2017年春季学期 微分方程II习题课勘误

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第一次习题课中, 我们讲到了Sobolev函数的链式法则, 也就是书上的习题17. 事实上, 更一般的定理叙述如下:

Chain rule for Sobolev functions If $f \in W^{1,p}(U), F \in C^1(\mathbb{R}), F' \in L^\infty(R), F(0) = 0$, then $F(f) \in W^{1,p}(U)$ and $\partial_i F(f) = F'(f)\partial_i f$, $\mathcal{L}^n$-a.e. for all $i = 1, \cdots, n$. When $\mathcal{L}^n(U) < \infty$, $F(0) = 0$ is not necessary.

$F(0) = 0$这个条件是用在U测度无限时证明$F(f) \in W^{1,p}$的过程中的, 具体如下.

Proof: Note that $|F(f) - F(0)| \leq \|F'\|_\infty |f|$, therefore we can only prove $|F(f) - F(0)| \in L^p$. When U is of finite n-dimensional Lebesgue measure, then we know any constant function is an L^p function. But when $\mathcal{L}^n(U) = \infty$, we must set $F(0) = 0$ since any nonzero constant is not an L^p function. Therefore we've proved $F(f) \in L^p$.

Note that the proof of the chain rule does not require $\partial_i F(f) \in L^p$. Therefore we can prove $\partial_i F(f) = F'(f)\partial_i f$. $\partial_i F(f) \in L^p$ is trivial since $F' \in L^\infty$ and $\partial_i f \in L^p$.

该叙述摘自Evans和Gariepy的Measure Theory and Fine Properties of Functions (Revised Version), Chapter 4, Theorem 4.4, on page 153.

第二次习题课中, 习题11我最后一步做的有问题. 11题的正确解法如下:

Proof: Let $u^\epsilon(x) = (\eta_\epsilon * u)(x), V \subset\subset U$. Then $Du^\epsilon = \eta_\epsilon * Du \ \ in \ \ V$. Since u^ϵ is smooth in V, then there exists a constant C_ϵ such that $u^\epsilon = C_\epsilon \ \ in \ \ V$.

Note that $\|u^\epsilon\|_p \leq \|\eta_\epsilon\|_1 \|u\|_p < \infty$ uniformly on ϵ. Therefore $\{C_\epsilon\}$ is a uniformly bounded sequence. Consequently, there exists a subsequence C_{ϵ_i} converges to some $C \in \mathbb{R}$ as $i \to \infty$. Finally $u^\epsilon \to u$ in $W^{1,p}(V)$ (also a.e.) yields that $u = C$ a.e. in any $V \subset\subset U$. Done.