Well-posedness and Incompressible Limit of Current-Vortex Sheets with Surface Tension in Compressible Ideal MHD

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April 30, 2024

Abstract

Current-vortex sheet is one of the characteristic discontinuities in ideal compressible magnetohydrodynamics (MHD). The motion of current-vortex sheets is described by a free-interface problem of two-phase MHD flows with magnetic fields tangential to the interface. This model has been widely used in both solar physics and controlled nuclear fusion. This paper is the first part of the two-paper sequence, which aims to present a comprehensive study for compressible current-vortex sheets with or without surface tension. We prove the local well-posedness and the incompressible limit of current-vortex sheets with surface tension. The key observation is a hidden structure of Lorentz force in the vorticity analysis which motivates us to establish the uniform estimates in anisotropic-type Sobolev spaces with weights of Mach number determined by the number of tangential derivatives. Besides, we develop a robust framework for iteration and approximation to prove the local existence of vortex-sheet problems. Our method does not rely on Nash-Moser iteration nor tangential smoothing.

Mathematics Subject Classification (2020): 35Q35, 76N10, 76W05, 35L65.
Keywords: Current-vortex sheets, Magnetohydrodynamics, Incompressible limit, Surface tension.

Contents

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1 Introduction

The equations of compressible ideal magnetohydrodynamics (MHD) in $\mathbb{R}^d$ ($d = 2, 3$) can be written in the following form

$$\begin{cases} \varrho D_t u = B \cdot \nabla B - \nabla Q, \quad Q := P + \frac{1}{2}|B|^2, \\ D_t \varrho + \varrho \nabla \cdot u = 0, \\ D_t B = B \cdot \nabla u - B \nabla \cdot u, \\ \nabla \cdot B = 0, \\ D_t \mathfrak{s} = 0. \end{cases} \tag{1.1}$$

Here $\nabla := (\partial_{x_1}, \cdots, \partial_{x_d})$ is the standard spatial derivative. $D_t := \partial_t + u \cdot \nabla$ is the material derivative. The fluid velocity, the magnetic field, the fluid density, the fluid pressure and the entropy are denoted by $u = (u_1, \cdots, u_d)$, $B = (B_1, \cdots, B_d)$, ϱ, P and $\mathfrak{s}$ respectively. The quantity $Q := P + \frac{1}{2}|B|^2$ is the total pressure. Note that the fourth equation in (1.1) is just an initial constraint instead of an independent equation. The last equation of (1.1) is derived from the equation of total energy and Gibbs relation and we refer to [17, Ch. 4.3] for more details. To close system (1.1), we need to introduce the equation of state

$$P = P(\varrho, \mathfrak{s}) \text{ satisfying } \frac{\partial P}{\partial \varrho} > 0. \tag{1.2}$$

A typical choice in this paper would be the polytropic gas parametrized by $\lambda > 0$:

$$P_\lambda(\varrho, \mathfrak{s}) = \lambda^2(\varrho^\gamma \exp(\mathfrak{s}/C_V) - 1), \quad \gamma > 1, \ C_V > 0. \tag{1.3}$$

We also need to assume $\varrho \geq \bar{\rho}_0 > 0$ for some constant $\bar{\rho}_0 > 0$, which together with $\frac{\partial P}{\partial \varrho} > 0$ guarantees the hyperbolicity of system (1.1).

1.1 Mathematical formulation of current-vortex sheets

Let $H > 10$ be a given real number, $x = (x_1, \cdots, x_d)$ and $x' := (x_1, \cdots, x_{d-1})$ and the space dimension $d = 2, 3$. We define the regions $\Omega^+(t) := \{x \in \mathbb{T}^{d-1} \times \mathbb{R} : \psi(t, x') < x_d < H\}$, $\Omega^-(t) := \{x \in \mathbb{T}^{d-1} \times \mathbb{R} : -H < x_d < \psi(t, x')\}$ and the moving interface $\Sigma(t) := \{x \in \mathbb{T}^{d-1} \times \mathbb{R} : x_d = \psi(t, x')\}$ between $\Omega^+(t)$ and $\Omega^-(t)$. We assume $U^\pm = (u^\pm, B^\pm, P^\pm, \mathfrak{s}^\pm)^\top$ to be a smooth solution to (1.1) in $\Omega^\pm(t)$ respectively. We say $\Sigma(t)$ is a *current-vortex sheet* (or an *MHD tangential discontinuity*) if the following conditions are satisfied:

$$[\![Q]\!] = \sigma \mathcal{H}, \quad B^\pm \cdot N = 0, \quad \partial_t \psi = u^\pm \cdot N \quad \text{on } \Sigma(t), \tag{1.4}$$

where $N := (-\partial_1 \psi, \cdots, -\partial_{d-1} \psi, 1)^\top$ is the normal vector to $\Sigma(t)$ (pointing towards $\Omega^+(t)$), $\sigma \geq 0$ is the constant coefficient of surface tension and the quantity $\mathcal{H} := \overline{\nabla} \cdot \left(\frac{\overline{\nabla} \psi}{\sqrt{1 + |\overline{\nabla} \psi|^2}} \right)$ is twice the mean curvature of $\Sigma(t)$ with $\overline{\nabla} = (\partial_1, \cdots, \partial_{d-1})$. The jump of a function f on $\Sigma(t)$ is denoted by $[\![f]\!] := f^+|_{\Sigma(t)} - f^-|_{\Sigma(t)}$ with $f^\pm := f|_{\Omega^\pm(t)}$. The first condition shows that the jump of total pressure is balanced by surface tension. The second condition shows that both plasmas are perfect conductors. The third condition shows that there is no mass flow across the interface and thus the two plasmas are physically contact and mutually impermeable. These conditions on $\Sigma(t)$ are given by the Rankine-Hugoniot conditions for ideal compressible MHD when the magnetic fields are tangential to the interface, and we refer to Trakhinin-Wang [63, Appendix A] for detailed derivation. Besides, we impose the slip boundary conditions on the rigid boundaries $\Sigma^\pm := \mathbb{T}^{d-1} \times \{\pm H\}$

$$u_d^\pm = B_d^\pm = 0 \quad \text{on } \Sigma^\pm. \tag{1.5}$$

Remark 1.1 (Initial constraints for the magnetic field). The conditions $\nabla \cdot B^\pm = 0$ in $\Omega^\pm(t)$, $B^\pm \cdot N|_{\Sigma(t)} = 0$ and $B_d^\pm = 0$ on $\Sigma^\pm$ are constraints for initial data so that system (1.1) with jump conditions (1.4) is not over-determined. One can use the continuity equation, the evolution equation of B and the kinematic boundary condition to show that $D_t^\pm(\frac{1}{\rho^\pm} \nabla \cdot B^\pm) = 0$ in $\Omega^\pm(t)$ and $D_t^\pm(\frac{B^\pm}{\rho^\pm} \cdot N) = 0$ on $\Sigma(t)$ and $\Sigma^\pm$ with $D_t^\pm := \partial_t + u^\pm \cdot \nabla$. Thus, the initial constraints can propagate within the lifespan of solutions if initially hold.

To make the initial-boundary-valued problem (1.1)-(1.5) solvable, we have to require the initial data to satisfy certain compatibility conditions. Let $(u_0^\pm, B_0^\pm, \varrho_0^\pm, \mathfrak{s}_0^\pm, \psi_0) := (u^\pm, B^\pm, \varrho^\pm, \mathfrak{s}^\pm, \psi)|_{t=0}$ be the initial data of system (1.1)-(1.4). We say the

initial data satisfies the compatibility condition up to m-th order ($m \in \mathbb{N}$) if

$$
\begin{aligned}
(D_t^\pm)^j \llbracket Q \rrbracket|_{t=0} &= \sigma (D_t^\pm)^j \mathcal{H}|_{t=0} \quad \text{on } \Sigma(0), \ \ 0 \le j \le m, \\
(D_t^\pm)^j \partial_t \psi|_{t=0} &= (D_t^\pm)^j (u^\pm \cdot N)|_{t=0} \quad \text{on } \Sigma(0), \ \ 0 \le j \le m, \\
\partial_t^j u_d^\pm &= 0 \quad \text{on } \Sigma^\pm, \ \ 0 \le j \le m.
\end{aligned}
\tag{1.6}
$$

With these compatibility conditions, one can show that the magnetic fields also satisfy (cf. [60, Section 4.1])

$$
(D_t^\pm)^j (B^\pm \cdot N)|_{t=0} = 0 \quad \text{on } \Sigma(0) \text{ and } \Sigma^\pm, \ \ 0 \le j \le m.
$$

We also note that the fulfillment of the first condition implicitly requires the fulfillment of the second one.

For $T > 0$, we denote $\Omega_T^\pm := \bigcup_{0 \le t \le T} \{t\} \times \Omega^\pm(t)$ and $\Sigma_T := \bigcup_{0 \le t \le T} \{t\} \times \Sigma(t)$. We consider the Cauchy problem of (1.1): Given the initial data $(u_0^\pm, B_0^\pm, \varrho_0^\pm, \mathfrak{s}_0^\pm, \psi_0)$ satisfying the compatibility conditions (1.6) up to certain order, the vortex-sheet condition $|\llbracket u_0 \cdot \tau \rrbracket|_\Sigma > 0$ for any vector τ tangential to $\Sigma(0)$, the constraints $\nabla \cdot B_0^\pm = 0$ in $\Omega^\pm(0)$, $(B_0^\pm \cdot N)|_{\Sigma(0)} = 0$ and $B_{0d}^\pm|_{\Sigma^\pm} = 0$, we want to study the well-posedness and the incompressible limit of the following system for the case $\sigma > 0$ in this paper. The zero-surface-tension limit under suitable stability conditions on Σ_T and further improvement of the incompressible limit will be discussed in the second part of the two-paper sequence.

$$
\begin{cases}
\varrho^\pm(\partial_t + u^\pm \cdot \nabla)u^\pm - B^\pm \cdot \nabla B^\pm + \nabla Q^\pm = 0, \quad Q^\pm := P^\pm + \frac{1}{2}|B^\pm|^2 & \text{in } \Omega_T^\pm, \\
(\partial_t + u^\pm \cdot \nabla)\varrho^\pm + \varrho^\pm \nabla \cdot u^\pm = 0 & \text{in } \Omega_T^\pm, \\
(\partial_t + u^\pm \cdot \nabla)B^\pm = B^\pm \cdot \nabla u^\pm - B^\pm \nabla \cdot u^\pm & \text{in } \Omega_T^\pm, \\
\nabla \cdot B^\pm = 0 & \text{in } \Omega_T^\pm, \\
(\partial_t + u^\pm \cdot \nabla)\mathfrak{s}^\pm = 0 & \text{in } \Omega_T^\pm, \\
P^\pm = P^\pm(\varrho^\pm, \mathfrak{s}^\pm), \quad \frac{\partial P^\pm}{\partial \varrho^\pm} > 0, \quad \varrho^\pm \ge \overline{\rho}_0 > 0 & \text{in } \overline{\Omega_T^\pm}, \\
\llbracket Q \rrbracket = \sigma \overline{\nabla} \cdot \left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}} \right) & \text{on } \Sigma_T, \\
B^\pm \cdot N = 0 & \text{on } \Sigma_T, \\
\partial_t \psi = u^\pm \cdot N & \text{on } \Sigma_T, \\
u_d^\pm = B_d^\pm = 0 & \text{on } [0,T] \times \Sigma^\pm, \\
(u^\pm, B^\pm, \varrho^\pm, \mathfrak{s}^\pm)|_{t=0} = (u_0^\pm, B_0^\pm, \varrho_0^\pm, \mathfrak{s}_0^\pm) \quad \text{in } \Omega^\pm(0), \qquad \psi|_{t=0} = \psi_0 \quad \text{on } \Sigma(0).
\end{cases}
\tag{1.7}
$$

System (1.7), as a hyperbolic conservation law, admits a conserved L^2 energy

$$
E_0(t) := \sum_\pm \frac{1}{2} \int_{\Omega^\pm(t)} \varrho^\pm |u^\pm|^2 + |B^\pm|^2 + 2\mathfrak{P}(\varrho^\pm, \mathfrak{s}^\pm) + \varrho^\pm |\mathfrak{s}^\pm|^2 \, \mathrm{d}x + \sigma \, \mathrm{Area}(\Sigma(t))
$$

where $\mathfrak{P}(\varrho^\pm, \mathfrak{s}^\pm) = \int_{\overline{\rho}_0}^{\varrho^\pm} \frac{P^\pm(z, \mathfrak{s}^\pm)}{z^2} \, \mathrm{d}z$. See Section 3.1 for proof.

1.2 Reformulation in flattened domains

1.2.1 Flattening the fluid domains

We shall convert (1.7) into a PDE system defined in fixed domains $\Omega^\pm := \mathbb{T}^{d-1} \times \{0 < \pm x_d < H\}$. One way to achieve this is to use the Lagrangian coordinates, but it would bring lots of unnecessary technical difficulties when analyzing the surface tension. Here, we consider a family of diffeomorphisms $\Phi(t, \cdot) : \Omega^\pm \to \Omega^\pm(t)$ characterized by the moving interface. In particular, let

$$
\Phi(t, x', x_d) = (x', \varphi(t, x_d)),
\tag{1.8}
$$

where

$$
\varphi(t, x) = x_d + \chi(x_d)\psi(t, x')
\tag{1.9}
$$

and $\chi \in C_c^\infty([-H, H])$ is a smooth cut-off function satisfying the following bounds:

$$
\|\chi'\|_{L^\infty(\mathbb{R})} \le \frac{1}{\|\psi_0\|_\infty + 20}, \quad \sum_{j=1}^8 \|\chi^{(j)}\|_{L^\infty(\mathbb{R})} \le C, \quad \chi = 1 \ \text{ on } (-1, 1)
\tag{1.10}
$$

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for some generic constant $C > 0$. We assume $|\psi_0|_{L^\infty(\mathbb{T}^2)} \leq 1$. One can prove that there exists some $T_0 > 0$ such that $\sup_{[0,T_0]} |\psi(t, \cdot)|_{L^\infty(\mathbb{T}^2)} < 10 < H$, the free interface is still a graph within the time interval $[0, T_0]$ and

$$\partial_d \varphi(t, x', x_d) = 1 + \chi'(x_d)\psi(t, x') = 1 - \frac{1}{20} \times 10 \geq \frac{1}{2}, \quad t \in [0, T_0],$$

which ensures that $\Phi(t)$ is a diffeomorphism in $[0, T_0]$.

Based on this, we introduce the new variables

$$v^\pm(t, x) = u^\pm(t, \Phi(t, x)), \quad b^\pm(t, x) = B^\pm(t, \Phi(t, x)), \quad \rho^\pm(t, x) = \varrho^\pm(t, \Phi(t, x)),$$
$$S^\pm(t, x) = \mathfrak{s}^\pm(t, \Phi(t, x)), \quad q^\pm(t, x) = Q^\pm(t, \Phi(t, x)), \quad p^\pm(t, x) = P(t, \Phi(t, x)) \tag{1.11}$$

that represent the velocity fields, the magnetic fields, the densities, the entropy functions, the total pressure functions and the fluid pressure functions defined in the fixed domains $\Omega^\pm$ respectively. Also, we introduce the differential operators

$$\nabla^\varphi = (\partial_1^\varphi, \cdots, \partial_d^\varphi), \quad \partial_a^\varphi = \partial_a - \frac{\partial_a \varphi}{\partial_d \varphi}\partial_d, \quad a = t, 1, \cdots, d-1; \quad \partial_d^\varphi = \frac{1}{\partial_d \varphi}\partial_d. \tag{1.12}$$

Moreover, setting the tangential gradient operator and the tangential derivatives as

$$\overline{\nabla} := (\partial_1, \cdots, \partial_{d-1}), \quad \overline{\partial}_i := \partial_i, \quad i = 1, \cdots, d-1,$$

then the boundary conditions (1.4) on the free interface $\Sigma(t)$ are turned into

$$[\![q]\!] = \sigma \mathcal{H}(\psi) := \sigma \overline{\nabla} \cdot \left(\frac{\overline{\nabla}\psi}{\sqrt{1 + |\overline{\nabla}\psi|^2}} \right) \quad \text{on } [0, T] \times \Sigma, \tag{1.13}$$

$$\partial_t \psi = v^\pm \cdot N, \quad N = (-\overline{\partial}_1 \psi, -\overline{\partial}_2 \psi, 1)^\top \quad \text{on } [0, T] \times \Sigma, \tag{1.14}$$

$$b^\pm \cdot N = 0 \quad \text{on } [0, T] \times \Sigma, \tag{1.15}$$

where $\Sigma = \mathbb{T}^{d-1} \times \{x_d = 0\}$.

Let $D_t^{\varphi\pm} := \partial_t^\varphi + v^\pm \cdot \nabla^\varphi$. Then system (1.7) is converted into

$$\begin{cases}
\rho^\pm D_t^{\varphi\pm} v^\pm - (b^\pm \cdot \nabla^\varphi)b^\pm + \nabla^\varphi q^\pm = 0, \quad q^\pm = p^\pm + \frac{1}{2}|b^\pm|^2 & \text{in } [0, T] \times \Omega^\pm, \\
D_t^{\varphi\pm} \rho^\pm + \rho^\pm \nabla^\varphi \cdot v^\pm = 0 & \text{in } [0, T] \times \Omega^\pm, \\
p^\pm = p^\pm(\rho^\pm, S^\pm), \quad \frac{\partial p^\pm}{\partial \rho^\pm} > 0, \quad \rho^\pm \geq \bar{\rho}_0 > 0 & \text{in } [0, T] \times \Omega^\pm, \\
D_t^{\varphi\pm} b^\pm - (b^\pm \cdot \nabla^\varphi)v^\pm + b^\pm \nabla^\varphi \cdot v^\pm = 0 & \text{in } [0, T] \times \Omega^\pm, \\
\nabla^\varphi \cdot b^\pm = 0 & \text{in } [0, T] \times \Omega^\pm, \\
D_t^{\varphi\pm} S^\pm = 0 & \text{in } [0, T] \times \Omega^\pm, \\
[\![q]\!] = \sigma \overline{\nabla} \cdot \left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}} \right) & \text{on } [0, T] \times \Sigma, \\
\partial_t \psi = v^\pm \cdot N & \text{on } [0, T] \times \Sigma, \\
b^\pm \cdot N = 0 & \text{on } [0, T] \times \Sigma, \\
v_d^\pm = b_d^\pm = 0 & \text{on } [0, T] \times \Sigma^\pm, \\
(v^\pm, b^\pm, \rho^\pm, S^\pm, \psi)|_{t=0} = (v_0^\pm, b_0^\pm, \rho_0^\pm, S_0^\pm, \psi_0).
\end{cases} \tag{1.16}$$

Invoking (1.12), we can alternatively write the material derivative D_t^φ as

$$D_t^{\varphi\pm} = \partial_t + \bar{v}^\pm \cdot \overline{\nabla} + \frac{1}{\partial_d \varphi}(v^\pm \cdot \mathbf{N} - \partial_t \varphi)\partial_d, \tag{1.17}$$

where $\bar{v}^\pm := (v_1^\pm, \cdots, v_{d-1}^\pm)^\top$ is the horizontal components of the fluid velocity, $\bar{v}^\pm \cdot \overline{\nabla} := \sum_{j=1}^{d-1} v_j^\pm \partial_j$, and $\mathbf{N} := (-\partial_1 \varphi, \cdots, -\partial_{d-1}\varphi, 1)^\top$ is the extension of the normal vector N into $\Omega^\pm$. This formulation will be helpful for us to define the linearized material derivative when using Picard iteration to construct the solution.

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1.2.2 Parametrization of the equation of state

We assume the fluids in Ω^+ and Ω^- satisfy the same equation of state of polytropic gases. Specifically, we parametrize the equation of state to be $\rho = \rho(p/\lambda^2, S)$ where $\lambda > 0$ is proportional to the sound speed $c_s := \sqrt{\partial_p \rho}$. For a polytropic gas, the equation of state is parametrized in terms of $\lambda > 0$:

$$p_\lambda(\rho, S) = \lambda^2 \left(\rho^\gamma \exp(S/C_V) - 1 \right), \quad \gamma > 1, \ C_V > 0. \tag{1.18}$$

When viewing the density as a function of the pressure and the entropy, this indicates

$$\rho_\lambda(p/\lambda^2, S) = \left(\left(1 + \frac{p}{\lambda^2} \right) e^{-\frac{S}{C_V}} \right)^{\frac{1}{\gamma}}, \quad \text{and} \ \log\left(\rho_\lambda(p/\lambda^2, S) \right) = \gamma^{-1} \log\left(\left(1 + \frac{p}{\lambda^2} \right) e^{-\frac{S}{C_V}} \right). \tag{1.19}$$

Let $\mathcal{F}^\pm(p^\pm, S^\pm) := \log \rho^\pm(p^\pm, S^\pm)$. Since $\frac{\partial p^\pm}{\partial \rho^\pm} > 0$ and $\rho^\pm > 0$ imply $\frac{\partial \mathcal{F}^\pm}{\partial p^\pm} = \frac{1}{\rho^\pm} \frac{\partial \rho^\pm}{\partial p^\pm} > 0$, then the second equation in (1.16) is equivalent to

$$\frac{\partial \mathcal{F}^\pm}{\partial p^\pm}(p^\pm, S^\pm) D_t^{\varphi\pm} p^\pm + \nabla^\varphi \cdot v^\pm = 0. \tag{1.20}$$

Also, we assume there exists a constant $C > 0$, such that the following inequality holds for $0 \le k \le 8$:

$$|\partial_p^k \mathcal{F}(p, S)| \le C, \quad |\partial_p^k \mathcal{F}(p, S)| \le C |\partial_p \mathcal{F}(p, S)|^k \le C \partial_p \mathcal{F}(p, S). \tag{1.21}$$

Hence, we can view $\mathcal{F} = \log \rho$ as a parametrized family $\{\mathcal{F}_\varepsilon(p, S)\}$ as well, where $\varepsilon = \frac{1}{\lambda}$. Indeed, we have

$$\frac{\partial \mathcal{F}_\varepsilon}{\partial p} = \gamma^{-1} \log\left((1 + \varepsilon^2 p) e^{-\frac{S}{C_V}} \right). \tag{1.22}$$

Since we work on the case when the entropy and velocity are both bounded (later we will assume $u, S \in H^4(\Omega)$), there exists $A > 0$ such that

$$\frac{\partial \mathcal{F}_\varepsilon}{\partial p}(p, S) = \frac{1}{\rho} \frac{\partial \rho}{\partial p}(p, S) \le A\varepsilon^2. \tag{1.23}$$

We slightly abuse the terminology and call λ the sound speed and call ε the Mach number. When $\lambda \gg 1$ ($\varepsilon \ll 1$), the constant A in (1.23) can be greater than 1 such that

$$A^{-1}\varepsilon^2 \le \frac{\partial \mathcal{F}_\varepsilon}{\partial p}(p, S) \le A\varepsilon^2. \tag{1.24}$$

We sometimes write $\mathcal{F}_p^\pm := \frac{\partial \mathcal{F}_\varepsilon}{\partial p}(p^\pm, S^\pm) = \varepsilon^2$ for simplicity when discussing the incompressible limit.

1.3 History and background

1.3.1 An overview of previous results

There have been a lot of studies about free-boundary problems in ideal MHD, of which the original models in physics are mainly three types: plasma-vacuum interface model, current-vortex sheets and MHD contact discontinuities. The plasma-vacuum problem is related to plasma confinement problems [17, Chap. 4] in laboratory plasma physics, which describes the motion of *one isolated* perfectly conducting fluid in an electro-magentic field confined in a vacuum region (in which there is another vacuum magnetic field satisfying the pre-Maxwell system). When the vacuum magnetic fields are neglected, the plasma-vacuum model is reduced the free-boundary problem of one-phase MHD flows and we refer to [24, 36, 22, 21, 20, 26] for local well-posedness (LWP) theory in incompressible ideal MHD. It should be noted that, when the surface tension is neglected, the Rayleigh-Taylor sign condition $-\nabla_N Q|_{\Sigma(t)} \ge c_0 > 0$ should be added as an initial constraint for LWP which is the analogue of Euler equations [14] and we refer to Hao-Luo [25] for the proof. For the full plasma-vacuum model without surface tension in incompressible ideal MHD, we refer to [18, 19, 56, 34]. As for the compressible case, in a series of works [52, 61, 62, 64], Secchi, Trakhinin and Wang used Nash-Moser iteration to construct the solution due to the derivative loss in the linearized problems. Very recently, Lindblad and the author [32] proved the LWP and a continuation criterion for the one-phase free-boundary problem in compressible ideal MHD without surface tension, which gave the first result about the energy estimates without loss of regularity.

A vortex sheet is an interface between two "impermeable" fluids across which there is a tangential discontinuity in fluid velocity. For incompressible inviscid fluids without surface tension, vortex sheets tend to be violently unstable, which exhibit the so-called Kelvin-Helmholtz instability. There have been numerous mathematical studies, especially for 2D irrotational flows, and we refer to [15, 68] and references therein. On the other hand, surface tension is expected to "suppress" the Kelvin-Helmholtz instability and we refer to [2, 8, 53]. When the compressibility is taken into account, we shall consider not only the motion of the interface of discontinuities but also its interaction with the wave propagation in the interior. Let $\mathbf{j} = \varrho(u \cdot N - \partial_t \psi)$ be the mass transfer flux. In view of hyperbolic conservation laws, strong discontinuities can be classified into shock waves ($\mathbf{j} \neq 0$, $[\![\varrho]\!] \neq 0$) and characteristic (contact) discontinuities ($\mathbf{j} = 0$). For compressible Euler equations, contact discontinuities are classified to be vortex sheets ($[\![u_\tau]\!] \neq \vec{0}$) and entropy waves ($[\![u]\!] = \vec{0}$, $[\![\varrho]\!], [\![\mathfrak{s}]\!] \neq 0$). The existence and the structural stability of multi-dimensional shocks for compressible Euler equations was proved by Majda [39, 40] (see also Blokhin [5]) provided that the uniform Kreiss-Lopatinskiĭ condition [28] is satisfied. Since compressible vortex sheets are characteristic discontinuities (the uniform Kreiss-Lopatinskiĭ condition is never satisfied), there is a potential loss of normal derivatives for compressible vortex sheets, which makes the proof of existence more difficult. For 3D Euler equations, compressible vortex sheets are always violently unstable [16, 44, 58] which exhibit an analogue of Kelvin-Helmholtz instability; whereas for 2D Euler equations, Coulombel-Secchi [11, 12] proved the existence of "supersonic" vortex sheets when the Mach number for the rectilinear background solution $(\pm v, \rho)$ exceeds $\sqrt{2}$ and the violent instability when the Mach number is lower than $\sqrt{2}$. Similarly as the incompressible case, surface tension again prevents such violent instability and we refer to Stevens [54] for the proof of structural stability.

As for MHD, there are three types of characteristic discontinuities: current-vortex sheets ($\mathbf{j} = 0$, $B^\pm \cdot N|_{\Sigma(t)} = 0$), MHD contact discontinuities ($\mathbf{j} = 0$, $B^\pm \cdot N|_{\Sigma(t)} \neq 0$) and Alfvén (rotational) discontinuities ($\mathbf{j} \neq 0$, $[\![\varrho]\!] = 0$). The Rankine-Hugoniot conditions for current-vortex sheets and MHD contact discontinuities (cf. [17, Chap. 4.5] and [63, Appendix A]) are

- (Current-vortex sheets/Tangential discontinuities) $[\![Q]\!] = \sigma \mathcal{H}$, $B^\pm \cdot N = 0$, $\partial_t \psi = u^\pm \cdot N$ on $\Sigma(t)$.
- (MHD contact discontinuities) $[\![P]\!] = \sigma \mathcal{H}$, $[\![u]\!] = [\![B]\!] = \vec{0}$, $B^\pm \cdot N \neq 0$, $\partial_t \psi = u^\pm \cdot N$ on $\Sigma(t)$.

MHD contact discontinuities usually arise from astrophysical plasmas [17], where the magnetic fields typically originate in a rotating object, such as a star or a dynamo operating inside, and intersect the surface of discontinuity. An example is the photosphere of the sun. In contrast, current-vortex sheets require the magnetic fields to be tangential to the interface. An example in laboratory plasma physics is that the discontinuities confine a high-density plasma by a lower-density one, which is isolated thermally from an outer rigid wall. In particular, when the plasma is liquid metal, the effect of surface tension cannot be neglected [45]. In astrophysics, a generally accepted model for compressible current-vortex sheets is the heliopause [4] (in some sense, the "boundary" of the solar system[1]) that separates the interstellar plasma from the solar wind plasma. The night-side magnetopause of the earth is also considered to be current-vortex sheets.

For MHD contact discontinuities, the transversality of magnetic fields could enhance the regularity of the free interface and avoid the possible normal derivative loss in the interior. We refer to Morando-Trakhinin-Trebeschi [47] for the 2D case under Rayleigh-Taylor sign condition $N \cdot \nabla [\![Q]\!]|_{\Sigma(t)} \geq c_0 > 0$, Trakhinin-Wang [63] for the case with nonzero surface tension, and Wang-Xin [67] for both 2D and 3D cases without surface tension or Rayleigh-Taylor sign condition. In other words, Wang-Xin [67] showed that transversal magnetic fields across the interface could suppress the Rayleigh-Taylor instability.

As for current-vortex sheets, Kelvin-Helmholtz instability can also be suppressed, but, unlike the transversal magnetic fields in MHD contact discontinuities, the tangential magnetic fields must satisfy certain constraints. For 3D incompressible ideal MHD, Syrovatskiĭ [57] introduced a stability condition by using normal mode analysis:

$$\varrho^+ |B^+ \times [\![u]\!]|^2 + \varrho^- |B^- \times [\![u]\!]|^2 < (\varrho^+ + \varrho^-)|B^+ \times B^-|^2, \tag{1.25}$$

which corresponds to the transition to violent instability, that is, ill-posedness of the linearized problem. Coulombel-Morando-Secchi-Trebeschi [10] proved the a priori estimate for the nonlinear problem under a more restrictive condition

$$\max \left\{ \left| \frac{B^+}{\sqrt{\varrho^+}} \times [\![u]\!] \right|, \left| \frac{B^-}{\sqrt{\varrho^-}} \times [\![u]\!] \right| \right\} < \left| \frac{B^+}{\sqrt{\varrho^+}} \times \frac{B^-}{\sqrt{\varrho^-}} \right|. \tag{1.26}$$

Sun-Wang-Zhang [55] proved local well-posedness of the nonlinear problem under the original Syrovatskiĭ condition (1.25) by adapting the framework of Shatah-Zeng [53]. Very recently, Liu-Xin [33] gave a comprehensive study for both $\sigma > 0$ and $\sigma = 0$ cases (see also Li-Li [30]).

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[1]On August 25, 2012, Voyager 1 flew beyond the heliopause and entered interstellar space. At the time, it was at a distance about 122 A.U. (around 18 billion kilometers) from the sun. On November 5, 2018, Voyager 2 also traversed the heliopause.

For compressible current-vortex sheets without surface tension, Trakhinin [59] showed that the uniform Kreiss-Lopatinskiĭ condition [28] for the linearized problem is never satisfied, so only the neutral stability can be expected for the linearized problem. However, the specific range for the neutral stability cannot be explicitly calculated [59, Section 4.2]. Thus, it is still unknown if there is any *necessary and sufficient condition* for the linear (neutral) stability. To avoid testing the Kreiss-Lopatinskiĭ condition, Trakhinin [59] used the method of "Friedrichs secondary symmetrizer" to raise a sufficient condition for the problem linearized around a background planar current-vortex sheet $(\hat{v}^{\pm}, \hat{b}^{\pm}, \hat{\rho}^{\pm}, \hat{S}^{\pm})$ in flattened domains $\Omega^{\pm}$, which reads

$$\max \left\{ |\hat{b}^{-} \times [\![\hat{v}]\!]| \sqrt{\hat{\rho}^{+} \left(1 + (c_A^{+}/c_s^{+})^2\right)}, |\hat{b}^{+} \times [\![\hat{v}]\!]| \sqrt{\hat{\rho}^{-} \left(1 + (c_A^{-}/c_s^{-})^2\right)} \right\} < |\hat{b}^{+} \times \hat{b}^{-}|. \tag{1.27}$$

where $c_A^{\pm} := |\hat{b}^{\pm}|/\sqrt{\hat{\rho}^{\pm}}$ represents the Alfvén speed and $c_s^{\pm} := \sqrt{\partial \tilde{p}^{\pm}/\partial \tilde{\rho}^{\pm}}$ represents the sound speed. If we formally take the incompressible limit $\hat{\rho}^{\pm} \to 1$ and $c_s^{\pm} \to +\infty$, then the above inequality exactly converges to (1.26) used in [10], and it is easy to see (1.26) implies (1.25). Under (1.27), Chen-Wang [6] and Trakhinin [60] proved the well-posedness for the 3D problem without surface tension and see also [66, 46] for the 2D case without surface tension. That is to say, non-collinear magnetic fields in 3D and sufficiently strong magnetic fields in 2D can also suppress the analogue of Kelvin-Helmholtz instability for compressible vortex sheets. **When the surface tension is taken into account, it is expected to drop the extra assumptions to establish the well-posedness of compressible current-vortex sheets, but so far there is no available result.** Besides, the local existence results were established by using Nash-Moser iteration in all these previous works which leads to an unavoiable loss of regularity from initial data to solution.

Apart from the local existence, the singular limits for both free-surface ideal MHD flows and compressible vortex sheets are far less developed. Ohno-Shirota [48] showed that the linearized problem in a fixed domain with magnetic fields tangential to the boundary is ill-posed in standard Sobolev spaces $H^l(l \geq 2)$, but the corresponding incompressible problem is well-posed in standard Sobolev spaces [22, 55, 56, 33, 34]. The anisotropic Sobolev spaces defined in Section 1.4.1, first introduced by Chen [7], have been adopted in previous works about ideal compressible MHD [69, 50, 51, 60, 6, 52, 61, 62]. In other words, there is no explanation for the mismatch of the function spaces for local existence yet. Besides, it is also unclear about the comparison between the stabilization mechanism brought by surface tension and the one brought by certain magnetic fields when the plasma is compressible. These questions should be answered by rigorously justifying the incompressible limit and the zero-surface-tension limit. In particular, the existing literature about the incompressible limit of free-boundary problems in inviscid fluids is only avaliable for the one-phase problems [31, 35, 13, 70, 71, 38, 23]. **The incompressible limits of free-boundary MHD and inviscid vortex sheets both remain completely open.**

1.3.2 Our goals

We aim to give a comprehensive study for the local-in-time solution to current-vortex sheets in ideal MHD and particularly give affirmative answers to the abovementioned questions. Specifically, in this paper, we prove well-posedness and incompressible limit of current-vortex sheets with surface tension, namely system (1.16), in both 2D and 3D. In the second part of the two-paper sequence, we will prove the zero-surface-tension limit of system (1.16) under certain stability conditions in 3D and 2D respectively; besides, we will also improve the incompressible limit result such that the uniform boundedness (with respect to Mach number) of high-order time derivatives can be dropped, which is a rather nontrivial improvement and relies on a new framework to prove the uniform estimates.

To our knowledge, this is *the first result* about the incompressible limit of compressible vortex sheets and free-boundary MHD and also the first result about compressible current-vortex sheets with surface tension. The incompressible limit also ties our result to the suppression effect on Kelvin-Helmholtz instability brought by either surface tension or suitable magnetic fields.

1.4 Main results

1.4.1 Anisotropic Sobolev spaces

Following the notations in [65], we first define the anisotropic Sobolev space $H_*^m(\Omega^{\pm})$ for $m \in \mathbb{N}$ and $\Omega^{\pm} = \mathbb{T}^{d-1} \times \{0 < \pm x_d < H\}$. Let $\omega = \omega(x_d) = (H^2 - x_d^2)x_d^2$ be a smooth function[2] on $[-H, H]$. Then we define $H_*^m(\Omega^{\pm})$ for $m \in \mathbb{N}^*$ as follows

$$H_*^m(\Omega^{\pm}) := \left\{ f \in L^2(\Omega^{\pm}) \middle| (\omega \partial_d)^{\alpha_{d+1}} \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} f \in L^2(\Omega^{\pm}), \ \forall \alpha \text{ with } \sum_{j=1}^{d-1} \alpha_j + 2\alpha_d + \alpha_{d+1} \leq m \right\},$$

---

[2]The choice of $\omega(x_d)$ is not unique, as we just need $\omega(x_d)$ vanishes on $\Sigma \cup \Sigma^{\pm}$ and is comparable to the distance function near the interface and the boundaries.

equipped with the norm

$$\|f\|_{H^m_*(\Omega^\pm)}^2 := \sum_{\sum_{j=1}^{d-1} \alpha_j + 2\alpha_d + \alpha_{d+1} \leq m} \|(\omega \partial_d)^{\alpha_{d+1}} \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} f\|_{L^2(\Omega)}^2. \tag{1.28}$$

For any multi-index $\alpha := (\alpha_0, \alpha_1, \cdots, \alpha_d, \alpha_{d+1}) \in \mathbb{N}^{d+2}$, we define

$$\partial_*^\alpha := \partial_t^{\alpha_0} (\omega \partial_d)^{\alpha_{d+1}} \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}, \quad \langle \alpha \rangle := \sum_{j=0}^{d-1} \alpha_j + 2\alpha_d + \alpha_{d+1},$$

and define the **space-time anisotropic Sobolev norm** $\| \cdot \|_{m,*,\pm}$ to be

$$\|f\|_{m,*,\pm}^2 := \sum_{\langle \alpha \rangle \leq m} \|\partial_*^\alpha f\|_{L^2(\Omega^\pm)}^2 = \sum_{\alpha_0 \leq m} \|\partial_t^{\alpha_0} f\|_{H^{m-\alpha_0}_*(\Omega^\pm)}^2. \tag{1.29}$$

We also write the interior Sobolev norm to be $\|f\|_{s,\pm} := \|f(t,\cdot)\|_{H^s(\Omega^\pm)}$ for any function $f(t,x)$ on $[0,T] \times \Omega^\pm$ and denote the boundary Sobolev norm to be $|f|_s := |f(t,x')|_{H^s(\Sigma)}$ for any function $f(t,x')$ on $[0,T] \times \Sigma$.

From now on, we assume the dimension $d = 3$, that is, $\Omega^\pm = \mathbb{T}^2 \times \{0 < \pm x_3 < H\}$, $\Sigma^\pm = \mathbb{T}^2 \times \{x_3 = \pm H\}$ and $\Sigma = \mathbb{T}^2 \times \{x_3 = 0\}$. We will see the 2D case follows in the same manner up to slight modifications in the vorticity analysis and we refer to Section 3.6.4 for details. Invoking (1.20) and writing $\mathcal{F}_p^\pm := \frac{\partial \mathcal{F}^\pm}{\partial p^\pm}$, system (1.16) is equivalent to

$$\begin{cases} \rho^\pm D_t^{\varphi\pm} v^\pm - (b^\pm \cdot \nabla^\varphi) b^\pm + \nabla^\varphi q^\pm = 0, \quad q^\pm = p^\pm + \frac{1}{2}|b^\pm|^2 & \text{in } [0,T] \times \Omega^\pm, \\ \mathcal{F}_p^\pm D_t^{\varphi\pm} p^\pm + \nabla^\varphi \cdot v^\pm = 0 & \text{in } [0,T] \times \Omega^\pm, \\ p^\pm = p^\pm(\rho^\pm, S^\pm), \quad \mathcal{F}^\pm = \log \rho^\pm, \quad \mathcal{F}_p^\pm > 0, \quad \rho^\pm \geq \bar{\rho}_0 > 0 & \text{in } [0,T] \times \Omega^\pm, \\ D_t^{\varphi\pm} b^\pm - (b^\pm \cdot \nabla^\varphi) v^\pm + b^\pm \nabla^\varphi \cdot v^\pm = 0 & \text{in } [0,T] \times \Omega^\pm, \\ \nabla^\varphi \cdot b^\pm = 0 & \text{in } [0,T] \times \Omega^\pm, \\ D_t^{\varphi\pm} S^\pm = 0 & \text{in } [0,T] \times \Omega^\pm, \\ [\![q]\!] = \sigma \overline{\nabla} \cdot \left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}} \right) & \text{on } [0,T] \times \Sigma, \\ \partial_t \psi = v^\pm \cdot N & \text{on } [0,T] \times \Sigma, \\ b^\pm \cdot N = 0 & \text{on } [0,T] \times \Sigma, \\ v_d^\pm = b_d^\pm = 0 & \text{on } [0,T] \times \Sigma^\pm, \\ (v^\pm, b^\pm, \rho^\pm, S^\pm, \psi)|_{t=0} = (v_0^\pm, b_0^\pm, \rho_0^\pm, S_0^\pm, \psi_0). \end{cases} \tag{1.30}$$

Since the material derivatives are tangential to the boundary, that is, $D_t^{\varphi\pm} = \overline{D_t^\pm} := \partial_t + \bar{v}^\pm \cdot \overline{\nabla}$ on Σ and $\Sigma^\pm$, the compatibility conditions (1.6) for initial data up to m-th order ($m \in \mathbb{N}$) are now written as:

$$\begin{aligned} [\![\partial_t^j q]\!]|_{t=0} &= \sigma \partial_t^j \mathcal{H}|_{t=0} \quad \text{on } \Sigma, \quad 0 \leq j \leq m, \\ \partial_t^{j+1} \psi|_{t=0} &= \partial_t^j (v^\pm \cdot N)|_{t=0} \quad \text{on } \Sigma, \quad 0 \leq j \leq m, \\ \partial_t^j v_d^\pm|_{t=0} &= 0 \quad \text{on } \Sigma^\pm, \quad 0 \leq j \leq m. \end{aligned} \tag{1.31}$$

Under (1.31), one can prove that $\partial_t^j (b^\pm \cdot N)|_{t=0} = 0$ is also satisfied on Σ and $\Sigma^\pm$ for $0 \leq j \leq m$ and we refer to Trakhinin [60, Section 4] for details.

1.4.2   Main result 1: Well-posedness and uniform estimates in Mach number

The first result shows the local well-posedness and the energy estimates of (1.30) for each fixed $\sigma > 0$.

Theorem 1.1 (**Well-posedness and uniform estimates for fixed** $\sigma > 0$). Fix the constant $\sigma > 0$. Let $\mathbf{U}_0^\pm := (v_0^\pm, b_0^\pm, \rho_0^\pm, S_0^\pm)^\top \in H_*^8(\Omega^\pm)$ and $\psi_0 \in H^{9.5}(\Sigma)$ be the initial data of (1.30) satisfying

- the compatibility conditions (1.31) up to 7-th order;
- the constraints $\nabla^{\varphi_0} \cdot b_0^\pm = 0$ in $\Omega^\pm$, $b^\pm \cdot N|_{\{t=0\}\times(\Sigma \cup \Sigma^\pm)} = 0$;

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- $|\,[\![\bar{v}_0]\!]\,| > 0$ on Σ, $|\psi_0|_{L^\infty(\Sigma)} \le 1$, and $E(0) \le M$ for some constant $M > 0$.

Then there exists $T_\sigma > 0$ depending only on M and σ, such that (1.30) admits a unique solution $(v^\pm(t), b^\pm(t), \rho^\pm(t), S^\pm(t), \psi(t))$ that verifies the energy estimate

$$\sup_{t\in[0,T]} E(t) \le C(\sigma^{-1}) P(E(0)) \tag{1.32}$$

and $\sup_{t\in[0,T_\sigma]} |\psi(t)| < 10 < H$, where $P(\cdots)$ is a generic polynomial in its arguments. The energy $E(t)$ is defined to be

$$E(t) := E_4(t) + E_5(t) + E_6(t) + E_7(t) + E_8(t),$$

$$E_{4+l}(t) := \sum_{\pm} \sum_{\langle\alpha\rangle=2l} \sum_{k=0}^{4-l} \left\| \left(\varepsilon^{2l}\mathcal{T}^\alpha \partial_t^k \left(v^\pm, b^\pm, S^\pm, (\mathcal{F}_p^\pm)^{\frac{(k+\alpha_0-l-3)_+}{2}} p^\pm \right) \right) \right\|_{4-k-l,\pm}^2$$
$$+ \sum_{k=0}^{4+l} \left| \sqrt{\sigma}\varepsilon^{2l}\partial_t^k\psi \right|_{5+l-k}^2 \quad 0 \le l \le 4, \tag{1.33}$$

where $k_+ := \max\{k, 0\}$ for $k \in \mathbb{R}$ and we denote $\mathcal{T}^\alpha := (\omega(x_3)\partial_3)^{\alpha_4}\partial_t^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}$ to be a high-order tangential derivative for the multi-index $\alpha = (\alpha_0, \alpha_1, \alpha_2, 0, \alpha_4)$ with length (for the anisotropic Sobolev spaces) $\langle\alpha\rangle = \alpha_0 + \alpha_1 + \alpha_2 + 2\times0 + \alpha_4$. The quantity ε is the parameter defined in (1.22). Moreover, the $H^{9.5}(\Sigma)$-regularity of ψ can be recovered in the sense that

$$\sum_{l=0}^{4} \sum_{k=0}^{3+l} \left| \sigma\varepsilon^{2l}\partial_t^k\psi \right|_{5.5+l-k}^2 \le P(E(t)), \quad \forall t \in [0, T_\sigma]. \tag{1.34}$$

Remark 1.2 (Correction of $E_4(t)$). The norm $\|p^\pm\|_{4,\pm}^2$ in $E_4(t)$ defined by (1.33) should be replaced by $\|(\mathcal{F}_p^\pm)^{\frac{1}{2}}p^\pm\|_{0,\pm}^2 + \|\nabla p^\pm\|_{3,\pm}^2$ because we do not have L^2 estimates of $p^\pm$ without $\mathcal{F}_p^\pm$-weight. We still write $\|p^\pm\|_{4,\pm}^2$ as above for simplicity of notations.

Remark 1.3 (Weights of Mach number of $p^\pm$). In (1.33), the weight of Mach number of p is slightly different from (v, b, S), but such difference only occurs when $\mathcal{T}^\alpha$ are full time derivatives and $k = 4 - l$. In fact, due to $k \le 4 - l$ and $\alpha_0 \le \langle\alpha\rangle = 2l$, we know $(k + \alpha_0 - l - 3)_+$ is always equal to zero unless $\alpha_0 = 2l$ and $k = 4 - l$ simultanously hold.

Remark 1.4 (Relations with anisotropic Sobolev space). The energy functional $E(t)$ above is considered as a variant of $\|\cdot\|_{8,*,\pm}$ norm at time $t > 0$. For different multi-index α, we set suitable weights of Mach number according to the number of tangential derivatives that appear in ∂_*^α, such that the energy estimates for the modified norms are uniform in ε.

Remark 1.5 (Nonlinear structural stability). System (1.30) is studied in a bounded domain $\mathbb{T}^2 \times (-H, H)$. Indeed, our proof also applies to the case of an unbounded domain, such as $\mathbb{T}^2 \times \mathbb{R}_\pm$, $\mathbb{R}^2 \times \mathbb{R}_\pm$, for non-localised initial data $\mathbf{U}_0^\pm$ satisfying $(\mathbf{U}_0^\pm - \underline{\mathbf{U}}^\pm, \psi_0) \in H_*^8(\Omega) \times H^{9.5}(\Sigma)$ where $\underline{\mathbf{U}}^\pm$ represents a given piecewise-smooth background solution of planar current-vortex sheet $(\underline{v}_1^\pm, \underline{v}_2^\pm, 0, \underline{b}_1^\pm, \underline{b}_2^\pm, 0, \underline{p}^\pm, \underline{S}^\pm)^\top$ in $\Omega^\pm$. The result corresponding to this initial data exactly justifies *the existence and the local-in-time nonlinear structural stability* of the piecewise-smooth planar current-vortex sheet $\underline{\mathbf{U}}^\pm$.

1.4.3 Main result 2: The incompressible limit

Next we are concerned with the incompressible limit. For any fixed $\sigma > 0$, the energy estimates obtained in Theorem 1.1 are already uniform in ε. Also, $\|\partial_t(v, b, S)\|_3 + |\psi_t|_{4.5}$ is uniformly bounded in ε. Thus, using compactness argument, we can prove the incompressible limit for current-vortex sheets with surface tension. Specifically, the motion of incompressible current-vortex sheets with surface tension are characterised by the equations of $(\xi^\sigma, w^{\pm,\sigma}, h^{\pm,\sigma})$ with incompressible initial data

$(\xi_0^\sigma, w_0^{\pm,\sigma}, h_0^{\pm,\sigma})$ and a transport equation of $\mathfrak{S}^{\pm,\sigma}$:

$$
\begin{cases}
\mathfrak{R}^{\pm,\sigma}(\partial_t + w^{\pm,\sigma} \cdot \nabla^{\Xi^\sigma})w^{\pm,\sigma} - (h^{\pm,\sigma} \cdot \nabla^{\Xi^\sigma})h^{\pm,\sigma} + \nabla^{\Xi^\sigma}\Pi^{\pm,\sigma} = 0 & \text{in } [0,T] \times \Omega, \\
\nabla^{\Xi^\sigma} \cdot w^{\pm,\sigma} = 0 & \text{in } [0,T] \times \Omega, \\
(\partial_t + w^{\pm,\sigma} \cdot \nabla^{\Xi^\sigma})h^{\pm,\sigma} = (h^{\pm,\sigma} \cdot \nabla^{\Xi^\sigma})w^{\pm,\sigma} & \text{in } [0,T] \times \Omega, \\
\nabla^{\Xi^\sigma} \cdot h^{\pm,\sigma} = 0 & \text{in } [0,T] \times \Omega, \\
(\partial_t + w^{\pm,\sigma} \cdot \nabla^{\Xi^\sigma})\mathfrak{S}^{\pm,\sigma} = 0 & \text{in } [0,T] \times \Omega, \\
[\![\Pi^\sigma]\!] = \sigma\overline{\nabla} \cdot \left(\dfrac{\overline{\nabla}\xi^\sigma}{\sqrt{1+|\overline{\nabla}\xi^\sigma|^2}} \right) & \text{on } [0,T] \times \Sigma, \\
\partial_t\xi^\sigma = w^{\pm,\sigma} \cdot N^\sigma & \text{on } [0,T] \times \Sigma, \\
h^{\pm,\sigma} \cdot N^\sigma = 0 & \text{on } [0,T] \times \Sigma, \\
w_3^\pm = h_3^\pm = 0 & \text{on } [0,T] \times \Sigma^\pm, \\
(w^{\pm,\sigma}, h^{\pm,\sigma}, \mathfrak{S}^{\pm,\sigma}, \xi^\sigma)|_{t=0} = (w_0^{\pm,\sigma}, h_0^{\pm,\sigma}, \mathfrak{S}_0^{\pm,\sigma}, \xi_0^\sigma),
\end{cases}
\tag{1.35}
$$

where $\Xi^\sigma(t,x) = x_3 + \chi(x_3)\xi^\sigma(t,x')$ is the extension of ξ^σ in Ω and $N^\sigma := (-\overline{\partial}_1\xi^\sigma, -\overline{\partial}_2\xi^\sigma, 1)^\top$. The quantity $\Pi^\pm := \overline{\Pi}^\pm + \frac{1}{2}|h^\pm|^2$ represent the total pressure functions for the incompressible equations with $\overline{\Pi}^\pm$ the fluid pressure functions. The quantity $\mathfrak{R}^\pm$ satisfies the evolution equation $(\partial_t + w^{\pm,\sigma} \cdot \nabla^{\Xi^\sigma})\mathfrak{R}^{\pm,\sigma} = 0$ with initial data $\mathfrak{R}_0^{\pm,\sigma} := \rho^{\pm,\sigma}(0, \mathfrak{S}_0^{\pm,\sigma})$.

Denoting $(\psi^{\varepsilon,\sigma}, v^{\pm,\varepsilon,\sigma}, b^{\pm,\varepsilon,\sigma}, \rho^{\pm,\varepsilon,\sigma}, S^{\pm,\varepsilon,\sigma})$ to be the solution of (1.30) indexed by σ and ε, we prove that $(\psi^{\varepsilon,\sigma}, v^{\pm,\varepsilon,\sigma}, b^{\pm,\varepsilon,\sigma}, \rho^{\pm,\varepsilon,\sigma}, S^{\pm,\varepsilon,\sigma})$ converges to $(\xi^\sigma, w^{\pm,\sigma}, h^{\pm,\sigma}, \mathfrak{R}^{\pm,\sigma}, \mathfrak{S}^{\pm,\sigma})$ as $\varepsilon \to 0$ provided the convergence of initial data.

Theorem 1.2 (Incompressible limit for fixed $\sigma > 0$). Fix $\sigma > 0$. Let $(\psi_0^{\varepsilon,\sigma}, v_0^{\pm,\varepsilon,\sigma}, b_0^{\pm,\varepsilon,\sigma}, \rho_0^{\pm,\varepsilon,\sigma}, S_0^{\pm,\varepsilon,\sigma})$ be the initial data of (1.30) for each fixed $(\varepsilon, \sigma) \in \mathbb{R}^+ \times \mathbb{R}^+$, satisfying

a. The sequence of initial data $(\psi_0^{\varepsilon,\sigma}, v_0^{\pm,\varepsilon,\sigma}, b_0^{\pm,\varepsilon,\sigma}, \rho_0^{\pm,\varepsilon,\sigma}, S_0^{\pm,\varepsilon,\sigma}) \in H^{9.5}(\Sigma) \times H_*^8(\Omega^\pm) \times H_*^8(\Omega^\pm) \times H_*^8(\Omega^\pm) \times H^8(\Omega^\pm)$ satisfies the compatibility conditions (1.31) up to 7-th order, and $|\psi_0^{\varepsilon,\sigma}|_{L^\infty} \leq 1$.
b. $(\psi_0^{\varepsilon,\sigma}, v_0^{\pm,\varepsilon,\sigma}, b_0^{\pm,\varepsilon,\sigma}, S_0^{\pm,\varepsilon,\sigma}) \to (\xi_0^\sigma, w_0^{\pm,\sigma}, h_0^{\pm,\sigma}, \mathfrak{S}_0^{\pm,\sigma})$ in $H^{5.5}(\Sigma) \times H^4(\Omega^\pm) \times H^4(\Omega^\pm) \times H^4(\Omega^\pm)$ as $\varepsilon \to 0$.
c. The incompressible initial data satisfies $|[\![\overline{w}_0^\sigma]\!]| > 0$ on Σ, the constraints $\nabla^{\xi_0^\sigma} \cdot h_0^{\pm,\sigma} = 0$ in $\Omega^\pm$, $h^{\pm,\sigma} \cdot N^\sigma|_{\{t=0\}\times\Sigma} = 0$.

Then it holds that

$$
(\psi^{\varepsilon,\sigma}, v^{\pm,\varepsilon,\sigma}, b^{\pm,\varepsilon,\sigma}, S^{\pm,\varepsilon,\sigma}) \to (\xi^\sigma, w^{\pm,\sigma}, h^{\pm,\sigma}, \mathfrak{S}^{\pm,\sigma}),
\tag{1.36}
$$

weakly-* in $L^\infty([0,T_\sigma]; H^{5.5}(\Sigma) \times (H^4(\Omega^\pm))^3)$ and strongly in $C([0,T_\sigma]; H_{\text{loc}}^{5.5-\delta}(\Sigma) \times (H_{\text{loc}}^{4-\delta}(\Omega^\pm))^3)$ after possibly passing to a subsequence, where T_σ is the time obtained in Theorem 1.1.

Remark 1.6 (The "compatibility conditions" for the incompressible problem). For the incompressible problem, there is no need to require the so-called "compatibility conditions" for the initial data. The convergence of compressible data automatically implies the fulfillment of time-differentiated kinematic boundary conditions and the time-differentiated slip conditions at $t = 0$. The time-differentiated jump conditions can also be easily fulfilled by adjusting the boundary values of $\Pi^\pm$, as the pressure function Π is NOT uniquely determined by the other variables for the incompressible problem.

List of Notations: In the rest of this paper, we sometimes write $\mathcal{T}^k$ to represent a tangential derivative $\mathcal{T}^\alpha$ in $\Omega^\pm$ with order $\langle\alpha\rangle = k$ when we do not need to specify what the derivative $\mathcal{T}^\alpha$ contains. We also list all the notations used in this manuscript.

- $\Omega^\pm := \mathbb{T}^{d-1} \times \{0 < \pm x_d < H\}$, $\Sigma := \mathbb{T}^{d-1} \times \{x_d = 0\}$ and $\Sigma^\pm := \mathbb{T}^{d-1} \times \{x_d = \pm H\}$, $d = 2, 3$.
- $\|\cdot\|_{s,\pm}$: We denote $\|f\|_{s,\pm} := \|f(t,\cdot)\|_{H^s(\Omega^\pm)}$ for any function $f(t,x)$ on $[0,T] \times \Omega^\pm$.
- $|\cdot|_s$: We denote $|f|_s := |f(t,\cdot)|_{H^s(\Sigma)}$ for any function $f(t,x')$ on $[0,T] \times \Sigma$.
- $\|\cdot\|_{m,*}$: For any function $f(t,x)$ on $[0,T] \times \Omega$, $\|f\|_{m,*,\pm}^2 := \sum\limits_{\langle\alpha\rangle \leq m} \|\partial_*^\alpha f(t,\cdot)\|_{0,\pm}^2$ denotes the m-th order space-time anisotropic Sobolev norm of f.
- $P(\cdots)$: A generic polynomial with positive coefficients in its arguments;
- $[T, f]g := T(fg) - fT(g)$, and $[T, f, g] := T(fg) - T(f)g - fT(g)$, where T denotes a differential operator and f, g are arbitrary functions.
- $\overline{\partial}$: $\overline{\partial} = \partial_1, \cdots, \partial_{d-1}$ denotes the spatial tangential derivative.
- $A \overset{L}{=} B$: A is equal to B plus some lower-order terms that are easily controlled.

2 Strategy of the proof

Before going to the detailed proof, we would like to briefly introduce the strategies to tackle this complicated problem. We will decompose the problem into the following parts:

1. Uniform-in-ε estimates for one-phase compressible ideal MHD in a fixed domain with boundary.
2. Generalization to the free-boundary setting by using Alinhac good unknowns and analysis of three crucial terms that contributes to the boundary regularity, shows a cancellation structure to reach the incompressible limit and exhibits the crucial difficulty caused by the tangential velocity jump in vortex sheets respectively.
3. Design an appropriate approximate system to prove the local well-posedness without using Nash-Moser or tangential smoothing.

Moreover, we will make comparison between the compressible problem and the incompressible problem, between the Lagrangian coordinates and the "flattened coordinates" among the vortex sheet problem, the one-phase problem and the MHD contact discontinuity.

2.1 Uniform estimates for one-phase MHD flows in a fixed domain

First, let us temporarily forget about the free-boundary setting and recall how to derive uniform estimates in Mach number for the one-phase problem of (1.1) in the fixed domain $\Omega = \mathbb{T}^2 \times (-H, H)$ with the slip conditions $u_3 = B_3 = 0$ on $\partial\Omega$ in the preparatory work [65] by Wang and the author.

2.1.1 Div-Curl analysis: a hidden structure of Lorentz force

The entropy is easy to control thanks to $D_t \mathfrak{s} = 0$, so it suffices to analyze the relations between (u, B) and $Q := P + \frac{1}{2}|B|^2$. Using div-curl decomposition, we shall prove the H^3-estimates for the divergence part and the curl part in order to control $\|u, B\|_4$. The divergence part is reduced to the tangential derivatives $\|\mathcal{F}_p D_t P\|_3$. To control the curl part, we take $\nabla \times$ in the momentum equation and invoke the evolution equation of B to get

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \varrho |\partial^3 (\nabla \times u)|^2 + |\partial^3 (\nabla \times B)|^2 \, \mathrm{d}x = - \int_\Omega \partial^3 \nabla \times (B(\nabla \cdot u)) \cdot \partial^3 (\nabla \times B) \, \mathrm{d}x + \text{ controllable terms}, \tag{2.1}$$

where we find that there is a normal derivative loss in the term $\partial^3 \nabla \times (B(\nabla \cdot u))$. Indeed, invoking $\nabla \cdot u = -\mathcal{F}_p D_t P$, commuting ∇ with D_t and inserting the momentum equation $-\nabla P = \varrho D_t u + B \times (\nabla \times B)$, we find a hidden structure of the Lorentz force $B \times (\nabla \times B)$ that eliminates the normal derivative in the curl operator:

$$\mathcal{F}_p B \times (\partial^3 \nabla D_t P) = -\mathcal{F}_p \varrho B \times (\partial^3 D_t^2 u) - \mathcal{F}_p B \times (B \times \partial^3 D_t (\nabla \times B)) + \text{ lower order terms},$$

in which the second term contributes to an energy term $-\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \mathcal{F}_p |B \times (\partial^3 \nabla \times B)|^2 \, \mathrm{d}x$ plus controllable remainder terms. Thus, the vorticity analysis for compressible ideal MHD motivates us to **trade one normal derivative (in curl) for two tangential derivatives together with square weights of Mach number, namely $\varepsilon^2 D_t^2$.** Furthermore, it can be seen that the anisotropic Sobolev spaces defined in Section 1.4.1 should be the appropriate function spaces to study compressible ideal MHD with magnetic fields tangential to the boundary. This structure was first observed by the author and Wang in the recent preparatory work [65] and gives a definitive explanation on the "mismatch" of fucntion spaces for the well-posedness of incompressible MHD (H^m) and compressible MHD (H_*^{2m}): the "anisotropic part", namely the part containing more than m derivatives, must have weight ε^2 or higher power which converges to zero when taking the incompressible limit. The 2D case can also be similarly treated.

2.1.2 Reduction of pressure: motivation to design the energy functional

We still need to reduce the normal derivative falling on Q (or $P = Q - \frac{1}{2}|B|^2$). To do this, we just need to use $-\nabla Q = \varrho D_t u - (B \cdot \nabla)B$ and the fact that D_t and $B \cdot \nabla$ are both tangential. Repeatedly, all normal derivatives are reduced to tangential derivatives, and the tangential estimates are expected to be parallel to the proof of L^2 energy conservation.

A remaining task is to determine the weights of Mach number assigned on u, B, P when we invoke the momentum equation to reduce ∇P. One thing we already know from the momentum equation is that $\nabla(P + \frac{1}{2}|B|^2) \sim (B \cdot \nabla)B - D_t u$, which suggests that $\partial_t^k \nabla P$ should share the same weights of Mach number as $\partial_t^{k+1} u$. Apart from this, we recall that the L^2 energy conservation shows that $u, B, \sqrt{\mathcal{F}_p}P, \mathfrak{s} \in L^2(\Omega)$, which suggest that $\partial_t^k(u, B, \mathfrak{s})$ should share the same weights of Mach number as $\varepsilon \partial_t^k P$ when doing tangential estimates.

Thus, we can conclude our reduction scheme as follows

a. Using div-curl analysis to reduce any normal derivatives on u, B. In this process, we have $(\nabla \cdot u, \nabla \cdot B) \to \varepsilon^2 \mathcal{T} P$ and $(\nabla \times u, \nabla \times B) \to \varepsilon^2 \mathcal{T}^2 u$, where $\mathcal{T}$ can be any one of the tangential derivatives $\partial_t, \bar{\partial}_1, \bar{\partial}_2, \omega(x_3)\partial_3$.

b. Using the momentum equation to reduce ∇P to $\mathcal{T}(u, B)$ and $\nabla(\frac{1}{2}|B|^2)$ (this term should be further reduced via div-curl analysis).

c. Tangential estimates: When estimating $E_{4+l}(t)$ (defined in (1.33)), $\mathcal{T}^\gamma(u, B)$ is controlled together with $\sqrt{\mathcal{F}_p}\mathcal{T}^\gamma P$ in the estimates of full tangential derivatives, i.e., when $\langle \gamma \rangle = 4 + l$.

Based on the above three properties, we design the energy functional $E(t)$ in (1.33) and we expect to establish uniform-in-ε estimates for this energy functional.

2.2 Analysis in the free-interface setting

For the current-vortex sheets problem, one has to take into account of the free-interface motion. The regularity of free interface is unknown a priori, σ-dependent and determined by the solutions. We only focus on the uniform a priori estimates of (1.30) in the following 2 subsections and postpone the solvability of the current-vortex sheets problem to Section 2.3.

Compared with the fixed-domain problem, we may not apply the same div-curl inequality which is not appropriate for us to derive the uniform estimates for $E(t)$, because the kinematic boundary condition $v \cdot N = \psi_t$ introduces one more time derivative. One may alternatively use the following one:

$$\forall s \geq 1, \ \|X\|_s^2 \lesssim C(|\psi|_s, |\bar{\nabla}\psi|_{W^{1,\infty}}) \left(\|X\|_0^2 + \|\nabla^\varphi \cdot X\|_{s-1}^2 + \|\nabla^\varphi \times X\|_{s-1}^2 + \|\bar{\partial}^s X\|_s^2 \right), \tag{2.2}$$

The div-curl analysis converts all normal derivatives falling on v, b to tangential derivatives. According to the reduction scheme, we need to control $\|\varepsilon^{2l}\mathcal{T}^\alpha \mathcal{T}^\beta \partial_t^k(v, b, \sqrt{\mathcal{F}_p}p, S)\|_0^2$ where $\mathcal{T}^\alpha = (\omega(x_3)\partial_3)^{\alpha_4}\partial_t^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}$ and α, β, k, l satisfy

$$\langle \alpha \rangle = 2l, \ \langle \beta \rangle = 4 - l - k, \ 0 \leq k \leq 4 - l, \ 0 \leq l \leq 4 \ \text{and} \ \beta_0 = 0. \tag{2.3}$$

In fact, the $\varepsilon^{2l}\mathcal{T}^\alpha$-part comes from the vorticity analysis for E_{4+l} and the $\mathcal{T}^\beta \partial_t^k$-part comes from the interior tangential derivatives in div-curl inequality (2.2).

When commuting $\mathcal{T}^\gamma$ with ∇^φ, the commutator $[\mathcal{T}^\gamma, \partial_i^\varphi]f$ contains the term $(\partial_3\varphi)^{-1}\mathcal{T}^\gamma \partial_i \varphi \partial_3 f$ whose $L^2(\Omega)$-norm is controlled by $|\mathcal{T}^\gamma \bar{\nabla}\psi|_0$. However, the regularity of ψ obtained in $\mathcal{T}^\gamma$-estimate is $|\sqrt{\sigma}\mathcal{T}^\gamma \bar{\nabla}\psi|_0$, which is σ-dependent. To overcome this difficulty, we introduce the Alinhac good unknown method which reveals that the "essential" leading order term in $\mathcal{T}^\gamma(\nabla^\varphi f)$ is not simply $\nabla^\varphi(\mathcal{T}^\gamma f)$, but the covariant derivative of the "Alinhac good unknown" $\mathbf{F}$. Namely, the Alinhac good unknown for a function f with respect to $\mathcal{T}^\gamma$ is defined by $\mathbf{F}^\gamma := \mathcal{T}^\gamma f - \mathcal{T}^\gamma \varphi \partial_3^\varphi f$ and satisfies

$$\mathcal{T}^\gamma \nabla_i^\varphi f = \nabla_i^\varphi \mathbf{F}^\gamma + \mathfrak{C}_i^\gamma(f), \ \ \mathcal{T}^\gamma D_t^\varphi f = D_t^\varphi \mathbf{F}^\gamma + \mathfrak{D}^\gamma(f), \tag{2.4}$$

where $\|\mathfrak{C}_i^\gamma(f)\|_0$ and $\|\mathfrak{D}^\gamma(f)\|_0$ can be directly controlled. Therefore, we can reformulate the $\mathcal{T}^\gamma$-differentiated current-vortex sheets system (1.30) in terms of $\mathbf{V}^{\gamma,\pm}, \mathbf{B}^{\gamma,\pm}, \mathbf{P}^{\gamma,\pm}, \mathbf{Q}^{\gamma,\pm}, \mathbf{S}^{\gamma,\pm}$ (the Alinhac good unknowns of $v^\pm, b^\pm, p^\pm, q^\pm, S^\pm$ in $\Omega^\pm$) and reduce the $\mathcal{T}^\gamma$-estimate of $(v^\pm, b^\pm, p^\pm, q^\pm, S^\pm)$ to the L^2-estimate of good unknowns, which is similar to the L^2 energy conservation and exhibits several important structures when proving the uniform-in-ε estimates. This fact was first observed by Alinhac [1] in the study of rarefaction waves and was applied (implicitly) to the study of free-surface fluids by Christodoulou-Lindblad [9]. See also Masmoudi-Roussét [42] for an explicit formulation that has been widely adopted by related works.

Dropping the superscript γ for convenience and applying L^2 estimates to the good unknowns, we get the following equality which includes four major terms

$$\sum_{\pm} \frac{d}{dt} \frac{1}{2} \int_{\Omega^\pm} \rho^\pm |\mathbf{V}^\pm|^2 + |\mathbf{B}^\pm|^2 + \mathcal{F}_p^\pm |\mathbf{P}^\pm|^2 \, d\mathcal{V}_t = \text{ST} + \text{RT} + \text{VS} + \sum_{\pm}(Z^\pm + ZB^\pm) + \cdots \tag{2.5}$$

where $d\mathcal{V}_t := \partial_3\varphi\,dx$. These four major terms are

$$\text{ST} := \varepsilon^{4l}\int_\Sigma \mathcal{T}^\gamma(\sigma\mathcal{H})\partial_t\mathcal{T}^\gamma\psi\,dx', \quad \text{RT} := -\varepsilon^{4l}\int_\Sigma [\![\partial_3 q]\!]\mathcal{T}^\gamma\psi\mathcal{T}^\gamma\partial_t\psi\,dx', \tag{2.6}$$

$$\text{VS} := \varepsilon^{4l}\int_\Sigma \mathcal{T}^\gamma q^-([\![\bar{v}]\!]\cdot\overline{\nabla})\mathcal{T}^\gamma\psi\,dx', \tag{2.7}$$

$$ZB^\pm := \mp\varepsilon^{4l}\int_\Sigma \mathcal{T}^\gamma q^\pm[\mathcal{T}^\gamma, v_i, N_i]\,dx', \quad Z^\pm := -\varepsilon^{4l}\int_{\Omega^\pm} \mathcal{T}^\gamma q^\pm[\mathcal{T}^\gamma, \partial_3 v_i, \mathbf{N}_i]\,d\mathcal{V}_t. \tag{2.8}$$

On the interface Σ, the weight function $\omega(x_3) = 0$, so it remains to consider $\mathcal{T}^\gamma = \partial_t^{k+\alpha_0}\overline{\partial}^{4-l-k+(\alpha_1+\alpha_2)} = \partial_t^{k+\alpha_0}\overline{\partial}^{4+l-(k+\alpha_0)}$. For simplicity of notations, we replace $k + \alpha_0$ by k. One can directly show that the term ST gives the $\sqrt{\sigma}\varepsilon^{2l}$-weighted boundary regularity in $E(t)$. The term RT is supposed to give us boundary regularity $|\varepsilon^{2l}\partial_t^k\psi|_{4+l-k}^2$ without σ-weight provided the Rayleigh-Taylor sign condition $[\![\partial_3 q]\!] \geq c_0 > 0$. However, in the presence of surface tension, we cannot impose the Rayleigh-Taylor sign condition. Thus, we have to use the $\sqrt{\sigma}$-weighted boundary energy, contributed by surface tension, to control RT.

2.2.1 A crucial term for vortex sheets

Let us consider the term VS that exhibits an essential difficulty in the study of vortex sheets.

$$\text{VS} := \varepsilon^{4l}\int_\Sigma \partial_t^k\overline{\partial}^{4+l-k}q^-([\![\bar{v}]\!]\cdot\overline{\nabla})\partial_t^k\overline{\partial}^{4+l-k}\psi\,dx'. \tag{2.9}$$

The difficulty is that we only have a jump condition for $[\![q]\!]$ but no conditions for $q^\pm$ individually. Thus, when $0 \leq k \leq 3 + l$, we integrate $\overline{\partial}^{1/2}$ by parts and control $q^\pm$ by using Lemma B.4

$$\text{VS} \leq |\varepsilon^{2l}\partial_t^k\overline{\partial}^{3.5+l-k}q^-|_0|\varepsilon^{2l}([\![\bar{v}]\!]\cdot\overline{\nabla})\partial_t^k\overline{\partial}^{4+l-k}\psi|_{1/2} \leq \|\varepsilon^{2l}\partial_t^k\overline{\partial}^{4+l-k}q^-\|_{0,-}^{1/2}\|\varepsilon^{2l}\partial_t^k\overline{\partial}^{3+l-k}\partial_3 q^-\|_{0,-}^{1/2}|\bar{v}|_2|\varepsilon^{2l}\partial_t^k\psi|_{5.5+l-k}.$$

This indicates us to seek for the control of $|\varepsilon^{2l}\partial_t^k\psi|_{5.5+l-k}$ for $0 \leq k \leq 3 + l$, which is exactly given by the surface tension. Indeed, the jump condition $\mathcal{H}(\psi) = \sigma^{-1}[\![q]\!]$ and the ellipticity of the mean curvature operator indicates that we can control $|\varepsilon^{2l}\partial_t^k\psi|_{5.5+l-k}$ by $|\sigma^{-1}\varepsilon^{2l}\partial_t^k[\![q]\!]|_{3.5+l-k}$ plus lower-order terms. Thus, **surface tension significantly enhances the regularity of the free interface such that VS is directly controlled.**

Remark 2.1 (**Comparison with one-phase problems and MHD contact discontinuities**). The above estimate of VS term is not uniform in σ as the elliptic estimate is completely contributed by surface tension. This corresponds to the fact that one cannot take the vanishing surface tension limit of vortex sheets for Euler equations as they are usually violently unstable (except the 2D supersonic case [11, 12]). In the absence of surface tension, the term VS loses control even if the Rayleigh-Taylor sign condition holds because the Rayleigh-Taylor sign condition $N \cdot \nabla[\![Q]\!]|_{\Sigma(t)} \geq c_0 > 0$ only gives the energy of $|\varepsilon^{2l}\partial_t^k\psi|_{4+l-k}$ which is 1.5-order lower than the desired regularity. *For one-phase problems, the term VS does not appear* because everything in Ω^- is assumed to be vanishing, so the Rayleigh-Taylor sign condition is usually enough to guarantee the well-posedness [61, 32]. *For MHD contact discontinuities, the jump condition $[\![v]\!] = \vec{0}$ also eliminates the term VS* and the transversality of magnetic fields automatically give the bound for $|\partial_t^k\psi|_{4-k}$ (cf. Wang-Xin [67]). However, the term VS must appear in the vortex sheet problems due to $|[\![\bar{v}]\!]| > 0$ on Σ. Thus, the appearance of the term VS shows an essential difference from one-phase flow problems and MHD contact discontinuities.

Remark 2.2 (**Treatment of full time derivatives**). It should be noted that when the tangential derivatives are the full time derivatives $\varepsilon^{2l}\partial_t^{4+l}$, the above analysis is no longer valid as we cannot integrate by part $\partial_t^{1/2}$. Instead, one has to replace one ∂_t by $D_t^{\varphi,-}$ and repeatedly use the Gauss-Green formula, the symmetric structure, the continuity equation. In fact, this is the most difficult step in the proof of uniform estimates and we refer to step 2 in Section 3.4.3 for those rather technical computations.

2.2.2 A crucial cancellation structure for incompressible limit

So far, it remains to control the term $Z^\pm + ZB^\pm$:

$$Z^\pm + ZB^\pm = \mp\varepsilon^{4l}\int_\Sigma \mathcal{T}^\gamma q^\pm[\mathcal{T}^\gamma, v_i, N_i]\,dx' - \varepsilon^{4l}\int_{\Omega^\pm} \mathcal{T}^\gamma q^\pm[\mathcal{T}^\gamma, \partial_3 v_i, \mathbf{N}_i]\,d\mathcal{V}_t \tag{2.10}$$

We only obtain the regularity for $\sqrt{\mathcal{F}_p^\pm}\mathcal{T}^\gamma q^\pm$ in tangential estimates, but the first term contains $\mathcal{T}^\gamma q^\pm$ without ε-weights. When $\mathcal{T}^\gamma$ contains at least one spatial derivative ($\gamma_0 < \langle\gamma\rangle$), one can invoke the momentum equation to replace $\mathcal{T}_i q$ ($i = 1, 2, 4$) by

tangential derivatives of v, b, as only the full time derivatives of $q^{\pm}$ require one more ε-weight. However, there may be a loss of ε-weight in this term when $\mathcal{T}^{\gamma}$ only contains time derivatives, e.g., in $\varepsilon^{2l}\partial_t^{4+l}$-estimates for $0 \le l \le 4$. To get rid of this, there is a cancellation structure that is observed by comparing the concrete forms of the commutators. Using Gauss-Green formula and integrating by parts in ∂_t, it is easy to see that the leading-order part is

$$
\begin{aligned}
ZB^{\pm} + Z^{\pm} \stackrel{\partial_t}{=} & \varepsilon^{4l}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^{\pm}}\partial_3\partial_t^{3+l}q^{\pm}\,\partial_t^{3+l}v^{\pm}\cdot\partial_t\mathbf{N}\,\mathrm{d}x \\
& - \varepsilon^{4l}\int_{\Omega^{\pm}}\partial_3\partial_t^{3+l}q^{\pm}\partial_t(\partial_t^{3+l}v^{\pm}\cdot\partial_t\mathbf{N})\,\mathrm{d}x - \varepsilon^{4l}\int_{\Omega^{\pm}}\partial_t^{3+l}q^{\pm}\partial_t(\partial_t^{3+l}v^{\pm}\cdot\partial_3\partial_t\mathbf{N})\,\mathrm{d}x + \cdots
\end{aligned}
\tag{2.11}
$$

where the first term can be controlled by using Young's inequality after integrating in time t and the other two terms can be directly controlled uniformlly in ε because the full time derivatives of q no longer appear. Hence, the problematic terms in (2.10) are controlled uniformly in ε.

Remark 2.3 (Calculations in anisotropic Sobolev space). We use anisotropic Sobolev spaces and the normal derivative ∂_3 should be considered as a second-order derivative. A new difficulty is mainly presented in the following two ways: the control of commutators $\mathfrak{C}$ and $\mathfrak{D}$ is more subtle in the analysis of $E_8(t)$ (purely tangential regularity), and the standard Sobolev trace lemma is no longer useful. The latter issue can be resolved by using Lemma B.3 and Lemma B.4, while the first issue required very delicate analysis on the commutators $\mathfrak{C}_i(v_i)$ and $\mathfrak{C}(q)$. We refer to Section 3.3.2 for detailed reduction procedures.

Remark 2.4 (Comparison with the Lagrangian setting). In the author's previous paper [32] about the one-phase MHD without surface tension under the setting of Lagrangian coordinates, the "modified Alinhac good unknowns" were introduced to avoid the derivative loss in these commutators, that is, lots of modification terms were added to $\mathbf{F}$ such that the corresponding $\mathfrak{C}(f)$ is L^2-controllable. Those modification terms are necessary when using Lagrangian coordinates but are redundant in the setting of this paper when the free interface is a graph. The precise reason is that, in the Lagrangian setting, the boundary regularity we obtain from tangential estimates has the form $|\bar{\partial}^r\eta\cdot N|_0^2$ where η represents the flow map of v, which is not enough to control the top-order derivatives of the co-factor matrix $A := [\partial\eta]^{-1}$ and the Eulerian normal vector $N = \bar{\partial}\eta \times \bar{\partial}\eta$. In contrast, the setting in this paper allows us to *explicitly express the Eulerian normal vector, the surface tension, the boundary energy in terms of* $\bar{\nabla}\psi$, and we can also *explicitly write the normal derivative of the "non-characteristic variables"* $(q, v\cdot\mathbf{N})$ *in terms of tangential derivatives of the other quantities*, which will be frequently used throughout this manuscript.

2.3   A robust method to solve the compressible vortex sheets problem

As pointed out in a series of the author's previous works [37, 70, 71, 21, 38], the local existence for inviscid fluids is not a direct consequence of the a priori estimates without loss of regularity. There is a loss of one tangential spatial derivative in ψ arising from the analogues of ST and RT terms when doing the Picard iteration. Besides, due to the presence of surface tension and compressibility, one has to control the full time derivatives of v, b, p, S which only belong to $L^2(\Omega^{\pm})$ and their boundary regularity is unknown due to the failure of trace lemma. The delicate cancellation structures for the original nonlinear problem (1.30) no longer exist for the linearized problem. **Therefore, we shall enhance the regularity of ψ in both tangential spatial variables x' and the time variable t.** There are mainly two methods to prove the existence in previous related works

1. **Nash-Moser iteration**. Although there may be some derivative loss for the linearized problem, the order of regularity loss is a fixed number, so one can use Nash-Moser iteration to prove the local existence of smooth solution or solution in Sobolev spaces with a loss of regularity from initial data to solution (cf. [12, 60, 52, 61, 62, 64]).
2. **Tangential smoothing**. This method has been widely used in the study of free-surface inviscid fluids in Lagrangian coordinate [8, 22, 37, 70, 71, 21]. In the paper [38], Luo and the author first introduced the tangential smoothing scheme for Euler equations in the "flattened coordinate". However, the constraint $b\cdot N|_{\Sigma} = 0$ no longer propagates from the initial data after doing tangential smoothing on N.

In this paper, we introduce a new and simpler approximation scheme, namely the nonlinear approximate problem (3.1), indexed by $\kappa > 0$, for (1.30) by adding two regularization terms to the jump condition for q as below:

$$
[\![q]\!] = \sigma\mathcal{H} - \kappa(1-\bar{\Delta})^2\psi - \kappa(1-\bar{\Delta})\partial_t\psi,
\tag{2.12}
$$

where $\bar{\Delta} := \bar{\partial}_1^2 + \bar{\partial}_2^2$ is the tangential Laplacian operator on Σ. These two regularization terms help us to get $\sqrt{\kappa}$-weighted enhanced regularity for both ψ and ψ_t which is enough for us to compensate the loss of derivatives in the Picard iteration process. Also, the constraint $b\cdot N|_{\Sigma} = 0$ can be recovered at the end of each iteration step by modifying the normal component

of the solution to the linearized system. So, we can solve the nonlinear approximate problem for each fixed $\kappa > 0$ and we refer to Section 4.1 for detailed construction of the linearization and the iteration scheme. As for the uniform-in-κ estimates for the nonlinear approximate problem, the appearance of these two regularization terms will not introduce any uncontrollable terms with the help of some delicate technical modifications. In particular, the term VS remains the same as (2.9), and the elliptic estimate for $[\![q]\!]$ in Section 2.2.1 is still vaild and uniform in κ for the approximate problem. Hence, the local existence of (1.30) is proven after passing the limit $\kappa \to 0$. The bounds obtained in both the iteration process (for fixed κ) and the uniform-in-κ nonlinear estimates have no loss of regularity and are uniform in Mach number.

Since the abovementioned difficulties arising in previous frameworks do not appear in our new approximation scheme, **we believe that the approximation scheme is a robust method to prove the local existence (and the incompressible limit) for a large class of free-boundary problems in inviscid fluids, especially the vortex sheets problem with surface tension.** Furthermore, we believe that taking zero-surface-tension limit seems to be an alternative way, other than Nash-Moser iteration, to prove the local existence of compressible vortex sheets problems under certain stability conditions. This will be presented in the second paper of this two-paper sequence.

Remark 2.5. We choose the "flattened coordinate" because of the reasons mentioned in Remark 2.4. It should be noted that the design of the linearized problem and the Picard iteration process in the "flattened coordinate" is much more difficult than in the Lagrangian coordinate because one has to "define" the free surface in each step of the iteration, whereas the free surface is not explicitly computed and the flow map η is completely determined by the velocity in Lagrangian coordinates.

3   Uniform estimates of the nonlinear approximate system

Now we introduce the approximate system of (1.30) indexed by $\kappa > 0$.

$$
\begin{cases}
\rho^\pm D_t^{\varphi\pm} v^\pm - (b^\pm \cdot \nabla^\varphi) b^\pm + \nabla^\varphi q^\pm = 0, \ \ q^\pm = p^\pm + \frac{1}{2}|b^\pm|^2 & \text{in } [0,T] \times \Omega^\pm, \\
\mathcal{F}_p D_t^{\varphi\pm} p^\pm + \nabla^\varphi \cdot v^\pm = 0 & \text{in } [0,T] \times \Omega^\pm, \\
p^\pm = p^\pm(\rho^\pm, S^\pm), \ \ \mathcal{F}^\pm = \log \rho^\pm, \ \ \mathcal{F}_p^\pm > 0, \ \rho^\pm \geq \bar{\rho}_0 > 0 & \text{in } [0,T] \times \Omega^\pm, \\
D_t^{\varphi\pm} b^\pm - (b^\pm \cdot \nabla^\varphi) v^\pm + b^\pm \nabla^\varphi \cdot v^\pm = 0 & \text{in } [0,T] \times \Omega^\pm, \\
\nabla^\varphi \cdot b^\pm = 0 & \text{in } [0,T] \times \Omega^\pm, \\
D_t^{\varphi\pm} S^\pm = 0 & \text{in } [0,T] \times \Omega^\pm, \\
[\![q]\!] = \sigma \overline{\nabla} \cdot \left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}} \right) - \kappa(1-\overline{\Delta})^2 \psi - \kappa(1-\overline{\Delta})\partial_t \psi & \text{on } [0,T] \times \Sigma, \\
\partial_t \psi = v^\pm \cdot N & \text{on } [0,T] \times \Sigma, \\
b^\pm \cdot N = 0 & \text{on } [0,T] \times \Sigma, \\
v_3^\pm = b_3^\pm = 0 & \text{on } [0,T] \times \Sigma^\pm, \\
(v^\pm, b^\pm, \rho^\pm, S^\pm, \psi)|_{t=0} = (v_0^{\kappa,\pm}, b_0^{\kappa,\pm}, \rho_0^{\kappa,\pm}, S_0^{\kappa,\pm}, \psi_0^\kappa).
\end{cases}
\tag{3.1}
$$

Note that this system is not over-determined: the continuity equation, the evolution equation of $b^\pm$ and the kinematic boundary condition stay unchanged, so one can still prove $\nabla^\varphi \cdot b^\pm = 0$, $b^\pm \cdot N|_\Sigma = 0$ and $b_3^\pm|_{\Sigma^\pm} = 0$ propagates from the initial data.

The energy functional associated with system (3.1) is defined by

$$
E^\kappa(t) := E_4^\kappa(t) + E_5^\kappa(t) + E_6^\kappa(t) + E_7^\kappa(t) + E_8^\kappa(t)
$$

$$
E_{4+l}^\kappa(t) := \sum_\pm \sum_{\langle\alpha\rangle=2l} \sum_{k=0}^{4-l} \left\| \left(\varepsilon^{2l} \mathcal{T}^\alpha \partial_t^k \left(v^\pm, b^\pm, S^\pm, (\mathcal{F}_p^\pm)^{\frac{(k+\alpha_0-l-3)_+}{2}} p^\pm \right) \right) \right\|_{4-k-l,\pm}^2
\tag{3.2}
$$

$$
+ \sum_{k=0}^{4+l} \left| \sqrt{\sigma} \varepsilon^{2l} \partial_t^k \psi \right|_{5+k-l}^2 + \left| \sqrt{\kappa} \varepsilon^{2l} \partial_t^k \psi \right|_{6+k-l}^2 + \int_0^t \left| \sqrt{\kappa} \varepsilon^{2l} \partial_t^{k+1} \psi(\tau) \right|_{5+k-l}^2 \, d\tau,
$$

where $0 \leq l \leq 4$ and we denote $\mathcal{T}^\alpha := (\omega(x_3)\partial_3)^{\alpha_4} \partial_t^{\alpha_0} \partial_1^{\alpha_1} \partial_2^{\alpha_2}$ to be a tangential derivative for the multi-index $\alpha = (\alpha_0, \alpha_1, \alpha_2, 0, \alpha_4)$ with length $\langle\alpha\rangle = \alpha_0 + \alpha_1 + \alpha_2 + 2 \times 0 + \alpha_4$. The quantity $(k + \alpha_0 - l - 3)_+ = 1$ only when $\alpha_0 = 2l$ and $k = 4 - l$ and it is equal to 0 otherwise.

We aim to establish the a priori estimates of system (3.1) that is uniform in $\kappa > 0$, which allows us taking the limit $\kappa \to 0_+$ to construct the local-in-time solution to the original system (1.30) for fixed $\sigma > 0$. Spefically, we want to prove the following proposition

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Proposition 3.1. There exists some $T_\sigma > 0$ independent of κ, ε such that

$$\sup_{0 \le t \le T_\sigma} E^\kappa(t) \le C(\sigma^{-1}) P(E^\kappa(0)). \tag{3.3}$$

Remark 3.1. The initial data of the approximate system (3.1) is not the same as the initial data of the original system (1.30) because of the different compatibility conditions. The compatibility conditions (up to 7-th order) for system (3.1) are

$$
\begin{aligned}
\left.\left[\!\left[\partial_t^j q\right]\!\right]\right|_{t=0} &= \left.\partial_t^j \left(\sigma \mathcal{H} - \kappa(1-\overline{\Delta})^2 \psi - \kappa(1-\overline{\Delta})\partial_t \psi\right)\right|_{t=0} \quad \text{on } \Sigma, \ \ 0 \le j \le 7, \\
\left.\partial_t^{j+1} \psi\right|_{t=0} &= \left.\partial_t^j (v^\pm \cdot N)\right|_{t=0} \quad \text{on } \Sigma, \ \ 0 \le j \le 7, \\
\left.\partial_t^j v_3^\pm\right|_{t=0} &= 0 \quad \text{on } \Sigma^\pm, \ \ 0 \le j \le 7.
\end{aligned}
\tag{3.4}
$$

In Appendix C, we construct the initial data of (3.1) satisfying the compatibility conditions (3.4) that is uniformly bounded in κ and converges to a given initial data of (1.30) satisfying the compatibility conditions (1.31) up to 7-th order.

3.1 L^2 energy conservation

Proposition 3.2. The approximate system (3.1) admits the following conserved quantity: Let

$$
\begin{aligned}
E_0^\kappa(t) := \sum_\pm \frac{1}{2} \int_{\Omega^\pm} \rho^\pm |v^\pm|^2 + |b^\pm|^2 + 2\mathfrak{P}(\rho^\pm, S^\pm) + \rho^\pm |S^\pm|^2 \, \mathrm{d}\mathcal{V}_t \\
+ \frac{1}{2} \int_\Sigma \sigma \sqrt{1 + |\overline{\nabla}\psi|^2} + \kappa|(1-\overline{\Delta})\psi|^2 \, \mathrm{d}x' + \int_0^t \int_\Sigma \kappa|\langle\overline{\partial}\rangle\psi_t|^2 \, \mathrm{d}x' \, \mathrm{d}\tau.
\end{aligned}
\tag{3.5}
$$

Then $\frac{\mathrm{d}}{\mathrm{d}t} E_0^\kappa(t) = 0$ with in the lifespan of the solution to (3.1). Here $\langle\overline{\partial}\rangle := \sqrt{1-\overline{\Delta}}$, that is, $\widehat{\langle\overline{\partial}\rangle f}(\xi) = \sqrt{1+|\xi|^2}\,\hat{f}(\xi)$ in $\mathbb{T}^2$ and $\mathrm{d}\mathcal{V}_t := \partial_3 \varphi \, \mathrm{d}x$.

Proof. The proof of L^2 estimate is straightforward. Taking $L^2(\Omega^\pm)$-inner product of v and the first equation in (3.1) and using Reynolds transport formula (A.3), we get

$$
\begin{aligned}
\sum_\pm \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_{\Omega^\pm} \rho^\pm |v^\pm|^2 \, \mathrm{d}\mathcal{V}_t &= \sum_\pm \int_{\Omega^\pm} (\rho^\pm D_t^{\varphi\pm} v^\pm) \cdot v^\pm \, \mathrm{d}\mathcal{V}_t \\
&= \int_\Sigma \left[\!\left[q\right]\!\right] \partial_t \psi \, \mathrm{d}x' + \sum_\pm \int_{\Omega^\pm} p^\pm (\nabla^\varphi \cdot v^\pm) \, \mathrm{d}\mathcal{V}_t - \int_{\Omega^\pm} (b^\pm \cdot \nabla^\varphi) v^\pm \cdot b^\pm \, \mathrm{d}\mathcal{V}_t + \int_{\Omega^\pm} \frac{1}{2} |b^\pm|^2 (\nabla^\varphi \cdot v^\pm) \, \mathrm{d}\mathcal{V}_t,
\end{aligned}
\tag{3.6}
$$

where the integral on $\Sigma^\pm$ vanishes thanks to the slip conditions. Let $\mathfrak{P}(\rho^\pm, S^\pm) = \int_{\overline{\rho}_0}^{\rho^\pm} \frac{p^\pm(z, S^\pm)}{z^2} \, \mathrm{d}z$. Then the first integral above together with $D_t^{\varphi\pm} S^\pm = 0$ gives

$$
\int_{\Omega^\pm} p^\pm (\nabla^\varphi \cdot v^\pm) \, \mathrm{d}\mathcal{V}_t = -\int_{\Omega^\pm} \frac{p^\pm}{(\rho^\pm)^2} D_t^{\varphi\pm} \rho^\pm \, \mathrm{d}\mathcal{V}_t = -\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega^\pm} \rho^\pm \mathfrak{P}(\rho^\pm) \, \mathrm{d}\mathcal{V}_t.
$$

The boundary term gives $\sqrt{\sigma}$-weighted and $\sqrt{\kappa}$-weighted regularity of ψ and ψ_t. One has

$$
\int_\Sigma \left[\!\left[q\right]\!\right] \partial_t \psi \, \mathrm{d}x' = -\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_\Sigma \sigma \sqrt{1 + |\overline{\nabla}\psi|^2} + \kappa|(1-\overline{\Delta})\psi|^2 \, \mathrm{d}x' - \int_\Sigma \kappa|\langle\overline{\partial}\rangle\psi_t|^2 \, \mathrm{d}x'.
$$

Then we insert the evolution equation of $b^\pm$ in the third term in (3.6) to get the energy of $b^\pm$.

$$
\begin{aligned}
-\int_{\Omega^\pm} (b^\pm \cdot \nabla^\varphi) v^\pm \cdot b^\pm \, \mathrm{d}\mathcal{V}_t &= -\int_{\Omega^\pm} D_t^{\varphi\pm} b^\pm \cdot b^\pm \, \mathrm{d}\mathcal{V}_t - \int_{\Omega^\pm} |b^\pm|^2 (\nabla^\varphi \cdot v^\pm) \, \mathrm{d}\mathcal{V}_t \\
&= -\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_{\Omega^\pm} |b^\pm|^2 \, \mathrm{d}\mathcal{V}_t + \frac{1}{2} \int_{\Omega^\pm} |b^\pm|^2 (\nabla^\varphi \cdot v^\pm) \, \mathrm{d}\mathcal{V}_t - \int_{\Omega^\pm} |b^\pm|^2 (\nabla^\varphi \cdot v^\pm) \, \mathrm{d}\mathcal{V}_t,
\end{aligned}
$$

where the last two terms exactly cancels with the last term in (3.6). Finally, $D_t^{\varphi\pm} S^\pm = 0$ and the Reynolds transport theorem shows that $\frac{d}{dt}\frac{1}{2}\int_{\Omega^\pm}\rho^\pm|S^\pm|^2\,d\mathcal{V}_t = 0$. Therefore, we conclude that system (3.1) admits the following conserved quantity

$$
\begin{aligned}
E_0^\kappa(t) := &\sum_\pm \frac{1}{2}\int_{\Omega^\pm}\rho^\pm|v^\pm|^2 + |b^\pm|^2 + 2\mathfrak{P}(\rho^\pm, S^\pm) + \rho^\pm|S^\pm|^2\,d\mathcal{V}_t \\
&+ \frac{1}{2}\int_\Sigma \sigma\sqrt{1 + |\overline{\nabla}\psi|^2} + \kappa|(1-\overline{\Delta})\psi|^2\,dx' + \int_0^t\int_\Sigma \kappa|\langle\overline{\partial}\rangle\psi_t|^2\,dx'\,d\tau,
\end{aligned}
\tag{3.7}
$$

which can also be inherited to the original current-vortex sheet system (1.30) after taking $\kappa \to 0_+$. $\qquad\square$

3.2 Reformulations in Alinhac good unknowns

Let $\mathcal{T}^\gamma := (\omega(x_3)\partial_3)^{\gamma_4}\partial_t^{\gamma_0}\partial_1^{\gamma_1}\partial_2^{\gamma_2}$ be a tangential derivative with $\langle\gamma\rangle = \gamma_0 + \gamma_1 + \gamma_2 + \gamma_4$. We define the Alinhac good unknown of a given function f with respect to $\mathcal{T}^\gamma$ by $\mathbf{F}^\gamma := \mathcal{T}^\gamma f - \mathcal{T}^\gamma\varphi\partial_3^\varphi f$. The good unknown $\mathbf{F}$ satisfies

$$
\mathcal{T}^\gamma\nabla_i^\varphi f = \nabla_i^\varphi\mathbf{F}^\gamma + \mathfrak{C}_i^\gamma(f), \quad \mathcal{T}^\gamma D_t^\varphi f = D_t^\varphi\mathbf{F}^\gamma + \mathfrak{D}^\gamma(f),
\tag{3.8}
$$

where the commutators $\mathfrak{C}_i^\gamma(f)$ and $\mathfrak{D}^\gamma(f)$ are defined by

$$
\begin{aligned}
\mathfrak{C}_i^\gamma(f) =& (\partial_3^\varphi\partial_i^\varphi f)\mathcal{T}^\gamma\varphi + \left[\mathcal{T}^\gamma, \frac{\mathbf{N}_i}{\partial_3\varphi}, \partial_3 f\right] + \partial_3 f\left[\mathcal{T}^\gamma, \mathbf{N}_i, \frac{1}{\partial_3\varphi}\right] + \mathbf{N}_i\partial_3 f\left[\mathcal{T}^{\gamma-\gamma'}, \frac{1}{(\partial_3\varphi)^2}\right]\mathcal{T}^{\gamma'}\partial_3\varphi \\
&+ \frac{\mathbf{N}_i}{\partial_3\varphi}[\mathcal{T}^\gamma, \partial_3]f - \frac{\mathbf{N}_i}{(\partial_3\varphi)^2}\partial_3 f[\mathcal{T}^\gamma, \partial_3]\varphi, \quad i = 1, 2, 3,
\end{aligned}
\tag{3.9}
$$

and

$$
\begin{aligned}
\mathfrak{D}^\gamma(f) =& (D_t^\varphi\partial_3^\varphi f)\mathcal{T}^\gamma\varphi + [\mathcal{T}^\gamma, \bar{v}]\cdot\overline{\partial}f + \left[\mathcal{T}^\gamma, \frac{1}{\partial_3\varphi}(v\cdot\mathbf{N} - \partial_t\varphi), \partial_3 f\right] + \left[\mathcal{T}^\gamma, v\cdot\mathbf{N} - \partial_t\varphi, \frac{1}{\partial_3\varphi}\right]\partial_3 f \\
&+ \frac{1}{\partial_3\varphi}[\mathcal{T}^\gamma, v]\cdot\mathbf{N}\partial_3 f - (v\cdot\mathbf{N} - \partial_t\varphi)\partial_3 f\left[\mathcal{T}^{\gamma-\gamma'}, \frac{1}{(\partial_3\varphi)^2}\right]\mathcal{T}^{\gamma'}\partial_3\varphi \\
&+ \frac{1}{\partial_3\varphi}(v\cdot\mathbf{N} - \partial_t\varphi)[\mathcal{T}^\gamma, \partial_3]f + (v\cdot\mathbf{N} - \partial_t\varphi)\frac{\partial_3 f}{(\partial_3\varphi)^2}[\mathcal{T}^\gamma, \partial_3]\varphi
\end{aligned}
\tag{3.10}
$$

with $\langle\gamma'\rangle = 1$. Here $\mathbf{N} := (-\overline{\partial}_1\varphi, -\overline{\partial}_2\varphi, 1)^\top$ is the extension of normal vector N in $\Omega^\pm$. The third term on the right side of (3.9) is zero when $i = 3$ because $\mathbf{N}_3 = 1$ is a constant.

Therefore, we can reformulate the $\mathcal{T}^\gamma$-differentiated current-vortex sheets system (3.1) in terms of $\mathbf{V}^{\gamma,\pm}, \mathbf{B}^{\gamma,\pm}, \mathbf{P}^{\gamma,\pm}, \mathbf{S}^{\gamma,\pm}$ (the Alinhac good unknowns of $v^\pm, b^\pm, p^\pm, S^\pm$ in $\Omega^\pm$) as follows

$$
\rho^\pm D_t^{\varphi\pm}\mathbf{V}^{\gamma,\pm} - (b^\pm\cdot\nabla^\varphi)\mathbf{B}^{\gamma,\pm} + \nabla^\varphi\mathbf{Q}^{\gamma,\pm} = \mathcal{R}_v^{\gamma,\pm} - \mathfrak{C}^\gamma(q^\pm) \quad \text{in } [0, T]\times\Omega^\pm,
\tag{3.11}
$$

$$
\mathcal{F}_p D_t^{\varphi\pm}\mathbf{P}^{\gamma,\pm} + \nabla^\varphi\cdot\mathbf{V}^{\gamma,\pm} = \mathcal{R}_p^{\gamma,\pm} - \mathfrak{C}_i^\gamma(v_i^\pm) \quad \text{in } [0, T]\times\Omega^\pm,
\tag{3.12}
$$

$$
D_t^{\varphi\pm}\mathbf{B}^{\gamma,\pm} - (b^\pm\cdot\nabla^\varphi)\mathbf{V}^{\gamma,\pm} + b^\pm(\nabla^\varphi\cdot\mathbf{V}^{\gamma,\pm}) = \mathcal{R}_b^{\gamma,\pm} - b^\pm\mathfrak{C}_i^\gamma(v_i^\pm) \quad \text{in } [0, T]\times\Omega^\pm,
\tag{3.13}
$$

$$
\nabla^\varphi\cdot b^\pm = 0 \quad \text{in } [0, T]\times\Omega^\pm,
\tag{3.14}
$$

$$
D_t^{\varphi\pm}\mathbf{S}^{\pm,\alpha} = \mathfrak{D}^\gamma(S^\pm) \quad \text{in } [0, T]\times\Omega^\pm,
\tag{3.15}
$$

with boundary conditions

$$
[\![\mathbf{Q}^\gamma]\!] = \sigma\mathcal{T}^\gamma\mathcal{H} - \kappa\mathcal{T}^\gamma(1-\overline{\Delta})^2\psi - \kappa\mathcal{T}^\gamma(1-\overline{\Delta})\partial_t\psi - [\![\partial_3 q]\!]\mathcal{T}^\gamma\psi \quad \text{on } [0, T]\times\Sigma,
\tag{3.16}
$$

$$
\mathbf{V}^{\gamma,\pm}\cdot N = \partial_t\mathcal{T}^\gamma\psi + \bar{v}^\pm\cdot\overline{\nabla}\mathcal{T}^\gamma\psi - \mathcal{W}^{\gamma,\pm} \quad \text{on } [0, T]\times\Sigma,
\tag{3.17}
$$

$$
b^\pm\cdot N = 0 \quad \text{on } [0, T]\times\Sigma,
\tag{3.18}
$$

$$
b_3^\pm = v_3^\pm = \mathbf{B}_3^\pm = \mathbf{V}_3^\pm = 0 \quad \text{on } [0, T]\times\Sigma^\pm,
\tag{3.19}
$$

where $\mathcal{R}_v, \mathcal{R}_p, \mathcal{R}_b$ terms consist of the following commutators

$$
\mathcal{R}_v^{\gamma,\pm} := [\mathcal{T}^\gamma, b^\pm]\cdot\nabla^\varphi b^\pm - [\mathcal{T}^\gamma, \rho^\pm]D_t^{\varphi\pm}v^\pm - \rho^\pm\mathfrak{D}^\gamma(v^\pm)
\tag{3.20}
$$

$$
\mathcal{R}_p^{\gamma,\pm} := -[\mathcal{T}^\gamma, \mathcal{F}_p^\pm]D_t^{\varphi\pm}p^\pm - \mathcal{F}_p^\pm\mathfrak{D}^\gamma(p^\pm)
\tag{3.21}
$$

$$
\mathcal{R}_b^{\gamma,\pm} := [\mathcal{T}^\gamma, b^\pm]\cdot\nabla^\varphi v^\pm - \mathfrak{D}^\gamma(b^\pm),
\tag{3.22}
$$

and the boundary term $\mathcal{W}^{\gamma,\pm}$ is

$$\mathcal{W}^{\gamma,\pm} := (\partial_3 v^\pm \cdot N)\mathcal{T}^\gamma \psi + [\mathcal{T}^\gamma, N_i, v_i^\pm], \tag{3.23}$$

Note that $\omega(x_3) = 0$ on $\Sigma \cup \Sigma^\pm$, so all boundary conditions are vanishing when $\gamma_4 > 0$. Thus, $\mathcal{T}^\gamma$ can be written as $\partial_t^{k+\alpha_0}\overline{\partial}^{(4+l)-(k+\alpha_0)}$ on Σ. We can replace $k + \alpha_0$ by k ($0 \le k \le 4 + l$) in the boundary energy terms. In the rest of Section 3, we aim to prove the following tangential estimates

Proposition 3.3 (Tangential estimates for the approximate system)**.** For fixed $l \in \{0, 1, 2, 3, 4\}$ and any $\delta \in (0, 1)$, the following uniform-in-(κ, ε) energy inequalities hold:

$$\sum_\pm \sum_{\langle\alpha\rangle=2l} \sum_{\substack{0 \le k \le 4-l \\ k+\alpha_0 < 4+l}} \left\| \left(\varepsilon^{2l}\overline{\partial}^{4-k-l}\mathcal{T}^\alpha \partial_t^k (v^\pm, b^\pm, S^\pm, p^\pm)\right) \right\|_{0,\pm}^2$$

$$+ \sum_{k=0}^{3+l} \left| \sqrt{\sigma}\varepsilon^{2l}\partial_t^k\psi \right|_{5-k-l}^2 + \left| \sqrt{\kappa}\varepsilon^{2l}\partial_t^k\psi \right|_{6+k-l}^2 + \int_0^t \left| \sqrt{\kappa}\varepsilon^{2l}\partial_t^{k+1}\psi(\tau) \right|_{5+k-l}^2 \, d\tau$$

$$\lesssim \delta E_{4+l}^\kappa(t) + \sum_{k=0}^{3+l} \left|\varepsilon^{2l}\partial_t^k\psi(0)\right|_{5.5+l-k}^2 + P\left(\sigma^{-1}, \sum_{j=0}^l E_{4+j}^\kappa(0)\right) + P\left(\sum_{j=0}^l E_{4+j}^\kappa(t)\right) \int_0^t P\left(\sigma^{-1}, \sum_{j=0}^l E_{4+j}^\kappa(\tau)\right) d\tau \tag{3.24}$$

and

$$\sum_\pm \sum_{k=0}^{4-l} \left\| \left(\varepsilon^{2l}\partial_t^{4+l}(v^\pm, b^\pm, S^\pm, (\mathcal{F}_p)^{\frac{1}{2}}p^\pm)\right) \right\|_{4-k-l,\pm}^2 + \left| \sqrt{\sigma}\varepsilon^{2l}\partial_t^{4+l}\psi \right|_1^2 + \left| \sqrt{\kappa}\varepsilon^{2l}\partial_t^{4+l}\psi \right|_2^2 + \int_0^t \left| \sqrt{\kappa}\varepsilon^{2l}\partial_t^{5+l}\psi(\tau) \right|_1^2 \, d\tau$$

$$\lesssim \delta E_{4+l}^\kappa(t) + \left|\varepsilon^{2l}\partial_t^{3+l}\psi(0)\right|_{2.5}^2 + P\left(\sigma^{-1}, \sum_{j=0}^l E_{4+j}^\kappa(0)\right) + P\left(\sum_{j=0}^l E_{4+j}^\kappa(t)\right) \int_0^t P\left(\sigma^{-1}, \sum_{j=0}^l E_{4+j}^\kappa(\tau)\right) d\tau. \tag{3.25}$$

Here the first inequality represents the case when there are at least one spatial tangential derivatives and the second inequality represents the case of full time derivatives. Moreover, the term $\left|\varepsilon^{2l}\partial_t^k\psi(0)\right|_{5.5+l-k}^2$ on the right side does not appear when $\kappa = 0$.

3.3   Tangential estimates: full spatial derivatives

We first study the case when all tangential derivatives are spatial derivatives $\overline{\partial}_1$ and $\overline{\partial}_2$, namely $\gamma_0 = \gamma_4 = 0$ in $\mathcal{T}^\gamma := (\omega(x_3)\partial_3)^{\gamma_4}\partial_t^{\gamma_0}\partial_1^{\gamma_1}\partial_2^{\gamma_2}$. In view of the definition of $E(t)$ and the div-curl decomposition, we need to prove the L^2 estimates for the $\varepsilon^{2l}\overline{\partial}^{4+l}$-differentiated system ($0 \le l \le 4$). We now consider the case $l = 0$, that is, the $\overline{\partial}^4$-estimate for the approximate system (3.1) and aim to prove the following estimate

Proposition 3.4. Fix $l \in \{0, 1, 2, 3, 4\}$. For the tangential derivative $\mathcal{T}^\gamma = \overline{\partial}^{4+l}$, ($\gamma_0 + \gamma_4 = 0$, $\gamma_1 + \gamma_2 = 4 + l$), the $\varepsilon^{2l}\overline{\partial}^{4+l}$-differentiated approximate system admits the following uniform-in-(κ, ε) estimate: For any $0 < \delta < 1$

$$\left\| \varepsilon^{2l}\left(\mathbf{V}^{\gamma,\pm}, \mathbf{B}^{\gamma,\pm}, \mathbf{S}^{\gamma,\pm}, \sqrt{\mathcal{F}_p^\pm}\mathbf{P}^{\gamma,\pm}\right)(t) \right\|_0^2 + \left| \sqrt{\sigma}\varepsilon^{2l}\overline{\partial}^{4+l}\psi(t) \right|_1^2 + \left| \sqrt{\kappa}\varepsilon^{2l}\overline{\partial}^{4+l}\psi(t) \right|_2^2 + \int_0^t \left| \sqrt{\kappa}\varepsilon^{2l}\overline{\partial}^{4+l}\partial_t\psi(\tau) \right|_1^2 \, d\tau$$

$$\lesssim \delta E_{4+l}^\kappa(t) + \left|\varepsilon^{2l}\psi_0\right|_{5.5+l}^2 + \sum_{j=0}^l \int_0^t P(\sigma^{-1}, E_{4+j}^\kappa(\tau)) \, d\tau, \quad 0 \le l \le 4. \tag{3.26}$$

3.3.1   The case $l = 0$: $\overline{\partial}^4$-estimates

As stated in Section 2.2, we introduce the Alinhac good unknowns for $\mathcal{T}^\gamma = \overline{\partial}^4$ and drop the script γ for simplicity of notations

$$\mathbf{V}^\pm := \overline{\partial}^4 v^\pm - \overline{\partial}^4\varphi\partial_3^\varphi v^\pm, \quad \mathbf{B}^\pm := \overline{\partial}^4 b^\pm - \overline{\partial}^4\varphi\partial_3^\varphi b^\pm, \quad \mathbf{P}^\pm := \overline{\partial}^4 p^\pm - \overline{\partial}^4\varphi\partial_3^\varphi p^\pm, \quad \mathbf{Q}^\pm := \overline{\partial}^4 q^\pm - \overline{\partial}^4\varphi\partial_3^\varphi q^\pm.$$

Note that we have

$$\mathbf{Q}^\pm = \mathbf{P}^\pm + b \cdot \mathbf{B}^\pm + \underbrace{\sum_{k=1}^3 c_k \overline{\partial}^k b^\pm \cdot \overline{\partial}^{4-k} b^\pm}_{=:\mathcal{R}_q^{\gamma,\pm}}$$

for some constants $c_k \in \mathbb{N}^*$.

Step 1: Interior energy structure.

We test the equation (3.11) by $\mathbf{V}^\pm$ in $\Omega^\pm$ and integrate by parts to get one boundary term and several interior terms

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}\rho^\pm|\mathbf{V}^\pm|^2\,\mathrm{d}\mathcal{V}_t &= \int_{\Omega^\pm}\rho^\pm D_t^{\varphi\pm}\mathbf{V}^\pm\cdot\mathbf{V}^\pm\,\mathrm{d}\mathcal{V}_t \\
&= \int_{\Omega^\pm}(b^\pm\cdot\nabla^\varphi)\mathbf{B}^\pm\cdot\mathbf{V}^\pm\,\mathrm{d}\mathcal{V}_t - \int_{\Omega^\pm}\mathbf{V}^\pm\cdot\nabla^\varphi\mathbf{Q}^\pm\,\mathrm{d}\mathcal{V}_t + \underbrace{\int_{\Omega^\pm}\mathbf{V}^\pm\cdot(\mathcal{R}_v^\pm-\mathfrak{C}(q^\pm))\,\mathrm{d}\mathcal{V}_t}_{=:R_1^\pm} \\
&= -\int_{\Omega^\pm}\mathbf{B}^\pm\cdot(b^\pm\cdot\nabla^\varphi)\mathbf{V}^\pm\,\mathrm{d}\mathcal{V}_t + \int_{\Omega^\pm}b^\pm\cdot\mathbf{B}^\pm(\nabla^\varphi\cdot\mathbf{V}^\pm)\,\mathrm{d}\mathcal{V}_t + \int_{\Omega^\pm}\mathbf{P}^\pm(\nabla^\varphi\cdot\mathbf{V}^\pm)\,\mathrm{d}\mathcal{V}_t \\
&\quad \pm\int_\Sigma \mathbf{Q}^\pm(\mathbf{V}^\pm\cdot N)\,\mathrm{d}x' + R_1^\pm + \int_{\Omega^\pm}\mathcal{R}_q^\pm(\nabla^\varphi\cdot\mathbf{V}^\pm)\,\mathrm{d}\mathcal{V}_t.
\end{aligned}
\tag{3.27}
$$

Invoking the equation (3.13) for the evolution of $\mathbf{B}$ in the first integral above, the energy of $\mathbf{B}^\pm$ is produced.

$$
\begin{aligned}
&-\int_{\Omega^\pm}\mathbf{B}^\pm\cdot(b^\pm\cdot\nabla^\varphi)\mathbf{V}^\pm\,\mathrm{d}\mathcal{V}_t \\
&= -\int_{\Omega^\pm}\mathbf{B}^\pm\cdot D_t^{\varphi\pm}\mathbf{B}^\pm\,\mathrm{d}\mathcal{V}_t - \int_{\Omega^\pm}(\mathbf{B}^\pm\cdot b^\pm)(\nabla^\varphi\cdot\mathbf{V}^\pm)\,\mathrm{d}\mathcal{V}_t + \int_{\Omega^\pm}\mathbf{B}^\pm\cdot\mathcal{R}_b^\pm\,\mathrm{d}\mathcal{V}_t - \int_{\Omega^\pm}(\mathbf{B}^\pm\cdot b^\pm)\mathfrak{C}_i(v_i^\pm)\,\mathrm{d}\mathcal{V}_t \\
&= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}|\mathbf{B}^\pm|^2\,\mathrm{d}\mathcal{V}_t \underbrace{-\frac{1}{2}\int_{\Omega^\pm}(\nabla^\varphi\cdot v^\pm)|\mathbf{B}^\pm|^2\,\mathrm{d}\mathcal{V}_t + \int_{\Omega^\pm}\mathbf{B}^\pm\cdot\mathcal{R}_b^\pm\,\mathrm{d}\mathcal{V}_t}_{R_2^\pm} \\
&\quad -\int_{\Omega^\pm}(\mathbf{B}^\pm\cdot b^\pm)(\nabla^\varphi\cdot\mathbf{V}^\pm)\,\mathrm{d}\mathcal{V}_t - \int_{\Omega^\pm}(\mathbf{B}^\pm\cdot b^\pm)\mathfrak{C}_i(v_i^\pm)\,\mathrm{d}\mathcal{V}_t,
\end{aligned}
\tag{3.28}
$$

where the first term in the last line is cancelled with the second integral in (3.27), and the analysis of the second term in the last line will be postponed.

The third term in (3.27) produces the energy of $(\mathcal{F}_p^\pm)^{\frac{1}{2}}\mathbf{P}^\pm$ with the help of equation (3.12).

$$
\begin{aligned}
&\int_{\Omega^\pm}\mathbf{P}^\pm(\nabla^\varphi\cdot\mathbf{V}^\pm)\,\mathrm{d}\mathcal{V}_t \\
&= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}\mathcal{F}_p^\pm(\mathbf{P}^\pm)^2\,\mathrm{d}\mathcal{V}_t \underbrace{-\frac{1}{2}\int_{\Omega^\pm}(D_t^{\varphi\pm}\mathcal{F}_p^\pm+\mathcal{F}_p^\pm\nabla^\varphi\cdot v^\pm)|\mathbf{P}^\pm|^2\,\mathrm{d}\mathcal{V}_t + \int_{\Omega^\pm}\mathbf{P}^\pm\mathcal{R}_p^\pm\,\mathrm{d}\mathcal{V}_t}_{R_3^\pm} - \int_{\Omega^\pm}\mathbf{P}^\pm\mathfrak{C}_i(v_i^\pm)\,\mathrm{d}\mathcal{V}_t.
\end{aligned}
\tag{3.29}
$$

The last term in (3.27) can be controlled by inserting again the continuity equation and integrating $D_t^{\varphi\pm}$ by parts. We have

$$
\begin{aligned}
\int_{\Omega^\pm}\mathcal{R}_q^\pm(\nabla^\varphi\cdot\mathbf{V}^\pm)\,\mathrm{d}\mathcal{V}_t &= -\int_{\Omega^\pm}\mathcal{F}_p^\pm\mathcal{R}_q^\pm D_t^{\varphi\pm}\mathbf{P}^\pm\,\mathrm{d}\mathcal{V}_t + \int_{\Omega^\pm}\mathcal{R}_q^\pm\mathcal{R}_p^\pm\,\mathrm{d}\mathcal{V}_t - \int_{\Omega^\pm}\mathcal{R}_q^\pm\mathfrak{C}_i(v_i^\pm)\,\mathrm{d}\mathcal{V}_t \\
&= -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}\left(\sqrt{\mathcal{F}_p^\pm}\mathcal{R}_q^\pm\right)\left(\sqrt{\mathcal{F}_p^\pm}\mathbf{P}^\pm\right)\,\mathrm{d}\mathcal{V}_t + \int_{\Omega^\pm}\left(\sqrt{\mathcal{F}_p^\pm}D_t^{\varphi\pm}\mathcal{R}_q^\pm\right)\left(\sqrt{\mathcal{F}_p^\pm}\mathbf{P}^\pm\right)\,\mathrm{d}\mathcal{V}_t + \int_{\Omega^\pm}\mathcal{R}_q^\pm\mathcal{R}_p^\pm\,\mathrm{d}\mathcal{V}_t \\
&\quad -\int_{\Omega^\pm}\mathcal{R}_q^\pm\mathfrak{C}_i(v_i^\pm)\,\mathrm{d}\mathcal{V}_t,
\end{aligned}
\tag{3.30}
$$

where the first term on the right side is controlled under time integral by

$$
\delta\left\|\sqrt{\mathcal{F}_p^\pm}\mathbf{P}^\pm(t)\right\|_0^2 + P(E_4^\kappa(0)) + \int_0^t P(E_4^\kappa(\tau))\,\mathrm{d}\tau,\ \ \forall 0<\delta\ll 1
$$

and the second term, the third term on the right side can be both controlled by $P(E_4(t))$ via direct computation because $\mathcal{R}_q$ only contains 3-rd order tangential derivative of b.

The entropy is directly bounded by testing the transport equation of $\mathbf{S}^\pm$ with $\mathbf{S}^\pm$ itself

$$
\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\Omega^\pm}\rho^\pm(\mathbf{S}^\pm)^2\,\mathrm{d}\mathcal{V}_t = \int_{\Omega^\pm}\rho^\pm\mathfrak{D}(S^\pm)\mathbf{S}^\pm\,\mathrm{d}\mathcal{V}_t \le \|\mathbf{S}^\pm\|_0\|\rho^\pm\|_{L^\infty}\sqrt{E_4^\kappa(t)}.
\tag{3.31}
$$

The remainder terms are controlled by direct computation. For the commutator $\mathfrak{C}, \mathfrak{D}$, we have $\|\mathfrak{C}(f^\pm)\|_{0,\pm} \lesssim C(|\psi|_4)\|f^\pm\|_{0,\pm}$ and $\|\mathfrak{D}(f^\pm)\|_{0,\pm} \lesssim C(|\psi|_4, |\partial_t\psi|_{L^\infty}|)\|f^\pm\|_{0,\pm}$ when $\mathcal{T}^\gamma = \overline{\partial}^4$ by straightforward computation. Note that the initial data is well-prepared in the sense that $\partial_t v|_{t=0} = O(1)$ with respect to Mach number, so there is no loss of ε-weight in $\mathcal{R}_v$ term. We have

$$R_1^\pm + R_2^\pm + R_3^\pm \leq P(E_4^\kappa(t)). \tag{3.32}$$

Step 2: The boundary regularity contributed by surface tension.

We denote $Z^\pm := -\int_{\Omega^\pm}(\mathbf{P}^\pm + b^\pm \cdot \mathbf{B}^\pm + \mathcal{R}_q^\pm)\mathfrak{C}_i(v_i^\pm)\,\mathrm{d}\mathcal{V}_t = -\int_{\Omega^\pm}\mathbf{Q}^\pm \mathfrak{C}_i(v_i^\pm)\,\mathrm{d}\mathcal{V}_t$ to be the remaining interior terms presented above which should be controlled together with some boundary terms involving $\mathcal{W}^\pm$. Now we analyze the boundary integral in (3.27). The sum of two boundary integrals can be written as

$$\begin{aligned}
&\int_\Sigma \mathbf{Q}^+(\mathbf{V}^+ \cdot N)\,\mathrm{d}x' - \int_\Sigma \mathbf{Q}^-(\mathbf{V}^- \cdot N)\,\mathrm{d}x' \\
&= \int_\Sigma (\overline{\partial}^4 q^+ - \overline{\partial}^4\psi\partial_3 q^+)(\partial_t\overline{\partial}^4\psi + (\bar{v}^+ \cdot \overline{\nabla})\overline{\partial}^4\psi - \mathcal{W}^+)\,\mathrm{d}x' \\
&\quad - \int_\Sigma (\overline{\partial}^4 q^- - \overline{\partial}^4\psi\partial_3 q^-)(\partial_t\overline{\partial}^4\psi + (\bar{v}^- \cdot \overline{\nabla})\overline{\partial}^4\psi - \mathcal{W}^-)\,\mathrm{d}x' \\
&= \int_\Sigma \overline{\partial}^4[\![q]\!]\,\overline{\partial}^4\partial_t\psi\,\mathrm{d}x' + \int_\Sigma \overline{\partial}^4[\![q]\!]\,(\bar{v}^+ \cdot \overline{\nabla})\overline{\partial}^4\psi\,\mathrm{d}x' + \int_\Sigma \overline{\partial}^4 q^-([\![\bar{v}]\!] \cdot \overline{\nabla})\overline{\partial}^4\psi\,\mathrm{d}x' \\
&\quad - \int_\Sigma [\![\partial_3 q]\!]\,\overline{\partial}^4\psi\partial_t\overline{\partial}^4\psi\,\mathrm{d}x' - \int_\Sigma \partial_3 q^+\overline{\partial}^4\psi(\bar{v}^+ \cdot \overline{\nabla})\overline{\partial}^4\psi\,\mathrm{d}x' + \int_\Sigma \partial_3 q^-\overline{\partial}^4\psi(\bar{v}^- \cdot \overline{\nabla})\overline{\partial}^4\psi\,\mathrm{d}x' \\
&\quad - \int_\Sigma \mathbf{Q}^+\mathcal{W}^+\,\mathrm{d}x' + \int_\Sigma \mathbf{Q}^-\mathcal{W}^-\,\mathrm{d}x' \\
&=: \mathrm{ST} + \mathrm{ST}' + \mathrm{VS} + \mathrm{RT} + \mathrm{RT}^+ + \mathrm{RT}^- + ZB^+ + ZB^-.
\end{aligned} \tag{3.33}$$

We will see that the term ST gives the $\sqrt{\sigma}$-weighted boundary regularity (contributed by surface tension) and the $\sqrt{\kappa}$-weighted boundary regularity (contributed by the two regularization terms) which help us control the terms ST$'$, VS, RT, RT$^\pm$. The terms $ZB^\pm$ will be controlled together with $Z^\pm$ by using Gauss-Green formula. Do note that the slip conditions imply $\mathbf{V}_3^\pm = \mathbf{B}_3^\pm = \overline{\partial}\psi = 0$ on $\Sigma^\pm$, which eliminates all integrals on $\Sigma^\pm$.

Inserting the jump condition $[\![q]\!] = \sigma\mathcal{H} - \kappa(1 - \overline{\Delta})^2\psi - \kappa(1 - \overline{\Delta})\psi_t$ into the term ST, we get

$$\begin{aligned}
\mathrm{ST} &= \sigma\int_\Sigma \overline{\partial}^4\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\psi}{\sqrt{1 + |\overline{\nabla}\psi|^2}}\right)\overline{\partial}^4\partial_t\psi\,\mathrm{d}x' - \int_\Sigma \kappa(1 - \overline{\Delta})^2\overline{\partial}^4\psi\overline{\partial}^4\partial_t\psi\,\mathrm{d}x' - \int_\Sigma \kappa(1 - \overline{\Delta})\overline{\partial}^4\partial_t\psi\overline{\partial}^4\partial_t\psi\,\mathrm{d}x' \\
&= \sigma\int_\Sigma \overline{\partial}^4\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\psi}{\sqrt{1 + |\overline{\nabla}\psi|^2}}\right)\overline{\partial}^4\partial_t\psi\,\mathrm{d}x' - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma\left|\sqrt{\kappa}\langle\overline{\partial}\rangle^2\overline{\partial}^4\psi\right|^2\,\mathrm{d}x' - \int_\Sigma\left|\sqrt{\kappa}\overline{\partial}^4\langle\overline{\partial}\rangle\psi_t\right|^2\,\mathrm{d}x'.
\end{aligned} \tag{3.34}$$

Integrating by parts in the mean curvature term and using

$$\overline{\partial}\left(\frac{1}{|N|}\right) = \frac{\overline{\nabla}\psi\cdot\overline{\nabla}\partial\psi}{|N|^3}, \quad |N| = \sqrt{1 + |\overline{\nabla}\psi|^2},$$

we get

$$\begin{aligned}
&\sigma\int_\Sigma \overline{\partial}^4\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\psi}{\sqrt{1 + |\overline{\nabla}\psi|^2}}\right)\overline{\partial}^4\partial_t\psi\,\mathrm{d}x' \\
&= -\sigma\int_\Sigma \frac{\overline{\partial}^4\overline{\nabla}\psi}{|N|}\cdot\partial_t\overline{\nabla}\overline{\partial}^4\psi\,\mathrm{d}x' + \sigma\int_\Sigma \frac{\overline{\partial}\psi\cdot\overline{\nabla}\overline{\partial}^4\psi}{|N|^3}\overline{\nabla}\psi\cdot\partial_t\overline{\nabla}\overline{\partial}^4\psi\,\mathrm{d}x' \\
&\quad \underbrace{-\sigma\int_\Sigma\left(\left[\overline{\partial}^3, \frac{1}{|N|}\right]\overline{\partial}\overline{\nabla}_i\psi + \left[\overline{\partial}^3, \frac{1}{|N|^3}\right](\overline{\nabla}_k\psi\cdot\overline{\partial}\overline{\nabla}_k\psi\overline{\nabla}_i\psi) - \frac{1}{|N|^3}[\overline{\partial}^3, \overline{\nabla}_i\psi\overline{\nabla}_k\psi]\overline{\partial}\overline{\nabla}_k\psi\right)\cdot\partial_t\overline{\nabla}_i\overline{\partial}^4\psi\,\mathrm{d}x'}_{=:\ \mathrm{ST}_1^R}
\end{aligned} \tag{3.35}$$

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which is further equal to

$$-\frac{\sigma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma \frac{|\overline{\partial}^4\overline{\nabla}\psi|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}} - \frac{|\overline{\nabla}\psi\cdot\overline{\partial}^4\overline{\nabla}\psi|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}^3}\,\mathrm{d}x'$$

$$+\underbrace{\frac{\sigma}{2}\int_\Sigma \partial_t\left(\frac{1}{|N|}\right)\left|\overline{\partial}^4\overline{\nabla}\psi\right|^2 - \partial_t\left(\frac{1}{|N|^3}\right)\left|\overline{\nabla}\psi\cdot\overline{\nabla}\overline{\partial}^4\psi\right|^2\,\mathrm{d}x'}_{=:\,\mathrm{ST}_2^R} + \mathrm{ST}_1^R \tag{3.36}$$

The control of ST_1^R, ST_2^R is straightforward which has been analyzed in the author's previous paper [38, (4.77)-(4.78)], so we only record the result here

$$\mathrm{ST}_1^R + \mathrm{ST}_2^R \lesssim P(|\overline{\nabla}\psi|_{L^\infty})|\overline{\nabla}\psi|_{W^{1,\infty}}\left|\sqrt{\sigma}\overline{\partial}^4\overline{\nabla}\psi\right|_0\left|\sqrt{\sigma}\partial_t\overline{\partial}^4\psi\right|_0 \le P(E_4^\kappa(t)).$$

Using Cauchy's inequality

$$\forall \mathbf{a}\in\mathbb{R}^2,\quad \frac{|\mathbf{a}|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}} - \frac{|\overline{\nabla}\psi\cdot\mathbf{a}|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}^3} \ge \frac{|\mathbf{a}|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}^3}, \tag{3.37}$$

we obtain the $\sqrt{\sigma}$-weighted boundary regularity

$$\int_0^t \mathrm{ST}\,\mathrm{d}\tau + \frac{\sigma}{2}\int_\Sigma \frac{|\overline{\nabla}\overline{\partial}^4\psi|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}^3}\,\mathrm{d}x' + \int_\Sigma\left|\sqrt{\kappa}\langle\overline{\partial}\rangle^2\overline{\partial}^4\psi\right|^2\,\mathrm{d}x' + \int_0^t\int_\Sigma\left|\sqrt{\kappa}\overline{\partial}^4\langle\overline{\partial}\rangle\psi_t\right|^2\,\mathrm{d}x'\,\mathrm{d}\tau$$

$$\le \int_0^t \mathrm{ST}_1^R + \mathrm{ST}_2^R\,\mathrm{d}x' \le \int_0^t P(E_4^\kappa(\tau))\,\mathrm{d}\tau. \tag{3.38}$$

So far, we already obtain the boundary regularity $\sqrt{\sigma}\psi\in H^5(\Sigma)$, $\sqrt{\kappa}\psi\in H^6(\Sigma)$ and $\sqrt{\kappa}\psi_t\in L_t^2 H_{x'}^5([0,T]\times\Sigma)$. Using this, we can easily control ST' term in (3.33). Invoking again the boundary condition for $[\![q]\!]$, we get

$$\mathrm{ST}' = \int_\Sigma \sigma\mathcal{H}(\bar{v}^+\cdot\overline{\nabla})\overline{\partial}^4\psi\,\mathrm{d}x' - \kappa\int_\Sigma(1-\overline{\Delta})^2\overline{\partial}^4\psi\,(\bar{v}^+\cdot\overline{\nabla})\overline{\partial}^4\psi\,\mathrm{d}x' - \kappa\int_\Sigma(1-\overline{\Delta})\overline{\partial}^4\psi_t(\bar{v}^+\cdot\overline{\nabla})\overline{\partial}^4\psi\,\mathrm{d}x'. \tag{3.39}$$

Integrating by parts $1-\overline{\Delta}$ in the second term and $\langle\overline{\partial}\rangle = \sqrt{1-\overline{\Delta}}$ in the third term above, we can easily use the $\sqrt{\kappa}$-weighted energy to control the last two terms.

$$-\kappa\int_\Sigma(1-\overline{\Delta})^2\overline{\partial}^4\psi\,(\bar{v}^+\cdot\overline{\nabla})\overline{\partial}^4\psi\,\mathrm{d}x'$$

$$= -\kappa\int_\Sigma\left((1-\overline{\Delta})\overline{\partial}^4\psi\right)(\bar{v}^+\cdot\overline{\nabla})(1-\overline{\Delta})\overline{\partial}^4\psi\,\mathrm{d}x' - \kappa\int_\Sigma\left((1-\overline{\Delta})\overline{\partial}^4\psi\right)[1-\overline{\Delta},\bar{v}^+\cdot\overline{\nabla}]\overline{\partial}^4\psi\,\mathrm{d}x', \tag{3.40}$$

where the first term is controlled by $|\bar{v}^+|_{W^{1,\infty}}|\sqrt{\kappa}(1-\overline{\Delta})\overline{\partial}^4\psi|_0^2$ after integrating $\bar{v}^+\cdot\overline{\nabla}$ by parts and using the symmetry, and the second term is directly controlled by $|\bar{v}^+|_{W^{2,\infty}}|\sqrt{\kappa}(1-\overline{\Delta})\overline{\partial}^4\psi|_0|\sqrt{\kappa}\overline{\partial}^4\psi|_2$. Similarly, we have for any $\delta\in(0,1)$

$$-\kappa\int_0^t\int_\Sigma(1-\overline{\Delta})\overline{\partial}^4\psi_t(\bar{v}^+\cdot\overline{\nabla})\overline{\partial}^4\psi\,\mathrm{d}x'\,\mathrm{d}\tau = -\kappa\int_0^t\int_\Sigma\langle\overline{\partial}\rangle\overline{\partial}^4\psi_t\,\langle\overline{\partial}\rangle((\bar{v}^+\cdot\overline{\nabla})\overline{\partial}^4\psi)\,\mathrm{d}x'\,\mathrm{d}\tau$$

$$\le \delta\left|\sqrt{\kappa}\langle\overline{\partial}\rangle\overline{\partial}^4\psi_t\right|_{L_t^2 L_{x'}^2}^2 + \frac{1}{4\delta}\int_0^t\int_\Sigma|\bar{v}^+|_{W^{1,\infty}}^2\left|\sqrt{\kappa}\overline{\partial}^4\psi\right|_2^2\,\mathrm{d}x'\,\mathrm{d}\tau \le \delta E_4^\kappa(t) + \int_0^t P(E_4^\kappa(\tau))\,\mathrm{d}\tau. \tag{3.41}$$

Picking $\delta>0$ to be sufficiently small, the δ-term can be absorbed by $E_4^\kappa(t)$. The first term in ST' is controlled in the same way if we integrating $\overline{\nabla}\cdot$ by parts. Here we only list the result and refer the details to [38, (4.87)-(4.89)]

$$\int_\Sigma \sigma\mathcal{H}(\bar{v}^+\cdot\overline{\nabla})\overline{\partial}^4\psi\,\mathrm{d}x' \le P(|\overline{\nabla}\psi|_{W^{1,\infty}}|\bar{v}^+|_{W^{1,\infty}})\left|\sqrt{\sigma}\overline{\nabla}\overline{\partial}^4\psi\right|_0^2 \le P(E_4^\kappa(t)). \tag{3.42}$$

Next we control the terms RT and $\mathrm{RT}^\pm$ in (3.33). Note that we do not have the Rayleight-Taylor sign condition $[\![\partial_3 q]\!]|_\Sigma \ge c_0 > 0$, so we have to use the $\sqrt{\sigma}$-weighted energy to control these terms, we have

$$\mathrm{RT} \le |\partial_3 q|_{L^\infty}|\psi|_4|\psi_t|_4 \le \sigma^{-1}P(E_4^\kappa(t)). \tag{3.43}$$

Similarly, integrating $\bar{v}^\pm\cdot\nabla$ by parts in $\mathrm{RT}^\pm$ and using symmetry, the terms $\mathrm{RT}^\pm$ can be directly controlled

$$\mathrm{RT}^\pm \le |\bar{v}^\pm\partial_3 q|_{W^{1,\infty}}|\psi|_4^2 \le \sigma^{-1}P(E_4^\kappa(t)). \tag{3.44}$$

Step 3: The crucial term for vortex sheets problem.

Now we study the term VS in (3.33) which appears to be the most problematic term for the vortex sheets problem. Note that we do not have any boundary condition for $q^\pm$ individually. Thus, we may alternatively integrate $\overline{\partial}^{1/2}$ by parts and use (B.5) to control VS.

$$\text{VS} = \int_\Sigma \overline{\partial}^4 q^- (\llbracket \overline{v} \rrbracket \cdot \overline{\nabla}) \overline{\partial}^4 \psi \, dx' \lesssim \|\overline{\partial}^4 q^-\|_{0,-}^{\frac{1}{2}} \|\partial_3 \overline{\partial}^3 q^-\|_{0,-}^{\frac{1}{2}} \|\overline{v}^\pm\|_{2,\pm} |\overline{\nabla}\overline{\partial}^4 \psi|_{1/2} \leq P(E_4^\kappa(t)) |\psi|_{5.5}, \tag{3.45}$$

where we have used the Kato-Ponce inequality (cf. Lemma B.6) for $s = 1/2$, $p_1 = 2$, $p_2 = \infty$, $q_1 = q_2 = 4$ and Sobolev embedding $H^{1/2}(\mathbb{T}^2) \hookrightarrow L^4(\mathbb{T}^2)$. Now we need to control $|\psi|_{5.5}$ via the jump condition of $\llbracket q \rrbracket$. Without the κ-regularization terms, we may use the ellipticity of the mean curvature operator to control $|\psi|_{5.5}$ by $\sigma^{-1} |\llbracket q \rrbracket|_{3.5}$. Now, we can still prove analogous result for the κ-regularized jump condition.

Lemma 3.5 (Elliptic estimate for the free interface). *For any $s \geq 0.5$ and $\kappa > 0$, we have the uniform-in-κ estimate*

$$|\psi|_{s+1.5} \leq |\psi_0|_{s+1.5} + \sigma^{-1} \left(P(|\overline{\nabla}\psi|_{L^\infty}) |\overline{\nabla}\psi|_{W^{1,\infty}} |\overline{\partial}\psi|_{s-0.5} + |\llbracket q \rrbracket|_{s-0.5} \right).$$

Moreover, when $\kappa = 0$, $|\psi_0|_{s+1.5}$ is not needed

$$|\sigma\psi|_{s+1.5} \leq P(|\overline{\nabla}\psi|_{L^\infty}) |\overline{\nabla}\psi|_{W^{1,\infty}} |\sigma\overline{\partial}\psi|_{s-0.5} + |\llbracket q \rrbracket|_{s-0.5}. \tag{3.46}$$

Proof. We take $\langle\overline{\partial}\rangle^{s+0.5}$ in the jump condition to get

$$-\langle\overline{\partial}\rangle^{s+0.5} \llbracket q \rrbracket = -\sigma\langle\overline{\partial}\rangle^{s+0.5}\overline{\nabla} \cdot \left(\frac{\overline{\nabla}\psi}{\sqrt{1 + |\overline{\nabla}\psi|^2}} \right) + \kappa(1 - \overline{\Delta})^2 \langle\overline{\partial}\rangle^{s+0.5}\psi + \kappa(1 - \overline{\Delta})\langle\overline{\partial}\rangle^{s+0.5}\psi_t.$$

Testing this equation with $\langle\overline{\partial}\rangle^{s+0.5}\psi$ in $L^2(\Sigma)$, we get

$$-\int_\Sigma \langle\overline{\partial}\rangle^{s+0.5} \llbracket q \rrbracket \ \langle\overline{\partial}\rangle^{s+0.5}\psi \, dx' \leq |\langle\overline{\partial}\rangle^{s-0.5} \llbracket q \rrbracket|_0 |\langle\overline{\partial}\rangle^{s+1.5}\psi|_0.$$

For the right side, we can mimic the treatment of ST term to obtain the boundary regularity. The two regularization terms can be directly controlled

$$\int_\Sigma \kappa(1 - \overline{\Delta})^2 \langle\overline{\partial}\rangle^{s+0.5}\psi \ \langle\overline{\partial}\rangle^{s+0.5}\psi \, dx' = \int_\Sigma \kappa(1 - \overline{\Delta})\langle\overline{\partial}\rangle^{s+0.5}\psi \ (1 - \overline{\Delta})\langle\overline{\partial}\rangle^{s+0.5}\psi \, dx' = \left| \sqrt{\kappa}\psi \right|_{s+2.5}^2,$$

$$\int_\Sigma \kappa(1 - \overline{\Delta})\langle\overline{\partial}\rangle^{s+0.5}\psi_t \ \langle\overline{\partial}\rangle^{s+0.5}\psi \, dx' \stackrel{\langle\overline{\partial}\rangle}{=} \frac{d}{dt} \left| \sqrt{\kappa}\psi \right|_{s+1.5}^2.$$

The term involving surface tension is controlled as follows

$$-\sigma \int_\Sigma \langle\overline{\partial}\rangle^{s+0.5}\overline{\nabla} \cdot \left(\frac{\overline{\nabla}\psi}{\sqrt{1 + |\overline{\nabla}\psi|^2}} \right) \langle\overline{\partial}\rangle^{s+0.5}\psi \, dx' = \sigma \int_\Sigma \langle\overline{\partial}\rangle^{s+0.5} \left(\frac{\overline{\nabla}\psi}{\sqrt{1 + |\overline{\nabla}\psi|^2}} \right) \cdot \langle\overline{\partial}\rangle^{s+0.5}\overline{\nabla}\psi \, dx'$$

$$= \sigma \int_\Sigma \frac{|\langle\overline{\partial}\rangle^{s+0.5}\overline{\nabla}\psi|^2}{\sqrt{1 + |\overline{\nabla}\psi|^2}} - \frac{|\overline{\nabla}\psi \cdot \langle\overline{\partial}\rangle^{s+0.5}\overline{\nabla}\psi|^2}{\sqrt{1 + |\overline{\nabla}\psi|^2}^3} \, dx'$$

$$+ \sigma \int_\Sigma \left(\left[\langle\overline{\partial}\rangle^{s-0.5}, \frac{1}{|N|} \right] \langle\overline{\partial}\rangle\overline{\nabla}_i\psi + \left[\langle\overline{\partial}\rangle^{s-0.5}, \frac{1}{|N|^3} \right] (\overline{\nabla}_k\psi \cdot \langle\overline{\partial}\rangle\overline{\nabla}_k\psi\overline{\nabla}_i\psi) - \frac{1}{|N|^3}[\langle\overline{\partial}\rangle^{s-0.5}, \overline{\nabla}\psi]\langle\overline{\partial}\rangle\overline{\nabla}_i\psi \right) \cdot \overline{\nabla}_i\langle\overline{\partial}\rangle^{s+0.5}\psi \, dx'$$

Using Kato-Ponce commutator estimate (cf. (B.8) in Lemma B.6), the commutators in the last line of the above identity are controlled by $P(|\overline{\nabla}\psi|_{L^\infty}) |\overline{\nabla}\psi|_{W^{1,\infty}} |\overline{\partial}\psi|_{s-0.5}$. Using again Cauchy's inequality (3.37), we conclude the elliptic estimate by

$$\sigma|\psi|_{s+1.5}^2 + \kappa|\psi|_{s+2.5}^2 + \kappa\frac{d}{dt}|\psi|_{s+1.5}^2 \leq \left(P(|\overline{\nabla}\psi|_{L^\infty}) |\overline{\nabla}\psi|_{W^{1,\infty}} |\sigma\overline{\partial}\psi|_{s-0.5} + |\llbracket q \rrbracket|_{s-0.5} \right) |\psi|_{s+1.5}.$$

In particular, Lemma B.7 suggests that we have

$$|\psi|_{s+1.5} \le |\psi_0|_{s+1.5} + \sigma^{-1}\left(P(|\overline{\nabla}\psi|_{L^\infty})|\overline{\nabla}\psi|_{W^{1,\infty}}|\sigma\overline{\partial}\psi|_{s-0.5} + |\, [\![q]\!]\,|_{s-0.5}\right).$$

Moreover, when $\kappa = 0$, $|\psi_0|_{s+1.5}$ no longer appears as we do not need Lemma B.7

$$|\sigma\psi|_{s+1.5} \le P(|\overline{\nabla}\psi|_{L^\infty})|\overline{\nabla}\psi|_{W^{1,\infty}}|\sigma\overline{\partial}\psi|_{s-0.5} + |\, [\![q]\!]\,|_{s-0.5}. \tag{3.47}$$

$$\square$$

Now we can easily obtain the control for the problematic term VS by setting $s = 4$ in Lemma 3.5

$$\text{VS} \lesssim |\psi_0|_{5.5} + \sigma^{-1}P(E_4^\kappa(t)). \tag{3.48}$$

Step 4: A cancellation structure for the incompressible limit.

It remains to control the term $Z^\pm$ and $ZB^\pm$. In $\overline{\partial}^4$-estimates, each of these terms can be directly controlled. However, in the control of $E_8^\kappa(t)$ and the control of full time derivatives, there will be extra technical difficulties due to the loss of Mach number or the anisotropy of the function spaces. Thus, we would like to present a robust approach to control these terms. We take $Z^- + ZB^-$ as an example and the "+" case is controlled in the same way by reversing the sign when integrating by parts. Recall that $\mathbf{Q}^- = \overline{\partial}^4 q^- - \overline{\partial}^4\varphi\partial_3^\varphi q^-$, so we have

$$ZB^- = \int_\Sigma \overline{\partial}^4 q^-(\partial_3 v^- \cdot N)\overline{\partial}^4\psi \, dx' - \int_\Sigma \overline{\partial}^4\psi\partial_3 q^-(\partial_3 v^- \cdot N)\overline{\partial}^4\psi \, dx' + \sum_{k=1}^3 \int_\Sigma \binom{4}{k}\mathbf{Q}^-\overline{\partial}^{4-k}v^- \cdot \overline{\partial}^k N \, dx'. \tag{3.49}$$

The first two terms in ZB^- can be directly controlled

$$\int_\Sigma \overline{\partial}^4 q^-(\partial_3 v^- \cdot N)\overline{\partial}^4\psi \, dx' - \int_\Sigma \overline{\partial}^4\psi\partial_3 q^-(\partial_3 v^- \cdot N)\overline{\partial}^4\psi \, dx'$$
$$\le \left(|\overline{\partial}^{7/2}q^-|_0|\psi|_{4.5} + |\psi|_4^2\right)|\partial_3 v \cdot N|_{1.5} \le \left(\|q^-\|_{4,-}|\psi|_{4.5} + |\psi|_4^2\right)\|\partial v^-\|_{2,-}|\psi|_{2.5} \le P(\sigma^{-1}, E_4^\kappa(t)).$$

The last term in ZB^- is controlled together with $Z^- := -\int_{\Omega^-} \mathbf{Q}^-\mathfrak{C}_i(v_i^-) \, d\mathcal{V}_t$. Recall that

$$\mathfrak{C}_i(v_i^-) = (\partial_3^\varphi\partial_i^\varphi v_i^-)\overline{\partial}^4\varphi - \left[\overline{\partial}^4, \frac{\partial_i\varphi}{\partial_3\varphi}, \partial_3 v_i^-\right] - \partial_3 v_i^-\left[\overline{\partial}^4, \partial_i\varphi, \frac{1}{\partial_3\varphi}\right] + \partial_i\varphi\partial_3 v_i^-\left[\overline{\partial}^3, \frac{1}{(\partial_3\varphi)^2}\right]\overline{\partial}\partial_3\varphi, \ i = 1, 2$$

and

$$\mathfrak{C}_3(f) = (\partial_3^\varphi)^2 v_3^-\overline{\partial}^4\varphi + \left[\overline{\partial}^4, \frac{1}{\partial_3\varphi}, \partial_3 v_3^-\right] - \partial_3 v_3^-\left[\overline{\partial}^3, \frac{1}{(\partial_3\varphi)^2}\right]\overline{\partial}\partial_3\varphi.$$

Note that $\mathbf{N}_i = -\partial_i\varphi$ for $i = 1, 2$, so we have

$$-\left[\overline{\partial}^4, \frac{\partial_i\varphi}{\partial_3\varphi}, \partial_3 v_i^-\right] = \left[\overline{\partial}^4, \frac{\mathbf{N}_i}{\partial_3\varphi}, \partial_3 v_i^-\right] = \sum_{k=1}^3 \binom{4}{k}\overline{\partial}^k\left(\frac{\mathbf{N}_i}{\partial_3\varphi}\right)\overline{\partial}^{4-k}\partial_3 v_i^-$$

$$= \sum_{k=1}^3 \binom{4}{k}\overline{\partial}^k\mathbf{N}_i\partial_3^\varphi\overline{\partial}^{4-k}v_i^- - \binom{4}{k}\left[\overline{\partial}^k, \frac{1}{\partial_3\varphi}\right]\partial_i\varphi\partial_3\overline{\partial}^{4-k}v_i^-,$$

where the contribution of the first term above gives us (using Gauss-Green formula)

$$
\begin{aligned}
ZB^- &- \sum_{k=1}^{3} \int_{\Omega^-} \binom{4}{k} \mathbf{Q}^- \overline{\partial}^k \mathbf{N}_i \partial_3^\varphi \overline{\partial}^{4-k} v_i^- \, \mathrm{d}\mathcal{V}_t \\
&= \sum_{k=1}^{3} \binom{4}{k} \left(\int_\Sigma \mathbf{Q}^- \overline{\partial}^{4-k} v^- \cdot \overline{\partial}^k N \, \mathrm{d}x' - \int_{\Omega^-} \mathbf{Q}^- \overline{\partial}^k \mathbf{N}_i \partial_3 \overline{\partial}^{4-k} v_i^- \, \mathrm{d}x \right) \\
&= \sum_{k=1}^{3} \binom{4}{k} \left(\int_{\Omega^-} \partial_3 \mathbf{Q}^- \overline{\partial}^k \mathbf{N}_i \overline{\partial}^{4-k} v_i^- \, \mathrm{d}x + \int_{\Omega^-} \mathbf{Q}^- \overline{\partial}^k \mathbf{N}_i \partial_3 \overline{\partial}^{4-k} v_i^- \, \mathrm{d}x - \int_{\Omega^-} \mathbf{Q}^- \overline{\partial}^k \mathbf{N}_i \partial_3 \overline{\partial}^{4-k} v_i^- \, \mathrm{d}x \right) \\
&= \sum_{k=1}^{3} \binom{4}{k} \int_{\Omega^-} \partial_3 \mathbf{Q}^- \overline{\partial}^k \mathbf{N}_i \overline{\partial}^{4-k} v_i^- \, \mathrm{d}x.
\end{aligned}
\tag{3.50}
$$

Now invoking $\mathbf{Q}^- = \overline{\partial}^4 q^- - \overline{\partial}^4 \varphi \partial_3^\varphi q^-$ and integrating one $\overline{\partial}$ by parts, we find that

$$
\sum_{k=1}^{3} \binom{4}{k} \int_{\Omega^-} \partial_3 \mathbf{Q}^- \overline{\partial}^k \mathbf{N}_i \overline{\partial}^{4-k} v_i^- \, \mathrm{d}x \lesssim (\|\overline{\partial}^3 \partial_3 q^-\|_{0,-} + |\psi|_4 \|\partial_3 q^-\|_{L^\infty(\Omega^-)}) |\psi|_4 \|v_i^-\|_{4,-}.
\tag{3.51}
$$

Among other terms in $\mathfrak{C}_i(v_i^-)$, we shall focus on the case when there are 4 derivatives falling on v_i^- and φ, and the control of these terms (lised below) appears to be easier.

$$
-\int_{\Omega^-} \mathbf{Q}^- \overline{\partial}^4 \varphi \partial_3^\varphi (\nabla^\varphi \cdot v^-) \, \mathrm{d}\mathcal{V}_t \quad \text{from the first term in } \mathfrak{C}_i(v_i^-)
$$

$$
4 \sum_{i=1}^{3} \int_{\Omega^-} \mathbf{Q}^- \partial_3 \overline{\partial} \varphi \, \partial_3^\varphi \overline{\partial}^3 v^- \cdot \mathbf{N} \, \mathrm{d}x \quad \text{from the second term in } \mathfrak{C}_i(v_i^-) \text{ when } \overline{\partial}^3 \text{ falls on } \partial_3 v_i^-.
\tag{3.52}
$$

Note that $\partial_3^\varphi v^- \cdot N = \nabla^\varphi \cdot v^- - \overline{\nabla} \cdot \overline{v}^-$, we have

$$
-\int_{\Omega^-} \mathbf{Q}^- \overline{\partial}^4 \varphi \partial_3^\varphi (\nabla^\varphi \cdot v) \, \mathrm{d}\mathcal{V}_t \lesssim \left\| \sqrt{\mathcal{F}_p^-} \mathbf{Q}^- \right\|_{0,-} \left\| \sqrt{\mathcal{F}_p^-} \partial_3 D_t^{\varphi,-} p^- \right\|_{0,-} |\psi|_4,
\tag{3.53}
$$

and

$$
\begin{aligned}
4 \int_{\Omega^-} \mathbf{Q}^- \partial_3 \overline{\partial} \varphi \, \partial_3^\varphi \overline{\partial}^3 v^- \cdot \mathbf{N} \, \mathrm{d}x &\overset{L}{=} 4 \int_{\Omega^-} \mathbf{Q}^- \partial_3 \overline{\partial} \varphi \, \overline{\partial}^3 (\nabla^\varphi \cdot v^-) \, \mathrm{d}x - 4 \int_{\Omega^-} \mathbf{Q}^- \partial_3 \overline{\partial} \varphi \, \overline{\partial}^3 (\overline{\nabla} \cdot \overline{v}^-) \, \mathrm{d}x \\
&\lesssim |\overline{\partial} \psi|_{L^\infty} \left(\left\| \sqrt{\mathcal{F}_p^-} \mathbf{Q}^- \right\|_{0,-} \left\| \sqrt{\mathcal{F}_p^-} \overline{\partial}^3 D_t^{\varphi,-} p^- \right\|_{0,-} + \|\overline{\nabla} \overline{\partial}^3 \mathbf{Q}^-\|_{0,-} \|\overline{\partial}^4 v^-\|_{0,-} \right).
\end{aligned}
\tag{3.54}
$$

Thus, combining the estimates in the above four steps, we conclude the $\overline{\partial}^4$-estimate by: For the tangential derivative $\mathcal{T}^\gamma = \overline{\partial}^4$ ($\gamma_0 = \gamma_4 = 0$, $\gamma_1 + \gamma_2 = 4$) and for any $0 < \delta < 1$, we have

$$
\begin{aligned}
&\left\| \left(\mathbf{V}^{\gamma,\pm}, \mathbf{B}^{\gamma,\pm}, \mathbf{S}^{\gamma,\pm}, \sqrt{\mathcal{F}_p^\pm} \mathbf{P}^{\gamma,\pm} \right)(t) \right\|_0^2 + \left| \sqrt{\sigma} \varepsilon^{2l} \overline{\partial}^{4+l} \overline{\nabla} \psi(t) \right|_0^2 + \left| \sqrt{\kappa} \varepsilon^{2l} \overline{\partial}^{4+l} \psi(t) \right|_2^2 + \int_0^t \left| \sqrt{\kappa} \varepsilon^{2l} \overline{\partial}^{4+l} \partial_t \psi(\tau) \right|_1^2 \, \mathrm{d}\tau \\
&\lesssim \delta E_{4+l}^\kappa(t) + \left| \varepsilon^{2l} \psi_0 \right|_{5.5+l}^2 + \sum_{j=0}^{l} \int_0^t P(\sigma^{-1}, E_{4+j}^\kappa(\tau)) \, \mathrm{d}\tau, \quad 0 \leq l \leq 4.
\end{aligned}
\tag{3.55}
$$

Remark 3.2. It should be noted that we only have the L^2 control of $\mathbf{V}, \mathbf{B}, \mathbf{S}$ and $(\mathcal{F}_p)^{\frac{1}{2}} \mathbf{P}$ in the tangential estimates, but the term $\mathbf{Q}$ without $\mathcal{F}_p$-weight does appear in tangential estimates. When $\mathcal{T}^\gamma$ contains at least one spatial derivative, that is, $\gamma_0 < \langle \gamma \rangle$, one can invoke the momentum equation to replace $\mathcal{T}q$ by $D_t^\varphi v$ and $(b \cdot \nabla^\varphi) b$ to avoid the loss of Mach number. This also suggests that we can actually control $\|\mathbf{P}\|_0$ instead of only $\|\mathcal{F}_p^{1/2} \mathbf{P}\|_0$ when there is at least one spatial derivatives. However, when $\mathcal{T}^\gamma$ only consists of time derivatives, we cannot do such substitution any longer. Thus, we have to use the above cancellation structure between ZB and Z to control these two terms together.

3.3.2 The case $l > 0$: No loss of regularity or weights of Mach number

Next we consider the tangential estimates for ε-weighted spatial derivatives, namely $\varepsilon^{2l}\overline{\partial}^{4+l}$ for $1 \leq l \leq 4$. The proof is parallel to the case $\mathcal{T}^\gamma = \overline{\partial}^4$, but we have to check the following aspects

- a. We have to guarantee that there is no loss of $\mathcal{F}_p$-weight in various commutators, especially those involving q.
- b. When $l = 4$, we only have tangential regularity for 8 derivatives. Due to the anisotropy of the function space H_*^8, we have to put extra efforts to reduce the terms involving the derivative $\overline{\partial}^7 \partial_3$.

We only show the detailed modifications for the case $l = 4$, that is, the $\varepsilon^8 \overline{\partial}^8$-estimate. When $1 \leq l \leq 3$, similar modifications can be made in the same way.

Commutators of type $\varepsilon^8 [\overline{\partial}^8, f] \mathcal{T} g$ for $\mathcal{T} = \overline{\partial}$ or D_t^φ

This type of commutator includes the following terms

$$-[\mathcal{T}^\gamma, \rho] D_t^\varphi v \text{ in } \mathcal{R}_v, \quad -[\mathcal{T}^\gamma, \mathcal{F}_p] D_t^\varphi p \text{ in } \mathcal{R}_p,$$
$$[\mathcal{T}^\gamma, \bar{v}] \cdot \overline{\partial} f \text{ and } \partial_3^\varphi f [\mathcal{T}^\gamma, v] \cdot \mathbf{N} \text{ in } \mathfrak{D}^\gamma(f)$$

It is controlled directly by expanding the commutator. We have

$$\varepsilon^8 [\overline{\partial}^8, f] \mathcal{T} g = (\varepsilon^8 \overline{\partial}^8 f) \mathcal{T} g + 8(\varepsilon^6 \overline{\partial}^7 f)(\varepsilon^2 \overline{\partial} \mathcal{T} g) + 28(\varepsilon^6 \overline{\partial}^6 f)(\varepsilon^2 \overline{\partial}^2 \mathcal{T} g) + 56(\varepsilon^6 \overline{\partial}^5 f)(\varepsilon^2 \overline{\partial}^3 \mathcal{T} g)$$
$$+ 70(\varepsilon^2 \overline{\partial}^4 f)(\varepsilon^6 \overline{\partial}^4 \mathcal{T} g) + 56(\varepsilon^2 \overline{\partial}^3 f)(\varepsilon^6 \overline{\partial}^5 \mathcal{T} g) + 28(\varepsilon^2 \overline{\partial}^2 f)(\varepsilon^6 \overline{\partial}^6 \mathcal{T} g) + 8(\overline{\partial} f)(\varepsilon^8 \overline{\partial}^7 \mathcal{T} g),$$

whose $L^2(\Omega)$ norm is controlled by

$$\|\varepsilon^8 \overline{\partial}^8 f\|_0 \|\mathcal{T} g\|_{L^\infty} + 8\varepsilon^2 \|\varepsilon^6 \overline{\partial}^7 f\|_0 \|\overline{\partial} \mathcal{T} g\|_{L^\infty} + 28\|\varepsilon^6 \overline{\partial}^6 f\|_{L^6} \|\varepsilon^2 \overline{\partial}^2 \mathcal{T} g\|_{L^3} + 56\|\varepsilon^6 \overline{\partial}^5 f\|_{L^6} \|\varepsilon^2 \overline{\partial}^3 \mathcal{T} g\|_{L^3}$$
$$+ 70\|\varepsilon^2 \overline{\partial}^4 f\|_{L^3} \|\varepsilon^6 \overline{\partial}^4 \mathcal{T} g\|_{L^6} + 56\|\varepsilon^6 \overline{\partial}^3 f\|_{L^3} \|\varepsilon^2 \overline{\partial}^5 \mathcal{T} g\|_{L^6} + 28\|\varepsilon^6 \overline{\partial}^6 g\|_{L^6} \|\varepsilon^2 \overline{\partial}^2 \mathcal{T} f\|_{L^3} + 8\varepsilon^2 \|\varepsilon^6 \overline{\partial}^7 g\|_0 \|\overline{\partial} \mathcal{T} f\|_{L^\infty}$$
$$\lesssim (1 + \varepsilon^2) \left(\sqrt{E_8^\kappa(t) E_4^\kappa(t)} + \sqrt{E_7^\kappa(t) E_4^\kappa(t)} + \sqrt{E_7^\kappa(t) E_5^\kappa(t)} \right),$$

where we use the Sobolev embedding $H^1 \hookrightarrow L^6$ and $H^1 \hookrightarrow H^{1/2} \hookrightarrow L^3$ in 3D. In 2D case, we can replace (L^6, L^3) by (L^4, L^4) and use Ladyženskaya's inequality $\|f\|_{L^4}^2 \leq \|f\|_{L^2} \|\partial f\|_{L^2} \leq \|f\|_1^2$ to obtain the same bound.

Commutator $\varepsilon^8 [\overline{\partial}^8, b] \cdot \nabla^\varphi f$ for $f = b, v$

This term appears when we commute $\mathcal{T}^\gamma$ with $(b \cdot \nabla^\varphi)$. Note that we can rewrite the directional derivative to be $(b \cdot \nabla^\varphi) = \bar{b} \cdot \overline{\nabla} + (\partial_3 \varphi)^{-1}(b \cdot \mathbf{N})\partial_3$. When commuting $\overline{\partial}^8$ with $\bar{b} \cdot \overline{\nabla}$, the estimate is exactly the same as $\varepsilon^8 [\overline{\partial}^8, f] \mathcal{T} g$. For the commutator $[\overline{\partial}^8, (\partial_3 \varphi)^{-1}(b \cdot \mathbf{N})]\partial_3 f$, we just need to put extra effort on the term $8\overline{\partial}((\partial_3 \varphi)^{-1}(b \cdot \mathbf{N}))\overline{\partial}^7 \partial_3 f$ because the length of the multi-index exceeds 8 when $|x_3| \lesssim 1$. (Recall that the weight function $\omega(x_3)$ is comparable to $|x_3|$ when $x_3 \lesssim 1$ and is comparable to 1 when $|x_3| \gg 1$.) In this case, we notice that $b \cdot \mathbf{N}|_\Sigma = 0$, and thus its interior value can be expressed via the fundamental theorem of calculus

$$(\partial_3 \varphi)^{-1}(b \cdot \mathbf{N})(x', x_3) = 0 + \int_0^{x_3} \partial_3 \left((\partial_3 \varphi)^{-1}(b \cdot \mathbf{N})(x', \xi_3)\right) d\xi_3,$$

whose $L^\infty(\Omega)$ norm is controlled by $C\omega(x_3)\|\partial_3(b \cdot \mathbf{N})\|_{L^\infty(\Omega)}$.

Commutator $\mathfrak{D}(f)$ for $f = v, p, b, S$

Among all terms in (3.10), we need to further analyze the third term, that is, the commutator $\varepsilon^8 \left[\overline{\partial}^8, \frac{1}{\partial_3 \varphi}(v \cdot \mathbf{N} - \partial_t \varphi), \partial_3 f\right]$ for $f = v, b, p$. The problem is the same as above, that is, $\overline{\partial}^7$ may fall on $\partial_3 f$ which is not directly controllable. Again, we notice that there is only one $\overline{\partial}$ falling on $\frac{1}{\partial_3 \varphi}(v \cdot \mathbf{N} - \partial_t \varphi)$ and $(v \cdot \mathbf{N} - \partial_t \varphi)|_\Sigma = 0$, so we can use the same method (as in the control of $\varepsilon^8 [\overline{\partial}^8, b] \cdot \nabla^\varphi f$) to control this commutator.

Commutator $\mathfrak{C}(q)$

The problematic term is $-8(\partial_3\varphi)^{-1}(\overline{\partial}\mathbf{N}_i)(\overline{\partial}^7\partial_3 q)$ arising from $[\mathcal{T}^\gamma, \mathbf{N}_i/\partial_3\varphi, \partial_3 q]$. To control this term, we can invoke the third component of the momentum equation to convert $\partial_3 q$ to tangential derivatives of other quantities

$$-\partial_3 q = (\partial_3\varphi)\left(\rho D_t^\varphi v_3 - (b\cdot\nabla^\varphi)b_3\right),$$

where $D_t^\varphi = \partial_t + \bar{v}\cdot\overline{\nabla} + (\partial_3\varphi)^{-1}(v\cdot\mathbf{N} - \partial_t\varphi)\partial_3$ and $(b\cdot\nabla^\varphi) = \bar{b}\cdot\overline{\nabla} + (\partial_3\varphi)^{-1}(b\cdot\mathbf{N})\partial_3$ are both tangential derivatives. Also, there is no loss of weight of Mach number in this term because one can always replace $\overline{\partial}q$ by $D_t^\varphi v$ and $(b\cdot\nabla^\varphi)b$.

Commutator $\mathfrak{C}_i(v_i)$

The problematic term is $-8(\partial_3\varphi)^{-1}(\overline{\partial}\mathbf{N}_i)(\overline{\partial}^7\partial_3 v_i)$ arising from $[\mathcal{T}^\gamma, \mathbf{N}_i/\partial_3\varphi, \partial_3 v_i]$. In fact, this term may not be controlled independently, but its contribution only appears in $-\int_\Omega \mathbf{Q}\mathfrak{C}_i(v_i)\,\mathrm{d}\mathcal{V}_t$ which has been analyzed in step 4 of Section 3.3.1. Specifically, its contribution in the term Z, after combining it with ZB term, is

$$8\varepsilon^{16}\int_\Omega \partial_3(\overline{\partial}^8 q - \overline{\partial}^8\varphi\partial_3^\varphi q)\,\overline{\partial}\mathbf{N}_i\,\overline{\partial}^7 v_i\,\mathrm{d}x,$$

which is controlled by $(\|\varepsilon^8\overline{\partial}^7\partial_3 q\|_0 + \|\varepsilon^8\overline{\partial}^8\psi\|_0\|\partial q\|_{L^\infty})|\overline{\partial}\psi|_{W^{1,\infty}}\|\varepsilon^8\overline{\partial}^8 v\|_0$ after integrating one $\overline{\partial}$ by parts. Then we convert $\partial_3 q$ to tangential derivatives of other quantities via the momentum equation, which has been presented in the control of $\mathfrak{C}(q)$.

Based on the above analysis, we can follow the same method as in $\overline{\partial}^4$-estimate to prove the following inequality for $\varepsilon^{2l}\overline{\partial}^{4+l}$-estimates ($1 \leq l \leq 4$) for the nonlinear κ-approximate problem (3.1): For any $0 < \delta < 1$ and fixed $l \in \{1, 2, 3, 4\}$.

$$\left\|\varepsilon^{2l}\left(\mathbf{V}^{\gamma,\pm}, \mathbf{B}^{\gamma,\pm}, \mathbf{S}^{\gamma,\pm}, \sqrt{\mathcal{F}_p^\pm}\mathbf{P}^{\gamma,\pm}\right)(t)\right\|_{0,\pm}^2 + \left|\sqrt{\sigma}\varepsilon^{2l}\overline{\partial}^{4+l}\overline{\nabla}\psi(t)\right|_0^2 + \left|\sqrt{\kappa}\varepsilon^{2l}\overline{\partial}^{4+l}\psi(t)\right|_2^2 + \int_0^t\left|\sqrt{\kappa}\varepsilon^{2l}\overline{\partial}^{4+l}\partial_t\psi(\tau)\right|_1^2\,\mathrm{d}\tau$$

$$\lesssim \delta E_{4+l}^\kappa(t) + \left|\varepsilon^{2l}\psi_0\right|_{5.5+l}^2 + \sum_{j=0}^l\int_0^t P(\sigma^{-1}, E_{4+j}^\kappa(\tau))\,\mathrm{d}\tau,$$
(3.56)

where $(\mathbf{V}^{\gamma,\pm}, \mathbf{B}^{\gamma,\pm}, \mathbf{S}^{\gamma,\pm}, \mathbf{P}^{\gamma,\pm})$ represent that Alinhac good unknowns of $(v^\pm, b^\pm, S^\pm, p^\pm)$ with respect to $\overline{\partial}^{4+l}$.

3.4 Tangential estimates: full time derivatives

Now we control the full time derivatives, that is, the $\varepsilon^{2l}\partial_t^{4+l}$ estimates for $0 \leq l \leq 4$. We will take the most difficult case $l = 4$ for an example, that is, the $\varepsilon^8\partial_t^8$-estimate. The other cases ($0 \leq l \leq 3$) can be treated in the same way.

3.4.1 Replacing one time derivative by a material derivative

Following the analysis in Section 3.3.1 and Section 3.3.2, we expect to control the following norms

$$\left\|\varepsilon^8\left(\mathbf{V}^\pm, \mathbf{B}^\pm, \sqrt{\mathcal{F}_p^\pm}\mathbf{P}^\pm, \mathbf{S}^\pm\right)\right\|_{0,\pm}^2 + \left|\varepsilon^8\sqrt{\sigma}\partial_t^8\psi\right|_1^2 + \left|\varepsilon^8\sqrt{\kappa}\partial_t^8\psi\right|_2^2 + \left|\varepsilon^8\sqrt{\kappa}\partial_t^9\psi\right|_{L_t^2 H_{x'}^1}^2,$$

which further gives the control of $\left\|\varepsilon^8\partial_t^8(v^\pm, b^\pm, \sqrt{\mathcal{F}_p^\pm}p^\pm, S^\pm)\right\|_0^2$. However, there are several extra difficulties that may make our previous method invalid.

a. We cannot substitute ∂q by $\mathcal{T}(v, b)$ because there is no spatial derivative.

b. $\partial_t^{4+l}p$ has weight $\sqrt{\mathcal{F}_p^\pm}\varepsilon^{2l} = O(\varepsilon^{1+2l})$ instead of ε^{2l}. There might be a loss of ε-weight.

c. $\sqrt{\mathcal{F}_p^\pm}\varepsilon^{2l}\partial_t^{4+l}q$ only has $L^2(\Omega)$ regularity, so the trace lemma is no longer valid.

d. We cannot integrate by parts for "half-order time derivative" $\partial_t^{1/2}$. Thus, the control of VS term will be rather different.

To overcome the abovementioned difficulties, especially (c) and (d) in the control of the crucial boundary term VS, we would like to replace the full-time derivative ∂_t^{4+l} by $D_t^{\varphi,-}\partial_t^{3+l}$ where $D_t^{\varphi,-} = \partial_t + \bar{v}^-\cdot\overline{\nabla} + (\partial_3\varphi)^{-1}(v^-\cdot\mathbf{N} - \partial_t\varphi)\partial_3$ and $v^-|_{\Omega^+}$ is defined to be the Sobolev extension of v^- in Ω^+. We aim to prove the following estimates.

Proposition 3.6. Fix $l \in \{0, 1, 2, 3, 4\}$. For $\varepsilon^{2l} D_t^{\varphi,-} \partial_t^{3+l}$-differentiated approximate system $(0 \le l \le 4)$, we have the following uniform-in-(κ, ε) estimate for any $0 < \delta < 1$

$$\left\| \varepsilon^{2l} \left(\mathbf{V}^{*,\gamma,\pm}, \mathbf{B}^{*,\gamma,\pm}, \mathbf{S}^{*,\gamma,\pm}, (\mathcal{F}_p^{\pm})^{1/2} \mathbf{P}^{*,\gamma,\pm} \right)(t) \right\|_0^2 + \left| \sqrt{\sigma} \varepsilon^{2l} D_t^{\varphi,-} \partial_t^{3+l} \overline{\nabla} \psi(t) \right|_0^2$$

$$+ \left| \sqrt{\kappa} \varepsilon^{2l} D_t^{\varphi,-} \partial_t^{3+l} \psi(t) \right|_2^2 + \int_0^t \left| \sqrt{\kappa} \varepsilon^{2l} D_t^{\varphi,-} \partial_t^{3+l} \partial_t \psi(\tau) \right|_1^2 \, d\tau \tag{3.57}$$

$$\lesssim \delta E_{4+l}^\kappa(t) + \sum_{j=0}^l P(E_{4+j}^\kappa(0)) + \int_0^t P(\sigma^{-1}, E_{4+j}^\kappa(\tau)) \, d\tau, \quad 0 \le l \le 4,$$

where $\left(\mathbf{V}^{*,\gamma,\pm}, \mathbf{B}^{*,\gamma,\pm}, \mathbf{S}^{*,\gamma,\pm}, (\mathcal{F}_p^{\pm})^{1/2} \mathbf{P}^{*,\gamma,\pm} \right)$ represent the Alinhac good unknowns of $(v^\pm, b^\pm, S^\pm, p^\pm)$ with respect to $D_t^{\varphi,-} \partial_t^{3+l}$, that is, $\mathbf{F}^{*,\gamma,\pm} = D_t^{\varphi,-} \partial_t^{3+l} f^\pm - (D_t^{\varphi,-} \partial_t^{3+l} \varphi) \partial_3^\varphi f^\pm$.

For the case $l = 4$, we introduce the Alinhac good unknowns with respect to $D_t^{\varphi,-} \partial_t^7$

$$(\mathbf{V}^{*,\pm}, \mathbf{B}^{*,\pm}, \mathbf{P}^{*,\pm}, \mathbf{Q}^{*,\pm}, \mathbf{S}^{*,\pm}) := D_t^{\varphi,-} \partial_t^7 (v^\pm, b^\pm, p^\pm, q^\pm, S^\pm) - (D_t^{\varphi,-} \partial_t^7 \varphi) \partial_3^\varphi (v^\pm, b^\pm, p^\pm, q^\pm, S^\pm).$$

They satisfy

$$D_t^{\varphi,-} \partial_t^7 \partial_i^\varphi f^\pm = \partial_i^\varphi \mathbf{F}^{*,\pm} + \mathfrak{C}_i^*(f^\pm), \quad D_t^{\varphi,-} \partial_t^7 D_t^{\varphi,-} f^\pm = D_t^{\varphi,-} \mathbf{F}^{*,\pm} + \mathfrak{D}_i^*(f^\pm) f,$$

where $\mathfrak{C}^*(f), \mathfrak{D}^*(f)$ are defined in the same way as (3.9)-(3.10) by replacing $\mathcal{T}^\gamma$ with $D_t^{\varphi,-} \partial_t^7$. The boundary conditions of these good unknowns are

$$\llbracket \mathbf{Q}^* \rrbracket = \sigma \overline{D_t^-} \partial_t^7 \mathcal{H} - \kappa \overline{D_t^-} \partial_t^7 (1 - \overline{\Delta})^2 \psi - \kappa \overline{D_t^-} \partial_t^7 (1 - \overline{\Delta}) \partial_t \psi - \llbracket \partial_3 q \rrbracket \overline{D_t^-} \partial_t^7 \psi \tag{3.58}$$

$$\mathbf{V}^{*,\pm} \cdot N = \partial_t \overline{D_t^-} \partial_t^7 \psi + (\bar{v}^\pm \cdot \overline{\nabla}) \overline{D_t^-} \partial_t^7 \psi - \overline{D_t^\pm} \bar{v}^- \cdot \overline{\nabla} \partial_t^7 \psi - \mathcal{W}^{*,\pm}, \tag{3.59}$$

with

$$\mathcal{W}^{*\pm} = (\partial_3 v^\pm \cdot N) \overline{D_t^-} \partial_t^7 \psi + [D_t^{\varphi,-} \partial_t^7, N_i, v_i^\pm], \tag{3.60}$$

where we use the fact that $D_t^{\varphi\pm}|_\Sigma = \overline{D_t^\pm} = \partial_t + \bar{v}^\pm \cdot \overline{\nabla}$. Note that $\overline{D_t^-}$ does not directly commute with ∂_t or ∂_i, so there is an extra term $-\overline{D_t^\pm} \bar{v}^- \cdot \overline{\nabla} \partial_t^7 \psi$ in the expression of $\mathbf{V}^{\pm,*} \cdot N$.

3.4.2 Analysis of the interior commutators

Since we replaced ∂_t^8 with $D_t^{\varphi,-} \partial_t^7$ and $D_t^{\varphi,-}$ does not directly commute with ∂_3, we need to further analyze the commutators $\mathfrak{C}_i(f)$ for $f = q$ and v_i and $\mathfrak{D}(f)$ for $f = v, b, p, S$. The problematic thing is that ∂_3 may fall on $(\partial_3 \varphi)^{-1} (v^- \cdot \mathbf{N} - \partial_t \varphi)$ (in $D_t^{\varphi,-}$) and produce a normal derivative without a weight function that vanishes on Σ, which may introduce a second-order derivative in the setting of anisotropic Sobolev space. This problem does not appear in $\mathfrak{D}(f)$, as we find that such commutator has the form $(\partial_3 \varphi)^{-1} (v \cdot \mathbf{N} - \partial_t \varphi) [D_t^{\varphi,-} \partial_t^7, \partial_3] f$ which already includes a weight $(v \cdot \mathbf{N} - \partial_t \varphi)$ that vanishes on Σ. In $\mathfrak{C}_i(f)$, according (3.9), we need to further analyze the term $\mathbf{N}_i (\partial_3 \varphi)^{-1} [D_t^{\varphi,-} \partial_t^7, \partial_3] f$ for $f = q, v_i$. Using $D_t^{\varphi,-} = \partial_t + \bar{v}^- \cdot \overline{\nabla} + (\partial_3 \varphi)^{-1} (v^- \cdot \mathbf{N} - \partial_t \varphi) \partial_3$, we have

$$\mathbf{N}_i (\partial_3 \varphi)^{-1} [D_t^{\varphi,-} \partial_t^7, \partial_3] f = \mathbf{N}_i (\partial_3 \varphi)^{-1} [D_t^{\varphi,-}, \partial_3] \partial_t^7 f$$

$$= -\mathbf{N}_i (\partial_3 \varphi)^{-1} \partial_3 \bar{v}^- \cdot \overline{\nabla} \partial_t^7 f + \mathbf{N}_i \partial_3 \left((\partial_3 \varphi)^{-1} (v^- \cdot \mathbf{N} - \partial_t \varphi) \right) \partial_3^\varphi \partial_t^7 f.$$

The first term above can be directly controlled in L^2 because only tangential derivative falls on $\partial_t^7 f$. For the second term, we can invoke the momentum equation and the continuity equation to convert this normal derivative to a tangential derivative.

- When $f = q$, we use $-\partial_3^\varphi q = \rho D_t^\varphi v_3 - (b \cdot \nabla^\varphi) b_3$.
- When $f = v_i$, using $\nabla^\varphi \cdot v = \overline{\nabla} \cdot \bar{v} + \partial_3^\varphi v \cdot \mathbf{N}$, the continuity equation becomes $\partial_3^\varphi v \cdot \mathbf{N} = -\varepsilon^2 D_t^\varphi p - \overline{\nabla} \cdot \bar{v}$. Thus we have $\partial_t^7 \partial_3^\varphi v \cdot \mathbf{N} = -\partial_t^7 (\varepsilon^2 D_t^\varphi p + \overline{\nabla} \cdot \bar{v}) + [\partial_t^7, \mathbf{N}] \cdot \partial_3 v$ in which both terms can be directly controlled in $\| \cdot \|_{8,*}$ norm.

Also note that there is no extra loss of Mach number even if $\partial_t^8 p$ requires one more ε-weight. In fact, the only term in the commutators $\mathfrak{C}, \mathfrak{D}$ that contains $\partial_t^8 p$ is $\mathcal{R}_p$, but there is an extra weight $\mathcal{F}_p = O(\varepsilon^2)$ multiplying on it. Therefore, we can follow the same strategy presented in Section 3.3.1 and Section 3.3.2 to analyze the interior part. We can prove the following energy identity

$$\sum_{\pm} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\varepsilon^{16}}{2} \int_{\Omega^{\pm}} \rho^{\pm} |\mathbf{V}^{*,\pm}|^2 + |\mathbf{B}^{*,\pm}|^2 + \mathcal{F}_p^{\pm}(\mathbf{P}^{*,\pm})^2 + \rho^{\pm}(\mathbf{S}^{*,\pm})^2 \, \mathrm{d}\mathcal{V}_t$$
$$= \mathrm{ST}^* + \mathrm{ST}^{*\prime} + \mathrm{VS}^* + \mathrm{RT}^* + \sum_{\pm} \mathrm{RT}^{*,\pm} + ZB^{*,\pm} + Z^{*,\pm} + R_{\Sigma}^{*,\pm} + R_{\Omega}^{*,\pm}, \tag{3.61}$$

where

$$\mathrm{ST}^* := \varepsilon^{16} \int_{\Sigma} \overline{D_t^-} \partial_t^7 \, [\![q]\!] \, \partial_t \overline{D_t^-} \partial_t^7 \psi \, \mathrm{d}x', \tag{3.62}$$

$$\mathrm{ST}^{*\prime} := \varepsilon^{16} \int_{\Sigma} \overline{D_t^-} \partial_t^7 \, [\![q]\!] \, (\bar{v}^+ \cdot \overline{\nabla}) \overline{D_t^-} \partial_t^7 \psi \, \mathrm{d}x', \tag{3.63}$$

$$\mathrm{VS}^* := \varepsilon^{16} \int_{\Sigma} \overline{D_t^-} \partial_t^7 q^- \, ([\![\bar{v}]\!] \cdot \overline{\nabla}) \overline{D_t^-} \partial_t^7 \psi \, \mathrm{d}x', \tag{3.64}$$

$$\mathrm{RT}^* := - \varepsilon^{16} \int_{\Sigma} [\![\partial_3 q]\!] \, \overline{D_t^-} \partial_t^7 \psi \, \partial_t \overline{D_t^-} \partial_t^7 \psi \, \mathrm{d}x', \tag{3.65}$$

$$\mathrm{RT}^{*,\pm} := \mp \varepsilon^{16} \int_{\Sigma} \partial_3 q^{\pm} \, \overline{D_t^-} \partial_t^7 \psi \, (\bar{v}^{\pm} \cdot \overline{\nabla}) \overline{D_t^-} \partial_t^7 \psi \, \mathrm{d}x', \tag{3.66}$$

$$R_{\Sigma}^{*,\pm} := \pm \varepsilon^{16} \int_{\Sigma} \mathbf{Q}^{*,\pm} \overline{D_t^{\pm}} \bar{v}^- \cdot \overline{\nabla} \partial_t^7 \psi \, \mathrm{d}x', \tag{3.67}$$

$$ZB^{*,\pm} := \mp \varepsilon^{16} \int_{\Sigma} \mathbf{Q}^{*,\pm} \mathcal{W}^{*,\pm} \, \mathrm{d}x', \quad Z^{*,\pm} = - \int_{\Omega^{\pm}} \varepsilon^{16} \mathbf{Q}^{\pm,*} \mathfrak{C}_i^*(v_i^{\pm}) \, \mathrm{d}\mathcal{V}_t, \tag{3.68}$$

and $R_{\Omega}^{*,\pm}$ represents the controllable terms in the interior containing the analogues of $R_1^{\pm}, R_2^{\pm}, R_3^{\pm}$. Specifically, we have

$$\varepsilon^{-16} R_{\Omega}^{*,\pm} = \int_{\Omega^{\pm}} \mathbf{V}^{*,\pm} \cdot (\mathcal{R}_v^{*,\pm} - \mathfrak{C}^*(q^{\pm})) \, \mathrm{d}\mathcal{V}_t + \int_{\Omega^{\pm}} \mathcal{R}_q^{*,\pm} (\nabla^{\varphi} \cdot \mathbf{V}^{*,\pm}) \, \mathrm{d}\mathcal{V}_t + \int_{\Omega^{\pm}} \mathbf{B}^{*,\pm} \cdot \mathcal{R}_b^{*,\pm} \, \mathrm{d}\mathcal{V}_t + \int_{\Omega^{\pm}} \mathbf{P}^{*,\pm} \mathcal{R}_p^{*,\pm} \, \mathrm{d}\mathcal{V}_t$$
$$- \frac{1}{2} \int_{\Omega^{\pm}} (\nabla^{\varphi} \cdot v^{\pm}) |\mathbf{B}^{*,\pm}|^2 \, \mathrm{d}\mathcal{V}_t - \frac{1}{2} \int_{\Omega^{\pm}} (D_t^{\varphi\pm} \mathcal{F}_p^{\pm} + \mathcal{F}_p^{\pm} \nabla^{\varphi} \cdot v^{\pm}) |\mathbf{P}^{*,\pm}|^2 \, \mathrm{d}\mathcal{V}_t + \int_{\Omega^{\pm}} \rho^{\pm} \mathfrak{D}^*(S^{\pm}) \mathbf{S}^{*,\pm} \, \mathrm{d}\mathcal{V}_t \tag{3.69}$$

where

$$\mathcal{R}_v^{*,\pm} := [D_t^{\varphi,-} \partial_t^7, b^{\pm}] \cdot \nabla^{\varphi} b^{\pm} - [D_t^{\varphi,-} \partial_t^7, \rho^{\pm}] D_t^{\varphi\pm} v^{\pm} - \rho^{\pm} \mathfrak{D}^*(v^{\pm}),$$
$$\mathcal{R}_p^{*,\pm} := - [D_t^{\varphi,-} \partial_t^7, \mathcal{F}_p^{\pm}] D_t^{\varphi\pm} p^{\pm} - \mathcal{F}_p^{\pm} \mathfrak{D}^*(p^{\pm}),$$
$$\mathcal{R}_b^{*,\pm} := [D_t^{\varphi,-} \partial_t^7, b^{\pm}] \cdot \nabla^{\varphi} v^{\pm} - \mathfrak{D}^*(b^{\pm}), \quad \mathcal{R}_q^{*,\pm} := \mathbf{Q}^{*,\pm} - \mathbf{P}^{*,\pm} - b^{\pm} \cdot \mathbf{B}^{*,\pm}.$$

These terms can be directly controlled in the same way as presented in Section 3.3.1, so we omit the details

$$\int_0^t R_{\Omega}^{*,\pm} \, \mathrm{d}\tau \lesssim P(E^{\kappa}(0)) + \int_0^t P(E_4^{\kappa}(t)) E_8^{\kappa}(t) \, \mathrm{d}\tau. \tag{3.70}$$

3.4.3 Analysis of the boundary integrals

Similarly as in Section 3.3.1, we can decompose the control of these terms in the following steps.

Step 1: Boundary regularity of full time derivatives given by surface tension.

Invoking the boundary condition (3.58) for $[\![\mathbf{Q}^*]\!]$, the term ST becomes

$$
\begin{aligned}
\mathrm{ST}^* = \sigma\varepsilon^{16} \int_\Sigma \overline{D_t^-}\partial_t^7\overline{\nabla}\cdot\left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}}\right)\partial_t\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x' \\
-\kappa\varepsilon^{16}\int_\Sigma \overline{D_t^-}\partial_t^7(1-\overline{\Delta})^2\psi\,\partial_t\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x' - \kappa\varepsilon^{16}\int_\Sigma \overline{D_t^-}\partial_t^7(1-\overline{\Delta})\partial_t\psi\,\partial_t\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x' \\
=: \mathrm{ST}_0^* + \mathrm{ST}_{1,\kappa}^* + \mathrm{ST}_{2,\kappa}^*.
\end{aligned}
\tag{3.71}
$$

Commuting $\overline{\nabla}\cdot$ with $\overline{D_t^-}$, we have

$$
\mathrm{ST}_0^* = \sigma\varepsilon^{16}\int_\Sigma \overline{\nabla}\cdot\overline{D_t^-}\partial_t^7(\overline{\nabla}\psi/|N|)\,\partial_t\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x' + \underbrace{\sigma\varepsilon^{16}\int_\Sigma \overline{\partial}_i\overline{v}_j^-\overline{\partial}_j\partial_t^7(\overline{\partial}_i\psi/|N|)\partial_t\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x'}_{\mathrm{ST}_0^{*,R}}
\tag{3.72}
$$

Integrating $\overline{\nabla}\cdot$ by parts in the mean curvature term, we get an analogous energy term contributed by surface tension as in Section 3.3.1

$$
\begin{aligned}
&\sigma\varepsilon^{16}\int_\Sigma \overline{\nabla}\cdot\overline{D_t^-}\partial_t^7\left(\frac{\overline{\nabla}\psi}{\sqrt{1+|\overline{\nabla}\psi|^2}}\right)\partial_t\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x' \\
&= -\sigma\varepsilon^{16}\int_\Sigma \frac{\overline{D_t^-}\partial_t^7\overline{\nabla}\psi}{|N|}\cdot\partial_t\overline{D_t^-}\partial_t^7\overline{\nabla}\psi\,\mathrm{d}x' + \sigma\varepsilon^{16}\int_\Sigma \frac{\overline{\nabla}\psi\cdot\overline{D_t^-}\partial_t^7\overline{\nabla}\psi}{|N|^3}\overline{\nabla}\psi\cdot\partial_t\overline{D_t^-}\partial_t^7\overline{\nabla}\psi\,\mathrm{d}x' \\
&\underbrace{-\sigma\varepsilon^{16}\int_\Sigma \frac{\overline{D_t^-}\partial_t^7\overline{\partial}_i\psi}{|N|}\cdot\partial_t(\overline{\partial}_i\overline{v}_j^-\overline{\partial}_j\partial_t^7\psi)\,\mathrm{d}x' - \sigma\varepsilon^{16}\int_\Sigma \frac{\overline{\nabla}\psi\cdot\overline{D_t^-}\partial_t^7\overline{\partial}_i\psi}{|N|^3}\overline{\nabla}\psi\,\partial_t(\overline{\partial}_i\overline{v}_j^-\overline{\partial}_j\partial_t^7\psi)\,\mathrm{d}x'}_{=:\,\mathrm{ST}_1^{*,R}} \\
&\underbrace{-\sigma\varepsilon^{16}\int_\Sigma\left(\left[\overline{D_t^-}\partial_t^6,\frac{1}{|N|}\right]\partial_t\overline{\nabla}\psi + \left[\overline{D_t^-}\partial_t^6,\frac{1}{|N|^3}\right]((\overline{\nabla}\psi\cdot\partial_t\overline{\nabla}\psi)\overline{\nabla}\psi) - \frac{1}{|N|^3}[\overline{D_t^-}\partial_t^6,\overline{\nabla}\psi]\partial_t\overline{\nabla}\psi\right)\cdot\partial_t\overline{\nabla D_t^-}\partial_t^7\psi\,\mathrm{d}x'}_{=:\,\mathrm{ST}_2^{*,R}}
\end{aligned}
\tag{3.73}
$$

where the right side is further equal to

$$
\begin{aligned}
&-\frac{\sigma\varepsilon^{16}}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma \frac{|\overline{D_t^-}\partial_t^7\overline{\nabla}\psi|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}} - \frac{|\overline{\nabla}\psi\cdot\overline{D_t^-}\partial_t^7\overline{\nabla}\psi|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}^3}\,\mathrm{d}x' \\
&+\underbrace{\frac{\sigma\varepsilon^{16}}{2}\int_\Sigma \partial_t\left(\frac{1}{|N|}\right)\left|\overline{D_t^-}\partial_t^7\overline{\nabla}\psi\right|^2 - \partial_t\left(\frac{1}{|N|^3}\right)\left|\overline{\nabla}\psi\cdot\overline{D_t^-}\partial_t^7\overline{\nabla}\psi\right|^2\,\mathrm{d}x'}_{=:\,\mathrm{ST}_3^{*,R}} + \mathrm{ST}_1^{*,R} + \mathrm{ST}_2^{*,R}.
\end{aligned}
\tag{3.74}
$$

The first line above together with the inequality (3.37) gives the $\sqrt{\sigma}$-weighted boundary regularity as in step 2 in Section 3.3.1. The term $\mathrm{ST}_1^{*,R}$ is generated by commuting $\overline{D_t^-}$ with $\overline{\nabla}$ (the one falling on $\partial_t^7\psi$) and is directly controlled by the energy. The term $\mathrm{ST}_3^{*,R}$ is controlled in the same way as ST_2^R in step 2 of Section 3.3. The term $\mathrm{ST}_2^{*,R}$ is controlled by integrating ∂_t by parts under time integral, which was also analyzed in [38, Section 4.6]. The term $\mathrm{ST}_0^{*,R}$ is controlled by integrating by parts in ∂_t and then in $\overline{\partial}_j$ under time integral (which is similar to $\mathrm{ST}_2^{*,R}$). Thus, we conclude their estimates by

$$
\int_0^t \mathrm{ST}_0^{*,R} + \mathrm{ST}_1^{*,R} + \mathrm{ST}_2^{*,R} + \mathrm{ST}_3^{*,R}\,\mathrm{d}\tau \lesssim P(E^\kappa(0)) + \int_0^t P(E^\kappa(\tau))\,\mathrm{d}\tau.
\tag{3.75}
$$

Next we analyze the terms $\mathrm{ST}_{1,\kappa}^*$, $\mathrm{ST}_{2,\kappa}^*$ involving the κ-regularization terms. Note that we have to commute $\overline{D_t^-}$ with $1 - \overline{\Delta}$ or $\langle\overline{\partial}\rangle = \sqrt{1 - \overline{\Delta}}$ when deriving the $\sqrt{\kappa}$-weighte terms. Integrating $1 - \overline{\Delta}$ by parts in $\mathrm{ST}_{1,\kappa}^*$

$$
\begin{aligned}
\mathrm{ST}_{1,\kappa}^* &= -\kappa\varepsilon^{16} \int_\Sigma \overline{D_t^-}\partial_t^7(1-\overline{\Delta})^2\psi \, \partial_t\overline{D_t^-}\partial_t^7\psi \, \mathrm{d}x' \\
&= -\kappa\varepsilon^{16} \int_\Sigma D_t^{\varphi,-}\partial_t^7(1-\overline{\Delta})\psi \, \partial_t\left(\overline{D_t^-}\partial_t^7(1-\overline{\Delta})\psi\right) \mathrm{d}x' \\
&\quad - \kappa\varepsilon^{16} \int_\Sigma [\overline{D_t^-}, 1-\overline{\Delta}]\left(\partial_t^7(1-\overline{\Delta})\psi\right)\partial_t\overline{D_t^-}\partial_t^7\psi \, \mathrm{d}x' - \kappa\varepsilon^{16}\int_\Sigma \overline{D_t^-}\partial_t^7(1-\overline{\Delta})\psi \, \partial_t\left([1-\overline{\Delta}, \overline{D_t^-}]\partial_t^7\psi\right)\mathrm{d}x' \\
&=: -\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\left|\sqrt{\kappa}\varepsilon^8\overline{D_t^-}\partial_t^7(1-\overline{\Delta})\psi\right|_0^2 + \mathrm{ST}_{11,\kappa}^{*,R} + \mathrm{ST}_{12,\kappa}^{*,R}
\end{aligned}
\tag{3.76}
$$

On Σ, the material derivative $D_t^{\varphi,-} = \overline{D_t^-} = \partial_t + \bar{v}^- \cdot \overline{\nabla}$, so the commutator is

$$
[\overline{D_t^-}, 1-\overline{\Delta}]f = [\overline{\Delta}, \bar{v}^- \cdot \overline{\nabla}]f = \overline{\Delta}\bar{v}^- \cdot \overline{\nabla}f + 2\overline{\partial}_i\bar{v}_j^-\overline{\partial}_j\overline{\partial}_i f.
$$

Then $\mathrm{ST}_{11,\kappa}^{*,R}$ is controlled under time integral by integrating $\overline{\partial}_j$ by parts in the second term

$$
\begin{aligned}
\int_0^t \mathrm{ST}_{11,\kappa}^{*,R}\,\mathrm{d}\tau &= -\kappa\varepsilon^{16}\int_0^t\int_\Sigma \overline{\Delta}\bar{v}_j^-\overline{\partial}_j\left(\partial_t^7(1-\overline{\Delta})\psi\right)\partial_t\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x'\,\mathrm{d}\tau \\
&\quad + 2\kappa\varepsilon^{16}\int_0^t\int_\Sigma \overline{\partial}_i\bar{v}_j^-\overline{\partial}_i\left(\partial_t^7(1-\overline{\Delta})\psi\right)\overline{\partial}_j\partial_t\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x'\,\mathrm{d}\tau + \text{ lower order terms} \\
&\lesssim \delta\left|\sqrt{\kappa}\varepsilon^8\overline{D_t^-}\partial_t^8\psi\right|_1^2 + \frac{1}{4\delta}\int_0^t\left|\sqrt{\kappa}\varepsilon^8\overline{\partial}\partial_t^7\psi\right|_2^2|\overline{\partial}\bar{v}^-|_{W^{1,\infty}}^2\,\mathrm{d}\tau \leq \delta E_8^\kappa(t) + \int_0^t P(E_8^\kappa(\tau), E_4^\kappa(\tau))\,\mathrm{d}\tau.
\end{aligned}
\tag{3.77}
$$

The control of $\mathrm{ST}_{12,\kappa}^{*,R}$ is easier because there is no term containing 9 time derivatives of ψ. It is directly controlled by using the $\sqrt{\kappa}$-weighted boundary energy obtained above.

$$
\mathrm{ST}_{12,\kappa}^{*,R} \lesssim \left|\sqrt{\kappa}\varepsilon^8\overline{D_t^-}\partial_t^7(1-\overline{\Delta})\psi\right|_0\left(\left|\sqrt{\kappa}\varepsilon^8\partial_t^8\psi\right|_2 + \left|\sqrt{\kappa}\varepsilon^8\partial_t^7\psi\right|_2\right)\sqrt{E_4^\kappa(t)} \leq E_8^\kappa(t)\sqrt{E_4^\kappa(t)}.
$$

The control of $\mathrm{ST}_{2,\kappa}^*$ is similar to $\mathrm{ST}_{1,\kappa}^*$. Using $\langle\overline{\partial}\rangle^2 = 1 - \overline{\Delta}$, we have

$$
\begin{aligned}
\mathrm{ST}_{2,\kappa}^* &= -\int_\Sigma\left|\sqrt{\kappa}\varepsilon^8 D_t^{\varphi,-}\partial_t^8\langle\overline{\partial}\rangle\psi\right|^2 \mathrm{d}x' \\
&\quad + \kappa\varepsilon^{16}\int_\Sigma[\overline{D_t^-}, \overline{\partial}_i]\left(\partial_t^8\overline{\partial}_i\psi\right)\partial_t\overline{D_t^-}\partial_t^7\psi\,\mathrm{d}x' + \kappa\varepsilon^{16}\int_\Sigma\overline{D_t^-}\partial_t^8\overline{\partial}_i\psi\left([\overline{\partial}_i\partial_t, \overline{D_t^-}]\partial_t^7\psi\right)\mathrm{d}x' \\
&=: -\int_\Sigma\left|\sqrt{\kappa}\varepsilon^8 D_t^{\varphi,-}\partial_t^8\langle\overline{\partial}\rangle\psi\right|^2 \mathrm{d}x' + \mathrm{ST}_{21,\kappa}^{*,R} + \mathrm{ST}_{22,\kappa}^{*,R},
\end{aligned}
\tag{3.78}
$$

where we use the concrete form of the commutators

$$
[\overline{D_t^-}, \overline{\partial}_i] = -\overline{\partial}_i\bar{v}_j^-\overline{\partial}_j f, \quad [\overline{\partial}_i\partial_t, \overline{D_t^-}]f = \partial_t\left(\overline{\partial}_i\bar{v}_j^-\overline{\partial}_j\overline{D_t^-}f\right) + \partial_t\bar{v}_j^-\overline{\partial}_j\overline{\partial}_i f
$$

to get estimates similar to $\mathrm{ST}_{11,\kappa}^{*,R}$ and $\mathrm{ST}_{12,\kappa}^{*,R}$

$$
\int_0^t \mathrm{ST}_{21,\kappa}^{*,R} + \mathrm{ST}_{22,\kappa}^{*,R}\,\mathrm{d}\tau \lesssim \delta E_8^\kappa(t) + \int_0^t P(E_8^\kappa(\tau), E_4^\kappa(\tau))\,\mathrm{d}\tau.
$$

Hence, the control of ST^* in (3.61) is concluded by

$$
\begin{aligned}
&\int_0^t \mathrm{ST}\,\mathrm{d}\tau + \frac{1}{2}\int_\Sigma\frac{|\sqrt{\sigma}\varepsilon^8\overline{D_t^-}\partial_t^7\overline{\nabla}\psi|^2}{\sqrt{1+|\overline{\nabla}\psi|^2}^{-3}}\,\mathrm{d}x' \\
&\quad + \int_\Sigma\left|\sqrt{\kappa}\varepsilon^8\overline{D_t^-}\partial_t^7(1-\overline{\Delta})\psi\right|^2 \mathrm{d}x' + \int_0^t\int_\Sigma\left|\sqrt{\kappa}\varepsilon^8\overline{D_t^-}\partial_t^8\langle\overline{\partial}\rangle\psi\right|^2 \mathrm{d}x'\,\mathrm{d}\tau \\
&\leq \delta E_8^\kappa(t) + P(E^\kappa(0)) + \int_0^t P(E^\kappa(\tau))\,\mathrm{d}\tau.
\end{aligned}
\tag{3.79}
$$

The term ST$^{*\prime}$ is controlled in the same way as ST* by replacing $\partial_t \overline{D_t^-} \partial_t^7 \psi$ with $(\bar{v}^+ \cdot \overline{\nabla}) \overline{D_t^-} \partial_t^7 \psi$. We no longer get energy terms, but we can integrate $(\bar{v}^+ \cdot \overline{\nabla})$ by parts and use symmetry and the above boundary regularity to control them. Invoking the jump condition, we have

$$\mathrm{ST}^{*\prime} = \sigma \varepsilon^{16} \overline{D_t^-} \partial_t^7 \mathcal{H} \, (\bar{v}^+ \cdot \overline{\nabla}) \overline{D_t^-} \partial_t^7 \psi \, \mathrm{d}x'$$

$$- \kappa \varepsilon^{16} \int_\Sigma \overline{D_t^-} \partial_t^7 (1 - \overline{\Delta})^2 \psi \, (\bar{v}^+ \cdot \overline{\nabla}) \overline{D_t^-} \partial_t^7 \psi \, \mathrm{d}x' - \kappa \varepsilon^{16} \int_\Sigma \overline{D_t^-} \partial_t^7 (1 - \overline{\Delta}) \partial_t \psi \, (\bar{v}^+ \cdot \overline{\nabla}) \overline{D_t^-} \partial_t^7 \psi \, \mathrm{d}x'$$

$$=: \mathrm{ST}_0^{*\prime} + \mathrm{ST}_{1,\kappa}^{*}{}' + \mathrm{ST}_{2,\kappa}^{*}{}'. \tag{3.80}$$

Following the analysis (3.72)-(3.75), the first term is controlled thanks to the boundary regularity and symmetric structure after integrating $(\bar{v}^+ \cdot \overline{\nabla})$ by parts.

$$\mathrm{ST}_0^{*\prime} \overset{L}{=} \frac{1}{2} \sigma \varepsilon^{16} \int_\Sigma (\overline{\nabla} \cdot \bar{v}^+) \left(\frac{|\overline{D_t^-} \partial_t^7 \psi|^2}{|N|} - \frac{|\overline{\nabla}\psi \cdot \overline{\nabla D_t^-} \partial_t^7 \psi|^2}{|N|^3} \right) \mathrm{d}x' \leq P(|\overline{\nabla}\psi|_{L^\infty}) |\bar{v}^+|_{W^{1,\infty}} \left| \sqrt{\sigma} \varepsilon^8 \overline{\nabla D_t^-} \partial_t^7 \psi \right|_0^2. \tag{3.81}$$

Similarly, we can use the symmetric structure to control $\mathrm{ST}_{1,\kappa}^{*}{}' + \mathrm{ST}_{2,\kappa}^{*}{}'$. We only check the commutators arising in the control of $\mathrm{ST}_{1,\kappa}^{*}{}'$ as an example.

$$\mathrm{ST}_{1,\kappa}^{*,R\prime} := - \kappa \varepsilon^{16} \int_\Sigma [\overline{D_t^-}, 1 - \overline{\Delta}] \left(\partial_t^7 (1 - \overline{\Delta}) \psi \right) (\bar{v}^+ \cdot \overline{\nabla}) \left(\overline{D_t^-} \partial_t^7 \psi \right) \mathrm{d}x'$$

$$- \kappa \varepsilon^{16} \int_\Sigma \left(\overline{D_t^-} \partial_t^7 (1 - \overline{\Delta}) \psi \right) (\bar{v}^+ \cdot \overline{\nabla}) \left([1 - \overline{\Delta}, \overline{D_t^-}] \partial_t^7 \psi \right) \mathrm{d}x'$$

$$- \kappa \varepsilon^{16} \int_\Sigma \overline{D_t^-} \left(\partial_t^7 (1 - \overline{\Delta}) \psi \right) [1 - \overline{\Delta}, (\bar{v}^+ \cdot \overline{\nabla})] \left(\overline{D_t^-} \partial_t^7 \psi \right) \mathrm{d}x'$$

$$=: \mathrm{ST}_{11,\kappa}^{*,R}{}' + \mathrm{ST}_{12,\kappa}^{*,R}{}' + \mathrm{ST}_{13,\kappa}^{*,R}{}'. \tag{3.82}$$

The control of $\mathrm{ST}_{11,\kappa}^{*,R}{}' + \mathrm{ST}_{12,\kappa}^{*,R}{}'$ is similar to $\mathrm{ST}_{11,\kappa}^{*,R} + \mathrm{ST}_{12,\kappa}^{*,R}$. We have

$$\mathrm{ST}_{11,\kappa}^{*,R}{}' \overset{L}{=} - \kappa \varepsilon^{16} \int_\Sigma \overline{\Delta} \bar{v}_j^- \overline{\partial}_j \left(\partial_t^7 (1 - \overline{\Delta}) \psi \right) (\bar{v}^+ \cdot \overline{\nabla}) \left(\overline{D_t^-} \partial_t^7 \psi \right) \mathrm{d}x'$$

$$+ 2\kappa \varepsilon^{16} \int_\Sigma \overline{\partial}_j \bar{v}_j^- \overline{\partial}_i \left(\partial_t^7 (1 - \overline{\Delta}) \psi \right) \overline{\partial}_j (\bar{v}^+ \cdot \overline{\nabla}) \left(\overline{D_t^-} \partial_t^7 \psi \right) \mathrm{d}x'$$

$$\lesssim |\bar{v}^-|_{W^{2,\infty}} |\bar{v}^+|_{L^\infty} \left| \sqrt{\kappa} \varepsilon^8 \partial_t^7 \psi \right|_3 \left(\left| \sqrt{\kappa} \varepsilon^8 \partial_t^7 \psi \right|_3 + \left| \sqrt{\kappa} \varepsilon^8 \partial_t^8 \psi \right|_2 \right) \lesssim E_4^\kappa(t) E_8^\kappa(t), \tag{3.83}$$

and

$$\mathrm{ST}_{12,\kappa}^{*,R}{}' \lesssim |\bar{v}^-|_{W^{2,\infty}} |\bar{v}^+|_{L^\infty} \left(\left| \sqrt{\kappa} \varepsilon^8 \partial_t^8 \psi \right|_2 + \left| \sqrt{\kappa} \varepsilon^8 \partial_t^7 \psi \right|_3 \right)^2 \lesssim E_4^\kappa(t) E_8^\kappa(t). \tag{3.84}$$

The extra term $\mathrm{ST}_{13,\kappa}^{*,R}{}'$ is also directly controlled

$$\mathrm{ST}_{13,\kappa}^{*,R}{}' = \kappa \varepsilon^{16} \int_\Sigma \overline{D_t^-} \left(\partial_t^7 (1 - \overline{\Delta}) \psi \right) \left(\overline{\Delta} \bar{v}_j^+ \overline{\partial}_j + 2\overline{\partial}_i \bar{v}_j^+ \overline{\partial}_j \overline{\partial}_i \right) \left(\overline{D_t^-} \partial_t^7 \psi \right) \mathrm{d}x'$$

$$\lesssim |\bar{v}^+|_{W^{2,\infty}} |\bar{v}^-|_{L^\infty} \left(\left| \sqrt{\kappa} \varepsilon^8 \partial_t^8 \psi \right|_2 + \left| \sqrt{\kappa} \varepsilon^8 \partial_t^7 \psi \right|_3 \right)^2 \lesssim E_4^\kappa(t) E_8^\kappa(t). \tag{3.85}$$

Thus we have

$$\mathrm{ST}_{1,\kappa}^{*}{}' \lesssim \frac{1}{2} \int_\Sigma (\overline{\nabla} \cdot \bar{v}^+) \left| \sqrt{\kappa} \varepsilon^8 \overline{D_t^-} \partial_t^7 (1 - \overline{\Delta}) \psi \right|^2 + E_4^\kappa(t) E_8^\kappa(t). \tag{3.86}$$

Similarly, we have

$$\int_0^t \overline{ST_{2,\kappa}^*}' \, d\tau = \kappa\varepsilon^{16} \int_0^t \int_\Sigma [\overline{D_t^-}, \overline{\partial}_i](\partial_t^8 \overline{\partial}_i \psi)(\bar{v}^+ \cdot \overline{\nabla})\overline{D_t^-}\partial_t^7 \psi \, dx' \, d\tau$$

$$- \kappa\varepsilon^{16} \int_0^t \int_\Sigma (\overline{D_t^-}\partial_t^8 \overline{\partial}_i \psi)\, \overline{\partial}_i\left((\bar{v}^+ \cdot \overline{\nabla})\overline{D_t^-}\partial_t^7 \psi\right) dx' \, d\tau$$

$$- \kappa\varepsilon^{16} \int_0^t \int_\Sigma \overline{D_t^-}\partial_t^7 \partial_t \psi \,(\bar{v}^+ \cdot \overline{\nabla})\overline{D_t^-}\partial_t^7 \psi \, dx'$$

$$\lesssim \delta \left| \sqrt{\kappa}\varepsilon^8 \overline{D_t^-}\partial_t^8 \langle\overline{\partial}\rangle \psi \right|_0^2 + \int_0^t |\bar{v}^\pm|_{W^{1,\infty}}^2 \left(\left| \sqrt{\kappa}\varepsilon^8 \partial_t^7 \psi \right|_3^2 + \left| \sqrt{\kappa}\varepsilon^8 \partial_t^8 \psi \right|_2^2\right) d\tau. \tag{3.87}$$

Hence, we have the estimate of $ST^{*\prime}$:

$$\int_0^t ST^{*\prime} \, d\tau \lesssim \delta E_8^\kappa(t) + \int_0^t E_8^\kappa(\tau)E_4^\kappa(\tau) \, d\tau. \tag{3.88}$$

What's more, we can also control the remainder term $R_\Sigma^{*,\pm} := \pm\varepsilon^{16} \int_\Sigma \mathbf{Q}^{*,\pm}\overline{D_t^\pm}\bar{v}^- \cdot \overline{\nabla}\partial_t^7 \psi \, dx'$. Indeed, we use Gauss-Green formula to write it to be an interior intergral.

$$R_\Sigma^{*,\pm} \overset{L}{=} -\varepsilon^{16} \int_{\Omega^\pm} \partial_3 \mathbf{Q}^{*,\pm}\, \overline{D_t^\pm}\bar{v}^- \cdot \overline{\nabla}\partial_t^7 \varphi \, dx \tag{3.89}$$

Recall that $\mathbf{Q}^{*,\pm} = D_t^{\varphi,-}\partial_t^7 q^\pm - D_t^{\varphi,-}\partial_t^7 \psi \partial_3 q^\pm$. Note that $[\partial_3^\varphi, D_t^{\varphi,-}]\partial_t^7 q = \partial_3^\varphi v_j^- \partial_j^\varphi q = (\partial_3^\varphi \bar{v}^- \cdot \overline{\nabla})\partial_t^7 q + (\partial_3^\varphi v^- \cdot \mathbf{N})\partial_t^7 \partial_3 q$, so one can still convert ∂q to a tangential derivative of v, b. We now integrate by parts $\overline{D_t^-}$ to get

$$\int_0^t R_\Sigma^{*,\pm} \, d\tau \overset{L}{=} \varepsilon^{16} \int_0^t \int_{\Omega^\pm} \partial_3(\partial_t^7 q^\pm - \partial_t^7 \varphi \partial_3 q^\pm)\overline{D_t^\pm}\bar{v}^- \cdot D_t^{\varphi,-}\overline{\nabla}\partial_t^7 \varphi \, dx \, d\tau$$

$$- \varepsilon^{16} \int_0^t \int_{\Omega^\pm} \partial_3(\partial_t^7 q^\pm - \partial_t^7 \varphi \partial_3 q^\pm)\overline{D_t^\pm}\bar{v}^- \cdot \overline{\nabla}\partial_t^7 \varphi \, dx. \tag{3.90}$$

Using the reduction for $\partial_3 q$ again, we can control the above integral by

$$\int_0^t R_\Sigma^{*,\pm} \, d\tau \lesssim \delta\|\varepsilon^8 \partial_3 \partial_t^7 q^\pm\|_{0,\pm}^2 + \int_0^t P(E_4^\kappa(\tau))E_8^\kappa(\tau) \, d\tau \lesssim \delta E_8^\kappa(t) + \int_0^t P(E_4^\kappa(\tau))E_8^\kappa(\tau) \, d\tau. \tag{3.91}$$

Step 2: Control of VS term.

Now we start to analyze the most difficult boundary term

$$VS^* := \varepsilon^{16} \int_\Sigma \overline{D_t^-}\partial_t^7 q^- \,(\llbracket\bar{v}\rrbracket \cdot \overline{\nabla})\overline{D_t^-}\partial_t^7 \psi \, dx'. \tag{3.92}$$

Note that there is no spatial derivative $\overline{\partial}$ in VS^*, so we cannot integrate $\overline{\partial}^{1/2}$ by parts as in step 3 in Section 3.3.1. To overcome this difficulty, we try to rewrite the term $\overline{D_t^-}\partial_t^7 \psi$ by invoking the kinematic boundary condition

$$\overline{D_t^-}\partial_t^7 \psi = \partial_t^8 \psi + \bar{v}^- \cdot \overline{\nabla}\partial_t^7 \psi = \partial_t^7(v^- \cdot N) - v^- \cdot \partial_t^7 N$$

$$= \partial_t^7 v^- \cdot N + [\partial_t^7, N_i, v_i^-],$$

and thus

$$VS^* = \varepsilon^{16} \int_\Sigma \overline{D_t^-}\partial_t^7 q^- \,(\llbracket\bar{v}\rrbracket \cdot \overline{\nabla})\partial_t^7 v^- \cdot N \, dx'$$

$$+ \varepsilon^{16} \int_\Sigma \overline{D_t^-}\partial_t^7 q^- \, \partial_t^7 v^- \cdot (\llbracket\bar{v}\rrbracket \cdot \overline{\nabla})N \, dx' + \varepsilon^{16} \int_\Sigma \overline{D_t^-}\partial_t^7 q^- \,(\llbracket\bar{v}\rrbracket \cdot \overline{\nabla})[\partial_t^7, N_i, v_i^-] \, dx'$$

$$=: VS_0^* + VS_1^{*,ZB} + VS_2^{*,ZB} \tag{3.93}$$

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Using divergence theorem, we convert VS_0^* to an interior integral in Ω^-

$$
\begin{aligned}
\mathrm{VS}_0^* &= \varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \partial_t^7 q^- \, \nabla^\varphi \cdot \left((\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) \partial_t^7 v^- \right) \, \mathrm{d}\mathcal{V}_t + \varepsilon^{16} \int_{\Omega^-} \partial_i^\varphi D_t^{\varphi,-} \partial_t^7 q^- \, (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) \partial_t^7 v_i^- \, \mathrm{d}\mathcal{V}_t \\
&=: \ \mathrm{VS}_{01}^* + \mathrm{VS}_{02}^*,
\end{aligned}
\tag{3.94}
$$

where $\llbracket \bar{v} \rrbracket = \bar{v}^+ - \bar{v}^-$ is defined via Sobolev extension in Ω^-. In VS_{01}^*, we want to commute $\nabla^\varphi \cdot$ with $(\llbracket \bar{v} \rrbracket \cdot \overline{\nabla})$ in order to get a similar cancellation structure as in $ZB + Z$. The commutator is

$$
\begin{aligned}
[\partial_i^\varphi, \llbracket \bar{v} \rrbracket \cdot \overline{\nabla}] f &= \bar{\partial}_i (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla} f) - \frac{\overline{\partial_i \varphi}}{\partial_3 \varphi} \partial_3 (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla} f) - (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) \left(\bar{\partial}_i f - \frac{\overline{\partial_i \varphi}}{\partial_3 \varphi} \partial_3 f \right) \\
&= (\bar{\partial}_i \llbracket \bar{v} \rrbracket_j)(\bar{\partial}_j f) - \frac{\overline{\partial_i \varphi}}{\partial_3 \varphi} (\partial_3 \llbracket \bar{v} \rrbracket_j)(\bar{\partial}_j f) + \llbracket \bar{v} \rrbracket_j \, \bar{\partial}_j \left(\frac{\overline{\partial_i \varphi}}{\partial_3 \varphi} \right) \partial_3 f \\
&= \partial_i^\varphi \llbracket \bar{v} \rrbracket \cdot \overline{\nabla} f - (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) \mathbf{N}_i \partial_3^\varphi f + \mathbf{N}_i \frac{(\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) \partial_3 \varphi}{\partial_3 \varphi} \partial_3^\varphi f, \quad i = 1, 2,
\end{aligned}
$$

$$
[\partial_3^\varphi, \llbracket \bar{v} \rrbracket \cdot \overline{\nabla}] f = \partial_3^\varphi \llbracket \bar{v} \rrbracket \cdot \overline{\nabla} f + \frac{(\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) \partial_3 \varphi}{\partial_3 \varphi} \partial_3^\varphi f.
$$

Commuting $\nabla^\varphi \cdot$ with $(\llbracket \bar{v} \rrbracket \cdot \overline{\nabla})$, we get

$$
\begin{aligned}
\mathrm{VS}_{01}^* &= \varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \partial_t^7 q^- \, \nabla^\varphi \cdot \left((\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) \partial_t^7 v^- \right) \, \mathrm{d}\mathcal{V}_t \\
&= \varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \partial_t^7 q^- \, (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) \left(\nabla^\varphi \cdot \partial_t^7 v^- \right) \, \mathrm{d}\mathcal{V}_t - \varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \partial_t^7 q^- \, \partial_3 \partial_t^7 v^- \cdot (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) \mathbf{N} \, \mathrm{d}x \\
&\quad + \varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \partial_t^7 q^- \, \partial_i^\varphi \llbracket \bar{v} \rrbracket \cdot \overline{\nabla} \partial_t^7 v_i^- \, \mathrm{d}\mathcal{V}_t + \varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \partial_t^7 q^- \, (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) \partial_3 \varphi \, \partial_3^\varphi \partial_t^7 v^- \cdot \mathbf{N} \, \mathrm{d}x \\
&=: \mathrm{VS}_{011}^* + \mathrm{VS}_{011}^{*,Z} + \mathrm{VS}_{011}^{*,R} + \mathrm{VS}_{012}^{*,R}
\end{aligned}
\tag{3.95}
$$

Next we introduce $\mathbf{F}^\sharp := \partial_t^7 f - \partial_t^7 \psi \partial_3^\varphi f$ to be the Alinhac good unknown of f with respect to ∂_t^7 in order to commute ∇^φ with ∂_t^7. Namely, we have

$$
\partial_t^7 \partial_i^\varphi f = \partial_i^\varphi \mathbf{F}^\sharp + \mathfrak{C}_i^\sharp(f), \quad \partial_t^7 D_t^\varphi f = D_t^\varphi \mathbf{F}^\sharp + \mathfrak{D}^\sharp(f),
$$

where $\mathfrak{C}^\sharp, \mathfrak{D}^\sharp$ are defined in the same way as (3.9)-(3.10) with $\mathcal{T}^\gamma = \partial_t^7$. With this formulation, we have

$$
\nabla^\varphi \cdot \partial_t^7 v^- = \nabla^\varphi \cdot \mathbf{V}^{\sharp,-} + \partial_i^\varphi (\partial_t^7 \varphi \partial_3^\varphi v_i^-) = \partial_t^7 (\nabla^\varphi \cdot v^-) - \mathfrak{C}_i^\sharp(v_i^-) + \partial_i^\varphi (\partial_t^7 \varphi \partial_3^\varphi v_i^-).
$$

Now we insert the good unknowns in VS_{011}^* to get

$$
\begin{aligned}
\mathrm{VS}_{011}^* &= \varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \partial_t^7 q^- \, (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) \partial_t^7 (\nabla^\varphi \cdot v^-) \, \mathrm{d}\mathcal{V}_t \underbrace{- \varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \partial_t^7 q^- \, (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) \left(\mathfrak{C}_i^\sharp(v_i^-) - \partial_i^\varphi (\partial_t^7 \varphi \partial_3^\varphi v_i^-) \right) \, \mathrm{d}\mathcal{V}_t}_{\mathrm{VS}_{012}^{*,Z}} \\
&= -\varepsilon^{16} \int_{\Omega^-} \mathcal{F}_p^\pm D_t^{\varphi,-} \partial_t^7 p^- \, (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) \partial_t^7 D_t^{\varphi,-} p^- \, \mathrm{d}\mathcal{V}_t + \varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \partial_t^7 (\tfrac{1}{2} |b^-|^2) \, (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) \partial_t^7 (\nabla^\varphi \cdot v^-) \, \mathrm{d}\mathcal{V}_t + \mathrm{VS}_{012}^{*,Z} \\
&=: \mathrm{VS}_{0111}^* + \mathrm{VS}_{0111}^{*,B} + \mathrm{VS}_{012}^{*,Z}.
\end{aligned}
\tag{3.96}
$$

By the definition of $\mathbf{P}^{\sharp,-}$

$$
D_t^{\varphi,-} \partial_t^7 p^- = D_t^{\varphi,-} \mathbf{P}^{\sharp,-} + D_t^{\varphi,-} (\partial_t^7 \varphi \partial_3^\varphi p^-), \quad \partial_t^7 D_t^{\varphi,-} p^- = D_t^{\varphi,-} \mathbf{P}^\sharp + \mathfrak{D}^\sharp(p^-).
$$

Then we integrate ($[\![\bar{v}]\!] \cdot \overline{\nabla}$) by parts and use symmetry to find

$$
\begin{aligned}
\mathrm{VS}^*_{0111} = &-\varepsilon^{16} \int_{\Omega^-} \mathcal{F}_p^{\pm} D_t^{\varphi,-} \mathbf{P}^{\sharp,-} \,([\![\bar{v}]\!] \cdot \overline{\nabla}) D_t^{\varphi,-} \mathbf{P}^{\sharp,-} \,\mathrm{d}\mathcal{V}_t \\
&\underbrace{+\varepsilon^{16} \int_{\Omega^-} \mathcal{F}_p^{\pm} D_t^{\varphi,-} \mathbf{P}^{\sharp,-} \left((\overline{\nabla} \cdot [\![\bar{v}]\!]) D_t^{\varphi,-} (\partial_t^7 \varphi \partial_3^\varphi p^-) - ([\![\bar{v}]\!] \cdot \overline{\nabla}) \left(\mathfrak{D}^{\sharp}(p^-) - D_t^{\varphi,-}(\partial_t^7 \varphi \partial_3^\varphi p^-) \right) \right) \mathrm{d}\mathcal{V}_t}_{\mathrm{VS}^{*,R}_{0111}} \\
&\underbrace{-\varepsilon^{16} \int_{\Omega^-} \mathcal{F}_p^{\pm} D_t^{\varphi,-}(\partial_t^7 \varphi \partial_3^\varphi p^-) \,([\![\bar{v}]\!] \cdot \overline{\nabla}) \mathfrak{D}^{\sharp}(p^-) \,\mathrm{d}\mathcal{V}_t}_{\mathrm{VS}^{*,R}_{0112}} \\
\overset{[\![\bar{v}]\!] \cdot \overline{\nabla}}{=\!=\!=} &-\frac{1}{2} \int_{\Omega^-} (\overline{\nabla} \cdot [\![\bar{v}]\!]) \left(\sqrt{\mathcal{F}_p^{\pm}} \varepsilon^8 D_t^{\varphi,-} \mathbf{P}^{\sharp,-} \right)^2 \mathrm{d}\mathcal{V}_t + \mathrm{VS}^{*,R}_{0111} + \mathrm{VS}^{*,R}_{0112},
\end{aligned}
\tag{3.97}
$$

where the first term on the right side is controlled by $\left\| (\mathcal{F}_p^{\pm})^{\frac{1}{2}} \varepsilon^8 \mathbf{P}^{*,-} \right\|_0^2 \|\overline{\nabla} [\![\bar{v}]\!]\|_{L^\infty}^2$. Next we adapt the analysis for $Z^{\pm} + ZB^{\pm}$ term to the control of $\mathrm{VS}^{*,ZB}_1 + \mathrm{VS}^{*,Z}_{011}$ and $\mathrm{VS}^{*,ZB}_2 + \mathrm{VS}^{*,Z}_{012}$. Using Gauss-Green formula, we have

$$
\begin{aligned}
\mathrm{VS}^{*,ZB}_1 + \mathrm{VS}^{*,Z}_{011} = &\,\varepsilon^{16} \int_{\Omega^-} (D_t^{\varphi,-} \partial_3^\varphi \partial_t^7 q^-)\,([\![\bar{v}]\!] \cdot \overline{\nabla}) \mathbf{N} \cdot \partial_t^7 v^- \,\mathrm{d}\mathcal{V}_t \\
&+ \varepsilon^{16} \int_{\Omega^-} [\partial_3^\varphi, D_t^{\varphi,-}](\partial_t^7 q^-)\,([\![\bar{v}]\!] \cdot \overline{\nabla}) \mathbf{N} \cdot \partial_t^7 v^- + D_t^{\varphi,-} \partial_t^7 q^- \,\partial_3^\varphi \left(([\![\bar{v}]\!] \cdot \overline{\nabla}) \mathbf{N} \right) \cdot \partial_t^7 v^- \,\mathrm{d}\mathcal{V}_t \\
=: &\,\varepsilon^{16} \int_{\Omega^-} (D_t^{\varphi,-} \partial_3^\varphi \partial_t^7 q^-)\,([\![\bar{v}]\!] \cdot \overline{\nabla}) \mathbf{N} \cdot \partial_t^7 v^- \,\mathrm{d}\mathcal{V}_t + \mathrm{VS}^{*,ZR}_1.
\end{aligned}
\tag{3.98}
$$

The main term is controlled by integrating $D_t^{\varphi,-}$ by parts under time integral and invoking the momentum equation to replace $\partial_3^\varphi q^-$ by tangential derivatives of v^-, b^-:

$$
\begin{aligned}
&\varepsilon^{16} \int_0^t \int_{\Omega^-} (D_t^{\varphi,-} \partial_3^\varphi \partial_t^7 q^-)\,([\![\bar{v}]\!] \cdot \overline{\nabla}) \mathbf{N} \cdot \partial_t^7 v^- \,\mathrm{d}\mathcal{V}_t \,\mathrm{d}\tau \\
\overset{L}{=} &-\varepsilon^{16} \int_0^t \int_{\Omega^-} (\partial_3^\varphi \partial_t^7 q^-) \, D_t^{\varphi,-} \left(([\![\bar{v}]\!] \cdot \overline{\nabla}) \mathbf{N} \cdot \partial_t^7 v^- \right) \mathrm{d}\mathcal{V}_t \,\mathrm{d}\tau + \varepsilon^{16} \int_{\Omega^-} (\partial_3^\varphi \partial_t^7 q^-)\,([\![\bar{v}]\!] \cdot \overline{\nabla}) \mathbf{N} \cdot \partial_t^7 v^- \,\mathrm{d}\mathcal{V}_t \Big|_0^t \\
\lesssim &\,\delta \|\varepsilon^8 \partial_3^\varphi \partial_t^7 q^-\|_{0,-}^2 + P(E_4^\kappa(0)) E_8^\kappa(0) + \int_0^t \|\varepsilon^8 \partial_3^\varphi \partial_t^7 q^-\|_{0,-} \|\varepsilon^8 (\overline{\partial}\partial_t^7 v^-, \partial_t^8 v^-)\|_0 P(E_4^\kappa(\tau)) \,\mathrm{d}\tau \\
\leq &\,\delta E_8^\kappa(t) + P(E_4^\kappa(0)) E_8^\kappa(0) + \int_0^t P(E_4^\kappa(\tau)) E_8^\kappa(\tau) \,\mathrm{d}\tau, \quad \forall \delta \in (0,1).
\end{aligned}
\tag{3.99}
$$

For $\mathrm{VS}^{*,ZB}_2 + \mathrm{VS}^{*,Z}_{012}$, we recall that the term $\mathfrak{C}_i^{\sharp}(v_i^-)$ in $\mathrm{VS}^{*,Z}_{012}$ includes a term $[\partial_t^7, \mathbf{N}_i/\partial_3\varphi, v_i^-]$ which also appears in $\mathrm{VS}^{*,ZB}_2$. Thus we can again use the Gauss-Green formula to analyze this term. Let us first compute the commutator in $\mathrm{VS}^{*,Z}_{012}$:

$$
\begin{aligned}
\mathfrak{C}_i^{\sharp}(v_i^-) - \partial_i^\varphi(\partial_t^7 \varphi \partial_3^\varphi v_i^-) = &-\partial_i^\varphi \partial_t^7 \varphi \partial_3^\varphi v_i + \left[\partial_t^7, \frac{\mathbf{N}_i}{\partial_3\varphi}, \partial_3 v_i^- \right] - \partial_3 v_i^- \left[\partial_t^7 \mathbf{N}_i, \frac{1}{\partial_3\varphi} \right] - \mathbf{N}_i \partial_3 v_i^- \left[\partial_t^6, \frac{1}{(\partial_3\varphi)^2} \right] \partial_t \partial_3\varphi \\
= &-\partial_i^\varphi \partial_t^7 \varphi \partial_3^\varphi v_i + \left[\partial_t^7, \mathbf{N}_i, \partial_3 v_i^- \right] + \sum_{k=1}^{6} \binom{7}{k} [\partial_t^k, (\partial_3\varphi)^{-1}] \mathbf{N}_i \, \partial_t^{7-k} \partial_3 v_i^- \\
&- \partial_3 v_i^- \left[\partial_t^7, \mathbf{N}_i, \frac{1}{\partial_3\varphi} \right] - \mathbf{N}_i \partial_3 v_i^- \left[\partial_t^6, \frac{1}{(\partial_3\varphi)^2} \right] \partial_t \partial_3\varphi \\
=: &\,\frac{1}{\partial_3\varphi} \left[\partial_t^7, \mathbf{N}_i, \partial_3 v_i^- \right] + \mathfrak{C}_i^{\sharp,R}(v_i^-)
\end{aligned}
$$

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Then

$$VS_2^{*,ZB} + VS_{012}^{*,Z} = \varepsilon^{16} \int_\Sigma D_t^{\varphi,-} \partial_t^7 q^- \left([\![\bar{v}]\!] \cdot \overline{\nabla}\right) [\partial_t^7, \mathbf{N}_i, v_i^-] \, dx' - \varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \partial_t^7 q^- \left([\![\bar{v}]\!] \cdot \overline{\nabla}\right) \left(\frac{1}{\partial_3\varphi} \left[\partial_t^7, \mathbf{N}_i, \partial_3 v_i^-\right]\right) dV_t$$

$$\underbrace{-\varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \partial_t^7 q^- \left([\![\bar{v}]\!] \cdot \overline{\nabla}\right) \mathfrak{C}_i^{\sharp,R}(v_i^-) \, dV_t}_{VS_2^{*,ZR}}$$

$$= \varepsilon^{16} \int_{\Omega^-} \partial_3^\varphi \left(D_t^{\varphi,-} \partial_t^7 q^-\right) \left([\![\bar{v}]\!] \cdot \overline{\nabla}\right) [\partial_t^7, \mathbf{N}_i, v_i^-] \, dx + VS_2^{*,ZR} + \text{lower order terms}, \tag{3.100}$$

where the first term on the right side is again controlled by integrating $D_t^{\varphi,-}$ by parts under time integral. We omit the details and just list the result

$$\int_0^t \int_{\Omega^-} \varepsilon^{16} \partial_3^\varphi \left(D_t^{\varphi,-} \partial_t^7 q^-\right) \left([\![\bar{v}]\!] \cdot \overline{\nabla}\right) [\partial_t^7, \mathbf{N}_i, v_i^-] \, dx \, d\tau \leq \delta E_8^\kappa(t) + P(E_4^\kappa(0))E_8^\kappa(0) + \int_0^t P(E_4^\kappa(\tau))E_8^\kappa(\tau) \, d\tau, \quad \forall \delta \in (0,1).$$

Now the term VS_1^* is controlled except for those remainder terms $VS_{011}^{*,R}$, $VS_{012}^{*,R}$, $VS_{0111}^{*,B}$, $VS_{0111}^{*,R}$, $VS_{0112}^{*,R}$, $VS_1^{*,ZR}$ and $VS_2^{*,ZR}$. In fact, apart from $VS_{0111}^{*,B}$, the other remainder terms can be directly controlled by counting the number of derivatives and invoking the reduction for $\partial_3^\varphi \partial_t^7 v^- \cdot \mathbf{N}$ and $\partial_3^\varphi \partial_t^7 q^-$. There is no loss of Mach number in these remainder terms. In fact, when $\partial_t^8 p^-$ appears in the remainder terms, either we have $\varepsilon^{16} \mathcal{F}_p^\pm$-weight to control it directly, or we can integrate by parts $D_t^{\varphi,-}$ and $([\![\bar{v}]\!] \cdot \overline{\nabla})$ under time integral to move one time derivative to v_i^-. Besides, the control of $\partial_t^7 \varphi, \partial_t^8 \varphi$ depends on the boundary regularity contributed by surface tension and so depends on σ^{-1}. Therefore, we can conclude the estimates of VS_1^* by

$$VS_{01}^* + VS_1^{*,ZB} + VS_2^{*,ZB} \leq VS_{0111}^{*,B} + \delta E_8^\kappa(t) + P(E_4^\kappa(0))E_8^\kappa(0) + \int_0^t P(\sigma^{-1}, E^\kappa(\tau))E_8^\kappa(\tau) \, d\tau \quad \forall \delta \in (0,1). \tag{3.101}$$

Next we control $VS_{02}^* = \varepsilon^{16} \int_{\Omega^-} \partial_i^\varphi D_t^{\varphi,-} \partial_t^7 q^- \left([\![\bar{v}]\!] \cdot \overline{\nabla}\right) \partial_t^7 v_i^- \, dV_t$. First, we commute $D_t^{\varphi,-}$ with ∂_i^φ to get

$$VS_{02}^* = \varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \partial_i^\varphi \partial_t^7 q^- \left([\![\bar{v}]\!] \cdot \overline{\nabla}\right) \partial_t^7 v_i^- \, dV_t + \varepsilon^{16} \int_{\Omega^-} \partial_i^\varphi v_j^- \partial_j^\varphi \partial_t^7 q^- \left([\![\bar{v}]\!] \cdot \overline{\nabla}\right) \partial_t^7 v_i^- \, dV_t$$

$$=: VS_{021}^* + VS_{021}^{*,R}. \tag{3.102}$$

In the first term, we integrate by parts $D_t^{\varphi,-}$ under time integral and commute $D_t^{\varphi,-}$ with $([\![\bar{v}]\!] \cdot \overline{\nabla})$ to get

$$\int_0^t VS_{021}^* \, d\tau \overset{L}{=} -\varepsilon^{16} \int_0^t \int_{\Omega^-} \partial_i^\varphi \partial_t^7 q^- \left([\![\bar{v}]\!] \cdot \overline{\nabla}\right) D_t^{\varphi,-} \partial_t^7 v_i^- \, dV_t \, d\tau + \varepsilon^{16} \int_{\Omega^-} \partial_i^\varphi \partial_t^7 q^- \left([\![\bar{v}]\!] \cdot \overline{\nabla}\right) \partial_t^7 v_i^- \, dV_t \Big|_0^t$$

$$- \varepsilon^{16} \int_0^t \int_{\Omega^-} \partial_i^\varphi \partial_t^7 q^- [D_t^{\varphi,-}, ([\![\bar{v}]\!] \cdot \overline{\nabla})] \partial_t^7 v_i^- \, dV_t \, d\tau$$

$$=: \int_0^t VS_{0211}^* \, d\tau + VS_{022}^{*,R} + \int_0^t VS_{023}^{*,R} \, d\tau. \tag{3.103}$$

Next we insert the good unknowns $\mathbf{Q}^{\sharp,-}$ and $\mathbf{V}^{\sharp,-}$ and invoke again the momentum equation $\rho^- D_t^{\varphi,-} \mathbf{V}^{\sharp,-} - (b^- \cdot \nabla^\varphi)\mathbf{B}^{\sharp,-} = -\nabla^\varphi \mathbf{Q}^{\sharp,-} + \mathcal{R}_v^{\sharp,-} - \mathfrak{C}^\sharp(q^-)$ to get

$$VS_{0211}^* = -\varepsilon^{16} \int_{\Omega^-} \partial_i^\varphi (\mathbf{Q}^{\sharp,-} - \partial_t^7 \varphi \partial_3^\varphi q^-) \left([\![\bar{v}]\!] \cdot \overline{\nabla}\right) D_t^{\varphi,-} (\mathbf{V}^{\sharp,-} - \partial_t^7 \varphi \partial_3^\varphi v_i) \, dV_t$$

$$= \varepsilon^{16} \int_{\Omega^-} \rho^- D_t^{\varphi,-} \mathbf{V}^{\sharp,-} \cdot \left([\![\bar{v}]\!] \cdot \overline{\nabla}\right) D_t^{\varphi,-} \mathbf{V}^{\sharp,-} \, dV_t - \varepsilon^{16} \int_{\Omega^-} (b^- \cdot \nabla^\varphi)\mathbf{B}^{\sharp,-} \cdot \left([\![\bar{v}]\!] \cdot \overline{\nabla}\right) D_t^{\varphi,-} \mathbf{V}^{\sharp,-} \, dV_t$$

$$+ \varepsilon^{16} \int_{\Omega^-} \left(\mathfrak{C}^\sharp(q^-) - \mathcal{R}_v^{\sharp,-}\right) \cdot \left([\![\bar{v}]\!] \cdot \overline{\nabla}\right) D_t^{\varphi,-} \mathbf{V}^{\sharp,-} \, dV_t + \varepsilon^{16} \int_{\Omega^-} \partial_i^\varphi (\partial_t^7 \varphi \partial_3^\varphi q^-) \left([\![\bar{v}]\!] \cdot \overline{\nabla}\right) D_t^{\varphi,-} \partial_t^7 v_i^- \, dV_t$$

$$=: \varepsilon^{16} \int_{\Omega^-} \rho^- D_t^{\varphi,-} \mathbf{V}^{\sharp,-} \left([\![\bar{v}]\!] \cdot \overline{\nabla}\right) D_t^{\varphi,-} \mathbf{V}^{\sharp,-} \, dV_t + VS_{0211}^{*,B} + VS_{0211}^{*,R} + VS_{0212}^{*,R}, \tag{3.104}$$

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where the first term is again controlled by integrating by parts in $(\llbracket \bar v \rrbracket \cdot \overline\nabla)$ and using symmetry

$$\varepsilon^{16} \int_{\Omega^-} \rho^- D_t^{\varphi,-} \mathbf{V}^{\sharp,-} \, (\llbracket \bar v \rrbracket \cdot \overline\nabla) D_t^{\varphi,-} \mathbf{V}^{\sharp,-} \, \mathrm{d}\mathcal{V}_t = \frac{\varepsilon^{16}}{2} \int_{\Omega^-} \left(\overline\nabla \cdot (\rho^- \, \llbracket \bar v \rrbracket) \right) \left| D_t^{\varphi,-} \mathbf{V}^{\sharp,-} \right|^2 \, \mathrm{d}\mathcal{V}_t \le P(E_4^\kappa(t)) E_8^\kappa(t). \tag{3.105}$$

Next we wish to combine $\mathrm{VS}_{0211}^{*,B}$ with $\mathrm{VS}_{0111}^{*,B} := \varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \partial_t^7 (\frac{1}{2}|b^-|^2)\,(\llbracket \bar v \rrbracket \cdot \overline\nabla) \partial_t^7 (\nabla^\varphi \cdot v^-) \, \mathrm{d}\mathcal{V}_t$ to get a cancellation structure. In $\mathrm{VS}_{0111}^{*,B}$, we invoke the evolution equation $D_t^{\varphi,-} b_j^- = (b^- \cdot \nabla^\varphi) v^- - b^- (\nabla^\varphi \cdot v^-)$ to get

$$\begin{aligned}
\mathrm{VS}_{0111}^{*,B} \overset{L}{=} {} & \varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \mathbf{B}_j^{\sharp,-} b_j^- \, (\llbracket \bar v \rrbracket \cdot \overline\nabla) \partial_t^7 (\nabla^\varphi \cdot v^-) \, \mathrm{d}\mathcal{V}_t \\
= {} & \varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \mathbf{B}_j^{\sharp,-} \, (\llbracket \bar v \rrbracket \cdot \overline\nabla) \partial_t^7 \left(b_j^- (\nabla^\varphi \cdot v^-) \right) \, \mathrm{d}\mathcal{V}_t + \underbrace{\varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \mathbf{B}_j^{\sharp,-} \left[b_j, (\llbracket \bar v \rrbracket \cdot \overline\nabla) \partial_t^7 \right] (\nabla^\varphi \cdot v^-) \, \mathrm{d}\mathcal{V}_t}_{\mathrm{VS}_{0111}^{*,BR}} \\
= {} & -\varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \mathbf{B}_j^{\sharp,-} \, (\llbracket \bar v \rrbracket \cdot \overline\nabla) D_t^{\varphi,-} \mathbf{B}_j^{\sharp,-} \, \mathrm{d}\mathcal{V}_t + \varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \mathbf{B}_j^{\sharp,-} \, (\llbracket \bar v \rrbracket \cdot \overline\nabla) \partial_t^7 ((b^- \cdot \nabla^\varphi) v_j^-) \, \mathrm{d}\mathcal{V}_t \\
& - \varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \mathbf{B}_j^{\sharp,-} \, (\llbracket \bar v \rrbracket \cdot \overline\nabla) \mathfrak{D}^\sharp(b_j^-) \, \mathrm{d}\mathcal{V}_t + \mathrm{VS}_{0111}^{*,BR},
\end{aligned} \tag{3.106}$$

where the first term on the right side is again controlled by integrating by parts in $(\llbracket \bar v \rrbracket \cdot \overline\nabla)$ and using symmetry, and the third term on the right side is controlled directly after inserting the expression of $\mathfrak{D}^\sharp(b)$. We denote

$$\mathrm{VS}_{0112}^{*,B} := \varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \mathbf{B}_j^{\sharp,-} \, (\llbracket \bar v \rrbracket \cdot \overline\nabla) \partial_t^7 ((b^- \cdot \nabla^\varphi) v_j^-) \, \mathrm{d}\mathcal{V}_t$$

to be the second term on the right side above. Inserting the good unknown $\mathbf{V}^{\sharp,-}$, the term $\mathrm{VS}_{0112}^{*,B}$ is equal to

$$\begin{aligned}
& \varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \mathbf{B}_i^{\sharp,-} \, (\llbracket \bar v \rrbracket \cdot \overline\nabla) \left((b^- \cdot \nabla^\varphi) \mathbf{V}_i^{\sharp,-} \right) \, \mathrm{d}\mathcal{V}_t + \underbrace{\varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \mathbf{B}_i^{\sharp,-} \, (\llbracket \bar v \rrbracket \cdot \overline\nabla) \left([\partial_t^7, b_j^-] \partial_j^\varphi v_i^- + b_j^- \mathfrak{C}_j^\sharp(v_i^-) \right) \, \mathrm{d}\mathcal{V}_t}_{\mathrm{VS}_{0112}^{*,BR}} \\
= {} & \varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \mathbf{B}_i^{\sharp,-} \, (b^- \cdot \nabla^\varphi) \left((\llbracket \bar v \rrbracket \cdot \overline\nabla) \mathbf{V}_i^{\sharp,-} \right) \, \mathrm{d}\mathcal{V}_t + \varepsilon^{16} \int_{\Omega^-} D_t^{\varphi,-} \mathbf{B}_i^{\sharp,-} \left[(b^- \cdot \nabla^\varphi), (\llbracket \bar v \rrbracket \cdot \overline\nabla) \right] \mathbf{V}_i^{\sharp,-} \, \mathrm{d}\mathcal{V}_t + \mathrm{VS}_{0112}^{*,BR} \\
=: {} & \mathrm{VS}_{0113}^{*,B} + \mathrm{VS}_{0113}^{*,BR} + \mathrm{VS}_{0112}^{*,BR}.
\end{aligned} \tag{3.107}$$

Now we can integrate by parts $(b^- \cdot \nabla^\varphi)$ and then $D_t^{\varphi,-}$ in $\mathrm{VS}_{0113}^{*,B}$ in order to produce the cancellation with $\mathrm{VS}_{0211}^{*,B}$. Under time integral, $\int_0^t \mathrm{VS}_{0113}^{*,B} \, \mathrm{d}\tau$ is equal to

$$\begin{aligned}
& \int_0^t \underbrace{\varepsilon^{16} \int_{\Omega^-} (b^- \cdot \nabla^\varphi) \mathbf{B}_i^{\sharp,-} \, (\llbracket \bar v \rrbracket \cdot \overline\nabla) D_t^{\varphi,-} \mathbf{V}_i^{\sharp,-} \, \mathrm{d}\mathcal{V}_t}_{= -\mathrm{VS}_{0211}^{*,B}} \, \mathrm{d}\tau + \varepsilon^{16} \int_{\Omega^-} (b^- \cdot \nabla^\varphi) \mathbf{B}_i^{\sharp,-} \, (\llbracket \bar v \rrbracket \cdot \overline\nabla) \mathbf{V}_i^{\sharp,-} \, \mathrm{d}\mathcal{V}_t \Big|_0^t \\
& + \varepsilon^{16} \int_0^t \int_{\Omega^-} [(b^- \cdot \nabla^\varphi), D_t^{\varphi,-}] \mathbf{B}_i^{\sharp,-} \, (\llbracket \bar v \rrbracket \cdot \overline\nabla) \mathbf{V}_i^{\sharp,-} \, \mathrm{d}\mathcal{V}_t \, \mathrm{d}\tau + \varepsilon^{16} \int_0^t \int_{\Omega^-} (b^- \cdot \nabla^\varphi) \mathbf{B}_i^{\sharp,-} \left[D_t^{\varphi,-}, (\llbracket \bar v \rrbracket \cdot \overline\nabla) \right] \mathbf{V}_i^{\sharp,-} \, \mathrm{d}\mathcal{V}_t \, \mathrm{d}\tau \\
=: {} & -\int_0^t \mathrm{VS}_{0211}^{*,B} \, \mathrm{d}\tau + \mathrm{VS}_{0211}^{*,BR} + \int_0^t \mathrm{VS}_{0212}^{*,BR} + \mathrm{VS}_{0213}^{*,BR} \, \mathrm{d}\tau.
\end{aligned} \tag{3.108}$$

Note that $[D_t^{\varphi,-}, (b^- \cdot \nabla^\varphi)] = -(\nabla^\varphi \cdot v^-)(b^- \cdot \nabla^\varphi) f$ and when we commute $(\llbracket \bar v \rrbracket \cdot \overline\nabla)$ with either $D_t^{\varphi,-}$ or $(b^- \cdot \nabla^\varphi)$,, no normal derivative will be generated because the weight functions in front of ∂_3 (namely, $b^- \cdot \mathbf{N}$ and $(v^- \cdot \mathbf{N} - \partial_t \varphi)$) are still vanishing on the interface Σ after taking $(\llbracket \bar v \rrbracket \cdot \overline\nabla)$. Therefore, the commutators above are all controllable in $\| \cdot \|_{8,*,-}$ norm and no loss of Mach number occurs. The following remainder terms are controlled directly

$$\mathrm{VS}_{022}^{*,R} + \mathrm{VS}_{0211}^{*,BR} + \int_0^t \mathrm{VS}_{021}^{*,R} + \mathrm{VS}_{023}^{*,R} + \mathrm{VS}_{0111}^{*,BR} + \mathrm{VS}_{0112}^{*,BR} + \mathrm{VS}_{0212}^{*,BR} + \mathrm{VS}_{0213}^{*,BR} \, \mathrm{d}\tau$$

$$\le \delta E_8^\kappa(t) + P(E^\kappa(0)) + \int_0^t P(E^\kappa(\tau)) \, \mathrm{d}\tau. \tag{3.109}$$

In the terms $VS^{*,R}_{0211} + VS^{*,R}_{0212}$, we can integrate $(\llbracket \bar{v} \rrbracket \cdot \overline{\nabla})$ by parts to get to get the desired control thanks to the $\sqrt{\sigma}$-weighted boundary regularity of ψ

$$VS^{*,R}_{0211} + VS^{*,R}_{0212} \lesssim_{\sigma^{-1}} \left(|\varepsilon^8 \partial_t^7 \psi|_2 + |\varepsilon^8 \partial_t^8 \psi|_1 \right) \| \varepsilon^8 D_t^{\varphi,-} \partial_t^7 v \|_{0,-} P(E_4^\kappa(t)). \tag{3.110}$$

Thus, the control of VS^*_{02} term is concluded by

$$\int_0^t VS^*_{02} + VS^{*,B}_{0111} \, d\tau \leq \delta E_8^\kappa(t) + P(E^\kappa(0)) + \int_0^t P(\sigma^{-1}, E^\kappa(\tau)) \, d\tau. \tag{3.111}$$

Finally, combining (3.92), (3.93), (3.101) and (3.111), we get the estimate of VS^* term

$$\int_0^t VS^* \, d\tau \leq \delta E_8^\kappa(t) + P(E^\kappa(0)) + \int_0^t P(\sigma^{-1}, E^\kappa(\tau)) \, d\tau. \tag{3.112}$$

Step 3: Control of RT term.

In step 3, we control the terms RT^* and $RT^{*,\pm}$ defined in (3.65)-(3.66), The latter one can be directly controlled by using symmetry

$$RT^{*,\pm} = \mp \frac{1}{2} \int_\Sigma (\overline{\nabla} \cdot (\partial_3 q^\pm \, v^\pm)) \left| \overline{D_t^-} \partial_t^7 \psi \right|^2 \, dx' \leq \sigma^{-1} E_4^\kappa(t) E_8^\kappa(t). \tag{3.113}$$

The term $RT^* = -\varepsilon^{16} \int_\Sigma \llbracket \partial_3 q \rrbracket \, \overline{D_t^-} \partial_t^7 \psi \, \partial_t \overline{D_t^-} \partial_t^7 \psi \, dx'$ cannot be controlled in the same way as in the estimates of spatial derivatives because we do not have $L^2(\Sigma)$-control for $\partial_t \overline{D_t^-} \partial_t^7 \psi$ without κ-weight nor can we integrate by parts $\partial_t^{1/2}$. To overcome this difficulty, we need to invoke the kinematic boundary condition to reduce the number of time derivatives. We have

$$\overline{D_t^-} \partial_t^7 \psi = \partial_t^7 v^- \cdot N + [\partial_t^7, v^-, N], \quad \partial_t \overline{D_t^-} \partial_t^7 \psi = \partial_t^8 v^- \cdot N + 8 \partial_t^7 v^- \cdot \partial_t N + \text{ lower order terms.}$$

Plugging it to RT^*, we find

$$RT^* \overset{L}{=} -\varepsilon^{16} \int_\Sigma \llbracket \partial_3 q \rrbracket \, \partial_t^7 v^- \cdot N \, \partial_t^8 v^- \cdot N \, dx' - 8\varepsilon^{16} \int_\Sigma \llbracket \partial_3 q \rrbracket \, \partial_t^7 v^- \cdot N \, \partial_t^7 v^- \cdot \partial_t N \, dx' =: RT_1^* + RT_2^*. \tag{3.114}$$

The term RT_2^* can be controlled by using Gauss-Green formula

$$RT_2^* \overset{L}{=} -8\varepsilon^{16} \int_{\Omega^-} \llbracket \partial_3 q \rrbracket (\partial_3^\varphi \partial_t^7 v^- \cdot \mathbf{N})(\partial_t^7 v^- \cdot \partial_t \mathbf{N}) \, d\mathcal{V}_t - 8\varepsilon^{16} \int_{\Omega^-} \llbracket \partial_3 q \rrbracket (\partial_t^7 v^- \cdot \mathbf{N})(\partial_3 \partial_t^7 v^- \cdot \partial_t \mathbf{N}) \, dx, \tag{3.115}$$

where $\llbracket \partial_3 q \rrbracket$ is defined via Sobolev extension. The first term above is directly controlled after invoking the reduction $\partial_3^\varphi \partial_t^7 v^- \cdot \mathbf{N} \overset{L}{=} -\partial_t^7 (\varepsilon^2 D_t^{\varphi,-} p^- + \overline{\nabla} \cdot \bar{v}^-)$. For the second term, it suffices to integrate ∂_t by parts under time integral

$$- 8\varepsilon^{16} \int_0^t \int_{\Omega^-} \llbracket \partial_3 q \rrbracket (\partial_t^7 v^\pm \cdot \mathbf{N})(\partial_3 \partial_t^7 v^- \cdot \partial_t \mathbf{N}) \, dx \, d\tau$$

$$\overset{L}{=} - 8\varepsilon^{16} \int_{\Omega^-} \llbracket \partial_3 q \rrbracket (\partial_t^7 v^- \cdot \mathbf{N})(\partial_3 \partial_t^6 v^- \cdot \partial_t \mathbf{N}) \, dx \Big|_0^t + 8 \int_0^t \int_{\Omega^-} \llbracket \partial_3 q \rrbracket (\partial_t^8 v^- \cdot \mathbf{N})(\partial_3 \partial_t^6 v^- \cdot \partial_t \mathbf{N}) \, dx$$

$$\lesssim \delta \| \varepsilon^8 \partial_3 \partial_t^6 v^- \|_{0,-}^2 + P(E^\kappa(0)) + \int_0^t P(E_4^\kappa(\tau)) E_8^\kappa(\tau) \, d\tau \tag{3.116}$$

Using the same trick as above, the term RT_1^* is directly controlled by repeated invoking $\partial_3^\varphi \partial_t^7 v^- \cdot \mathbf{N} \overset{L}{=} -\partial_t^7 (\varepsilon^2 D_t^{\varphi,-} p^- + \overline{\nabla} \cdot \bar{v}^-)$

$$\int_0^t RT_1^* \, d\tau \overset{L}{=} - \varepsilon^{16} \int_0^t \int_{\Omega^-} \llbracket \partial_3 q \rrbracket \left((\partial_3^\varphi \partial_t^7 v^- \cdot \mathbf{N})(\partial_t^8 v^- \cdot \mathbf{N}) + (\partial_t^7 v^- \cdot \mathbf{N})(\partial_3^\varphi \partial_t^8 v^- \cdot \mathbf{N}) \right) d\mathcal{V}_t \, d\tau$$

$$\overset{\partial_t, L}{=\!=\!=} \int_0^t \int_{\Omega^-} \llbracket \partial_3 q \rrbracket \left((\partial_3^\varphi \partial_t^7 v^- \cdot \mathbf{N})(\partial_t^8 v^- \cdot \mathbf{N}) - \partial_t (\partial_t^7 v^- \cdot \mathbf{N})(\partial_3^\varphi \partial_t^7 v^- \cdot \mathbf{N}) \right) d\mathcal{V}_t \, d\tau$$

$$- \varepsilon^{16} \int_{\Omega^-} \llbracket \partial_3 q \rrbracket (\partial_t^7 v^- \cdot \mathbf{N})(\partial_3^\varphi \partial_t^7 v^- \cdot \mathbf{N}) \, d\mathcal{V}_t \Big|_0^t$$

$$\lesssim \delta E_8^\kappa(t) + P(E^\kappa(0)) + \int_0^t P(E_4^\kappa(\tau)) E_8^\kappa(\tau) \, d\tau. \tag{3.117}$$

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Hence, we conclude the estimate of RT* by

$$\int_0^t RT^* \, d\tau \lesssim \delta E_8^\kappa(t) + P(E^\kappa(0)) + \int_0^t P(E_4^\kappa(\tau)) E_8^\kappa(\tau) \, d\tau. \tag{3.118}$$

Step 4: The cancellation structure between ZB^* and Z^*.

Now we control the term $ZB^{*,\pm} + Z^{*,\pm}$. Note that we cannot integrate by parts $\overline{\partial}^{1/2}$ due to the lack of spatial derivatives. First, $ZB^{*,\pm}$ can be written as

$$ZB^{*,\pm} = \mp \varepsilon^{16} \int_\Sigma \overline{D_t^-} \partial_t^7 q^\pm (\partial_3 v^\pm \cdot N) \overline{D_t^-} \partial_t^7 \psi \, dx' \pm \varepsilon^{16} \int_\Sigma \overline{D_t^-} \partial_t^7 \psi \, \partial_3 q^\pm (\partial_3 v^\pm \cdot N) \overline{D_t^-} \partial_t^7 \psi \, dx'$$

$$\mp \varepsilon^{16} \int_\Sigma \mathbf{Q}^{*,\pm} \left[D_t^{\varphi,-} \partial_t^7, N_i, v_i^\pm \right] dx'$$

$$=: ZB_1^{*,R,\pm} + ZB_2^{*,R,\pm} + ZB_0^{*,\pm}. \tag{3.119}$$

The second term on the right side can be directly controlled. We have

$$ZB_2^{*,R,\pm} \le \left| D_t^{\varphi,-} \partial_t^7 \psi \right|_0^2 \left| \partial_3 q^\pm (\partial_3 v^\pm \cdot N) \right|_{L^\infty} \le P(\sigma^{-1}, E_4^\kappa(t)) E_8^\kappa(t). \tag{3.120}$$

For the first term, using again $\overline{D_t^-} \partial_t^7 \psi = \partial_t^7 v \cdot N$+lower order terms, we can convert it to an interior integral.

$$\int_0^t ZB_1^{*,R,\pm} \, d\tau \overset{L}{=} \varepsilon^{16} \int_0^t \int_{\Omega^\pm} (\partial_3 v^\pm \cdot \mathbf{N}) \left(\partial_3^\varphi D_t^{\varphi,-} \partial_t^7 q^\pm \, \partial_t^7 v^\pm \cdot \mathbf{N} + D_t^{\varphi,-} \partial_t^7 q^\pm \, \partial_3^\varphi \partial_t^7 v^\pm \cdot \mathbf{N} \right) d\mathcal{V}_t$$

$$\overset{D_t^{\varphi,-}}{=} \varepsilon^{16} \int_{\Omega^\pm} (\partial_3 v^\pm \cdot \mathbf{N}) \partial_3^\varphi \partial_t^7 q^\pm \, \partial_t^7 v^\pm \cdot \mathbf{N} \, d\mathcal{V}_t \Big|_0^t + \varepsilon^{16} \int_0^t \int_{\Omega^\pm} (\partial_3 v^\pm \cdot \mathbf{N}) \left([\partial_3^\varphi, D_t^{\varphi,-}] \partial_t^7 q^\pm \right) \partial_t^7 v^\pm \cdot \mathbf{N} \, d\mathcal{V}_t \, d\tau \tag{3.121}$$

$$+ \varepsilon^{16} \int_0^t \int_{\Omega^\pm} (\partial_3 v^\pm \cdot \mathbf{N}) \left(\partial_3^\varphi \partial_t^7 q^\pm \, D_t^{\varphi,-} \partial_t^7 v^\pm \cdot \mathbf{N} + D_t^{\varphi,-} \partial_t^7 q^\pm \, \partial_3^\varphi \partial_t^7 v^\pm \cdot \mathbf{N} \right) d\mathcal{V}_t + \text{l.o.t}$$

Now we can invoke the reduction for $\partial_3^\varphi q$ and $\partial_3^\varphi v \cdot \mathbf{N}$ to convert ∂_3^φ to a tangential derivative

$$\partial_3^\varphi \partial_t^7 q \overset{L}{=} \partial_t^7 (\rho D_t^\varphi v - (b \cdot \nabla^\varphi) b), \quad \partial_3^\varphi \partial_t^7 v \cdot \mathbf{N} \overset{L}{=} -\partial_t^7 (\mathcal{F}_p D_t^\varphi p + \overline{\nabla} \cdot \bar{v}).$$

Note that the second equation above produces an extra $\mathcal{F}_p = O(\varepsilon^2)$ weight, so there is no loss of Mach number when $D_t^{\varphi,-} \partial_t^7 q$ appears. When $D_t^{\varphi,-} \partial_t^7 q$ is multiplied by $\partial_t^7 \overline{\nabla} \cdot \bar{v}$, we can further integrate by parts in ∂_t and then in $\overline{\nabla} \cdot$ to move one time derivative to v. Hence, $ZB_1^{*,R,\pm}$ is controlled in $\| \cdot \|_{8,*}$ norm without loss of ε-weights

$$\int_0^t ZB_1^{*,R,\pm} \, d\tau \lesssim \delta E_8^\kappa(t) + P(E^\kappa(0)) + \int_0^t P(E^\kappa(\tau)) \, d\tau. \tag{3.122}$$

Next we will see again the cancellation structure in $ZB_0^{*,\pm} + Z^{*,\pm}$. From (3.9), we have

$$\sum_{i=1}^3 \mathfrak{C}_i^*(v_i) = \partial_3^\varphi (\nabla^\varphi \cdot v) D_t^{\varphi,-} \partial_t^7 \varphi + \left[D_t^{\varphi,-} \partial_t^7, \frac{\mathbf{N}_i}{\partial_3 \varphi}, \partial_3 v_i \right] + \partial_3 v_i \left[D_t^{\varphi,-} \partial_t^7, \mathbf{N}_i, (\partial_3 \varphi)^{-1} \right]$$

$$+ (\partial_3 v \cdot \mathbf{N}) \left[D_t^{\varphi,-} \partial_t^6, \frac{1}{(\partial_3 \varphi)^2} \right] \partial_t \partial_3 \varphi + (\partial_3 \varphi)^{-1} \mathbf{N} \cdot [D_t^{\varphi,-} \partial_t^7, \partial_3] v + (\partial_3 \varphi)^{-1} (\partial_3^\varphi v \cdot \mathbf{N}) [D_t^{\varphi,-} \partial_t^7, \partial_3] \varphi,$$

where we have further analysis on the second term and the fifth term

$$\left[D_t^{\varphi,-} \partial_t^7, \frac{\mathbf{N}_i}{\partial_3 \varphi}, \partial_3 v_i \right] \overset{L}{=} (\partial_3 \varphi)^{-1} \left[D_t^{\varphi,-} \partial_t^7, \mathbf{N}_i, \partial_3 v_i \right] - \partial_3^\varphi \partial_t \varphi \partial_3^\varphi D_t^{\varphi,-} \partial_t^6 v \cdot \mathbf{N}, \tag{3.123}$$

$$(\partial_3 \varphi)^{-1} \mathbf{N} \cdot [D_t^{\varphi,-} \partial_t^7, \partial_3] v = \left(\partial_3^\varphi \bar{v}^- \cdot \overline{\nabla} \right) \partial_t^7 v \cdot \mathbf{N} + \partial_3 \left((\partial_3 \varphi)^{-1} (v^- \cdot \mathbf{N} - \partial_t \varphi) \right) \partial_3^\varphi \partial_t^7 v \cdot \mathbf{N}. \tag{3.124}$$

Thus, we find that, apart from the term $(\partial_3 \varphi)^{-1} \left[D_t^{\varphi,-} \partial_t^7, \mathbf{N}_i, \partial_3 v_i \right]$, all the other terms in $\mathfrak{C}_i^*(v_i)$ include either a tangential derivative falling on the leading order term or the term $\partial_3^\varphi v \cdot \mathbf{N}$ (possibly with some derivatives) such that $\mathcal{F}_p D_t^\varphi p$ and $\overline{\nabla} \cdot \bar{v}$ are

produced by invoking the continuity equation. Thus, when $\mathbf{Q}^*$ is multiplied with these terms, its contribution in $Z^{*,\pm}$ can be directly controlled without any loss of weights of Mach number.

It now remains to control $ZB_0^{*,\pm} + Z^{*,\pm}$ with $Z_0^{*,\pm} := \varepsilon^{16} \int_{\Omega^\pm} \mathbf{Q}^{*,\pm} (\partial_3 \varphi)^{-1} \left[D_t^{\varphi,-} \partial_t^7, \mathbf{N}_i, \partial_3 v_i \right] d\mathcal{V}_t$. Using $d\mathcal{V}_t = \partial_3 \varphi \, dx$ and Gauss-Green formula, we have

$$ZB_0^{*,\pm} + Z_0^{*,\pm} = \mp \varepsilon^{16} \int_\Sigma \mathbf{Q}^{*,\pm} \left[D_t^{\varphi,-} \partial_t^7, \mathbf{N}_i, v_i^\pm \right] dx' + \varepsilon^{16} \int_{\Omega^\pm} \mathbf{Q}^{*,\pm} \left[D_t^{\varphi,-} \partial_t^7, \mathbf{N}_i, \partial_3 v_i^\pm \right] dx$$

$$= \sum_{j=0}^1 \sum_{k=1}^6 \varepsilon^{16} \binom{7}{k} \left(\pm \int_\Sigma \mathbf{Q}^{*,\pm} (D_t^{\varphi,-})^j \partial_t^k v_i^\pm \, (D_t^{\varphi,-})^{1-j} \partial_t^{6-k} N_i \, dx' + \int_{\Omega^\pm} \mathbf{Q}^{*,\pm} (D_t^{\varphi,-})^j \partial_t^k \partial_3 v_i^\pm \, (D_t^{\varphi,-})^{1-j} \partial_t^{6-k} \mathbf{N}_i \, dx \right)$$

$$\overset{L}{=} - \sum_{j=0}^1 \sum_{k=1}^6 \varepsilon^{16} \binom{7}{k} \int_{\Omega^\pm} \partial_3^\varphi \mathbf{Q}^{*,\pm} (D_t^{\varphi,-})^j \partial_t^k v_i^\pm \, (D_t^{\varphi,-})^{1-j} \partial_t^{6-k} \mathbf{N}_i \, d\mathcal{V}_t. \tag{3.125}$$

Recall that $\mathbf{Q}^* = D_t^{\varphi,-} \partial_t^7 q^- - D_t^{\varphi,-} \partial_t^7 \varphi \partial_3^\varphi q$, we can integrate by parts this $D_t^{\varphi,-}$ under time integral and invoke the momentum equation to reduce $\partial_3^\varphi \mathbf{Q}^*$ to $-\rho D_t^\varphi \mathbf{V}^* + (b \cdot \nabla^\varphi) \mathbf{B}^*$+lower order terms. Note that $[\partial_3^\varphi, D_t^{\varphi,-}] \partial_t^7 q = \partial_3^\varphi v_j^- \partial_j^\varphi q = (\partial_3^\varphi \bar{v}^- \cdot \overline{\nabla}) \partial_t^7 q + (\partial_3^\varphi v^- \cdot \mathbf{N}) \partial_t^7 \partial_3 q$, so one can still convert the normal derivative ∂q to a tangential derivative of v, b. Thus, we have

$$\int_0^t ZB_0^{*,\pm} + Z_0^{*,\pm} \, d\tau \overset{L}{=} \sum_{j=0}^1 \sum_{k=1}^6 \varepsilon^{16} \binom{7}{k} \int_0^t \int_{\Omega^\pm} \partial_3^\varphi \mathbf{Q}^{\sharp,\pm} D_t^{\varphi,-} \left((D_t^{\varphi,-})^j \partial_t^k v_i^\pm \cdot, (D_t^{\varphi,-})^{1-j} \partial_t^{6-k} \mathbf{N} \right) d\mathcal{V}_t \, d\tau$$

$$+ \varepsilon^{16} \binom{7}{k} \int_0^t \int_{\Omega^\pm} \partial_3^\varphi \mathbf{Q}^{\sharp,\pm} \left((D_t^{\varphi,-})^j \partial_t^k v_i^\pm \cdot, (D_t^{\varphi,-})^{1-j} \partial_t^{6-k} \mathbf{N} \right) d\mathcal{V}_t \Big|_0^t$$

$$\lesssim \delta \| \partial_3^\varphi \mathbf{Q}^{\sharp,\pm} \|_0^2 + P(E^\kappa(0)) + \int_0^t P(\sigma^{-1}, E^\kappa(\tau)) \, d\tau, \quad \forall \delta \in (0,1). \tag{3.126}$$

Combining this with the control of remainder terms and commutators, we can easily obtain that

$$\int_0^t ZB^{*,\pm} + Z^{*,\pm} \, d\tau \lesssim \delta E_8^\kappa(0) + P(E^\kappa(0)) + \int_0^t P(\sigma^{-1}, E^\kappa(\tau)) \, d\tau, \quad \forall \delta \in (0,1). \tag{3.127}$$

3.5 Tangential estimates: general cases and summary

Let $\mathcal{T}^\gamma = (\omega(x_3)\partial_3)^{\gamma_4} \partial_t^{\gamma_0} \overline{\partial}_1^{\gamma_1} \overline{\partial}_2^{\gamma_2}$ be a tangential derivative with length of the multi-index $\langle \gamma \rangle := \gamma_0 + \gamma_1 + \gamma_2 + 2 \times 0 + \gamma_4$. Section 3.3.1-Section 3.4 are devoted to the control of full spatial derivatives ($\gamma_1 + \gamma_2 = \langle \gamma \rangle$) and full time derivatives ($\gamma_0 = \langle \gamma \rangle$). Now we analyze how to handle the general case.

Space-time mixed derivatives: $\gamma_0 > 0$ and $\gamma_1 + \gamma_2 > 0$

Let us temporarily assume $\gamma_4 = 0$. In this case, the tangential derivatives that we need to consider have the form $\overline{\partial}^{4-l-k} \partial_t^k \mathcal{T}^\alpha$ with $\langle \alpha \rangle = 2l$, $\alpha_4 = 0$ and weights of Mach number ε^{2l}. That is, we need to consider $\varepsilon^{2l} \partial_t^{k+\alpha_0} \overline{\partial}^{4+l-k-\alpha_0}$-estimates. Following the previous paper [38] by Luo and the author, the control of space-time mixed tangential derivatives ($0 < k + \alpha_0 < 4 + l$) is the same as the control of purely spatial tangential derivatives. In particular, compared with the one-phase fluid problem [38], we only need one spatial derivative to do integration by parts in order for the control of the extra problematic term

$$\mathrm{VS} := \varepsilon^{4l} \int_\Sigma \partial_t^{k+\alpha_0} \overline{\partial}^{4+l-k-\alpha_0} q^- \, (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla}) \partial_t^{k+\alpha_0} \overline{\partial}^{4+l-k-\alpha_0} \psi \, dx'$$

in which we need to integrate by parts $\overline{\partial}^{1/2}$ and seek for the control of $\varepsilon^{2l} \left| \partial_t^{k+\alpha_0} \overline{\partial}^{4+l-k-\alpha_0} \right|_{1.5}$. Mimicking the proof of Lemma 3.5, we can show that (replacing $k + \alpha_0$ by k)

Lemma 3.7 (Elliptic estimate for the time derivatives of the free interface). Fix $l \in \{0, 1, 2, 3, 4\}$. For $0 < k < 4 + l$, we have the following uniform-in-(ε, κ) inequality, in which the first term on the right side disappears when $\kappa = 0$.

$$\left| \varepsilon^{2l} \partial_t^k \psi \right|_{5.5+l-k} \leq \left| \varepsilon^{2l} \partial_t^k \psi(0) \right|_{5.5+l-k} + \sigma^{-1} \left| \varepsilon^{2l} \partial_t^k \llbracket q \rrbracket \right|_{3.5+l-k}$$

$$+ P\left(\sigma^{-1}, |\overline{\nabla}\psi|_{L^\infty}, \sum_{j=0}^l E_{4+j}^\kappa(t) \right) \left(\left| \varepsilon^{2l} \partial_t^k \psi \right|_{4.5+l-k} + \left| \varepsilon^{2l} \partial_t^{k-1} \psi \right|_{5.5+l-k} \right).$$

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Weighted normal derivatives: $\gamma_4 > 0$

In the most general case, $\mathcal{T}^\gamma$ may contain weighted normal derivative $\omega(x_3)\partial_3$, so we have to analyze the commutator involving $[\mathcal{T}^\gamma, \partial_3]$ in $\mathfrak{C}(f)$ and $\mathfrak{D}(f)$ defined in (3.9)-(3.10). The problematic thing is that ∂_3 may fall on $\omega(x_3)$ which converts a "tangential" derivative $\omega(x_3)\partial_3$ (a first-order derivative) to a normal derivative ∂_3 (considered to be second-order under the setting of anisotropic Sobolev spaces). Such terms in $\mathfrak{D}(f)$ are

$$(\partial_3\varphi)^{-1}(v\cdot\mathbf{N} - \partial_t\varphi)[\mathcal{T}^\gamma, \partial_3]f + (v\cdot\mathbf{N} - \partial_t\varphi)\frac{\partial_3 f}{(\partial_3\varphi)^2}[\mathcal{T}^\gamma, \partial_3]\varphi.$$

They can be directly controlled because an extra weight $(v\cdot\mathbf{N} - \partial_t\varphi)$, which vanishes on Σ, is automatically generated to compensate the possible loss of weight function. As for $\mathfrak{C}(f)$, we notice that the terms involving $[\mathcal{T}^\gamma, \partial_3]$ can be written to be

$$\frac{\mathbf{N}_i}{\partial_3\varphi}[\mathcal{T}^\gamma, \partial_3]f - \frac{\mathbf{N}_i}{\partial_3\varphi}\partial_3^\varphi f[\mathcal{T}^\gamma, \partial_3]\varphi, \quad f = q \text{ or } v_i.$$

The second term above is easy to control because $\varphi(t,x) = x_3 + \chi(x_3)\psi(t,x')$ implies the C^∞-regularity of φ in x_3 direction. For the first term, it may generate a term $\mathcal{T}^\beta\partial_3 f\mathbf{N}_i$ with $\beta_i = \gamma_i(i = 0,1,2)$, $\beta_4 = \gamma_4 - 1$, whose $L^2(\Omega)$ norm may be not directly bounded. Luckily, for $f = q$ or v_i, we can again invoke the momentum equation or the continuity equation to reduce $-\partial_3^\varphi q$ and $\partial_3^\varphi v\cdot\mathbf{N}$ to tangential derivatives $\rho D_t^\varphi v - (b\cdot\nabla^\varphi)b$ and $-\mathcal{F}_p D_t^\varphi p - \overline{\nabla}\cdot\bar{v}$ respectively. Therefore, there is no extra loss of derivative in the commutators $\mathfrak{C}(f)$ and $\mathfrak{D}(f)$ when $\gamma_4 > 0$.

Summary of tangential estimates

Finally, we need to recover the estimates of $\mathcal{T}^\gamma(v, b, S, \sqrt{\mathcal{F}_p}p)$ from the L^2-estimates of their Alinhac good unknowns. By definition, we have

$$\left\|\mathcal{T}^\gamma f^\pm\right\|_{0,\pm}^2 \leq \left\|\mathbf{F}^{\gamma,\pm}\right\|_{0,\pm}^2 + |\mathcal{T}^\gamma\psi|_0^2 \|\partial_3^\varphi f^\pm\|_{L^\infty(\Omega^\pm)}^2,$$

in which $\|\mathbf{F}^{\gamma,\pm}\|_{0,\pm}$ and $|\mathcal{T}^\gamma\psi|_0$ have been controlled by $\delta E^\kappa(t) + \int_0^t P(\sigma^{-1}, E^\kappa(\tau))\,d\tau$. When $\mathcal{T}^\gamma$ contains at least one spatial derivative, we can use $-\mathcal{T}q \sim D_t^\varphi v + (b\cdot\nabla^\varphi)b$ to get the control of $\mathcal{T}q$ instead of $\sqrt{\mathcal{F}_p}\mathcal{T}q$. For the full time derivatives, we use $D_t^{\varphi,-} = \partial_t + (\bar{v}^-\cdot\overline{\nabla}) + (\partial_3\varphi)^{-1}(v^-\cdot\mathbf{N} - \partial_t\varphi)\partial_3$ to convert the $\varepsilon^{2l}\partial_t^{4+l}$-estimate to $\varepsilon^{2l}D_t^{\varphi,-}\partial_t^{3+l}$-estimate, $\varepsilon^{2l}\overline{\partial}\partial_t^{3+l}$-estimate and $\varepsilon^{2l}(\omega\partial_3)\partial_t^{3+l}$-estimate, in the second part of which the norm $|\varepsilon^{2l}\partial_t^{3+l}\psi(0)|_{2.5}$ is needed to control the VS term. Also, since $\omega(x_3) = 0$ on the interface, $\mathcal{T}^\gamma$ can be expressed as $\overline{\partial}^{4+l-k}\partial_t^k$ for $0 \leq k \leq 4+l$, $0 \leq l \leq 4$. Hence, we conclude the tangential estimates by the following inequalities

$$\sum_\pm \sum_{\langle\alpha\rangle=2l} \sum_{\substack{0\leq k\leq 4-l \\ k+\alpha_0<4+l}} \left\|\left(\varepsilon^{2l}\overline{\partial}^{4-k-l}\mathcal{T}^\alpha\partial_t^k(v^\pm, b^\pm, S^\pm, p^\pm)\right)\right\|_{0,\pm}^2 + \sum_{k=0}^{4+l}\left|\sqrt{\sigma}\varepsilon^{2l}\partial_t^k\psi\right|_{5+l-k}^2 + \left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^k\psi\right|_{6+l-k}^2 + \int_0^t\left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^{k+1}\psi(\tau)\right|_{5+l-k}^2\,d\tau$$

$$\lesssim \delta E_{4+l}^\kappa(t) + \sum_{k=0}^{3+l}\left|\varepsilon^{2l}\partial_t^k\psi(0)\right|_{5.5+l-k}^2 + \sum_{j=0}^{l} P\left(\sigma^{-1}, E_{4+j}^\kappa(0)\right)$$

$$+ P\left(\sum_{j=0}^{l} E_{4+j}^\kappa(t)\right)\int_0^t P\left(\sigma^{-1}, \sum_{j=0}^{l} E_{4+j}^\kappa(\tau)\right)\,d\tau \tag{3.128}$$

and

$$\sum_\pm \sum_{k=0}^{4-l}\left\|\left(\varepsilon^{2l}\partial_t^{4+l}(v^\pm, b^\pm, S^\pm, (\mathcal{F}_p)^{\frac{1}{2}}p^\pm)\right)\right\|_{4-k-l,\pm}^2 + \left|\sqrt{\sigma}\varepsilon^{2l}\partial_t^{4+l}\psi\right|_1^2 + \left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^{4+l}\psi\right|_2^2 + \int_0^t\left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^{5+l}\psi(\tau)\right|_1^2\,d\tau$$

$$\lesssim \delta E_{4+l}^\kappa(t) + \left|\varepsilon^{2l}\partial_t^{3+l}\psi(0)\right|_{2.5}^2 + P\left(\sigma^{-1}, \sum_{j=0}^{l} E_{4+j}^\kappa(0)\right) + P\left(\sum_{j=0}^{l} E_{4+j}^\kappa(t)\right)\int_0^t P\left(\sigma^{-1}, \sum_{j=0}^{l} E_{4+j}^\kappa(\tau)\right)\,d\tau. \tag{3.129}$$

These are exactly the desired uniform-in-(κ, ε) energy estimates in Proposition 3.3.

3.6 Div-Curl analysis and reduction of pressure

The tangential derivatives of the variables (v, b, p) are analyzed in Section 3.3-Section 3.5. Here we show the reduction of normal derivatives of pressure and the analysis for the divergence and vorticity. We use the div-curl decomposition (cf. Lemma B.1) such that the normal derivatives of (v, b) are controlled via their divergence and curl parts. For $0 \le l \le 3$, $0 \le k \le 3 - l$, $\langle \alpha \rangle = 2l$, $\alpha_3 = 0$, we have

$$\left\| \varepsilon^{2l} \partial_t^k \mathcal{T}^\alpha (v^\pm, b^\pm) \right\|_{4-k-l,\pm}^2 \le C \left(\left\| \varepsilon^{2l} \partial_t^k \mathcal{T}^\alpha (v^\pm, b^\pm) \right\|_{0,\pm}^2 + \left\| \varepsilon^{2l} \nabla^\varphi \cdot \partial_t^k \mathcal{T}^\alpha (v^\pm, b^\pm) \right\|_{3-k-l,\pm}^2 \right.$$

$$\left. + \left\| \varepsilon^{2l} \nabla^\varphi \times \partial_t^k \mathcal{T}^\alpha (v^\pm, b^\pm) \right\|_{3-k-l,\pm}^2 + \left\| \varepsilon^{2l} \overline{\partial}^{4-k-l} \partial_t^k \mathcal{T}^\alpha (v^\pm, b^\pm) \right\|_{0,\pm}^2 \right) \quad (3.130)$$

with

$$C = C \left(\sum_{j=0}^{l} \sum_{k=0}^{3+j} |\varepsilon^{2j} \partial_t^j \psi|_{4+l-j}^2, |\overline{\nabla} \psi|_{W^{1,\infty}} \right) > 0$$

a positive continuious function linear in $|\varepsilon^{2j} \partial_t^j \psi|_{4+l-j}^2$. The conclusion for the div-curl analysis is

Proposition 3.8. Fix $l \in \{0, 1, 2, 3\}$. For any $0 \le k \le l - 1$, any multi-index α satisfying $\langle \alpha \rangle = 2l$ and any constant $\delta \in (0, 1)$, we can prove the following estimates for the curl part

$$\left\| \varepsilon^{2l} \nabla^\varphi \times \partial_t^k \mathcal{T}^\alpha v^\pm \right\|_{3-k-l,\pm}^2 + \left\| \varepsilon^{2l} \nabla^\varphi \times \partial_t^k \mathcal{T}^\alpha b^\pm \right\|_{3-k-l,\pm}^2$$

$$\lesssim \delta E_{4+l}^\kappa(t) + P \left(\sigma^{-1}, \sum_{j=0}^{l} E_{4+j}^\kappa(0) \right) + P(E_4^\kappa(t)) \int_0^t P \left(\sigma^{-1}, \sum_{j=0}^{l} E_{4+j}^\kappa(\tau) \right) + E_{4+l+1}^\kappa(\tau) \, d\tau, \quad (3.131)$$

and for the divergence part

$$\left\| \varepsilon^{2l} \nabla^\varphi \cdot \partial_t^k \mathcal{T}^\alpha v^\pm \right\|_{3-k-l,\pm}^2 + \left\| \varepsilon^{2l} \nabla^\varphi \cdot \partial_t^k \mathcal{T}^\alpha v^\pm \right\|_{3-k-l,\pm}^2$$

$$\lesssim \delta E_{4+l}^\kappa(t) + P \left(\sigma^{-1}, \sum_{j=0}^{l} E_{4+j}^\kappa(0) \right) + P(E_4^\kappa(t)) \int_0^t P \left(\sigma^{-1}, \sum_{j=0}^{l} E_{4+j}^\kappa(\tau) \right) d\tau. \quad (3.132)$$

3.6.1 Reduction of pressure and divergence

Let us start with $l = 0$. The spatial derivative of q is controlled by invoking the momentum equation:

$$-\partial_3 q = (\partial_3 \varphi) \left(\rho D_t^\varphi v_3 - (b \cdot \nabla^\varphi) b_3 \right); \quad (3.133)$$

$$-\overline{\partial}_i q = - (\partial_3 \varphi)^{-1} \overline{\partial}_i \varphi \, \partial_3 q + \rho D_t^\varphi v_i - (b \cdot \nabla^\varphi) b_i, \quad i = 1, 2. \quad (3.134)$$

Let $\mathcal{T}$ be ∂_t or $\overline{\partial}$ or $\omega(x_3)\partial_3$. Then we have

$$\|\partial_t^k \partial_3 q\|_{3-k} \lesssim \|\partial_t^k (\rho \mathcal{T} v_3)\|_{3-k} + \|\partial_t^k (b \mathcal{T} b_3)\|_{3-k} \quad (3.135)$$

$$\|\partial_t^k \overline{\partial}_i q\|_{3-k} \lesssim \|\partial_t^k (\overline{\partial}_i \varphi \partial_3 q)\|_{3-k} + \|\partial_t^k (\rho \mathcal{T} v_i)\|_{3-k} + \|\partial_t^k (b \mathcal{T} b_i)\|_{3-k}, \quad (3.136)$$

in which the leading order terms are $\|\partial_t^k \mathcal{T}(v, b)\|_{3-k}$ and $|\partial_t^k \psi|_{4-k}$. This shows that we can convert the control of spatial derivative of q to *tangential estimates* of v and b.

Next we turn to the div-curl analysis for v, b. Let us first analyze $E_4(t)$. For $0 \le k \le 3$, we have

$$\|\partial_t^k v\|_{4-k}^2 \le C(|\psi|_{4-k}, |\overline{\nabla} \psi|_{W^{1,\infty}}) \left(\|\partial_t^k v\|_0^2 + \|\nabla^\varphi \cdot \partial_t^k v\|_{3-k}^2 + \|\nabla^\varphi \times \partial_t^k v\|_{3-k}^2 + \|\overline{\partial}^{4-k} \partial_t^k v\|_0^2 \right), \quad (3.137)$$

$$\|\partial_t^k b\|_{4-k}^2 \le C(|\psi|_{4-k}, |\overline{\nabla} \psi|_{W^{1,\infty}}) \left(\|\partial_t^k b\|_0^2 + \|\nabla^\varphi \cdot \partial_t^k b\|_{3-k}^2 + \|\nabla^\varphi \times \partial_t^k b\|_{3-k}^2 + \|\overline{\partial}^{4-k} \partial_t^k b\|_0^2 \right). \quad (3.138)$$

For the divergence, we can directly invoke the continuity equation to convert $\nabla^\varphi \cdot v$ to time derivative of p together with square weights of Mach number. When $k = 0$, we have

$$\|\nabla^\varphi \cdot v\|_3^2 = \|\mathcal{F}_p D_t^\varphi p\|_3^2, \quad (3.139)$$

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which is further reduced to the tangential derivatives of v and b by using the above reduction of q. Note that the magnetic tension term $\frac{1}{2}|b|^2$ in the total pressure q does not involve extra normal derivatives thanks to $\mathcal{T}(\frac{1}{2}|b|^2) = b \cdot \mathcal{T}b$. Taking ∂_t in the continuity equation and omitting lower order terms, we have

$$\nabla^\varphi \cdot \partial_t^k v \overset{L}{=} -\mathcal{F}_p \partial_t^k D_t^\varphi p + (\partial_3\varphi)^{-1}\overline{\partial}\partial_t^k\varphi \cdot \partial_3 v,$$

which gives

$$\|\nabla^\varphi \cdot \partial_t^k v^\pm\|_{3-k,\pm}^2 \lesssim C(\|v^\pm\|_{W^{1,\infty}(\Omega^\pm)})\left(\left\|\mathcal{F}_p^\pm\partial_t^k\mathcal{T}p^\pm\right\|_{3-k,\pm}^2 + \left|\partial_t^k\psi\right|_{4-k}^2\right) + \text{ lower order terms.} \tag{3.140}$$

Again, this can be reduced to tangential derivatives of v, b until there is no spatial derivative falling on p. As for the divergence of magnetic fields, we can invoke the div-free constraint to convert it to lower order terms. Namely, using $\nabla^\varphi \cdot b = 0$, we have

$$\nabla^\varphi \cdot \partial_t^k b \overset{L}{=} \underbrace{\partial_t^k(\nabla^\varphi \cdot b)}_{=0} + (\partial_3\varphi)^{-1}\overline{\partial}\partial_t^k\varphi \cdot \partial_3 b$$

and thus

$$\|\nabla^\varphi \cdot \partial_t^k b^\pm\|_{3-k,\pm}^2 \lesssim C(\|b^\pm\|_{W^{1,\infty}(\Omega^\pm)})\left|\partial_t^k\psi\right|_{4-k}^2 + \text{ lower order terms.} \tag{3.141}$$

The term $\left|\partial_t^k\psi\right|_{4-k}^2$ has been controlled in tangential estimates of $E_4^\kappa(t)$. Combining the result of tangential estimates in Proposition 3.3, the control of divergence of time derivatives is concluded by

$$\left\|\nabla^\varphi \cdot \partial_t^k(v^\pm, b^\pm)\right\|_{3-k,\pm}^2 \lesssim C(\|v^\pm\|_{W^{1,\infty}(\Omega^\pm)})\left\|\mathcal{F}_p^\pm\partial_t^k\mathcal{T}p^\pm\right\|_{3-k,\pm}^2 + C(\|v^\pm, b^\pm\|_{W^{1,\infty}(\Omega^\pm)})\left|\partial_t^k\psi\right|_{4-k}^2$$
$$\lesssim C(\|v^\pm\|_{W^{1,\infty}(\Omega^\pm)})\left\|\mathcal{F}_p^\pm\partial_t^k\mathcal{T}p^\pm\right\|_{3-k,\pm}^2 + \delta E_4^\kappa(t) + P(E_4^\kappa(0)) + P(E_4^\kappa(t))\int_0^t P(E_4^\kappa(\tau))\,\mathrm{d}\tau, \tag{3.142}$$

where the term involving $p^\pm$ can be further reduced to $\mathcal{T}(v^\pm, b^\pm)$ when $3 - k > 0$ so that one can further apply the div-curl analysis to it.

3.6.2 Vorticity analysis for E_4

Taking $\nabla^\varphi\times$ in the momentum equation of v and the evolution equation of b, we get the evolution equation for the vorticity $\nabla^\varphi \times v$ and the current density $\nabla^\varphi \times b$

$$\rho D_t^\varphi(\nabla^\varphi \times v) - (b \cdot \nabla^\varphi)(\nabla^\varphi \times b) = -(\nabla^\varphi\rho) \times (D_t^\varphi v) - \rho(\nabla^\varphi v_j) \times (\partial_j^\varphi v) + (\nabla^\varphi b_j) \times (\partial_j^\varphi b), \tag{3.143}$$

$$D_t^\varphi(\nabla^\varphi \times b) - (b \cdot \nabla^\varphi)(\nabla^\varphi \times v) - b \times \nabla^\varphi(\nabla^\varphi \cdot v) = -(\nabla^\varphi \times b)(\nabla^\varphi \cdot v) - (\nabla^\varphi v_j) \times (\partial_j^\varphi b) + (\nabla^\varphi b_j) \times (\partial_j^\varphi v), \tag{3.144}$$

and taking ∂^3 gives

$$\rho D_t^\varphi(\partial^3\nabla^\varphi \times v) - (b \cdot \nabla^\varphi)(\partial^3\nabla^\varphi \times b) = RK_v, \tag{3.145}$$

$$D_t^\varphi(\partial^3\nabla^\varphi \times b) - (b \cdot \nabla^\varphi)(\partial^3\nabla^\varphi \times v) - b \times \partial^3\nabla^\varphi(\nabla^\varphi \cdot v) = RK_b, \tag{3.146}$$

where

$$RK_v := -[\partial^3, \rho D_t^\varphi](\nabla^\varphi \times v) + [\partial^3, (b \cdot \nabla^\varphi)](\nabla^\varphi \times b) + \partial^3(\text{right side of (3.143)}),$$

$$RK_b := -[\partial^3, D_t^\varphi](\nabla^\varphi \times b) + [\partial^3, (b \cdot \nabla^\varphi)](\nabla^\varphi \times v) + \partial^3(\text{right side of (3.144)}).$$

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Direct computation shows that the highest-order terms in RK_v, RK_b only have 4 spatial derivatives and do not contain time derivative of q. Therefore, we can prove the H^3-control of the voriticity and current density by standard energy estimates.

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}\rho^\pm\left|\partial^3(\nabla^\varphi\times v^\pm)\right|^2\mathrm{d}\mathcal{V}_t = \int_{\Omega^\pm}\rho^\pm D_t^{\varphi\pm}(\partial^3\nabla^\varphi\times v^\pm)\cdot(\partial^3\nabla^\varphi\times v^\pm)\,\mathrm{d}\mathcal{V}_t$$

$$= \int_{\Omega^\pm}(b^\pm\cdot\nabla^\varphi)(\partial^3\nabla^\varphi\times b^\pm)\cdot(\partial^3\nabla^\varphi\times v^\pm)\,\mathrm{d}\mathcal{V}_t + \underbrace{\int_{\Omega^\pm}RK_v^\pm\cdot(\partial^3\nabla^\varphi\times v^\pm)\,\mathrm{d}\mathcal{V}_t}_{=:L_1^\pm}$$

$$\overset{(b^\pm\cdot\nabla^\varphi)}{=\!=\!=} -\int_{\Omega^\pm}(\partial^3\nabla^\varphi\times b^\pm)\cdot D_t^{\varphi\pm}(\partial^3\nabla^\varphi\times b^\pm)\,\mathrm{d}\mathcal{V}_t + \underbrace{\int_{\Omega^\pm}(\partial^3\nabla^\varphi\times b^\pm)\cdot\left(b^\pm\times(\partial^3\nabla^\varphi(\nabla^\varphi\cdot v^\pm))\right)\mathrm{d}\mathcal{V}_t}_{=:K_1^\pm} \tag{3.147}$$

$$+ \underbrace{\int_{\Omega^\pm}(\partial^3\nabla^\varphi\times b^\pm)\cdot RK_b^\pm\,\mathrm{d}\mathcal{V}_t}_{L_2^\pm} +L_1^\pm,$$

where $L_1^\pm$, $L_2^\pm$ are directly controlled

$$L_1^\pm + L_2^\pm \le P(\|(v^\pm,b^\pm)\|_{4,\pm}, |\psi|_4). \tag{3.148}$$

It remains to analyze the term $K_1^\pm$ in which there is a key observation for the energy structure of compressible MHD system. We invoke the continuity equation $\nabla^\varphi\cdot v^\pm = \mathcal{F}_p^\pm D_t^{\varphi\pm}p^\pm$ and commute $D_t^{\varphi\pm}$ with ∇^φ to get

$$\nabla^\varphi(\nabla^\varphi\cdot v^\pm) = -\mathcal{F}_p^\pm D_t^{\varphi\pm}\nabla^\varphi p^\pm + \mathcal{F}_p^\pm(\nabla^\varphi v_j^\pm)(\partial_j^\varphi p^\pm).$$

Next, we rewrite the momentum equation to be $\rho^\pm D_t^{\varphi\pm}v^\pm + b^\pm\times(\nabla^\varphi\times b^\pm) = -\nabla^\varphi p^\pm$ and plug it into the highest-order term $-\mathcal{F}_p^\pm D_t^{\varphi\pm}\nabla^\varphi p^\pm$ to get

$$-\mathcal{F}_p^\pm D_t^{\varphi\pm}\nabla^\varphi p^\pm = \mathcal{F}_p^\pm D_t^{\varphi\pm}(\rho^\pm D_t^{\varphi\pm}v^\pm) + \mathcal{F}_p^\pm D_t^{\varphi\pm}(b^\pm\times(\nabla^\varphi\times b^\pm))$$

$$= \mathcal{F}_p^\pm\rho^\pm(D_t^{\varphi\pm})^2v^\pm + \mathcal{F}_p^\pm D_t^{\varphi\pm}(b^\pm\times(\nabla^\varphi\times b^\pm)) + \mathcal{F}_p^\pm(D_t^{\varphi\pm}\rho^\pm)(D_t^{\varphi\pm}v^\pm).$$

Thus, the term $K_1^\pm$ becomes

$$K_1^\pm = \int_{\Omega^\pm}(\partial^3\nabla^\varphi\times b^\pm)\cdot\left(b^\pm\times(\mathcal{F}_p^\pm\rho^\pm\partial^3(D_t^{\varphi\pm})^2v^\pm)\right)\mathrm{d}\mathcal{V}_t$$

$$- \int_{\Omega^\pm}\mathcal{F}_p^\pm\left(b^\pm\times(\partial^3\nabla^\varphi\times b^\pm)\right)\cdot D_t^{\varphi\pm}\left(b^\pm\times(\partial^3\nabla^\varphi\times b^\pm)\right)\mathrm{d}\mathcal{V}_t \tag{3.149}$$

$$+ \int_{\Omega^\pm}(\partial^3\nabla^\varphi\times b^\pm)\cdot RK_p^\pm\,\mathrm{d}\mathcal{V}_t,$$

where

$$RK_p^\pm := \mathcal{F}_p^\pm\partial^3\left((\nabla^\varphi v_j^\pm)(\partial_j^\varphi p^\pm) + (D_t^{\varphi\pm}\rho^\pm)(D_t^{\varphi\pm}v^\pm)\right) + [\partial^3,\mathcal{F}_p^\pm\rho^\pm](D_t^{\varphi\pm})^2v^\pm$$

$$+ \mathcal{F}_p^\pm[\partial^3,D_t^{\varphi\pm}](b^\pm\times(\nabla^\varphi\times b^\pm)) + \mathcal{F}_p^\pm D_t^{\varphi\pm}\left([\partial^3,b^\pm\times](\nabla^\varphi\times b^\pm)\right)$$

consists of ≤ 4 derivatives in each term and its contribution can be directly controlled

$$L_3^\pm := \int_{\Omega^\pm}(\partial^3\nabla^\varphi\times b^\pm)\cdot RK_p^\pm\,\mathrm{d}\mathcal{V}_t \le P(\|b^\pm,v^\pm,\mathcal{F}_p^\pm p^\pm\|_{4,\pm}, \|\mathcal{F}_p^\pm D_t^{\varphi\pm}(v^\pm,b^\pm,p^\pm)\|_{3,\pm}) \tag{3.150}$$

Note that the second term on the right side of $K_1^\pm$ is obtained by using the vector identity $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = -\mathbf{c}\cdot(\mathbf{b}\times\mathbf{a})$:

$$(\partial^3\nabla^\varphi\times b^\pm)\cdot\left(b^\pm\times D_t^{\varphi\pm}\left(b^\pm\times(\partial^3\nabla^\varphi\times b^\pm)\right)\right) = -D_t^{\varphi\pm}\left(b^\pm\times(\partial^3\nabla^\varphi\times b^\pm)\right)\cdot\left(b^\pm\times(\partial^3\nabla^\varphi\times b^\pm)\right).$$

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Therefore, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}\rho^\pm\left|\partial^3(\nabla^\varphi\times v^\pm)\right|^2\,\mathrm{d}\mathcal{V}_t$$

$$=-\int_{\Omega^\pm}(\partial^3\nabla^\varphi\times b^\pm)\cdot D_t^{\varphi\pm}(\partial^3\nabla^\varphi\times b^\pm)\,\mathrm{d}\mathcal{V}_t-\int_{\Omega^\pm}\mathcal{F}_p^\pm\left(b^\pm\times(\partial^3\nabla^\varphi\times b^\pm)\right)\cdot D_t^{\varphi\pm}\left(b^\pm\times(\partial^3\nabla^\varphi\times b^\pm)\right)\mathrm{d}\mathcal{V}_t$$

$$+\underbrace{\int_{\Omega^\pm}(\partial^3\nabla^\varphi\times b^\pm)\cdot\left(b^\pm\times(\mathcal{F}_p\rho^\pm\partial^3(D_t^{\varphi\pm})^2v^\pm)\right)\mathrm{d}\mathcal{V}_t}_{K^\pm}+L_1^\pm+L_2^\pm+L_3^\pm \tag{3.151}$$

$$=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}\left|\partial^3(\nabla^\varphi\times b^\pm)\right|^2+\mathcal{F}_p^\pm\left|b^\pm\times\partial^3(\nabla^\varphi\times b^\pm)\right|^2\,\mathrm{d}\mathcal{V}_t$$

$$+\frac{1}{2}\int_{\Omega^\pm}(\nabla^\varphi\cdot v^\pm)\left(\left|\partial^3(\nabla^\varphi\times b^\pm)\right|^2+\mathcal{F}_p^\pm\left|b^\pm\times\partial^3(\nabla^\varphi\times b^\pm)\right|^2\right)\mathrm{d}\mathcal{V}_t+K^\pm+L_1^\pm+L_2^\pm+L_3^\pm,$$

which further gives the control of vorticity and current density simultaneously

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}\rho^\pm\left|\partial^3(\nabla^\varphi\times v^\pm)\right|^2+\left|\partial^3(\nabla^\varphi\times b^\pm)\right|^2+\mathcal{F}_p^\pm\left|b^\pm\times\partial^3(\nabla^\varphi\times b^\pm)\right|^2\,\mathrm{d}\mathcal{V}_t$$

$$\lesssim P(E_4^\kappa(t))+P(|\psi|_4)\|b^\pm\|_{4,\pm}\|b^\pm\rho^\pm\|_{L^\infty(\Omega^\pm)}\left\|\varepsilon^2(D_t^{\varphi\pm})^2v^\pm\right\|_3\le P(E_4^\kappa(t))+E_5^\kappa(t). \tag{3.152}$$

Hence, the vorticity analysis for compressible ideal MHD cannot be closed in standard Sobolev space because of the term $\varepsilon^2\partial^3(D_t^{\varphi\pm})^2v^\pm$ in $K^\pm$. Instead, the appearance of this term indicates us to **trade one normal derivative (in the curl operator) for two tangential derivatives $(D_t^{\varphi\pm})^2$ together with square weights of Mach number ε^2. Besides, the normal derivative part involving $\partial^3 D_t^{\varphi\pm}(\nabla^\varphi\times b^\pm)$ contributes to the energy of current density thanks to the special structure of Lorentz force $-b^\pm\times(\nabla^\varphi\times b^\pm)$. This is exactly the motivation for us to define the energy functional $E(t)$ under the setting the anisotropic Sobolev spaces instead of standard Sobolev spaces.**

Similarly, the curl estimates for the time derivatives (in $E_4(t)$) can be proven in the same way by replacing ∂^3 with $\partial^{3-k}\partial_t^k$ ($1\le k\le3$). We omit the details and list the conclusion

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}\rho^\pm\left|\partial^{3-k}\partial_t^k(\nabla^\varphi\times v^\pm)\right|^2+\left|\partial^{3-k}\partial_t^k(\nabla^\varphi\times b^\pm)\right|^2+\mathcal{F}_p^\pm\left|b^\pm\times\partial^{3-k}\partial_t^k(\nabla^\varphi\times b^\pm)\right|^2\,\mathrm{d}\mathcal{V}_t$$

$$\lesssim P(E_4^\kappa(t))+\left\|\varepsilon^2\partial_t^k(D_t^{\varphi\pm})^2v^\pm\right\|_{3-k,\pm}^2\le P(E_4^\kappa(t))+E_5^\kappa(t). \tag{3.153}$$

Finally, we need to commute ∂_t^k with $\nabla^\varphi\times$ when $k\ge1$. We have

$$(\nabla^\varphi\times\partial_t^k v)_i\overset{L}{=}\partial_t^k(\nabla^\varphi\times v)_i+\epsilon_{ijl}(\partial_3\varphi)^{-1}(\bar\partial_j\partial_t^k\varphi)(\partial_3 v_l),$$

where ϵ_{ijl} is the sign of permutation $(ijl)\in S_3$. This gives

$$\|\nabla^\varphi\times\partial_t^k v^\pm\|_{3-k,\pm}^2\lesssim C(\|v^\pm\|_{W^{1,\infty}(\Omega^\pm)})\left(\left\|\partial_t^k(\nabla^\varphi\times v^\pm)\right\|_{3-k,\pm}^2+\left|\partial_t^k\psi\right|_{4-k}^2\right)+\text{ lower order terms}, \tag{3.154}$$

where both leading order terms have been controlled in tangential estimates of $E_4^\kappa(t)$. The same result holds for $b^\pm$. Using the result of tangential estimates of $E_4^\kappa(t)$, we have: for any $k\in\{0,1,2,3\}$ and any $\delta\in(0,1)$

$$\left\|\nabla^\varphi\times\partial_t^k v^\pm\right\|_{3-k,\pm}^2+\left\|\nabla^\varphi\times\partial_t^k b^\pm\right\|_{3-k,\pm}^2$$

$$\lesssim P(E_4^\kappa(0))+\int_0^t P(E_4^\kappa(\tau))+E_5^\kappa(\tau)\,\mathrm{d}\tau+P(\|v^\pm,b^\pm\|_{W^{1,\infty}(\Omega^\pm)})\left|\partial_t^k\psi\right|_{4-k}^2 \tag{3.155}$$

$$\lesssim \delta E_4^\kappa(t)+P(\sigma^{-1},E_4^\kappa(0))+P(E_4^\kappa(t))\int_0^t P(\sigma^{-1},E_4^\kappa(\tau))+E_5^\kappa(\tau)\,\mathrm{d}\tau,$$

3.6.3 Further div-curl analysis for $E_5\sim E_7$

The vorticity analysis for $E_4(t)$ requires the control of $\left\|\varepsilon^2\partial_t^k(D_t^{\varphi\pm})^2v^\pm\right\|_{3-k}^2$ for $0\le k\le3$. When $0\le k\le2$, there are still normal derivatives in this term. Thus, we shall do further div-curl analysis on $\left\|\varepsilon^2\partial_t^k(D_t^{\varphi\pm})^2v^\pm\right\|_{3-k}^2$ for $0\le k\le2$. Let $\mathcal{T}^\alpha=$

$\partial_t^{\alpha_0} \overline{\partial}_1^{\alpha_1} \overline{\partial}_2^{\alpha_2} (\omega(x_3)\partial_3)^{\alpha_4}$ with $\langle \alpha \rangle = 2$. The divergence part can be reduced in the same way as in Section 3.6.1. We take $\partial_t^k \mathcal{T}^\alpha$ in the continuity equation and omit the lower order terms to get

$$\nabla^\varphi \cdot \partial_t^k \mathcal{T}^\alpha v \overset{L}{=} -\varepsilon^2 \partial_t^k \mathcal{T}^\alpha D_t^\varphi p + (\partial_3 \varphi)^{-1} \overline{\partial} \partial_t^k \mathcal{T}^\alpha \varphi \cdot \partial_3 v,$$

which gives

$$\|\varepsilon^2 \nabla^\varphi \cdot \partial_t^k \mathcal{T}^\alpha v^\pm\|_{2-k}^2 \lesssim C(\|v\|_{W^{1,\infty}}) \left(\|\varepsilon^4 \partial_t^k \mathcal{T}^\alpha \mathcal{T} p^\pm\|_{2-k,\pm}^2 + |\varepsilon^2 \partial_t^k \mathcal{T}^\alpha \psi|_{2-k}^2 \right) + \text{ lower order terms.} \tag{3.156}$$

Remark 3.3. The term generated when commuting $\mathcal{T}^\alpha$ with ∇^φ is actually of lower order. One can check that (see also [65, (3.24)-(3.25)])

$$[(\omega\partial_3)^m, \partial_3]f = \underbrace{(\omega\partial_3)^m \partial_3 f - \partial_3((\omega\partial_3)^m f)}_{\text{both are (m+1)-th order terms}} = \sum_{k \le m-1} c_{m,k} (\omega\partial_3)^k \partial_3 f = \sum_{k \le m-1} d_{m,k} \partial_3 (\omega\partial_3)^k f$$

for some smooth functions $c_{m,k}, d_{m,k}$ depending on m, k and the derivatives (up to order m) of ω, and the right side only contains $\le m$-th order terms.

Similarly, using $\nabla^\varphi \cdot b = 0$, we have

$$\nabla^\varphi \cdot \partial_t^k \mathcal{T}^\alpha b \overset{L}{=} \underbrace{\partial_t^k \mathcal{T}^\alpha (\nabla^\varphi \cdot b)}_{=0} + (\partial_3 \varphi)^{-1} \overline{\partial} \partial_t^k \mathcal{T}^\alpha \varphi \cdot \partial_3 b$$

and thus

$$\|\varepsilon^2 \nabla^\varphi \cdot \partial_t^k b^\pm\|_{2-k,\pm}^2 \lesssim C(\|b^\pm\|_{W^{1,\infty}(\Omega^\pm)}) |\varepsilon^2 \partial_t^k \mathcal{T}^\alpha \psi|_{2-k}^2 + \text{ lower order terms.} \tag{3.157}$$

The control of divergence part in the analysis of $E_5^\kappa(t)$ is concluded by the following energy inequality. For any $k \in \{0, 1, 2\}$, any multi-index α with $\langle \alpha \rangle = 2$ and any $\delta \in (0, 1)$

$$\left\| \varepsilon^2 \nabla^\varphi \cdot \partial_t^k \mathcal{T}^\alpha (v^\pm, b^\pm) \right\|_{2-k,\pm}^2 \lesssim C(\|v^\pm\|_{W^{1,\infty}(\Omega^\pm)}) \left\| \varepsilon^4 \partial_t^k \mathcal{T}^\alpha \mathcal{T} p^\pm \right\|_{2-k,\pm}^2 + C(\|v^\pm, b^\pm\|_{W^{1,\infty}(\Omega^\pm)}) \left| \partial_t^k \mathcal{T}^\alpha \psi \right|_{3-k}^2$$

$$\lesssim C(\|v^\pm\|_{W^{1,\infty}(\Omega^\pm)}) \left\| \varepsilon^4 \partial_t^k \mathcal{T}^\alpha \mathcal{T} p^\pm \right\|_{2-k,\pm}^2 + \delta E_5^\kappa(t) + P(E_4^\kappa(0), E_5^\kappa(0)) + P(E_4^\kappa(t)) \int_0^t P(E_4^\kappa(\tau), E_5^\kappa(\tau)) \, d\tau, \tag{3.158}$$

where the term involving $\mathcal{T} p^\pm$ can be further reduced to $\mathcal{T}(v^\pm, b^\pm)$ when $2 - k > 0$ so that one can further apply the div-curl analysis to it.

As for the curl part, we can still mimic the proof in Section 3.6.2 to get the control of $\left\| \varepsilon^2 \partial_t^k \mathcal{T}^\alpha (\nabla^\varphi \times (v, b)) \right\|_{2-k}$ for $0 \le k \le 2$ and $\langle \alpha \rangle = 2$ with $\alpha_3 = 0$

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega^\pm} \rho^\pm \left| \varepsilon^2 \partial^{2-k} \partial_t^k \mathcal{T}^\alpha (\nabla^\varphi \times v^\pm) \right|^2 + \left| \varepsilon^2 \partial^{2-k} \partial_t^k \mathcal{T}^\alpha (\nabla^\varphi \times b^\pm) \right|^2 + \mathcal{F}_p^\pm \left| \varepsilon^2 b^\pm \times \partial^{2-k} \partial_t^k \mathcal{T}^\alpha (\nabla^\varphi \times b^\pm) \right|^2 \, d\mathcal{V}_t$$

$$\lesssim P(E_4^\kappa(t), E_5^\kappa(t)) + \left\| \varepsilon^4 \partial_t^k \mathcal{T}^\alpha (D_t^{\varphi\pm})^2 v^\pm \right\|_{2-k}^2 \le P(E_4^\kappa(t), E_5^\kappa(t)) + E_6^\kappa(t). \tag{3.159}$$

Then we commute $\partial^{2-k} \partial_t^k \mathcal{T}^\alpha$ with $\nabla^\varphi \times$ to get: for any $k \in \{0, 1, 2\}$, any multi-index α with $\langle \alpha \rangle = 2$ and $\alpha_3 = 0$, and any $\delta \in (0, 1)$

$$\left\| \nabla^\varphi \times \partial_t^k \mathcal{T}^\alpha v^\pm \right\|_{2-k,\pm}^2 + \left\| \nabla^\varphi \times \partial_t^k \mathcal{T}^\alpha b^\pm \right\|_{2-k,\pm}^2$$

$$\lesssim P(E_4^\kappa(0), E_5^\kappa(0)) + \int_0^t P(E_4^\kappa(\tau), E_5^\kappa(\tau)) + E_6^\kappa(\tau) \, d\tau + P(\|v^\pm, b^\pm\|_{W^{1,\infty}(\Omega^\pm)}) \left| \varepsilon^2 \partial_t^k \mathcal{T}^\alpha \psi \right|_{2-k}^2 \tag{3.160}$$

$$\lesssim \delta E_5^\kappa(t) + P(\sigma^{-1}, E_4^\kappa(0), E_5^\kappa(0)) + P(E_4^\kappa(t)) \int_0^t P(\sigma^{-1}, E_4^\kappa(\tau), E_5^\kappa(\tau)) + E_6^\kappa(\tau) \, d\tau,$$

where we use the result of tangential estimates to control $\left| \varepsilon^2 \partial_t^k \mathcal{T}^\alpha \psi \right|_{2-k}$. When $k \le 1$ in the above energy estimate, we shall continue to apply the div-curl analysis to $\left\| \varepsilon^4 \partial_t^k \mathcal{T}^\alpha (D_t^{\varphi\pm})^2 v^\pm \right\|_{2-k}^2$.

For E_6^κ and E_7^κ, we have analogous div-curl inequalities. For $l = 2, 3$, we continue to analyze the divergence and the curl according to (3.130). Similarly as above, we have the following estimates for any $k \in \{0, 1\}$, any multi-index α with $\langle \alpha \rangle = 4$, $\alpha_3 = 0$ and any $\delta \in (0, 1)$

$$\left\| \varepsilon^4 \nabla^\varphi \times \partial_t^k \mathcal{T}^\alpha v^\pm \right\|_{1-k,\pm}^2 + \left\| \varepsilon^4 \nabla^\varphi \times \partial_t^k \mathcal{T}^\alpha b^\pm \right\|_{1-k,\pm}^2$$
$$\lesssim \delta E_6^\kappa(t) + P\left(\sigma^{-1}, \sum_{l=0}^2 E_{4+l}^\kappa(0) \right) + P(E_4^\kappa(t)) \int_0^t P\left(\sigma^{-1}, \sum_{l=0}^2 E_{4+l}^\kappa(\tau) \right) + E_7^\kappa(\tau) \, d\tau, \tag{3.161}$$

where this E_7^κ term is contributed by $\left\| \varepsilon^6 \partial_t^k \mathcal{T}^\alpha (D_t^{\varphi\pm})^2 v^\pm \right\|_{1-k,\pm}$. When $k = 0$ in this term, we again apply the div-curl analysis to it in order to eliminate all normal derivatives falling on v, b. For any multi-index α with $\langle \alpha \rangle = 6$ and $\alpha_3 = 0$ and any $\delta \in (0, 1)$, we have

$$\left\| \varepsilon^6 \nabla^\varphi \times \mathcal{T}^\alpha v^\pm \right\|_{0,\pm}^2 + \left\| \varepsilon^6 \nabla^\varphi \times \mathcal{T}^\alpha b^\pm \right\|_{0,\pm}^2$$
$$\lesssim \delta E_7^\kappa(t) + P\left(\sigma^{-1}, \sum_{l=0}^3 E_{4+l}^\kappa(0) \right) + P(E_4^\kappa(t)) \int_0^t P\left(\sigma^{-1}, \sum_{l=0}^3 E_{4+l}^\kappa(\tau) \right) + E_8^\kappa(\tau) \, d\tau. \tag{3.162}$$

The control of divergence part for $E_6^\kappa(t)$, $E_7^\kappa(t)$ also follows the same way as $E_4^\kappa(t)$, $E_5^\kappa(t)$. For any $k \in \{0, 1\}$, any multi-index α with $\langle \alpha \rangle = 4$, $\alpha_3 = 0$, we have

$$\left\| \varepsilon^4 \nabla^\varphi \cdot \partial_t^k \mathcal{T}^\alpha (v^\pm, b^\pm) \right\|_{1-k,\pm}^2 \lesssim C(\|v^\pm\|_{W^{1,\infty}(\Omega^\pm)}) \left\| \varepsilon^6 \partial_t^k \mathcal{T}^\alpha \mathcal{T} p^\pm \right\|_{1-k,\pm}^2 + C(\|v^\pm, b^\pm\|_{W^{1,\infty}(\Omega^\pm)}) \left| \partial_t^k \mathcal{T}^\alpha \psi \right|_{2-k}^2$$
$$\lesssim \delta E_6^\kappa(t) + P\left(\sigma^{-1}, \sum_{l=0}^2 E_{4+l}^\kappa(0) \right) + P(E_4^\kappa(t)) \int_0^t P\left(\sigma^{-1}, \sum_{l=0}^2 E_{4+l}^\kappa(\tau) \right) d\tau, \tag{3.163}$$

where the term $\left\| \varepsilon^6 \partial_t^k \mathcal{T}^\alpha \mathcal{T} p^\pm \right\|_{1-k,\pm}^2$ does not appear because it can be converted to full tangential derivatives (part of $E_6^\kappa(t)$) which has been controlled in Section 3.3-Section 3.5. For any multi-index α with $\langle \alpha \rangle = 6$, $\alpha_3 = 0$, we have

$$\left\| \varepsilon^6 \nabla^\varphi \cdot \partial_t^k \mathcal{T}^\alpha (v^\pm, b^\pm) \right\|_{0,\pm}^2 \lesssim C(\|v^\pm\|_{W^{1,\infty}(\Omega^\pm)}) \left\| \varepsilon^8 \partial_t^k \mathcal{T}^\alpha \mathcal{T} p^\pm \right\|_{0,\pm}^2 + C(\|v^\pm, b^\pm\|_{W^{1,\infty}(\Omega^\pm)}) \left| \varepsilon^6 \mathcal{T}^\alpha \psi \right|_1^2$$
$$\lesssim \delta E_7^\kappa(t) + P\left(\sigma^{-1}, \sum_{l=0}^3 E_{4+l}^\kappa(0) \right) + P(E_4^\kappa(t)) \int_0^t P\left(\sigma^{-1}, \sum_{l=0}^3 E_{4+l}^\kappa(\tau) \right) d\tau, \tag{3.164}$$

where the term $\left\| \varepsilon^8 \mathcal{T}^\alpha \mathcal{T} p^\pm \right\|_{0,\pm}^2$ does not appear because it has been included in tangential estimates for $E_7^\kappa(t)$.

3.6.4 Modifications for the 2D case

In the case of 2D, the equations of vorticity $\nabla^{\varphi,\perp} \cdot v$ and current density $\nabla^{\varphi,\perp} \cdot b$ are

$$\rho D_t^\varphi (\nabla^{\varphi,\perp} \cdot v) - (b \cdot \nabla^\varphi)(\nabla^{\varphi,\perp} \cdot b) = -(\nabla^{\varphi,\perp}\rho) \cdot (D_t^\varphi v) - \rho(\nabla^{\varphi,\perp} v_j) \cdot (\nabla_j^\varphi v) + (\nabla^{\varphi,\perp} b_j) \cdot (\nabla_j^\varphi b), \tag{3.165}$$

$$D_t^\varphi(\nabla^{\varphi,\perp} \cdot b) - (b \cdot \nabla^\varphi)(\nabla^{\varphi,\perp} \cdot v) - b \cdot \nabla^{\varphi,\perp}(\nabla^\varphi \cdot v) = -(\nabla^{\varphi,\perp} \cdot b)(\nabla^\varphi \cdot v) - (\nabla^{\varphi,\perp} v_j) \cdot (\nabla_j^\varphi b) + (\nabla^{\varphi,\perp} b_j) \cdot (\nabla_j^\varphi v), \tag{3.166}$$

which has the same structure as (3.143)-(3.144). Thus, we expect to adopt the strategy in Section 3.6 to prove the div-curl estimates. The only slight difference is the structure of Lorentz force. Let us take the ∂^3-estimate of $\nabla^{\varphi,\perp} \cdot (v, b)$ for an example. In this case, the problematic term (in the analogue of $K_1^\pm$ in (3.147)) becomes

$$K_1^{\pm\prime} = \int_{\Omega^\pm} (\partial^3 \nabla^{\varphi,\perp} \cdot b^\pm) \left(b^\pm \cdot \nabla^{\varphi,\perp}(\partial^3 \nabla^\varphi \cdot v^\pm) \right) d\mathcal{V}_t.$$

Again, we invoke the continuity equation, commute ∇^φ with $D_t^{\varphi\pm}$ to get

$$b^\pm \cdot \nabla^{\varphi,\perp}(\partial^3 \nabla^\varphi \cdot v^\pm) \overset{L}{=} \varepsilon^2 (b_1^\pm \partial^3 \partial_2^\varphi D_t^{\varphi\pm} p^\pm - b_2^\pm \partial^3 \partial_1^\varphi D_t^{\varphi\pm} p^\pm) \overset{L}{=} \varepsilon^2 (b_1^\pm \partial^3 D_t^{\varphi\pm}(\partial_2^\varphi p^\pm) - b_2^\pm \partial^3 D_t^{\varphi\pm}(\partial_1^\varphi p^\pm)).$$

Then we plug the momentum equation

$$-\partial_1^\varphi p = \rho D_t^\varphi v_1 - b_1 \partial_1^\varphi b_1 - b_2 \partial_2^\varphi b_1 + b_1 \partial_1^\varphi b_1 + b_2 \partial_1^\varphi b_2 = \rho D_t^\varphi v_1 + b_2(\nabla^{\varphi,\perp} \cdot b)$$
$$-\partial_2^\varphi p = \rho D_t^\varphi v_2 - b_1 \partial_1^\varphi b_2 - b_2 \partial_2^\varphi b_2 + b_1 \partial_2^\varphi b_1 + b_2 \partial_2^\varphi b_2 = \rho D_t^\varphi v_2 - b_1(\nabla^{\varphi,\perp} \cdot b)$$

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to get

$$b^\pm \cdot \nabla^{\varphi,\perp}(\partial^3 \nabla^\varphi \cdot v^\pm) \overset{L}{=} \mathcal{F}_p^\pm \rho^\pm (b^{\pm,\perp} \cdot \partial^3 (D_t^{\varphi\pm})^2 v^\pm) - \mathcal{F}_p^\pm \left((b_1^\pm)^2 + (b_2^\pm)^2\right) \partial^3 D_t^{\varphi\pm}(\nabla^{\varphi,\perp} \cdot b), \quad b^\perp := (-b_2, b_1).$$

Thus, the term $K_1^{\pm\prime}$ can be controlled in a similar manner as in Section 3.6

$$\begin{aligned}
K_1^{\pm\prime} \overset{L}{=} & \int_{\Omega^\pm} (\partial^3 \nabla^{\varphi,\perp} \cdot b^\pm) \left(\mathcal{F}_p^\pm \rho^\pm b^{\pm,\perp} \cdot \partial^3 (D_t^{\varphi\pm})^2 v^\pm\right) \, \mathrm{d}\mathcal{V}_t \\
& - \int_{\Omega^\pm} \mathcal{F}_p^\pm |b^\pm|^2 (\partial^3 \nabla^{\varphi,\perp} \cdot b^\pm)(D_t^{\varphi\pm} \partial^3 \nabla^{\varphi,\perp} \cdot b^\pm) \, \mathrm{d}\mathcal{V}_t \\
= & \int_{\Omega^\pm} (\partial^3 \nabla^{\varphi,\perp} \cdot b^\pm) \left(\mathcal{F}_p^\pm \rho^\pm b^{\pm,\perp} \cdot \partial^3 (D_t^{\varphi\pm})^2 v^\pm\right) \, \mathrm{d}\mathcal{V}_t - \frac{1}{2} \int_{\Omega^\pm} (\nabla^\varphi \cdot v^\pm) \mathcal{F}_p^\pm |b^\pm|^2 |\partial^3 \nabla^{\varphi,\perp} \cdot b^\pm|^2 \, \mathrm{d}\mathcal{V}_t \\
& - \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega^\pm} \mathcal{F}_p^\pm |b^\pm|^2 |\partial^3 \nabla^{\varphi,\perp} \cdot b^\pm|^2 \, \mathrm{d}\mathcal{V}_t.
\end{aligned}$$

Hence, the curl estimate (3.152) should be modified to be

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega^\pm} \rho^\pm \left|\partial^3 (\nabla^{\varphi,\perp} \cdot v^\pm)\right|^2 + (1 + \mathcal{F}_p^\pm |b^\pm|^2) \left|\partial^3 (\nabla^{\varphi,\perp} \cdot b^\pm)\right|^2 \, \mathrm{d}\mathcal{V}_t \lesssim P(\widetilde{E}_4(t)) + \widetilde{E}_5(t). \tag{3.167}$$

3.7 Uniform estimates for the nonlinear approximate system

3.7.1 Control of the entropy

It remains to control the full (anisotropic) Sobolev norms of the entropy functions $S^\pm$. This can be easily proven thanks to $D_t^{\varphi\pm} S^\pm = 0$. In the control of $E_{4+l}^\kappa(t)$ for fixed $0 \le l \le 4$, we need to take the derivative $\partial_*^\alpha := \partial^{4-l-k} \partial_t^k \mathcal{T}^\gamma = \partial_3^{\gamma_3'}(\omega \partial_3)^{\gamma_4} \partial_t^{k+\gamma_0} \overline{\partial}_1^{\gamma_1+\gamma_1'} \overline{\partial}_2^{\gamma_2+\gamma_2'}$ with $\gamma_0 + \gamma_1 + \gamma_2 + \gamma_4 = 2l$, $\gamma_1' + \gamma_2' + \gamma_3' = 4 - k - l$ and $0 \le k \le 4 - l$ and also multiply the weight ε^{2l}. Then we can introduce the Alinhac good unknown $\mathbf{S}^\partial$ with respect to this general derivative ∂_*^α by

$$\mathbf{S}^{\partial,\pm} := \partial_*^\alpha S^\pm - \partial_*^\alpha \varphi \partial_3^\varphi S^\pm,$$

which satisfies the evolution equation $D_t^{\varphi\pm} \mathbf{S}^{\partial,\pm} = \mathfrak{D}^\partial(S^\pm)$ in $\Omega^\pm$ where $\mathfrak{D}^\partial(S^\pm)$ is defined by (3.10) after replacing $\mathcal{T}^\gamma$ with ∂_*^α. We will get

$$\begin{aligned}
\left\|\varepsilon^{2l} \partial_t^k \mathcal{T}^\gamma S^\pm\right\|_{4-k-l,\pm}^2 &\lesssim \left\|\varepsilon^{2l} \mathbf{S}^{\partial,\pm}\right\|_{0,\pm}^2 + \left|\partial_*^\alpha \psi\right|_0^2 \|\partial_3 S^\pm\|_{L^\infty(\Omega^\pm)}^2 \\
&\lesssim \delta E_{4+l}^\kappa(t) + P\left(\sigma^{-1}, \sum_{j=0}^l E_{4+j}^\kappa(0)\right) + E_4^\kappa(t) \int_0^t P\left(\sigma^{-1}, \sum_{j=0}^l E_{4+j}^\kappa(\tau)\right) \mathrm{d}\tau,
\end{aligned} \tag{3.168}$$

where we again invoke the estimate of $\left|\partial_*^\alpha \psi\right|_0$ that has been proven in Section 3.3-Section 3.5.

3.7.2 Uniform-in-κ estimates for the nonlinear approximate system

Summarizing Proposition 3.2 (L^2-energy conservation), Proposition 3.3 (tangential estimates), Proposition 3.8 (div-curl estimates) and (3.168) (entropy estimates), we conclude the estimates of the energy functional $E^\kappa(t)$ for the nonlinear approximate system (3.1) by

$$E^\kappa(t) \lesssim \delta E^\kappa(t) + P(E^\kappa(0)) + P(E^\kappa(t)) \int_0^t P(\sigma^{-1}, E^\kappa(\tau)) \, \mathrm{d}\tau, \quad \forall \delta \in (0,1) \tag{3.169}$$

Thus, choosing δ suitably small such that $\delta E^\kappa(t)$ can be absorbed by the left side and then using Gronwall-type argument, we find that there exists a time $T_\sigma > 0$ that depends on σ and the initial data and is independent of κ and ε, such that

$$\sup_{0 \le t \le T_\sigma} E^\kappa(t) \le C(\sigma^{-1}) P(E^\kappa(0)), \tag{3.170}$$

which is exactly the conclusion of Proposition 3.1.

4 Well-posedness of the nonlinear approximate system

We already prove the uniform-in-κ estimates for the nonlinear approximate problem (3.1). If we can prove the well-posedness of (3.1) for each fixed $\kappa > 0$, then the uniform estimates allow us to take the limit $\kappa \to 0_+$ and prove the local existence of system (1.30) for the compressible current-vortex sheets with surface tension. Since there is no loss of regularity in the estimate of $E^\kappa(t)$, we would like to use Picard iteration to construct the solution to (3.1) for each fixed κ.

4.1 Construction of the linearized problem and the iteration scheme

We start with $\psi^{[-1]} = \psi^{[0]} = 0$ and $(v^{[0],\pm}, b^{[0],\pm}, \rho^{[0],\pm}, S^{[0],\pm}) = (\vec{0}, \vec{0}, \rho^\pm, 0)$ for some constants $\rho^\pm \geq \bar{\rho}_0$. Then for any $n \geq 0, n \in \mathbb{N}$, given $\{(v^{[k],\pm}, b^{[k],\pm}, \rho^{[k],\pm}, S^{[k],\pm})\}_{k \leq n}$, we define $(v^{[n+1],\pm}, b^{[n+1],\pm}, q^{[n+1],\pm}, S^{[n+1],\pm}, \psi^{[n+1]})$ by the following linear system with variable coefficients only depending on $(v^{[n],\pm}, b^{[n],\pm}, q^{[n],\pm}, S^{[n],\pm}, \psi^{[n]}, \psi^{[n-1]})$

$$
\begin{cases}
\rho^{[n],\pm} D_t^{\varphi^{[n]},\pm} v^{[n+1],\pm} - (b^{[n],\pm} \cdot \nabla^{\varphi^{[n]},\pm}) b^{[n+1],\pm} + \nabla^{\varphi^{[n]},\pm} q^{[n+1],\pm} = 0 & \text{in } [0,T] \times \Omega^\pm, \\
(\mathcal{F}_p^\pm)^{[n]} D_t^{\varphi^{[n]},\pm} q^{[n+1],\pm} - (\mathcal{F}_p^\pm)^{[n]} D_t^{\varphi^{[n]},\pm} b^{[n+1],\pm} \cdot b^{[n],\pm} + \nabla^{\varphi^{[n]},\pm} \cdot v^{[n+1],\pm} = 0 & \text{in } [0,T] \times \Omega^\pm, \\
D_t^{\varphi^{[n]},\pm} b^{[n+1],\pm} - (b^{[n],\pm} \cdot \nabla^{\varphi^{[n]},\pm}) v^{[n+1],\pm} + b^{[n],\pm} \nabla^{\varphi^{[n]},\pm} \cdot v^{[n+1],\pm} = 0 & \text{in } [0,T] \times \Omega^\pm, \\
D_t^{\varphi^{[n]},\pm} S^{[n+1],\pm} = 0 & \text{in } [0,T] \times \Omega^\pm, \\
[\![q^{[n+1]}]\!] = \sigma \mathcal{H}(\psi^{[n]}) - \kappa(1 - \overline{\Delta})^2 \psi^{[n+1]} - \kappa(1 - \overline{\Delta}) \partial_t \psi^{[n+1]} & \text{on } [0,T] \times \Sigma, \\
\partial_t \psi^{[n+1]} = v^{[n+1],\pm} \cdot N^{[n]} & \text{on } [0,T] \times \Sigma, \\
v_3^{[n+1],\pm} = 0, & \text{on } [0,T] \times \Sigma^\pm, \\
(v^{[n+1],\pm}, b^{[n+1],\pm}, q^{[n+1],\pm}, S^{[n+1],\pm}, \psi^\pm)|_{t=0} = (v_0^{\kappa,\pm}, b_0^{\kappa,\pm}, q_0^{\kappa,\pm}, S_0^{\kappa,\pm}, \psi_0^\kappa),
\end{cases}
\tag{4.1}
$$

where $b_i^{[n],\pm} := b_i^{[n],\pm}$ for $i = 1, 2$ and $b_3^{[n],\pm}$ is defined by

$$
b_3^{[n],\pm} := b_3^{[n],\pm} + \Re_T^\pm \left(b_1^{[n],\pm} \overline{\partial}_1 \psi^{[n]} + b_2^{[n],\pm} \overline{\partial}_2 \psi^{[n]} - b_3^{[n],\pm} \right) \Big|_\Sigma
\tag{4.2}
$$

where $\Re_T^\pm$ is the lifting operator defined in Lemma B.3. The initial data $(v_0^{\kappa,\pm}, b_0^{\kappa,\pm}, \rho_0^{\kappa,\pm}, S_0^{\kappa,\pm}, \psi_0^\kappa)$ is the same as (3.1). The basic state $(v^{[n],\pm}, b^{[n],\pm}, \rho^{[n],\pm}, S^{[n],\pm}, \psi^{[n]}, \psi^{[n-1]})$ satisfies

1. (The hyperbolicity assumption) $\rho^{[n],\pm} > 0$ is determined by the equation of state (1.18) where $p^{[n],\pm}$ is defined by $p^{[n],\pm} := q^{[n],\pm} - \frac{1}{2} |b^{[n],\pm}|^2$. Then define $\mathcal{F}^{[n]} = \log \rho^{[n]}$, $\mathcal{F}_p^{[n],\pm} := \frac{\partial \mathcal{F}^{[n],\pm}}{\partial p} (p^{[n+1],\pm}, S^{[n+1],\pm}) > 0$.
2. (Tangential magnetic fields) $b^{[n],\pm} \cdot N^{[n]} = 0$ on Σ, and $b_3^{[n],\pm} = 0$ on $\Sigma^\pm$.
3. (Linearized material derivatives and covariant derivatives)

$$
D_t^{\varphi^{[n]},\pm} := \partial_t + \bar{v}^{[n]} \cdot \overline{\nabla} + \frac{1}{\partial_3 \varphi^{[n]}} (v^{[n]} \cdot \mathbf{N}^{[n-1]} - \partial_t \varphi^{[n]}) \partial_3,
\tag{4.3}
$$

$$
\partial_t^{\varphi^{[n]}} := \partial_t - \frac{\partial_t \varphi^{[n]}}{\partial_3 \varphi^{[n]}} \partial_3, \quad \nabla_a^{\varphi^{[n]}} = \partial_a^{\varphi^{[n]}} := \partial_a - \frac{\partial_a \varphi^{[n]}}{\partial_3 \varphi^{[n]}} \partial_3, \ a = 1, 2, \quad \nabla_a^{\varphi^{[n]}} = \partial_3^{\varphi^{[n]}} := \frac{1}{\partial_3 \varphi^{[n]}} \partial_3
\tag{4.4}
$$

where $N^{[n]} := (-\partial_1 \psi^{[n]}, -\partial_2 \psi^{[n]}, 1)^\top$ and $\mathbf{N}^{[n]}$ is the extension of $N^{[n]}$ with $\varphi^{[n]} = x_3 + \chi(x_3) \psi^{[n]}(t, x')$.

After solving the linear problem (4.1), we define $p^{[n+1],\pm} = q^{[n+1],\pm} - \frac{1}{2} |b^{[n+1],\pm}|^2$ and use the equation of state $p^{[n+1]} = p^{[n+1]}(\rho^{[n+1]}, S^{[n+1]})$ to determine the density $\rho^{[n+1]} > 0$. We shall also define the "modified magnteic fields" $b^{[n+1],\pm}$ as follows in order to guarantee $b^{[n+1],\pm} \cdot N^{[n+1]} = 0$ on Σ and $\Sigma^\pm$:

$$
b_1^{[n+1],\pm} = b_1^{[n+1],\pm}, \quad b_2^{[n+1],\pm} = b_2^{[n+1],\pm},
$$

$$
b_3^{[n+1],\pm} = b_3^{[n+1],\pm} + \Re_T^\pm \left(b_1^{[n+1],\pm} \overline{\partial}_1 \psi^{[n+1]} + b_2^{[n+1],\pm} \overline{\partial}_2 \psi^{[n+1]} - b_3^{[n+1],\pm} \right) \Big|_\Sigma.
\tag{4.5}
$$

Remark 4.1 (The boundary constraint of magnetic fields). The modified basic state $\mathring{b}$ is necessary here, because the quantity $b^{[n+1]}$ solved from (4.1) may not be tangential on Σ and so integrating $(b \cdot \nabla^\varphi)$ by parts produces uncontrollable boundary terms. When taking the limit $n \to \infty$, we can show that the limit function $b^{[\infty]}$ also satisfy the boundary constraint $b^{[\infty]} \cdot N^{[\infty]}|_\Sigma = 0$ which then indicates that $b_3^{[\infty]} = b_3^{[\infty]}$ in order to recover the nonlinear approximate system (3.1). We refer to Section 4.1.1 for details.

Remark 4.2 (The divergence constraint of magnetic fields). Notice that the divergence-free condition for $b^\pm$ no longer propagates from the initial data for the linear problem, but we will show that the contribution of the divergence of part of $b^\pm$ is still controllable and does not introduce extra substantial difficulty. After solving the nonlinear problem (3.1) for each fixed $\kappa > 0$, $\nabla^\varphi \cdot b^\pm = 0$ in (3.1) is automatically recovered from the initial constraint $\nabla^\varphi \cdot b_0^{\kappa,\pm} = 0$.

For simplicity of notations, given any $n \in \mathbb{N}$, we denote $(v^{[n+1],\pm}, b^{[n+1],\pm}, q^{[n+1],\pm}, p^{[n+1],\pm}, \rho^{[n+1],\pm}, S^{[n+1],\pm}, \psi^{[n+1]})$, $(v^{[n],\pm}, b^{[n],\pm}, \boldsymbol{b}^{[n],\pm}, q^{[n],\pm}, \rho^{[n],\pm}, p^{[n],\pm}, S^{[n],\pm}, \psi^{[n]})$, $\psi^{[n-1]}$ respectively by $(v^\pm, b^\pm, q^\pm, p^\pm, \rho^\pm, S^\pm)$, $(\mathring{v}^\pm, \mathring{b}^\pm, \mathring{\boldsymbol{b}}^\pm, \mathring{q}^\pm, \mathring{\rho}^\pm, \mathring{p}^\pm, \mathring{S}^\pm, \mathring{\psi})$ and $\dot{\psi}$. Also, we denote $D_t^{\varphi^{[n]},\pm}$ and $\partial_i^{\varphi^{[n]}}$, $\nabla_i^{\varphi^{[n]}}$ by $D_t^{\mathring{\varphi}\pm}$ and $\partial_i^{\mathring{\varphi}}$, $\nabla^{\mathring{\varphi}}$. Thus, the linear problem above becomes

$$
\begin{cases}
\mathring{\rho} D_t^{\mathring{\varphi}\pm} v^\pm - (\mathring{\boldsymbol{b}} \cdot \nabla^{\mathring{\varphi}}) b^\pm + \nabla^{\mathring{\varphi}} q^\pm = 0 & \text{in } [0,T] \times \Omega^\pm, \\
\mathring{\mathcal{F}}_p^\pm D_t^{\mathring{\varphi}\pm} q^\pm - \mathring{\mathcal{F}}_p^\pm D_t^{\mathring{\varphi}\pm} b^\pm \cdot \mathring{\boldsymbol{b}}^\pm + \nabla^{\mathring{\varphi}} \cdot v^\pm = 0 & \text{in } [0,T] \times \Omega^\pm, \\
D_t^{\mathring{\varphi}\pm} b^\pm - (\mathring{\boldsymbol{b}}^\pm \cdot \nabla^{\mathring{\varphi}}) v^\pm + \mathring{b}^\pm \nabla^{\mathring{\varphi}} \cdot v^\pm = 0 & \text{in } [0,T] \times \Omega^\pm, \\
D_t^{\mathring{\varphi}\pm} S^\pm = 0 & \text{in } [0,T] \times \Omega^\pm, \\
[\![q]\!] = \sigma \mathcal{H}(\mathring{\psi}) - \kappa(1-\overline{\Delta})^2 \psi - \kappa(1-\overline{\Delta})\partial_t \psi & \text{on } [0,T] \times \Sigma, \\
\partial_t \psi = v^\pm \cdot \mathring{N} & \text{on } [0,T] \times \Sigma, \\
v_3^\pm = 0 & \text{on } [0,T] \times \Sigma^\pm, \\
(v^\pm, b^\pm, q^\pm, S^\pm, \psi)|_{t=0} = (v_0^{\kappa,\pm}, b_0^{\kappa,\pm}, q_0^{\kappa,\pm}, S_0^{\kappa,\pm}, \psi_0^\kappa),
\end{cases}
\tag{4.6}
$$

where $D_t^{\mathring{\varphi}\pm} = \partial_t + \mathring{v} \cdot \overline{\nabla} + \frac{1}{\partial_3 \mathring{\varphi}}(\mathring{v} \cdot \mathring{N} - \partial_t \mathring{\varphi})\partial_3$ and $\mathcal{H}(\mathring{\psi}) = \overline{\nabla} \cdot (\overline{\nabla}\mathring{\psi}/|\mathring{N}|)$.

4.1.1 The way to recover the nonlinear approximate system

Before going to those tedious heavy calculations, let us first see how to construct the solution to the nonlinear system (3.1) for fixed $\kappa > 0$ if the existence of (4.6) and the strong convergence result has been proven. We assume that the expected limit is denoted by $(v^{[\infty],\pm}, b^{[\infty],\pm}, \boldsymbol{b}^{[\infty],\pm}, q^{[\infty],\pm}, \psi^{[\infty]})$. Then the limit functions satisfy the following system

$$
\begin{cases}
\rho^{[\infty],\pm} D_t^{\varphi^{[\infty]},\pm} v^{[\infty],\pm} - (b^{[\infty],\pm} \cdot \nabla^{\varphi^{[\infty]}}) b^{[\infty],\pm} + \nabla^{\varphi^{[\infty]}} q^{[\infty],\pm} = 0 & \text{in } [0,T] \times \Omega^\pm, \\
(\mathcal{F}_p^\pm)^{[\infty]} D_t^{\varphi^{[\infty]},\pm} q^{[\infty],\pm} - (\mathcal{F}_p^\pm)^{[\infty]} D_t^{\varphi^{[\infty]},\pm} b^{[\infty],\pm} \cdot b^{[\infty],\pm} + \nabla^{\varphi^{[\infty]}} \cdot v^{[\infty],\pm} = 0 & \text{in } [0,T] \times \Omega^\pm, \\
D_t^{\varphi^{[\infty]},\pm} b^{[\infty],\pm} - (b^{[\infty],\pm} \cdot \nabla^{\varphi^{[\infty]}}) v^{[\infty],\pm} + b^{[\infty],\pm}\nabla^{\varphi^{[\infty]}} \cdot v^{[\infty],\pm} = 0 & \text{in } [0,T] \times \Omega^\pm, \\
D_t^{\varphi^{[\infty]},\pm} S^{[\infty],\pm} = 0 & \text{in } [0,T] \times \Omega^\pm, \\
[\![q^{[\infty]}]\!] = \sigma \mathcal{H}(\psi^{[\infty]}) - \kappa(1-\overline{\Delta})^2 \psi^{[\infty]} - \kappa(1-\overline{\Delta})\partial_t \psi^{[\infty]} & \text{on } [0,T] \times \Sigma, \\
\partial_t \psi^{[\infty]} = v^{[\infty],\pm} \cdot N^{[\infty]} & \text{on } [0,T] \times \Sigma, \\
v_3^{[\infty],\pm} = 0 & \text{on } [0,T] \times \Sigma^\pm, \\
(v^{[\infty],\pm}, b^{[\infty],\pm}, q^{[\infty],\pm}, S^{[\infty],\pm}, \psi^{[\infty]})|_{t=0} = (v_0^{\kappa,\pm}, b_0^{\kappa,\pm}, q_0^{\kappa,\pm}, S_0^{\kappa,\pm}, \psi_0^\kappa),
\end{cases}
\tag{4.7}
$$

where $\rho^{[\infty]}$ is defined via the equation of state $\rho = \rho(p, S)$ and $p^{[\infty]} := q^{[\infty]} - \frac{1}{2}|b^{[\infty]}|^2$. Also we have

$$
D_t^{\varphi^{[\infty]},\pm} = \partial_t + \bar{v}^{[\infty],\pm} \cdot \overline{\nabla} + \frac{1}{\partial_3 \varphi^{[\infty]}}(v^{[\infty],\pm} \cdot N^{[\infty]} - \partial_t \varphi^{[\infty]})\partial_3,
$$

$$
b^{[\infty],\pm} \cdot \nabla^{\varphi^{[\infty]}} = \bar{b}^{[\infty],\pm} \cdot \overline{\nabla} + \frac{1}{\partial_3 \varphi^{[\infty]}}(b^{[\infty],\pm} \cdot N^{[\infty]})\partial_3.
$$

We must prove that $b_3^{[\infty],\pm} = \boldsymbol{b}_3^{[\infty],\pm}$ in $\Omega^\pm$. According to the definition of $\boldsymbol{b}^{[n]}$, the limit function satisfies

$$
b_3^{[\infty],\pm} = \boldsymbol{b}_3^{[\infty],\pm} + \mathfrak{R}_T^\pm \left(b_1^{[\infty],\pm}\overline{\partial}_1 \psi^{[\infty]} + b_2^{[\infty],\pm}\overline{\partial}_2 \psi^{[\infty]} - b_3^{[\infty],\pm} \right)\Big|_\Sigma \Rightarrow b_3^{[\infty],\pm} \cdot N^{[\infty]}|_\Sigma = 0.
$$

Since Lemma B.3 implies that $\mathfrak{R}_T^\pm(0) = 0$, then the remaining step is to show $b^{[\infty],\pm} \cdot N^{[\infty]}|_\Sigma = 0$ holds with in the lifespan of the

solution to (4.7) provided $b^{[\infty],\pm} \cdot N^{[\infty]}|_{t=0} = 0$ on Σ. On Σ, we compute that

$$D_t^{\varphi^{[\infty]},\pm}(b^{[\infty],\pm} \cdot N^{[\infty]}) = D_t^{\varphi^{[\infty]},\pm}b^{[\infty],\pm} \cdot N^{[\infty]} + b^{[\infty],\pm} \cdot D_t^{\varphi^{[\infty]},\pm}N^{[\infty]}$$

$$= \underbrace{(\bar{b}^{[\infty],\pm} \cdot \overline{\nabla})}_{=\bar{b}^{[\infty],\pm}\cdot\overline{\nabla}} v^{[\infty],\pm} \cdot N^{[\infty]} + \underbrace{(b^{[\infty],\pm} \cdot N^{[\infty]})}_{=0 \text{ on } \Sigma} \partial_3 v^{[\infty],\pm} \cdot N^{[\infty]} + \underbrace{(b^{[\infty],\pm} \cdot N^{[\infty]})}_{=0 \text{ on } \Sigma}(\nabla^{\varphi^{[\infty]}} \cdot v^{[\infty],\pm})$$

$$- \bar{b}_i^{[\infty],\pm}\overline{\partial}_i\partial_t\psi^{[\infty]} - \bar{b}_i^{[\infty],\pm}\bar{v}_j^{[\infty],\pm}\overline{\partial}_j\overline{\partial}_i\psi^{[\infty]}$$

$$= (\bar{b}^{[\infty],\pm} \cdot \overline{\nabla})\underbrace{\left(v_3^{[\infty],\pm} - \bar{v}_j^{[\infty],\pm}\overline{\partial}_j\psi^{[\infty]}\right)}_{=\partial_t\psi^{[\infty]}} + \bar{b}_i^{[\infty],\pm}\bar{v}_j^{[\infty],\pm}\overline{\partial}_j\overline{\partial}_i\psi^{[\infty]} - \bar{b}_i^{[\infty],\pm}\overline{\partial}_i\partial_t\psi^{[\infty]} - \bar{b}_i^{[\infty],\pm}\bar{v}_j^{[\infty],\pm}\overline{\partial}_j\overline{\partial}_i\psi^{[\infty]} = 0.$$

Thus standard L^2 energy estimate shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma \left|b^{[\infty],\pm} \cdot N^{[\infty]}\right|^2 \, \mathrm{d}x' = \int_\Sigma (\overline{\nabla} \cdot \bar{v}^{[\infty],\pm})\left|b^{[\infty],\pm} \cdot N^{[\infty]}\right|^2 \, \mathrm{d}x' \leq |\overline{\partial}v^{[\infty],\pm}|_{L^\infty}\left|b^{[\infty],\pm} \cdot N^{[\infty]}\right|_0^2. \tag{4.8}$$

Since $b^{[\infty],\pm} \cdot N^{[\infty]}|_{t=0} = 0$ on Σ, we conclude that $b^{[\infty],\pm} \cdot N^{[\infty]} = 0$ always holds on Σ by using Grönwall's inequality. Plugging it back to the expression of $\boldsymbol{b}_3^{[\infty],\pm}$, we find $\boldsymbol{b}_3^{[\infty],\pm} = b_3^{[\infty],\pm}$ in $\Omega^\pm$ as desired. Then we can replace $\boldsymbol{b}$ by b in the limit system (4.7)

$$\begin{cases} \rho^{[\infty],\pm}D_t^{\varphi^{[\infty]},\pm}v^{[\infty],\pm} - (b^{[\infty],\pm} \cdot \nabla^{\varphi^{[\infty]}})b^{[\infty],\pm} + \nabla^{\varphi^{[\infty]}}q^{[\infty],\pm} = 0 & \text{in } [0,T] \times \Omega^\pm, \\ (\mathcal{F}_p^\pm)^{[\infty]}D_t^{\varphi^{[\infty]},\pm}q^{[\infty],\pm} - (\mathcal{F}_p^\pm)^{[\infty]}D_t^{\varphi^{[\infty]},\pm}b^{[\infty],\pm} \cdot b^{[\infty],\pm} + \nabla^{\varphi^{[\infty]}} \cdot v^{[\infty],\pm} = 0 & \text{in } [0,T] \times \Omega^\pm, \\ D_t^{\varphi^{[\infty]},\pm}b^{[\infty],\pm} - (b^{[\infty],\pm} \cdot \nabla^{\varphi^{[\infty]}})v^{[\infty],\pm} + b^{[\infty],\pm}\nabla^{\varphi^{[\infty]}} \cdot v^{[\infty],\pm} = 0 & \text{in } [0,T] \times \Omega^\pm, \\ D_t^{\varphi^{[\infty]},\pm}S^{[\infty],\pm} = 0 & \text{in } [0,T] \times \Omega^\pm, \\ [\![q^{[\infty]}]\!] = \sigma\mathcal{H}(\psi^{[\infty]}) - \kappa(1-\overline{\Delta})^2\psi^{[\infty]} - \kappa(1-\overline{\Delta})\partial_t\psi^{[\infty]} & \text{on } [0,T] \times \Sigma, \\ \partial_t\psi^{[\infty]} = v^{[\infty],\pm} \cdot N^{[\infty]}, \quad b^{[\infty],\pm} \cdot N^{[\infty]} = 0 & \text{on } [0,T] \times \Sigma, \\ v_3^{[\infty],\pm} = b_3^{[\infty],\pm} = 0 & \text{on } [0,T] \times \Sigma^\pm, \\ (v^{[\infty],\pm}, b^{[\infty],\pm}, q^{[\infty],\pm}, S^{[\infty],\pm}, \psi^{[\infty]})|_{t=0} = (v_0^{\kappa,\pm}, b_0^{\kappa,\pm}, q_0^{\kappa,\pm}, S_0^{\kappa,\pm}, \psi_0^\kappa), \end{cases} \tag{4.9}$$

in which $(v^{[\infty],\pm}, b^{[\infty],\pm}, q^{[\infty],\pm}, \psi^{[\infty]})$ exactly gives the solution to the nonlinear approximate system (3.1). Finally, the divergence constraint $\nabla^{\varphi^{[\infty]}} \cdot b^{[\infty],\pm} = 0$ in $\Omega^\pm$ automatically holds thanks to the second equation, the fourth equation in (4.9) and $\nabla^{\varphi_0^\kappa} \cdot b_0^{\kappa,\pm} = 0$ in $\Omega^\pm$.

4.2   Well-posedness of the linearized approximate problem

Although (4.6) is a first-order linear symmetric hyperbolic system with characteristic boundary conditions, it is still not easy to apply the duality argument in Lax-Phillips [29] to prove the well-posedness of (4.6) in $L^2([0,T] \times \Omega^\pm)$ because the boundary condition for the dual system of (4.6) may not be explicitly calculated due to the appearance of the regularization term $\kappa(1-\overline{\Delta})^2\psi$. Instead, we apply the classical Galerkin's method to construct the weak solution in $L^2([0,T] \times \Omega^\pm)$ to the linearized problem (4.6). Once this is done, the weak solution is actually a strong solution by the argument in [43, Chapter 2.2.3].

We assume the basic state $(\mathring{v}, \mathring{b}, \mathring{q}, \mathring{\rho}, \mathring{p}, \mathring{S}, \mathring{\psi})$ and $\dot{\psi}$ satisfy the following bounds: There exists some $\mathring{K}_0 > 0$ and a time $T_\kappa > 0$ (depending on $\kappa > 0$) such that

$$\sup_{0 \leq t \leq T_\kappa} \sum_{l=0}^4 \Big(\sum_\pm \sum_{\langle\alpha\rangle=2l} \sum_{k=0}^{4-l} \left\|\left(\varepsilon^{2l}\mathcal{T}^\alpha\partial_t^k(\mathring{v}^\pm, \mathring{b}^\pm, \mathring{S}^\pm, (\mathcal{F}_p^\pm)^{\frac{(k+\alpha_0-l-3)_+}{2}}\mathring{q}^\pm)\right)\right\|_{4-k-l,\pm}^2$$

$$+ \sum_{k=0}^{4+l} \left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^k\mathring{\psi}\right|_{6+l-k}^2 + \int_0^t \left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^{5+l}\mathring{\psi}\right|_1^2 \, \mathrm{d}\tau\Big) \leq \mathring{K}_0, \tag{4.10}$$

where $\mathcal{T}^\alpha := (\omega(x_3)\partial_3)^{\alpha_4}\partial_t^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}$ with the multi-index $\alpha = (\alpha_0, \alpha_1, \alpha_2, 0, \alpha_4)$, $\langle\alpha\rangle = \alpha_0 + \alpha_1 + \alpha_2 + 2 \times 0 + \alpha_4$. Moreover, we have

$$\forall 0 \leq T \leq T_\kappa, \quad \int_0^T \left\|\varepsilon^{2l}\mathcal{T}^\alpha\partial_t^k\mathring{b}^\pm\right\|_{4-k-l,\pm}^2 \, \mathrm{d}t \leq C(\mathring{K}_0).$$

Remark 4.3. The L_t^2-type bound of $\mathring{b}$ is obtained by using the second part of Lemma B.3 and the $\sqrt{\kappa}$-weighted enhanced regularity for the free interface. Indeed, the modification term $\mathfrak{R}_T^\pm \left(\mathring{b}_1^\pm \overline{\partial}_1 \mathring{\psi} + \mathring{b}_2^\pm \overline{\partial}_2 \mathring{\psi} - \mathring{b}_3^\pm \right)\big|_\Sigma$ has *vanishing initial value* thank to $b_0^{\kappa,\pm} \cdot N_0^\kappa = 0$ on Σ. Thus, one can extend this function to $(-\infty, T] \times \Omega^\pm$ and then apply the trace lemma for anisotropic Sobolev spaces (cf. Trakhinin-Wang [61, Lemma 3.4] or Lemma B.3 in this paper) to show that

$$\int_0^T \left\| \mathcal{T}^\alpha \partial_t^k (\mathring{b}^\pm - \check{b}^\pm) \right\|_{4-k-l,\pm}^2 \, \mathrm{d}t \leq \left\| \mathring{b}^\pm - \check{b}^\pm \right\|_{8,*,T,\pm}^2 \lesssim \left| \mathring{b}_1^\pm \overline{\partial}_1 \mathring{\psi} + \mathring{b}_2^\pm \overline{\partial}_2 \mathring{\psi} - \mathring{b}_3^\pm \right|_{7,T}^2$$

$$\lesssim \left\| \mathring{b}_1^\pm \overline{\partial}_1 \mathring{\varphi} + \mathring{b}_2^\pm \overline{\partial}_2 \mathring{\varphi} - \mathring{b}_3^\pm \right\|_{8,*,T,\pm}^2 = \int_0^T \left\| \mathring{b}_1^\pm \overline{\partial}_1 \mathring{\varphi} + \mathring{b}_2^\pm \overline{\partial}_2 \mathring{\varphi} - \mathring{b}_3^\pm \right\|_{8,*,\pm}^2 \, \mathrm{d}t \lesssim T \mathring{K}_0, \quad \forall T \in [0, T_\kappa],$$

where $\|\cdot\|_{m,*,T,\pm}, |\cdot|_{m,T}$ norms are defined in Appendix B. Notice that this $\sqrt{\kappa}$-weighted enhanced regularity is necessary here, otherwise we lose the control of $|\overline{\partial}\psi(t)|_8$ and a loss of tangential derivative occurs as in lots of previous works [6, 60, 61, 62] and references therein.

4.2.1 Construction of Galerkin sequence

Since our domain $\Omega := \mathbb{T}^2 \times (-H, H)$ is bounded, there exists an orthonormal basis $\{e_j\}_{j=1}^\infty \subset C^\infty(\Omega)$ for $L^2(\Omega)$ which is also an orthogonal basis of $H_0^1(\Omega)$. To construct the Galerkin sequence, we first write the linearized system (4.6) into a symmetric hyperbolic system of $U^\pm := (q^\pm, v^\pm, b^\pm, S^\pm)^\top \in \mathbb{R}^8$:

$$A_0(\mathring{U}^\pm)\partial_t U^\pm + A_1(\mathring{U}^\pm)\partial_1 U^\pm + A_2(\mathring{U}^\pm)\partial_2 U^\pm + A_3(\mathring{U}^\pm)\partial_3 U^\pm = 0 \quad \text{in } \Omega^\pm \tag{4.11}$$

where the coefficient matrices are

$$A_0(\mathring{U}) := \begin{bmatrix} \mathring{\mathcal{F}}_p & \vec{0}^\top & -\mathring{\mathcal{F}}_p \mathring{b}^\top & 0 \\ \vec{0} & \mathring{\rho} I_3 & O_3 & \vec{0} \\ -\mathring{\mathcal{F}}_p \mathring{b} & O_3 & I_3 + \mathring{\mathcal{F}}_p \mathring{b} \otimes \mathring{b} & \vec{0} \\ 0 & \vec{0}^\top & \vec{0}^\top & 1 \end{bmatrix}, \quad A_i(\mathring{U}) := \begin{bmatrix} \mathring{\mathcal{F}}_p \mathring{v}_i & \vec{e}_i^\top & -\mathring{\mathcal{F}}_p \mathring{v}_i \mathring{b}^\top & 0 \\ \vec{e}_i & \mathring{\rho} \mathring{v}_i I_3 & -\mathring{b}_i I_3 & \vec{0} \\ -\mathring{\mathcal{F}}_p \mathring{v}_i \mathring{b} & -\mathring{b}_i I_3 & \mathring{v}_i I_3 + \mathring{\mathcal{F}}_p \mathring{v}_i (\mathring{b} \otimes \mathring{b}) & \vec{0} \\ 0 & \vec{0}^\top & \vec{0}^\top & \mathring{v}_i \end{bmatrix} \quad (i = 1, 2),$$

$$A_3(\mathring{U}) := \frac{1}{\partial_3 \mathring{\varphi}} \begin{bmatrix} \mathring{\mathcal{F}}_p (\mathring{v} \cdot \mathbf{N} - \partial_t \mathring{\varphi}) & \mathbf{N}^\top & -\mathring{\mathcal{F}}_p (\mathring{v} \cdot \mathbf{N} - \partial_t \mathring{\varphi}) \mathring{b}^\top & 0 \\ \mathbf{N} & \mathring{\rho}(\mathring{v} \cdot \mathbf{N} - \partial_t \mathring{\varphi}) I_3 & -(\mathring{b} \cdot \mathbf{N}) I_3 & \vec{0} \\ -\mathring{\mathcal{F}}_p (\mathring{v} \cdot \mathbf{N} - \partial_t \mathring{\varphi}) \mathring{b} & -(\mathring{b} \cdot \mathbf{N}) I_3 & (\mathring{v} \cdot \mathbf{N} - \partial_t \mathring{\varphi}) I_3 + \mathring{\mathcal{F}}_p (\mathring{v} \cdot \mathbf{N} - \partial_t \mathring{\varphi})(\mathring{b} \otimes \mathring{b}) & \vec{0} \\ 0 & \vec{0}^\top & \vec{0}^\top & \mathring{v} \cdot \mathbf{N} - \partial_t \mathring{\varphi} \end{bmatrix}.$$

Also notice that the matrix $A_3(\mathring{U})$ is equal to the following matrix on the boundary

$$A_3(\mathring{U})|_{\Sigma, \Sigma^\pm} := \begin{bmatrix} 0 & \mathring{\mathbf{N}}^\top & \mathbf{0}_4^\top \\ \mathring{\mathbf{N}} & O_3 & \\ \mathbf{0}_4 & & O_4 \end{bmatrix},$$

which has constant rank 2 and has one negative eigenvalue and one positive eigenvalue, so the correct number of boundary conditions to solve $U^\pm$ in (4.11) is $1 + 1 = 2$ (the jump conditions for q and $v \cdot \mathring{\mathbf{N}}$), and we need one more equation, namely $\partial_t \psi = v^\pm \cdot \mathring{\mathbf{N}}$, to determine the free interface.

Given $2 \leq m \in \mathbb{N}^*$, we introduce the Galerkin sequence $\{U^{m,\pm}(t, x), \psi^m(t, x')\}$ by

$$U_j^{m,\pm}(t, x) := \sum_{l=1}^m U_{lj}^{m,\pm}(t) e_l(x) \quad 1 \leq j \leq 8, \tag{4.12}$$

$$\psi^m(t, x') := \sum_{l=1}^m \psi_l^m(t) e_l(x', 0). \tag{4.13}$$

The Galerkin sequence is assumed to satisfy the boundary conditions

$$\partial_t \psi^m = U_4^{m,\pm} - U_2^{m,\pm} \overline{\partial}_1 \mathring{\psi} - U_3^{m,\pm} \overline{\partial}_2 \mathring{\psi} \tag{4.14}$$

$$\llbracket U_1^m \rrbracket = \sigma \mathcal{H}(\mathring{\psi}) - \kappa(1 - \overline{\Delta})^2 \psi^m - \kappa(1 - \overline{\Delta})\partial_t \psi^m \tag{4.15}$$

Now we introduce an ODE system as the "truncated version" of (4.11) in $\text{Span}\{e_1, \cdots, e_m\}$ by testing the Galerkin sequence by a vector field $\phi := (\phi_1, \cdots, \phi_8)^\top$ with

$$\phi_i := \sum_{l=1}^{m} \phi_{il}(t) e_l(x) \in \text{Span}\{e_1, \cdots, e_m\}.$$

$$\int_{\Omega^\pm} A_0^{ij}(\mathring{U}^\pm)(\partial_t U_j^{m,\pm})\phi_i \, d\mathring{\mathcal{V}}_t + \sum_{k=1}^{2} A_k^{ij}(\mathring{U}^\pm)(\overline{\partial}_k U_j^{m,\pm})\phi_i \, d\mathring{\mathcal{V}}_t + \int_{\Omega^\pm} A_3^{ij}(\mathring{U}^\pm)(\partial_3 U_j^{m,\pm})\phi_i \, d\mathring{\mathcal{V}}_t = 0 \tag{4.16}$$

where $d\mathring{\mathcal{V}}_t := \partial_3 \mathring{\varphi} \, dx$ Integrating by parts in $\overline{\partial}_k$ and ∂_3, we get

$$\int_{\Omega^\pm} A_0^{ij}(\mathring{U}^\pm)(\partial_t U_j^{m,\pm})\phi_i - \sum_{k=1}^{3} U_j^{m,\pm} \partial_k(A_k^{ij}(\mathring{U}^\pm)\phi_i) \, d\mathring{\mathcal{V}}_t \mp \int_\Sigma A_3^{ij}(\mathring{U}^\pm)U_j^{m,\pm}\phi_i \, dx' = 0. \tag{4.17}$$

Plugging the Galerkin sequence into the above identity, we get

$$\int_{\Omega^\pm} A_0^{ij}(\mathring{U}^\pm)e_l(x)\phi_i(U_{lj}^{m,\pm})'(t) - \sum_{k=1}^{3} \partial_k(A_k^{ij}(\mathring{U}^\pm)\phi_i)e_l(x)U_{lj}^{m,\pm}(t) \, d\mathcal{V}_t = \pm \int_\Sigma A_3^{ij}(\mathring{U}^\pm)U_j^{m,\pm}\phi_i \, dx'. \tag{4.18}$$

Taking sum for the two parts in $\Omega^\pm$, setting $\phi_i(x) = e_i(x)$ and using the jump condition for $[\![q]\!]$ and $v^\pm \cdot \mathring{N}$, we obtain a first-order linear ODE system for $\{U_{lj}^\pm(t)\}$

$$\sum_{\pm} \left(\int_{\Omega^\pm} A_0^{ij}(\mathring{U}^\pm)e_l(x)e_i(x) \, d\mathring{\mathcal{V}}_t \right) (U_{lj}^{m,\pm})'(t) - \left(\int_{\Omega^\pm} \partial_k(A_k^{ij}(\mathring{U}^\pm)e_i(x))e_l(x) \, d\mathring{\mathcal{V}}_t \right) U_{lj}^{m,\pm}(t)$$

$$= \int_\Sigma [\![q]\!] \underbrace{\left(-e_2(x',0)\overline{\partial}_1 \mathring{\psi}(t,x') - e_3(x',0)\overline{\partial}_2 \mathring{\psi}(t,x') + e_4(x',0) \right)}_{=: \phi_v \cdot \mathring{N}} \, dx'$$

$$= -\sigma \int_\Sigma (\overline{\nabla}\mathring{\psi}/|\mathring{N}|) \cdot \overline{\nabla} \left(\phi_v \cdot \mathring{N} \right) \, dx' - \kappa \int_\Sigma (1 - \overline{\Delta})\psi^m (1 - \overline{\Delta}) \left(\phi_v \cdot \mathring{N} \right) \, dx'$$

$$- \kappa \int_\Sigma \partial_t \psi^m \left(\phi_v \cdot \mathring{N} \right) \, dx' - \kappa \int_\Sigma \overline{\nabla}\partial_t \psi^m \cdot \overline{\nabla} \left(\phi_v \cdot \mathring{N} \right) \, dx'. \tag{4.19}$$

Since the basis $\{e_l\}$ are smooth and the coefficients $(\mathring{U}^{m,\pm}, \mathring{\psi})$ are sufficiently regular, standard ODE theory guarantees the local existence and uniqueness of the above ODE system (4.19) with initial data

$$U_{lj}^{m,\pm}(0) := \int_{\Omega^\pm} U_j^{m,\pm}(0,x)e_k(x)\partial_3\mathring{\varphi}_0 \, dx.$$

4.2.2 Existence of solutions to the linearized problem

The existence of weak solution is guaranteed by uniform-in-m estimates for the Galerkin sequence $\{U^{m,\pm}(t,x), \psi^m(t,x')\}$. Now we let the test function $\phi = U^{m,\pm}$ in $\Omega^\pm$ respectively to obtain the standard L^2-type energy estimates thanks to the symmetric property of the coefficient matrices and the concrete form of $A_3(\mathring{U})$ on the boundary

$$\sum_{\pm} \frac{d}{dt} \frac{1}{2} \int_{\Omega^\pm} (U^{m,\pm})^\top \cdot A_0(\mathring{U}^\pm)U^{m,\pm} \, d\mathcal{V}_t$$

$$= \sum_{\pm} \frac{1}{2} \int_{\Omega^\pm} (U^{m,\pm})^\top \cdot \partial_t(A_0(\mathring{U}^\pm))U^{m,\pm} \, d\mathcal{V}_t + \frac{1}{2} \int_{\Omega^\pm} (U^{m,\pm})^\top \cdot \partial_k(A_k(\mathring{U}^\pm))U^{m,\pm} \, d\mathcal{V}_t$$

$$+ \int_\Sigma [\![U_1^m]\!] (U_4^{m,\pm} - U_2^{m,\pm}\overline{\partial}_1\mathring{\psi} - U_3^{m,\pm}\overline{\partial}_2\mathring{\psi}) \, dx' \tag{4.20}$$

where the interior term can be controlled directly by $C(\mathring{K}_0)\|U^{m,\pm}\|^2_{0,\pm}$. For the boundary term, using the boundary conditions (4.14)-(4.15), we get the energy bounds under time integral

$$
\int_0^t \int_\Sigma [\![U_1^m]\!] \left(U_4^{m,\pm} - U_2^{m,\pm}\overline{\partial}_1\mathring{\psi} - U_3^{m,\pm}\overline{\partial}_2\mathring{\psi}\right) \mathrm{d}x' \, \mathrm{d}\tau
$$

$$
= \int_0^t \int_\Sigma \left(\sigma\mathcal{H}(\mathring{\psi}) - \kappa(1-\overline{\Delta})^2\psi^m - \kappa(1-\overline{\Delta})\partial_t\psi^m\right)\partial_t\psi^m \, \mathrm{d}x' \, \mathrm{d}\tau
$$

$$
= -\sigma\int_0^t \int_\Sigma (\overline{\nabla}\mathring{\psi}/|\mathring{\mathbf{N}}|) \cdot \overline{\nabla}\partial_t\psi^m \, \mathrm{d}x' \, \mathrm{d}\tau - \kappa\int_0^t \int_\Sigma (1-\overline{\Delta})\psi^m \, (1-\overline{\Delta})\partial_t\psi_m \, \mathrm{d}x' \, \mathrm{d}\tau
$$

$$
\quad - \kappa\int_0^t \int_\Sigma \partial_t\psi^m\partial_t\psi_m \, \mathrm{d}x' \, \mathrm{d}\tau - \kappa\int_0^t \int_\Sigma \overline{\nabla}\partial_t\psi^m \cdot \overline{\nabla}\partial_t\psi_m \, \mathrm{d}x' \, \mathrm{d}\tau
$$

$$
\lesssim \frac{\sigma}{\sqrt{\kappa}}\int_0^t P(|\overline{\nabla}\mathring{\psi}|_{L^\infty})|\overline{\nabla}\mathring{\psi}|_0^2 \, \mathrm{d}\tau + \delta|\sqrt{\kappa}\psi^m|^2_{L_t^2H^1_{x'}} - \frac{1}{2}\left|\sqrt{\kappa}\psi^m\right|^2_2\Big|_0^t - \int_0^t \left|\sqrt{\kappa}\partial_t\psi^m\right|^2_1 \, \mathrm{d}\tau. \tag{4.21}
$$

We define

$$
N^m(t) := \sum_\pm \left\|\left(\sqrt{\mathring{\mathcal{F}}_p^\pm}U_1^{m,\pm}, U_2^{m,\pm}, \cdots, U_8^{m,\pm}\right)\right\|^2_{0,\pm} + \left|\sqrt{\kappa}\psi^m\right|^2_2 + \int_0^t \left|\sqrt{\kappa}\partial_t\psi^m\right|^2_1 \, \mathrm{d}\tau. \tag{4.22}
$$

Since $A_0(\mathring{U}^\pm) > 0$, we obtain the uniform-in-m estimate for the Galerkin sequence $\{U^{m,\pm}(t,x), \psi^m(t,x')\}$.

$$
N^m(t) \leq N^m(0) + \int_0^t C(\mathring{K}_0, \kappa^{-1})N^m(\tau) \, \mathrm{d}\tau, \tag{4.23}
$$

and thus there exists a time $T_N > 0$ (depending on κ and $N^m(0)$, independent of m) such that

$$
\sup_{0\leq t\leq T_N} N^m(t) \leq C'(\mathring{K}_0, \kappa^{-1})N^m(0).
$$

Because $L^\infty([0, T_N]; L^2(\Omega^\pm))$ is not reflexive, we alternatively consider the weak convergence in $L^2([0, T_N]; L^2(\Omega^\pm))$. By Eberlein-Šmulian theorem and the uniqueness of expansion in Galerkin basis $\{e_l\}_{l=1}^\infty$, there exists a subsequence $\{U^{m_k,\pm}(t,x), \psi^{m_k}(t,x')\}_{k=1}^\infty$ such that

$$
\left(\sqrt{\mathring{\mathcal{F}}_p^\pm}U_1^{m_k,\pm}, U_2^{m_k,\pm}, \cdots, U_8^{m_k,\pm}\right) \rightharpoonup \left(\sqrt{\mathring{\mathcal{F}}_p^\pm}q^\pm, v^\pm, \cdots, b^\pm, S^\pm\right) \text{ in } L^2([0, T_N]; L^2(\Omega^\pm)) \tag{4.24}
$$

$$
\psi^{m_k} \rightharpoonup \psi \text{ in } L^2([0, T_N]; H^2(\Sigma)), \quad \partial_t\psi^{m_k} \rightharpoonup \partial_t\psi \text{ in } L^2([0, T_N]; H^1(\Sigma)). \tag{4.25}
$$

This proves the existence of weak solution to (4.11) (and equivalently (4.6)). The uniqueness easily follows from the estimate of $N(t)$ and the linearity of (4.11). The weak solution is actually a strong solution according to the argument in [43, Chapter 2.2.3].

4.3 Higher-order estimates of the linearized approximate problem

To proceed the Picard iteration, we shall prove that the bounds (4.10) for the coefficients $(\mathring{U}, \mathring{\psi}, \mathring{\dot{\psi}})$ can be preserved by the solution to (4.6). Fix $\kappa > 0$, we define the energy functional for (4.6) to be

$$
\mathring{E}^\kappa(t) := \mathring{E}_4^\kappa(t) + \cdots + \mathring{E}_8^\kappa(t)
$$

$$
\mathring{E}_{4+l}^\kappa(t) := \sum_\pm \sum_{\langle\alpha\rangle=2l} \sum_{k=0}^{4-l} \left\|\left(\varepsilon^{2l}\mathcal{T}^\alpha\partial_t^k(v^\pm, b^\pm, S^\pm, (\mathring{\mathcal{F}}_p^\pm)^{\frac{(k+\alpha_0-l-3)_+}{2}}q^\pm)\right)\right\|^2_{4-k-l,\pm} \tag{4.26}
$$

$$
\quad + \sum_{k=0}^{4+l}\left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^k\psi\right|^2_{6+l-k} + \int_0^t\left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^{5+l}\psi\right|^2_1 \, \mathrm{d}\tau
$$

where $\mathcal{T}^\alpha := (\omega(x_3)\partial_3)^{\alpha_4}\partial_t^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}$ with the multi-index $\alpha = (\alpha_0, \alpha_1, \alpha_2, 0, \alpha_4)$, $\langle\alpha\rangle = \alpha_0 + \alpha_1 + \alpha_2 + 2\times 0 + \alpha_4$. We aim to prove that

Proposition 4.1. There exists some $T_\kappa > 0$ depending on κ and $\mathring{K}_0$, such that

$$\sup_{0 \le t \le T_\kappa} \mathring{E}^\kappa(t) \le C(\kappa^{-1}, \mathring{K}_0)\mathring{E}^\kappa(0).$$

A large part of the proof of proposition (4.1) is similar to the analysis in Section 3. Moreover, since $\kappa > 0$ is fixed, we obtain higher boundary regularity for the free interface ψ, which allows us to avoid some technical steps (such as the analysis in Section 3.4). We will skip some details for the part substantially similar to Section 3 and emphasize the different part. Now we start with div-curl analysis.

4.3.1 Div-Curl analysis

We start with $\mathring{E}_4(t)$. Using (B.1) and the boundary conditions for v, b, we get

$$\left\| v^\pm, b^\pm \right\|_{4,\pm}^2 \lesssim C(|\mathring{\psi}|_4, |\overline{\nabla}\mathring{\psi}|_{W^{1,\infty}}) \left(\|(v^\pm, b^\pm)\|_{0,\pm}^2 + \|\nabla^{\mathring{\varphi}} \cdot (v^\pm, b^\pm)\|_{3,\pm}^2 + \|\nabla^{\mathring{\varphi}} \times (v^\pm, b^\pm)\|_{3,\pm}^2 + \|\overline{\partial}^4 (v^\pm, b^\pm)\|_0^2 \right). \tag{4.27}$$

Remark 4.4. Here we cannot use the div-curl inequality (B.2) to estimate the normal traces because the boundary constraint $b \cdot \mathring{N} = 0$ no longer holds for the linearized problem.

The L^2-estimates are already proven in the uniform estimates of Galerkin sequence, so we no longer repeat it. The treatment of $\nabla^{\mathring{\varphi}} \cdot v$ is also the same as in Section 3.3.1, that is, invoking the continuity equation. For $\nabla^{\mathring{\varphi}} \cdot b$, we no longer have the div-free constraint. Instead, we can take $\nabla^{\mathring{\varphi}} \cdot$ in the linearized evolution equation of b to get

$$D_t^{\mathring{\varphi}\pm}(\nabla^{\mathring{\varphi}} \cdot b^\pm) = \partial_i^{\mathring{\varphi}}(\mathring{b}_j^\pm \partial_j^{\mathring{\varphi}} v_i^\pm) - \partial_i^{\mathring{\varphi}}(\mathring{b}_i^\pm \partial_j^{\mathring{\varphi}} v_j^\pm) + [D_t^{\mathring{\varphi}\pm}, \nabla^{\mathring{\varphi}} \cdot]b^\pm \tag{4.28}$$

$$= (\partial_i^{\mathring{\varphi}}\mathring{b}_j^\pm)(\partial_j^{\mathring{\varphi}} v_i^\pm) - (\nabla^{\mathring{\varphi}} \cdot b^\pm)(\nabla^{\mathring{\varphi}} \cdot v^\pm) + [D_t^{\mathring{\varphi}\pm}, \nabla^{\mathring{\varphi}} \cdot]b^\pm. \tag{4.29}$$

Direct calculation shows that $[D_t^{\mathring{\varphi}\pm}, \partial_i^{\mathring{\varphi}}](\cdot) = -(\partial_i^{\mathring{\varphi}}\mathring{v}_j)\partial_j^{\mathring{\varphi}}(\cdot) - (\partial_i^{\mathring{\varphi}}\partial_t(\mathring{\varphi} - \varphi))\partial_3^{\mathring{\varphi}}(\cdot)$. On the other hand, the κ-regularization term provides extra regularity for $\varphi_t, \mathring{\varphi}_t, \dot{\varphi}_t$. Thus, standard H^3 estimates give the control of divergence

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\| \nabla^{\mathring{\varphi}} \cdot b^\pm \right\|_{3,\pm}^2 \lesssim C(\mathring{K}_0, \kappa^{-1})\left(\|b^\pm\|_{4,\pm}\|v^\pm\|_{4,\pm}\right) \le C(\mathring{K}_0, \kappa^{-1})\mathring{E}_4^\kappa(t). \tag{4.30}$$

The vorticity part is analyzed in a similar way as in Section 3.6. The evolution equations are

$$\mathring{\rho}D_t^{\mathring{\varphi}}(\nabla^{\mathring{\varphi}} \times v) - (\mathring{b} \cdot \nabla^{\mathring{\varphi}})(\nabla^{\mathring{\varphi}} \times b) = (\nabla^{\mathring{\varphi}}\mathring{\rho}) \times (D_t^{\mathring{\varphi}}v) - (\nabla^{\mathring{\varphi}}\mathring{b}_j) \times (\partial_j^{\mathring{\varphi}}b)$$
$$- \mathring{\rho}\left((\nabla^{\mathring{\varphi}}\mathring{v}_j) \times (\partial_j^{\mathring{\varphi}}v) + \nabla^\varphi(\partial_t\mathring{\varphi} - \partial_t\dot{\varphi}) \times \partial_3^{\mathring{\varphi}}v\right),$$
$$D_t^{\mathring{\varphi}}(\nabla^{\mathring{\varphi}} \times b) - (\mathring{b} \cdot \nabla^{\mathring{\varphi}})(\nabla^{\mathring{\varphi}} \times v) - \mathring{b} \times \nabla^{\mathring{\varphi}}(\nabla^{\mathring{\varphi}} \cdot v) = -(\nabla^{\mathring{\varphi}} \times \mathring{b})(\nabla^{\mathring{\varphi}} \cdot v) - (\nabla^{\mathring{\varphi}}\mathring{b}_j) \times (\partial_j^{\mathring{\varphi}}v)$$
$$- (\nabla^{\mathring{\varphi}}\mathring{v}_j) \times (\partial_j^{\mathring{\varphi}}b) - \nabla^\varphi(\partial_t\mathring{\varphi} - \partial_t\dot{\varphi}) \times \partial_3^{\mathring{\varphi}}b,$$

on the right side of which the highest-order derivative is 1 (except the mismatch term). Thus, we can still follow the analysis in Section 3.6.2 to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\Omega^\pm} \mathring{\rho}^\pm \left|\partial^3\nabla^{\mathring{\varphi}} \times v^\pm\right|^2 + \left|\partial^3\nabla^{\mathring{\varphi}} \times b^\pm\right|^2 \, \mathrm{d}\mathring{\mathcal{V}}_t \le P(\mathring{E}_4^\kappa(t), \mathring{K}_0) + \mathring{K}_1^\pm, \tag{4.31}$$

where

$$\mathring{K}_1^\pm := \int_{\Omega^\pm} (\partial^3\nabla^{\mathring{\varphi}} \times b^\pm) \cdot \left(\mathring{b}^\pm \times (\partial^3\nabla^{\mathring{\varphi}}(\nabla^{\mathring{\varphi}} \cdot v^\pm))\right) \, \mathrm{d}\mathring{\mathcal{V}}_t. \tag{4.32}$$

Again, we invoke the continuity equation and the momentum equation to get

$$\mathring{b} \times (\partial^3\nabla^{\mathring{\varphi}}(\nabla^{\mathring{\varphi}} \cdot v)) \stackrel{L}{=} -\mathring{\mathcal{F}}_p\mathring{b} \times (\partial^3\nabla^{\mathring{\varphi}}D_t^{\mathring{\varphi}}q) + \mathring{\mathcal{F}}_p\mathring{b} \times ((\partial^3\nabla^{\mathring{\varphi}}D_t^{\mathring{\varphi}}b_j)\mathring{b}_j)$$
$$\stackrel{L}{=} \mathring{\mathcal{F}}_p\mathring{\rho}\mathring{b} \times (\partial^3(D_t^{\mathring{\varphi}})^2 v) - \mathring{\mathcal{F}}_p\mathring{b} \times D_t^{\mathring{\varphi}}(\mathring{b}_j\partial^3\partial_j^{\mathring{\varphi}}b) + \mathring{\mathcal{F}}_p\mathring{b} \times D_t^{\mathring{\varphi}}((\partial^3\nabla^{\mathring{\varphi}}b_j)\mathring{b}_j)$$
$$\stackrel{L}{=} \mathring{\mathcal{F}}_p\mathring{\rho}\mathring{b} \times (\partial^3(D_t^{\mathring{\varphi}})^2 v) + \mathring{\mathcal{F}}_p\mathring{b} \times D_t^{\mathring{\varphi}}(\mathring{b} \times (\partial^3\nabla^{\mathring{\varphi}} \times b))$$

where we use the vector identity $(\mathbf{a} \times (\nabla^{\mathring{\varphi}} \times \mathbf{b}))_i = (\partial_i^{\mathring{\varphi}} \mathbf{b}_j)\mathbf{a}_j - \mathbf{a}_j \partial_j^{\mathring{\varphi}} \mathbf{b}_i$, and the omitted terms are directly controlled by $P(\mathring{E}_4^\kappa(t), \mathring{K}_0)$. Thus, we have

$$\mathring{K}_1^\pm \overset{L}{=} \int_{\Omega^\pm} \mathring{\mathcal{F}}_p^\pm \mathring{\rho}(\partial^3 \nabla^{\mathring{\varphi}} \times b) \cdot \left(\mathring{b} \times (\partial^3 (D_t^{\mathring{\varphi}})^2 v)\right) \, \mathrm{d}\mathring{\mathcal{V}}_t + \int_{\Omega^\pm} \mathring{\mathcal{F}}_p^\pm (\partial^3 \nabla^{\mathring{\varphi}} \times b) \cdot \left(\mathring{b} \times D_t^{\mathring{\varphi}}(\mathring{b} \times (\partial^3 \nabla^{\mathring{\varphi}} b))\right) \, \mathrm{d}\mathcal{V}_t$$

$$= \int_{\Omega^\pm} \mathring{\mathcal{F}}_p^\pm \mathring{\rho}(\partial^3 \nabla^{\mathring{\varphi}} \times b) \cdot \left(\mathring{b} \times (\partial^3 (D_t^{\mathring{\varphi}})^2 v)\right) \, \mathrm{d}\mathring{\mathcal{V}}_t - \int_{\Omega^\pm} \mathring{\mathcal{F}}_p^\pm D_t^{\mathring{\varphi}} \left(\mathring{b} \times (\partial^3 \nabla^{\mathring{\varphi}} \times b)\right) \cdot \left(\mathring{b} \times (\partial^3 \nabla^{\mathring{\varphi}} \times b)\right) \, \mathrm{d}\mathcal{V}_t$$

$$\lesssim -\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega^\pm} \mathring{\mathcal{F}}_p^\pm \left|\mathring{b} \times (\partial^3 \nabla^{\mathring{\varphi}} \times b)\right|_0^2 + P(\mathring{K}_0)\mathring{E}_4^\kappa(t) + \mathring{E}_5^\kappa(t).$$

So, we have

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega^\pm} \mathring{\rho}^\pm \left|\partial^3 \nabla^{\mathring{\varphi}} \times v^\pm\right|^2 + \left|\partial^3 \nabla^{\mathring{\varphi}} \times b^\pm\right|^2 + \mathcal{F}_p^\pm \left|\mathring{b}^\pm \times (\partial^3 \nabla^{\mathring{\varphi}} \times b^\pm)\right|_0^2 \, \mathrm{d}\mathring{\mathcal{V}}_t$$
$$\lesssim P(\mathring{K}_0)\mathring{E}_4^\kappa(t) + \mathring{E}_5^\kappa(t). \tag{4.33}$$

Similarly as in Section 3.6.2 and Section 3.6.3, we can prove the div-curl estimates for time-differentiated system and $\mathcal{T}^\alpha$-differentiated system. For $0 \le l \le 3, 0 \le k \le 3 - l, \langle \alpha \rangle = 2l, \ \alpha_3 = 0$, we have

$$\left\|\varepsilon^{2l} \partial_t^k \mathcal{T}^\alpha(v^\pm, b^\pm)\right\|_{4-l-k,\pm}^2 \le C(|\mathring{\psi}|_3)\left(\left\|\varepsilon^{2l} \partial_t^k \mathcal{T}^\alpha(v^\pm, b^\pm)\right\|_{0,\pm}^2 + \left\|\varepsilon^{2l} \nabla^{\mathring{\varphi}} \cdot \partial_t^k \mathcal{T}^\alpha(v^\pm, b^\pm)\right\|_{3-k-l,\pm}^2 \right.$$
$$\left. + \left\|\varepsilon^{2l} \nabla^{\mathring{\varphi}} \times \partial_t^k \mathcal{T}^\alpha(v^\pm, b^\pm)\right\|_{3-k-l,\pm}^2 + \left\|\varepsilon^{2l} \overline{\partial}^{4-k-l} \partial_t^k \mathcal{T}^\alpha(v^\pm, b^\pm)\right\|_{0,\pm}^2 \right). \tag{4.34}$$

Then the curl part has the following control

$$\left\|\varepsilon^{2l} \nabla^{\mathring{\varphi}} \times \partial_t^k \mathcal{T}^\alpha v^\pm\right\|_{3-l-k,\pm}^2 + \left\|\varepsilon^{2l} \nabla^{\mathring{\varphi}} \times \partial_t^k \mathcal{T}^\alpha b^\pm\right\|_{3-l-k,\pm}^2 + \left\|\varepsilon^{2l} \mathring{\mathcal{F}}_p^\pm \mathring{b} \times (\partial_t^k \mathcal{T}^\alpha b^\pm)\right\|_{3-l-k,\pm}^2$$
$$\lesssim P(\mathring{K}_0)\left(\sum_{j=0}^l E_{4+j}^\kappa(0) \right) + P(\mathring{K}_0) \int_0^t \sum_{j=0}^l \mathring{E}_{4+j}^\kappa(\tau) + \mathring{E}_{4+l+1}^\kappa(\tau) \, \mathrm{d}\tau. \tag{4.35}$$

Similarly, the divergence part is controlled by

$$\left\|\varepsilon^{2l} \nabla^{\mathring{\varphi}} \cdot \partial_t^k \mathcal{T}^\alpha v^\pm\right\|_{3-l-k,\pm}^2 + \left\|\varepsilon^{2l} \nabla^{\mathring{\varphi}} \cdot \partial_t^k \mathcal{T}^\alpha b^\pm\right\|_{3-l-k,\pm}^2$$
$$\lesssim \left\|\varepsilon^{2l} \mathring{\mathcal{F}}_p^\pm \partial_t^k \mathcal{T}^\alpha D_t^{\mathring{\varphi}\pm}(q^\pm, b^\pm)\right\|_{3-l-k,\pm}^2 + C(\mathring{K}_0)\left(\sum_{j=0}^l E_{4+j}^\kappa(0) \right) + P(\mathring{K}_0) \int_0^t \sum_{j=0}^l \mathring{E}_{4+j}^\kappa(\tau) \, \mathrm{d}\tau, \tag{4.36}$$

in which the first term will be controlled via tangential estimates.

For the pressure q, we still use the linearized momentum equation to convert it to tangential derivatives of v and b. This step is exactly the same as Section 3.6.1, so we do not repeat the details here.

4.3.2 Tangential estimates

The tangential estimates are also proved in the same way as Section 3.3-Section 3.5. What's more, the rather technical part in the estimates of full time derivatives can be simplified a lot thanks to the $\sqrt{\kappa}$-weighted extra regularity of the free interface. For $\mathcal{T}^\gamma$-differentiated linearize system (4.6), we introduce the corresponding Alinhac good unknown $\mathring{\mathbf{F}}^\gamma := \mathcal{T}^\gamma f - \mathcal{T}^\gamma \mathring{\varphi} \partial_3^{\mathring{\varphi}} f$ which satisfies

$$\mathcal{T}^\gamma(\partial_i^{\mathring{\varphi}} f) = \partial_i^{\mathring{\varphi}} \mathring{\mathbf{F}}^\gamma + \mathring{\mathfrak{C}}_i^\gamma(f), \quad \mathcal{T}^\gamma(D_t^{\mathring{\varphi}} f) = D_t^{\mathring{\varphi}} \mathring{\mathbf{F}}^\gamma + \mathring{\mathfrak{D}}^\gamma(f),$$

where

$$\mathring{\mathfrak{C}}_i^\gamma(f) = (\partial_3^{\mathring{\varphi}} \partial_i^{\mathring{\varphi}} f)\mathcal{T}^\gamma \mathring{\varphi} + \left[\mathcal{T}^\gamma, \frac{\mathring{\mathbf{N}}_i}{\partial_3 \mathring{\varphi}}, \partial_3 f\right] + \partial_3 f\left[\mathcal{T}^\gamma, \mathring{\mathbf{N}}_i, \frac{1}{\partial_3 \mathring{\varphi}}\right] + \mathring{\mathbf{N}}_i \partial_3 f\left[\mathcal{T}^{\gamma-\gamma'}, \frac{1}{(\partial_3 \mathring{\varphi})^2}\right]\mathcal{T}^{\gamma'} \partial_3 \mathring{\varphi}$$
$$+ \frac{\mathring{\mathbf{N}}_i}{\partial_3 \mathring{\varphi}}[\mathcal{T}^\gamma, \partial_3]f - \frac{\mathring{\mathbf{N}}_i}{(\partial_3 \mathring{\varphi})^2} \partial_3 f[\mathcal{T}^\gamma, \partial_3]\mathring{\varphi}, \tag{4.37}$$

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and

$$\mathring{\mathfrak{D}}^\gamma(f) = (D_t^{\mathring{\varphi}}\partial_3^{\mathring{\varphi}}f)\mathcal{T}^\gamma\varphi + [\mathcal{T}^\gamma,\mathring{\bar v}]\cdot\bar\partial f + \left[\mathcal{T}^\gamma,\frac{1}{\partial_3\mathring{\varphi}}(\mathring{v}\cdot\dot{\mathbf{N}}-\partial_t\mathring{\varphi}),\partial_3 f\right] + \left[\mathcal{T}^\gamma,(\mathring{v}\cdot\dot{\mathbf{N}}-\partial_t\mathring{\varphi}),\frac{1}{\partial_3\mathring{\varphi}}\right]\partial_3 f$$

$$+\frac{1}{\partial_3\varphi}[\mathcal{T}^\gamma,\mathring{v}]\cdot\dot{\mathbf{N}}\partial_3 f - (\mathring{v}\cdot\dot{\mathbf{N}}-\partial_t\mathring{\varphi})\partial_3 f\left[\mathcal{T}^{\gamma-\gamma'},\frac{1}{(\partial_3\mathring{\varphi})^2}\right]\mathcal{T}^{\gamma'}\partial_3\mathring{\varphi}$$

$$+\frac{1}{\partial_3\mathring{\varphi}}(\mathring{v}\cdot\dot{\mathbf{N}}-\partial_t\mathring{\varphi})[\mathcal{T}^\gamma,\partial_3]f + (\mathring{v}\cdot\dot{\mathbf{N}}-\partial_t\mathring{\varphi})\frac{\partial_3 f}{(\partial_3\mathring{\varphi})^2}[\mathcal{T}^\gamma,\partial_3]\mathring{\varphi} + \mathcal{T}^\gamma\partial_t(\varphi-\mathring{\varphi})\partial_3^{\mathring{\varphi}}f \tag{4.38}$$

with $\langle\gamma'\rangle = 1$. Since $\dot{\mathbf{N}}_3 = 1$, the third term in $\mathring{\mathfrak{C}}_i^\gamma(f)$ does not appear when $i = 3$. Under this setting, the $\mathcal{T}^\gamma$-differentiated linearized system is reformulated as follows

$$\mathring\rho^\pm D_t^{\mathring\varphi\pm}\mathring{\mathbf{V}}^{\gamma,\pm} - (\mathring{\boldsymbol{b}}^\pm\cdot\nabla^{\mathring\varphi})\mathring{\mathbf{B}}^{\gamma,\pm} + \nabla^{\mathring\varphi}\mathring{\mathbf{Q}}^{\gamma,\pm} = \mathring{\mathcal{R}}_v^{\gamma,\pm} - \mathring{\mathfrak{C}}^\gamma(q^\pm) \quad\text{in } [0,T]\times\Omega^\pm, \tag{4.39}$$

$$\mathring{\mathcal{F}}_p^\pm D_t^{\mathring\varphi\pm}\mathring{\mathbf{Q}}^{\gamma,\pm} - \mathring{\mathcal{F}}_p^\pm D_t^{\mathring\varphi\pm}\mathbf{B}^{\gamma,\pm}\cdot\mathring{\boldsymbol{b}}^\pm + \nabla^{\mathring\varphi}\cdot\mathring{\mathbf{V}}^{\gamma,\pm} = \mathring{\mathcal{R}}_p^{\gamma,\pm} - \mathring{\mathfrak{C}}_i^\gamma(v_i^\pm) \quad\text{in } [0,T]\times\Omega^\pm, \tag{4.40}$$

$$D_t^{\mathring\varphi\pm}\mathring{\mathbf{B}}^{\gamma,\pm} - (\mathring{\boldsymbol{b}}^\pm\cdot\nabla^{\mathring\varphi})\mathring{\mathbf{V}}^{\gamma,\pm} + \mathring{\boldsymbol{b}}^\pm(\nabla^{\mathring\varphi}\cdot\mathring{\mathbf{V}}^{\gamma,\pm}) = \mathring{\mathcal{R}}_b^{\gamma,\pm} - \mathring{\boldsymbol{b}}^\pm\mathring{\mathfrak{C}}_i^\gamma(v_i^\pm) \quad\text{in } [0,T]\times\Omega^\pm, \tag{4.41}$$

$$D_t^{\mathring\varphi\pm}\mathring{\mathbf{S}}^{\pm,\alpha} = \mathring{\mathfrak{D}}^\gamma(S^\pm) \quad\text{in } [0,T]\times\Omega^\pm, \tag{4.42}$$

with boundary conditions

$$\left[\!\left[\mathring{\mathbf{Q}}^\gamma\right]\!\right] = \sigma\mathcal{T}^\gamma\mathcal{H}(\mathring\psi) - \kappa\mathcal{T}^\gamma(1-\overline{\Delta})^2\psi - \kappa\mathcal{T}^\gamma(1-\overline{\Delta})\partial_t\psi - \left[\!\left[\partial_3 q\right]\!\right]\mathcal{T}^\gamma\mathring\psi \quad\text{on } [0,T]\times\Sigma, \tag{4.43}$$

$$\mathring{\mathbf{V}}^{\gamma,\pm}\cdot\mathring{N} = \partial_t\mathcal{T}^\gamma\psi + \bar v^\pm\cdot\overline\nabla\mathcal{T}^\gamma\mathring\psi - \mathcal{W}^{\gamma,\pm} \quad\text{on } [0,T]\times\Sigma, \tag{4.44}$$

where $\mathring{\mathcal{R}}_v, \mathring{\mathcal{R}}_p, \mathring{\mathcal{R}}_b$ terms consist of the following commutators

$$\mathring{\mathcal{R}}_v^{\gamma,\pm} := [\mathcal{T}^\gamma,\mathring{\boldsymbol{b}}^\pm]\cdot\nabla^{\mathring\varphi}b^\pm - [\mathcal{T}^\gamma,\mathring\rho^\pm]D_t^{\mathring\varphi\pm}v^\pm - \mathring\rho^\pm\mathring{\mathfrak{D}}^\gamma(v^\pm) \tag{4.45}$$

$$\mathring{\mathcal{R}}_p^{\gamma,\pm} := -[\mathcal{T}^\gamma,\mathring{\mathcal{F}}_p^\pm]D_t^{\mathring\varphi\pm}q^\pm - \mathring{\mathcal{F}}_p^\pm\mathring{\mathfrak{D}}^\gamma(q^\pm)$$

$$+ [\mathcal{T}^\gamma,\mathring{\mathcal{F}}_p^\pm]D_t^{\mathring\varphi\pm}b^\pm\cdot\mathring{\boldsymbol{b}}^\pm + \mathring{\mathcal{F}}_p^\pm\mathring{\mathfrak{D}}^\gamma(b^\pm)\cdot\mathring{\boldsymbol{b}}^\pm + [\mathcal{T}^\gamma,\mathring{\mathcal{F}}_p^\pm\mathring{\boldsymbol{b}}^\pm]\cdot D_t^{\mathring\varphi\pm}b^\pm \tag{4.46}$$

$$\mathring{\mathcal{R}}_b^{\gamma,\pm} := [\mathcal{T}^\gamma,\mathring{\boldsymbol{b}}^\pm]\cdot\nabla^{\mathring\varphi}v^\pm - \mathring{\mathfrak{D}}^\gamma(b^\pm), \tag{4.47}$$

and the boundary term $\mathcal{W}^{\gamma,\pm}$ is

$$\mathring{\mathcal{W}}^{\gamma,\pm} := (\partial_3 v^\pm\cdot\mathring{N})\mathcal{T}^\gamma\mathring\psi + [\mathcal{T}^\gamma,\mathring{N}_i,v_i^\pm]. \tag{4.48}$$

Given $0 \le l \le 4$, we shall consider the tangential estimates for $\overline\partial^{4-k-l}\partial_t^k\mathcal{T}^\alpha$ for $0 \le k \le 4-l$ and $\langle\alpha\rangle = 2l$, $\alpha_3 = 0$. Following the analysis in Section 3.3-Section 3.5, using the linearized Reynolds transport theorem (Lemma A.7), dropping γ for simplicity of notations, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\Omega^\pm}\mathring\rho|\varepsilon^{2l}\mathring{\mathbf{V}}^\pm|^2\,\mathrm{d}\mathring{\mathcal{V}}_t = \int_{\Omega^\pm}\varepsilon^{4l}(\mathring{\boldsymbol{b}}^\pm\cdot\nabla^{\mathring\varphi})\mathring{\mathbf{B}}^\pm\cdot\mathring{\mathbf{V}}^\pm\,\mathrm{d}\mathring{\mathcal{V}}_t - \int_{\Omega^\pm}\varepsilon^{4l}\mathring{\mathbf{V}}^\pm\cdot\nabla^{\mathring\varphi}\mathring{\mathbf{Q}}^\pm\,\mathrm{d}\mathring{\mathcal{V}}_t$$

$$-\int_{\Omega^\pm}\varepsilon^{4l}(\mathring{\mathcal{R}}_v^\pm - \mathring{\mathfrak{C}}(q^\pm))\cdot\mathring{\mathbf{V}}^\pm\,\mathrm{d}\mathring{\mathcal{V}}_t$$

$$+\frac{1}{2}\int_{\Omega^\pm}\varepsilon^{4l}\left(D_t^{\mathring\varphi\pm}\mathring\rho + \mathring\rho\nabla^{\mathring\varphi}\cdot\mathring{v}^\pm + \mathring\rho\partial_3^{\mathring\varphi}(\mathring{\bar v}\cdot\overline\nabla)(\mathring\varphi-\dot\varphi)\right)|\mathring{\mathbf{V}}^\pm|^2\,\mathrm{d}\mathring{\mathcal{V}}_t, \tag{4.49}$$

where the last two terms can be directly controlled by $C(\mathring{K}_0)\mathring{E}^\kappa(t)$. In the rest of this section, we will use the notation $\overset{L}{=}$ to skip some of these remainder terms. We then analyze the first line. Integrating $(\mathring{\boldsymbol{b}}^\pm\cdot\nabla^{\mathring\varphi})$ and $\nabla^{\mathring\varphi}$ by parts and using $\mathring{b}\cdot\mathring{N}|_\Sigma = 0$, we get

$$\int_{\Omega^\pm}\varepsilon^{4l}(\mathring{\boldsymbol{b}}^\pm\cdot\nabla^{\mathring\varphi})\mathring{\mathbf{B}}^\pm\cdot\mathring{\mathbf{V}}^\pm\,\mathrm{d}\mathring{\mathcal{V}}_t = -\int_{\Omega^\pm}\varepsilon^{4l}\mathring{\mathbf{B}}^\pm\cdot(\mathring{\boldsymbol{b}}^\pm\cdot\nabla^{\mathring\varphi})\mathring{\mathbf{V}}^\pm\,\mathrm{d}\mathring{\mathcal{V}}_t - \int_{\Omega^\pm}\varepsilon^{4l}(\nabla^{\mathring\varphi}\cdot\mathring{\boldsymbol{b}}^\pm)\mathring{\mathbf{B}}^\pm\cdot\mathring{\mathbf{V}}^\pm\,\mathrm{d}\mathring{\mathcal{V}}_t$$

$$= -\int_{\Omega^\pm}\varepsilon^{4l}\mathring{\mathbf{B}}^\pm\cdot D_t^{\mathring\varphi\pm}\mathring{\mathbf{B}}^\pm\,\mathrm{d}\mathring{\mathcal{V}}_t - \int_{\Omega^\pm}\varepsilon^{4l}(\mathring{\mathbf{B}}^\pm\cdot\mathring{\boldsymbol{b}}^\pm)(\nabla^{\mathring\varphi}\cdot\mathring{\mathbf{V}}^\pm)\,\mathrm{d}\mathring{\mathcal{V}}_t - \int_{\Omega^\pm}\varepsilon^{4l}(\nabla^{\mathring\varphi}\cdot\mathring{\boldsymbol{b}}^\pm)\mathring{\mathbf{B}}^\pm\cdot\mathring{\mathbf{V}}^\pm\,\mathrm{d}\mathring{\mathcal{V}}_t$$

$$\overset{L}{=} -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}|\varepsilon^{2l}\mathring{\mathbf{B}}^\pm|^2\,\mathrm{d}\mathring{\mathcal{V}}_t - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}\mathring{\mathcal{F}}_p^\pm(\varepsilon^{2l}\mathring{\mathbf{B}}^\pm\cdot\mathring{\boldsymbol{b}}^\pm)^2\,\mathrm{d}\mathcal{V}_t + \int_{\Omega^\pm}\varepsilon^{4l}\mathring{\mathcal{F}}_p^\pm(\mathring{\mathbf{B}}^\pm\cdot\mathring{\boldsymbol{b}}^\pm)D_t^{\mathring\varphi\pm}\mathring{\mathbf{Q}}^\pm\,\mathrm{d}\mathring{\mathcal{V}}_t \tag{4.50}$$

and

$$-\int_{\Omega^\pm} \varepsilon^{4l}\mathring{\mathbf{V}}^\pm \cdot \nabla^{\mathring\varphi}\mathring{\mathbf{Q}}^\pm \, \mathrm{d}\mathring{\mathcal{V}}_t = \pm \underbrace{\int_\Sigma \varepsilon^{4l}(\mathring{\mathbf{V}}^\pm \cdot \mathring{N})\mathring{\mathbf{Q}}^\pm \, \mathrm{d}x'}_{=:\mathring{I}^\pm} + \int_{\Omega^\pm} \varepsilon^{4l}\mathring{\mathbf{Q}}^\pm(\nabla^{\mathring\varphi}\cdot\mathring{\mathbf{V}}^\pm)\,\mathrm{d}\mathring{\mathcal{V}}_t$$

$$\overset{L}{=} \mathring{I}^\pm - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}\mathring{\mathcal{F}}_p^\pm(\varepsilon^{2l}\mathring{\mathbf{Q}}^\pm)^2\,\mathrm{d}\mathring{\mathcal{V}}_t + \int_{\Omega^\pm}\mathring{\mathcal{F}}_p^\pm\varepsilon^{4l}\mathring{\mathbf{Q}}^\pm D_t^{\mathring\varphi\pm}(\mathring{\mathbf{B}}^\pm\cdot\mathring{b}^\pm)\,\mathrm{d}\mathring{\mathcal{V}}_t - \underbrace{\int_{\Omega^\pm}\varepsilon^{4l}\mathring{\mathbf{Q}}^\pm\mathring{\mathfrak{C}}_i(v_i^\pm)\,\mathrm{d}\mathring{\mathcal{V}}_t}_{=:\mathring{Z}^\pm}. \tag{4.51}$$

Notice that

$$\int_{\Omega^\pm}\varepsilon^{4l}\mathring{\mathcal{F}}_p^\pm(\mathring{\mathbf{B}}^\pm\cdot\mathring{b}^\pm)D_t^{\mathring\varphi\pm}\mathring{\mathbf{Q}}^\pm\,\mathrm{d}\mathring{\mathcal{V}}_t + \int_{\Omega^\pm}\varepsilon^{4l}\mathring{\mathcal{F}}_p^\pm\mathring{\mathbf{Q}}^\pm D_t^{\mathring\varphi\pm}(\mathring{\mathbf{B}}^\pm\cdot\mathring{b}^\pm)\,\mathrm{d}\mathring{\mathcal{V}}_t \overset{L}{=} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}\varepsilon^{4l}\mathring{\mathcal{F}}_p^\pm\mathring{\mathbf{Q}}^\pm(\mathring{\mathbf{B}}^\pm\cdot\mathring{b}^\pm)\,\mathrm{d}\mathring{\mathcal{V}}_t, \tag{4.52}$$

we find that

$$\int_{\Omega^\pm}\varepsilon^{4l}(\mathring{b}^\pm\cdot\nabla^{\mathring\varphi})\mathring{\mathbf{B}}^\pm\cdot\mathring{\mathbf{V}}^\pm\,\mathrm{d}\mathring{\mathcal{V}}_t - \int_{\Omega^\pm}\varepsilon^{4l}\mathring{\mathbf{V}}^\pm\cdot\nabla^{\mathring\varphi}\mathring{\mathbf{Q}}^\pm\,\mathrm{d}\mathring{\mathcal{V}}_t$$

$$\overset{L}{=} \mathring{I}^\pm + \mathring{Z}^\pm - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}|\varepsilon^{2l}\mathring{\mathbf{B}}^\pm|^2\,\mathrm{d}\mathring{\mathcal{V}}_t - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\pm}\mathring{\mathcal{F}}_p^\pm\left|\varepsilon^{2l}\left(\mathring{\mathbf{Q}}^\pm - \mathring{\mathbf{B}}^\pm\cdot\mathring{b}^\pm\right)\right|^2\,\mathrm{d}\mathring{\mathcal{V}}_t. \tag{4.53}$$

Thus, we already get the energy terms for $\mathring{\mathbf{V}}$, $\mathring{\mathbf{B}}$ and $\mathring{\mathbf{Q}}$, and it remains to analyze the boundary term $\mathring{I}^\pm$. Again, following the analysis in Section 3.3-Section 3.5, we have

$$\mathring{I}^+ + \mathring{I}^- = \mathring{\mathrm{S}}\mathrm{T} + \mathring{\mathrm{S}}\mathrm{T}' + \mathring{\mathrm{V}}\mathrm{S} + \mathring{\mathrm{R}}\mathrm{T} + \mathring{\mathrm{R}}\mathrm{T}^+ + \mathring{\mathrm{R}}\mathrm{T}^- + \mathring{Z}B^+ + \mathring{Z}B^- \tag{4.54}$$

where

$$\mathring{\mathrm{S}}\mathrm{T} := \varepsilon^{4l}\int_\Sigma \mathcal{T}^\gamma\,[\![q]\!]\,\partial_t\mathcal{T}^\gamma\psi\,\mathrm{d}x', \tag{4.55}$$

$$\mathring{\mathrm{S}}\mathrm{T}' := \varepsilon^{4l}\int_\Sigma \mathcal{T}^\gamma\,[\![q]\!]\,(\bar{v}^+\cdot\overline{\nabla})\mathcal{T}^\gamma\mathring\psi\,\mathrm{d}x', \tag{4.56}$$

$$\mathring{\mathrm{V}}\mathrm{S} := \varepsilon^{4l}\int_\Sigma \mathcal{T}^\gamma q^-\,([\![\bar{v}]\!]\cdot\overline{\nabla})\mathcal{T}^\gamma\mathring\psi\,\mathrm{d}x', \tag{4.57}$$

$$\mathring{\mathrm{R}}\mathrm{T} := -\varepsilon^{4l}\int_\Sigma [\![\partial_3 q]\!]\,\mathcal{T}^\gamma\mathring\psi\,\partial_t\mathcal{T}^\gamma\psi\,\mathrm{d}x', \tag{4.58}$$

$$\mathring{\mathrm{R}}\mathrm{T}^\pm := \mp\,\varepsilon^{4l}\int_\Sigma \partial_3 q^\pm\,\mathcal{T}^\gamma\mathring\psi\,(\bar{v}^\pm\cdot\overline{\nabla})\mathcal{T}^\gamma\mathring\psi\,\mathrm{d}x', \tag{4.59}$$

$$\mathring{Z}B^\pm := \mp\,\varepsilon^{4l}\int_\Sigma \mathring{\mathbf{Q}}^\pm\mathring{W}^\pm\,\mathrm{d}x',\quad \mathring{Z}^\pm = -\int_{\Omega^\pm}\varepsilon^{4l}\mathring{\mathbf{Q}}^\pm\mathring{\mathfrak{C}}_i(v_i^\pm)\,\mathrm{d}\mathring{\mathcal{V}}_t. \tag{4.60}$$

4.3.3 Analysis of the boundary integrals

Since the weight function $\omega(x_3)$ vanishes on Σ, we can alternatively write $\mathcal{T}^\alpha = \partial_t^{\alpha_0}\overline{\partial}^{2l-\alpha_0}$ and $\mathcal{T}^\gamma = \partial_t^{k+\alpha_0}\overline{\partial}^{4+l-(k+\alpha_0)}$. Replacing $k+\alpha_0$ by k, it suffices to analyze the case $\mathcal{T}^\gamma = \partial_t^k\overline{\partial}^{4+l-k}$ for $0 \le k \le 4+l$, $0 \le l \le 4$. First, there is no need to analyze $\mathring{\mathrm{R}}\mathrm{T}$ and $\mathring{\mathrm{R}}\mathrm{T}^\pm$ because they can be directly controlled by using the energy bounds (4.10) for the basic state. For the term $\mathring{\mathrm{S}}\mathrm{T}$, the boundary regularity is given by the κ-regularization terms instead of the surface tension because we do not need a uniform-in-κ estimate for the linearized problem. Using the jump conditions for $[\![q]\!]$ and integrating by parts, we have

$$\int_0^t \mathring{\mathrm{S}}\mathrm{T}\,\mathrm{d}\tau = \sigma\int_0^t\int_\Sigma \varepsilon^{4l}\partial_t^k\overline{\partial}^{4+l-k}\left(\frac{\overline{\nabla}\mathring\psi}{\sqrt{1+|\overline{\nabla}\mathring\psi|^2}}\right)\cdot\overline{\nabla}\partial_t^{k+1}\overline{\partial}^{4+l-k}\psi\,\mathrm{d}x'\,\mathrm{d}\tau$$

$$-\kappa\int_0^t\int_\Sigma \varepsilon^{4l}\partial_t^k\overline{\partial}^{4+l-k}(1-\overline{\Delta})\psi\,\partial_t^{k+1}\overline{\partial}^{4+l-k}(1-\overline{\Delta})\psi\,\mathrm{d}x'\,\mathrm{d}\tau - \kappa\int_0^t\int_\Sigma\left|\varepsilon^{2l}\partial_t^{k+1}\overline{\partial}^{4+l-k}\langle\overline{\partial}\rangle\psi\right|^2\,\mathrm{d}x'\,\mathrm{d}\tau$$

$$\lesssim -\left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^k\overline{\partial}^{4+l-k}\psi\right|_2^2\Big|_0^t - \left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^{k+1}\overline{\partial}^{4+l-k}\psi\right|_{L_t^2 H_{x'}^1}^2 + \delta\left|\sqrt{\kappa}\varepsilon^{2l}\partial_t^{k+1}\overline{\partial}^{4+l-k}\psi\right|_{L_t^2 H_{x'}^1}^2 + \frac{\sigma}{\kappa}\int_0^t C(\mathring{K}_0)\,\mathrm{d}\tau. \tag{4.61}$$

For the term $\mathring{S}T'$, we have

$$\int_0^t \mathring{S}T'\,d\tau = \sigma \int_0^t \int_\Sigma \varepsilon^{4l} \partial_t^k \overline\partial^{4+l-k} \left(\frac{\overline\nabla \mathring\psi}{\sqrt{1+|\overline\nabla \mathring\psi|^2}} \right) \cdot \overline\nabla \left((\bar v^+ \cdot \overline\nabla) \partial_t^k \overline\partial^{4+l-k} \mathring\psi \right) dx'\,d\tau$$

$$- \kappa \int_0^t \int_\Sigma \varepsilon^{4l} \partial_t^k \overline\partial^{4+l-k}(1-\overline\Delta)\psi\,(1-\overline\Delta)\left((\bar v^+ \cdot \overline\nabla)\partial_t^k \overline\partial^{4+l-k}\mathring\psi \right) dx'\,d\tau$$

$$- \kappa \int_0^t \int_\Sigma \varepsilon^{4l} \partial_t^{k+1}\overline\partial^{4+l-k}\langle\overline\partial\rangle\psi\,\langle\overline\partial\rangle\left((\bar v^+ \cdot \overline\nabla)\partial_t^k \overline\partial^{4+l-k}\mathring\psi \right) dx'\,d\tau$$

$$\lesssim \sigma C(\mathring K_0, \kappa^{-1})t + C(\mathring K_0)\int_0^t \left| \sqrt\kappa \varepsilon^{2l}\partial_t^k \overline\partial^{4+l-k}\psi \right|_2 \|v^+\|_{4,\pm}\,d\tau$$

$$+ \delta \left| \sqrt\kappa \varepsilon^{2l}\partial_t^{k+1}\overline\partial^{4+l-k}\psi \right|^2_{L_t^2 H_{x'}^1} + C(\mathring K_0)\int_0^t \|v^+\|^2_{4,\pm}\,d\tau$$

$$\lesssim \sigma C(\mathring K_0, \kappa^{-1})t + C(\mathring K_0)\int_0^t \mathring E^\kappa_{4+l}(\tau) + \mathring E^\kappa_4(\tau)\,d\tau. \tag{4.62}$$

The term $\mathring V S$ can also be directly controlled even if $\mathcal T^\gamma$ only contains time derivatives. When $k < 4+l$, we can use the κ-weighted energy to control it after integrating $\overline\partial^{\frac12}$ by parts and using Lemma B.4

$$\mathring V S = \int_\Sigma \varepsilon^{4l}\partial_t^k \overline\partial^{3.5+l-k}q^- \,\overline\partial^{\frac12}\left(([\![\bar v]\!]\cdot\overline\nabla)\partial_t^k \overline\partial^{4+l-k}\mathring\psi \right) dx'$$

$$\lesssim \|\varepsilon^{2l}\partial_t^k \overline\partial^{4+l-k}q^-\|^{\frac12}_{0,-}\|\varepsilon^{2l}\partial_t^k \overline\partial^{3+l-k}\partial_3 q^-\|^{\frac12}_{0,-}|\bar v^\pm|_{W^{\frac12,\infty}}\left|\varepsilon^{2l}\partial_t^k \mathring\psi\right|_{5.5+l-k}$$

$$\lesssim (\mathring E^\kappa_4(t) + \mathring E^\kappa_{4+l}(t))C(\mathring K_0). \tag{4.63}$$

When $k = 4+l$, we can first integrate ∂_t by parts and then integrate $\overline\partial^{\frac12}$ by parts

$$\int_0^t \mathring V S\,d\tau \stackrel{L}{=} \int_0^t \int_\Sigma \varepsilon^{4l}\overline\partial^{\frac12}([\![\bar v]\!]\,\partial_t^{3+l}q^-)\,\overline\partial^{\frac12}\partial_t^{5+l}\mathring\psi\,dx'\,d\tau + \int_\Sigma \varepsilon^{4l}\partial_t^{3+l}q^-\,([\![\bar v]\!]\cdot\overline\nabla)\partial_t^{4+l}\mathring\psi\,dx'\Big|_0^t$$

$$\lesssim \int_0^t \|\varepsilon^{2l}\overline\partial\partial_t^{3+l}q^-\|^{\frac12}_{0,-}\|\varepsilon^{2l}\partial_t^{3+l}\partial_3 q^-\|^{\frac12}_{0,-}|\bar v^\pm|_{W^{\frac12,\infty}}\left|\varepsilon^{2l}\partial_t^{5+l}\mathring\psi\right|_{0.5}\,d\tau$$

$$+ \delta\|\varepsilon^{2l}\partial_t^{3+l}q^-\|^2_{1,-} + |\bar v^\pm|^2_{L^\infty}\left|\varepsilon^{2l}\partial_t^4\mathring\psi\right|^2_1 + C(\mathring K_0)\mathring E^\kappa(0)$$

$$\lesssim \delta\mathring E^\kappa_{4+l}(t) + C(\mathring K_0, \kappa^{-1})\left(\mathring E^\kappa(0) + \int_0^t \mathring E^\kappa(\tau)\,d\tau \right). \tag{4.64}$$

For $\mathring Z B + \mathring Z$, the cancellation structure obtained in Section 3.3.1 and Section 3.4 still holds. Following step 4 in Section 3.4, we have

$$\mathring Z B^\pm + \mathring Z^\pm = \mp \int_\Sigma \varepsilon^{4l}(\partial_t^k \overline\partial^{4+l-k}q^\pm - \partial_t^k \overline\partial^{4+l-k}\mathring\psi\partial_3 q^\pm)(\partial_3 v^\pm\cdot \mathring N)\partial_t^k \overline\partial^{4+l-k}\mathring\psi\,dx'$$

$$\mp \int_\Sigma \varepsilon^{4l}\mathring Q^\pm\left[\partial_t^k \overline\partial^{4+l-k}, \mathring N_i, v_i^\pm \right]dx' - \int_{\Omega^\pm}\varepsilon^{4l}\mathring Q^\pm\mathring C_i(v_i^\pm)\,d\mathring V_t, \tag{4.65}$$

where the first line is controlled in the same way as $\mathring V S$. Mimicking the proof in step 4 in Section 3.4, we have

$$\mp \int_\Sigma \varepsilon^{4l}\mathring Q^\pm\left[\partial_t^k \overline\partial^{4+l-k}, \mathring N_i, v_i^\pm \right]dx' - \int_{\Omega^\pm}\varepsilon^{4l}\mathring Q^\pm\mathring C_i(v_i^\pm)\,d\mathring V_t$$

$$\stackrel{L}{=} \int_{\Omega^\pm}\varepsilon^{4l}\partial_3^{\mathring c}\mathring Q^\pm\left[\partial_t^k \overline\partial^{4+l-k}, \mathring N_i, v_i^\pm \right]d\mathring V_t, \tag{4.66}$$

whose time integral can be directly controlled by

$$\delta\mathring E^\kappa_{4+l}(t) + C(\mathring K_0, \kappa^{-1})\left(\mathring E^\kappa(0) + \int_0^t \mathring E^\kappa(\tau)\,d\tau \right)$$

after integrating by parts one tangential derivative in $\partial_t^k \overline\partial^{4+l-k}$.

4.3.4 Uniform-in-n estimates for the linearized approximate system

Summarizing the estimates obtained in Section 4.3.1-Section 4.3.3, we prove that for any $\delta \in (0,1)$,

$$\mathring{E}^\kappa(t) \lesssim \delta \mathring{E}^\kappa(t) + C(\mathring{K}_0, \kappa^{-1})\left(\mathring{E}^\kappa(0) + \int_0^t \mathring{E}^\kappa(\tau)\,d\tau \right).$$

Choosing $\delta > 0$ suitably small such that the δ-term can be absorbed by the left side and using Grönwall's inequality, we find that there exists a time $T_\kappa > 0$ (independent of ε and n), such that

$$\sup_{0 \le t \le T_\kappa} \mathring{E}^\kappa(t) \le C'(\mathring{K}_0, \kappa^{-1})\mathring{E}^\kappa(0)$$

for some positive function C' continuous in its arguments. Following the argument in remark 4.3, it is straightforward to show that

$$\sum_{\pm} \sum_{l=0}^{4} \sum_{\langle \alpha \rangle = 2l} \sum_{k=0}^{4-l} \int_0^t \left\| \varepsilon^{2l} \mathcal{T}^\alpha \partial_t^k b^\pm \right\|_{4-k-l,\pm}^2 \, d\tau < P(\mathring{E}^\kappa(t)) \quad \forall t \in [0, T_\kappa].$$

4.4 Picard iteration: well-posedness of the nonlinear approximate problem

We already establish the local existence of the linear system (4.1) for each n and the uniform-in-n estimates for the solution to (4.1). It suffices to prove $\{(v^{[n],\pm}, b^{[n],\pm}, b^{[n],\pm}, q^{[n],\pm}, \psi^{[n]})\}$ has a strongly convergent subsequence (in certain anisotropic Sobolev norms). For a function sequence $\{f^{[n],\pm}\}$, we define $[f]^{[n],\pm} := f^{[n+1],\pm} - f^{[n],\pm}$. Then we can write the linear system of $\{([v]^{[n],\pm}, [b]^{[n],\pm}, [q]^{[n],\pm}, [\psi]^{[n]})\}$ as follows

$$
\begin{cases}
\rho^{[n],\pm} D_t^{\varphi^{[n]}}[v]^{[n],\pm} - (b^{[n],\pm} \cdot \nabla^{\varphi^{[n]}})[b]^{[n],\pm} + \nabla^{\varphi^{[n]}}[q]^{[n],\pm} + \nabla^{[\varphi]^{[n-1]}} q^{[n],\pm} = -\mathring{f}_v^{[n],\pm} & \text{in } [0,T] \times \Omega^\pm \\
\mathcal{F}_p^{[n],\pm} D_t^{\varphi^{[n]}}[q]^{[n],\pm} - \mathcal{F}_p^{[n],\pm} D_t^{\varphi^{[n]}}[b]^{[n],\pm} \cdot b^{[n],\pm} + \nabla^{\varphi^{[n]}} \cdot [v]^{[n],\pm} + \nabla^{[\varphi]^{[n-1]}} \cdot v^{[n],\pm} = -\mathring{f}_p^{[n],\pm} & \text{in } [0,T] \times \Omega^\pm \\
D_t^{\varphi^{[n]}}[b]^{[n],\pm} - (b^{[n],\pm} \cdot \nabla^{\varphi^{[n]}})[v]^{[n],\pm} + b^{[n],\pm}(\nabla^{\varphi^{[n]}} \cdot [v]^{[n],\pm} + \nabla^{[\varphi]^{[n-1]}} \cdot v^{[n],\pm}) = -\mathring{f}_b^{[n],\pm} & \text{in } [0,T] \times \Omega^\pm \\
D_t^{\varphi^{[n]}}[S]^{[n],\pm} = -\mathring{f}_S^{[n],\pm} & \text{in } [0,T] \times \Omega^\pm \\
\left[\!\left[[q]^{[n],\pm} \right]\!\right] = \sigma(\mathcal{H}(\psi^{[n]}) - \mathcal{H}(\psi^{[n-1]})) - \kappa(1 - \overline{\Delta})^2[\psi]^{[n]} - \kappa(1 - \overline{\Delta})\partial_t[\psi]^{[n]} & \text{on } [0,T] \times \Sigma \\
\partial_t[\psi]^{[n]} = [v]^{[n],\pm} \cdot N^{[n]} + v^{[n],\pm} \cdot [N]^{[n-1]} & \text{on } [0,T] \times \Sigma \\
v_3^{[n],\pm} = v_3^{[n-1],\pm} = b_3^{[n],\pm} = b_3^{[n-1],\pm} = 0 & \text{on } [0,T] \times \Sigma^\pm \\
([v]^{[n]}, [b]^{[n]}, [q]^{[n]}, [\psi]^{[n]})|_{t=0} = (\vec{0}, \vec{0}, 0, 0),
\end{cases} \tag{4.67}
$$

where the source terms are defined by

$$
\begin{aligned}
\mathring{f}_v^{[n],\pm} &:= [\rho]^{[n-1],\pm}\partial_t v^{[n],\pm} + [\rho\bar{v}]^{[n-1],\pm} \cdot \overline{\nabla} v^{[n],\pm} + [\rho V_{\mathbf{N}}]^{[n-1],\pm}\partial_3 v^{[n],\pm} \\
&\quad - [b]^{[n-1],\pm} \cdot \overline{\nabla} b^{[n],\pm} - [B_{\mathbf{N}}]^{[n-1],\pm}\partial_3 b^{[n],\pm},
\end{aligned} \tag{4.68}
$$

$$
\begin{aligned}
\mathring{f}_q^{[n],\pm} &:= [\mathcal{F}_p]^{[n-1],\pm}\partial_t q^{[n],\pm} + [\mathcal{F}_p\bar{v}]^{[n-1],\pm} \cdot \overline{\nabla} q^{[n],\pm} + [\mathcal{F}_p V_{\mathbf{N}}]^{[n-1],\pm}\partial_3 q^{[n],\pm} \\
&\quad - ([\mathcal{F}_p]^{[n-1],\pm}\partial_t b^{[n],\pm} + [\mathcal{F}_p\bar{v}]^{[n-1],\pm} \cdot \overline{\nabla} b^{[n],\pm} + [\mathcal{F}_p V_{\mathbf{N}}]^{[n-1],\pm}\partial_3 b^{[n],\pm}) \cdot b^{[n],\pm} \\
&\quad - (\mathcal{F}_p^\pm)^{[n-1]} D_t^{\varphi^{[n-1]}} b^{[n],\pm} \cdot [b]^{[n-1],\pm},
\end{aligned} \tag{4.69}
$$

$$
\begin{aligned}
\mathring{f}_p^{[n],\pm} &:= [\bar{v}]^{[n-1],\pm} \cdot \overline{\nabla} b^{[n],\pm} + [V_{\mathbf{N}}]^{[n-1],\pm}\partial_3 b^{[n],\pm} - [b]^{[n-1],\pm} \cdot \overline{\nabla} v^{[n],\pm} - [B_{\mathbf{N}}]^{[n-1],\pm}\partial_3 v^{[n],\pm} \\
&\quad + [b]^{[n-1],\pm}(\nabla^{\varphi^{[n-1]}} \cdot v^{[n],\pm}),
\end{aligned} \tag{4.70}
$$

$$
\mathring{f}_S^{[n],\pm} := [\bar{v}]^{[n-1],\pm} \cdot \overline{\nabla} S^{[n],\pm} + [V_{\mathbf{N}}]^{[n-1],\pm}\partial_3 S^{[n],\pm}, \tag{4.71}
$$

with

$$V_{\mathbf{N}}^{[n]} := \frac{1}{\partial_3 \varphi^{[n]}}(v^{[n]} \cdot \mathbf{N}^{[n-1]} - \partial_t \varphi^{[n]}), \quad B_{\mathbf{N}}^{[n]} := \frac{1}{\partial_3 \varphi^{[n]}}(b^{[n]} \cdot \mathbf{N}^{[n]}), \quad \nabla^{[\varphi]^{[n-1]}} f^{[n]} := -[\mathbf{N}/\partial_3\varphi]^{[n-1]}\partial_3 f^{[n]}$$

For $1 \leq n \in \mathbb{N}^*$, we define the energy for the linear system (4.67) as follows

$$[\mathring{E}^\kappa]^{[n]}(t) := [\mathring{E}^\kappa]_3^{[n]}(t) + \cdots + [\mathring{E}^\kappa]_6^{[n]}(t),$$

$$[\mathring{E}^\kappa]_{3+l}^{[n]}(t) := \sum_\pm \sum_{k=0}^{3} \sum_{\langle\alpha\rangle=2l} \left\| \varepsilon^{2l} \partial_t^k \mathcal{T}^\alpha([v]^{[n],\pm}, [b]^{[n],\pm}, [q]^{[n],\pm}, [S]^{[n],\pm}) \right\|_{3-k-l}^2$$

$$+ \sum_{k=0}^{3+l} \left| \sqrt{\kappa} \varepsilon^{2l} \partial_t^k [\psi]^{[n]} \right|_{5+l-k}^2 + \int_0^t \left| \sqrt{\kappa} \varepsilon^{2l} \partial_t^{4+l} [\psi]^{[n]} \right|_1^2 \, d\tau, \quad 0 \leq l \leq 3, \tag{4.72}$$

where $\mathcal{T}^\alpha := (\omega(x_3)\partial_3)^{\alpha_4} \partial_t^{\alpha_0} \partial_1^{\alpha_1} \partial_2^{\alpha_2}$ with the multi-index $\alpha = (\alpha_0, \alpha_1, \alpha_2, 0, \alpha_4)$, $\langle\alpha\rangle = \alpha_0 + \alpha_1 + \alpha_2 + 2 \times 0 + \alpha_4$. It should be noted that the initial value of $[\mathring{E}^\kappa]^{[n]}$. Thus, we shall prove the following proposition in order for the strong convergence.

Proposition 4.2. There exists a time $T'_\kappa > 0$ depending on κ and $\mathring{K}_0$, such that

$$\forall 2 \leq n \in \mathbb{N}^*, \quad \sup_{0 \leq t \leq T'_\kappa} [\mathring{E}^\kappa]^{[n]}(t) \leq \frac{1}{4} \left(\sup_{0 \leq t \leq T'_\kappa} [\mathring{E}^\kappa]^{[n-1]}(t) + \sup_{0 \leq t \leq T'_\kappa} [\mathring{E}^\kappa]^{[n-2]}(t) \right). \tag{4.73}$$

The proof of proposition is substantially similar to the estimates of $\mathring{E}(t)$ in Section 4.3, so we will not go into every detail but only write the sketch of the proof.

Step 1: Div-Curl analysis and reduction of pressure

The reduction of pressure follows in the same way as in Section 3.6.1. Invoking the momentum equation, we have

$$-(\partial_3 \varphi^{[n]})^{-1} \partial_3 [q]^{[n],\pm} = \rho^{[n],\pm} D_t^{\varphi^{[n]}} [v]^{[n],\pm} - (b^{[n],\pm} \cdot \nabla^{\varphi^{[n]}})[b]^{[n],\pm} + \mathring{f}_v^{[n],\pm} + (\partial_3 \varphi^{[n]})^{-1} \partial_3 q^{[n],\pm}.$$

Then using $\partial_i^{\mathring{\varphi}} = \overline{\partial}_i - \overline{\partial}_i \mathring{\varphi} \partial_3^{\mathring{\varphi}}$, we can convert ∂q to a spatial derivative of v and b plus the given term $\partial_3 q^{[n],\pm}$.

For the div-curl analysis, using (B.1), we have for $0 \leq l \leq 2$, $0 \leq k \leq 2 - l$

$$\left\| \varepsilon^{2l} \partial_t^k \mathcal{T}^\alpha([v]^{[n],\pm}, [b]^{[n],\pm}) \right\|_{3-k-l,\pm}^2$$

$$\leq C(\mathring{K}_0) \Big(\|\varepsilon^{2l} \partial_t^k \mathcal{T}^\alpha([v]^{[n],\pm}, [b]^{[n],\pm})\|_{0,\pm}^2 + \left\| \varepsilon^{2l} \nabla^{\varphi^{[n]}} \times \partial_t^k \mathcal{T}^\alpha([v]^{[n],\pm}, [b]^{[n],\pm}) \right\|_{2-k-l,\pm}^2$$

$$+ \left\| \varepsilon^{2l} \nabla^{\varphi^{[n]}} \cdot \partial_t^k \mathcal{T}^\alpha([v]^{[n],\pm}, [b]^{[n],\pm}) \right\|_{2-k-l,\pm}^2 + \left\| \overline{\partial}^{3-k-l} \partial_t^k \mathcal{T}^\alpha([v]^{[n],\pm}, [b]^{[n],\pm}) \right\|_{0,\pm}^2 \Big). \tag{4.74}$$

The L^2 estimate is straightforward, so we skip the proof. For the curl part, we again analyze the evolution equations of vorticity and current

$$\rho^{[n]} D_t^{\varphi^{[n]}} (\nabla^{\varphi^{[n]}} \times [v]^{[n]}) - (b^{[n]} \cdot \nabla^{\varphi^{[n]}})(\nabla^{\varphi^{[n]}} \times [b]^{[n]})$$

$$= - \nabla^{\varphi^{[n]}} \times \mathring{f}_v^{[n]} - \nabla^{\varphi^{[n]}} \rho^{[n]} \times D_t^{\varphi^{[n]}} [v]^{[n]} + (\nabla^{\varphi^{[n]}} b_j^{[n]}) \times (\partial_j^{\varphi^{[n]}} [b]^{[n]}) + \rho^{[n]} [D_t^{\varphi^{[n]}}, \nabla^{\varphi^{[n]}} \times][v]^{[n]}, \tag{4.75}$$

$$D_t^{\varphi^{[n]}} (\nabla^{\varphi^{[n]}} \times [b]^{[n]}) - (b^{[n]} \cdot \nabla^{\varphi^{[n]}})(\nabla^{\varphi^{[n]}} \times [v]^{[n]}) - b^{[n]} \times (\nabla^{\varphi^{[n]}}(\nabla^{\varphi^{[n]}} \cdot [v]^{[n]}))$$

$$= - \nabla^{\varphi^{[n]}} \times \mathring{f}_b^{[n]} + [D_t^{\varphi^{[n]}}, \nabla^{\varphi^{[n]}} \times][b]^{[n]} + (\nabla^{\varphi^{[n]}} b_j^{[n]}) \times (\partial_j^{\varphi^{[n]}} [v]^{[n]})$$

$$- \nabla^{\varphi^{[n]}} \times (b^{[n]} \nabla^{[\varphi]^{[n-1]}} \cdot v^{[n],\pm}) - (\nabla^{\varphi^{[n]}} \times b^{[n]})(\nabla^{\varphi^{[n]}} \cdot [v]^{[n]}). \tag{4.76}$$

Mimicing the proof in Section 4.3.1 and using the vanishing initial value of system (4.67), we can prove

$$\left\| \varepsilon^{2l} \nabla^{\varphi^{[n]}} \times \partial_t^k \mathcal{T}^\alpha [v]^{[n],\pm} \right\|_{2-k-l,\pm}^2 + \left\| \varepsilon^{2l} \nabla^{\varphi^{[n]}} \times \partial_t^k \mathcal{T}^\alpha [b]^{[n],\pm} \right\|_{2-k-l,\pm}^2 + \left\| \varepsilon^{2l} \sqrt{(\mathcal{F}_p^\pm)^{[n]}} b^{[n],\pm} \times (\partial_t^k \mathcal{T}^\alpha [b]^\pm) \right\|_{2-l-k,\pm}^2$$

$$\lesssim C(\mathring{K}_0) \int_0^t \sum_{j=0}^{l} [E]_{3+j}^{[n]}(\tau) + [E]_{3+l+1}^{[n]}(\tau) \, d\tau. \tag{4.77}$$

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Similarly, the divergence of $[v]^{[n]}$ can be converted to tangential derivatives of $[q]^{[n]}$ and $[b]^{[n]}$ by invoking the continuity equation, and the evolution equation of $\nabla^{\mathring{\varphi}} \cdot [b]^{[n]}$ is

$$D_t^{\varphi^{[n]}}(\nabla^{\varphi^{[n]}} \cdot [b]^{[n]}) = (\partial_i^{\varphi^{[n]}} b_j^{[n]})(\partial_j^{\varphi^{[n]}}[v]_i^{[n]}) - (\nabla^{\varphi^{[n]}} \cdot b^{[n]})(\nabla^{\varphi^{[n]}} \cdot [v]^{[n]}) + [D_t^{\varphi^{[n]}}, \nabla^{\mathring{\varphi}} \cdot][b]^{[n]}$$
$$- \nabla^{\varphi^{[n]}} \cdot (f_b^{\mathring{\partial}[n]} + b^{[n]} \nabla^{[\varphi]^{[n-1]}} \cdot v^{[n],\pm}), \tag{4.78}$$

so the divergence part is controlled by

$$\left\| \varepsilon^{2l} \nabla^{\varphi^{[n]}} \cdot \partial_t^k \mathcal{T}^\alpha [v]^{[n],\pm} \right\|_{2-k-l,\pm}^2 + \left\| \varepsilon^{2l} \nabla^{\varphi^{[n]}} \cdot \partial_t^k \mathcal{T}^\alpha [b]^{[n],\pm} \right\|_{2-k-l,\pm}^2$$

$$\lesssim \left\| \varepsilon^{2l} (\mathcal{F}_p^\pm)^{[n]} \partial_t^k \mathcal{T}^\alpha D_t^{\varphi^{[n]}}([q]^\pm, [b]^\pm) \right\|_{2-l-k,\pm}^2 + C(\mathring{K}_0) \int_0^t \sum_{j=0}^l [E]_{3+j}^{[n]}(\tau) \, d\tau, \tag{4.79}$$

in which the first term will be reduced to tangential estimates.

Step 2: Tangential estimates

It remains to prove the tangential estimates for $\mathcal{T}^\gamma$-differentiated system where $\mathcal{T}^\gamma = \overline{\partial}^{3-l-k} \partial_t^k \mathcal{T}^\alpha$ satisfies $\alpha_3 = 0, \langle \alpha \rangle = 2l, \ 0 \leq k \leq 3-l, \ 0 \leq l \leq 3$. We shall introduce the Alinhac good unknowns $([V], [B], [Q])$ as below instead of directly taking tangential derivatives in (4.67).

$$[\mathbf{F}]^{[n]} := \mathring{\mathbf{F}}^{[n+1]} - \mathring{\mathbf{F}}^{[n]} = \mathcal{T}^\gamma [f]^{[n]} - \mathcal{T}^\gamma \varphi^{[n]} \partial_3^{\varphi^{[n]}} [f]^{[n]} - \mathcal{T}^\gamma \varphi^{[n]} \partial_3^{[\varphi]^{[n-1]}} q^{[n]} - \mathcal{T}^\gamma [\varphi]^{[n-1]} \partial_3^{\varphi^{[n-1]}} q^{[n]}$$

and it satisfies

$$\mathcal{T}^\gamma(\partial_i^{\varphi^{[n]}}[f]^{[n]} + \partial_i^{[\varphi]^{[n-1]}} f^{[n]}) = \partial_i^{\varphi^{[n]}}[\mathbf{F}]^{[n]} + [\mathfrak{C}]_i^{[n]}(f), \quad \mathcal{T}^\gamma(D_t^{\varphi^{[n]}}[f]^{[n]} + D_t^{[\varphi]^{[n-1]}} f^{[n]}) = D_t^{\varphi^{[n]}}[\mathbf{F}]^{[n]} + [\mathfrak{D}]^{[n]}(f)$$

with

$$[\mathfrak{C}]_i^{[n]}(f) = \mathfrak{C}_i^{[n]}(f^{[n+1]}) - \mathfrak{C}_i^{[n-1]}(f^{[n]}) + \text{lower-order controllable terms}$$
$$[\mathfrak{D}]^{[n]}(f) = \mathfrak{D}^{[n]}(f^{[n+1]}) - \mathfrak{D}^{[n-1]}(f^{[n]}) + \text{lower-order controllable terms}$$

where $\mathfrak{C}_i^{[n]}(f^{[m]}), \mathfrak{D}^{[n]}(f^{[m]})$ are defined by setting $\mathring{\varphi} = \varphi^{[n]}, \dot{\varphi} = \varphi^{[n-1]}, f^{[n+1]} = f, f^{[n]} = \mathring{f}$ in (4.37)-(4.38). This can be seen by substracting the corresponding identities of $\mathring{\mathbf{F}}$ with superscript $[n-1]$ from the ones with superscript $[n]$. The evolution equations of the good unknowns are (with $\pm$ dropped)

$$\rho^{[n]} D_t^{\varphi^{[n]}}[\mathbf{V}]^{[n]} - (b^{[n]} \cdot \nabla^{\varphi^{[n]}})[\mathbf{B}]^{[n]} + \nabla^{\varphi^{[n]}}[\mathbf{Q}]^{[n]} = -\mathfrak{C}^{[n]}(q^{[n+1]}) + \mathfrak{C}^{[n-1]}(q^{[n]}) + [\mathcal{R}]_v \quad \text{in } \Omega \tag{4.80}$$

$$\mathcal{F}_p^{[n]} D_t^{\varphi^{[n]}}[\mathbf{Q}]^{[n]} - \mathcal{F}_p^{[n]} D_t^{\varphi^{[n]}}[\mathbf{B}]^{[n]} \cdot b^{[n]} + \nabla^{\varphi^{[n]}} \cdot [\mathbf{V}]^{[n]} = -\mathfrak{C}_i^{[n]}(v_i^{[n+1]}) + \mathfrak{C}_i^{[n-1]}(v_i^{[n]}) + [\mathcal{R}]_q \quad \text{in } \Omega \tag{4.81}$$

$$D_t^{\varphi^{[n]}}[\mathbf{B}]^{[n]} - (b^{[n]} \cdot \nabla^{\varphi^{[n]}})[\mathbf{V}]^{[n]} + b^{[n]}(\nabla^{\varphi^{[n]}} \cdot [\mathbf{V}]^{[n]}) = -b^{[n]} \left(\mathfrak{C}_i^{[n]}(v_i^{[n+1]}) - \mathfrak{C}_i^{[n-1]}(v_i^{[n]}) \right) + [\mathcal{R}]_b \quad \text{in } \Omega \tag{4.82}$$

where $[\mathcal{R}]$ terms are controllable in $L^2(\Omega)$ by

$$\|[\mathcal{R}]\|_0^2 \leq C(\mathring{K}_0)([\mathring{E}^\kappa]^{[n]}(t) + [\mathring{E}^\kappa]^{[n-1]}(t) + [\mathring{E}^\kappa]^{[n-2]}(t)).$$

The boundary conditions of these good unknowns on the interface Σ are

$$[\mathbf{Q}]^{[n]} := \sigma \mathcal{T}^\gamma \left(\mathcal{H}(\psi^{[n]}) - \mathcal{H}(\psi^{[n-1]}) \right) - \kappa(1 - \overline{\Delta})^2 \mathcal{T}^\gamma [\psi]^{[n]} - \kappa(1 - \overline{\Delta}) \partial_t \mathcal{T}^\gamma [\psi]^{[n]}$$
$$- \mathcal{T}^\gamma \psi^{[n]} \left[\!\left[\partial_3 [q]^{[n]} \right]\!\right] - \mathcal{T}^\gamma [\psi]^{[n-1]} \left[\!\left[\partial_3 q^{[n]} \right]\!\right] \tag{4.83}$$

$$[\mathbf{V}]^{[n]} \cdot N^{[n]} := \mathcal{T}^\gamma \partial_t [\psi]^{[n]} + [\bar{v}]^{[n]} \cdot \overline{\nabla} \mathcal{T}^\gamma \psi^{[n]} + (\bar{v}^{[n]} \cdot \overline{\nabla}) \mathcal{T}^\gamma [\psi]^{[n-1]} + \mathcal{T}^\gamma \bar{v}^{[n]} \cdot \overline{\nabla} [\psi]^{[n-1]} - [\mathcal{W}]^{[n]} \tag{4.84}$$

$$[\mathcal{W}]^{[n]} := (\partial_3 [v]^{[n]} \cdot N^{[n]}) \mathcal{T}^\gamma \psi^{[n]} + (\partial_3 v^{[n]} \cdot N^{[n]}) \mathcal{T}^\gamma [\psi]^{[n-1]} + \left[\mathcal{T}^\gamma, N_i^{[n]}, v_i^{[n+1]} \right] - \left[\mathcal{T}^\gamma, N_i^{[n-1]}, v_i^{[n]} \right] \tag{4.85}$$

Given $0 \leq l \leq 3$, following Section 4.3.2, we can similarly prove that

$$\sum_{\pm} \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_{\Omega^{\pm}} \varepsilon^{2l} \rho^{[n]} |[\mathbf{V}]^{[n],\pm}|^2 + |[\mathbf{B}]^{[n],\pm}|^2 + (\mathcal{F}_p^{\pm})^{[n],\pm} ([\mathbf{Q}]^{[n],\pm} - [\mathbf{B}]^{[n],\pm} \cdot b^{[n],\pm})^2 \, \mathrm{d}\mathcal{V}_t^{[n]}$$

$$= [\,\mathrm{ST}]^{[n]} + [\,\mathrm{ST}']^{[n]} + [\,\mathrm{VS}]^{[n]} + [\,\mathrm{RT}]^{[n]} + \sum_{\pm} [\,\mathrm{RT}]^{[n],\pm} + ([ZB]^{[n],\pm} + [Z]^{[n],\pm}) \tag{4.86}$$

$$+ C(\mathring{K}_0)([\mathring{E}^{\kappa}]^{[n]}(t) + [\mathring{E}^{\kappa}]^{[n-1]}(t) + [\mathring{E}^{\kappa}]^{[n-2]}(t))$$

where the term $[\mathring{E}^{\kappa}]^{[n-1]} + [\mathring{E}^{\kappa}]^{[n-2]}$ is produced from the estimates of $[\varphi]^{[n-1]}, [\varphi]^{[n-2]}$. The above terms on the right side are defined by

$$[\,\mathrm{ST}]^{[n]} := \varepsilon^{4l} \int_{\Sigma} \mathcal{T}^{\gamma} \left[\!\!\left[[q]^{[n]} \right]\!\!\right] \partial_t \mathcal{T}^{\gamma} [\psi]^{[n]} \, \mathrm{d}x', \tag{4.87}$$

$$[\,\mathrm{ST}']^{[n]} := \varepsilon^{4l} \int_{\Sigma} \mathcal{T}^{\gamma} \left[\!\!\left[[q]^{[n]} \right]\!\!\right] ([\bar{v}^+]^{[n]} \cdot \overline{\nabla}) \mathcal{T}^{\gamma} \psi^{[n]} \, \mathrm{d}x' + \varepsilon^{4l} \int_{\Sigma} \mathcal{T}^{\gamma} \left[\!\!\left[[q]^{[n]} \right]\!\!\right] (\bar{v}^{[n],+} \cdot \overline{\nabla}) \mathcal{T}^{\gamma} [\psi]^{[n-1]} \, \mathrm{d}x', \tag{4.88}$$

$$[\,\mathrm{VS}]^{[n]} := \varepsilon^{4l} \int_{\Sigma} \mathcal{T}^{\gamma} [q]^{[n],-} ([\![\bar{v}]\!]^{[n]} \cdot \overline{\nabla}) \mathcal{T}^{\gamma} \psi^{[n]} \, \mathrm{d}x' + \varepsilon^{4l} \int_{\Sigma} \mathcal{T}^{\gamma} [q]^{[n],-} ([\![\bar{v}^{[n]}]\!] \cdot \overline{\nabla}) \mathcal{T}^{\gamma} [\psi]^{[n-1]} \, \mathrm{d}x', \tag{4.89}$$

$$[\,\mathrm{RT}]^{[n]} := - \varepsilon^{4l} \int_{\Sigma} \left([\![\partial_3 [q]^{[n]}]\!] \mathcal{T}^{\gamma} \psi^{[n]} + \mathcal{T}^{\gamma} [\psi]^{[n-1]} [\![\partial_3 q^{[n]}]\!] \right) \partial_t \mathcal{T}^{\gamma} \psi \, \mathrm{d}x', \tag{4.90}$$

$$[\,\mathrm{RT}]^{[n],\pm} := \mp \varepsilon^{4l} \int_{\Sigma} \left(\partial_3 [q]^{[n],\pm} \mathcal{T}^{\gamma} \psi^{[n]} + \mathcal{T}^{\gamma} [\psi]^{[n-1]} \partial_3 q^{[n],\pm} \right) \left((\bar{v}^{\pm} \cdot \overline{\nabla}) \mathcal{T}^{\gamma} \mathring{\psi} + (\bar{v}^{[n],\pm} \cdot \overline{\nabla}) \mathcal{T}^{\gamma} [\psi]^{[n-1]} \right) \mathrm{d}x', \tag{4.91}$$

$$[ZB]^{[n],\pm} := \mp \varepsilon^{4l} \int_{\Sigma} [\mathbf{Q}]^{[n],\pm} [\mathcal{W}]^{[n],\pm} \, \mathrm{d}x', \quad [Z]^{[n],\pm} = - \int_{\Omega^{\pm}} \varepsilon^{4l} [\mathbf{Q}]^{[n],\pm} (\mathfrak{C}_i^{[n]}(v_i^{[n+1],\pm}) - \mathfrak{C}_i^{[n-1]}(v_i^{[n],\pm})) \, \mathrm{d}\mathcal{V}_t^{[n]}. \tag{4.92}$$

Step 3: Boundary regularity of $[\psi]$

The analysis of the boundary integrals is still similar to Section 4.3.3. Since $\omega(x_3) = 0$ on Σ, we can rewrite $\partial_t^k \mathcal{T}^{\alpha}$ to be $\partial_t^k \overline{\partial}^{3+l-k}$ for $0 \leq k \leq 3 + l$, $0 \leq l \leq 3$. Then the term $[\,\mathrm{ST}]^{[n]}$ gives the regularity of $[\psi]^{[n]}$ after inserting the jump condition for $[q]^{[n]}$

$$\int_0^t [\,\mathrm{ST}]^{[n]} \, \mathrm{d}\tau \lesssim - \left| \sqrt{\kappa} \varepsilon^{2l} \partial_t^k [\psi]^{[n]} \right|^2_{5+l-k} \Big|_0^t - \int_0^t \left| \sqrt{\kappa} \varepsilon^{2l} \partial_t^{k+1} [\psi]^{[n]} \right|^2_{4+l-k} + \frac{\sigma}{\kappa} C(\mathring{K}_0) \int_0^t [\mathring{E}^{\kappa}]^{[n]}(\tau) + [\mathring{E}^{\kappa}]^{[n-1]}(\tau) \, \mathrm{d}\tau. \tag{4.93}$$

The term $[\,\mathrm{ST}']^{[n]}$ can be controlled by inserting the jump condition for $[q]^{[n]}$ and then integrating by parts $\overline{\nabla}\cdot, 1 - \overline{\Delta}, \sqrt{1 - \overline{\Delta}}$ in the three terms in $[q]^{[n]}$ respectively. This is essentially the same as shown in Section 4.3.3, so we only list the result

$$\int_0^t [\,\mathrm{ST}']^{[n]} \, \mathrm{d}\tau \lesssim \sigma C(\mathring{K}_0, \kappa^{-1}) t + C(\mathring{K}_0) \int_0^t [\mathring{E}^{\kappa}]^{[n]}(\tau) + [\mathring{E}^{\kappa}]^{[n-1]}(\tau) \, \mathrm{d}\tau \tag{4.94}$$

The terms $[\,\mathrm{RT}]^{[n]}, [\,\mathrm{RT}]^{[n],\pm}$ are also controlled directly with the help of κ-weighted enhanced regularity. The term $[\,\mathrm{VS}]^{[n]}$ is also controlled directly by integrating by parts for one tangential derivative in $\partial_t^k \overline{\partial}^{3+l-k}$ as in Section 4.3.3. Finally, for the term $([ZB]^{[n],\pm} + [Z]^{[n],\pm})$, we still have the previously-used cancellation structure

$$[ZB]^{[n],\pm} + [Z]^{[n],\pm} \overset{L}{=} \mp \int_{\Sigma} \varepsilon^{4l} [\mathbf{Q}]^{[n],\pm} \left[\partial_t^k \overline{\partial}^{3+l-k}, N_i^{[n]}, v_i^{[n],\pm} \right] \mathrm{d}x' - \int_{\Omega^{\pm}} \varepsilon^{4l} [\mathbf{Q}]^{\pm} \mathfrak{C}_i^{[n]}(v_i^{[n],\pm}) \, \mathrm{d}\mathcal{V}_t^{[n]}$$

$$\pm \int_{\Sigma} \varepsilon^{4l} [\mathbf{Q}]^{[n],\pm} \left[\partial_t^k \overline{\partial}^{3+l-k}, N_i^{[n-1]}, v_i^{[n+1],\pm} \right] \mathrm{d}x' + \int_{\Omega^{\pm}} \varepsilon^{4l} [\mathbf{Q}]^{[n],\pm} \mathfrak{C}_i^{[n-1]}(v_i^{[n],\pm}) \, \mathrm{d}\mathcal{V}_t^{[n]} \tag{4.95}$$

where the omitted terms are controlled in the same way as $[\,\mathrm{VS}]^{[n]}$. Mimicking the proof in step 4 in Section 3.4, we have

$$\mp \int_{\Sigma} \varepsilon^{4l} [\mathbf{Q}]^{[n],\pm} \left[\partial_t^k \overline{\partial}^{3+l-k}, N_i^{[n]}, v_i^{[n],\pm} \right] \mathrm{d}x' - \int_{\Omega^{\pm}} \varepsilon^{4l} [\mathbf{Q}]^{\pm} \mathfrak{C}_i^{[n]}(v_i^{[n],\pm}) \, \mathrm{d}\mathcal{V}_t^{[n]}$$

$$\overset{L}{=} \int_{\Omega^{\pm}} \varepsilon^{4l} \partial_3^{\varphi^{[n]}} [\mathbf{Q}]^{[n],\pm} \left[\partial_t^k \overline{\partial}^{3+l-k}, \mathbf{N}_i^{[n]}, v_i^{[n+1],\pm} \right] \mathrm{d}\mathring{\mathcal{V}}_t, \tag{4.96}$$

whose time integral can be directly controlled by

$$\delta[\mathring{E}^\kappa]^{[n]}(t) + C(\mathring{K}_0, \kappa^{-1}) \int_0^t [\mathring{E}^\kappa]^{[n]}(\tau) + [\mathring{E}^\kappa]^{[n-1]}(\tau) \, d\tau$$

after integrating by parts for one tangential derivative in $\partial_t^k \overline{\partial}^{3+l-k}$. Similar estimate applies to the second line of $[ZB]^{[n],\pm} + [Z]^{[n],\pm}$:

$$\int_0^t \left(\pm \int_\Sigma \varepsilon^{4l} [\mathbf{Q}]^{[n],\pm} \left[\partial_t^k \overline{\partial}^{3+l-k}, N_i^{[n-1]}, v_i^{[n+1],\pm} \right] dx' + \int_{\Omega^\pm} \varepsilon^{4l} [\mathbf{Q}]^{[n],\pm} \mathfrak{C}_i^{[n-1]}(v_i^{[n],\pm}) \, d\mathcal{V}_t^{[n]} \right) d\tau$$

$$\lesssim \delta[\mathring{E}^\kappa]^{[n]}(t) + C(\mathring{K}_0, \kappa^{-1}) \int_0^t [\mathring{E}^\kappa]^{[n]}(\tau) + [\mathring{E}^\kappa]^{[n-1]}(\tau) + [\mathring{E}^\kappa]^{[n-2]}(\tau) \, d\tau$$

Step 4: Convergence

Summarizing the above estimates and using $[\mathring{E}^\kappa]^{[n]}(0) = 0$, we obtain the energy inequality

$$[\mathring{E}^\kappa]^{[n]}(t) \lesssim \delta[\mathring{E}^\kappa]^{[n]}(t) + C(\mathring{K}_0, \kappa^{-1}) \int_0^t [\mathring{E}^\kappa]^{[n]}(\tau) + [\mathring{E}^\kappa]^{[n-1]}(\tau) + [\mathring{E}^\kappa]^{[n-2]}(\tau) \, d\tau.$$

Choosing $0 < \delta \ll 1$ suitably small, the δ-term can be absorbed by the left side. Thus, there exists a time $T_\kappa' > 0$ depending on $\kappa, \mathring{K}_0$ and independent of n, such that

$$\sup_{0 \le t \le T_\kappa'} [\mathring{E}^\kappa]^{[n]}(t) \le \frac{1}{4} \left(\sup_{0 \le t \le T_\kappa'} [\mathring{E}^\kappa]^{[n-1]}(t) + \sup_{0 \le t \le T_\kappa'} [\mathring{E}^\kappa]^{[n-2]}(t) \right), \tag{4.97}$$

and thus we know by induction that

$$\sup_{0 \le t \le T_\kappa'} [\mathring{E}^\kappa]^{[n]}(t) \le C(\mathring{K}_0, \kappa^{-1}) / 2^{n-1} \to 0 \text{ as } n \to +\infty. \tag{4.98}$$

Hence, for any fixed $\kappa > 0$, the sequence of approximate solutions $\{(v^{[n],\pm}, b^{[n],\pm}, \boldsymbol{b}^{[n],\pm}, q^{[n],\pm}, \psi^{[n]})\}_{n \in \mathbb{N}^*}$ has a strongly convergent subsequence. We write the limit function to be $\{(v^{[\infty],\pm}, b^{[\infty],\pm}, \boldsymbol{b}^{[\infty],\pm} q^{[\infty],\pm}, \psi^{[\infty]})\}_{n \in \mathbb{N}^*}$ to keep consistent with the notations in Section 4.1.1.

Step 5: Well-posedness of the nonlinear approximate problem for each fixed κ

According to the argument about the limiting process in Section 4.1.1, we know the limit of $b^{[n],\pm}$ coincides with the limit of $\boldsymbol{b}^{[n],\pm}$. Thus, the limit functions $\{(v^{[\infty],\pm}, b^{[\infty],\pm}, q^{[\infty],\pm}, \psi^{[\infty]})\}_{n \in \mathbb{N}^*}$ introduced in Section 4.1.1 exactly give the solution to the nonlinear κ-problem (3.1) in the time interval $[0, T_\kappa']$ for each fixed $\kappa > 0$. The uniqueness follows from a parallel argument.

5   Well-posedness and incompressible limit

5.1   Well-posedness of compressible current-vortex sheets with surface tension

We are ready to prove the local well-posedness of the original system (1.30) for 3D compressible current-vortex sheets with fixed surface tension coefficient $\sigma > 0$. Recall that we introduce the nonlinear approximate system (3.1) indexed by $\kappa > 0$. In Section 4, we use Galerkin approximation and Picard iteration to prove the well-posedness of (3.1) for each fixed $\kappa > 0$. The lifespan for (3.1) may rely on $\kappa > 0$. Then we prove the uniform-in-κ estimates for (3.1) *without loss of regularity* so that we can extend the solution of (3.1) to a κ-independent lifespan $[0, T]$. In Appendix C, we construct the initial data of (3.1) that converges to the given initial data of (1.30) as $\kappa \to 0$. Thus, by taking $\kappa \to 0$, we obtain the local existence of the original system (1.30) and the energy estimates for $E(t)$ defined in (1.33) without loss of regularity.

It remains to prove the uniqueness. Namely, we assume $(v^{[1],\pm}, b^{[1],\pm}, q^{[1],\pm}, \psi^{[1]})$ and $(v^{[2],\pm}, b^{[2],\pm}, q^{[2],\pm}, \psi^{[2]})$ are two solutions to (1.30) *with the same initial data*. Define $[f] := f^{[1]} - f^{[2]}$, and we need to prove $([v]^\pm, [b]^\pm, [q]^\pm, [\psi])$ are identically zero. In fact, the argument for uniqueness is quite similar to the analysis in Section 4.4. The only difference is that the boundary regularity is now given by the surface tension instead of the κ-regularization terms. This has been studied in the previous paper [38, Section 6] by Luo and the author, so we refer to [38, Section 6] and omit the details here.

5.2 Incompressible limit of compressible current-vortex sheets with surface tension

Next, we justify the incompressible limit of the solution obtained above, that is the limiting behavior of the local-in-time solution of (1.30) as $\varepsilon \to 0$. Given $\sigma > 0$, we introduce the equations of $(\xi^\sigma, w^{\pm,\sigma}, h^{\pm,\sigma})$ describing the motion of incompressible non-uniform current-vortex sheets together with a transport equation of entropy $\mathfrak{S}^\sigma$

$$
\begin{cases}
\mathfrak{R}^{\pm,\sigma}(\partial_t + w^{\pm,\sigma} \cdot \nabla^{\Xi^\sigma})w^{\pm,\sigma} - (h^{\pm,\sigma} \cdot \nabla^{\Xi^\sigma})h^{\pm,\sigma} + \nabla^{\Xi^\sigma}\Pi^{\pm,\sigma} = 0 & \text{in } [0,T] \times \Omega^\pm, \\
\nabla^{\Xi^\sigma} \cdot w^{\pm,\sigma} = 0 & \text{in } [0,T] \times \Omega^\pm, \\
(\partial_t + w^{\pm,\sigma} \cdot \nabla^{\Xi^\sigma})h^{\pm,\sigma} = (h^{\pm,\sigma} \cdot \nabla^{\Xi^\sigma})w^{\pm,\sigma} & \text{in } [0,T] \times \Omega^\pm, \\
\nabla^{\Xi^\sigma} \cdot h^{\pm,\sigma} = 0 & \text{in } [0,T] \times \Omega^\pm, \\
(\partial_t + w^{\pm,\sigma} \cdot \nabla^{\Xi^\sigma})\mathfrak{S}^{\pm,\sigma} = 0 & \text{in } [0,T] \times \Omega^\pm, \\
[\![\Pi^\sigma]\!] = \sigma\overline{\nabla} \cdot \left(\dfrac{\overline{\nabla}\xi^\sigma}{\sqrt{1+|\overline{\nabla}\xi^\sigma|^2}}\right) & \text{on } [0,T] \times \Sigma, \\
\partial_t \xi^\sigma = w^{\pm,\sigma} \cdot N^\sigma & \text{on } [0,T] \times \Sigma, \\
h^{\pm,\sigma} \cdot N^\sigma = 0 & \text{on } [0,T] \times \Sigma, \\
(w^{\pm,\sigma}, h^{\pm,\sigma}, \mathfrak{S}^{\pm,\sigma}, \xi^\sigma)|_{t=0} = (w_0^{\pm,\sigma}, h_0^{\pm,\sigma}, \mathfrak{S}_0^{\pm,\sigma}, \xi_0^\sigma),
\end{cases}
\tag{5.1}
$$

where $\Xi^\sigma(t,x) = x_3 + \chi(x_3)\xi^\sigma(t,x')$ to be the extension of ξ^σ in Ω and $N^\sigma := (-\overline{\partial}_1\xi^\sigma, -\overline{\partial}_2\xi^\sigma, 1)^\top$. The quantity $\Pi^\pm := \overline{\Pi}^\pm + \frac{1}{2}|h^\pm|^2$ represent the total pressure for the incompressible equations with $\overline{\Pi}^\pm$ the fluid pressure functions. The quantity $\mathfrak{R}^\pm$ satisfies the evolution equation $(\partial_t + w^\pm \cdot \nabla^\varphi)\mathfrak{R}^\pm = 0$ with initial data $\mathfrak{R}_0^\pm := \rho^\pm(0, \mathfrak{S}_0^\pm)$.

Denote $(\psi^{\varepsilon,\sigma}, v^{\pm,\varepsilon,\sigma}, b^{\pm,\varepsilon,\sigma}, \rho^{\pm,\varepsilon,\sigma}, S^{\pm,\varepsilon,\sigma})$ to be the solution of (1.30) (indexed by σ and ε) with initial data $(\psi_0^{\varepsilon,\sigma}, v_0^{\pm,\varepsilon,\sigma}, b_0^{\pm,\varepsilon,\sigma}, \rho_0^{\pm,\varepsilon,\sigma}, S_0^{\pm,\varepsilon,\sigma})$. For fixed $\sigma > 0$, we want to show the convergence from the solutions to (1.30) to the solution to (5.1) as $\varepsilon \to 0$ provided the convergence of initial data. We assume

1. (Constraints for compressible initial data) The sequence of initial data $(\psi_0^{\varepsilon,\sigma}, v_0^{\pm,\varepsilon,\sigma}, b_0^{\pm,\varepsilon,\sigma}, \rho_0^{\pm,\varepsilon,\sigma}, S_0^{\pm,\varepsilon,\sigma}) \in H^{9.5}(\Sigma) \times (H_*^8(\Omega^\pm))^4$ of (1.30) satisfy the constraints $\nabla^\varphi \cdot b_0^{\pm,\varepsilon,\sigma} = 0$ in $\Omega^\pm$, $b^{\pm,\varepsilon,\sigma} \cdot N^\sigma|_{t=0} = 0$ on $\Sigma \cup \Sigma^\pm$, the compatibility conditions (1.31) up to 7-th order, $|\psi_0^{\varepsilon,\sigma}| \leq 1$ and $|[\![\overline{v}_0]\!]| > 0$.
2. (Convergence of initial data) $(\psi_0^{\varepsilon,\sigma}, v_0^{\pm,\varepsilon,\sigma}, b_0^{\pm,\varepsilon,\sigma}, \rho_0^{\pm,\varepsilon,\sigma}, S_0^{\pm,\varepsilon,\sigma}) \to (\xi_0^\sigma, w_0^{\pm,\sigma}, h_0^{\pm,\sigma}, \mathfrak{R}_0^{\pm,\sigma}, \mathfrak{S}_0^{\pm,\sigma})$ in $H^{5.5}(\Sigma) \times (H^4(\Omega^\pm))^4$.
3. (Constraints for incompressible initial data) The incompressible data $(\xi_0^\sigma, w_0^{\pm,\sigma}, h_0^{\pm,\sigma}, \mathfrak{R}_0^{\pm,\sigma}, \mathfrak{S}_0^{\pm,\sigma}) \in H^5(\Sigma) \times (H^4(\Omega^\pm))^4$ satisfies the constraints $\nabla^{\xi_0^\sigma} \cdot h_0^\pm = 0$ in $\Omega^\pm$, $h^{\pm,\sigma} \cdot N^\sigma|_{t=0} = 0$ on $\Sigma \cup \Sigma^\pm$, $|\xi_0^\sigma| \leq 2$ and $[\![\overline{w}_0]\!] > 0$.

Under these assumptions, we can prove that there exists a time $T_\sigma > 0$ that depends on σ and initial data and is independent of Mach number ε, such that the corresponding solutions to (1.30) converge to the solution to (5.1) as the Mach number $\varepsilon \to 0$

$$(\psi^{\varepsilon,\sigma}, v^{\pm,\varepsilon,\sigma}, b^{\pm,\varepsilon,\sigma}, \rho^{\pm,\varepsilon,\sigma}, S^{\pm,\varepsilon,\sigma}) \to (\xi^\sigma, w^{\pm,\sigma}, h^{\pm,\sigma}, \mathfrak{R}^{\pm,\sigma}, \mathfrak{S}^{\pm,\sigma})$$

strongly in $C([0,T_\sigma]; H_{\text{loc}}^{5.5-\delta}(\Sigma) \times (H_{\text{loc}}^{4-\delta}(\Omega^\pm))^4)$, and weakly-* in $L^\infty([0,T_\sigma]; H^{5.5}(\Sigma) \times (H^4(\Omega^\pm))^4)$.

In fact, according to estimates obtained in Theorem 1.1, we already have the uniform-in-ε boundedness for $\psi^{\varepsilon,\sigma}$, $v^{\pm,\varepsilon,\sigma}$, $b^{\pm,\varepsilon,\sigma}$, $S^{\pm,\varepsilon,\sigma}$ as well as their first-order time derivatives. Thus, using Aubin-Lions compactness lemma, the above convergence is a straightforward result of uniform-in-ε estimates. Theorem 1.2 is proven.

Conflict of interest. On behalf of all authors, the corresponding author states that there is no conflict of interest.

Data avaliability. Data sharing is not applicable as no datasets were generated or analyzed during the current study.

A   Reynolds transport theorems

We record the Reynolds transport theorems used in this paper. For the proof, we refer to Luo-Zhang [38, Appendix A]

Lemma A.1. Let f, g be smooth functions defined on $[0,T] \times \Omega$. Then:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega fg\partial_3\varphi \,\mathrm{d}x = \int_\Omega (\partial_t^\varphi f)g\partial_3\varphi \,\mathrm{d}x + \int_\Omega f(\widetilde{\varphi_t}g)\partial_3\varphi \,\mathrm{d}x + \int_{x_3=0} fg\partial_t\psi \,\mathrm{d}x', \tag{A.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega fg\partial_3\mathring{\varphi} \,\mathrm{d}x = \int_\Omega (\partial_t^{\mathring{\varphi}} f)g\partial_3\mathring{\varphi} \,\mathrm{d}x + \int_\Omega f(\partial_t^{\mathring{\varphi}}g)\partial_3\mathring{\varphi} \,\mathrm{d}x + \int_{x_3=0} fg\partial_t\mathring{\psi} \,\mathrm{d}x'. \tag{A.2}$$

Lemma A.2 (**Integration by parts for covariant derivatives**). Let f, g be defined as in Lemma A.1. Then:

$$\int_\Omega (\partial_i^\varphi f) g \partial_3 \varphi \, dx = - \int_\Omega f(\partial_i^\varphi g) \partial_3 \varphi \, dx + \int_{x_3=0} fg N_i \, dx', \tag{A.3}$$

$$\int_\Omega (\partial_i^{\mathring{\varphi}} f) g \partial_3 \mathring{\varphi} \, dx = - \int_\Omega f(\partial_i^{\mathring{\varphi}} g) \partial_3 \mathring{\varphi} \, dx + \int_{x_3=0} fg \mathring{N}_i \, dx'. \tag{A.4}$$

The following theorem holds.

Theorem A.3 (**Reynolds transport theorem**). Let f be a smooth function defined on $[0, T] \times \Omega$. Then:

$$\frac{d}{dt} \int_\Omega \rho |f|^2 \partial_3 \varphi \, dx = \int_\Omega \rho(D_t^\varphi f) f \partial_3 \varphi \, dx. \tag{A.5}$$

Theorem A.3 leads to the following two corollaries. The first one records the integration by parts formula for D_t^φ.

Corollary A.4 (**Reynolds transport theorem - a variant**). It holds that

$$\frac{d}{dt} \int_\Omega fg \partial_3 \varphi \, dx = \int_\Omega (D_t^\varphi f) g \partial_3 \varphi \, dx + \int_\Omega f(D_t^\varphi g) \partial_3 \varphi \, dx + \int_\Omega (\nabla^\varphi \cdot v) fg \partial_3 \varphi \, dx. \tag{A.6}$$

The second corollary concerns the transport theorem as well as the integration by parts formula for the linearized material derivative $D_t^{\mathring{\varphi}}$.

Corollary A.5 (**Reynolds transport theorem for linearized κ-problem**). Let $D_t^{\mathring{\varphi}} := \partial_t + (\mathring{\bar{v}} \cdot \overline{\nabla}) + \frac{1}{\partial_3 \mathring{\varphi}}(\mathring{v} \cdot \mathbf{N} - \partial_t \mathring{\varphi}) \partial_3$ be the linearized material derivative. Then:

$$\frac{1}{2}\frac{d}{dt} \int_\Omega \mathring{\rho} |f|^2 \partial_3 \mathring{\varphi} \, dx = \int_\Omega \mathring{\rho}(D_t^{\mathring{\varphi}} f) f \partial_3 \mathring{\varphi} \, dx + \frac{1}{2}\int_\Omega \left(D_t^{\mathring{\varphi}} \mathring{\rho} + \mathring{\rho} \nabla^{\mathring{\varphi}} \cdot \mathring{v} \right) |f|^2 \partial_3 \mathring{\varphi} \, dx \tag{A.7}$$

$$+ \frac{1}{2}\int_\Omega \mathring{\rho} |f|^2 \left(\partial_3 (\mathring{\bar{v}} \cdot \overline{\nabla})(\mathring{\varphi} - \varphi) \right) dx.$$

$$\frac{1}{2}\frac{d}{dt} \int_\Omega |f|^2 \partial_3 \mathring{\varphi} \, dx = \int_\Omega (D_t^{\mathring{\varphi}} f) f \partial_3 \mathring{\varphi} \, dx + \frac{1}{2}\int_\Omega \nabla^{\mathring{\varphi}} \cdot \mathring{v} |f|^2 \partial_3 \mathring{\varphi} \, dx \tag{A.8}$$

$$+ \frac{1}{2}\int_\Omega |f|^2 \left(\partial_3 (\mathring{\bar{v}} \cdot \overline{\nabla})(\mathring{\varphi} - \varphi) \right) dx.$$

B   Preliminary lemmas about Sobolev inequalities

Lemma B.1 (Hodge-type elliptic estimates). For any sufficiently smooth vector field X and $s \geq 1$, one has

$$\|X\|_s^2 \leq C(|\psi|_s, |\overline{\nabla}\psi|_{W^{1,\infty}}) \left(\|X\|_0^2 + \|\nabla^\varphi \cdot X\|_{s-1}^2 + \|\nabla^\varphi \times X\|_{s-1}^2 + \|\overline{\partial}^\alpha X\|_0^2 \right), \tag{B.1}$$

$$\|X\|_s^2 \leq C'(|\psi|_{s+\frac{1}{2}}, |\overline{\nabla}\psi|_{W^{1,\infty}}) \left(\|X\|_0^2 + \|\nabla^\varphi \cdot X\|_{s-1}^2 + \|\nabla^\varphi \times X\|_{s-1}^2 + |X \cdot N|_{s-\frac{1}{2}}^2 \right), \tag{B.2}$$

$$\|X\|_s^2 \leq C''(|\psi|_{s+\frac{1}{2}}, |\overline{\nabla}\psi|_{W^{1,\infty}}) \left(\|X\|_0^2 + \|\nabla^\varphi \cdot X\|_{s-1}^2 + \|\nabla^\varphi \times X\|_{s-1}^2 + |X \times N|_{s-\frac{1}{2}}^2 \right), \tag{B.3}$$

for any multi-index α with $|\alpha| = s$. The constant $C(|\psi|_s, |\overline{\nabla}\psi|_{W^{1,\infty}}) > 0$ depends linearly on $|\psi|_s^2$ and the constants $C'(|\psi|_{s+\frac{1}{2}}, |\overline{\nabla}\psi|_{W^{1,\infty}}) > 0$ and $C'(|\psi|_{s+\frac{1}{2}}, |\overline{\nabla}\psi|_{W^{1,\infty}}) > 0$ depend linearly on $|\psi|_{s+\frac{1}{2}}^2$.

Lemma B.2 (Normal trace lemma). For any sufficiently smooth vector field X and $s \geq 0$, one has

$$|X \cdot N|_{s-\frac{1}{2}}^2 \lesssim C'''(|\psi|_{s+\frac{1}{2}}, |\overline{\nabla}\psi|_{W^{1,\infty}}) \left(\|\langle \overline{\partial} \rangle^s X\|_0^2 + \|\nabla^\varphi \cdot X\|_{s-1}^2 \right) \tag{B.4}$$

where the constant $C'''(|\psi|_{s+\frac{1}{2}}, |\overline{\nabla}\psi|_{W^{1,\infty}}) > 0$ depends linearly on $|\psi|_{s+\frac{1}{2}}^2$.

We list two lemmas for the estimates of traces in the anisotropic Sobolev spaces. Define

$$L_T^2(H_*^m(\Omega^\pm)) = \bigcap_{k=0}^{m} H^k((-\infty, T]; H_*^{m-k}(\Omega^\pm))$$

with the norm $\|u\|_{m,*,T,\pm} := \int_{-\infty}^{T} \|u(t)\|_{m,*,\pm}^2 \, dt$. Similarly, we define

$$L_T^2(H^m(\Sigma)) = \bigcap_{k=0}^{m} H^k((-\infty, T]; H^{m-k}(\Sigma))$$

with the norm $|u|_{m,T} := \int_{-\infty}^{T} |u(t)|_m^2 \, dt$.

Lemma B.3 (Trace lemma for anisotropic Sobolev spaces, [61, Lemma 3.4]). *Let $m \geq 1$, $m \in \mathbb{N}^*$, then we have the following trace lemma for the anisotropic Sobolev space.*

1. *If $f \in L_T^2(H_*^{m+1}(\Omega^\pm))$, then its trace $f|_\Sigma$ belongs to $L_T^2(H^m(\Omega^\pm))$ and satisfies*

$$|f|_{m,T} \lesssim \|f\|_{m+1,*,T,\pm}.$$

2. *There exists a linear continuous operator $\mathfrak{R}_T^\pm : L_T^2(H^m(\Sigma)) \to L_T^2(H_*^{m+1}(\Omega^\pm))$ such that $(\mathfrak{R}_T^\pm g)|_\Sigma = g$ and*

$$\|\mathfrak{R}_T^\pm g\|_{m+1,*,T,\pm} \lesssim |g|_{m,T}.$$

Proof. The proof for the above lemma can be found in [49, Theorem 1] when we replace $(-\infty, T)$ by $(-\infty, \infty)$. In our case, we can prove the same result by doing Sobolev extension. Namely, given $f \in L_T^2(H_*^{m+1}(\Omega^+))$, we can extend it to $F(t, x) : R \times \Omega^+ \to \mathbb{R}$ such that

$$\|f\|_{m+1,*,T,+} \lesssim \|F(t,x)\|_{H_*^{m+1}(\mathbb{R} \times \Omega^+)} \lesssim \|f\|_{m+1,*,T,+}.$$

We can apply [49, Theorem 1] to F, and then do the truncation in $(-\infty, T]$

$$|f|_{m,T} \lesssim |F|_{H^m(\mathbb{R} \times \Sigma)} \lesssim \|F(t,x)\|_{H_*^{m+1}(\mathbb{R} \times \Omega^+)} \lesssim \|f\|_{m+1,*,T,+}.$$

$\square$

There is one derivative loss in the above trace lemma, which is 1/2-order more than the trace lemma for standard Sobolev spaces. Indeed, for $\Omega^\pm$ defined in this paper, we have the following estimate that will be applied to control the non-characteristic variables q, $v \cdot \mathbf{N}$ and $b \cdot \mathbf{N}$.

Lemma B.4 (An estimate for traces of non-characteristic variables). *Let $\Omega^\pm := \mathbb{T}^{d-1} \times \{0 \leqslant x_d \leqslant \pm H\}$, $\Sigma = \mathbb{T}^{d-1} \times \{x_d = 0\}$ and $\Sigma^\pm = \mathbb{T}^{d-1} \times \{\pm H\}$. Let $\mathcal{T}^\alpha = (\omega(x_d)\partial_d)^{\alpha_{d+1}} \partial_t^{\alpha_0} \overline{\partial}_1^{\alpha_1} \cdots \overline{\partial}_{d-1}^{\alpha_{d-1}} \partial_d^{\alpha_d}$ with $\langle \alpha \rangle := \alpha_0 + \cdots + \alpha_{d-1} + 2\alpha_d + \alpha_{d+1} = m - 1$, $m \in \mathbb{N}^*$. Let $q^\pm(t, x) \in H_*^m(\Omega)$ satisfy $\|q^\pm(t)\|_{m,*,\pm} + \|\partial_d q^\pm(t)\|_{m-1,*,\pm} < \infty$ for any $0 \leq t \leq T$ and let $f^\pm \in H_*^2(\Omega^\pm) \cap H^{\frac{3}{2}}(\Omega^\pm)$ be a function vanishing on $\Sigma^\pm$. Then we have*

$$\int_\Sigma (\langle \overline{\partial} \rangle^{\frac{1}{2}} \mathcal{T}^\gamma q^\pm)(\langle \overline{\partial} \rangle f^\pm) \, dx' \leq (\|\partial_d q^\pm\|_{m-1,*,\pm} + \|q^\pm\|_{m,*,\pm})\|\langle \overline{\partial} \rangle^{\frac{1}{2}} f^\pm\|_{1,\pm} \tag{B.5}$$

In particular, for $s \geq 1$, we have the following inequality for any $g^\pm \in H_*^s(\Omega^\pm)$ with $g^\pm|_{\Sigma^\pm} = 0$.

$$|g^\pm|_{s-1/2}^2 \leq \|\langle \overline{\partial} \rangle^s g^\pm\|_{0,\pm}\|\langle \overline{\partial} \rangle^{s-1} \partial_d g^\pm\|_{0,\pm} \leq \|g^\pm\|_{s,*,\pm}\|\partial_d g^\pm\|_{s-1,*,\pm}.$$

Proof. This is a direct consequence of Gauss-Green formula. Note that the unit exterior normal vectors for $\Omega^\pm$ are $(0, \cdots, 0, \mp 1)^\top$ respectively, so we have

$$\int_\Sigma (\langle \overline{\partial} \rangle^{\frac{1}{2}} \mathcal{T}^\gamma q^\pm)(\langle \overline{\partial} \rangle f^\pm) \, dx' = \mp \int_{\Omega^\pm} (\partial_d \mathcal{T}^\gamma q^\pm)(\langle \overline{\partial} \rangle^{\frac{3}{2}} f^\pm) + (\langle \overline{\partial} \rangle \mathcal{T}^\gamma q^\pm)(\langle \overline{\partial} \rangle^{\frac{1}{2}} \partial_d f^\pm) \, dx$$
$$\leq (\|\partial_d q^\pm\|_{m-1,*,\pm} + \|q^\pm\|_{m,*,\pm})\|\langle \overline{\partial} \rangle^{\frac{1}{2}} f^\pm\|_{1,\pm} \tag{B.6}$$

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In particular, let $q^\pm = g^\pm$ and $f^\pm = \langle \bar{\partial} \rangle^{s-\frac{3}{2}} g^\pm$ in (B.5) and we get

$$|g^\pm|^2_{s-1/2} = \int_\Sigma (\langle \bar{\partial} \rangle^{s-1/2} g^\pm)(\langle \bar{\partial} \rangle^{s-1/2} g^\pm) \, dx' = \mp 2 \int_{\Omega^\pm} (\partial_d \langle \bar{\partial} \rangle^{s-1/2} g^\pm)(\langle \bar{\partial} \rangle^{s-1/2} g^\pm) \, dx$$

$$\stackrel{\langle \bar{\partial} \rangle^{1/2}}{=} \mp 2 \int_{\Omega^\pm} (\partial_d \langle \bar{\partial} \rangle^{s-1} g^\pm)(\langle \bar{\partial} \rangle^s g^\pm) \, dx.$$

$\square$

The following lemma concerns the Sobolev embeddings.

Lemma B.5 ([61, Lemma 3.3]). We have the following inequalities

$$H^m(\Omega^\pm) \hookrightarrow H^m_*(\Omega^\pm) \hookrightarrow H^{\lfloor m/2 \rfloor}(\Omega^\pm), \quad \forall m \in \mathbb{N}^*$$

$$\|u\|_{L^\infty(\Omega^\pm)} \lesssim \|u\|_{H^3_*(\Omega^\pm)}, \quad \|u\|_{W^{1,\infty}(\Omega^\pm)} \lesssim \|u\|_{H^5_*(\Omega^\pm)}, \quad |u|_{W^{1,\infty}(\Omega^\pm)} \lesssim \|u\|_{H^5_*(\Omega^\pm)}.$$

We also need the following Kato-Ponce type multiplicative Sobolev inequality.

Lemma B.6 ([27]). Let $J = (1 - \Delta)^{1/2}$, $s \geq 0$. Then the following estimates hold:

$$\|J^s(fg)\|_{L^2} \lesssim \|f\|_{W^{s,p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{q_1}} \|g\|_{W^{s,q_2}}, \tag{B.7}$$

where $1/2 = 1/p_1 + 1/p_2 = 1/q_1 + 1/q_2$ and $2 \leq p_1, q_2 < \infty$.

$$\|[J^s, f]g\|_{L^p} \lesssim \|\partial f\|_{L^\infty} \|J^{s-1} g\|_{L^p} + \|J^s f\|_{L^p} \|g\|_{L^\infty} \tag{B.8}$$

where $s \geq 0$ and $1 < p < \infty$.

We also need the following transport-type estimate in order to close the uniform estimates for the nonlinear approximate system.

Lemma B.7. Let $f(t) \in W^{1,1}(0, T)$ and $g \in L^1(0, T)$ and $\kappa > 0$. Assume that

$$f(t) + \kappa f'(t) \leq g(t) \quad \text{a.e. } t \in (0, T).$$

Then for any $t \in (0, T)$,

$$\sup_{\tau \in [0,t]} f(\tau) \leq f(0) + \operatorname*{ess\,sup}_{\tau \in (0,t)} |g(\tau)|.$$

C   Construction of initial data satisfying the compatibility conditions

Given initial data $(v_0^\pm, b_0^\pm, q_0^\pm, S_0^\pm, \psi_0)$ of the original current-vortex sheets problem (1.30) satisfying the compatibility conditions (1.31) up to 7-th order, we need to construct a sequence of initial data $(v_0^{\kappa,\pm}, b_0^{\kappa,\pm}, q_0^{\kappa,\pm}, S_0^{\kappa,\pm}, \psi_0^\kappa)$ to the nonlinear κ-approximate system (3.1) satisfying the compatibility conditions (3.4) up to 7-th order that converge to the given data as $\kappa \to 0_+$.

C.1   Reformulation of the compatibility conditions

Let us first ignore the κ-regularization terms and consider the compatibility conditions (1.31) for the original system. Also, let us omit the fixed boundaries $\Sigma^\pm$, omit the density functions, consider the isentropic case and write $\varepsilon^2 = \mathcal{F}_p^\pm$ for convenience. The heuristic idea is that the odd ($m = 2r + 1$) order compatibility condition is rewritten to be

$$-\left[\!\left[\Lambda_{\varepsilon,b_0}^{r+1} (\Delta^{\varphi_0})^r (\nabla^{\varphi_0} \cdot v_0) \right]\!\right] = \cdots \quad \text{on } \Sigma$$

and the even ($m = 2r$) order compatibility condition is rewritten to be

$$\left[\!\left[\Lambda_{\varepsilon,b_0}^r (\Delta^{\varphi_0})^r q_0 \right]\!\right] = \cdots \quad \text{on } \Sigma$$

with $\Lambda_{\varepsilon,b_0} := \varepsilon^{-2} + |b_0|^2$. Such reformulation is convenient for us to add κ-perturbation terms to construct the desired data for (3.1). More specifically, let us start with the zero-th order compatibility conditions:

$$[\![q_0]\!] = \sigma\mathcal{H}(\psi_0), \quad \psi_t|_{t=0} = v_0^{\pm} \cdot N_0 = v_{03}^{\pm} - \bar{v}_0^{\pm} \cdot \bar{\partial}\psi_0. \tag{C.1}$$

The first-order compatibility conditions are

$$\partial_t [\![q]\!]|_{t=0} = \sigma\partial_t\mathcal{H}(\psi)|_{t=0}, \quad \psi_{tt}|_{t=0} = \partial_t(v^{\pm} \cdot N)|_{t=0}, \tag{C.2}$$

which are not easy to compute, especially the first one. The left side is equal to

$$\partial_t q^+ - \partial_t q^- = \overline{D_t^+}q^+ - \overline{D_t^-}q^- - (\bar{v}^+ \cdot \overline{\nabla})[\![q]\!] - ([\![\bar{v}]\!] \cdot \overline{\nabla})q^-.$$

Using the continuity equation, the evolution equation of b, we get

$$\overline{D_t}q = -\varepsilon^{-2}(\nabla^{\varphi} \cdot v) + \overline{D_t}b \cdot b = -\underbrace{(\varepsilon^{-2} + |b|^2)}_{=:\Lambda_{\varepsilon,b}}(\nabla^{\varphi} \cdot v) + (\bar{b} \cdot \overline{\nabla})v \cdot b \text{ on } \Sigma,$$

and thus the time-differentiated jump condition becomes

$$[\![\Lambda_{\varepsilon,b}(\nabla^{\varphi_0} \cdot v_0)]\!] = [\![(\bar{b}_0 \cdot \overline{\nabla})v_0 \cdot b_0]\!] - ([\![\bar{v}_0]\!] \cdot \overline{\nabla})q_0^- + \overline{D_t^+}(\sigma\mathcal{H}(\psi))|_{t=0} \text{ on } \Sigma.$$

Here and thereafter, we will repeatedly use $\overline{D_t^{\pm}}\psi = v_3^{\pm}$ on Σ and omit lots of redundant terms in order for simplicity of notations. For example, we will write $\mathcal{H}(\psi) \sim \overline{\Delta}\psi$, write $(1 - \overline{\Delta})$ to be $-\overline{\Delta}$, and omit the commutators between $\overline{D_t^+}$ and $\mathcal{H}, (1 - \overline{\Delta})$, the density function ρ. Indeed, later we will see that the concrete form of those omitted term is not important, and we just need to find out the major term as in [32, Appendix A]. Under this setting, we have

$$[\![\Lambda_{\varepsilon,b}(\nabla^{\varphi_0} \cdot v_0)]\!] \sim [\![(\bar{b}_0 \cdot \overline{\nabla})v_0 \cdot b_0]\!] - ([\![\bar{v}_0]\!] \cdot \overline{\nabla})q_0^- + \sigma\overline{\Delta}v_{03}^+ \text{ on } \Sigma. \tag{C.3}$$

For higher-order compatibility conditions, we invoke the wave equation for total pressure $q^{\pm}$ to get (cf. [32, Appendix A.1])

$$(\overline{D_t})^2 q = \Lambda_{\varepsilon,b}\Delta^{\varphi}q + \mathcal{M}_0(v,b) + \mathcal{N}_0(v,b) \quad \text{on } \Sigma, \tag{C.4}$$

where

$$\mathcal{M}_0(v,b) = -(\bar{b} \cdot \overline{\nabla})^2 q + (\bar{b} \cdot \overline{\nabla})^2 b \cdot b + \mathcal{R}_0(v,b), \quad \mathcal{N}_0(v,b) = \partial_i^{\varphi}v^j\partial_j^{\varphi}v^i - \partial_i^{\varphi}b^j\partial_j^{\varphi}b^i$$

and $\mathcal{R}_0(v,b)$ only contains the first-order derivatives of b, v with the form

$$\mathcal{R}_0(v,b) = P_0(b)((\partial^{i_1}v)(\partial^{i_2}v) + (\partial^{j_1}b)(\partial^{j_2}b))$$

where $P_0(b)$ is a polynomial of b only containing cubic and quadratic terms and $(i_1, i_2, j_1, j_2) = (0, 0, 1, 1)$ or $(1, 1, 0, 0)$. Taking substraction between the equation of q^+ and the equation of q^-, we get

$$[\![(\overline{D_t})^2 q]\!]|_{t=0} = [\![\Lambda_{\varepsilon,b_0}\Delta^{\varphi_0}q_0]\!] + [\![\mathcal{M}_0(v_0,b_0) + \mathcal{N}_0(v_0,b_0)]\!] \quad \text{on } \Sigma.$$

Then using $\overline{D_t^+} = \overline{D_t^-} + ([\![\bar{v}]\!] \cdot \overline{\nabla})$, we get

$$[\![(\overline{D_t})^2 q]\!]|_{t=0} = (\overline{D_t^+})^2(\sigma\mathcal{H}(\psi))|_{t=0} + \mathcal{T}_{[\![v]\!]}^2 q^-|_{t=0},$$

where each $\mathcal{T}_{[\![v]\!]}$ represents either of $\overline{D_t^-}$ and $([\![\bar{v}]\!] \cdot \overline{\nabla})$. So, the second-order compatibility condition is reformulated as

$$[\![\Lambda_{\varepsilon,b_0}\Delta^{\varphi_0}q_0]\!] = (\overline{D_t^+})^2(\sigma\mathcal{H}(\psi))|_{t=0} + \mathcal{T}_{[\![v]\!]}^2 q^-|_{t=0} - [\![\mathcal{M}_0(v_0,b_0) + \mathcal{N}_0(v_0,b_0)]\!]$$
$$\sim -\sigma\overline{\Delta}\partial_3 q_0^+ + \sigma\overline{\Delta}(\bar{b}^+ \cdot \overline{\nabla})b_{03}^+ + \mathcal{T}_{[\![v]\!]}^2 q^-|_{t=0} - [\![\mathcal{M}_0(v_0,b_0) + \mathcal{N}_0(v_0,b_0)]\!] \quad \text{on } \Sigma. \tag{C.5}$$

Taking one more material derivative in the wave equation and again use the continuity equation, we get

$$(\overline{D_t})^3 q \sim -\Lambda_{\varepsilon,b}^2\Delta^{\varphi}(\nabla^{\varphi} \cdot v) + \varepsilon^{-2}(\bar{b} \cdot \overline{\nabla})^2(\nabla^{\varphi} \cdot v) + \mathcal{M}_1(v,b,q) + \mathcal{N}_1(v,b,q) \tag{C.6}$$

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where the concrete form of $\mathcal{M}_1, \mathcal{N}_1$ will be specified later. Recursively, after long and tedious calculations (cf. [32, (A.4)-(A.7)]), we find that the time-differentiated wave equation (restricted on $\{t = 0\} \times \Sigma$) can be expressed as

$$m = 2r + 1, \quad -\Lambda_{\varepsilon,b_0}^{r+1}(\Delta^{\varphi_0})^r(\nabla^{\varphi_0} \cdot v_0) = (\overline{D_t})^{2r+1}q + \sum_{j=0}^{r}(\Delta^{\varphi_0})^j(\mathcal{M}_{2r-1-2j}(v_0, b_0, q_0) + \mathcal{N}_{2r-1-2j}(v_0, b_0, q_0)) \text{ on } \Sigma, \qquad \text{(C.7)}$$

$$m = 2r, \quad \Lambda_{\varepsilon,b_0}^{r}(\Delta^{\varphi_0})^r q_0 = (\overline{D_t})^{2r}q + \sum_{j=0}^{r-1}(\Delta^{\varphi_0})^j(\mathcal{M}_{2r-2-2j}(v_0, b_0, q_0) + \mathcal{N}_{2r-2-2j}(v_0, b_0, q_0)) \text{ on } \Sigma, \qquad \text{(C.8)}$$

where $\mathcal{M}_{-1}(v_0, b_0) := -(\bar{b}_0 \cdot \overline{\nabla})v_0 \cdot b_0$ and $\mathcal{N}_{-1} := 0$, and for $r \geq 1$ we define

$$m = 2r - 1, \quad \mathcal{M}_{2r-1}(v_0, b_0, q_0) = (\bar{b}_0 \cdot \overline{\nabla})^2(\Delta^{\varphi_0})^{r-1}(\nabla^{\varphi_0} \cdot v_0) + \sum_{l=2}^{r+1}\underbrace{b_0^{i_1} \cdots b_0^{i_{2l}}(\nabla^{2r+1}v_0)}_{<2^l \text{ terms}} + \mathcal{R}_{2r-1}(v_0, b_0, q_0), \qquad \text{(C.9)}$$

$$m = 2r, \quad \mathcal{M}_{2r}(v_0, b_0, q_0) = -(\bar{b}_0 \cdot \overline{\nabla})^2(\Delta^{\varphi_0})^r q_0 + \mathcal{R}_{2r}(v_0, b_0, q_0),$$

$$+ \sum_{l=2}^{r+1}\underbrace{(\bar{b}_0 \cdot \overline{\nabla})^{r+2}(\nabla^r b_0)b_0^{i_1} \cdots b_0^{i_{2l}} + (\bar{b}_0 \cdot \overline{\nabla})^2(\nabla^{2r}q_0)b_0^{j_1} \cdots b_0^{j_{2l}}}_{<2^l \text{ terms}}; \qquad \text{(C.10)}$$

and the term $\mathcal{R}_m$, where every top-order term has $(m + 1)$-th order derivative, has the following form

$$\mathcal{R}_m(v_0, b_0, q_0) = P_k(b_0)\left(C_{i_1\cdots i_p, j_1\cdots j_n, k_1\cdots k_l}^m(\nabla^{i_1}v_0)\cdots(\nabla^{i_p}v_0)(\nabla^{j_1}b_0)\cdots(\nabla^{j_n}b_0)(\nabla^{k_1}q_0)\cdots(\nabla^{k_l}q_0)\right),$$

where ∇ may represent either of ∇^{φ_0} or ∂, and $P_k(\cdot)$ is a polynomial of its arguments and the lowest power is 4 and the indices above satisfy

$$1 \leq i_1, \cdots, i_p, j_1, \cdots, j_n \leq k + 1, 0 \leq k_1, \cdots, k_l \leq m + 1,$$
$$i_1 + \cdots + i_p + j_1 + \cdots + j_n + k_1 + \cdots + k_l = m + 1.$$

The term $\mathcal{N}_m(v_0, b_0, q_0)$ has the following form

$$\mathcal{N}_m(v_0, b_0, q_0) = P_{m,1}(b_0)(\nabla^{1+2\lfloor\frac{m}{2}\rfloor}v_0)(\nabla v_0) + P_{m,2}(b_0)(\nabla^{2\lceil\frac{m}{2}\rceil}q_0)(\nabla v_0) + P_{k,0}(b_0)(\nabla^{m+1}b_0)(\nabla v_0)$$

$$+ P_m'(b_0)D_{i_1\cdots i_p, j_1\cdots j_n, k_1\cdots k_l}^m\left((\nabla^{i_1}v_0)\cdots(\nabla^{i_p}v_0)(\nabla^{j_1}b_0)\cdots(\nabla^{j_n}b_0)(\nabla^{k_1}q_0)\cdots(\nabla^{k_l}q_0)\right), \qquad \text{(C.11)}$$

where $P_{m,1}(\cdot), P_{m,2}(\cdot), P_m'(\cdot)$ are polynomials of their arguments and $P_{m,0}(\cdot)$ is a polynomial of its arguments and the lowest power is 2. The indices above satisfy

$$1 \leq i_1, \cdots, i_p, j_1, \cdots, j_n \leq k, 0 \leq k_1, \cdots, k_l \leq m,$$
$$i_1 + \cdots + i_p + j_1 + \cdots + j_n + k_1 + \cdots + k_l = m + 1.$$

Next we take the difference between the equations (C.7)-(C.8) in Ω^+ and those in Ω^- and restrict the equation on $\{t = 0\} \times \Sigma$ to get the jump condition in the m-th order compatibility conditions

$$m = 2r + 1, \quad -\left[\!\left[\Lambda_{\varepsilon,b_0}^{r+1}(\Delta^{\varphi_0})^r(\nabla^{\varphi_0} \cdot v_0)\right]\!\right] = \left[\!\left[(\overline{D_t})^{2r+1}q\right]\!\right] + \sum_{j=0}^{r}\left[\!\left[(\Delta^{\varphi_0})^j(\mathcal{M}_{2r-1-2j}(v_0, b_0, q_0) + \mathcal{N}_{2r-1-2j}(v_0, b_0, q_0))\right]\!\right] \text{ on } \Sigma, \quad \text{(C.12)}$$

$$m = 2r, \quad \left[\!\left[\Lambda_{\varepsilon,b_0}^{r}(\Delta^{\varphi_0})^r q_0\right]\!\right] = \left[\!\left[(\overline{D_t})^{2r}q\right]\!\right] + \sum_{j=0}^{r-1}\left[\!\left[(\Delta^{\varphi_0})^j(\mathcal{M}_{2r-2-2j}(v_0, b_0, q_0) + \mathcal{N}_{2r-2-2j}(v_0, b_0, q_0))\right]\!\right] \text{ on } \Sigma. \qquad \text{(C.13)}$$

Then using $\overline{D_t^+} = \overline{D_t^-} + (\llbracket \bar{v} \rrbracket \cdot \overline{\nabla})$, we get

$$\left[\!\left[(\overline{D_t})^m q\right]\!\right] = (\overline{D_t^+})^m \llbracket q \rrbracket + \mathcal{T}_{\llbracket \bar{v} \rrbracket}^m q^-|_{t=0},$$

where each $\mathcal{T}_{\llbracket \bar{v} \rrbracket}$ represents either $(\overline{D_t^-})$ or $(\llbracket \bar{v} \rrbracket \cdot \overline{\nabla})$. Using the jump condition for $\llbracket q \rrbracket$, we have

$$m = 2r : (\overline{D_t^+})^{2r} \llbracket q \rrbracket \sim \sigma \overline{\Delta}(\overline{D_t^+})^{2r-1}v_3^+ \sim \sigma \Lambda_{\varepsilon,b_0}^{r-1}\overline{\Delta}(\Delta^{\varphi_0})^{r-1}\partial_3 q_0^+ + \sigma \overline{\Delta}\mathcal{S}_{2r-1}(v_0^+, b_0^+, q_0^+) \qquad \text{(C.14)}$$

$$m = 2r + 1 : (\overline{D_t^+})^{2r+1} \llbracket q \rrbracket \sim \sigma \overline{\Delta}(\overline{D_t^+})^{2r}v_3^+ \sim -\sigma \Lambda_{\varepsilon,b_0}^{r}\overline{\Delta}(\Delta^{\varphi_0})^{r-1}\partial_3(\nabla^{\varphi_0} \cdot v_0^+) + \sigma \overline{\Delta}\mathcal{S}_{2r}(v_0^+, b_0^+, q_0^+) \qquad \text{(C.15)}$$

where the leading-order terms in $\mathcal{S}_m$ are

$$\mathcal{S}_{2r-1} \overset{L}{=} (\Lambda_{\varepsilon,b_0})^{r-2}(\bar{b}_0 \cdot \overline{\nabla})^2(\Delta^{\varphi_0})^{r-2}\partial_3 q_0^+, \quad \mathcal{S}_{2r} \overset{L}{=} -(\Lambda_{\varepsilon,b_0})^{r-1}(\bar{b}_0 \cdot \overline{\nabla})^2(\Delta^{\varphi_0})^{r-2}\partial_3(\nabla^{\varphi_0} \cdot v_0^+). \tag{C.16}$$

Thus, the compatibility conditions for the original current-vortex sheets system (1.30) are reformulated as

$$m = 2r + 1, \quad -\left[\!\left[\Lambda_{\varepsilon,b_0}^{r+1}(\Delta^{\varphi_0})^r(\nabla^{\varphi_0} \cdot v_0)\right]\!\right] \sim \sum_{j=0}^{r}\left[\!\left[(\Delta^{\varphi_0})^j(\mathcal{M}_{2r-1-2j}(v_0, b_0, q_0) + \mathcal{N}_{2r-1-2j}(v_0, b_0, q_0))\right]\!\right] \tag{C.17}$$

$$+ \mathcal{T}_{[\![\bar{v}]\!]}^{2r+1}q^-|_{t=0} - \sigma\Lambda_{\varepsilon,b_0}^r\overline{\Delta}(\Delta^{\varphi_0})^{r-1}\partial_3(\nabla^{\varphi_0} \cdot v_0^+) + \sigma\overline{\Delta}\mathcal{S}_{2r}(v_0^+, b_0^+, q_0^+) \quad \text{on } \Sigma,$$

$$m = 2r, \quad \left[\!\left[\Lambda_{\varepsilon,b_0}^r(\Delta^{\varphi_0})^r q_0\right]\!\right] \sim \sum_{j=0}^{r-1}\left[\!\left[(\Delta^{\varphi_0})^j(\mathcal{M}_{2r-2-2j}(v_0, b_0, q_0) + \mathcal{N}_{2r-2-2j}(v_0, b_0, q_0))\right]\!\right] \tag{C.18}$$

$$+ \mathcal{T}_{[\![\bar{v}]\!]}^{2r}q^-|_{t=0} + \sigma\Lambda_{\varepsilon,b_0}^{r-1}\overline{\Delta}(\Delta^{\varphi_0})^{r-1}\partial_3 q_0^+ + \sigma\overline{\Delta}\mathcal{S}_{2r-1}(v_0^+, b_0^+, q_0^+)) \quad \text{on } \Sigma.$$

Note that the time-differentiated kinematic boundary condition is already implicitly used when deriving the above compatibility conditions. Similarly, the compatibility conditions for the κ-approximate problem (3.1) are reformulated as

$$m = 2r + 1, \quad -\left[\!\left[\Lambda_{\varepsilon,b_0}^{r+1}(\Delta^{\varphi_0})^r(\nabla^{\varphi_0} \cdot v_0^\kappa)\right]\!\right] \sim \sum_{j=0}^{r}\left[\!\left[(\Delta^{\varphi_0})^j(\mathcal{M}_{2r-1-2j}(v_0^\kappa, b_0^\kappa, q_0^\kappa) + \mathcal{N}_{2r-1-2j}(v_0^\kappa, b_0^\kappa, q_0^\kappa))\right]\!\right]$$

$$+ \mathcal{T}_{[\![\bar{v}]\!]}^{2r+1}q^-|_{t=0} - \sigma\Lambda_{\varepsilon,b_0}^r\overline{\Delta}(\Delta^{\varphi_0})^{r-1}\partial_3(\nabla^{\varphi_0} \cdot v_0^{\kappa,+}) + \kappa\Lambda_{\varepsilon,b_0}^r\overline{\Delta}^2(\Delta^{\varphi_0})^{r-1}\partial_3(\nabla^{\varphi_0} \cdot v_0^{\kappa,+}) + \kappa\Lambda_{\varepsilon,b_0}^r\overline{\Delta}(\Delta^{\varphi_0})^r\partial_3 q_0^+$$

$$+ (\sigma\overline{\Delta} - \kappa\overline{\Delta}^2)\mathcal{S}_{2r}(v_0^{\kappa,+}, b_0^{\kappa,+}, q_0^{\kappa,+}) + \kappa\overline{\Delta}\mathcal{S}_{2r+1}(v_0^{\kappa,+}, b_0^{\kappa,+}, q_0^{\kappa,+}) \quad \text{on } \Sigma, \tag{C.19}$$

$$m = 2r, \quad \left[\!\left[\Lambda_{\varepsilon,b_0}^r(\Delta^{\varphi_0})^r q_0^\kappa\right]\!\right] \sim \sum_{j=0}^{r-1}\left[\!\left[(\Delta^{\varphi_0})^j(\mathcal{M}_{2r-2-2j}(v_0^\kappa, b_0^\kappa, q_0^\kappa) + \mathcal{N}_{2r-2-2j}(v_0^\kappa, b_0^\kappa, q_0^\kappa))\right]\!\right]$$

$$+ \mathcal{T}_{[\![\bar{v}]\!]}^{2r}q^-|_{t=0} + \sigma\Lambda_{\varepsilon,b_0}^{r-1}\overline{\Delta}(\Delta^{\varphi_0})^{r-1}\partial_3 q_0^+ - \kappa\Lambda_{\varepsilon,b_0}^{r-1}\overline{\Delta}^2(\Delta^{\varphi_0})^{r-1}\partial_3 q_0^+ - \kappa\Lambda_{\varepsilon,b_0}^r\overline{\Delta}(\Delta^{\varphi_0})^{r-1}\partial_3(\nabla^{\varphi_0} \cdot v_0^+)$$

$$+ (\sigma\overline{\Delta} - \kappa\overline{\Delta}^2)\mathcal{S}_{2r-1}(v_0^{\kappa,+}, b_0^{\kappa,+}, q_0^{\kappa,+}) + \kappa\overline{\Delta}\mathcal{S}_{2r}(v_0^{\kappa,+}, b_0^{\kappa,+}, q_0^{\kappa,+}) \quad \text{on } \Sigma. \tag{C.20}$$

C.2 Construction of the converging initial data

Given initial data $(v_0^\pm, b_0^\pm, q_0^\pm, S_0^\pm, \psi_0)$ of (1.30) satisfying the compatibility conditions (C.17)-(C.18) up to 7-th order, we now construct the initial data $(v_0^{\kappa,\pm}, b_0^{\kappa,\pm}, q_0^{\kappa,\pm}, S_0^{\kappa,\pm}, \psi_0^\kappa)$ to (3.1) satisfying the compatibility conditions (C.19)-(C.20) up to 7-th order that converge to the given data as $\kappa \to 0_+$. To do this, we just need to **equally distribute the κ-term to the solution in Ω^+ and the solution in Ω^-**.

C.2.1 Recover the 0-th order and the 1-st order compatibility conditions

First, we pick $b_0^{\kappa,\pm} = b_0^\pm$, $\psi_0^\kappa = \psi_0$. We define $\partial_t\psi^\kappa|_{t=0} := v_0^\pm \cdot N_0$ and $\partial_t b^\pm|_{t=0} = (b_0^\pm \cdot \nabla^{\varphi_0})v_0^\pm - b_0^\pm(\nabla^{\varphi_0} \cdot v_0^\pm)$ in $\Omega^\pm$. Then the constraints for the magnetic field are automatically satisfied. Now, we construct $q_0^{(0)}$ such that $(v_0^\pm, b_0^\pm, q_0^{(0),\pm}, \psi_0)$ satisfies the 0-th order compatibility condition (C.20). The function $q_0^{(1),\pm}$ is set to be the solution to the poly-harmonic equation

$$\begin{cases} \Delta^2 q_0^{(0),\pm} = \Delta^2 q_0^\pm & \text{in } \Omega^\pm \\ q_0^{(0),\pm} = q_0^\pm \mp \frac{1}{2}\kappa\overline{\Delta}^2\psi_0 \pm \frac{1}{2}\kappa\overline{\Delta}(v_0^\pm \cdot N_0) & \text{on } \Sigma \\ \partial_3 q_0^{(0),\pm} = \partial_3 q_0^\pm & \text{on } \Sigma \\ \partial_3^j q_0^{(0),\pm} = \partial_3^j q_0^\pm, \ 0 \leq j \leq 1 & \text{on } \Sigma^\pm. \end{cases} \tag{C.21}$$

Then for $s \geq 4$, we have

$$\|q_0^{(0),\pm} - q_0^\pm\|_{s,\pm} \lesssim \kappa|\overline{\Delta}^2\psi_0|_{s-0.5} + \kappa|\overline{\Delta}(v_0^\pm \cdot N_0)|_{s-0.5} \to 0 \quad \text{as } \kappa \to 0.$$

With this $q_0^{(0)}$, we define $\partial_t^2\psi|_{t=0} = \partial_t(v^+ \cdot N)|_{t=0}$ via $(v_0^\pm, b_0^\pm, q_0^{(0),\pm}, \psi_0)$ on Σ. (Note that $\partial_t v \cdot N|_{t=0}$ already includes $\partial_3 q_0$. Only when we have $\partial_3 q_0^{(0),\pm} = \partial_3 q_0^\pm$ on Σ can we keep the jump condition $[\![\partial_t(v \cdot N)]\!] = 0$.) and also define the corresponding $\partial_t^2 b|_{t=0}$ in $\Omega^\pm$ via the evolution equation of b. Thus, the ∂_t-differentiated boundary constraint for $b \cdot N$ is also satisfied.

Now we introduce $v_0^{(0),\pm}$ such that $(v_0^{(0),\pm}, b_0^\pm, q_0^{(0),\pm}, \psi_0)$ satisfies the 1-st order compatibility condition (C.19). We define $\bar{v}_{0i}^{(0),\pm} = \bar{v}_{0i}^\pm$ for $i = 1, 2$ and define $v_{03}^{(0),\pm}$ via the following poly-harmonic equation

$$
\begin{cases}
\Delta^3 v_{03}^{(0),\pm} = \Delta^3 v_{03}^\pm & \text{in } \Omega^\pm \\
\Lambda_{\varepsilon,b_0}(\nabla^{\varphi_0} \cdot v_{03}^{(0),\pm}) = (\nabla^{\varphi_0} \cdot \Lambda_{\varepsilon,b_0} v_{03}^\pm) \mp \frac{1}{2}(\llbracket \bar{v}_0 \rrbracket \cdot \overline{\nabla})(q_0^{(0),\pm} - q_0^\pm) \mp \frac{\kappa}{2}\overline{\Delta}^2 v_{03}^+ \pm \frac{\kappa}{2}\overline{\Delta}\partial_3 q_0^{(0),+} & \text{on } \Sigma \\
v_{03}^{(0),\pm} = v_{03}^\pm, \quad \partial_3^2 v_{03}^{(0),\pm} = \partial_3^2 v_{03}^\pm & \text{on } \Sigma \\
\partial_3^j v_{03}^{(0),\pm} = \partial_3^j v_{03}^\pm, \quad 0 \le j \le 2 & \text{on } \Sigma^\pm.
\end{cases}
\tag{C.22}
$$

It is also straightforward to see the convergece for $s \ge 6$

$$
\|v_0^{(0),\pm} - v_0^\pm\|_{s,\pm} \lesssim |q_0^{(0),\pm} - q_0^\pm|_{s-0.5} + \kappa(|v_{03}^+|_{s+2.5} + |\partial_3 q_0^+|_{s+0.5}).
$$

C.2.2 Higher-order compatibility conditions

For $r \ge 1$, we can inductively define $q_0^{(r),\pm}$ such that $(v_0^{(r-1),\pm}, b_0^\pm, q_0^{(r),\pm}, \psi_0)$ satisfies the compatibility condition up to $2r$-th order

$$
\begin{cases}
\Delta^{2r+2} q_0^{(r),\pm} = \Delta^{2r+2} q_0^{(r-1),\pm} & \text{in } \Omega^\pm \\
\Lambda_{\varepsilon,b_0}^r (\Delta^{\varphi_0})^r q_0^{(r),\pm} = \Lambda_{\varepsilon,b_0}^r (\Delta^{\varphi_0})^r q_0^{(r-1),\pm} \\
\quad + \sum_{j=0}^{r-1}(\Delta^{\varphi_0})^j \Big((\mathcal{M}_{2r-2-2j} + \mathcal{N}_{2r-2-2j})(v_0^{(r-1),\pm}, b_0^\pm, q_0^{(r),\pm}) - (\mathcal{M}_{2r-2-2j} + \mathcal{N}_{2r-2-2j})(v_0^{(r-2),\pm}, b_0^\pm, q_0^{(r-1),\pm}) \Big) \\
\quad \pm \frac{1}{2}\Big((\mathcal{T}_{\llbracket \bar{v}^{(r-1)} \rrbracket}^{2r} q^{(r),-} - \mathcal{T}_{\llbracket \bar{v}^{(r-2)} \rrbracket}^{2r} q^{(r-1),-}) + \sigma \Lambda_{\varepsilon,b_0}^{r-1} \underbrace{\overline{\Delta}(\Delta^{\varphi_0})^{r-1}\partial_3(q_0^{(r),+} - q_0^{(r-1),+})}_{=0} \\
\qquad + \sigma\overline{\Delta}(\mathcal{S}_{2r-1}(v_0^{(r-1),+}, b_0^+, q_0^{(r),+}) - \mathcal{S}_{2r-1}(v_0^{(r-2),+}, b_0^+, q_0^{(r-1),+})) \Big) \\
\quad \mp \frac{\kappa}{2}\Big(\Lambda_{\varepsilon,b_0}^{r-1} \underbrace{\overline{\Delta}^2(\Delta^{\varphi_0})^{r-1}\partial_3(q_0^{(r),+} - q_0^{(r-1),+})}_{=0} - \overline{\Delta}\Lambda_{\varepsilon,b_0}^r (\Delta^{\varphi_0})^{r-1}\partial_3\nabla^{\varphi_0} \cdot (v_0^{(r-1),+} - v_0^{(r-2),+}) \Big) \\
\quad \mp \frac{\kappa}{2}\Big((\overline{\Delta}^2\mathcal{S}_{2r-1} - \overline{\Delta}\mathcal{S}_{2r})(v_0^{(r-1),+}, b_0^+, q_0^{(r),+}) - (\overline{\Delta}^2\mathcal{S}_{2r-1} - \overline{\Delta}\mathcal{S}_{2r})(v_0^{(r-2),+}, b_0^+, q_0^{(r-1),+}) \Big) & \text{on } \Sigma \\
\partial_3^j q_0^{(r),\pm} = \partial_3^j q_0^{(r-1),\pm}, \quad 0 \le j \le 2r+1, j \ne 2r & \text{on } \Sigma \\
\partial_3^j q_0^{(r),\pm} = \partial_3^j q_0^{(r-1),\pm}, \quad 0 \le j \le 2r+1 & \text{on } \Sigma^\pm,
\end{cases}
\tag{C.23}
$$

and define $\bar{v}_0^{(r),\pm} = \bar{v}_0^{(r-1),\pm}$ and $v_{03}^{(r),\pm}$ such that $(v_0^{(r),\pm}, b_0^\pm, q_0^{(r),\pm}, \psi_0)$ satisfies the compatibility condition up to $(2r+1)$-th order

$$
\begin{cases}
\Delta^{2r+3} v_{03}^{(r),\pm} = \Delta^{2r+3} v_{03}^{(r-1),\pm} & \text{in } \Omega^\pm \\
-\Lambda_{\varepsilon,b_0}^r (\Delta^{\varphi_0})^r (\nabla^{\varphi_0} \cdot v_0^{(r),\pm}) = -\Lambda_{\varepsilon,b_0}^r (\Delta^{\varphi_0})^r (\nabla^{\varphi_0} \cdot v_0^{(r-1),\pm}) \\
\quad + \sum_{j=0}^r (\Delta^{\varphi_0})^j \Big((\mathcal{M}_{2r-1-2j} + \mathcal{N}_{2r-1-2j})(v_0^{(r),\pm}, b_0^\pm, q_0^{(r),\pm}) - (\mathcal{M}_{2r-1-2j} + \mathcal{N}_{2r-1-2j})(v_0^{(r-1),\pm}, b_0^\pm, q_0^{(r-1),\pm}) \Big) \\
\quad \pm \frac{1}{2}\Big((\mathcal{T}_{\llbracket \bar{v}^{(r)} \rrbracket}^{2r+1} q^{(r),-} - \mathcal{T}_{\llbracket \bar{v}^{(r-1)} \rrbracket}^{2r} q^{(r-1),-}) - \sigma \Lambda_{\varepsilon,b_0}^{r-1} \underbrace{\overline{\Delta}(\Delta^{\varphi_0})^{r-1}\partial_3\nabla^{\varphi_0} \cdot (v_{03}^{(r),+} - v_{03}^{(r-1),+})}_{=0} \\
\qquad + \sigma\overline{\Delta}(\mathcal{S}_{2r}(v_0^{(r),+}, b_0^+, q_0^{(r),+}) - \mathcal{S}_{2r}(v_0^{(r-1),+}, b_0^+, q_0^{(r-1),+})) \Big) \\
\quad \pm \frac{\kappa}{2}\Big(\overline{\Delta}^2\Lambda_{\varepsilon,b_0}^{r-1} \underbrace{(\Delta^{\varphi_0})^{r-1}\partial_3\nabla^{\varphi_0} \cdot (v_0^{(r),+} - v_0^{(r-1),+})}_{=0} - \overline{\Delta}\Lambda_{\varepsilon,b_0}^r (\Delta^{\varphi_0})^{r-1}\partial_3(q_0^{(r),+} - q_0^{(r-1),+}) \Big) \\
\quad \mp \frac{\kappa}{2}\Big((\overline{\Delta}^2\mathcal{S}_{2r} - \overline{\Delta}\mathcal{S}_{2r+1})(v_0^{(r),+}, b_0^+, q_0^{(r),+}) - (\overline{\Delta}^2\mathcal{S}_{2r} - \overline{\Delta}\mathcal{S}_{2r+1})(v_0^{(r-1),+}, b_0^+, q_0^{(r-1),+}) \Big) & \text{on } \Sigma \\
\partial_3^j v_{03}^{(r),\pm} = \partial_3^j v_{03}^{(r-1),\pm}, \quad 0 \le j \le 2r+2, j \ne 2r+1 & \text{on } \Sigma \\
\partial_3^j v_{03}^{(r),\pm} = \partial_3^j v_{03}^{(r-1),\pm}, \quad 0 \le j \le 2r+2 & \text{on } \Sigma^\pm.
\end{cases}
\tag{C.24}
$$

Since we require the compatibility conditions up to 7-th order, we can stop at $r = 3$ and define $(v_0^{\kappa,\pm}, b_0^{\kappa,\pm}, q_0^{\kappa,\pm}, S_0^{\kappa,\pm}, \psi_0^\kappa)$ to be $(v_0^{(3),\pm}, b_0^\pm, q_0^{(3),\pm}, S_0^\pm, \psi_0)$. It is also straightforward to see the convergence after long and tedious calculations: For $s \ge 2 \times (2r+3) = 18$, we have the convergence as $\kappa \to 0$

$$
\left\|(v_0^{\kappa,\pm}, q_0^{\kappa,\pm}) - (v_0^\pm, q_0^\pm)\right\|_{s,\pm} \lesssim P(\|v_0^\pm, b_0^\pm, q_0^\pm, S_0^\pm\|_{s+1,\pm}) \left(\kappa|\psi_0|_{s+3.5} + \sum_{j=0}^r \kappa|(\Delta^{\varphi_0})^j v_0^\pm|_{s+1.5-2j} + \kappa|(\Delta^{\varphi_0})^{(j-1)+}\partial_3 q_0^\pm|_{s+0.5-2j} \right) \to 0,
$$

provided that the given initial data is sufficiently regular. Specifically, picking $s = 18$, the given data is required to satisfy $\|(v_0^\pm, b_0^\pm, q_0^\pm, S_0^\pm)\|_{20,\pm} + |\psi_0|_{21.5} < +\infty$. We may assume the given data belongs to C^∞-class for convenience.

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