# 实分析(H)第四次习题课讲稿

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## 1 课本第三章习题

3. **Proof:** (1)Since 0 is the Lebesgue density point of $E$, then

$$\forall n\geq 3, \exists\{r_n\}\ \ with\ \ r_n \xrightarrow{decreasingly} 0, s.t.: m(B(0,r_n)\cap E)\geq (1-\frac{1}{n})m(B(0,r_n)).$$

Since $B(0,r_n)$ is symmetric, then

$$\forall n\geq 3, \exists\{r_n\}\ \ with\ \ r_n \xrightarrow{decreasingly} 0, s.t.: m(B(0,r_n)\cap -E)\geq (1-\frac{1}{n})m(B(0,r_n)).$$

By the Inclusion-Exclusion Principle(容斥原理),

$$m(B(0,r_n)\cap E\cap -E)\geq (1-\frac{2}{n})m(B(0,r_n))>0.$$

Thus, for each distinct $n$, we can choose an $x_n$ from $E\cap -E\cap B(0,r_n)$. It is easily to see that $x_n, -x_n\in E$ and $x_n\to 0$(since $r_n\to 0$).

(2)Similarly as in (1), we can choose a sequence of $\{r_n\}$ which decreasingly tends to 0 such that

$$\forall n\geq 4, m(B(0,r_n)\cap E)\geq (1-\frac{1}{n})m(B(0,r_n)).$$

$$\forall n\geq 4, m(B(0,r_n)\cap -E)\geq (1-\frac{1}{n})m(B(0,r_n)).$$

$$\forall n\geq 4, m(B(0,r_n)\cap \frac{E}{2})\geq (1-\frac{1}{n})m(B(0,r_n)).$$

Then

$$m(B(0,r_n)\cap E\cap -E\cap \frac{E}{2})\geq (1-\frac{3}{n})m(B(0,r_n))>0.$$

Thus, for each distinct $n$, we can choose an $x_n$ from $E\cap -E\cap\frac{E}{2}\cap B(0,r_n)$. It is easily to see that $x_n, -x_n, 2x_n\in E$ and $x_n\to 0$(since $r_n\to 0$).

□

6. **Proof:** Note that

$$\begin{aligned} x\in E_\alpha^+ &\Leftrightarrow \exists h>0, s.t. \frac{1}{h}\int_x^{x+h}|f(y)|dy>\alpha \\ &\Leftrightarrow \int_x^{x+h}|f(y)|dy>h\alpha \\ &\Leftrightarrow (\int_0^{x+h}-\int_0^x)|f(y)|dy-\alpha(x+h)+\alpha x>0 \\ &\Leftrightarrow F(x+h)>F(x), \exists h>0,\ \ where\ \ F(x)=\int_0^x|f(y)|dy-\alpha x \end{aligned}$$

Thus $E_\alpha^+=\{x:\exists h>0, s.t.\ \ F(x+h)>F(x)\}$. Since F is continuous, by Rising Sum Lemma(太阳升引理) $E_\alpha^+=\bigcup_{j=1}^\infty(a_j,b_j)$ (disjoint) and $F(a_j)=F(b_j)$ implies $\int_{a_j}^{b_j}|f(y)|dy=\alpha(b_j-a_j)$. Take the summation with respect to $j$, we have

$$\int_{E_\alpha^+}|f(y)|=\alpha m(E_\alpha^+).$$

□

7. **Proof:** $m(E) > 0$ is quite trivial. Suppose $m(E) < 1$ and we want to deduce a contradiction. Denote $E^c$ as $[0,1] - E$.

Since $E^c$ has positive measure, then there exists $x \in E^c$ to be the Lebesgue density point of $E^c$. Fix $\alpha > 0$, and then $\forall \beta > 1 - 0.5\alpha$, there exists an interval $I$, s.t. $m(I \cap E^c) \geq \beta m(I)$.

By the hypothesis of the problem, for the interval $I$ chosen above, $m(I \cap E) \geq \alpha m(I)$. Take the summation of the two formulas above, we have $m(I) \geq (1 + 0.5\alpha)m(I)$. This contradicts with $m(I) > 0$. Thus $m(E) = 1$.

□

9. **Proof:** Note that $\delta$ is Lipschitz continuous and thus it is absolutely continuous(see Exercise 32). Then $\delta'$ a.e. exists and $\delta'(x) = 0$ a.e. $x \in F$, since $\delta = 0$ in $F$. Thus a.e. $x \in F$ has the following property:

$$0 = \lim_{|y| \to 0} \frac{|\delta(x+y) - \delta(x)|}{|y|} = \lim_{|y| \to 0} \frac{|\delta(x+y)|}{|y|}.$$

□

11. **Proof:** (1)Suppose $f \in BV[0,1]$, but $a \leq b$, we want to deduce a contradiction. Set

$$x_n = \frac{1}{(n\pi + \pi/2)^{1/b}}.$$

Then

$$T_f[0,1] \geq \sum_{n=1}^{\infty} |f(x_n) - f(x_{n-1})| \geq \sum_{n=1}^{\infty} \frac{1}{(n\pi + \pi/2)^{a/b}} = \infty (since \;\; a \leq b).$$

This contradicts with $f \in BV[0,1]$. Thus $a > b$.

(2)Suppose $a > b$. Note the $f$ is differentiable in (0,1). Then for each partition $\pi : 0 = x_0 < \cdots < x_n = 1$, we have

$$\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \leq \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} |f'(t)| dt \leq \int_0^1 a x^{a-1} + b x^{a-b-1} < \infty (since \;\; -1 < a - b - 1 < a - 1).$$

Take the supremum for all partition $\pi$ and the LHS of the last formula is $T_f[0,1]$, which implies $f$ is of bounded variation.

(3)(i)$x^{a+1} < h < 1$, then

$$|f(x+h) - f(x)| \leq x^a + (x+h)^a \leq h^{a/a+1} + (2h^{1/a+1})^a \leq 2^{a+1} h^{a/a+1}$$

.

(ii)$x^{a+1} \geq h$, then by mean value formula,

$$|f(x+h) - f(x)| = |f'(x+p)|h = ah(|(x+p)^{a-1} \sin \frac{1}{(x+p)^a}| + |\frac{1}{x+p}|) \leq \frac{2ah}{x} \leq 2ah^{a/a+1}$$

.

Thus $|f(x+h) - f(x)| \leq \max\{2^{a+1}, 2a\} h^{a/a+1}$.

□

14. **Proof:** (1)By the definition of lim sup, we know

$$D^+F(x) = \inf_{\delta > 0} \sup_{0 < h < \delta} \frac{F(x+h) - F(x)}{h} = \inf_{n \in \mathbb{N}} \sup_{0 < h < \frac{1}{n}} \frac{F(x+h) - F(x)}{h} = \inf_{n \in \mathbb{N}} \sup_{\mathbb{Q} \cap (0, \frac{1}{n})} \frac{F(x+h) - F(x)}{h}.$$

This is obviously measurable. Note that the last step is correct, since for each $h < 1/n$, we can "insert" a rational number into the "gap".

(2)Follow the hint, we have

$$\limsup_{h\to 0} \frac{J(x+h)-J(x)}{h} = \lim_{m\to\infty}\lim_{k\to\infty}\lim_{N\to\infty}\sup_{1/k\leq|h|\leq 1/m} |\frac{J_N(x+h)-J_N(x)}{h}|$$

, where $J_n(x)=\sum_{n=1}^{N}\alpha_n j_n(x)$ is the partial sum of the jump function.

□

15. **Proof:** Write $F = G_1 - G_2$ where $G_1$ and $G_2$ are increasing. As shown in Lemmas 3.12, 3.13, an increasing function is a continuous increasing function plus a jump function. Hence $G_1 = F_1 + J_1$ where $F_1$ is continuous and increasing, and $J_1$ is a jump function; similarly, $G_2 = F_2 + J_2$. Then $F = (F_1 - F_2) + (J_1 - J_2)$. But $J_1 - J_2$ is a jump function, and jump functions are continuous only if they're constant. Since $F$ is continuous, this implies that $J_1 - J_2$ is constant; WLOG, $J_1 - J_2 = 0$. (Otherwise we could re-define $F_1' = F_1 + (J_1 - J_2)$ and $F_1'$ would also be continuous and increasing.) Hence $F = F_1 - F_2$.

□

16. **Proof:**(1)$F(x) = P_F(a,x) - N_F(a,x) + F(a)$, then $F'(x) = P_F'(a,x) - N_F'(a,x)$. Thus

$$\int_a^b |F'(x)|dx \leq \int_a^b P_F'(a,x) + N_F'(a,x)dx \leq P_F(a,b) + N_F(a,b) = T_F(a,b).$$

(2)(i)Suppose $F \in AC[a,b]$, then we can write $F(x) = \int_a^x F'(t)dt + F(a)$. Thus for each partition $\pi : a = x_0 < \cdots < x_n = b$, we have

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \leq \int_a^b |F'(t)|dt \leq (by\ \ (1))T_F(a,b).$$

Take the supremum for all partition $\pi$ we know LHS=$T_F(a,b)$. Thus we have $\int_a^b |F'(t)|dt = T_F(a,b)$.

(ii)Conversely, suppose the equality holds. Define $G(x) = \int_a^x |F'(t)|dt - T_F(a,x)$. Then $G(a) = G(b) = 0$. And $\forall x < y, G(y)-G(x) = \int_x^y |F'(t)|dt - T_F(x,y) \leq 0(by\ \ (1))$. Thus $G(x) = 0$ in $[a,b]$ and $T_F(x) = \int_a^x |F'(t)|dt$ is absolutely continuous. Thus $\forall\epsilon > 0, \exists\delta > 0$, we have $\sum_{i=1}^{n} |T_F(x,b_i) - T_F(x,a_i)| < \epsilon$ whenever $\sum_{i=1}^{n}(b_i - a_i) < \delta$. Note that $|F(b_i) - F(a_i)| \leq T_F(a_i, b_i)$. We know $F$ satisfies the definition of absolutely continuous.

□

17. **Proof:**

$$I_1 := \int_{|y|<\epsilon} |f(x-y)||K_\epsilon(y)|dy \leq \frac{A}{|\epsilon|^d}\int_{|y|<\epsilon} |f(x-y)|dy \leq Cf^*(x).$$

Note that we take the supremum over all the balls containing $x$ to "construct" the Hardy-Littlewood Maximal Function in the last step.

$$\begin{aligned}
I_2 &:= \int_{|y|\geq\epsilon} |f(x-y)||K_\epsilon(y)|dy = \sum_{k=0}^{\infty}\int_{2^k\epsilon\leq|y|<2^{k+1}\epsilon} |f(x-y)||K_\epsilon(y)|dy \\
&\leq \sum_{k=0}^{\infty}\frac{A'\epsilon}{(2^k\epsilon)^{d+1}}\int_{|y|<2^{k+1}\epsilon} |f(x-y)|dy \\
&= \sum_{k=0}^{\infty}\frac{A'}{2^{k-d}}\frac{1}{(2^{k+1}\epsilon)^d}\int_{|y|<2^{k+1}\epsilon} |f(x-y)|dy \\
&\leq A''f^*(x).
\end{aligned}$$

And $\sup_\epsilon |K_\epsilon * f| \le I_1 + I_2$. Done.

□

19. **Proof:** We first admit (1) is correct. Now we use (1) to prove (2). Note that each measurable set can be decomposed as $E = F \cup Z$, where $F$ is an $F_\sigma$-set and $Z$ measures 0. By (1), f(Z) measures 0. As for $f(F)$, F can be decomposed as $F = \bigcup_{k=1}^\infty F_k = \bigcup_{k=1}^\infty \bigcup_{n=1}^\infty (F_k \bigcap B(0,n))$. This is a countable union of compave sets. Note that a continuous function maps a compact set to be a compact set. Thus f(F) is also a countable union of compace sets (an $F_\sigma$-set also). Thus $f(E)$ is measurable.

Now we prove (1). Suppose $Z$ measures 0. $\forall \delta > 0, \exists$ open set $O, Z \subseteq O, m(O) < \epsilon$. $O$ can be decomposed as $\bigcup_{j=1}^\infty (a_j, b_j)$. Set $m_j = \inf_{[a_j,b_j]} f(x), M_j = \sup_{[a_j,b_j]} f(x)$. Then

$$m(f(O)) = \sum_{j=1}^\infty |f(M_j) - F(m_j)| = \sum_j |\int_{m_j}^{M_j} f'(t)dt| \le \sum_j \int_{a_j}^{b_j} |f'(t)|dt = \int_O |f'(t)|dt.$$

Since $f'$ is integrable and $m(O)$ can be arbitrarily small, we know the RHS of last formula $< \epsilon$ by the absolute continuity of integral. Let $\epsilon$ tend to 0 and we are done.

□

20. **Proof:** (1)Suppose $C$ is a Cantor-like set of $[a,b]$ with positive measure and $K := [a,b] - C$. Define $F(x) := \int_a^x \chi_K(t)dt$. Then $F' = 0$ on a positive measure set. To see the strict monotonicity we merely note that $K$ intersects a open interval in positive measure.

(2)Write $K = \bigcup_{j=1}^\infty (a_j, b_j)$. Then $F(K) = \bigcup_j (F(a_j), F(b_j))$ and $m(F(K)) = \sum_{j=1}^\infty (F(b_j) - F(a_j))$. Note that $m(F([a,b])) = F(b) - F(a) = \int_K 1dx = m(K)$. Thus $\int_C \chi_K = 0$. $m(F(C)) = 0$. Note that $C$ is of positive measure. Thus there exists unmeasurable subset $N \subseteq C$ and $F(N) \subseteq F(C)$ measures 0. Set $E = F(N)$ and we are done.

(3)Each measurable set can be decomposed as the union of an $F_\sigma$-set $D$ and a set $Z$ measuring 0. $F^{-1}(D)$ is measurable since $F^{-1}$ maps closed set to closed set. As for $F^{-1}(Z) \cap \{F'(x) > 0\}$, you should prove the hint $m(O) = \int_{F^{-1}(O)} F'(x)dx$ by decomposing $O = \bigcup_{j=1}^\infty (a_j, b_j)$ and the rest is quite easy. After proving the hint, we choose a decreasing sequence of open sets $\{O_n\}$ covering $Z$ with $m(O_n) < 1/n$. Then

$$m(O_n) = \int_{F^{-1}(O_n)} F'(x)dx = \int_{F^{-1}(O_n)\cap\{F'>0\}} F'(x)dx.$$

Use DCT or MCT we know

$$0 = m(\bigcap_{n=1}^\infty O_n) = \int_{\bigcap_n F^{-1}(O_n)\cap\{F'>0\}} F'(x)dx.$$

Thus $\{F' > 0\} \cap \bigcap_n F^{-1}(O_n)$ measures 0 and its subset $F^{-1}(Z) \cap \{F' > 0\}$ measures 0. Done.

□

24. (1)**Add an extra condition "$F$ is bounded"!**

Set $F_J$ as the jump function of $F$. Then $G = F - F_J$ is continuous and increasing. Set $F_A(x) = \int_a^x F_1'(t)dt$ and $F_c = F_1 - F_A$. $F_A \in AC[a,b]$ is quite trivial. $F_c$ is continuous and $F_C' = F_1' - F_A' = F_1' - F_1' = 0$ a.e. Next we need the check the fact that $F_C$ is increasing. Since $\forall y > x$, $F_1(y) - F_1(x) \ge \int_x^y F_1'(t)dt$ we know $F_C$ is increasing. Done.

(2)Suppose also $F = G_A + G_C + G_J$. Since the "jump" of a function is uniquely defined by $F$ itself. Thus $F_J = G_J + C$. Thus $F_A' = G_A'$ a.e., but $F_A, G_A$ are absolutely continuous. Thus we have $F_A = G_A + C'$ and $F_C = G_C - C - C'$. Done.

□

23, 32:Suppose $F \in C[a,b]$.

(1)Prove that if $D^+F \geq 0, \forall x \in [a,b]$, then $F$ is increasing.

(2)$F$ is Lipschitz continuous with Lip-const $M$, iff $F \in AC[a,b]$ and $|F'| \leq M$.

**Proof:** (1)It suffices to prove $F(b) \geq F(a)$, then use any sub-interval $[a', b'] \subseteq [a,b]$ to replace $[a,b]$ to get the result. We prove this by contradiction. Suppose $F(b) < F(a)$, set $G_r(x) = F(x) - F(a) + r(x-a)$. For $r$ sufficiently small, We have $G_r(b) < 0 = G_r(a)$. Set $x_0 = \sup\{x : G_r(x) \geq 0\}$. Since $D^+G_r = r + D^+F > 0$, then we know there exists $x_1 > x_0$ s.t. $G_r(x_1) > 0$. Thus $\exists x_2 > x_1, G_r(x_2) = 0$ since $G_r$ is continuous and $G_r(b) < 0$. So the existence of $x_2$ contradicts with the definition of $x_0$. Therefore $F(b) \geq F(a)$. Done.

(2)If RHS holds, then $|f(x) - F(y)| \leq \int_x^y |F'(t)|dt \leq M|x-y|$.

If LHS holds, $\forall \epsilon > 0$, take $\delta = \frac{\epsilon}{M}$. Then $\sum_{i=1}^n |F(b_i) - F(a_i)| \leq \epsilon$ whenever $\sum_{i=1}^n (b_i - a_i) < \delta$. Thus $F$ is absolutely continuous and $F'$ a.e. exists. For the points where $F'$ exists, $|\lim_{h\to 0} \frac{F(x+h)-F(x)}{h}| \leq M$ by the given Lipschitz condition. Done.

□

## 2 Two Problems in 2015 Final Exam

7. 假设有以下命题正确：设$f$是$\mathbb{R}$上的连续函数, $2\pi$, 1都是$f$的周期，则$f$恒为某个常数$C$.

现在假设$f$仅是$\mathbb{R}$上的局部可积函数，且$2\pi$, 1都是它的周期，证明$f$ a.e.是个常数.

**证明：**令$f_h(x) = \frac{1}{h}\int_x^{x+\frac{1}{h}} f(t)dt$. 则$f_h$连续, 且周期与$f$相同, 那么则$f_h(x) = C$. 据Lebegsue微分定理, 令$h \to 0$, 我们就有$f(x) = C$,a.e.

□

8. 设$\forall \epsilon \in (0,1), f \in AC[\epsilon, 1]$. 且满足

$$\int_0^1 x|f'(x)|^p dx < +\infty, (p > 2).$$

证明:$\lim_{x\to 0+} f(x)$存在.

**证明：**只需要证明$\lim_{a,b\to 0+} |f(b) - f(a)| = 0$, 再用柯西列的方法证明即可. 为此不妨$b > a$.

$$|f(b) - f(a)| \leq \int_a^b |f'(t)|dt = \int_a^b x^{-1/p}(x^{1/p}|f'(x)|)dx$$

. 用Holder不等式,

$$RHS \leq (\int_a^b x^{-p'/p} dx)^{1/p'} \cdot (\int_0^1 x|f'(x)|^p dx)^{1/p} \to 0$$

as $a, b \to 0$. 这是因为上式右边第一个积分是趋于0的, 因为$p > 2$, 则$1 - \frac{p'}{p} > 0$.

□