中国科学技术大学2014–2015学年第一学期考试参考答案

考试科目:多元统计分析

考试时间:2015年1月16日下午14:30–16:30

一、

(1) $\hat{\mu} = \dfrac{\sum_{n=1}^{20} X_n}{20} \qquad \hat{\Sigma} = \dfrac{1}{20}\sum_{n=1}^{20}(X_n - \hat{\mu})(X_n - \hat{\mu})^T$

(2) $\boldsymbol{\chi}_6^2$

(3) $\bar{X} \ \sim \ N_6\left(\mu, \dfrac{\Sigma}{20}\right) \qquad \sqrt{n}(\bar{X} - \mu) \ \sim \ N_6(0, \Sigma)$

(4) 服从自由度为n-1的威沙特分布

二、

(1)

$$det(\lambda I - A) = \begin{vmatrix} \lambda - 9 & 2 \\ 2 & \lambda - 6 \end{vmatrix} = (\lambda - 5)(\lambda - 10)$$
$$\therefore \lambda_1 = 10, \quad \lambda_2 = 5$$

解$(A - \lambda_1 I)x = 0$得

$$e_1 = \big(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\big)^T$$

解$(A - \lambda_2 I)x = 0$得

$$e_2 = \big(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\big)^T$$

$$\therefore A = \lambda_1 e_1 e_1^T + \lambda_2 e_2 e_2^T$$

(2)

$$\begin{aligned} A^{-1} &= \lambda_1^{-1} e_1 e_1^T + \lambda_2^{-1} e_2 e_2^T \\ &= \begin{pmatrix} \frac{9}{50} & -\frac{1}{25} \\ -\frac{1}{25} & \frac{3}{25} \end{pmatrix} \end{aligned}$$

特征值为

$$\mu_1 = \lambda_1^{-1} = \frac{1}{10} \qquad \mu_2 = \lambda_2^{-1} = \frac{1}{5}$$

对应特征向量仍为

$$e_1=(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}})^T \qquad e_2=(-\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}})^T$$

三、

(1)

$$Y=C^TX=\begin{pmatrix}1&1&1\end{pmatrix}\begin{pmatrix}X_1\\X_2\\X_3\end{pmatrix} \qquad , \qquad \text{其中}C=\begin{pmatrix}1&1&1\end{pmatrix}^T$$

$$Y\ \sim\ N(C^T\mu,C^T\Sigma C)$$

即

$$Y\ \sim\ N(2,11)$$

(2)

$$\begin{pmatrix}X_1\\X_2+X_3\end{pmatrix}=AX \qquad , \qquad \text{其中}A=\begin{pmatrix}1&0&0\\0&1&1\end{pmatrix}$$

$$\therefore\begin{pmatrix}X_1\\X_2+X_3\end{pmatrix}\ \sim\ N(A\mu,A\Sigma A^T)$$

其中

$$A\Sigma A^T=\begin{pmatrix}4&0\\0&7\end{pmatrix}$$

$\therefore$相互独立

(3)

$$\begin{pmatrix}X_1\\X_2\end{pmatrix}\ \sim\ N\left(\begin{pmatrix}1\\7\end{pmatrix},\begin{pmatrix}4&1\\1&5\end{pmatrix}\right)$$

$$\therefore X_1\Big|X_2=0.5\ \sim\ N\left(\frac{13}{10},\frac{19}{5}\right)$$

(4)

$$E\big[X_1\Big|(X_2=0.5,X_3=3)\big]=1+\begin{pmatrix}1&-1\end{pmatrix}\begin{pmatrix}0.2&0\\0&0.5\end{pmatrix}\left[\begin{pmatrix}0.5\\3\end{pmatrix}-\begin{pmatrix}-1\\2\end{pmatrix}\right]=-\frac{1}{5}$$

$$Var\big[X_1\Big|(X_2=0.5,X_3=3)\big]=4-\begin{pmatrix}1&-1\end{pmatrix}\begin{pmatrix}0.2&0\\0&0.5\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}=\frac{33}{10}$$

$$\therefore X_1\Big|(X_2=0.5,X_3=3)\ \sim\ N\left(-\frac{1}{5},\frac{33}{10}\right)$$

(5)

$$\max_{||\alpha||=1}Var(\alpha^TX)=\max_{||\alpha||=1}\alpha^T\Sigma\alpha=\lambda_1(\Sigma)=5.700$$

四、

(1)

$$det(\lambda I-\Sigma)=\begin{vmatrix}\lambda-8 & -2\\ -2 & \lambda-5\end{vmatrix}=(\lambda-9)(\lambda-4)$$
$$\therefore \lambda_1=9 \qquad \lambda_2=4$$

解$(\Sigma-\lambda_1 I)x=0$得

$$e_1=(\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}})^T$$

$$\therefore Y_1=e_1^T X=\frac{2}{\sqrt{5}}X_1+\frac{1}{\sqrt{5}}X_2$$

贡献率为

$$\frac{\lambda_1}{\lambda_1+\lambda_2}=0.69$$

(2)

$$\Lambda=\sqrt{\lambda_1}e_1=\begin{pmatrix}\frac{6}{\sqrt{5}} & \frac{3}{\sqrt{5}}\end{pmatrix}^T$$
$$\Sigma-\Lambda\Lambda^T=\begin{pmatrix}8 & 2\\ 2 & 5\end{pmatrix}-\begin{pmatrix}\frac{36}{5} & \frac{18}{5}\\ \frac{18}{5} & \frac{9}{5}\end{pmatrix}=\begin{pmatrix}\frac{4}{5} & -\frac{8}{5}\\ -\frac{8}{5} & \frac{16}{5}\end{pmatrix}$$
$$\therefore \Psi=\begin{pmatrix}\frac{4}{5} & 0\\ 0 & \frac{16}{5}\end{pmatrix} \qquad E=\begin{pmatrix}0 & -\frac{8}{5}\\ -\frac{8}{5} & 0\end{pmatrix}$$

五、

(1)

$$n(\bar{x}-\mu_0)^T S^{-1}(\bar{x}-\mu_0)=18.54$$
$$\frac{(n-1)p}{n-p}F_{p,n-p}(\alpha)=\frac{29\times 2}{28}F_{2,28}(0.1)=5.18$$
$$n(\bar{x}-\mu_0)^T S^{-1}(\bar{x}-\mu_0)>\frac{(n-1)p}{n-p}F_{p,n-p}(\alpha)$$

$\therefore$在显著性水平$\alpha=0.1$下拒绝H_0

(2)

μ的90%置信椭球为

$$n(\bar{x}-\mu)^T S^{-1}(\bar{x}-\mu)\le\frac{(n-1)p}{n-p}F_{p,n-p}(\alpha)=5.18$$

S的特征值为

$$\lambda_1 = 9 \qquad \lambda_2 = 4$$

则长短半轴的长度分别为

$$\sqrt{\lambda_1}\sqrt{\frac{(n-1)p}{n-p}F_{p,n-p}(\alpha)} = 6.83$$
$$\sqrt{\lambda_2}\sqrt{\frac{(n-1)p}{n-p}F_{p,n-p}(\alpha)} = 4.55$$

六、

$$\Sigma_{11}^{-\frac{1}{2}} = \begin{pmatrix} \frac{1}{8} & 0 \\ 0 & 1 \end{pmatrix} \qquad \Sigma_{22}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{64} \end{pmatrix}$$
$$\Sigma_{11}^{-\frac{1}{2}}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-\frac{1}{2}} = \begin{pmatrix} 0 & 0 \\ 0 & 0.36 \end{pmatrix}$$
$$\lambda_1 = 0.36 \qquad \lambda_2 = 0$$
$$e_1 = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$$
$$f_1 \propto \Sigma_{22}^{-\frac{1}{2}}\Sigma_{21}\Sigma_{11}^{-\frac{1}{2}}e_1$$
$$\therefore f_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$$

所以第一典型相关系数及相应的典型相关变量为

$$\therefore U_1 = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{8} & 0 \\ 0 & 1 \end{pmatrix}X^{(1)} = X_2^{(1)}$$
$$V_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{8} \end{pmatrix}X^{(2)} = X_1^{(2)}$$
$$Corr(U_1, V_1) = \sqrt{\lambda_1} = 0.6$$