\alpha\beta} = -\partial_{\alpha\beta} w - \tau_{\alpha n} \partial_\beta w$$

$$\Rightarrow \partial_{\alpha\beta} u = \sigma_{\alpha\beta}^{\nu} \partial_\nu v + \tau_{\alpha\beta} v \qquad \begin{array}{l} \sigma_{\alpha\beta}^{\nu} \in C^1(\Sigma) \\ \tau_{\alpha\beta} \in C(\Sigma) \end{array}$$

Step3: 完成估计:

如今, 可以用一些 $C^1(\Sigma)$ 上函数 g^α 与 $C^0(\Sigma)$ 上函数 h.

表示 I 如下:

$$I = \int_\Sigma v (g^\alpha \partial_\alpha v + hv) \, d\mathcal{H}^{n-1} \cdot |\underbrace{\det w}_{=1}|$$

$$|I| = \cancel{\left|\int_\Sigma \frac{\partial v (g^\alpha \partial_\alpha v + hv^2)}{2}\right.}$$

$$\left|\int_\Sigma \left(\frac{1}{2} g^\alpha \partial_\alpha (v^2) + hv^2\right) d\mathcal{H}^{n-1}\right|$$

分部积分地

$$= \left|\int_\Sigma \left(h - \frac{1}{2}\partial_\alpha g^\alpha\right) v^2 \, d\mathcal{H}^{n-1}\right| \qquad \text{迹定理}$$

$$\lesssim \int_\Sigma v^2 \, d\mathcal{H}^{n-1} \lesssim \int_{\partial U} \left|\frac{\partial u}{\partial n}\right|^2 dS \lesssim \int_U |Du|^2 dx$$

再由 Ch5. T9 有 $\int_U |Du|^2 dx \lesssim \|u\|_{L^2} \|D^2 u\|_{L^2} \lesssim \varepsilon \|D^2 u\|_{L^2}^2 + C(\varepsilon) \|u\|_{L^2}^2$ □

双曲方程的能量法：

11. $$\begin{cases} \partial_t^2 u + \partial_x^4 u = 0 & \text{in } (0,1)\times(0,T). \\ u = \partial_x u = 0 & \text{on } (\{0\}\times[0,T]) \cup (\{1\}\times[0,T]) \\ u = g. \quad u_t = h. & \text{on } [0,1]\times\{t=0\} \end{cases}$$

<u>存在至多一个光滑解</u>.

<u>Proof</u>:

$v = u_1 - u_2.$

$$\Rightarrow \partial_t^2 v + \partial_x^4 v = 0.$$

乘以 v_t $\Rightarrow$ $\partial_t v \partial_t^2 v + \partial_t v \, \partial_x^4 v = 0.$

而：$\partial_t(\partial_t v)^2 = 2\partial_t^2 v \partial_t v.$

$\partial_t(\partial_x^2 v) = 2(\partial_x^2 v \cdot \partial_t \partial_x^2 v)$

所以上式 $\Rightarrow$ $\frac{1}{2}\partial_t\left(\|v_t\|_{L^2}^2 + \|v_{xx}\|_{L^2}^2\right) = 0.$

积分.

(分部2次.对x).

$\Rightarrow \|v_t\|^2 + \|v_{xx}\|^2 = \text{const.} = t=0$ 时的值 $=0$

$\Rightarrow v = 0.$

□.

$$\begin{cases} \partial_t u + Lu = f & \text{in } U_T \\ u(0) = g & \text{on } \partial U \end{cases}$$ $\square$

X.1.3 正则性

对(*) $\begin{cases} \partial_t u + Lu = f & \text{in } U_T \\ u(0) = g & \text{on } \partial U \end{cases}$

在下定正则性的结论之前，我们先要通过一些形式推导来预测最终的结论。

对热方程：$\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times [0, T] \\ u(0) = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$

$$\Rightarrow \int_{\mathbb{R}^n} f^2 = \int_{\mathbb{R}^n} (u_t - \Delta u)^2 dx.$$

$$= \int_{\mathbb{R}^n} u_t^2 - 2\Delta u \cdot u_t + (\Delta u)^2 dx.$$

$$= \int_{\mathbb{R}^n} u_t^2 - 2\nabla u_t \cdot \nabla u + (\Delta u)^2 dx$$

$$\Rightarrow \frac{d}{dt}\int |\nabla u|^2 dx + \int u_t^2 + \int |\Delta u|^2 = \int f^2 dx.$$

对t积分可得:

$$\operatorname*{ess\,sup}_{0\le t\le T}\int_{\mathbb{R}^n}|\nabla u|^2+\int_0^T\int_{\mathbb{R}^n}u_t^2+|\Delta u|^2\,dx\,dt$$

$$\le\int|\nabla u(0)|^2\,dx+\int_0^T\int_{\mathbb{R}^n}|f|^2\,dx\,dt$$

$$\lesssim\int|\nabla g|^2dx+\int_0^T\int|f|^2\,dx\,dt$$

所以，对弱解,我们预测: $g\in H_0^1,\ f\in L_t^2L_x^2 \Rightarrow u\in L_t^2H_0^1,\ u'\in L_t^2H_x^{-1}$

$$\Rightarrow u\in L^2(0,T;H^2)\cap L^\infty(0,T;H_0^1)$$

$$\uparrow\quad u'\in L^2(0,T;L^2)$$

观察上面的式子表示什么存在!

进一步地,~~若 g 再对~~ 令 $\tilde{u}=u_t$.

$$\Rightarrow\begin{cases}\tilde{u}_t-\Delta\tilde{u}=\tilde{f} & \text{in } \mathbb{R}^n\times(0,T]\\ \tilde{u}=\tilde{g} & \text{in } \mathbb{R}^n\times\{t=0\}\end{cases}$$

$\tilde{f}=\partial_t f$.

$\tilde{g}=f(\cdot,0)+\Delta g$
$=\partial_t u(\cdot,0)$.

两边乘 $\tilde{u}$.

$$\Rightarrow \underline{\tilde{u}\cdot\partial_t\tilde{u}}-\underline{\tilde{u}\Delta\tilde{u}}=\tilde{f}\tilde{u}$$

和 $\frac{1}{2}\frac{d}{dt}\|\tilde{u}\|_{L^2}^2$ 分部积分后会变成 $\int|\nabla\tilde{u}|^2$

对t积分, x积分.

$$\Rightarrow\frac{1}{2}\int_0^T\frac{d}{dt}\|\tilde{u}\|_{L^2}^2\,dt+\int_0^T\int|\nabla u_t|^2\,dx\,dt\lesssim\int_0^T\int_{\mathbb{R}^n}f_t^2\,dx\,dt+\int_{\mathbb{R}^n}|\nabla^2 g|^2+f(\cdot,0)^2\,dx)$$

~~又: $\sup_{0\le}$~~

$$\Rightarrow\sup_{0\le t\le T}\int_{\mathbb{R}^n}|u_t|^2\,dx+\int_0^T\int_{\mathbb{R}^n}|\nabla u_t|^2\,dx\,dt$$

$$\le C\left(\int_0^T\int_{\mathbb{R}^n}f_t^2\,dx\,dt+\int_{\mathbb{R}^n}|\nabla^2 g|^2+f(\cdot,0)^2\,dx\right)$$

但 $\sup_{0\le t\le T}\|f(\cdot,t)\|_{L^2}\le C\|f\|_{L^2(\mathbb{R}^n\times(0,T))}+\|f_t\|_{L^2(\mathbb{R}^n\times(0,T))}$ $\Leftarrow$ (Thm 7.0.3).

又：$-\Delta u = f - \partial_t u$

所以. $\int_{\mathbb{R}^n} |\nabla^2 u|^2 \, dx \leq C \int f^2 + (\partial_t u)^2 \, dx$ (拉有同子知性).

$$\Rightarrow \sup_{0\le t\le T} \int_{\mathbb{R}^n} |u_t|^2 + |\nabla^2 u|^2 \, dx + \int_0^T \int_{\mathbb{R}^n} |\nabla u_t|^2 \, dx\, dt$$

$$\leq C \left(\int_0^T \int_{\mathbb{R}^n} f_t^2 + f^2 \, dx\, dt + \int_{\mathbb{R}^n} |\nabla^2 g|^2 \, dx \right)$$

□.

由此，我们可以预测.

$$g \in H^2,\quad f' \in L^2(0,T;L^2) \Rightarrow \left.\begin{array}{l} u \in L_t^\infty H_x^2 \\ u' \in L_t^\infty L_x^2 \cap L_t^2 H_0^1 \\ u'' \in L^2(0,T;H^{-1}) \end{array}\right\} \text{观察上人的结论}.$$

关于热方程，本身具有良好的光滑效应.

$$\begin{cases} \partial_t u - \Delta u = 0 \\ u_{t=0} = g \in L^2. \end{cases}$$

由 Fourier 变换有：$\hat{u}(t,\xi) = e^{-t|\xi|^2} \hat{g}(\xi)$.

$$\Rightarrow \|u\|_{H^s(\mathbb{R}^d)} \underset{\uparrow}{:=} \| \langle\xi\rangle^s \hat{u}(t,\xi) \|_{L^2(\mathbb{R}^d)}.$$

$(\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}})$

$$= \| \underbrace{(e^{-t|\xi|^2} \langle\xi\rangle^s)}_{\text{有界Fourier乘子}} \hat{g} \|_{L^2}.$$

Mikhlin-Hörmander 乘子定理

$$\lesssim \|g\|_{L^2}. \quad \forall s>0$$

对自由热方程. $g \in L^2 \Rightarrow u \in C^\infty$

□.

(选)
抛物方程 衰减估计.

$$\partial_t u - \Delta u = 0 \qquad u \in C^{1,2}(\mathbb{R}_{\geq 0} \times \mathbb{R}^d).$$
$$u(0,x) = f(x) \in C^2(\mathbb{R}^d)$$
$$\Rightarrow \|u\|_{L^p(\mathbb{R}^d)} \leq t^{-\frac{d}{2}(\frac{1}{r}-\frac{1}{p})} \|f\|_{L^r(\mathbb{R}^d)} \qquad 1 \leq p \leq r < \infty$$

Pf: 直接解方程:

$$u(x,t) = \int_{\mathbb{R}^d} \underbrace{\frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{4t}}}_{\Phi(x-y,t)} f(y)\, dy.$$
$$= (\Phi * f)(x)$$

$$\|u\|_{L^p} = \|\Phi * f\|_{L^p}$$

卷积不等式 $\leq \underbrace{\|\Phi\|_{L^q}}_{\downarrow} \|f\|_{L^r}$ $\qquad 1+\frac{1}{p} = \frac{1}{q}+\frac{1}{r}$

直接计算有 $\lesssim t^{-\frac{d}{2}(\frac{1}{r}-\frac{1}{p})}$.

卷积 Young 不等式: $\|f*g\|_q \leq \|f\|_p \|g\|_r \qquad 1+\frac{1}{q} = \frac{1}{p}+\frac{1}{r}$

$$|f*g| = \left|\int f(y)\, g(x-y)\, dy\right|$$

$f, g \geq 0$.

$$= \left|\int f(y)^a\, g(x-y)^b \left(f(y)^{1-a} g(x-y)^{1-b}\right) dy\right|$$

$$\overset{\text{三元Hölder}}{\leq} \|f^a\|_{p_1} \|g^b\|_{p_2} \|f(y)^{1-a} g(x-y)^{1-b}\|_{p_3}.$$

凑: $a = 1-\frac{p}{q}$. $\quad b = 1-\frac{r}{q}$.

$$p_1 = \frac{p}{1-\frac{p}{q}} = \frac{pq}{q-p} \qquad p_2 = \frac{r}{1-\frac{r}{q}} = \frac{rq}{q-r}.$$

即有:

$$|f*g| \leq \|f\|_p^{1-\frac{p}{q}} \|g\|_r^{1-\frac{r}{q}} \left(\int |f(y)|^p |g(x-y)|^r dy\right)^{\frac{1}{q}}$$

取 L^q 范数

$$\|f*g\|_q \leq \|f\|_p^{1-\frac{p}{q}} \|g\|_r^{1-\frac{r}{q}} \left(\int\left(\int |f(y)|^p |g(x-y)|^r dy\right) dx\right)^{\frac{1}{q}}$$
$$= \|f\|_p \|g\|_r.$$

TT^* 方法：（对称算子的应用）

若是 Hilbert 空间 $T \in \mathcal{L}(\mathcal{H} \to \mathcal{H})$

证明：$\|TT^*\| = \|T^*T\| = \|T\|^2$.

令 TT^*、T^*T 对称、

证：对称算子

又：$(TT^*x, y) = (T^*x, T^*y)$

$$\xRightarrow{x=y} (TT^*x, x) = \|T^*x\|^2.$$

$$\Rightarrow \|TT^*\| = \sup_{|x|=1} |(TT^*x, x)| = \sup_{\|x\|=1} \|T^*x\|^2$$

$$= \|T^*\|^2 = \|T\|^2.$$

另一边、同理

□.

用处：对 ~~基本方程~~ 色散方程（Schrödinger 方程，KdV）Schrödinger 基本解的估计.

Schrödinger 方程：

$$\begin{cases} i\partial_t u + \Delta u = f \\ u(0) = u_0 \end{cases}$$

$$\Rightarrow u = e^{it\Delta} u_0 \pm i\int_0^t e^{i(t-s)\Delta} f(s)\, ds.$$

$$(e^{it\Delta} f)^\wedge = e^{-it|\xi|^2} \hat{f}(\xi).$$

$\|e^{it\Delta} f\|_{L^2} \lesssim \|f\|_{L^2}$

$\|e^{it\Delta} f\|_{L^\infty} \lesssim t^{-\frac{d}{2}} \|f\|_{L^1}$

再由 Riesz-Thorin 插值即可

衰减估计：$\|e^{it\Delta} f\|_{L^p} \lesssim t^{-d(\frac{1}{2} - \frac{1}{p})} \|f\|_{L^{p'}}$ $p \geq 2$.

Strichartz 估计：$\|e^{it\Delta} u_0\|_{L_t^q L_x^r} \lesssim \|u_0\|_{L^2}$.

$$\left\|\int_0^t e^{i(t-s)\Delta} f(s)\, ds\right\|_{L_x^2} \leq \|f\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}.$$

$$\frac{2}{\tilde{q}} + \frac{d}{\tilde{r}} = \frac{d}{2}, \quad \tilde{q}, \tilde{r} \geq 2$$

先證第2个对.

第一个: $\|e^{it\Delta}u_0\|_{L_t^q L_x^r} = \sup\limits_{\|\varphi\|_{L_t^{q'}L_x^{r'}}\le 1} \left|\int_{\mathbb{R}} \langle e^{it\Delta}u_0, \varphi\rangle\, dt\right|$

$$= \sup_{\varphi} \left|\langle u_0, \int e^{it\Delta}\varphi_t\rangle\right|$$

$$\le \sup \|u_0\|_{L^2} \left\|\underbrace{\int e^{it\Delta}\varphi}\right\|_{L^2}.$$

$$\|\underbrace{\int e^{it\Delta}\varphi}\|_{L^2}^2 = \langle \int e^{it\Delta}\varphi(t)\, dt, \int e^{is\Delta}\varphi(s)\, ds\rangle$$

从 $\|T\|_2^2$

$$= \iint \langle \underline{e^{i(t-s)\Delta}\varphi(s)}, \varphi(t)\rangle\, dt\, ds$$

化成了 TT^*

$$\le \|\varphi\|_{L_t^{\tilde q'} L_x^{\tilde r'}} \underline{\left\|\int e^{i(t-s)\Delta}\varphi(s)\, ds\right\|_{L_t^{\tilde q} L_x^{\tilde r}}}$$

Minkowski

$$\le \left\|\int_{\mathbb{R}} \|e^{i(t-s)\Delta}f(s)\|_{L_x^{\tilde r}}\, ds\right\|_{L_t^{\tilde q}}$$

衰减估计

$$\le \left\|\int_{\mathbb{R}} |t-s|^{-d(\frac{1}{2}-\frac{1}{\tilde r})} \|f(\cdot)\|_{L_x^{\tilde r'}}\, ds\right\|_{L_t^{\tilde q}}$$

$$= \left\| |\cdot|^{-\overset{\gamma}{\overbrace{d(\frac{1}{2}-\frac{1}{\tilde r})}}} * \|f(\cdot)\|_{L^{\tilde r'}} \right\|_{L_t^{\tilde q'}}$$

Hardy-Littlewood-Sobolev

$$\lesssim \|f\|_{L_t^{\tilde q'} L_x^{\tilde r'}}$$

$$\frac{2}{\tilde q} + \frac{d}{\tilde r} = \frac{d}{2}$$

$$\Downarrow$$

$$\begin{cases} \frac{1}{\tilde q'} + 1 = \frac{1}{\tilde q} + \frac{\gamma}{d} \\ \gamma = d\left(\frac{1}{2} - \frac{1}{\tilde r}\right) \end{cases}$$

这些估计称作 Schrödiger 方程的 <u>Strichartz</u> 估计.